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A class of Drinfeld doubles that are ribbon algebras ☆

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Abstract

Andruskiewitsch and Schneider classify a large class of pointed Hopf algebras with abelian coradical. The Drinfeld double of each such Hopf algebra is investigated. The Drinfeld doubles of a family of Hopf algebras from the above classification are ribbon Hopf algebras.

Keywords: Pointed Hopf algebras; Drinfeld double; Ribbon elements

Introduction

Quasitriangular Hopf algebras have an universal R-matrix which is a solution of the Yang–Baxter equation and their modules can be used to determine invariants of braids, knots and links. Drinfeld's quantum double construction gives a method to produce a quasitriangular Hopf algebra from a Hopf algebra and its dual.

The concept of ribbon categories was introduced by Joyal and Street. Their definition requires the notion of duality and provides isotopy invariants of framed links. Through their representations, ribbon Hopf algebras give rise to ribbon categories. They were introduced by Turaev and Reshetikhin in [17] who also showed that the quantum groups of Drinfeld and Jimbo are ribbon algebras. A ribbon Hopf algebra is a quasitriangular Hopf algebra which possesses an invertible central element known as the ribbon element.

Kauffman and Radford [12] have shown that the Drinfeld double $D(A_l)$ of a Taft algebra A_l (of dimension l^2) has a ribbon element if and only if l is odd. The ribbon element of $D(A_l)$ for

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l odd, provides an important invariant of 3-manifolds (see [10]). In [12] the authors also gave a criterion for a general Drinfeld double to possess a ribbon element. Benkart and Witherspoon investigated the structure of two parameter quantum groups of sl_n and gl_n [6]. In [7] they have shown that the restricted two parameter quantum groups $u_{r,s}(sl_n)$ are Drinfeld doubles of certain pointed Hopf algebras and possess ribbon elements under certain compatibility conditions between the parameters r and s. Gelaki and Westreich in [9] determined when Lusztig's small quantum group $U_q(sl_n)'$ is ribbon, unimodular and factorizable.

In this paper we provide a new class of Drinfeld doubles which possess ribbon elements. They are the Drinfeld doubles of a family of pointed Hopf algebras constructed by Andruskiewitsch and Schneider in [2]. The pointed Hopf algebras from [2] are liftings of Radford's biproducts of Nichols algebras with group algebras. The Radford biproducts are their associated graded algebras with respect to the coradical filtration. Andruskiewitsch and Schneider [2] showed that under some restrictions on the group order, all finite dimensional pointed Hopf algebras having an abelian group of grouplike elements are this type of liftings. The definition by generators and relations of these pointed Hopf algebras is very similar to that of quantum groups and it includes Lusztig's small quantum groups.

If G is an abelian finite group and V a Yetter–Drinfeld module over kG with a braiding of finite Cartan type (see [4]) then let $A = kG \# B(V)$, where $B(V)$ is the Nichols algebra of V. We show that $D(A)$ is a ribbon Hopf algebra.

In Section 1 we present the construction of the finite dimensional pointed Hopf algebras with abelian coradical constructed in [2].

In Section 2 the dual Hopf algebra of such a pointed Hopf algebra is investigated. If there are no linking relations it is shown that the dual Hopf algebra contains a subalgebra isomorphic to a Nichols algebra. A pointed Hopf algebra whose root vectors are nilpotent is called a Hopf algebra of *nilpotent type*. In the situation of a Hopf algebra of nilpotent type and no linking relations the structure of the dual algebra is completely determined in this section. This recovers a result from [5]. If the Hopf algebra is not of nilpotent type in the above sense, then its dual might not be anymore pointed and/or of nilpotent type. It will be interesting to completely determine the Hopf structure of the dual Hopf algebra in this situation. This would give new examples of Hopf algebras similar to that determined for rank one in [13].

Section 3 investigates the algebra structure of the Drinfeld double of a pointed Hopf algebra from Andruskiewitsch and Schneider's classification when there are no linking relations. In the nilpotent type situation, namely $A = kG \# B(V)$ the Drinfeld double structure of $D(A)$ is completely determined. They have the same defining relations as the restricted two parameter quantum groups but with more grouplike elements. As an example, (see 3.13) it is shown that the Drinfeld doubles of certain pointed Hopf algebras are quotients of two parameter quantum groups of type $U_{r,r^{-1}}(\mathbf{g})$ for a suitable choice of the root of unity r and of the semisimple Lie algebra $\mathbf{g}$. This can be regarded as a generalization of the fact that the Drinfeld double of a Taft algebra is a quotient of $U_{q,q^{-1}}(sl_2)$.

In Section 4 some notions about Hopf algebras in a braided category are recalled. The integrals and distinguished grouplike elements of the bosonization algebra are given. In [12] the authors gave a criterion to decide when a Drinfeld double is a ribbon Hopf algebra. Using this criterion a sufficient condition for the Drinfeld doubles of biproduct Hopf algebras to be ribbon is given.

Section 5 describes the integrals and the distinguished grouplike elements for the class of Hopf algebras of nilpotent type with no linking relations, as well as for their dual Hopf algebras. It is shown that the Drinfeld doubles corresponding to the pointed Hopf algebras of the form $B(V) \# kG$ where $V \in {}^G_G\mathcal{YD}$ are ribbon algebras.

Appendix A contains some quantum binomial formulae taken from [3] and a crucial lemma that is used in Section 2.

Throughout this paper we work over an algebraically closed field k of characteristic zero. For an abelian group G and an element $g \in G$ by $\langle g\rangle$ is denoted the cyclic subgroup of G generated by g, and by $\widehat{G}$ the group of linear characters of G. For $g \in G$, the element $\hat{\hat{g}} \in \widehat{\widehat{G}} \cong G$ is defined as $\hat{\hat{g}}(\chi) = \chi(g)$ for all $\chi \in \widehat{G}$.

The standard Hopf algebraic notations from [15] are used. For a Hopf algebra A, by A_{ad} is denoted the A-module which has the underlying vector space A and for which the action of A is given by the adjoint action $ad_A(x)(y) = \sum x_1 y S x_2$.

1. The pointed Hopf algebras with abelian coradical

Let $\mathcal{D} = (G, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta})$ be a datum of finite Cartan type associated to an abelian group G. That is $g_i \in G$ and $\chi_i \in \widehat{G}$ such that $\chi_i(g_i) \neq 1$ for all $1 \leqslant i \leqslant \theta$ and the Cartan condition

$$\chi_j(g_i)\chi_i(g_j) = \chi_i(g_i)^{a_{ij}}$$

for all $1 \leqslant i, j \leqslant \theta$. The matrix $(a_{ij})_{1\leqslant i,j\leqslant\theta}$ is a Cartan matrix of finite type. Let $\Pi = \{\alpha_1, \dots, \alpha_\theta\}$ be a set of simple roots for the Cartan matrix $(a_{ij})_{1\leqslant i,j\leqslant\theta}$ and let Φ be the root system corresponding to Π. Let also Φ^+ be the set of positive roots of the root system Φ. For $\alpha_i, \alpha_j \in \Pi$ one writes $i \sim j$ if the corresponding nodes in the Dynkin diagram are connected. Let $\lambda = (\lambda_{ij})_{1\leqslant i,j\leqslant\theta,\, i\nsim j}$ be a set of linking parameters, that is $\lambda_{ij} \in k$ and

$$\lambda_{ij} = 0, \quad \text{if } g_i g_j = 1 \text{ or } \chi_i\chi_j \neq \epsilon.$$

Let V be a finite dimensional Yetter–Drinfeld module over the group algebra kG. Suppose V has a basis $(x_i)_{1\leqslant i\leqslant\theta}$ with $x_i \in V_{g_i}^{\chi_i}$, where $V_{g_i}^{\chi_i} := \{gv = \chi_i(g)v,\ \delta(v) = g_i \otimes v\}$ and δ is the comodule structure of V. The group G acts by automorphisms on the tensor algebra $T(V)$ where $g(x_i) = \chi_i(g)x_i$. The braided commutators $[x_i, y]_c = ad_c(x_i)(y)$ are defined by

$$ad_c(x_i)(y) = x_i y - g_i(y)x_i$$

for all $y \in T(V)$. The induced map $c: T(V) \otimes T(V) \to T(V) \otimes T(V)$ given by $c(x_i \otimes y) = g_i(y) \otimes x_i$ is a braiding and $T(V)$ becomes a braided Hopf algebra in the category of Yetter–Drinfeld modules.

Andruskiewitsch and Schneider [2] introduced the following infinite dimensional Hopf algebra $U(\mathcal{D}, \lambda)$ defined by the generators $g \in G$ and $x_1, \dots, x_\theta$. As an algebra, the relations in $U(\mathcal{D}, \lambda)$ are those of G and

$$g x_i g^{-1} = \chi_i(g)x_i,$$
$$ad_c(x_i)^{1-a_{ij}}(x_j) = 0 \quad (i \neq j,\ i \sim j),$$
$$ad_c(x_i)(x_j) = \lambda_{ij}(1 - g_i g_j) \quad (i < j,\ i \nsim j).$$

The coalgebra structure of $U(\mathcal{D}, \lambda)$ is given by

$$\Delta(g) = g \otimes g, \qquad \Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i$$

for all $g \in G$ and $1 \leqslant i \leqslant \theta$.

Recall that $ad_c(x_i)(y) = ad(x_i)(y)$ for all $y \in U(\mathcal{D}, \lambda)$.

Assume that the order N_i of $\chi_i(g_i)$ is odd for all i and is prime to 3 for all i in a connected component of type G_2. The order of $\chi_i(g_i)$ is constant in each connected component J; denote this common order by N_J or N_α if α is a positive root belonging to the component J.

For any $\alpha \in \Phi^+$, $\alpha = \sum_{i=1}^{\theta} n_i \alpha_i$, let $\mathrm{ht}(\alpha) = \sum_{i=1}^{\theta} n_i$. Put

$$g_\alpha = g_1^{n_1} \cdots g_\theta^{n_\theta} \quad \text{and} \quad \chi_\alpha = \chi_1^{n_1} \cdots \chi_\theta^{n_\theta}.$$

Let $(\mu_\alpha)_{\alpha \in \Phi^+}$ be a system of root vector parameters, this means that $\mu_\alpha \in k$ and

$$\mu_\alpha = 0 \quad \text{if } g_\alpha^{N_\alpha} = 1 \text{ or } \chi_\alpha^{N_\alpha} \neq \epsilon.$$

Consider $(x_\alpha)_{\alpha \in \Phi^+}$ the root vectors corresponding to the positive roots $\alpha \in \Phi^+$. They are iterated braided commutators of x_i [2].

The finite dimensional Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ is the quotient of $U(\mathcal{D}, \lambda)$ by the Hopf ideal generated by

$$x_\alpha^{N_\alpha} - u_\alpha(\mu) \quad \left(\alpha \in \Phi^+\right)$$

where the elements $u_\alpha(\mu) \in kG$ are defined in [2]. It will be later used the fact that $u_\alpha(\mu)$ are central in $u(\mathcal{D}, \lambda, \mu)$ and they lie in the augmented ideal generated by $g_i^{N_i} - 1$ (see [2]).

We say that $A = u(\mathcal{D}, \lambda, \mu)$ is of *nilpotent type* if $\mu_\alpha = 0$ for all $\alpha \in \Phi^+$. It follows from [2] that in this situation $u_\alpha(\mu) = 0$ for all $\alpha \in \Phi^+$ and we abbreviate $A = u(\mathcal{D}, \lambda)$.

Over a field of characteristic zero any pointed finite dimensional Hopf algebra with an abelian group G of grouplike elements such that the order of G is not divisible by primes less than 11 is isomorphic to some $u(\mathcal{D}, \lambda, \mu)$ [2].

1.1. PBW-bases of $U(\mathcal{D}, \lambda)$

Let $y_1, \ldots, y_p$ the ordering of $(x_\alpha)_{\alpha \in \Phi^+}$ corresponding to the convex ordering $\beta_1, \ldots, \beta_p$ of the positive roots. In the paper [2] it has been shown that $\{y_1^{u_1} \cdots y_p^{u_p} g \mid u_i \geqslant 0,\ g \in G\}$ form a PBW-basis of $U(\mathcal{D}, \lambda)$. The images of y_i in the quotient $u(\mathcal{D}, \lambda, \mu)$ are also denoted by y_i. Then $\{y_1^{u_1} \cdots y_p^{u_p} g \mid 0 \leqslant u_i \leqslant N_{\beta_i} - 1,\ g \in G\}$ form a basis for $A = u(\mathcal{D}, \lambda, \mu)$.

1.2. Grading of $U(\mathcal{D}, 0)$

Let $\underline{e}_1, \ldots, \underline{e}_\theta$ be the standard basis of $\mathbb{Z}^\theta$. Then $U(\mathcal{D}, 0)$ is a $\mathbb{Z}^\theta$-graded Hopf algebra [2] where the degree of x_i is $\underline{e}_i$ and any group element $g \in G$ has degree zero. Given a homogeneous element x in $U(\mathcal{D}, \lambda)$ we denote its degree by $\dim(x)$.

If $\underline{u} \in \mathbb{N}^p$, let

$$y_{\underline{u}} = y_1^{u_1} \cdots y_p^{u_p},$$
$$g_{\underline{u}} = g_{\beta_1}^{u_1} \cdots g_{\beta_p}^{u_p},$$
$$\chi_{\underline{u}} = \chi_{\beta_1}^{u_1} \cdots \chi_{\beta_p}^{n_p}.$$

Note that if $\underline{u} = 0$ then $y_{\underline{u}} = g_{\underline{u}} = 1$ and $\chi_{\underline{u}} = \epsilon$.

For any positive root $\beta_i = \sum_{j=1}^{\theta} m_{ij}\alpha_j$ one has $\dim(y_i) = \sum_{j=1}^{\theta} m_{ij}\underline{e}_j > 0$ and if $\underline{u} \in \mathbb{N}^p$ then $\dim(y_{\underline{u}}) = \sum_{i=1}^{p} u_i \dim(y_i)$.

Since $gx_ig^{-1} = \chi_i(g)x_i$ one has that $gy_{\underline{u}}g^{-1} = \chi_{\underline{u}}(g)y_{\underline{u}}$ for all $\underline{u} \in \mathbb{N}^p$ and $g \in G$. From [19] one knows that if $1 \leqslant i < j \leqslant p$ then

$$y_jy_i = \chi_{\beta_i}(g_{\beta_j})y_iy_j + \sum_{I(i,j)} c(a_{i+1}, \ldots, a_{j-1})y_{i+1}^{a_{i+1}} \cdots y_{j-1}^{a_{j-1}}$$

where

$$I(i,j) = \left\{(a_{i+1}, \ldots, a_{j-1}) \in \mathbb{N}^{j-i-1} \;\middle|\; \sum_{s=i+1}^{j-1} a_s \dim(y_s) = \dim(y_i) + \dim(y_j)\right\}$$

and $c(a_{i+1}, \ldots, a_{j-1}) \in k$.

It follows that in $U(\mathcal{D}, 0)$ one has

$$y_{\underline{u}}y_{\underline{v}} = \sum_{\underline{w} \in \mathbb{N}^p} y_{\underline{w}}a_{\underline{w}}(\underline{u}, \underline{v}) \tag{1.1}$$

such that $a_{\underline{w}}(\underline{u}, \underline{v}) \in k$ and $\dim(y_{\underline{w}}) = \dim(y_{\underline{u}}) + \dim(y_{\underline{v}})$ whenever $a_{\underline{w}}(\underline{u}, \underline{v}) \neq 0$.

Let

$$\Delta(y_{\underline{u}}) = \sum_{\underline{v}, \underline{w} \in \mathbb{N}^p} y_{\underline{v}}c_{\underline{v},\underline{w}}^{\underline{u}} \otimes y_{\underline{w}}d_{\underline{v},\underline{w}}^{\underline{u}} \tag{1.2}$$

in $U(\mathcal{D}, \lambda)$ where $c_{\underline{v},\underline{w}}^{\underline{u}}, d_{\underline{v},\underline{w}}^{\underline{u}} \in k$. Since $U(\mathcal{D}, \lambda)$ is a $\mathbb{Z}^{\theta}$-graded coalgebra one has that $\dim(y_{\underline{u}}) = \dim(y_{\underline{v}}) + \dim(y_{\underline{w}})$ whenever both $c_{\underline{v},\underline{w}}^{\underline{u}}$ and $d_{\underline{v},\underline{w}}^{\underline{u}}$ are not zero.

1.3. The situation $A = u(\mathcal{D}, 0, \mu)$

Consider now $A = u(\mathcal{D}, 0, \mu)$ as a quotient of $U(\mathcal{D}, 0)$.

Then the multiplication relation (1.1) becomes

$$y_{\underline{u}}y_{\underline{v}} = \sum_{\underline{w} \in \mathbb{N}^p} y_{\underline{w}}a_{\underline{w}}(\underline{u}, \underline{v}) \tag{1.3}$$

where now $a_{\underline{w}}(\underline{u}, \underline{v}) \in kG$ and $\dim(y_{\underline{w}}) \leqslant \dim(y_{\underline{u}}) + \dim(y_{\underline{v}})$.

The comultiplication in $u(\mathcal{D}, 0, \mu)$ is given by

$$\Delta(y_{\underline{u}}) = \sum_{\underline{v}, \underline{w} \in \mathbb{N}^p} y_{\underline{v}}c_{\underline{v},\underline{w}}^{\underline{u}} \otimes y_{\underline{w}}d_{\underline{v},\underline{w}}^{\underline{u}} \tag{1.4}$$

where now $c_{\underline{v},\underline{w}}^{\underline{u}}, d_{\underline{v},\underline{w}}^{\underline{u}} \in kG$ and $\dim(y_{\underline{u}}) > \dim(y_{\underline{v}}) + \dim(y_{\underline{w}})$.

Let $\mathcal{I}$ be the ideal of kG generated by the elements $u_\alpha(\mu)$, $\alpha \in \Phi^+$. Then $\epsilon(\mathcal{I}) = 0$ and also $\chi_j(\mathcal{I}) = 0$ for any $1 \leqslant j \leqslant \theta$. Indeed, the elements $u_\alpha(\mu)$ lie in the augmented ideal generated by $g_i^{N_i} - 1$ (see [2]) therefore $\epsilon(\mathcal{I}) = 0$. On the other hand $\mu_i \neq 0$ implies that $\chi_i^{N_i} = \epsilon$ from the definition of μ_i. The equation $\chi_j(g_i)\chi_i(g_j) = \chi_i(g_i)^{a_{ij}}$ raised to the power N_i gives that $\chi_j(g_i)^{N_i} = 1$, thus $\chi_j(g_i^{N_i} - 1) = 0$.

If $\dim(y_{\underline{w}}) < \dim(y_{\underline{u}}) + \dim(y_{\underline{v}})$ in (1.3) then $a_{\underline{w}}(\underline{u}, \underline{v}) \in \mathcal{I}$ since the only way to get a smaller degree in a product of type $y_{i_1} y_{i_2} \cdots y_{i_s}$ is by using the root vector relations $x_\alpha^{N_\alpha} = u_\alpha(\mu)$. Then $\epsilon(a_{\underline{w}}(\underline{u}, \underline{v})) = 0$.

On the other hand if $\dim(y_{\underline{u}}) > \dim(y_{\underline{v}}) + \dim(y_{\underline{w}})$ in the comultiplication formula (1.4) then by the same argument as above one has that $c^{\underline{u}}_{\underline{v},\underline{w}} \in \mathcal{I}$ or $d^{\underline{u}}_{\underline{v},\underline{w}} \in \mathcal{I}$.

In this situation $\epsilon(c^{\underline{u}}_{\underline{v},\underline{w}}) = 0$ or $\epsilon(d^{\underline{u}}_{\underline{v},\underline{w}}) = 0$. Moreover, since $c^{\underline{u}}_{\underline{v},\underline{w}}$ or $d^{\underline{u}}_{\underline{v},\underline{w}}$ is in the ideal $\mathcal{I}$ of kG generated by $u_\alpha(\mu)$ one also has $\chi_i(c^{\underline{u}}_{\underline{v},\underline{w}}) = 0$ or $\chi_i(d^{\underline{u}}_{\underline{v},\underline{w}}) = 0$, for all $1 \leqslant i \leqslant \theta$.

1.4. The situation $A = u(\mathcal{D}, 0, 0)$

If A is of nilpotent type then the root vector relations are $x_\alpha^{N_\alpha} = 0$ and the degree is preserved by multiplication and comultiplication. Thus in this situation $A = u(\mathcal{D}, 0, 0)$ is also a $\mathbb{Z}^\theta$-graded Hopf algebra and $A \cong kG \# B(V)$ (see [2]).

2. The dual Hopf algebra

Let $A = u(\mathcal{D}, 0, \mu)$ be a Hopf algebra as above. For $1 \leqslant l \leqslant p$, let $\underline{f}_l \in \mathbb{N}^p$ be the element $(0, \ldots, 1, \ldots, 0)$ with 1 on the lth position. Consider the numbers m_i with $1 \leqslant m_i \leqslant p$ such that $\alpha_i = \beta_{m_i}$ for all $1 \leqslant i \leqslant \theta$. Thus $y_{m_i} = x_i = y_{\underline{f}_{m_i}}$. Notice that the elements of the previous section $\underline{e}_l$ are elements of $\mathbb{N}^\theta$ while the new introduced elements $\underline{f}_l$ are elements of $\mathbb{N}^p$.

Extend any linear characters $\chi \in \widehat{G}$ to an element of A^* such that $\chi(y_{\underline{u}} g) = 0$ if $\underline{u} \neq 0$. Consider also the following elements $\xi_i \in A^*$ defined by $\xi_i(y_{\underline{u}} g) = \delta_{\underline{u}, \underline{f}_{m_i}}$ for all $\underline{u} \in \mathbb{N}^p$. One has that $\xi_i(x_i a) = \epsilon(a)$ for all $a \in kG$.

The following lemma ([13], Lemma 1) will be used in the proof of the third relation of the next proposition.

Lemma 2.1. *Let H be a bialgebra over the field k and suppose that K is a sub-bialgebra of H with antipode. Suppose that $a \in G(K)$ and $x \in H \setminus K$ satisfy $xa = qax$ for some nonzero $q \in k$ and $\Delta(x) = x \otimes a + 1 \otimes x$. Let $V = K + Kx + \cdots$. Then*:

(1) *V is a free left K-module under left multiplication with basis $\{1, x, x^2, \ldots\}$ or $\{1, x, x^2, \ldots, x^{n-1}\}$ for some $n \geqslant 2$.*
(2) *Suppose that k has characteristic zero and V has left K-module basis $\{1, x, x^2, \ldots, x^{n-1}\}$ for some $n \geqslant 2$. Then q is a primitive nth root of unity and $x^n = c$ for some $c \in K$ which satisfies $\Delta(c) = c \otimes a^n + 1 \otimes c$. In particular $a \neq 1$.*
(3) *Suppose that $g \in G(K)$ and $z \in K + Kx$ satisfy $\Delta(z) = z \otimes g + 1 \otimes z$. If $z \notin K$ then $g = a$ and $z = \alpha x + b$ where $\alpha \in K$ is not zero and $b \in K$ satisfy $\Delta(z) = b \otimes a + 1 \otimes b$.*

Let $\overline{A}$ the subalgebra (with unit) of A generated by x_i, $1 \leqslant i \leqslant \theta$. Some algebra and coalgebra relations for A^* are given in the next proposition.

Proposition 2.2. *The following relations hold in* A^*:

(1) $\Delta(\xi_i) = \xi_i \otimes 1 + \chi_i \otimes \xi_i$.
(2) $\chi\xi_i\chi^{-1} = \chi(g_i)\xi_i$, *if* $\chi \in G(A^*)$. *In particular* $\chi_j\xi_i = \chi_j(g_i)\xi_i\chi_j$.
(3) $\xi_i^{N_i} = 0$.
(4) $ad(\xi_i)^{1-a_{ij}}(\xi_j) = 0$ *for all* $1 \leqslant i, j \leqslant \theta$.

Proof. (1) From definition of ξ_i it can be seen that $\xi_i(yg) = \xi_i(y)$ for all $y \in \overline{A}$ and $g \in G$. One has to show that $\xi_i(ab) = \xi_i(a)\epsilon(b) + \chi_i(a)\xi_i(b)$ for all $a, b \in A$. It is enough to check the last relation on the basis elements of A. Thus one has to show that:

$$\xi_i\big((y_{\underline{u}}g)(y_{\underline{v}}h)\big) = \xi_i(y_{\underline{u}}g)\epsilon(y_{\underline{v}}h) + \chi_i(y_{\underline{u}}g)\xi_i(y_{\underline{v}}h) \tag{2.3}$$

for all $\underline{u}, \underline{v} \in \mathbb{N}^p$ and all $g, h \in G$.

Since $gy_{\underline{v}} = \chi_{\underline{v}}(g)y_{\underline{v}}g$ it follows that $\xi_i((y_{\underline{u}}g)(y_{\underline{v}}h)) = \chi_{\underline{v}}(g)\xi_i(y_{\underline{u}}y_{\underline{v}}gh) = \chi_{\underline{v}}(g)\xi_i(y_{\underline{u}}y_{\underline{v}})$. On the other hand $\xi_i(y_{\underline{u}}g)\epsilon(y_{\underline{v}}h) + \chi_i(y_{\underline{u}}g)\xi_i(y_{\underline{v}}h) = \xi_i(y_{\underline{u}})\epsilon(y_{\underline{v}}) + \chi_i(y_{\underline{u}}g)\xi_i(y_{\underline{v}})$. Thus one has to show that:

$$\chi_{\underline{v}}(g)\xi_i(y_{\underline{u}}y_{\underline{v}}) = \xi_i(y_{\underline{u}})\epsilon(y_{\underline{v}}) + \chi_i(y_{\underline{u}}g)\xi_i(y_{\underline{v}}).$$

If $\underline{u} \neq 0$ and $\underline{v} \neq 0$ then $\dim(y_{\underline{u}}) > 0$ and $\dim(y_{\underline{v}}) > 0$. The right-hand side of the above equation is zero since $\epsilon(y_{\underline{v}}) = \chi_i(y_{\underline{u}}g) = 0$. On the other hand if $y_{\underline{u}}y_{\underline{v}}$ written with respect to the above basis of A contains a term of the type x_ia_i with $a_i \in kG$ then since $\dim(y_{\underline{u}}y_{\underline{v}}) \neq \dim(x_i)$ it follows from the discussion of the previous section that $\epsilon(a_i) = 0$ and then $\xi_i(x_ia_i) = 0$. Thus in this situation both terms of the above equation are zero. (Note that $\dim(y_{\underline{u}}y_{\underline{v}}) = \dim(x_i)$ implies that $\underline{u} = \underline{f}_{m_i}$ and $\underline{v} = 0$ or $\underline{u} = 0$ and $\underline{v} = \underline{f}_{m_i}$.)

Suppose now that $\underline{u} = 0$ which means that $y_{\underline{u}} = 1$. Eq. (2.3) becomes $\chi_{\underline{v}}(g)\xi_i(y_{\underline{v}}) = \chi_i(g)\xi_i(y_{\underline{v}})$. From the definition of ξ_i the only possibility for both terms to be nonzero is that of $\underline{v} = \underline{f}_i$ which means $y_{\underline{v}} = x_i$. In this situation the left-hand side is $\chi_{\underline{f}_i}(g)\xi_i(x_i) = \chi_i(g)$ which is the same value as the one of the right-hand side term.

The last possibility to discuss is when $\underline{v} = 0$ which means that $y_{\underline{v}} = 1$. Then Eq. (2.3) becomes $\xi_i(y_{\underline{u}}) = \xi_i(y_{\underline{u}})$.

(2) If $\chi \in G(A^*)$ then $\chi(u_\alpha(\mu)) = \chi(x_\alpha^{N_\alpha}) = \chi(x_\alpha)^{N_\alpha} = 0$. Since $\chi(u_\alpha(\mu)) = 0$ it follows that χ is zero on the ideal $\mathcal{I}$ of kG.

One has to prove that

$$\chi\xi_i(y_{\underline{u}}g) = \chi(g_i)\xi_i\chi(y_{\underline{u}}g) \tag{2.4}$$

for all $\underline{u} \in \mathbb{N}^p$ and $g \in G$.

As in the previous section, let

$$\Delta(y_{\underline{u}}) = \sum_{\underline{v}, \underline{w} \in \mathbb{N}^p} y_{\underline{v}}c^{\underline{u}}_{\underline{v},\underline{w}} \otimes y_{\underline{w}}d^{\underline{u}}_{\underline{v},\underline{w}}$$

where $c^{\underline{u}}_{\underline{v},\underline{w}}, d^{\underline{u}}_{\underline{v},\underline{w}} \in kG$. Then the first term of Eq. (2.4) becomes

$$\chi\xi_i(y_{\underline{u}}g) = \sum_{\underline{v},\underline{w}\in\mathbb{N}^p} \chi\left(y_{\underline{v}}c^{\underline{u}}_{\underline{v},\underline{w}}g\right)\xi_i\left(y_{\underline{w}}d^{\underline{u}}_{\underline{v},\underline{w}}g\right).$$

The only possibility for the right-hand side term of the previous equality to be nonzero is when $\dim(y_{\underline{v}}) = 0$ and $\dim(y_{\underline{w}}) = \underline{e}_i$ which means $\underline{v} = 0$ and $\underline{w} = \underline{f}_i$. If $\dim(y_{\underline{u}}) \neq \underline{e}_i$ then this is possible only by reduction via the root vector relations and as in the discussion from the previous section it follows that either $c^{\underline{u}}_{\underline{v},\underline{w}}$ or $d^{\underline{u}}_{\underline{v},\underline{w}}$ are in the ideal $\mathcal{I}$ generated by $u_\alpha(\mu)$. Then either $\chi(y_{\underline{v}}c^{\underline{u}}_{\underline{v},\underline{w}}g) = 0$ (if $c^{\underline{u}}_{\underline{v},\underline{w}} \in \mathcal{I}$) or $\xi_i(y_{\underline{w}}d^{\underline{u}}_{\underline{v},\underline{w}}g) = 0$ (if $d^{\underline{u}}_{\underline{v},\underline{w}} \in \mathcal{I}$). Thus if $\dim(y_{\underline{u}}) \neq \underline{e}_i$ the left-hand side of Eq. (2.4) is zero.

If $\dim(y_{\underline{u}}) = \underline{e}_i$, which is equivalent to $y_{\underline{u}} = x_i$, then $\Delta(x_ig) = x_ig \otimes g + g_ig \otimes x_ig$ and $\chi\xi_i(x_ig) = \chi(g_ig)$.

For the second term of Eq. (2.4) one has that

$$\chi(g_i)\xi_i\chi(y_{\underline{u}}g) = \chi(g_i)\sum_{\underline{v},\underline{w}\in\mathbb{N}^p} \xi_i\left(y_{\underline{v}}c^{\underline{u}}_{\underline{v},\underline{w}}g\right)\chi\left(y_{\underline{w}}d^{\underline{u}}_{\underline{v},\underline{w}}g\right).$$

A similar discussion shows that the only possibility for this term to be nonzero is when $\dim(y_{\underline{v}}) = \underline{e}_i$ and $\dim(y_{\underline{w}}) = 0$ which are equivalent to $\underline{v} = \underline{f}_{m_i}$ and $\underline{w} = 0$. If $\dim(y_{\underline{u}}) \neq \underline{e}_i$ then as in the discussion from the previous paragraph it follows that either $c^{\underline{u}}_{\underline{v},\underline{w}}$ or $d^{\underline{u}}_{\underline{v},\underline{w}}$ are in the ideal $\mathcal{I}$ generated by $u_\alpha(\mu)$ and then the value of the term is still 0.

If $y_{\underline{u}} = x_i$ then, using the formula for $\Delta(x_i)$, one has that $\chi(g_i)\xi_i\chi(x_ig) = \chi(g_ig)$, thus Eq. (2.4) is true in this situation too.

Computing $(\Delta \otimes \mathrm{Id})\Delta$ and $(\mathrm{Id} \otimes \Delta)\Delta$ for ξ_i in the formula from (1) it follows that $\Delta(\chi_i) = \chi_i \otimes \chi_i$, thus χ_i are grouplike elements of A^* for any $1 \leqslant i \leqslant \theta$. Then the second relation of (2) follows from the first one.

(3) Let H be the Hopf subalgebra of $A^{*\,\mathrm{coop}}$ generated by ξ_i and χ_i. One has $\chi_i\xi_i = \chi_i(g_i)\xi_i\chi_i$ and the order of $\chi_i(g_i)$ is N_i. The second statement of Lemma 2.1 applied for $K = k\langle\chi_i\rangle$ and $x = \xi_i$ gives that $\xi_i^{N_i} \in k\langle\chi_i\rangle$. But since $\xi_i^{N_i}(g) = 0$ for all $g \in G$ it follows that $\xi_i^{N_i} = 0$.

(4) Let $z = ad(\xi_i)^{1-a_{ij}}(\xi_j)$. Clearly $z(g) = 0$ for all $g \in G(A)$ since $\xi_j(g) = 0$. From Corollary A.6 from Appendix A one knows that z is a skew primitive element of A^*, that is

$$\Delta(z) = z \otimes 1 + \chi \otimes z$$

where $\chi = \chi_i^{1-a_{ij}}\chi_j$. Then $z(gy) = \chi(g)z(y)$ for all $g \in G$ and $y \in \overline{A}$. On the other hand $z(x_ix_j) = z(x_i)\epsilon(x_j) + \chi(x_i)z(x_j) = 0$ for all $1 \leqslant i, j \leqslant \theta$ and by induction on r one has $z(x_{i_1}x_{i_2}\cdots x_{i_r}) = 0$ for all $r \geqslant 2$.

In order to show that $z = 0$ it is enough to check that $z(x_m) = 0$ for all $1 \leqslant m \leqslant \theta$.

Let $f, f' \in A^*$. Then

$$\big(ad(f)(f')\big)(x) = \big(f_1f'S(f_2)\big)(x) = f_1(x_1)f'(x_2)f_2(Sx_3)$$

for all $x \in A$. Since

$$\Delta^2(x_m) = x_m \otimes 1 \otimes 1 + g_m \otimes x_m \otimes 1 + g_m \otimes g_m \otimes x_m$$

one has

$$\begin{aligned}\big(ad(f)(f')\big)(x_m) &= f_1(x_m)f'(1)f_2(1) + f_1(g_m)f'(x_m)f_2(1) + f_1(g_m)f'(g_m)f_2(Sx_m)\\ &= f(x_m)\epsilon(f') + f(g_m)f'(x_m) + f\big(g_mS(x_m)\big)f'(g_m).\end{aligned}$$

Suppose moreover that $f(g_m) = f'(g_m) = 0$ and $\epsilon(f') = 0$. Then $(ad(f)(f'))(x_m) = 0$. Clearly $f = \xi_i$ and $f' = ad(\xi_i)^{-a_{ij}}(\xi_j)$ satisfy the above conditions, thus $z(x_m) = 0$. □

Proposition 2.5. *Let $A = u(\mathcal{D}, 0, \mu)$ as above and H be the subgroup of G generated by the elements $\langle g_i^{N_i} \mid \mu_i \neq 0\rangle$. Then $G(A^*) = \widehat{G/H}$.*

Proof. If $\chi \in G(A^*)$ then relation $gx_ig^{-1} = \chi_i(g)x_i$ implies $\chi(x_i) = 0$ for all $1 \leqslant i \leqslant \theta$. Thus $\chi(\mu_i(g_i^{N_i} - 1)) = \chi(x_i^{N_i}) = 0$ and since $\mu_i \neq 0$ it follows that $\chi(g_i^{N_i}) = 1$ and $\chi \in \widehat{G/H}$. Conversely, suppose $\chi \in \widehat{G/H} \subset \widehat{G}$ and extend χ to an element in A^* as at the beginning of this section. The equation $\chi(ab) = \chi(a)\chi(b)$ will be verified on the basis elements of A. Suppose $a = y_{\underline{u}}g$ and $b = y_{\underline{v}}h$. If $\underline{u} \neq 0$ or $\underline{v} \neq 0$ then clearly $\chi(a)\chi(b) = 0$. On the other hand $\chi(ab) = \chi(y_{\underline{u}}y_{\underline{v}}gh)\chi_{\underline{v}}(g) = 0$ since the part of degree zero of the product $y_{\underline{u}}y_{\underline{v}}$ is in $\mathcal{I}$ and by its definition $\chi|_{\mathcal{I}} = 0$. If $\underline{u} = 0$ and $\underline{v} = 0$ then the equation $\chi(ab) = \chi(a)\chi(b)$ is satisfied since χ is a character of G. □

Let $\overline{A^*}$ the subalgebra of A^* generated by $(\xi_i)_{1\leqslant i\leqslant\theta}$. It follows that $\overline{A^*}$ is the Nichols algebra of the $\widehat{G}$-braided vector space W with a basis given by $Y_i \in W_{\chi_i}^{\hat{\hat{g}}_i}$. Similarly to the construction for A, for any $\alpha \in \Phi^+$ let Y_α be the corresponding iterated commutators of ξ_i. Denote these elements with $Y_1, \ldots, Y_p$ using the convex ordering of the positive roots. Clearly $Y_{m_i} = \xi_i$ for all $1 \leqslant i \leqslant \theta$.

Let $\widetilde{\mathcal{D}} = (\widehat{G}, (\chi_i)_{1\leqslant i\leqslant\theta}, (\hat{g}_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta})$. It can be verified that is a datum of finite Cartan type associated to the abelian group $\widehat{G}$.

Corollary 2.6. *If $A \cong u(\mathcal{D}, 0, 0)$ is a pointed Hopf algebra of nilpotent type then $A^* = u(\widetilde{\mathcal{D}}, 0, 0)$ is also a pointed Hopf algebra. A basis for A^* is given by $\{\chi Y_{\underline{u}} \mid \underline{u} \in \mathbb{N}^p,\ 0 \leqslant u_i \leqslant N_{\beta_i} - 1,\ \chi \in \widehat{G}\}$.*

Proof. If $\mu = 0$ and $\lambda = 0$ then A is a $\mathbb{Z}^\theta$-graded Hopf algebra and any $\chi \in \widehat{G}$ extended to A^* as in the beginning of this section becomes a grouplike element of A^*. The previous theorem implies that $A^* \cong B(W) \# k\widehat{G}$ and the basis description follows from [2]. □

Remark 2.7. In a recent paper [14] it was proved that all the liftings of $B(V) \# kG$ where G is an abelian group whose order has no prime divisors < 11 are monoidally Morita–Takeuchi equivalent and therefore cocycle deformations of $B(V) \# kG$. Thus their dual algebras are the same and Corollary 2.6 remains true for any lifting of $B(V) \# kG$. Thus if $A = u(\mathcal{D}, \lambda, \mu)$ then $A^* \cong u(\widetilde{\mathcal{D}}, 0, 0)$ as algebras for any λ and μ.

3. The Drinfeld double of A

Let $A = u(\mathcal{D}, 0, \mu)$ as in the previous section.

Proposition 3.1. *The following relations hold in $D(A)$:*

(1) $g\xi_i g^{-1} = \chi_i^{-1}(g)\xi_i$ *for all* $g \in G$.
(2) $g\gamma = \gamma g$ *for any* $g \in G$ *and* $\gamma \in \widehat{G}$.
(3) $x_i\xi_j = \xi_j x_i$ *for* $i \neq j$.
(4) $[x_i, \xi_i] = \chi_i - g_i$ *for all* $1 \leqslant i \leqslant \theta$.
(5) *If* $\gamma \in G(A^*)$ *then* $\gamma^{-1}x_i\gamma = \gamma(g_i)x_i$ *for all* $1 \leqslant i \leqslant \theta$.

Proof. One has that

$$af = \left(a_1 \rightharpoonup f \leftharpoonup S^{-1}a_3\right)a_2$$

for all $a \in A$ and $f \in A^*$. For the first formula notice that $g\xi_i = (g \rightharpoonup \xi_i \leftharpoonup g^{-1})g$ and $g \rightharpoonup \xi_i \leftharpoonup g^{-1} = \chi_i^{-1}(g)\xi_i$. Similarly $g\gamma = (g \rightharpoonup \gamma \leftharpoonup g^{-1})g$ and $g \rightharpoonup \gamma = \gamma(g)\gamma$ while $\gamma \leftharpoonup g^{-1} = \gamma(g^{-1})\gamma$. Thus the second formula is proved.

To prove relations (3) and (4) notice that

$$\Delta^2(x_i) = g_i \otimes x_i \otimes 1 + x_i \otimes 1 \otimes 1 + g_i \otimes g_i \otimes x_i.$$

Then $x_i f = (g_i \rightharpoonup f)x_i + x_i \rightharpoonup f + (g_i \rightharpoonup f \leftharpoonup S^{-1}x_i)g_i$, for all $f \in A^*$.

Since $S^{-1}x_i = -x_i g_i^{-1}$ this last formula becomes

$$x_i f = (g_i \rightharpoonup f)x_i + x_i \rightharpoonup f - \left(g_i \rightharpoonup f \leftharpoonup x_i \leftharpoonup g_i^{-1}\right)g_i. \tag{3.2}$$

If $f = \xi_j$ with $j \neq i$ then $g_i \rightharpoonup \xi_j = \xi_j$ and the first term of the above equality is $\xi_j x_i$. On the other hand the other two terms are zero since $x_i \rightharpoonup \xi_j = \xi_j \leftharpoonup x_i = 0$. Indeed $(x_i \rightharpoonup \xi_j)(y_{\underline{u}}g) = \xi_j(y_{\underline{u}}gx_i) = \chi_i(g)\xi_j(y_{\underline{u}}x_i g) = \chi_i(g)\xi_j(y_{\underline{u}}x_i)$. Since $i \neq j$ one has $\dim(y_{\underline{u}}x_i) \neq \dim(x_j)$. Then the product $y_{\underline{u}}x_i$ has a term of the type $x_j a_j$ with $a_j \in kG$ in its writing as linear combination of the standard basis (after putting all terms $x_j g$ together) only by using the root vector relations. Thus in this situation $a_j \in \mathcal{I}$ and $\epsilon(a_j) = 0$ which implies that $\xi_j(x_j a_j) = 0$. Similarly, $\xi_j \leftharpoonup x_i = 0$ and the third relation is proved.

For the next relation suppose that $f = \xi_i$. Then $g_i \rightharpoonup \xi_i = \xi_i$ and the first term of the above equality is $\xi_i x_i$. On the other hand $x_i \rightharpoonup \xi_i = \chi_i$ since $(x_i \rightharpoonup \xi_i)(y_{\underline{u}}g) = \xi_i(y_{\underline{u}}gx_i) = \chi_i(g)\xi_i(y_{\underline{u}}x_i g) = \chi_i(g)\xi_i(y_{\underline{u}}x_i)$ and if $\underline{u} \neq 0$ (which means that $\dim(y_{\underline{u}}) \neq 0$) then as before this term is zero. If $\underline{u} = 0$ which means $y_{\underline{u}} = 1$ then $(x_i \rightharpoonup \xi_i)(y_{\underline{u}}g) = (x_i \rightharpoonup \xi_i)(g) = \xi_i(gx_i) = \chi_i(g)$. Thus the second term of Eq. (3.2) is χ_i. The last term, $-(g_i \rightharpoonup \xi_i \leftharpoonup x_i \leftharpoonup g_i^{-1})g_i$ is equal to $-g_i$ since $g_i \rightharpoonup \xi_i = \xi_i$, $\xi_i \leftharpoonup x_i = \epsilon$ and $\epsilon \leftharpoonup g_i^{-1} = \epsilon$. The proof for $\xi_i \leftharpoonup x_i = \epsilon$ is similar to the one of $x_i \rightharpoonup \xi_i = \chi_i$. The proof of (4) is now complete.

For the last relation put $f = \gamma$ in (3.2). One has $g_i \leftharpoonup \gamma = \gamma(g_i)\gamma$. The other two terms are zero since $x_i \rightharpoonup \gamma = \gamma \leftharpoonup x_i = 0$. The proof of these facts is similar to the one in part (3). One uses that γ is zero on $\mathcal{I}$ since $\gamma \in G(A^*)$. $\square$

Let $A = u(\mathcal{D}, 0, 0)$ with $\mathcal{D} = (G, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta})$ a Cartan datum of finite type. Using Proposition 3.1 and formula (A.7) from Appendix A the following relations hold in $D(A)$:

$$x_i^{N_i} = \left(\xi_i \chi_i^{-1}\right)^{N_i} = 0, \tag{3.3}$$

$$(g\chi)x_i(g\chi)^{-1} = \langle \chi_i \hat{g_i}^{-1}, g\chi\rangle x_i, \tag{3.4}$$

$$(g\chi)\left(\xi_i \chi_i^{-1}\right)(g\chi)^{-1} = \langle \chi_i^{-1}\hat{g}_i, g\chi\rangle \xi_i, \tag{3.5}$$

$$ad(x_i)^{1-a_{ij}}(x_j) = 0, \tag{3.6}$$

$$ad\left(\xi_i \chi_i^{-1}\right)^{1-a_{ij}}\left(\xi_j \chi_j^{-1}\right) = 0, \tag{3.7}$$

$$ad(x_i)\left(\xi_i \chi_i^{-1}\right) = \left(1 - g_i \chi_i^{-1}\right), \tag{3.8}$$

$$\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \tag{3.9}$$

$$\Delta\left(\xi_i \chi_i^{-1}\right) = \chi_i^{-1} \otimes \xi_i \chi_i^{-1} + \xi_i \chi_i^{-1} \otimes 1. \tag{3.10}$$

To verify the relation (3.8) one has

$$ad(x_i)\left(\xi_i \chi_i^{-1}\right) = x_i \xi_i \chi_i^{-1} - g_i \xi_i \chi_i^{-1} g_i^{-1} x_i = (x_i \xi_i - \xi_i x_i)\chi_i^{-1} = \left(1 - g_i \chi_i^{-1}\right).$$

Consider $\mathcal{D}' = (G \times \widehat{G}, (a_i)_{1\leqslant i\leqslant 2\theta}, (\nu_i)_{1\leqslant i\leqslant 2\theta}, (b_{ij})_{1\leqslant i,j\leqslant 2\theta})$ where $a_i = g_i$, $a_{\theta+i} = \chi_i^{-1}$ and $\nu_i = \chi_i a_i^{-1}$, $\nu_{\theta+i} = \chi_i^{-1} a_i$ for all $1 \leqslant i \leqslant \theta$.

The matrix (b_{ij}) consists of two diagonal copies of the matrix a_{ij}. It can easily be verified that $\mathcal{D}'$ is also a datum of finite Cartan type associated to the abelian group $G \times \widehat{G}$. (The character group of $G \times \widehat{G}$ is identified with $\widehat{G} \times G$.)

Define the linking parameters λ given by

$$\lambda_{ij} = \begin{cases} 1, & j = i + \theta, \\ 0, & j \neq i + \theta. \end{cases}$$

If the generating variables of $U(\mathcal{D}', \lambda)$ are denoted by z_i then define

$$\phi : U(\mathcal{D}', \lambda) \to D(A)$$

by

$$\phi(g\chi) = g\chi, \qquad \phi(z_i) = x_i, \qquad \phi(z_{\theta+i}) = \xi_i \chi_i^{-1}$$

for all $g \in G$ and $\chi \in \widehat{G}$ and for all $1 \leqslant i \leqslant \theta$.

Relations (3.3)–(3.8) show that ϕ is a well defined algebra map and relations (3.9), (3.10) imply that ϕ is a Hopf algebra map. In the next corollary it is proved that ϕ induces an isomorphism of Hopf algebras $\phi : u(\mathcal{D}', \lambda, 0) \to D(A)$. (See also [5].)

Corollary 3.11. *Let $A = u(\mathcal{D}, 0, 0)$ be a pointed Hopf algebra with $\mathcal{D} = (G, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta})$ a Cartan datum of finite type.*

(1) *Let $\mathcal{D}' = (G \times \widehat{G}, (a_i)_{1\leqslant i\leqslant 2\theta}, (\nu_i)_{1\leqslant i\leqslant 2\theta}, (b_{ij})_{1\leqslant i,j\leqslant 2\theta})$ and λ defined as above. Then $D(A) \cong u(\mathcal{D}', \lambda, 0)$.*
(2) *The Drinfeld double $D(A)$ is generated by G, $\widehat{G}$, x_i, ξ_i ($i = \overline{1, \theta}$), with the defining relations given by those of A, A^* and the relations from Proposition* 3.1.

Proof. If A is of nilpotent type with no linking relations then $\widehat{G} = G(A^*)$ and the last relation of the Proposition 3.1 holds for any $\gamma \in \widehat{G}$. Then it follows from Corollary 2.6 and the PBW-basis description of A that as an algebra $D(A)$ is generated by $\{g, \chi, x_i, \xi_i, \mid g \in G,\ \chi \in \widehat{G}, 1 \leqslant i \leqslant \theta\}$. This implies that the above map ϕ is surjective and it factors through a map $\phi : u(\mathcal{D}', \lambda) \to D(A)$ also denoted by ϕ. Let s be the number of connected components of the Dynkin diagram of the Lie algebra g. Since $\dim u(\mathcal{D}', \lambda, 0) = |G|^2 \prod_{i=1}^{s} N_i^{2p_i} = \dim D(A)$ it follows that ψ is an isomorphism. $\square$

Let Γ be an abelian group $n \geqslant 1$, $K_i, L_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$ for all $1 \leqslant i \leqslant n$, and $(a_{ij})_{1 \leqslant i,j \leqslant n}$ a Cartan matrix of finite type. A *reduced datum of Cartan finite type* was defined in [16].

It consists of a datum $\mathcal{D}_{\mathrm{red}} = \mathcal{D}_{\mathrm{red}}(\Gamma, (L_i)_{1 \leqslant i \leqslant n}, (K_i)_{1 \leqslant i \leqslant n}, (\chi_i)_{1 \leqslant i \leqslant n}, (a_{ij})_{1 \leqslant i,j \leqslant n})$ such that:

$$\chi_j(K_i)\chi_i(K_j) = \chi_i(K_i)^{a_{ij}},$$

$$\chi_i(L_j) = \chi_j(K_i),$$

$$K_i L_i \neq 1, \quad \text{and} \quad \chi_i(K_i) \neq 1$$

for all $1 \leqslant i, j \leqslant n$.

Let $\mathcal{D}_{\mathrm{red}}$ be a reduced datum of finite Cartan type and X a Yetter–Drinfeld module over $k[\Gamma]$ with basis $x_1, \ldots, x_n, y_1, \ldots, y_n$ where $x_i \in X_{L_i}^{\chi_i^{-1}}$ and $y_i \in X_{K_i}^{\chi_i}$. Let $(l_i)_{1 \leqslant i \leqslant n}$ be a family of nonzero parameters in k.

Let Γ act on the free algebra $k\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle$ by $\gamma x_i = \chi_i^{-1}(\gamma)x_i$ and $\gamma y_i = \chi(\gamma) y_i$, for all $\gamma \in \Gamma$ and $1 \leqslant i \leqslant n$.

The Hopf algebra $U(\mathcal{D}_{\mathrm{red}}, l)$ is defined [16] as the quotient of the smash product $k\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle \# k[\Gamma]$ modulo the ideal generated by

$$ad_c(x_i)^{1-a_{ij}}(x_j) \quad \text{for all } 1 \leqslant i, j \leqslant n,\ i \neq j,$$

$$ad_c(y_i)^{1-a_{ij}}(y_j) \quad \text{for all } 1 \leqslant i, j \leqslant n,\ i \neq j$$

$$x_i y_j - \chi_j(L_i) y_j x_i - \delta_{ij} l_i (1 - K_i L_i) \quad \text{for all } 1 \leqslant i, j \leqslant n.$$

Example 3.12. This example shows that $D(A)$ is a quotient Hopf algebra of $U(\mathcal{D}_{\mathrm{red}}, l)$ whose representations were studied in [16].

Let $A = u(\mathcal{D}, 0, 0)$ with $\mathcal{D} = (G, (g_i)_{1 \leqslant i \leqslant \theta}, (\chi_i)_{1 \leqslant i \leqslant \theta}, (a_{ij})_{1 \leqslant i,j \leqslant \theta})$ a Cartan datum of finite type.

Let $\mathcal{D}_{\mathrm{red}} = \mathcal{D}_{\mathrm{red}}(\Gamma, (L_i)_{1 \leqslant i \leqslant \theta}, (K_i)_{1 \leqslant i \leqslant \theta}, (\vartheta_i)_{1 \leqslant i \leqslant \theta}, (a_{ij})_{1 \leqslant i,j \leqslant \theta})$ where $\Gamma = G \times \widehat{G}$, $L_i = g_i$, $K_i = \chi_i^{-1}$ and $\vartheta_i = \chi_i^{-1}\hat{g}_i$. ($\widehat{\Gamma}$ is again identified with $\widehat{G} \times G$.) Let $l_i = \lambda_{i,i+\theta} = 1$ for all $1 \leqslant i \leqslant n$. Then from [16], p. 27, it follows that $U(\mathcal{D}_{\mathrm{red}}, l) = U(\mathcal{D}', \lambda)$ where $U(\mathcal{D}', \lambda)$ was defined above. Thus $D(A)$ is a quotient of $U(\mathcal{D}_{\mathrm{red}}, l)$.

Example 3.13. In the next example we will show that certain Drinfeld doubles can be realized as quotients of the two parameter quantum groups $U_{r,r^{-1}}(\mathbf{g})$ for a suitable choice of the root of unity r and of the semisimple Lie algebra $\mathbf{g}$. This can be regarded as a generalization of the well known fact (for type A_1) that the Drinfeld double of a Taft algebra is a quotient of $U_{q,q^{-1}}(sl_2)$.

Let $C = (a_{ij})_{1 \leqslant i,j \leqslant \theta}$ be a Cartan matrix of finite type and $\mathbf{g}$ be the associated semisimple Lie algebra over $\mathbb{Q}$. Let $d_i \in \{1,2,3\}$ be a set of relatively prime positive integers such that $d_i a_{ij} = d_j a_{ji}$ for all $1 \leqslant i, j \leqslant \theta$. Let r, s be two numbers such that rs^{-1} is a root of unity of odd order N and prime with 3 if $\mathbf{g}$ has components of type G_2.

Let $\langle -,- \rangle$ be the Euler form of $\mathbf{g}$ which is the bilinear form on the root lattice Q defined by

$$\langle i, j \rangle := \langle \alpha_i, \alpha_j \rangle = \begin{cases} d_i a_{ij}, & i < j, \\ d_i, & i = j, \\ 0, & i > j. \end{cases}$$

To the semisimple Lie algebra $\mathbf{g}$ and the numbers r, s one can associate a two parameter quantum group $U := U_{r,s}(\mathbf{g})$ as in [11]. $U_{r,s}(\mathbf{g})$ is generated by $e_i, f_i, \omega_i^{\pm 1}, {\omega_i}'^{\pm 1}$ subject to the following relations:

$$\begin{aligned}
&(\mathrm{R1}) \quad \omega_i^{\pm 1}\omega_j^{\pm 1} = \omega_j^{\pm 1}\omega_i^{\pm 1}, \qquad \omega_i'^{\pm 1}\omega_j'^{\pm 1} = \omega_j'^{\pm 1}\omega_i'^{\pm 1}, \\
&\phantom{(\mathrm{R1}) \quad} \omega_i^{\pm 1}\omega_j'^{\pm 1} = \omega_j'^{\pm 1}\omega_i^{\pm 1}, \qquad \omega_i^{\pm 1}\omega_i^{\mp 1} = \omega_i'^{\pm 1}\omega_i'^{\mp 1} = 1; \\
&(\mathrm{R2}) \quad \omega_i e_j \omega_i^{-1} = r^{\langle j,i \rangle} s^{-\langle i,j \rangle} e_j, \qquad \omega_i' e_j \omega_i'^{-1} = r^{-\langle i,j \rangle} s^{\langle j,i \rangle} e_j; \\
&(\mathrm{R3}) \quad \omega_i f_j \omega_i^{-1} = r^{-\langle j,i \rangle} s^{\langle i,j \rangle} f_j, \qquad \omega_i' f_j \omega_i'^{-1} = r^{\langle i,j \rangle} s^{-\langle j,i \rangle} f_j; \\
&(\mathrm{R4}) \quad e_i f_j - f_j e_i = \delta_{i,j} \frac{\omega_i - \omega_i'}{r_i - s_i}; \\
&(\mathrm{R5}) \quad \sum_{k=0}^{1-a_{ij}} \binom{1-a_{ij}}{k}_{r_i s_i^{-1}} c_{ij}^{(k)} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \quad \text{if } i \neq j; \\
&(\mathrm{R6}) \quad \sum_{k=0}^{1-a_{ij}} \binom{1-a_{ij}}{k}_{r_i s_i^{-1}} c_{ij}^{(k)} f_i^k f_j f_i^{1-a_{ij}-k} = 0 \quad \text{if } i \neq j;
\end{aligned}$$

where $c_{ij}^{(k)} = (r_i s_i^{-1})^{k(k-1)/2} r^{k\langle j,i \rangle} s^{-k\langle i,j \rangle}$, for $i \neq j$ and $\binom{n}{k}_q$ is the quantum binomial coefficient, see Appendix A. $U_{r,s}(\mathbf{g})$ is a Hopf algebra with the comultiplication given by

$$\Delta\left(\omega_i^{\pm 1}\right) = \omega_i^{\pm 1} \otimes \omega_i^{\pm 1}, \qquad \Delta\left(\omega_i'^{\pm 1}\right) = \omega_i'^{\pm 1} \otimes \omega_i'^{\pm 1},$$
$$\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \qquad \Delta(f_i) = f_i \otimes \omega_i' + 1 \otimes f_i.$$

The counit is given by

$$\epsilon\left(\omega_i^{\pm 1}\right) = 1, \qquad \epsilon\left(\omega_i'^{\pm 1}\right) = 1, \qquad \epsilon(e_i) = \epsilon(f_i) = 0$$

and the antipode is given by

$$S\left(\omega_i^{\pm 1}\right) = \omega_i^{\mp 1}, \qquad S\left(\omega_i'^{\pm 1}\right) = \omega_i'^{\mp 1}, \qquad S(e_i) = -\omega_i^{-1} e_i, \qquad S(f_i) = -f_i \omega_i'^{-1}.$$

Since the matrix $C = (a_{ij})_{1 \leqslant i \leqslant \theta}$ is of finite Cartan type one has that $\det(d_i a_{ij}) \neq 0$ so there are odd numbers N such that the matrix $(d_i a_{ij})_{1 \leqslant i \leqslant \theta}$ is invertible in $\mathbb{Z}_N$. Suppose further that $s = r^{-1}$ is a primitive root of unity of order N.

We will show that under these assumptions certain Drinfeld doubles are Hopf quotients of $U_{r,r^{-1}}(\mathbf{g})$.

Let $G = \prod_{i=1}^{\theta} \mathbb{Z}_N$ and $g_1, g_2, \ldots, g_\theta$ be generators of each component of the product. Define $\chi_i \in \widehat{G}$ by $\chi_i(g_j) = r^{\langle i,j\rangle} s^{-\langle j,i\rangle}$. It follows that $\chi_i(g_j) = r^{\langle i,j\rangle + \langle j,i\rangle} = r^{d_i a_{ij}}$. It can be checked that the characters χ_i are well defined and $\mathcal{D} = (G, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta})$ is a Cartan datum of finite type. Let $A = u(\mathcal{D}) := u(\mathcal{D}, 0, 0)$. We will show that $D(A)$ is a quotient of the two parameter quantum group $U_{r,r^{-1}}(\mathbf{g})$. Define $\psi : U_{r,r^{-1}}(\mathbf{g}) \to D(A)$ by $\psi(e_i) = \frac{1}{(s_i - r_i)^{1/2}} x_i$, $\psi(f_i) = \frac{1}{(s_i - r_i)^{1/2}} \xi_i$, $\psi(\omega_i) = g_i$, $\psi(\omega_i') = \chi_i$. Using formula (A.2) from Appendix A it can be checked that ψ is well defined. The relations (R5) and (R6) are sent to 0 by ϕ since $ad_A(x_i)^{1-a_{ij}}(x_j) = 0$ and respectively $ad_{A^*}(\xi_i)^{1-a_{ij}}(\xi_j) = 0$. It can be checked that ψ is a Hopf algebra map. In order to see that ψ is surjective one has to check that the set of the grouplike elements of $U_{r,r^{-1}}(\mathbf{g})$ is mapped onto $G \times \widehat{G}$. Thus one has to see that the subgroup generated by $\langle \chi_i \rangle_{1\leqslant i\leqslant\theta}$ is the entire $\widehat{G}$. This fact follows from the assumption that the matrix $(d_i a_{ij})_{1\leqslant i\leqslant\theta}$ is invertible in $\mathbb{Z}_N$.

4. Braided Hopf algebras

Let H be a finite dimensional Hopf algebra and $R \in {}^H_H\mathcal{YD}$ be a Yetter–Drinfeld module over H.

Recall that R is called a braided Hopf algebra in ${}^H_H\mathcal{YD}$ if it is an algebra and coalgebra such that the comultiplication and counit are morphisms in ${}^H_H\mathcal{YD}$.

Let A and H be Hopf algebras and $p : A \to H$ and $j : H \to A$ Hopf algebra homomorphisms such that $pj = \mathrm{id}_H$. Let

$$R := A^{\mathrm{co}\,H} = \{ a \in A \mid (\mathrm{id} \otimes p)\Delta(a) = a \otimes 1 \}. \tag{4.1}$$

Then R is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ with the following structures:

(1) H acts on R via the adjoint action.
(2) The coaction of H is $(p \otimes \mathrm{id})\Delta$.
(3) R is a subalgebra of A.
(4) The comultiplication on R is given by $\Delta_R(r) = r_1 jpS(r_2) \otimes r_3 \in R \otimes R$.

For all $r \in R$ one has $p(r) = \epsilon(r)1_H$. This can be seen applying $m.(p \otimes S_H)$ to the identity $r_1 \otimes p(r_2) = r \otimes 1$.

Define $\nu : A \to R$ by $\nu(a) = a_1 jpS(a_2)$. Then $\nu(ab) = a_1 \nu(b) jpS(a_2)$ and $\nu(aj(h)) = \nu(a)j(h)$ for all $a, b \in A$ and $h \in H$. It can be proved that ν is a coalgebra map and it induces a coalgebra isomorphism $\nu : A/Aj(H)^+ \cong R$ [4]. Thus $\nu^* : R^* \to A^*$ is an algebra embedding and

$$\nu^*(R^*) = \{ f \in A^* \mid f(aj(h)) = f(a)\epsilon(h) \text{ for all } a \in A,\ h \in H \}. \tag{4.2}$$

4.1. The map $\phi : A \to R \# H$ given by $a \mapsto \nu(a_1) \# p(a_2)$ is an isomorphism of algebras with the inverse given by $r \# h \mapsto rj(h)$.

4.2. *Dual Hopf algebra*

One has that $p^*: H^* \to A^*$ and $j^*: A^* \to H^*$ are Hopf algebra homomorphisms such that $j^* p^* = \mathrm{id}_{H^*}$ Then

$$\begin{aligned} A^{*\operatorname{co} H^*} &:= \bigl\{ f \in A^* \bigm| (\mathrm{id} \otimes j^*)\Delta(f) = f \otimes \epsilon \bigr\} \\ &= \bigl\{ f \in A^* \bigm| f\bigl(aj(h)\bigr) = f(a)\epsilon(h) \text{ for all } a \in A,\ h \in H \bigr\} \\ &= \nu^*(R^*). \end{aligned}$$

Thus $A^* \cong R^* \# H^*$ via $f \mapsto f_1 p^* j^*(Sf_2) \# j^*(f_3)$ with the inverse given by $r^* \# f \mapsto \nu^*(r^*) p^*(f)$.

4.3. Under this isomorphism one has that

$$\begin{aligned} (r^* \# f)(r \# h) &= \bigl(\nu^*(r^*)p^*(f)\bigr)\bigl(rj(h)\bigr) = \nu^*(r^*)\bigl(r_1 j(h_1)\bigr)p^*(f)\bigl(r_2 j(h_2)\bigr) \\ &= \nu^*(r^*)(r_1)p^*(f)\bigl(r_2 j(h)\bigr) = r^*(r_1)(f)\bigl(p(r_2)h\bigr) \\ &= r^*(r)f(h). \end{aligned}$$

4.4. By duality one has that $A = A^{**} \cong R^{**} \# H^{**} = R \# H$ and it can be checked that the isomorphism obtained in such a way is just the isomorphism from 4.1.

For all $r \in R$ and $h \in H$ it follows that $S^{-1} j(h_2) r j(h_1) \in R$ since

$$\begin{aligned} \bigl((\mathrm{id} \otimes p)\Delta\bigr)\bigl(S^{-1} j(h_2) r j(h_1)\bigr) &= S^{-1} j(h_4) r_1 j(h_1) \otimes p\bigl(S^{-1} j(h_3) r_2 j(h_2)\bigr) \\ &= S^{-1} j(h_4) r j(h_1) \otimes p\bigl(S^{-1} j(h_3) j(h_2)\bigr) \\ &= S^{-1} j(h_1) r j(h_2) \otimes 1. \end{aligned}$$

4.5. *Definition*

Let A be a finite dimensional Hopf algebra. An element $z \in A$ is called a *left integral of* A (respectively *right integral of* A) if $az = \epsilon(a)z$ (respectively $za = \epsilon(a)z$) for all $a \in A$. The space of left (respectively right) integrals of A is a one dimensional ideal $\int_A^l$ (respectively $\int_A^r$) of A and $S(\int_A^l) = \int_A^r$ where S is the antipode of A (see [15]).

If $z \in \int_A^l$ is a nonzero left integral of A, then there is a unique grouplike element $\gamma \in G(A^*)$, called *the distinguished grouplike element of* A^* such that $za = \gamma(a)z$, for all $a \in A$. If $z' \in \int_A^r$ then $az' = \gamma^{-1} z'$, for all $a \in A$.

If $\lambda \in \int_{A^*}^r$ is nonzero, then there exists a unique grouplike element $g \in G(A)$ such that $f\lambda = f(g)\lambda$ for all $f \in A^*$. The element $g \in G(A)$ is called *the distinguished grouplike element of* A.

4.6. *Left integrals in A*

Let x be a left integral of R and Λ be a left integral of H. Then $\Lambda x = \Lambda_1.x \# \Lambda_2$ is a left integral of A. Clearly $h(\Lambda x) = \epsilon(h)\Lambda x$ and $r(\Lambda x) = \Lambda_2(S^{-1}(\Lambda_1).r)x = \Lambda_2\epsilon(S^{-1}(\Lambda_1).r)x = \epsilon(r)\Lambda x$ for all $h \in H$ and $r \in R$.

4.7. There is $\gamma \in G(H^*)$ such that $h.x = \gamma(h)x$ for all $h \in H$. Indeed

$$\begin{aligned} r(h.x) = r\big(j(h_1)xj(Sh_2)\big) &= j(h_3)\big(S^{-1}j(h_2)rj(h_1)\big)xS(h_4) \\ &= j(h_3)\epsilon\big(S^{-1}j(h_2)rj(h_1)\big)xS(h_4) \\ &= \epsilon(r)\big(j(h_1)xj(Sh_2)\big). \end{aligned}$$

Thus $h.x$ is an integral in R and since the space of integrals is one dimensional it follows that $h.x = \gamma(h)x$ for some $\gamma \in G(H^*)$.

4.8. The distinguished grouplike element of A^*

Let α_R and α_H be the distinguished grouplike elements of R and H. Then $xr = \alpha_R(r)x$ and $\Lambda h = \alpha_H(h)\Lambda$ for all $r \in R$ and $h \in H$. Let α_A be the distinguished grouplike element of A. Then α_A is given by the following equation: $(\Lambda x)(r \# h) = \alpha_A(r \# h)(\Lambda x)$.

On the other hand

$$\begin{aligned} (\Lambda x)(r \# h) = \alpha_R(r)(\Lambda x h) &= \alpha_R(r)\Lambda h_2\big(S^{-1}(h_1).x\big) \\ &= \alpha_R(r)\alpha_H(h_2)\gamma^{-1}(h_1)\Lambda x = \alpha_R(r)\big(\gamma^{-1}\alpha_H\big)(h)\Lambda x. \end{aligned}$$

Using 4.3 it can be shown that $\alpha_A = \alpha_R \# \gamma^{-1}\alpha_H$.

4.9. Right integral in A^*

If t is a right integral in R^* and λ a right integral in H^* it can be similarly checked that $t \# \lambda$ is a right integral in A^* (see also [8]).

Similarly to 4.7 it can be proved that there is a grouplike element $g \in H$ such that $f.t = f(g)t$ for all $f \in H^*$. Indeed

$$\begin{aligned} \big(f_1 t s(f_2)\big)r^* &= f_1 t\big(Sf_4 r^* S^2 f_3\big)Sf_2 \\ &= f_1 t\epsilon\big(Sf_4 r^* S^2 f_3\big)Sf_2 \\ &= r^*(1) f_1 t S(f_2) \end{aligned}$$

for all $r^* \in R^*$ and $f \in H^*$. Thus $f.t$ is an integral in R^* and since the space of integrals is one dimensional it follows that $f.t = f(g)t$ for some $g \in G(H)$.

4.10. The distinguished grouplike element of A

Let g_R and g_H be the distinguished grouplike elements of R^* and H^*. Thus $r^*t = r^*(g_R)t$ and $h^*\lambda = h^*(g_H)\lambda$ for all $r^* \in R^*$ and $h^* \in H^*$. It follows that the distinguished grouplike element of A is $g_A = g_R \# gg_H$. Indeed,

$$(r^* \# f)(t \# \lambda) = r^*(f_1.t) \# f_2\lambda = f_1(g)r^*t \# f_2(g_H)\lambda = r^*(g_R)f(gg_H)$$

for all $r^* \in R^*$ and $f \in H^*$. Using 4.1 and 4.4 it follows that $g_A = g_R \# gg_H$.

4.11. The situation when R is a graded braided Hopf algebra

Suppose $R = \bigoplus_{i=0}^{N} R(i)$ is a graded braided Hopf algebra with $R(0) = k$. Then $R(N) = k$ is the space of left and right integrals in R [1]. Thus R is unimodular and $\alpha_R = \epsilon$. If $R = \bigoplus_{i=0}^{N} R(i)$ is a graded braided Hopf algebra in ${}_H^H\mathcal{YD}$ with $R(0) = k$ then $R^* = \bigoplus_{i=0}^{N} R(i)^*$ is a graded braided Hopf algebra in ${}_{H^*}^{H^*}\mathcal{YD}$. Thus R^* is also unimodular and $g_R = 1$. Thus in this situation $\alpha_A = \epsilon \# \gamma^{-1}\alpha_H$ and $g_A = 1 \# g g_H$.

4.12. Ribbon elements

A Hopf algebra A is called quasitriangular if there is an invertible element $R = \sum x_i \otimes y_i \in A \otimes A$ such that $\Delta(a) = R\Delta(a)R^{-1}$ for all $a \in A$, and R satisfies the following relations $(\Delta \otimes \mathrm{id})R = R_{13}R_{23}$, $(\mathrm{id} \otimes \Delta)R = R_{13}R_{12}$ where $R_{12} = \sum x_i \otimes y_i \otimes 1$, $R_{13} = \sum x_i \otimes 1 \otimes y_i$, $R_{23} = \sum 1 \otimes x_i \otimes y_i$. Let $u = \sum S(y_i)x_i$. Then $uS(u)$ is central in A and is referred to as the Casimir element.

An element $v \in A$ is called *quasiribbon* element of a quasitriangular Hopf algebra (A, R) if:

(1) $v^2 = c$,
(2) $S(v) = v$,
(3) $\epsilon(v) = 1$,
(4) $\Delta(v) = R\widetilde{R}^{-1}(v \otimes v)$ where $\widetilde{R} = \sum y_i \otimes x_i$ if $R = \sum x_i \otimes y_i$.

If v is central in A then v is called *ribbon element* of A and (A, R, v) is called *ribbon Hopf algebra*. Ribbon elements are used to construct invariants of knots and links [12,17,18].

The Drinfeld double $D(A)$ of a finite dimensional Hopf algebra A is a quasitriangular Hopf algebra with $R = \sum (1 \otimes e_i) \otimes (f_i \otimes 1)$ where e_i and f_i are dual bases of A and A^*. Kauffman and Radford provided the following criterion for a Drinfeld double $D(A)$ to be a ribbon Hopf algebra.

Theorem 4.3. *(See [12].) Assume A is a finite dimensional Hopf algebra and let g and γ be the distinguished grouplike elements of A and A^* respectively. Then*:

(i) *$D(A)$ has a quasiribbon element if and only if there exist grouplike elements $h \in A$, $\delta \in A^*$ such that $h^2 = g$ and $\delta^2 = \gamma$.*
(ii) *$(D(A), R)$ has a ribbon element if and only if there exist h and δ as in* (i) *such that*

$$S^2(a) = h\big(\delta \rightharpoonup a \leftharpoonup \delta^{-1}\big)h^{-1}$$

for all $a \in A$.

4.13. Condition for $D(A)$ to be ribbon

According to 4.3 $D(A)$ is a ribbon algebra if and only if there are grouplike elements δ_A and h_A in A^* and A, respectively such that $\delta_A^2 = \epsilon_R \# \gamma^{-1}\alpha_H$ and $h_A^2 = 1 \# g g_H$ and

$$S^2(a) = h_A\big(\delta_A \rightharpoonup a \leftharpoonup {\delta_A}^{-1}\big){h_A}^{-1} \tag{4.4}$$

for all $a \in A$.

It is enough to check this conditions on the algebra generators of A namely, $r \in R$ and $j(h)$ with $h \in H$.

4.14. Suppose that there are grouplike elements $\delta \in G(H^*)$ and $h \in G(H)$, respectively such that $\delta^2 = \alpha_H \gamma^{-1}$ and $h^2 = g g_H$. Consider $\delta_A = \epsilon \# \delta$ and $h_A = 1 \# h$. These are grouplike elements of A^* and A, respectively and $\delta_A^2 = \epsilon_R \# \alpha_H \gamma^{-1}$ and $h_A^2 = 1 \# g g_H$.

For $a = j(x)$ with $x \in H$ the condition (4.4) becomes

$$S^2(x) = h\big(\delta \rightharpoonup x \leftharpoonup \delta^{-1}\big)h^{-1}. \tag{4.5}$$

For $a = r$ one has that

$$\begin{aligned}\Delta_{R\#H}(r \# 1) &= \phi(r_1) \otimes \phi(r_2)\\ &= \big(\nu(r_1) \# p(r_2)\big) \otimes \big(\nu(r_3) \# p(r_4)\big)\\ &= \big(\nu(r_1) \# p(r_2)\big) \otimes \big(\nu(r_3) \# 1\big)\end{aligned}$$

since $r_1 \otimes r_2 \otimes r_3 \otimes p(r_4) = r_1 \otimes r_2 \otimes r_3 \otimes 1$ for all $r \in R$.

Thus

$$\begin{aligned}(\epsilon \# \delta) \rightharpoonup (r \# 1) &= \big(\nu(r_1) \# p(r_2)\big)\big\langle \epsilon \# \delta, \nu(r_3) \# 1\big\rangle\\ &= \nu(r_1) \# p(r_2) = \nu(r) \# 1\\ &= r \# 1.\end{aligned}$$

Similarly

$$\begin{aligned}(r \# 1) \leftharpoonup \big(\epsilon \# \delta^{-1}\big) &= \big(\nu(r_3) \# 1\big)\big\langle \epsilon \# \delta^{-1}, \nu(r_1) \# p(r_2)\big\rangle\\ &= \big(\nu(r_2) \# 1\big)\delta^{-1}\big(p(r_1)\big).\end{aligned}$$

Thus the condition (4.4) for $a = r$ is

$$S^2(r) = \big(h.\nu(r_2)\big)\delta^{-1}\big(p(r_1)\big) \tag{4.6}$$

for all $r \in R$.

5. Drinfeld doubles which are ribbon

Using the results from the previous section, we determine the left and right integrals of $A \cong u(\mathcal{D}, 0, 0)$ and its distinguished grouplike element. By duality, the integrals of A^* and its distinguished grouplike element are also described. The condition obtained in the previous section for $D(A)$ to be a ribbon algebra will be verified for $A = u(\mathcal{D}, 0, 0)$.

Consider $H = kG$ for an abelian group G and V be a finite dimensional Yetter–Drinfeld module over the group algebra kG. Then V has a basis $(x_i)_{1 \leqslant i \leqslant \theta}$ with $x_i \in V_{g_i}^{\chi_i}$, where $V_{g_i}^{\chi_i} := \{v \in V \mid gv = \chi_i(g)v,\ \delta(v) = g_i \otimes v\}$ and δ is the comodule structure of V.

Suppose that $V \in {}^H_H\mathcal{YD}$ of finite Cartan type, which means $\chi_i(g_i) \neq 1$ for all $1 \leqslant i \leqslant \theta$ and there is a Cartan matrix of finite type $(a_{ij})_{1 \leqslant i,j \leqslant \theta}$ such that

$$\chi_j(g_i)\chi_i(g_j) = \chi_i(g_i)^{a_{ij}}$$

for all $1 \leqslant i, j \leqslant \theta$.

Let $R = B(V)$ the Nichols algebra of a finite dimensional braided vector space $V \in {}^H_H\mathcal{YD}$ of finite Cartan type. Then $A = u(\mathcal{D}, 0, 0) = B(V) \# kG$ and from Corollary 2.6 it follows that $A^* = u(\widetilde{\mathcal{D}}, 0, 0)$.

Proposition 5.1. *Let $\Lambda_G = \frac{1}{|G|}\sum_{g \in G} g$ be the integral of kG and $x = \prod_{i=1}^p y_i^{N_i - 1}$. Then $t_l = \Lambda_G x$ is a left integral of A and $t_r = x\Lambda_G$ is a right integral of A.*

Proof. Since x is a homogeneous element with maximal degree in $B(V)$, it follows from [1] that x is a left and right integral of R. Then 4.6 implies that $\Lambda_G x$ is a left integral in A and 4.9 implies that $x\Lambda_G$ is a right integral in A. $\square$

Proposition 5.2. *The element $\gamma \in G(A^*)$ defined by $\gamma(g) = \prod_{i=1}^p \chi_{\beta_i}^{-(N_i-1)}(g)$ and $\gamma(x_i) = 0$ is the distinguished grouplike element of A^*.*

Proof. Using 4.8 the element $\alpha_A = \alpha_R \# \gamma^{-1}\alpha_H$ is distinguished grouplike element of A^*. In the situation $R = B(V)$ and $H = kG$ one has that $\alpha_R = \epsilon_R$ and $\alpha_H = \epsilon_H$. On the other hand γ is given by the equation $g.x = \gamma(g)x$ for all $g \in G$. Since $g.y_i = \chi_{\beta_i}(g)y_i$ it follows that $g.x = \prod_{i=1}^p \chi_{\beta_i}(g)^{N_i-1}x$ and thus $\gamma = \prod_{i=1}^p \chi_{\beta_i}^{-(N_i-1)}$. $\square$

Proposition 5.3. *Let $\Lambda_{\widehat{G}} = \frac{1}{|G|}\sum_{\vartheta \in \widehat{G}} \vartheta$ be the integral of kG^* and $Y = \prod_{i=1}^p Y_i^{N_i-1}$. Then $T_l = \Lambda_{\widehat{G}} Y$ is a left integral of A^* and $T_r = Y\Lambda_{\widehat{G}}$ is a right integral of A^*. Moreover the element $g = \prod_{i=1}^p g_{\beta_i}^{(N_i-1)}$ is the distinguished grouplike element of A.*

Proof. Using Corollary 2.6 one has that $A^* = u(\widetilde{\mathcal{D}}, 0, 0)$ where $\widetilde{\mathcal{D}}$ was defined in Section 2. Then Propositions 5.1, 5.2 applied to $u(\widetilde{\mathcal{D}}, 0, 0)$ give the integrals and the distinguished grouplike element of A^*. $\square$

Theorem 5.4. *Let $\mathcal{D} = (G, (g_i)_{1 \leqslant i \leqslant \theta}, (\chi_i)_{1 \leqslant i \leqslant \theta}, (a_{ij})_{1 \leqslant i,j \leqslant \theta})$ be a datum of finite Cartan type and $A = u(\mathcal{D}, 0, 0)$ the pointed Hopf algebra associated to it. Assume that the order N_i of $\chi_i(g_i)$ is odd for all i and is prime to 3 for all i in a connected component of type G_2. Then $D(A)$ is a ribbon Hopf algebra.*

Proof. One has to verify relations (4.5) and (4.6) from 4.14. Using the above notations it follows that $x = \prod_{\alpha \in \Phi^+} x_\alpha^{N_\alpha - 1} = \prod_{i=1}^p y_i^{N_i-1}$ is a left integral in $R = B(V)$. On the other hand from Proposition 5.2 $g.x = (\prod_{\alpha \in \Phi^+} \chi_\alpha^{N_\alpha - 1})(g)$ which means that

$$\gamma = \prod_{\alpha \in \Phi^+} \chi_\alpha^{N_\alpha - 1}.$$

Similarly, Proposition 5.3 implies that $t = \prod_{\alpha \in \Phi^+} Y_\alpha^{N_\alpha - 1} = \prod_{i=1}^{p} Y_i^{N_i - 1}$ is a right integral in R^*. One has that $\chi.Y = \chi(\prod_{\alpha \in \Phi^+} g_\alpha^{N_\alpha - 1})Y$ for all $\chi \in kG^*$, which shows that

$$g = \prod_{\alpha \in \Phi^+} g_\alpha^{N_\alpha - 1}.$$

Since N_α is odd consider in 4.14 $\delta = \prod_{\alpha \in \Phi^+} \chi_\alpha^{-\frac{(N_\alpha - 1)}{2}}$ and $h = \prod_{\alpha \in \Phi^+} g_\alpha^{\frac{(N_\alpha - 1)}{2}}$.

Condition (4.5) is automatically satisfied since $H = kG$ is cocommutative. Indeed, if $g \in G$ then $S^2(g) = g$ and $h(\delta \rightharpoonup g \leftharpoonup \delta^{-1})h^{-1} = \delta(g)\delta^{-1}(g)hgh^{-1} = g$.

On the other hand condition (4.6) has to be checked only on a set of algebra generators of R, for example x_i with $1 \leqslant i \leqslant \theta$. Since $\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i$ this condition can be written as $h.\nu(x_i)\delta^{-1}(g_i) = S^2(x_i)$. Since $S^2(x_i) = \chi_i(g_i)^{-1}x_i$ and $\nu(x_i) = x_i$ this condition becomes $\chi_i(h)\delta^{-1}(g_i) = \chi_i(g_i)^{-1}$.

For $1 \leqslant i \leqslant \theta$ let s_i, given by $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$, be the reflection corresponding to the simple root α_i. If $\beta = \sum_{s=1}^{\theta} c_s\alpha_s$ is a root then

$$s_i(\beta) = \sum_{s=1}^{\theta} c_s s_i(\alpha_s) = \sum_{s=1}^{\theta} c_s(\alpha_s - a_{is}\alpha_i) = \beta - \left(\sum_{s=1}^{\theta} c_s a_{is}\right)\alpha_i$$

and therefore

$$\left(\sum_{s=1}^{\theta} c_s a_{is}\right)\alpha_i = \beta - s_i(\beta). \tag{5.5}$$

Then

$$\delta^{-1}(g_i)\chi_i(h) = \prod_{j=1}^{p} \chi_{\beta_j}^{\frac{(N_j-1)}{2}}(g_i)\chi_i\left(\prod_{j=1}^{p} g_{\beta_j}^{\frac{(N_j-1)}{2}}\right) = \prod_{j=1}^{p} \left(\chi_{\beta_j}(g_i)\chi_i(g_{\beta_j})\right)^{\frac{N_j-1}{2}}.$$

Suppose $\beta_j = \sum_{s=1}^{\theta} c_{js}\alpha_s$ with $c_{js} \in \mathbb{Z}_{\geqslant 0}$, for all $1 \leqslant j \leqslant p$. Then

$$\chi_{\beta_j}(g_i)\chi_i(g_{\beta_j}) = \prod_{s=1}^{\theta} \chi_s^{c_{js}}(g_i)\chi_i(g_s)^{c_{js}} = \prod_{s=1}^{\theta} \left(\chi_i(g_i)\right)^{\sum_{s=1}^{\theta} a_{is}c_{js}}.$$

Suppose that $\alpha_i \in J$, the connected component of the Dynkin diagram that contains α_i. Without loss of generality one may suppose that $\alpha_1, \ldots, \alpha_{\theta_1}$ are the simple roots of J and $\{\beta_1, \ldots, \beta_{p_1}\}$ are the corresponding positive roots. It follows that $\chi_{\beta_m}(g_i)\chi_i(g_{\beta_m}) = 1$ if $m \notin \{1, \ldots, p_1\}$ since $a_{im} = 0$.

Thus $\delta^{-1}(g_i)\chi_i(h) = \prod_{j=1}^{p_1} \chi_i(g_i)^{(\sum_{s=1}^{\theta_1} a_{is}c_{js})\frac{N_i-1}{2}}$. Since $\chi_i(g_i)^{N_i} = 1$ one has that

$$\delta^{-1}(g_i)\chi_i(h) = \prod_{j=1}^{p_1} \chi_i(g_i)^{-\frac{\sum_{s=1}^{\theta_1} a_{is}c_{js}}{2}} = \chi_i(g_i)^{-\frac{\sum_{j=1}^{p_1}\sum_{s=1}^{\theta_1} a_{is}c_{js}}{2}}.$$

Thus, in order to show that $D(A)$ has a ribbon element one has to check that

$$\sum_{j=1}^{p_1}\sum_{s=1}^{\theta_1} a_{is}c_{js} = 2.$$

Let $\rho_J = \sum_{j=1}^{p_1} \beta_j/2$ half sum of the positive roots of the connected component J. Using Eq. (5.5) one has $(\sum_{s=1}^{\theta_1} a_{is}c_{js})\alpha_i = \beta_j - s_i(\beta_j)$ for any $1 \leqslant j \leqslant p_1$. Therefore

$$\left(\sum_{j=1}^{p_1}\sum_{s=1}^{\theta_1} a_{is}c_{js}\right)\alpha_i = \sum_{j=1}^{p_1}\big(\beta_j - s_i(\beta_j)\big) = 2\big(\rho_J - s_i(\rho_J)\big).$$

Since $s_i(\rho_J) = \rho_J - \alpha_i$ one gets that $\sum_{j=1}^{p_1}\sum_{s=1}^{\theta_1} a_{is}c_{js} = 2$ for all $1 \leqslant i \leqslant \theta$. $\square$

Appendix A

For $n \in \mathbb{N}$ and $q \neq 0$, let $(n)_q = 1 + q + \cdots + q^{n-1}$ for $n \geqslant 1$ and $(0)_q = 1$. Define $(n)_q! = (1)_q(2)_q\cdots(n)_q$ and let

$$\binom{n}{i}_q = \frac{(n)_q!}{(i)_q!(n-i)_q!}$$

be the quantum binomial coefficients.

Note that if $q \neq 0$ then

$$\binom{n}{k}_{q^{-1}} = \binom{n}{k}_q q^{k(k-n)} \tag{A.1}$$

for all $0 \leqslant k \leqslant n$. The proof of this is deduced from the equalities: $(n)_{q^{-1}} = q^{-(n-1)}(n)_q$ and $(n)_{q^{-1}}! = q^{-n(n-1)/2}(n)_q!$.

If $ba = qab$ then

$$(a+b)^n = \sum_{i=0}^{n}\binom{n}{i}_q a^i b^{n-i}$$

for all $n \in \mathbb{N}$.

Let A be a finite dimensional Hopf algebra. Suppose $x, y \in A$ such that $\Delta(x) = x \otimes 1 + a \otimes x$, $\Delta(y) = y \otimes 1 + b \otimes y$ where $a, b \in G := G(A)$. Moreover, suppose that $gxg^{-1} = \chi(g)x$ and $gyg^{-1} = \mu(g)y$ for all $g \in G$ where $\chi, \mu \in \widehat{G}$. Let $z_N = ad(x)^N(y)$. Then

$$z_N = \sum_{i=0}^{N}(-1)^i\binom{N}{i}_{\chi(a)}\chi(a)^{i(i-1)/2}\mu(a)^i x^{N-i} y x^i. \tag{A.2}$$

The proof of the above formula is by induction on N. One has

$$z_1 = ad(x)(y) = xy - ayS(x) = xy - aya^{-1}x = xy - \mu(a)yx$$

and

$$z_{N+1} = ad(x)(z_N) = xz_N - az_Na^{-1}x = xz_N - \chi^N(a)\mu(a)z_Nx.$$

Thus

$$\begin{aligned}
z_{N+1} &= \sum_{i=0}^{N}(-1)^i\binom{N}{i}_{\chi(a)}\chi(a)^{i(i-1)/2}\mu(a)^ix^{N-i+1}yx^i \\
&\quad - \sum_{i=0}^{N}(-1)^i\binom{N}{i}_{\chi(a)}\chi(a)^{i(i-1)/2}\mu(a)^ix^{N-i}yx^{i+1}\chi(a)^N\mu(a) \\
&= x^{N+1}y + (-1)^{N+1}\chi(a)^{N(N+1)/2}\mu(a)^Nyx^{N+1} \\
&\quad + \sum_{i=1}^{N}(-1)^i\left[\binom{N}{i}_{\chi(a)}\chi(a)^{i(i-1)/2}\mu(a)^i\right. \\
&\quad \left. + \binom{N}{i-1}_{\chi(a)}\chi(a)^{(i-2)(i-1)/2}\mu(a)^{i-1}\chi(a)^N\mu(a)\right]x^{N+1-i}yx^i \\
&= \sum_{i=0}^{N+1}(-1)^i\binom{N+1}{i}_{\chi(a)}\chi(a)^{i(i-1)/2}\mu(a)^ix^{N+1-i}yx^i
\end{aligned}$$

since

$$\binom{N}{i}_{\chi(a)} + \binom{N}{i-1}_{\chi(a)}\chi(a)^{N-i+1} = \binom{N+1}{i}_{\chi(a)}.$$

We see that z_N has the same formula as in [3], formula (A.8), p. 33. In Lemma A.1, p. 33 of the same paper it is proved that if $\chi(b)\mu(a) = \chi^{1-r}(a)$ and $z_r = \sum_{i=0}^{r}\alpha_ix^iyx^{r-i}$ then α_i satisfy the following system:

$$\sum_{l\leqslant i\leqslant r-h}\alpha_i\binom{i}{l}_{\chi(a)}\binom{r-i}{h}_{\chi(a)}\mu(a)^{i-l}\chi(a)^{h(i-l)} = 0, \tag{A.3}$$

$$\sum_{u\leqslant i\leqslant r-v}\alpha_i\binom{i}{u}_{\chi(a)}\binom{r-i}{v}_{\chi(a)}\chi(b)^{r-i-v}\chi(a)^{u(r-i-v)} = 0. \tag{A.4}$$

The following lemma and its proof is similar to Lemma A.1 from [3].

Lemma A.5. *Let A be a finite dimensional Hopf algebra. Suppose $x, y \in A$ such that $\Delta(x) = x\otimes 1 + a\otimes x$, $\Delta(y) = y\otimes 1 + b\otimes y$ where $a, b \in G := G(A)$ and $ab = ba$. Moreover, suppose that $gxg^{-1} = \chi(g)x$ and $gyg^{-1} = \mu(g)y$ for all $g \in G$ where $\chi, \mu \in \widehat{G}$. Assume that $\chi(b)\mu(a) = \chi^{1-r}(a)$ for some $r \geqslant 0$ and let $z = ad(x)^{1-r}(y)$. Then z is a skew primitive element of A, $\Delta(z) = z\otimes 1 + a^{1-r}b\otimes z$.*

Proof. It can be shown that $z = \sum_{u=1}^{r} \alpha_u x^u y x^{r-u}$ where α_u are the scalars corresponding to formula (A.2) and they are the same as in [3], Lemma A.1. One has $\Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i}_q x^i a^{n-i} \otimes x^{n-i}$ for all $n \geqslant 0$, where $q = \chi(a)$. Thus

$$\begin{aligned}
\Delta(z) &= \sum_{u=0}^{r} \alpha_u \left(\sum_{i=0}^{u} \binom{u}{i}_q x^i a^{u-i} \otimes x^{u-i} \right) \times (y \otimes 1 + b \otimes y) \\
&\quad \times \left(\sum_{j=0}^{r-u} \binom{r-u}{j}_q x^j a^{r-u-j} \otimes x^{r-u-j} \right) \\
&= \sum_{u=0}^{r} \sum_{i=0}^{u} \sum_{j=0}^{r-u} \alpha_u \binom{u}{i}_q \binom{r-u}{j}_q x^i a^{u-i} y x^j a^{r-u-j} \otimes x^{u-i} x^{r-u-j} \\
&\quad + \sum_{u=0}^{r} \sum_{i=0}^{u} \sum_{j=0}^{r-u} \alpha_u \binom{u}{i}_q \binom{r-u}{j}_q x^i a^{u-i} b x^j a^{r-u-j} \otimes x^{u-i} y x^{r-u-j} \\
&= \sum_{u=0}^{r} \sum_{i=0}^{u} \sum_{j=0}^{r-u} \alpha_u \binom{u}{i}_q \binom{r-u}{j}_q \mu^{u-i}(a) \chi^j\left(a^{u-i}\right) x^i y x^j a^{r-i-j} \otimes x^{r-i-j} \\
&\quad + \sum_{u=0}^{r} \sum_{i=0}^{u} \sum_{j=0}^{r-u} \alpha_u \binom{u}{i}_q \binom{r-u}{j}_q \chi^j\left(a^{u-i} b\right) x^{i+j} a^{r-i-j} b \otimes x^{u-i} y x^{r-u-j} \\
&= \sum_{i+j=r} \alpha_i x^i y x^j \otimes 1 \\
&\quad + \sum_{0 \leqslant i+j < r} \left(\sum_{i \leqslant u \leqslant r-j} \alpha_u \binom{u}{i}_q \binom{r-u}{j}_q \mu^{u-i}(a) \chi^j\left(a^{u-i}\right) \right) x^i y x^j a^{r-i-j} \otimes x^{r-i-j} \\
&\quad + \sum_{u=0}^{r} \alpha_u a^r b \otimes x^u y x^{r-u} \\
&\quad + \sum_{0 < i+j \leqslant r} \left(\sum_{i \leqslant u \leqslant r-j} \alpha_u \binom{u}{i}_q \binom{r-u}{j}_q \chi^j\left(a^{u-i} b\right) \right) x^{i+j} a^{r-i-j} b \otimes x^{u-i} y x^{r-u-j} \\
&= z \otimes 1 + a^r b \otimes z.
\end{aligned}$$

The last equality is true since the first term of the above sum is $z \otimes 1$ and the third term is $a^r b \otimes z$. By formula (A.3) for each $i + j < r$ the coefficient of the $x^i y x^j a^{r-i-j} \otimes x^{r-i-j}$ in the second term is zero. Similarly, using that $\binom{u}{i}_q = \binom{u}{u-i}_q$, formula (A.4) implies that the coefficient of $x^{i+j} a^{r-i-j} b \otimes x^{u-i} y x^{r-u-j}$ in the last term is zero. □

Corollary A.6. *With the notations from Section* 2*, the element* $z = ad_{A^*}(\xi_i)^{1-a_{ij}}(\xi_j)$ *is skew primitive in* A^**:*

$$\Delta(z) = z \otimes 1 + \chi_i^{1-a_{ij}} \chi_j \otimes z$$

for all $1 \leqslant i, j \leqslant \theta$.

Proposition A.7. *With the notations from Section* 3 *one has that* $ad_{D(A)}(\xi_i \chi^{-1})^{1-a_{ij}}(\xi_j \chi_j^{-1}) = 0$ *for all* $1 \leqslant i, j \leqslant \theta$.

Proof. Note that Proposition 2.2 implies that the relations

$$\chi \xi_i^s \chi^{-1} = \chi^s(g_i)\xi_i, \qquad (\xi_i \chi)^n = \chi(g_i)^{\frac{n(n-1)}{2}} \xi_i^n \chi^n \tag{A.8}$$

hold in A^* for all $\chi \in G(A^*)$, $1 \leqslant i \leqslant \theta$ and any $n, s \geqslant 0$.

Using formula (A.2) the relation $ad_{A^*}(\xi_i)^{1-a_{ij}}(\xi_j) = 0$ can be written as:

$$\sum_{s=0}^{N} (-1)^i \binom{N}{s}_{\chi_i(g_i)} \chi_i(g_i)^{s(s-1)/2} \chi_j(g_i)^s \xi_i^{N-s} \xi_j \xi_i^s = 0$$

where $N = 1 - a_{ij}$. Since $\Delta_{A^*}(\xi_i) = \xi_i \otimes 1 + \chi_i \otimes \xi_i$ one has $\Delta_{D(A)^*}(\xi_i) = 1 \otimes \xi_i + \xi_i \otimes \chi_i$ and thus $\Delta_{D(A)}(\xi_i \chi_i^{-1}) = \chi_i^{-1} \otimes \xi_i \chi_i^{-1} + \xi_i \chi_i^{-1} \otimes 1$.

Using again formula (A.2) one has

$$\begin{aligned}
&ad_{D(A)}\big(\xi_i \chi^{-1}\big)^N \big(\xi_j \chi_j^{-1}\big) \\
&\quad = \sum_{s=0}^{N} (-1)^s \binom{N}{s}_{\chi_i^{-1}(g_i)} \chi_i(g_i)^{-s(s-1)/2} \chi_i^{-1}(g_j)^s \cdot \big(\xi_i \chi_i^{-1}\big)^{N-s} \xi_j \chi_j^{-1} \big(\xi_i \chi_i^{-1}\big)^s \\
&\quad = \sum_{s=0}^{N} (-1)^s \binom{N}{s}_{\chi_i^{-1}(g_i)} \chi_i(g_i)^{-s(s-1)/2} \chi_i^{-1}(g_j)^s \\
&\qquad \cdot \xi_i^{N-s} \chi_i^{-(N-s)} \xi_j \chi_j^{-1} \xi_i^s \chi_i^{-s} \chi_i^{-1}(g_i)^{(N-s)(N-s-1)/2 + s(s-1)/2} \\
&\quad = \chi_i(g_i)^{(N-N^2)/2} \sum_{s=0}^{N} (-1)^s \binom{N}{s}_{\chi_i^{-1}(g_i)} \chi_i(g_i)^{-s(s-1)/2} \chi_i^{-1}(g_j)^s \\
&\qquad \cdot \xi_i^{N-s} \chi_i^{-(N-s)} \xi_j \chi_j^{-1} \xi_i^s \chi_i^{-s} \chi_i(g_i)^{-s(s-N)} \\
&\quad = \chi_i(g_i)^{(N-N^2)/2} \sum_{s=0}^{N} (-1)^s \binom{N}{s}_{\chi_i(g_i)} \chi_i(g_i)^{s(s-n)} \chi_i(g_i)^{-s(s-1)/2} \chi_i^{-1}(g_j)^s \\
&\qquad \cdot \xi_i^{N-s} \xi_j \chi_i^{-(N-s)} \xi_i^s \chi_j^{-1} \chi_i^{-s} \chi_i(g_j)^{-(N-s)} \chi_j(g_i)^{-s} \chi_i(g_i)^{-s(s-N)} \\
&\quad = \chi_i(g_j)^{-N} \chi_i(g_i)^{(N-N^2)/2} \sum_{s=0}^{N} (-1)^s \binom{N}{s}_{\chi_i(g_i)} \chi_i(g_j)^s \chi_i(g_i)^{-s(s-1)/2} \chi_i^{-1}(g_j)^s \\
&\qquad \cdot \xi_i^{N-s} \xi_j \xi_i^s \chi_j(g_i)^{-s} \chi_i(g_i)^{-(N-s)s} \chi_j^{-1} \chi_i^{-N}
\end{aligned}$$

$$= \chi_j^{-1}\chi_i^{-N}\chi_i(g_j)^{-N}\chi_i(g_i)^{(N-N^2)/2}\sum_{s=0}^{N}(-1)^s\binom{N}{s}_{\chi_i(g_i)}\chi_i(g_i)^{-s(\frac{s-1}{2}+N-s)}\chi_j(g_i)^{-s}$$
$$\cdot\, \xi_i^{N-s}\xi_j\xi_i^s$$
$$= \chi_j^{-1}\chi_i^{-N}\chi_i(g_j)^{-N}\chi_i(g_i)^{(N-N^2)/2}\sum_{s=0}^{N}(-1)^s\binom{N}{s}_{\chi_i(g_i)}\chi_i(g_i)^{-s\frac{2N-s-1}{2}}\chi_j(g_i)^{-s}$$
$$\cdot\, \xi_i^{N-s}\xi_j\xi_i^s$$
$$= \chi_j^{-1}\chi_i^{-N}\chi_i(g_j)^{-N}\chi_i(g_i)^{(N-N^2)/2}\sum_{s=0}^{N}(-1)^s\binom{N}{s}_{\chi_i(g_i)}\chi_i(g_i)^{s(s-1)/2}\chi_i(g_i)^{-Ns+s}\chi_j(g_i)^{-s}$$
$$\cdot\, \xi_i^{N-s}\xi_j\xi_i^s.$$

If $N = 1 - a_{ij}$ then $-Ns + s = a_{ij}s$ and $\chi_i(g_i)^{sa_{ij}}\chi_j(g_i)^{-s} = \chi_i(g_j)^s$.
Thus

$$ad\big(\xi_i\chi^{-1}\big)^{1-a_{ij}}\big(\xi_j\chi_j^{-1}\big)$$
$$= \chi_j^{-1}\chi_i^{-N}\chi_i(g_j)^{-(1-a_{ij})}\chi_i(g_i)^{(N-N^2)/2}$$
$$\cdot \sum_{s=0}^{N}(-1)^s\binom{N}{s}_{\chi_i(g_i)}\chi_i(g_i)^{s(s-1)/2}\chi_i(g_j)^s\xi_i^{N-s}\xi_j\xi_i^s$$
$$= \chi_j^{-1}\chi_i^{-N}\chi_i(g_j)^{-(1-a_{ij})}\chi_i(g_i)^{(N-N^2)/2}ad_{A^*}^{1-a_{ij}}(\xi_i)(\xi_j) = 0.$$

Formula (A.1) for $q = \chi_i(g_i)$ and relations (A.8) were used in the above computations. □

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