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Representations of degree three for semisimple Hopf algebras☆

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Abstract

Let H be a cosemisimple Hopf algebra over an algebraically closed field. It is shown that if H has a simple subcoalgebra of dimension 9 and has no simple subcoalgebras of even dimension, then H contains either a grouplike element of order 2 or 3, or a family of simple subcoalgebras whose dimensions are the squares of each positive odd integer. In particular, if H is odd dimensional, then its dimension is divisible by 3.

MSC: 16W30

1. Introduction

Let H be a finite dimensional semisimple Hopf algebra over an algebraically closed field k. Kaplansky conjectured that if H has a simple module of dimension n, then n divides the dimension of H [2, Appendix 2].

Several special cases of Kaplansky's conjecture have been proved: Etingof and Gelaki [1] proved it under the additional assumption that H is quasi-triangular and k has characteristic 0 (for another proof see [9]); M. Lorenz and S. Zhu proved it independently under the additional assumption that the character of the simple module is central in H^* [5,10].

Dually, the Kaplansky conjecture says that if H is a finite dimensional cosemisimple Hopf algebra that contains a simple subcoalgebra of dimension n^2, then n divides the

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dimension of H. This was verified by Nichols and Richmond [7], for $n=2$. They proved that if H is cosemisimple and contains a simple subcoalgebra of dimension 4, then H contains either a Hopf subalgebra of dimension 2, 12 or 60, or a simple subcoalgebra of dimension n^2 for each positive integer n. Their approach is based on the study of the Grothendieck group of H.

In this paper, we give a similar treatment for the case $n=3$. Assume that H is cosemisimple and contains a simple subcoalgebra of dimension 9. If H has no simple subcoalgebras of even dimension we prove that H contains either a grouplike element of order 2 or 3, or a simple subcoalgebra of dimension n^2 for each positive odd integer n. In particular, if H is odd dimensional, then its dimension is divisible by 3. We remark that if H is bisemisimple, then the assumption that H has odd dimension automatically implies that the dimension of each simple subcoalgebra is odd (see the corollary on p. 95 of [3].)

The basic properties of the Grothendieck group of the category of right H-comodules are recalled in Section 2. Section 3 contains our main result, namely Theorem 4, together with the required lemmas.

We follow the standard notation found in [6]. Algebras and coalgebras are defined over k; comultiplication, counit and antipode are denoted by Δ, ε and S respectively; and the category of right comodules over H is denoted by $\mathcal{M}^H$.

2. The Grothendieck group $\mathcal{G}(H)$

Let H be a k-coalgebra. For $V\in\mathcal{M}^H$ define $[V]$ to be the isomorphism class of the comodule V. The Grothendieck group of the category of right comodules over H is the free abelian group generated by the isomorphism classes of simple right H-comodules. We denote this group by $\mathcal{G}(H)$. The following result appears in [7].

Proposition 1. *Let Γ denote the set of simple subcoalgebras of the coalgebra H. For each $C\in\Gamma$, let V_C be a simple right C-comodule. Then $\mathcal{G}(H)$ is the free abelian group with basis $B=\{[V_C]:C\in\Gamma\}$.*

Here $[V_C]$ is called a basic element and B is called the standard basis. Any basic element x is associated with a simple subcoalgebra x_C of H, and reciprocally, any simple subcoalgebra C of H is associated with a basic element C_x, as in the previous proposition. Every $z\in\mathcal{G}(H)$ may be written uniquely as $z=\sum_{x\in B}m(x,z)x$ where $m(x,z)\in\mathbb{Z}$. The integer $m(x,z)$ is called the multiplicity of x in z. If $m(x,z)\neq 0$, then x is called a basic component of z. The multiplicity function may be extended to a biadditive function $m:\mathcal{G}(H)\times\mathcal{G}(H)\to\mathbb{Z}$ by defining $m(w,z)=\sum_{x\in B}m(x,w)m(x,z)$. Note that there is a bijection between Γ and B, the set of isomorphism classes of simple right H-comodules.

If H is a bialgebra, $\mathcal{G}(H)$ becomes a ring. Recall that if M and N are right H-comodules then $M\otimes N$ is also a right H-comodule via

$$\rho(m\otimes n)=\sum_{(m),(n)} m_0\otimes n_0\otimes m_1n_1.$$

Let 1 denote $[k1]$, the unit of $\mathcal{G}(H)$. If $V \in \mathcal{M}^H$ is a simple comodule then the degree of the element $[V] \in \mathcal{G}(H)$ is defined to be the dimension of the comodule V and it is denoted by $|[V]|$. Since $B = \{[V_C] : C \in \Gamma\}$ is a basis for $\mathcal{G}(H)$ we get a linear map called the degree map, defined as above on the canonical basis of $\mathcal{G}(H)$ and then extended by linearity. By convention, an element $x \in \mathcal{G}(H)$ is said to be n-dimensional if its degree is equal to n. Note that $|xy| = |x||y|$ for all $x, y \in B$, and so $|wz| = |w||z|$ for all $w, z \in \mathcal{G}(H)$ which shows that the degree map is a ring homomorphism. Thus $|w||z| = \sum_x m(x, wz)|x|$ for all $w, z \in \mathcal{G}(H)$. Let $\mathcal{G}(H)_+$ be the subset of $\mathcal{G}(H)$ consisting of all elements $z \in \mathcal{G}(H)$ with the property that $m(x, z) \geqslant 0$ for any basic element $x \in \mathcal{G}(H)$. When H is cosemisimple, the elements of $\mathcal{G}(H)_+$ are in a bijective correspondence with all the isomorphism classes of H-comodules $V \in \mathcal{M}^H$ and $\mathcal{G}(H)_+$ is closed under multiplication.

Suppose H is a Hopf algebra with antipode S. If $U \in \mathcal{M}^H$ is a right comodule with the comodule structure $\rho : U \to U \otimes H$ then $U^* \in \mathcal{M}^H$ in the following way. Let $\{u_j\}$ be a basis of U, and write $\rho(u_j) = \sum_i u_i \otimes a_{ij}$, $a_{ij} \in H$. Then if $\{u_i^*\}$ is the dual basis of $\{u_i\}$ the map $\check{\rho}(u_i^*) = \sum_j u_j^* \otimes S(a_{ij})$ defines a right H-comodule structure on U^*. It is easy to see that $C_{U^*} = S(C_U)$ and if S is injective then U^* is simple whenever U is a simple comodule. Moreover $U^{**} \cong U$ if $S^2(C) = C$. The map ${}^* : \mathcal{G}(H) \to \mathcal{G}(H)$ given by $[M]^* = [M^*]$ is a group homomorphism and a ring antihomomorphism. If H is cosemisimple then * is an involution on $\mathcal{G}(H)$. A standard subring in $\mathcal{G}(H)$ is a subring of $\mathcal{G}(H)$ which is spanned as an abelian group by a subset of B.

Theorem 2 (Nichols and Richmond [7, Theorem 6]). *Let H be a bialgebra. There is a one-to-one correspondence between standard subrings of $\mathcal{G}(H)$ and subbialgebras of H generated as algebras by their simple subcoalgebras, given by: the subbialgebra A generated by its simple subcoalgebras corresponds to the standard subring spanned by $\{x_C : C$ is a simple subcoalgebra of $A\}$*

Furthermore, if H is a cosemisimple Hopf algebra this one-to-one correspondence induces a one-to-one correspondence between standard subrings closed under "" and Hopf subalgebras of H.*

The following facts about the multiplicity in $\mathcal{G}(H)$ will be used in the sequel.

Theorem 3 (Nichols and Richmond [7, Theorem 10]). *Let H be a Hopf algebra.*

(1) *If the antipode of H is injective then $m(x, y) = m(x^*, y^*)$ for all $x, y \in \mathcal{G}(H)$.*
(2) *If H is cosemisimple and k is algebraically closed, then*
  (a) *$m(x, yz) = m(y^*, zx^*) = m(y, xz^*)$ for all $x, y, z \in \mathcal{G}(H)$*
  (b) *For each grouplike element g of H, we have $m(g, xy) = 1$, if $y = x^*g$ and 0 otherwise.*
  (c) *Let $x \in \mathcal{G}(H)$ be a basic element. Then for any grouplike element g of H, $m(g, xx^*) > 0$ iff $m(g, xx^*) = 1$ iff $gx = x$. The set of such grouplike elements forms a group, of order at most $|x|^2$.*

For a basic element x, the grouplike elements entering in the basic decomposition of xx^* form a subgroup G of the grouplike elements of H (by 2(c) of the previous theorem). Then kG is a Hopf subalgebra of H, and by the freeness theorem [8] the order of G divides the dimension of H.

3. The 3-dimensional case

The goal of this section is to prove the following theorem.

Theorem 4. *Let H be a cosemisimple Hopf algebra over an algebraically closed field. Assume that H contains a simple subcoalgebra C of dimension 9 and has no simple subcoalgebras of even dimension. Then one of the following conditions must hold*:

(i) *H contains a grouplike element of order 2 or 3,*
(ii) *H has two families of subcoalgebras $\{C_{2n+1} : n \geqslant 1\}$ and $\{D_{2n+1} : n \geqslant 1\}$ with $\dim C_{2n+1} = \dim D_{2n+1} = (2n+1)^2$ such that*

$$C_{2n+1}C_3 = C_{2n-1} + D_{2n+1} + C_{2n+3}.$$

for each $n \geqslant 1$.

We need the following lemmas.

Lemma 5. *Let k be an algebraically closed field and H a cosemisimple Hopf algebra over k with a simple subcoalgebra C of dimension 9. Assume that H has no simple subcoalgebras of even dimension. Then one of the following conditions holds*:

(i) *H contains a grouplike element of order 2 or 3,*
(ii) *for any basic element $x_3 \in \mathscr{G}(H)$ with $|x_3| = 3$ and $x_3x_3^* = 1 + u + v$ where u and v are basic elements of $\mathscr{G}(H)$ with $|u| = 3$ and $|v| = 5$.*

Proof. Suppose that x_3 is a basic element of degree 3. Then $x_3x_3^* = 1 + y$ where $|y| = 8$. Note that since H has no simple subcoalgebras of even dimension, y is not a basic element. We consider the decomposition of y into basic elements in $\mathscr{G}(H)$. If y has 1, 2, 3, 5 or 8 1-dimensional representations in this decomposition, then together with 1 they form a subgroup of $\mathscr{G}(H)$ with 2, 3, 4, 6 or 9 elements respectively, and so (i) holds.

Since $|y| = 8$ and there are no 2-dimensional simple comodules of H, it is clear that y cannot have exactly 6 grouplike elements. A similar argument shows that y cannot have exactly 4 grouplike elements.

It follows that if (i) does not hold then y has no grouplike elements in its basic decomposition. Since any other element of $\mathscr{G}(H)$ has degree at least 3, $y = u + v$, where u and v are basic elements with $|u| = 3$ and $|v| = 5$. In this case (ii) holds. □

Lemma 6. *If H satisfies the assumptions of Theorem 4, then one of the following conditions holds*:

(i) *H contains a grouplike element of order 2 or 3,*

(ii) *there is a* 3*-dimensional self dual basic element* x_3 *with* $x_3^2 = 1 + x_3 + x_5$, *where* x_5 *is a* 5*-dimensional basic element*.

Proof. Assume (i) does not hold. By the previous lemma there is a 3-dimensional basic element with $x_3x_3^* = 1 + u + v$, where u and v are basic elements of $\mathscr{G}(H)$ with $|u| = 3$ and $|v| = 5$. Since $x_3x_3^*$ is self adjoint, the last relation implies that $u = u^*$ and $v = v^*$. If $u = x_3$, we are done. Otherwise the previous lemma applied to u instead of x_3 gives that $uu^* = 1 + u_1 + v_1$ with $|u_1| = 3$, $|v_1| = 5$, and u_1, v_1 self-adjoint. If $u \neq u_1$, Lemma 5 again gives $u_1u_1^* = 1 + u_2 + v_2$ where u_2, v_2 are self-adjoint with $|u_2| = 3$ and $|v_2| = 5$. It suffices to show that $u_1 = u_2$.

Since $m(u_1, u^2) = 1$, it follows from Theorem 3 that $m(u, uu_1) = 1$. Suppose $uu_1 = u + y$. Since $|y| = 6$, y is not a basic element of $\mathscr{G}(H)$. Therefore $y = w + \xi$ where $w, \xi \neq 0$.

Using the previous relations we have

$$u^2u_1 = (1 + u_1 + v_1)u_1 = u_1 + 1 + u_2 + v_2 + v_1u_1.$$

On the other hand

$$u^2u_1 = u(u + w + \xi) = 1 + u_1 + v_1 + uw + u\xi.$$

Hence

$$u_2 + v_2 + v_1u_1 = v_1 + uw + u\xi. \tag{1}$$

But $1 \leqslant m(w, uu_1) = m(u_1, uw)$. Assume $u_1 \neq u_2$. In this case, from the last two relations, u_1 must enter in the basic decomposition of v_1u_1 and $1 \leqslant m(u_1, v_1u_1) = m(v_1, u_1^2)$. Therefore $v_1 = v_2$ and $m(u_1, v_1u_1) = m(v_1, u_1^2) = 1$. Then (1) becomes $v_1u_1 + u_2 = uw + u\xi$. This relation is impossible in the case $u_1 \neq u_2$. Indeed, if $u_1 \neq u_2$ then the multiplicity of u_1 on the left hand side is equal to 1 whereas on the other side the multiplicity of u_1 is at least 2 since u_1 enters in both terms of the sum.

Before giving the proof of Theorem 4, one more lemma is needed.

Lemma 7. *Suppose H satisfies the assumptions of Theorem* 4 *and* $a, a', b \in \mathscr{G}(H)$ *with* $|a| = |a'| < |b|$ *and* a, b *basic elements. Let* x_3 *be a* 3*-dimensional basic element of* $\mathscr{G}(H)$ *with* $ax_3 = b + c + u = a' + c + v$, *for some* $c, u, v \in \mathscr{G}(H)_+$ *with* $c \neq 0$. *Then* $v = b$ *and* $u = a'$.

Proof. It is easy to see that any basic component of ax_3 has degree at least $|a|/3$. Indeed, if $1 \leqslant m(z, ax_3) = m(a, zx_3)$ then $|a| \leqslant |zx_3|$ and $|z| \geqslant |a|/3$. We have $b + u = a' + v$. Since $|a'| < |b|$ and b is a basic element, it follows that b is a basic component of v. Therefore $v = v_1 + b$ and $u = a' + v_1$. We want to show that $v_1 = 0$. If $v_1 \neq 0$ then $c + v_1$ must have at least three basic components since every basic element has odd degree. These are also basic components of ax_3. But this is impossible, since in that case $|c + v_1| \geqslant 3|a|/3 = |a|$ and therefore the degree of $b + c + u$ is strictly greater than $3|a|$. Thus $v_1 = 0$. □

We are ready to prove our main result.

Proof of Theorem 4. Assume (i) does not hold. By Lemma 6 there is a 3-dimensional basic element x_3 such that $x_3^2 = 1 + x_3 + x_5$.

It will be shown that (ii) holds in this case. For (ii) it suffices to prove the existence of two families of basic elements $\{x_{2n+1} : n \geqslant 1\}$, $\{x'_{2n+1} : n \geqslant 1\}$ corresponding to the two families of simple subcoalgebras $\{C_{2n+1} : n \geqslant 1\}$, $\{D_{2n+1} : n \geqslant 1\}$ satisfying $|x_{2n+1}| = |x'_{2n+1}| = 2n + 1$ and

$$x_{2n+1}x_3 = x_{2n-1} + x'_{2n+1} + x_{2n+3}$$

for all $n \geqslant 0$.

For $x_1 = 1$ and $x'_3 = x_3$ the first relation of (ii) is satisfied. Suppose we have found $x_3, x_5, \dots x_{2n+1}, x_{2n+3}$ and $x'_3, x'_5, \dots x'_{2n+1}$ such that:

$$x_{2k+1}x_3 = x_{2k-1} + x'_{2k+1} + x_{2k+3}$$

for any natural number k with $1 \leqslant k \leqslant n$.

We want to show that there are another two basic elements x'_{2n+3}, x_{2n+5} such that

$$x_{2n+3}x_3 = x_{2n+1} + x'_{2n+3} + x_{2n+5}.$$

Since $m(x_{2n+1}, x_{2n+3}x_3) = m(x_{2n+3}, x_{2n+1}x_3) = 1$ we may write

$$x_{2n+3}x_3 = x_{2n+1} + y_0 + z_0,$$

where y_0 is a basic component of $x_{2n+3}x_3$, different from x_{2n+1} and with the smallest possible degree. If $|y_0| \geqslant 2n + 3$ we are done. Indeed any other irreducible that enters in z_0 has dimension at least $2n + 3$. If z_0 is not basic then it contains at least three basic elements since $|z_0|$ is odd. In this case $|z_0| \geqslant 3(2n + 3)$ which is not possible. Hence z_0 is basic. Since $|y_0| + |z_0| = 4n + 8$ and $|z_0| \geqslant |y_0| \geqslant 2n + 3$ the following equalities are satisfied $|y_0| = 2n + 3$ and $|z_0| = 2n + 5$. Then let $x'_{2n+3} = y_0$ and $x_{2n+5} = z_0$.

The case $|y_0| < 2n + 3$ will be shown to be impossible. Note that $m(x_{2n+3}, y_0x_3) = m(y_0, x_{2n+3}x_3) = 1$ and if $y_0 \neq x_{2n+3}x_3$ we may write again

$$y_0x_3 = x_{2n+3} + y_1 + z_1,$$

where y_1 is a basic component of y_0x_3 different from x_{2n+3} and with the smallest possible dimension. The degree of $y_0x_3 - x_{2n+3}$ is even and therefore there are at least three basic components in the decomposition of y_0x_3. Since $|y_0| < 2n + 3$ we have $|y_1| < |y_0|$. The same procedure gives $y_1x_3 = y_0 + y_2 + z_2$ where y_2 is again a basic component of y_1x_3, different from y_0 and with the smallest possible dimension. Similarly $|y_2| < |y_1|$.

In this manner we construct a sequence of basic elements $y_0, y_1, \dots y_k$ with $|y_k| < |y_{k-1}| < \cdots < |y_2| < |y_1| < |y_0| < 2n + 3$ and

$$\begin{aligned} y_0x_3 &= x_{2n+3} + y_1 + z_1 \\ y_1x_3 &= y_0 + y_2 + z_2 \\ &\vdots \\ y_{k-1}x_3 &= y_{k-2} + y_k + z_k, \end{aligned}$$

where the z_i are not necessarily irreducible.

Since the dimension of y_k is decreasing this process must stop. Therefore we may suppose y_kx_3 is basic, so $y_kx_3 = y_{k-1}$. Note that the case $y_0x_3 = x_{2n+3}$ simply means that the process stops after the first stage.

It will be shown that $k=n=1$. First note that $k < n+1$ since $|y_k|<|y_{k-1}|< \cdots <|y_1| < |y_0| < 2n+3$ and all the elements have odd degree. Next we will prove that

$$y_{k-t} = y_kx_{2t+1} \tag{2}$$

for $1 \leqslant t \leqslant k+1$. For consistency of notation we put $y_{-1} = x_{2n+3}$.

For $t=1$, $y_kx_3 = y_{k-1}$ from above. Suppose

$$y_{k-t} = y_kx_{2t+1}$$

for $1 \leqslant t \leqslant s$. We need to prove that

$$y_{k-s-1} = y_kx_{2s+3}.$$

For $t=s$ we have $y_{k-s} = y_kx_{2s+1}$. Multiplication by x_3 gives $y_{k-s}x_3 = y_kx_{2s+1}x_3$. Then

$$y_{k-s-1} + y_{k-s+1} + z_{k-s+1} = y_kx_{2s-1} + y_kx'_{2s+1} + y_kx_{2s+3}$$

(We used that $2s+1 \leqslant 2k+1 \leqslant 2n+1$). Since $y_{k-s+1} = y_kx_{2s-1}$ it follows that

$$y_{k-s-1} + z_{k-s+1} = y_kx'_{2s+1} + y_kx_{2s+3}.$$

But $|y_kx'_{2s+1}|=|y_kx_{2s+1}|=|y_{k-s}| < |y_{k-s-1}|$. Lemma 7 applied to $a=y_{k-s}$, $a'=y_kx'_{2s+1}$ and $b=y_{k-s-1}$ implies that $z_{k-s+1} = y_kx'_{2s+1}$ and $y_{k-(s+1)} = y_kx_{2s+3}$ as required. Note that y_{-2} represents x_{2n+1} in the case $t=k+1$.

For $t=k+1$ relation (2) becomes

$$x_{2n+3} = y_kx_{2k+3}. \tag{3}$$

We show that this is impossible if $k<n$. Indeed, $2k+3 < 2n+3$ and relation (3) multiplied by x_3 gives

$$x_{2n+1} + y_0 + z_0 = y_kx_{2k+1} + y_kx'_{2k+3} + y_kx_{2k+5}.$$

Since $y_0 = y_kx_{2k+1}$ it follows that

$$x_{2n+1} + z_0 = y_kx'_{2k+3} + y_kx_{2k+5}. \tag{4}$$

It will be shown that x_{2n+1} cannot be a basic component of either term of the right hand side. The degree of the first term is

$$|y_kx'_{2k+3}| = |y_kx_{2k+3}| = |x_{2n+3}| = 2n+3$$

and the difference between its degree and the degree of x_{2n+1} is 2. The other components of $y_kx'_{2k+3}$ cannot be 1-dimensional since they are also components of $x_{2n+3}x_3$. If x_{2n+1} is a basic component of the second term, y_kx_{2k+5}, then relation 4 implies that $y_kx_{2k+5} = x_{2n+1} + u + v$, $z_0 = y_kx'_{2k+3} + u + v$ and $x_{2n+3}x_3 = x_{2n+1} + y_0 + y_kx'_{2k+3} + u + v$. Therefore $|y_0| + |u| + |v| = 2n+5$. Since any basic components of $x_{2n+3}x_3$ has degree at least $(2n+3)/3$ and y_0 is the smallest component different from x_{2n+1}, we have $(2n+3)/3 \leqslant |y_0| \leqslant (2n+5)/3$. But y_0x_3 contains x_{2n+3} and $|y_0x_3| - |x_{2n+3}| \leqslant 2$. So if $y_0x_3 - x_{2n+3} \neq 0$ then it has to be the sum of two grouplike elements, which is impossible. Indeed, if g and h were these two grouplike elements then $gy_0 = hy_0 = x_3$ and $gh^{-1}x_3 = x_3$ which implies $g = h$ and contradicts part 2 (c) of Theorem 3. We conclude that $y_0x_3 = x_{2n+3}$, and hence $|y_0| = (2n+3)/3$. Moreover, $|u| = |y_0| = (2n+3)/3$ and $|v| = |y_0| + 2$. Thus $ux_3 = x_{2n+3}$ and $vx_3 = x_{2n+3} + w$, where $|w| = 6$. But $n \geqslant 1$, $|v| \geqslant 4$ and so vx_3 cannot have any 1-dimensional basic components. Therefore w is the sum of two 3-dimensional basic components w_1 and w_2. Since $1 \leqslant m(w_i, vx_3) = m(v, w_ix_3)$ and $|w_ix_3| = 9$, it follows that $w_ix_3 - v$ has dimension at most 5. Multiplying on the right by x_3, it is easy to see that $w_ix_3 - v \neq 0$. So each product w_ix_3 has a 1-dimensional component g_i with $w_i = g_ix_3$. Therefore $vx_3^2 = x_{2n+3}x_3 + w_1x_3 + w_2x_3 = x_{2n+3}x_3 + g_1x_3^2 + g_2x_3^2$ has two components of degree 1, namely g_1 and g_2. On the other hand, $vx_3^2 = v + vx_3 + vx_5$ and the only 1-dimensional components of vx_3^2 appear in the basic decomposition of vx_5. Part 2(b) of Theorem 3 implies that $v = g_1x_5 = g_2x_5$. So for $i = 1, 2$, $vx_3 = g_ix_5x_3$ and its basic components have dimension at most 7. Since $2n + 3$ is divisible by 3 and $n \geqslant 1$ we have $|x_{2n+3}| = 2n + 3 \geqslant 9$. Hence x_{2n+3} cannot be a basic component of vx_3. This means that relation (3) cannot hold for $k < n$.

Now suppose $k = n$. Then $x_{2n+3} = y_nx_{2n+3}$ implies that $|y_n| = 1$ and $y_n = g$, a grouplike element of H. Moreover, relation 2 implies that $y_{n-t} = gx_{2t+1}$, for $0 \leqslant t \leqslant n$. Then

$$x_{2n+3}x_3 = x_{2n+1} + gx_{2n+1} + z_0, \tag{5}$$

which gives $|z_0| = 2n + 7$. We have to consider two cases, whether z_0 is a basic element or not.

If z_0 is basic then g has order 2, which would imply (i), contrary to our assumption. Indeed, the last relation multiplied by g becomes $x_{2n+3}x_3 = gx_{2n+3}x_3 = gx_{2n+1} + g^2x_{2n+1} + gz_0$. Therefore $g^2x_{2n+1} = x_{2n+1}$. The decomposition formula for $x_{2n+1}x_3$ multiplied by x_3 gives $g^2x_{2n-1} = x_{2n-1}$. Similarly, $g^2x_{2t+1} = x_{2t+1}$ for any $0 \leqslant t \leqslant n$. In particular $g^2x_1 = x_1$ gives $g^2 = 1$.

Hence z_0 cannot be a basic element. In this case, in its basic decomposition there are at least three terms, since every basic element has odd degree. By the choice of y_0, any of these basic elements has degree at least $|y_0| = 2n + 1$; therefore $2n + 7 \geqslant 3(2n + 1)$ which implies $n = 1$.

It will be shown that this is impossible. Write $z_0 = u_1 + u_2 + u_3$. Then (5) becomes

$$x_5x_3 = x_3 + gx_3 + u_1 + u_2 + u_3$$

and 1-dimensional basic elements cannot appear on the right-hand side. Hence $|u_i| = 3$ for $1 \leqslant i \leqslant 3$. It is easy to see that each of the products u_ix_3 has a 1-dimensional basic component since each of them has degree 9 and all of them have x_5 as a component. Hence, by part 2(b) of Theorem 3, $u_i = h_ix_3$ where each h_i is a grouplike element.

Consider the set V of grouplike elements h such that $hx_5 = x_5$. By part 2(c) of Theorem 3, V is a group. It is easy to see that $1, g, h_1, h_2, h_3 \in V$. If $h \in V$ then $m(hx_3, x_5x_3) = m(hx_5, x_3^2) = m(x_5, x_3^2) = 1$, implying that hx_3 is one of the basic elements u_i. But if $hx_3 = h_ix_3$ then multiplication by x_3 on the right gives $h + hx_3 + hx_5 = h_i + h_ix_3 + h_ix_5$ and so $h = h_i$. Therefore $V = \{1, g, h_1, h_2, h_3\}$. The relation $x_3^2 = 1 + x_3 + x_5$ multiplied by x_5 on the right gives $x_5^2 = 4x_5 + 1 + g + h_1 + h_2 + h_3$. This shows that $\{x_3, x_5\} \cup V$ generates a standard subring R_1 of $\mathcal{G}(H)$. We have $x_3^3 = x_3^2x_3 = x_3 + x_3^2 + x_5x_3$. On the other hand, $x_3^3 = x_3x_3^2 = x_3 + x_3^2 + x_3x_5$. Therefore $x_5x_3 = x_3x_5$. But $x_5x_3 = (x_3x_5)^*$ and

R_1 is closed under *. By Theorem 2, it corresponds to a Hopf subalgebra K_1 of H with dimension equal to $5*1^2+5*3^2+5^2=75$. In a similar way it can be seen that $\{x_5\}\cup V$ generates a standard subring R_2. It is also closed under * and by the same argument it corresponds to a Hopf subalgebra K_2 of H with dimension 30. Clearly $R_2\subseteq R_1$. Then K_2 is a Hopf subalgebra of K_1 contradicting the freeness theorem. □

Corollary 8. *Let H be an odd dimensional Hopf algebra over an algebraically closed field k. If H contains a simple subcoalgebra of dimension* 9 *and no simple subcoalgebras of even dimension then* $\dim_k H$ *is divisible by* 3.

Proof. Since $\dim_k H$ is odd the freeness theorem implies that H cannot have a grouplike element of order 2. Therefore H has a grouplike element of order 3 and the same theorem gives the divisibility relation. □

A similar result was obtained by Kashina et al. [4]. They assume that the characteristic of the base field is 0 in which case the assumption that H is odd dimensional automatically implies that H has no even dimensional simple subcoalgebras.

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