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New examples of the Green functors arising from representation theory of semisimple Hopf algebras

Sebastian Burciu Q1

Abstract

A general Mackey-type decomposition for representations of semisimple Hopf algebras is investigated. We show that such a decomposition occurs in the case that the module is induced from an arbitrary Hopf subalgebra and it is restricted back to a group subalgebra. Some other examples when such a decomposition occurs are also constructed. They arise from gradings on the category of corepresentations of a semisimple Hopf algebra and provide new examples of the Green functors in the literature.

1. *Introduction and main results*

Mackey's decomposition theorem of induced modules from subgroups is a very important tool in the representations theory of finite groups. This decomposition describes the process of an induction composed with a restriction in terms of the reverse processes consisting of restrictions followed by inductions. More precisely, if G is a finite group, M and N are two subgroups of G and V a finite-dimensional k-linear representation of M, then the well-known Mackey's decomposition states that there is an isomorphism of kN-modules: Q3

$$V \uparrow_{kM}^{kG} \downarrow_{kN}^{kG} \xrightarrow{\delta_V} \bigoplus_{x\in M\backslash G/N} k[N] \otimes_{k[\,{}^xM\cap N]} {}^xV. \qquad (1.1)$$

Here ${}^xM := xMx^{-1}$ is the conjugate subgroup and ${}^xV := V$ is the conjugate ${}^xM$-representation defined by $(xmx^{-1}) \cdot v := m \cdot v$ for all $m \in M$ and $v \in V$. The direct sum is indexed by a set of representative group elements of G for all double cosets $M\backslash G/N$ of G relative to the two subgroups M and N. Note that the inverse isomorphism of δ_V is given on each direct summand by the left multiplication operator $n \otimes_{kN\cap k\,{}^xM} v \mapsto nx \otimes_{kM} v$; see [**20**, Proposition 22].

The goal of this paper is to investigate a similar Mackey-type decomposition for the induced modules from Hopf subalgebras of semisimple Hopf algebras and restricted back to other Hopf subalgebras. In order to do this, we use the corresponding notion of a double coset relative to a pair of Hopf subalgebras of a semisimple Hopf algebra that was introduced by the author in [**3**]. We also have to define a conjugate Hopf subalgebra corresponding to the notion of a conjugate subgroup. For any Hopf subalgebra $K \subseteq H$ of a semisimple Hopf algebra H and any simple subcoalgebra C of H, we define the conjugate Hopf subalgebra ${}^CK$ of K in Proposition 3.2. This notion corresponds to the notion of conjugate subgroup from the above decomposition. In order to deduce that ${}^CK$ is a Hopf subalgebra of H, we use several crucial results from [**18**] concerning the product of two subcoalgebras of a semisimple Hopf algebra as well as Frobenius–Perron theory for nonnegative matrices.

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Using these tools, we can prove one of the following main results of this paper.

Theorem 1.1. *Let $K \subseteq H$ be a Hopf subalgebra of a semisimple Hopf algebra and M be a finite-dimensional K-module. Then for any subgroup $G \subseteq G(H)$ one has a canonical isomorphism of kG-modules*

$$M \uparrow_K^H \downarrow_{kG}^H \xrightarrow{\delta_M} \bigoplus_{C \in kG\backslash H/K} (kG \otimes_{kG_C} {}^C M). \tag{1.2}$$

Here $G(H)$ is the group of group-like elements of H and the subgroup $G_C \subseteq G$ is determined by $kG \cap {}^C K = kG_C$. The conjugate module ${}^C M$ is defined by ${}^C M := CK \otimes_K M$.

As in the classical group case, the homomorphism δ_M is the inverse of a natural homomorphism π_M, which is constructed by the left multiplication on each direct summand. It is not difficult to check (see Theorem 1.2) that in general, for any two Hopf subalgebras $K, L \subseteq H$ the left multiplication homomorphism π_M is always an epimorphism.

Theorem 1.2. *Let K and L be two Hopf subalgebras of a semisimple Hopf algebra H. For any finite-dimensional left K-module M, there is a canonical epimorphism of L-modules*

$$\bigoplus\nolimits_{C \in L\backslash H/K} (L \otimes_{L \cap {}^C K} {}^C M) \xrightarrow{\pi_M} M \uparrow_K^H \downarrow_L^H \tag{1.3}$$

given on components by $l \otimes_{L \cap {}^C K} v \mapsto lv$ for any $l \in L$ and any $v \in {}^C M$. Here the conjugate module ${}^C M$ is defined as above by ${}^C K := CK \otimes_K M$.

We remark that there is a similar direction in the literature in the paper [**8**]. In this paper, the author considers a similar decomposition, but for pointed Hopf algebras instead of semisimple Hopf algebras. Also, in [**14**] the author proves a similar result for some special Hopf subalgebras of quantum groups at roots of 1.

Another particular situation of Mackey's decomposition can be found in [**3**]. In this paper, it is proved that for pairs of Hopf subalgebras that generate just one double coset subcoalgebra, the above epimorphism π_M from Theorem 1.2 is in fact an isomorphism. In both papers, the above homomorphism π_M is given by left multiplication.

Definition 1.3. We say that (L, K) is a Mackey pair of Hopf subalgebras of H if the above left multiplication homomorphism π_M from Theorem 1.2 is an isomorphism for any finite-dimensional left K-module M.

Then Theorem 1.1 states that (kG, K) is a Mackey pair for any Hopf subalgebra $K \subset H$ and any subgroup $G \subset G(H)$. Moreover, in Theorem 6.1 it is shown that for any normal Hopf subalgebra K of H, the pair (K, K) is a Mackey pair. This allows us to prove a new formula (see Proposition 6.3) for the restriction of an induced module from a normal Hopf subalgebra, which substantially improves [**3**, Proposition 5.12]. It also gives a criterion for an induced module from a normal Hopf subalgebra to be irreducible, generalizing a well-known criterion for group representations; see, for example, [**20**, Corollary 7.1].

For any semisimple Hopf algebra H, using the universal grading of the fusion category $\mathrm{Rep}(H^*)$ we construct in Section 5 new Mackey pairs of Hopf subalgebras of H. In turn, this allows us to define a Green functor on the universal group G of the category of representations of H^*. For $H = kG$, one obtains in this way the usual Green functor [**11**]. As in group theory, this new Green functor can be used to determine new properties of the Grothendieck ring of a semisimple Hopf algebra.

In the last section, we prove the following tensor product formula for two induced modules from a Mackey pair of Hopf subalgebras.

THEOREM 1.4. *Suppose that (L, K) is a Mackey pair of Hopf subalgebras of a semisimple Hopf algebra H. Then for any K-module M and any L-module N one has a canonical isomorphism*:

$$M\uparrow_K^H \otimes N\uparrow_L^H \xrightarrow{\cong} \bigoplus_{C\in L\backslash H/K} ((CK\otimes_K M)\downarrow^{^C K}_{L\cap {^C K}} \otimes N\downarrow^L_{L\cap {^C K}})\uparrow^H_{L\cap {^C K}}. \qquad (1.4)$$

This generalizes a well-known formula for the tensor product of two induced group representations given, for example, in [2].

This paper is structured as follows. In Section 1, we recall the basic results on coset decomposition for Hopf algebras. Section **??** contains the construction for the conjugate Q6 Hopf subalgebra generalizing the conjugate subgroup of a finite group. These results are inspired from the treatment given in [5]. A general characterization for the conjugate Hopf subalgebra is given in Theorem 3.8. This theorem is automatically satisfied in the group case. In Section 4 we prove Theorem 1.2. Also in Section 4, we prove Theorem 1.1. We Q6 also show that for any semisimple Hopf algebra, there are some canonical associated Mackey pairs arising from the universal grading of the category of finite-dimensional corepresentations (see Theorem 5.5). Necessary and sufficient conditions for a given pair to be a Mackey pair are given in terms of the dimensions of the two Hopf subalgebras of the pair and their conjugate Hopf subalgebras. In Section 6, we prove that for a normal Hopf subalgebra K the pair (K, K) is always a Mackey pair. In Subsection 6.2, we prove the tensor product formula from Theorem 1.4.

We work over an algebraically closed field k of characteristic zero. We use Sweedler's notation Δ for comultiplication, but with the sigma symbol dropped. All the other Hopf algebra notations of this paper are the standard ones, used, for example, in [15].

2. *Double coset decomposition for Hopf subalgebras of semisimple Hopf algebras*

2.1. *Conventions*

Throughout this paper, H will be a semisimple Hopf algebra over k and $\Lambda_H \in H$ denotes its idempotent integral ($\epsilon(\Lambda_H) = 1$). It follows that H is also cosemisimple [13]. If K is a Hopf subalgebra of H, then K is also a semisimple and cosemisimple Hopf algebra [15]. For any two subcoalgebras C and D of H, we denote by CD the subcoalgebra of H generated as a k-vector space by all elements of the type cd with $c \in C$ and $d \in D$.

Let $G_0(H)$ be the Grothendieck group of the category of left H-modules. Then, since H is a Hopf algebra the group $G_0(H)$ has a ring structure under the tensor product of modules. Then the character ring $C(H) := G_0(H) \otimes_{\mathbb{Z}} k$ is a semisimple subalgebra of H^* (see [22]). Denote by $\mathrm{Irr}(H)$ the set of all irreducible characters of H. Then $C(H)$ has a basis consisting of the irreducible characters $\chi \in \mathrm{Irr}(H)$. Also, $C(H)$ coincides to the space $\mathrm{Cocom}(H^*)$ of cocommutative elements of H^*. By duality, the character ring $C(H^*)$ of the dual Hopf algebra H^* is a semisimple Hopf subalgebra of H and $C(H^*) = \mathrm{Cocom}(H)$. If M is a finite-dimensional H-module with character χ, then the linear dual M^* becomes a left H-module with character $\chi^* := \chi \circ S$.

2.2. *The subcoalgebra associated to a comodule*

Let W be any right H-comodule. Since H is finite-dimensional, it follows that W is a left H^*-module via the module structure $f \cdot w = f(w_1)w_0$, where $\rho(w) = w_0 \otimes w_1$ is the given right H-comodule structure of W. Then one can associate to W the coefficient subcoalgebra denoted by C_W (see [**12**]). Recall that C_W is the minimal subcoalgebra C of H with the property that $\rho(W) \subset W \otimes C$. Moreover, it can be shown that $C_W = (\mathrm{Ann}_{H^*}(W))^{\perp}$ and C_W is called the subcoalgebra of H associated to the right H-comodule W. If W is a simple right H-comodule (or equivalently W is an irreducible H^*-module), then the associated subcoalgebra C_W is a co-matrix coalgebra. More precisely, if $\dim W = q$, then $\dim C_W = q^2$ and it has a k-linear basis given by x_{ij} with $1 \leqslant i, j \leqslant q$. The coalgebra structure of C_W is then given by $\Delta(x_{ij}) = \sum_l x_{il} \otimes x_{lj}$ for all $1 \leqslant i, j \leqslant q$. Moreover, the irreducible character $d \in C(H^*)$ of W is given by formula $d = \sum_{i=0}^{q} x_{ii}$. It is easy to check that W is an irreducible H^*-module if and only if C_W is a simple subcoalgebra of H. This establishes a canonical bijection between the set $\mathrm{Irr}(H^*)$ of simple right H^*-comodules and the set of simple subcoalgebras of H. For any irreducible character $d \in \mathrm{Irr}(H^*)$, we also use the notation C_d for the simple subcoalgebra of H associated to the character d (see [**12**]).

Recall also that if M and N are two right H-comodules, then $M \otimes N$ is also a comodule with $\rho(m \otimes n) = m_0 \otimes n_0 \otimes m_1 n_1$.

REMARK 2.1. For a simple subcoalgebra $C \subset H$, we denote by M_C the simple H-comodule associated to C. Following [**18**], if C and D are simple subcoalgebras of a semisimple Hopf algebra H, then the simple comodules entering in the decomposition of $M_C \otimes M_D$ are in bijection with the set of all simple subcoalgebras of the product subcoalgebra CD of H. Moreover, this bijection is given by $W \mapsto C_W$ for any simple subcomodule W of $M_C \otimes M_D$.

2.3. *Double coset decomposition for Hopf subalgebras*

In this subsection, we recall the basic facts on double cosets of semisimple Hopf algebras developed in [**3**]. Let L and K be two Hopf subalgebras of H. As in [**3**], one can define an equivalence relation $r^H_{L,K}$ on the set of simple subcoalgebras of H as following: $C \sim D$ if $C \subset LDK$. The fact that $r^H_{L,K}$ is an equivalence relation is proved in [**3**]. In this paper, it is shown that $C \sim D$ if and only if $LDK = LCK$ as subcoalgebras of H. We also have the following proposition.

PROPOSITION 2.2. *If C and D are two simple subcoalgebras of H, then the following are equivalent*:

(1) $C \sim D$,
(2) $LCK = LDK$,
(3) $\Lambda_L C \Lambda_K = \Lambda_L D \Lambda_K$.

Proof. First assertion is equivalent to the second from [**3**, Corollary 2.5]. Clearly, (2) $\Rightarrow$ (3) by left multiplication with Λ_K and right multiplication with Λ_L. It will be shown that (3) $\Rightarrow$ (1). One has the following decomposition:

$$H = \bigoplus_{i=1}^{l} LC_iK,$$

where $C_1, \ldots, C_l$ are representative subcoalgebras for each equivalence class of $r^H_{K,L}$.

It follows that $\Lambda_L H \Lambda_K = \bigoplus_{i=1}^{l} \Lambda_L C_i \Lambda_K$. Thus, if $C \nsim D$, then $\Lambda_L C \Lambda_K \cap \Lambda_L D \Lambda_K = 0$, which proves (1). □

Remark 2.3. The above proposition shows that for any two simple subcoalgebras C and D of H, then either $LCK = LDK$ or $LCK \cap LDK = 0$. Therefore, for any subcoalgebra $D \subset LCK$, one has that $LCK = LDK$. In particular, for $L = k$, the trivial Hopf subalgebra, one has that $D \subset CK$ if and only if $DK = CK$.

2.3.1. *Notation* For the rest of the paper, we denote by $L\backslash H/K$ the set of double cosets LCK of H with respect to L and K. Thus, the elements LCK of $L\backslash H/K$ are given by a choice of representative of simple subcoalgebras in each equivalence class of $r^H_{L,\,K}$. Similarly, we denote by H/K be the set of right cosets CK of H with respect to K. This corresponds to a choice of a representative simple subcoalgebra in each equivalence class of $r^H_{k,\,K}$.

Remark 2.4. As noted in [3], one has that $LCK \in \mathcal{M}^H_K$ and therefore LCK is a free right K-module. Similarly, $LCK \in {}^H_L\mathcal{M}$ and therefore LCK is also a free left L-module.

By Burciu [3, Corollary 2.6], it follows that two simple subcoalgebras C and D are in the same double coset of H with respect to L and K if and only if

$$\Lambda_L \frac{c}{\epsilon(c)} \Lambda_K = \Lambda_L \frac{d}{\epsilon(d)} \Lambda_K, \tag{2.1}$$

where c and d are the irreducible characters of H^* associated to the simple subcoalgebras C and D. In particular, for $L = k$, the trivial Hopf subalgebra, it follows that $CK = DK$ if and only if

$$c\Lambda_K = \frac{\epsilon(c)}{\epsilon(d)} d\Lambda_K. \tag{2.2}$$

2.4. *Principal eigenspace for $\langle C \rangle$*

For a simple subcoalgebra C, we denote by $\langle C \rangle$ the Hopf subalgebra of H generated by C. If d is the character associated to C, then we also denote this Hopf subalgebra by $\langle d \rangle$.

2.4.1. *Frobenius–Perron theory for nonnegative matrices* Next, we will use the Frobenius–Perron theorem for matrices with nonnegative entries (see [9]). If $A \geqslant 0$ is such a matrix, then A has a positive eigenvalue λ, which has the biggest absolute value among all the other eigenvalues of A. The eigenspace corresponding to λ has a unique vector with all entries positive. λ is called Q4 the principal eigenvalue of A and the corresponding positive vector is called the principal vector of A. Also, the eigenspace of A corresponding to λ is called the principal eigenspace of the matrix A.

For an irreducible character $d \in \mathrm{Irr}(H^*)$, let L_d be the linear operator on $C(H^*)$ given by left multiplication by d. Recall [3] that $\epsilon(d)$ is the Frobenius–Perron eigenvalue of the nonnegative matrix associated to the operator L_d with respect to the basis given by the irreducible characters of H^*. In analogy with Frobenius–Perron theory, for a subcoalgebra C with associated character d, we call the space of eigenvectors of L_d corresponding to the eigenvalue $\epsilon(d)$ as the principal eigenspace for L_d.

The following corollary is a particular case of [3, Theorem 2.4].

COROLLARY 2.5. *The principal eigenspace of* L_{Λ_K} *is* $\Lambda_K C(H^*)$ *and it has a* k*-linear basis given by* $\Lambda_K d$*, where* d *are the characters of a set of representative simple coalgebras for the right cosets of* K *inside* H*.*

Using this, we can prove the following theorem.

THEOREM 2.6. *Let* C *be a subcoalgebra of a semisimple Hopf algebra* H *with associated character* $d \in C(H^*)$*. Then the principal eigenspaces of* L_d *and* $L_{\Lambda_{\langle d\rangle}}$ *coincide.*

Proof. Let V be the principal eigenspace of $L_{\Lambda_{\langle d\rangle}}$ and W be the principal eigenspace of L_d. Then by Corollary 2.5, one has that $V = \Lambda_{\langle d\rangle} C(H^*)$. Since $d\Lambda_{\langle d\rangle} = \epsilon(d)\Lambda_{\langle d\rangle}$, it follows that clearly $V \subseteq W$. On the other hand, since $\Lambda_{\langle d\rangle}$ is a polynomial with rational coefficients in d (see [17, Corollary 19]) it also follows that $W \subseteq V$. □

2.5. *Rank of cosets*

Let K be a Hopf subalgebra of a semisimple Hopf algebra H. Consider the equivalence relation $r^H_{k,K}$ on the set $\mathrm{Irr}(H^*)$ of simple subcoalgebras of H. As above, one has $C \sim D$ if and only if $CK = DK$. Therefore,

$$H = \bigoplus\nolimits_{C \in H/K} CK. \tag{2.3}$$

LEMMA 2.7. *The equivalence class under* $r^H_{k,K}$ *of the trivial subcoalgebra* k *is the set of all simple subcoalgebras of* K*.*

Proof. Indeed, suppose that C is a simple subcoalgebra of H equivalent to the trivial subcoalgebra k. Then $CK = kK = K$ by Proposition 2.2. Therefore, $C \subset CK = K$. Conversely, if $C \subset K$, then $CK \subset K$ and, since $CK \in \mathcal{M}^H_K$, it follows that $CK = K$. Thus, $C \sim k$. □

PROPOSITION 2.8. *If* D *is a simple subcoalgebra of a semisimple Hopf algebra* H *and* $e \in K$ *is an idempotent, then*

$$DK \otimes_K Ke \cong DKe,$$

as vector spaces.

Proof. Since H is free right K-module, one has that the map

$$\phi : H \otimes_K Ke \longrightarrow He, \quad h \otimes_K re \longmapsto hre$$

is an isomorphism of H-modules. Using the above decomposition (2.3) of H and the fact that DK is a free right K-module, note that ϕ sends $DK \otimes_K Ke$ to DKe. □

COROLLARY 2.9. *Let* K *be a Hopf subalgebra of a semisimple Hopf algebra* H*. For any simple subcoalgebra* C *of* H*, one has that the rank of* CK *as right* K*-module is* $\dim_k C\Lambda_K$*.*

Proof. Put $e = \Lambda_K$, the idempotent integral of K in the above proposition. □

2.6. *Frobenius–Perron eigenvectors for cosets*

Let T be the linear operator given by right multiplication with Λ_K on the character ring $C(H^*)$.

Remark 2.10. Using [3, Proposition 2.5], it follows that the largest (in absolute value) eigenvalue of T equals $\dim K$. Moreover, a basis of eigenvectors corresponding to this eigenvalue is given by $c\Lambda_K$, where the character $c \in \mathrm{Irr}(H^*)$ runs through a set of irreducible characters representative for all the right cosets $CK \in H/K$.

3. *The conjugate Hopf subalgebra* ${}^CK$

Let, as above, K be a Hopf subalgebra of a semisimple Hopf algebra H. For any simple subcoalgebra C of H in this section, we construct the conjugate Hopf subalgebra ${}^CK$ appearing in Theorem 1.3. If $c \in \mathrm{Irr}(H^*)$ is the associated irreducible character of C, then consider the following subset of $\mathrm{Irr}(H^*)$:

$$ {}^cK = \{d \in \mathrm{Irr}(H^*) \mid dc\Lambda_K = \epsilon(d)c\Lambda_K\}, \tag{3.1}$$

where, as above, $\Lambda_K \in K$ is the idempotent integral of K.

Recall from [18] that a subset $X \subset \mathrm{Irr}(H^*)$ is closed under multiplication if for every two elements $c, d \in X$ in the decomposition of the product $cd = \sum_{e\in \mathrm{Irr}(H^*)} m^e_{c,d}e$, then one has $e \in X$ whenever $m_e \neq 0$. Also, a subset $X \subset \mathrm{Irr}(H^*)$ is closed under '*' if $x^* \in X$ for all $x \in X$.

Following [18], any subset $X \subset \mathrm{Irr}(H^*)$ closed under multiplication generates a subbialgebra $H(X)$ of H defined by

$$H(X) := \bigoplus\nolimits_{x\in X} C_x. \tag{3.2}$$

Moreover, if the set X is also closed under '*', then $H(X)$ is a Hopf subalgebra of H.

Remark 3.1. Since in our case H is finite-dimensional, it is well known that any subbialgebra of H is also a Hopf subalgebra. Therefore, in this case any set X of irreducible characters closed under product is also closed under '*'.

Proposition 3.2. *The set* ${}^cK \subset \mathrm{Irr}(H^*)$ *is closed under multiplication and '*' and it generates a Hopf subalgebra* ${}^CK$ *of* H*. Thus,*

$$ {}^CK = \bigoplus\nolimits_{d\in {}^cK} C_d. \tag{3.3}$$

Proof. Suppose that D and D' are two simple subcoalgebras of H whose irreducible characters satisfy $d, d' \in {}^cK$. Then one has $dd'c\Lambda_K = \epsilon(dd')c\Lambda_K$. On the other hand, suppose that

$$dd' = \sum_{e\in \mathrm{Irr}(H^*)} m^e_{d,d'} e. \tag{3.4}$$

Then $\epsilon(dd')c\Lambda_K = dd'c\Lambda_K = \sum_{e\in\mathrm{Irr}(H^*)} m^e_{d,d'} ec\Lambda_K$ and Remark 2.10 implies that $ec\Lambda_K$ is a scalar multiple of $c\Lambda_K$ for any e with $m^e_{d,d'} \neq 0$. Therefore, $ec\Lambda_K = \epsilon(e)c\Lambda_K$ and $e \in {}^cK$. This shows that ${}^CK$ is a subbialgebra of H and by Remark 3.1 a Hopf subalgebra of H. □

Sometimes the notation ${}^CK$ will also be used for ${}^cK$, where $c \in \mathrm{Irr}(H^*)$ is the irreducible character associated to the simple subcoalgebra C.

The notion of conjugate Hopf subalgebra ${}^{C}K$ is motivated by the following proposition.

Proposition 3.3. *Let H be a semisimple Hopf algebra over k. If the simple subcoalgebra C is of the form $C = kg$ with $g \in G(H)$ a group-like element of H, then ${}^{C}K = gKg^{-1}$.*

Proof. Indeed, suppose that $D \in {}^{C}K$. If d is the associated irreducible character of D, then by definition it follows that $dg\Lambda_K = g\Lambda_K$. Thus, $g^{-1}dg\Lambda_K = \Lambda_K$. Therefore, the simple subcoalgebra $g^{-1}Dg$ of H is equivalent to the trivial subcoalgebra k. Then, using Lemma 2.7, one has that $g^{-1}Dg \subset K$ and therefore ${}^{C}K \subset gKg^{-1}$. The other inclusion $gKg^{-1} \subset {}^{C}K$ is obvious. □

Remark 3.4. In particular, for $H = kG$, one has that ${}^{C}k[M] = k[{}^{x}M]$, where $x \in G$ is given by $C = kx$.

Remark 3.5. (i) Using Remark 2.1, it follows from the definition of conjugate Hopf subalgebra that CK is always a left ${}^{C}K$-module.

(ii) Note that if $C(H^*)$ is commutative, then ${}^{C}K \supseteq K$. Indeed, for any $d \in \mathrm{Irr}(K^*)$ one has $d\Lambda_K = \epsilon(d)\Lambda_K$, and therefore $dc\Lambda_K = cd\Lambda_K = \epsilon(d)c\Lambda_K$.

(iii) If K is a normal Hopf subalgebra of H, then since Λ_K is a central element in H, by the same argument it also follows that ${}^{C}K \supseteq K$.

3.1. *Some properties of the conjugate Hopf subalgebra*

Proposition 3.6. *Let H be a semisimple Hopf algebra and K be a Hopf subalgebra of H. Then for any simple subcoalgebra C of H, one has that ${}^{C}K$ coincides to the maximal Hopf subalgebra L of H with the property $LCK = CK$.*

Proof. The equality ${}^{C}KCK = CK$ follows from the character equality $\Lambda_{{}^{C}K}c\Lambda_K = \epsilon(\Lambda_{{}^{C}K})c\Lambda_K$ and Remark 2.1. Conversely, if $LCK = CK$ by passing to the regular H^*-characters and using equation (2.1), then it follows that $\Lambda_L c\Lambda_K = \epsilon(\Lambda_L)c\Lambda_K$, which shows that $L \subset {}^{C}K$. □

Note that Remark 2.3, together with the previous proposition, implies that ${}^{C}KC \subseteq CK$.

Corollary 3.7. *One has that ${}^{C}K \subseteq CKC^*$.*

Proof. Since $S(C) = C^*$ by applying the antipode S to the above inclusion, one obtains that $C^*\,{}^{C}K \subseteq KC^*$. Therefore, $CC^*\,{}^{C}K \subseteq CKC^*$ and then one has ${}^{C}K \subseteq CC^*\,{}^{C}K \subseteq CKC^*$. □

Theorem 3.8. *One has that ${}^{C}K$ is the largest Hopf subalgebra L of H with the property $LC \subseteq CK$.*

Proof. We have seen above that ${}^{C}KC \subseteq CK$. Suppose now that $LC \subseteq CK$ for some Hopf subalgebra L of H. Then, by Remark 2.3, it follows that $LCK = CK$. Thus, by passing to regular characters, one has that $\Lambda_L c\Lambda_K = \epsilon(\Lambda_L)c\Lambda_K$, which shows the inclusion $L \subseteq {}^{C}K$. □

PROPOSITION 3.9. *Let H be a semisimple Hopf algebra and K be a Hopf subalgebra of H. Then, for any subcoalgebra D with $DK = CK$, one has that ${}^{D}K = {}^{C}K$.*

Proof. One has that ${}^{C}KCK = CK$. If $D \subset CK$, then by Remark 2.3, one has that ${}^{C}KDK = {}^{C}KCK = CK = DK$, which shows that ${}^{C}K = {}^{D}K$. □

4. *Mackey-type decompositions for representations of Hopf algebras*

Let K be a Hopf subalgebra of a semisimple Hopf algebra H and M be a finite-dimensional K-module. Note that for any simple subcoalgebra C of H, one has by Proposition 3.6 that ${}^{C}M := CK \otimes_K M$ is a left ${}^{C}K$-module via the left multiplication with elements of ${}^{C}K$.

REMARK 4.1. Let $H = kG$ be a group algebra of a finite group G and $K = kA$ for some subgroup A of G. Then note that ${}^{C}M := CK \otimes_K M$ coincides to the usual conjugate module ${}^{g}M$ if $C = kg$ for some $g \in G$. Recall that ${}^{g}M = M$ as vector spaces and $(gag^{-1}) \cdot m = a \cdot m$ for all $a \in A$ and all $m \in M$.

4.1. *Proof of Theorem* 1.2

Proof. Since by definition of the double cosets, one has $H = \bigoplus_{C \in L\backslash H/K} LCK$ and each LCK is a free K-module, the following decomposition of L-modules follows:

$$M \uparrow_K^H \downarrow_L^H = H \otimes_K M \cong \bigoplus_{C \in L\backslash H/K} (LCK \otimes_K M). \tag{4.1}$$

Consider now the k-linear map $\pi_M^{(C)} : L \otimes_{L \cap {}^{C}K} (CK \otimes_K M) \to LCK \otimes_K M$ given by

$$l \otimes_{L \cap {}^{C}K} (cx \otimes_K m) \longmapsto lcx \otimes_K m,$$

for all $l \in L$, $x \in K$, $c \in C$ and $m \in M$. It is easy to see that $\pi_M^{(C)}$ is a well-defined map and clearly a surjective morphism of L-modules. Then $\pi_M := \bigoplus_{C \in L\backslash H/K} \pi_M^{(C)}$ is surjective morphism of L-modules and the proof is complete. □

REMARK 4.2. Suppose that for $M = k$ one has that π_k isomorphism in Theorem 1.2. Then, using a dimension argument, it follows that the same epimorphism π_M from Theorem 1.2 is in fact an isomorphism for any finite-dimensional left H-module M.

4.2. *Mackey pairs*

It follows from the proof above that (L, K) is a Mackey pair if and only if π_k is an isomorphism, that is, if and only if each $\pi_k^{(C)}$ is isomorphism for any simple subcoalgebra C of H. Since $\pi_k^{(C)}$ is surjective passing to dimensions, one has that (L, K) is a Mackey pair if and only if

$$\dim LCK = \frac{(\dim L)\,(\dim CK)}{\dim L \cap {}^{C}K}, \tag{4.2}$$

for any simple subcoalgebra C of H.

Note that for $C = k1$, the above condition can be written as

$$\dim\ LK = \frac{(\dim L)(\dim K)}{\dim(L \cap K)}.$$

Remark 4.3. Note also that for any Mackey pair, it follows that

$$\frac{\dim L \cap {}^D K}{\dim DK} = \frac{\dim L \cap {}^C K}{\dim CK}, \tag{4.3}$$

if $LCK = LDK$.

Example 4.4. Suppose that L, K are Hopf subalgebras of H with $LK = KL$. Then (L, K) is a Mackey pair of Hopf subalgebras of LK by Burciu [**3**, Proposition 3.3].

4.3. *Proof of Theorem* 1.1

Let L, K be two Hopf subalgebras of a semisimple Hopf algebra H and let C be a simple subcoalgebra of H. Note that equations (2.1) and (2.2) imply that LCK can be written as a direct sum of right K-cosets,

$$LCK = \bigoplus_{DK \in \mathcal{S}} DK, \tag{4.4}$$

for a subset $\mathcal{S} \subset H/K$ of right cosets of K inside H. Note that always one has $CK \in \mathcal{S}$.

Next, we give a proof for the main result of Theorem 1.1.

Proof. Suppose that $L = kG$. By equation (4.2), one has to verify

$$\dim(kG)CK = \frac{|G|\,(\dim CK)}{\dim kG \cap {}^C K}, \tag{4.5}$$

for any subcoalgebra C of H. Since $kG \cap {}^C K$ is a Hopf subalgebra of kG, it follows that $kG \cap {}^C K = kG_C$ for some subgroup G_C of G. By equation (3.1), it follows that $G_C = \{g \in G \,|\, gd\Lambda_K = d\Lambda_K\}$, where $d \in \mathrm{Irr}(H^*)$ is the character associated to C. In terms of subcoalgebras, this can be written as $G_C = \{g \in G \,|\, gCK = CK\}$.

With the above notation, equation (4.5) becomes

$$\dim(kG)CK = \frac{|G|}{|G_C|} \dim CK. \tag{4.6}$$

Note that the group G acts transitively on the set $\mathcal{S}$ from equation (4.4). The action is given by $g \cdot DK = gDK$ for any $g \in G$ and any $DK \in \mathcal{S}$. Let St_C be the stabilizer of the right coset CK. Thus, the subgroup St_C of G is defined by $\mathrm{St}_C = \{g \in G \,|\, gCK = CK\}$, which shows that $\mathrm{St}_C = G_C$. Note that $\dim DK = \dim CK$ for any $DK \in \mathcal{S}$ since $DK = gCK$ for some $g \in G$. Thus, $\dim(kG)CK = |\mathcal{S}|(\dim CK)$ and equation (4.6) becomes

$$|\mathcal{S}| = \frac{|G|}{|G_C|}, \tag{4.7}$$

which is the same as the formula for the size of the orbit $\mathcal{S}$ of CK under the action of the finite group G. □

5. *New examples of the Green functors*

In this section, we construct new examples of the Green functors arising from gradings on the category of corepresentations of semisimple Hopf algebras.

5.1. *Gradings of fusion categories*

In this subsection, we recall a few basic results on gradings of fusion categories from [**10**] that will be further used in the paper. For an introduction to fusion categories, one might consult [7]. Let $\mathcal{C}$ be a fusion category and $\mathcal{O}(\mathcal{C})$ be the set of isomorphism classes of simple objects of $\mathcal{C}$. Recall that the fusion category $\mathcal{C}$ is graded by a finite group G if there is a function $\deg : \mathcal{O}(\mathcal{C}) \to G$ such that for any two simple objects $X, Y \in \mathcal{O}(\mathcal{C})$, then one has that $\deg(Z) = \deg(X)\deg(Y)$ whenever $Z \in \mathcal{O}(\mathcal{C})$ is a simple object such that Z is a constituent of $X \otimes Y$. Alternatively, there is a decomposition $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ such that the tensor functor of $\mathcal{C}$ sends $\mathcal{C}_g \otimes \mathcal{C}_h$ into $\mathcal{C}_{gh}$. Here $\mathcal{C}_g$ is defined as the full abelian subcategory of $\mathcal{C}$ generated by the simple objects X of $\mathcal{C}$ satisfying $\deg(X) = g$. Recall that a grading is called universal if any other grading of $\mathcal{C}$ is arising as a quotient of the universal grading. The universal grading always exists and its grading group denoted by $U_{\mathcal{C}}$ is called the universal grading group.

Remark 5.1. If $\mathcal{C} = \mathrm{Rep}(H)$ for a semisimple Hopf algebra H, then by Gelaki and Nikshych [10, Theorem 3.8] it follows that the Hopf center (that is, the largest central Hopf subalgebra) of H is kG^*, where G is the universal grading group of $\mathcal{C}$. We denote this Hopf center by $\mathcal{HZ}(H)$. Therefore, one has $\mathcal{HZ}(H) = kG^*$, where $G = U_{\mathrm{Rep}(H)}$. Moreover, in this case, by the universal property any other grading on $\mathcal{C} = \mathrm{Rep}(H)$ is given by a quotient group G/N of G. The corresponding graded components of $\mathcal{C}$ are given by

$$\mathcal{C}_{\bar{g}} = \{M \in \mathrm{Irr}(H) \mid M \downarrow^{H}_{k^{G/N}} = (\dim M)\bar{g}\}, \tag{5.1}$$

for all $g \in G$. Here $k^{G/N} \subset k^G$ is regarded as a central Hopf subalgebra of H. Also note that in this situation one has a central extension of Hopf algebras:

$$k \longrightarrow k^{G/N} \longrightarrow H \longrightarrow H//k^{G/N} \longrightarrow k. \tag{5.2}$$

5.2. *Gradings on* $\mathrm{Rep}(H^*)$ *and cocentral extensions*

Suppose that H is a semisimple Hopf algebra such that the fusion category $\mathrm{Rep}(H^*)$ is graded by a finite group G. Then the dual version of Remark 5.1 implies that H fits into a cocentral extension

$$k \longrightarrow B \longrightarrow H \xrightarrow{\pi} kG \longrightarrow k. \tag{5.3}$$

Recall that such an exact sequence of Hopf algebras is called cocentral if $kG^* \subset \mathcal{Z}(H^*)$ via the dual map π^*. On the other hand, using the reconstruction theorem from [1] it follows that

$$H \cong B^{\tau}\#_{\sigma} kF, \tag{5.4}$$

for some cocycle $\sigma : B \otimes B \to kF$ and some dual cocycle $\tau : kF \to B \otimes B$.

For any such cocentral sequence, it follows that G acts on $\mathrm{Rep}(B)$ and by Natale [16, Proposition 3.5] that $\mathrm{Rep}(H) = \mathrm{Rep}(B)^G$, the equivariantized fusion category. For the main properties of group actions and equivariantized fusion categories, one can consult, for example, [19]. Recall that the above action of G on $\mathrm{Rep}(B)$ is given by $T : G \to \mathrm{Aut}_{\otimes}(\mathrm{Rep}(B))$, $g \mapsto T^g$. For any $M \in \mathrm{Rep}(B)$, one has that $T^g(M) = M$ as vector spaces and the action of B is given by $b \cdot^g m := (g \cdot b) \cdot m$ for all $g \in G$ and all $b \in B$, $m \in M$. Here the weak action of G on B is the action used in the crossed product from equation (5.4).

For any subgroup M of G, it is easy to check that $H(M) = B\#_{\sigma}kM$, that is, $H(M)$ is the unique Hopf subalgebra of H fitting the exact cocentral sequence

$$k \longrightarrow B \longrightarrow H(M) \longrightarrow kM \longrightarrow k. \tag{5.5}$$

Lemma 5.2. *Let H be a semisimple Hopf algebra. Then gradings on the fusion category $\mathrm{Rep}(H^*)$ are in one-to-one correspondence with cocentral extensions*

$$k \longrightarrow B \longrightarrow H \xrightarrow{\pi} kG \longrightarrow k. \tag{5.6}$$

Proof. We have shown at the beginning of this subsection how to associate a cocentral extension to any G-grading on $\operatorname{Rep}(H^*)$.

Conversely, suppose that one has a cocentral exact sequence as in equation (5.13). Then $\operatorname{Rep}(H^*)$ is graded by G, where the graded component of degree $g \in G$ is given by

$$\operatorname{Rep}(H^*)_g = \{d \in \operatorname{Irr}(H^*) \mid \pi(d) = \epsilon(d)g\}. \tag{5.7}$$

Indeed, since $k^G \subset \mathcal{Z}(H^*)$ via π^* it follows that k^G acts by scalars on each irreducible representation of H^*. Therefore, for any $d \in \operatorname{Irr}(H^*)$ one has $d \downarrow_{k^G}^{H^*} = \epsilon(d)g$ for some $g \in G$. It follows then by Gelaki and Nikshych [**10**, Theorem 3.8] that $\operatorname{Rep}(H^*)$ is G-graded and

$$\operatorname{Rep}(H^*)_g = \{d \in \operatorname{Irr}(H^*) \mid d \downarrow_{k^G}^{H^*} = \epsilon(d)g\}. \tag{5.8}$$

On the other hand, it is easy to check that one has $\pi(d) = d \downarrow_{k^G}^{H^*}$ for any $d \in \operatorname{Irr}(H^*)$ (see also [**4**, Remark 3.2]).

Clearly, the two constructions are inverse one to the other. □

5.3. *New examples of Mackey pairs of Hopf subalgebras*

Let H be a semisimple Hopf algebra and $\mathcal{C} = \operatorname{Rep}(H^*)$. Since H^* is also a semisimple Hopf algebra [**13**], it follows that $\mathcal{C}$ is a fusion category. For the rest of this section, fix an arbitrary G-grading $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ on $\mathcal{C}$.

For any subset $M \subset G$, define $\mathcal{C}_M := \bigoplus_{m\, \in M} \mathcal{C}_m$ as a full abelian subcategory of $\mathcal{C}$. Thus, $\mathcal{O}(\mathcal{C}_M) = \bigsqcup_{m\in M} \mathcal{O}(\mathcal{C}_m)$. Let also $H(M)$ be the subcolagebra of H generated by all the simple subcoalgebras of H whose irreducible H^*-characters belong to $\mathcal{O}(\mathcal{C}_M)$.

For any subcoalgebra C of H, denote by $\operatorname{Irr}(C^*)$ the irreducible characters of the dual algebra C^*. Therefore, by its definition $H(M)$ verifies the equality $\operatorname{Irr}(H(M)^*) = \mathcal{O}(\mathcal{C}_M)$ and as a coalgebra can be written as $H(M) = \bigoplus_{\{d\in\mathcal{O}(\mathcal{C}_m)\,|\, m\in M\}} C_d$. Note that if M is a subgroup of G, then $H(M)$ is a Hopf subalgebra of H by Remark 2.1.

For any simple subcoalgebra C of H whose associated irreducible character $d \in \operatorname{Irr}(H^*)$ has degree g, we will also write for shortness that $\deg(C) = g$.

Proposition 5.3. *Let H be semisimple Hopf algebra and G be the universal grading group of $\operatorname{Rep}(H^*)$. Then, for any arbitrary two subgroups M and N of G, the set of double cosets $H(M)\backslash H/H(N)$ is canonically bijective to the set of group double cosets $M\backslash N/G$. Moreover, the bijection is given by $H(M)CH(N) \mapsto M\deg(C)N$.*

Proof. By Remark 2.1, one has the following equality in terms of irreducible H^*-characters:

$$\operatorname{Irr}(H(M)CH(N)^*) = \mathcal{O}(\mathcal{C}_{M\deg(C)N}).$$

Thus, if $H(M)CH(N) = H(M)DH(N)$, then $\deg(C) = \deg(D)$, which shows that the above map is well defined. Clearly, the map $H(M)CH(N) \mapsto M\deg(C)N$ is surjective. The injectivity of this map also follows from Remark 2.1. □

Note that the proof of the previous proposition implies that the coset $H_x = H(M)CH(N)$ with $\deg(C) = x$ is given by

$$H_x = \bigoplus_{\{d\in\mathcal{O}(\mathcal{C}_{mxn})\,|\, m\in M,\, n\in N\}} C_d. \tag{5.9}$$

Proposition 5.4. *Suppose that $V \in H(M)$-mod, that is, V is a $B\#_\sigma kM$-module. Then as B-modules, one has that ${}^C V \cong T^{g^{-1}}(\operatorname{Res}_B^{H(M)}(V))$, where $g \in G$ is chosen such that $\deg(C) = g$. Moreover, ${}^C(V \otimes W) \cong {}^C V \otimes {}^C W$ for any two left $H(M)$-modules V and W.*

Proof. Note that in this situation, one has that ${}^{C}H(M) = H({}^{g}M) = B\#_{\sigma}k^{g}M$. By definition, one has ${}^{C}V = CH(M) \otimes_{H(M)} V = H(gM) \otimes_{H(M)} V$. Thus,

$$ {}^{C}V = (B\#kgM) \otimes_{B\#kM} V \cong kgM \otimes_{kM} V, \tag{5.10}$$

where the inverse of the last isomorphism is given by $g \otimes_{kM} v \mapsto (1\#g) \otimes_{B\#kM} v$. Note that B acts on $kgM \otimes_{kM} V$ via $b \cdot (g \otimes_{kM} v) = g \otimes_{kM} (g^{-1} \cdot b)m$ for all $b \in B$, $v \in V$. This shows that indeed ${}^{C}V \cong T^{g^{-1}}(\mathrm{Res}_{B}^{H(M)}(V))$ as B-modules. Moreover, it follows that ${}^{C}V$ can be identified to V as vector spaces with the $B\#_{\sigma}kgMg^{-1}$-module structure given by $b \cdot v = (g^{-1} \cdot b)v$ and $(ghg^{-1}) \cdot v = ([g^{-1} \cdot (\sigma(ghg^{-1}, g)\sigma^{-1}(g, h))]\#_{\sigma}h) \cdot v$ for all $g \in G$, $h \in M$ and $v \in V$. Then it can be checked by direct computation that the map $v \otimes w \mapsto \tau^{-1}(g)(v \otimes w)$ from [**16**, Proposition 3.5] is in this case a morphism of $B\#_{\sigma}kgMg^{-1}$-modules. In order to do that one has to use the compatibility conditions from [**1**, Theorem 2.20]. □

5.4. *Examples of Mackey pairs arising from group gradings on the category* Rep(H^*)

Let, as above, H be a semisimple Hopf algebra with $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ be a group grading of $\mathcal{C} := \mathrm{Rep}(H^*)$. It follows that

$$\mathrm{FPdim}(\mathcal{C}_g) = \frac{\dim H^*}{\dim \mathcal{HZ}(H^*)}, \tag{5.11}$$

for all $g \in G$, where $\mathrm{FPdim}(\mathcal{C}_g) := \sum_{V \in \mathcal{O}(\mathcal{C}_G)} (\dim V)^2$ is the Perron–Frobenius dimension of the full abelian subcategory $\mathcal{C}_g$ of $\mathcal{C}$.

Theorem 5.5. *Let H be a semisimple Hopf algebra and M, N be any two subgroups of G. With the above notation, the pair $(H(M), H(N))$ is a Mackey pair of Hopf subalgebras of H.*

Proof. Put $L := H(M)$ and $K := H(N)$. Therefore, $\mathrm{Irr}(L^*) = \mathcal{O}(\mathcal{C}(M))$ and $\mathrm{Irr}(K^*) = \mathcal{O}(\mathcal{C}(N))$. Then we have to verify equation (4.2) for any simple subcoalgebra C. Fix a simple subcoalgebra C of H with $\deg(C) = x$. As above, one has ${}^{C}H(M) = H({}^{x}M)$.

It is easy to verify that $H(M) \cap H(N) = H(M \cap N)$ for any two subgroups M and N of G. This implies that $L \cap {}^{C}K = H(N \cap {}^{x}M)$. On the other hand from equation (5.9), note that $\dim LCK = |MxN| \mathrm{FPdim}(\mathcal{C}_1)$.

Then equation (4.2) is equivalent to the well-known formula for the size of a double coset relative to two subgroups:

$$|MxN| = \frac{|M||N|}{|M \cap {}^{x}N|}, \tag{5.12}$$

for any $x \in G$. □

Remark 5.6. The fact that $(H(M), H(N))$ is a Mackey pair also followed in this case from a more general version of Mackey's decomposition theorem that holds for the action of any finite group on a fusion category. These results will be contained in a future paper of the author.

Remark 5.7. It also should be noted that the author is not aware of any pair of Hopf subalgebras that is not a Mackey pair. It would be interesting to construct such counterexamples if they exist.

5.5. *Mackey and the Green functors*

For a finite group G, denote by $\mathcal{S}(G)$ the lattice of all subgroups of G. Following [**21**], a Mackey functor for G over a ring R can be regarded as a collection of vector spaces $M(H)$ for any $H \subset \mathcal{S}(G)$ together with a family of morphisms $I_K^L : M(K) \to M(L)$, $R_K^L : M(L) \to M(K)$ and $c_{K,g} : M(K) \to M({}^gK)$ for all subgroups K and L of G with $K \subset L$ and for all $g \in G$. This family of morphisms has to satisfy the following compatibility conditions:

(1) I_H^H, $R_H^H, c_{H,h} : M(H) \to M(H)$ are the identity morphisms for all subgroups H of G and any $h \in H$;
(2) $R_K^J R_H^K = R_K^J$, for all subgroups $J \subset K \subset H$;
(3) $I_H^K I_J^H = I_J^K$, for all subgroups $J \subset K \subset H$;
(4) $c_{K,g}c_{K,h} = c_{K,gh}$ for all elements $g, h \in G$;
(5) for any three subgroups $J, L \subseteq K$ of G and any $a \in M(J)$, one has the following Mackey axiom:
$$R_L^J(I_J^K(a)) = \sum_{x \in J\backslash K/L} I^L_{L\cap {}^xJ}(R^{{}^xJ}_{{}^xJ\cap L}(c_{J,x}(a))).$$

Moreover, a Green functor is a Mackey functor M such that for any subgroup K of G, one has that $M(K)$ is an associative R-algebra with identity and the following conditions are satisfied:

(6) R_K^L and $c_{K,g}$ are always unitary R-algebra homomorphisms;
(7) $I_K^L(aR_K^L(b)) = I_K^L(a)b$;
(8) $I_K^L(R_K^L(b)a) = bI_K^L(a)$ for all subgroups $K \subseteq L \subseteq G$ and all $a \in M(K)$ and $b \in M(L)$.

The Green functors play an important role in the representation theory of finite groups (see, for example, [**21**]).

5.6. *New examples of the Green functors*

The following theorem allows us to construct new examples of the Green functors from semisimple Hopf algebras.

THEOREM 5.8. *Let H be a semisimple Hopf algebra and G be a grading group for the fusion category* Rep(H^*). *Then the functor $M \mapsto K_0(H(M))$ is a Green functor.*

Proof. By Proposition 5.2 there is a cocentral extension
$$k \longrightarrow B \longrightarrow H \xrightarrow{\pi} kG \longrightarrow k, \tag{5.13}$$
for some Hopf subalgebra $B \subset H$. Then as above, for a simple subcoalgebra C of H with associated character $d \in H^*$, one has that if $\pi(d) = g$ for some $g \in G$, then $\pi(C) = kg$.

The map $R_K^L : K_0(H(L)) \to K_0(H(K))$ is induced by the restriction map $\mathrm{Res}^{H(L)}_{H(K)}$: $H(L)$-mod $\to H(K)$-mod. Similarly, the map I_K^L is induced by the induction functor between the same two categories of modules. Clearly, R_K^L is a unital algebra map and the compatibility conditions (7) and (8) follow from the adjunction of the two functors. Moreover, conditions (2) and (3) are automatically satisfied.

Define $c_{L,g} : K_0(L) \to K_0({}^gL)$ by $[M] \mapsto [{}^CM]$, where C is any simple subcoalgebra of H chosen with the property that $\deg(C) = g$. It follows by Proposition 5.4 that $c_{L,g}$ is a well-defined algebra map. Condition (4) is equivalent to $T^{gh}(M) \cong T^gT^h(M)$, which is automatically satisfied for a group action on a fusion category.

It is easy to see that all other axioms from the definition of a Green functor are satisfied. For example, the Mackey decomposition axiom (5) is satisfied by Theorem 5.5. □

6. *On normal Hopf subalgebras of semisimple Hopf algebras*

Recall that a Hopf subalgebra L of a Hopf algebra H is called a normal Hopf subalgebra if it is stable under the left and right adjoint action of H on itself. When H is a semisimple Hopf algebra, since $S^2 = \mathrm{id}$, in order for L to be normal, it is enough to be closed only under the left adjoint action, that is, $h_1LS(h_2) \subset L$ for any $h \in H$. Let $L^+ := L \cap \ker \epsilon$ and set $H//L := H/HL^+$. Since HL^+ is a Hopf ideal of H (see, for example, [**15**]), it follows that $H//L$ is a quotient Hopf algebra of H. Moreover, $(H//L)^*$ can be regarded as a Hopf subalgebra of H^* via the dual map of the canonical Hopf projection $\pi_L : H \to H//L$.

Proposition 6.1. *Suppose that H is a semisimple Hopf algebra. Then for any normal Hopf subalgebra K of H, one has that (K, K) is a Mackey pair of Hopf algebras.*

Proof. Note $KC = CK$ for any subcoalgebra C of K since K is a normal Hopf subalgebra of H. Then for any simple subcoalgebra C of H, the dimension condition from equation (4.2) can be written as

$$\dim CK = \frac{(\dim K)(\dim CK)}{\dim K \cap {}^C K}, \tag{6.1}$$

which is equivalent to $K \cap {}^C K = K$. This equality follows by the third item of Remark 3.5. □

6.1. *Irreducibility criterion for an induced module*

Remark 6.2. Let G be a finite group and H be a normal subgroup H of G. Then [**20**, Corollary 7.1] implies that an induced module $M\uparrow_H^G$ is irreducible if and only if M is irreducible and M is not isomorphic to any of its conjugate module ${}^gM$.

The previous theorem allows us to prove the following proposition, which is an improvement of [**3**, Proposition 5.12]. The second item is also a generalization of [**20**, Corollary 7.1].

Proposition 6.3. *Let K be a normal Hopf subalgebra of a semisimple Hopf algebra H and M be a finite-dimensional K-module.*

(i) *Then*

$$M\uparrow_K^H\downarrow_K^H \cong \bigoplus_{C \in H/K} {}^C M,$$

as K-modules.

(ii) *$M\uparrow_K^H$ is irreducible if and only if M is an irreducible K-module, which is not a direct summand of any conjugate module ${}^CM$ for any simple subcoalgebra C of H with $C \not\subset K$.* Q4

Proof. (i) Previous proposition implies that

$$M\uparrow_K^H\downarrow_K^H \cong \bigoplus_{C \in K\backslash H/K} K \otimes_{K \cap {}^C K} {}^C M, \tag{6.2}$$

as K-modules. On the other hand, since K is normal note that $CK = KC$ and therefore the space $K\backslash H/K$ of double cosets coincides to the space H/K of left (right) cosets (see also Paragraph 2.3.1 for the notation). In the proof of the same Proposition 6.1, it was also remarked that $K \cap {}^C K = K$.

(ii) One has that $M\uparrow_K^H$ is an irreducible H-module if and only if

$$\dim_k \mathrm{Hom}_H(M\uparrow_K^H,\, M\uparrow_K^H) = 1.$$

Note that by the Frobenius reciprocity, one has the following $\mathrm{Hom}_H(M\uparrow_K^H,\, M\uparrow_K^H) = \mathrm{Hom}_K(M,\, M\uparrow_K^H\downarrow_K^H)$. Then previous item implies that

$$\mathrm{Hom}_K(M,\, M\uparrow_K^H) \cong \bigoplus_{C\in H/K} \mathrm{Hom}_K(M,\, {}^{C}M). \tag{6.3}$$

Since for $C = k$, one has ${}^{k}M = M$ it follows that $\mathrm{Hom}_K(M,\, {}^{C}M) = 0$ for all $C \not\subset K$. □

6.2. *A tensor product formula for induced representations*

We need the following preliminary tensor product formula for induced representations which appeared in [6].

Proposition 6.4. *Let K be a Hopf subalgebra of a semisimple Hopf algebra H. Then for any K-module M and any H-module V, one has that*

$$M\uparrow_K^H \otimes V \cong (M \otimes V\downarrow_K^H)\uparrow_K^H. \tag{6.4}$$

Proof of Theorem 1.4. Applying Proposition 6.4, one has that

$$M\uparrow_K^H \otimes N\uparrow_L^H \cong (M\uparrow_K^H\downarrow_L^H \otimes N)\uparrow_L^H. \tag{6.5}$$

On the other hand, by Theorem 1.2 one has

$$M\uparrow_K^H\downarrow_L^H \cong \bigoplus\nolimits_{C\in L\backslash H/K}(L\otimes_{L\cap {}^{C}K}(CK\otimes_K M)). \tag{6.6}$$

Thus,

$$\begin{aligned} M\uparrow_K^H \otimes N\uparrow_L^H &\cong (M\uparrow_K^H\downarrow_L^H \otimes N)\uparrow_L^H \\ &\xrightarrow{\cong} \bigoplus\nolimits_{C\in L\backslash H/K}((L\otimes_{L\cap {}^{C}K}(CK\otimes_K M))\otimes N)\uparrow_L^H. \end{aligned}$$

Applying again Proposition 1.4 for the second tensor product, one obtains that

$$\begin{aligned} M\uparrow_K^H \otimes N\uparrow_L^H &\xrightarrow{\cong} ((CK\otimes_K M)\otimes N\downarrow_{L\cap {}^{C}K}^{L})\uparrow_{L\cap {}^{C}K}^{L}\uparrow_L^H \\ &\xrightarrow{\cong} \bigoplus\nolimits_{C\in L\backslash H/K} H\otimes_{L\cap {}^{C}K}((CK\otimes_K M)\otimes N\downarrow_{L\cap {}^{C}K}^{L}). \end{aligned}$$

□

Remark 6.5. Note that the above theorem always applies for $K = L$ a normal Hopf subalgebra of H.

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