HARMONIC MORPHISMS BETWEEN WEYL SPACES

RADU PANTILIE

Abstract

This is a brief survey on harmonic morphisms between Weyl spaces and twistorial maps.

Introduction

A 'Weyl connection' is a torsion free linear connection, on a conformal manifold, compatible with the conformal structure; a conformal manifold endowed with a Weyl connection is called a 'Weyl space' (see [3] and the references therein).

A function f on a Weyl space (M, c, D) is 'harmonic' if $\mathrm{trace}_c(D\mathrm{d}f) = 0$. A 'harmonic morphism' is a map between Weyl spaces which pulls-back germs of harmonic functions to germs of harmonic functions. By a basic result, harmonic morphisms are harmonic maps which are horizontally weakly conformal (see [7], [2], [15]).

For any almost Hermitian manifold (M, c, J) there exists a unique Weyl connection D on (M, c) such that $\mathrm{trace}_c(DJ) = 0$ (see [3], [7]); then any holomorphic map from (M, c, J) to an oriented two-dimensional conformal manifold is a harmonic morphism [7]. Conversely, any harmonic morphism between two-dimensional orientable conformal manifolds is holomorphic, with respect to suitable orientations (see [2], [15]).

Other similar results can be obtained by working with the more general notion of 'twistorial map' [12]. In this paper, we work with twistorial structures and maps on smooth manifolds (cf. [13]). Roughly speaking, a map $\varphi : M \to N$ is twistorial if it admits a holomorphic lift $\Phi : P_M \to P_N$ to the total spaces of some bundles $\pi_M : P_M \to M$ and $\pi_N : P_N \to N$, endowed with almost F-structures (see Definitions 2.2 and 4.1, below). Then, we have the following facts ([7]; see Proposition 5.1, below) for a submersive map $\varphi : (M^m, c_M, D^M) \to (N^n, c_N, D^N)$ between Weyl spaces of dimensions m and n, respectively, endowed with the almost twistorial structures of Examples 3.2, 3.3 or 3.4, below:

• If $m = 3$ and $n = 2$, then φ is a harmonic morphism if and only if it is a twistorial map.

• If $m = 4$ and $n = 2$, then φ is a twistorial harmonic morphism if and only if φ is horizontally conformal and the fibres of Φ are tangent to the connection induced by D^M on P_M.

• If $m = 4$ and $n = 3$, then any two of the following assertions imply the third:

(i) φ is a harmonic morphism.

2000 *Mathematics Subject Classification.* 53C43, 53C28.

Key words and phrases. harmonic morphism, Weyl space, twistorial map.

(ii) φ is twistorial.

(iii) The fibres of Φ are tangent to the connection induced by D^M on P_M.

Furthermore, any harmonic morphism from a four-dimensional Einstein–Weyl space to a two-dimensional conformal manifold is twistorial, and the same conclusion holds for harmonic morphisms between Einstein–Weyl spaces of dimensions four and three ([7] , see Theorem 5.3 , below). Moreover, in the latter case, the Weyl connection of the domain (M^4, c_M, D^M) is, locally, either the Obata connection of a hyper-Hermitian structure on (M^4, c_M) or it is the Levi-Civita connection of an Einstein representative of c_M ([7] , see Theorem 5.4 , below).

Recall that a Weyl space of dimension four (or three) can also be endowed with a natural almost twistorial structure which is always nonintegrable [4] . The relations between the resulting twistorial maps and harmonic morphisms are discussed in [8] .

I am grateful to John C. Wood, and to Eric Loubeau for useful comments.

1. Harmonic morphisms

In this paper, unless otherwise stated, all the manifolds and maps are assumed to be smooth; also, the manifolds are assumed to be connected.

Let M^m be a manifold of dimension m. If m is even then we denote by L the line bundle associated to the frame bundle of M^m through the morphism of Lie groups $\rho_m : \mathrm{GL}(m, \mathbb{R}) \to (0, \infty)$, $\rho_m(a) = |\det a|^{1/m}$, $\big(a \in \mathrm{GL}(m, \mathbb{R})\big)$; obviously, L is oriented. If m is odd then we denote by L the line bundle associated to the frame bundle of M^m through the morphism of Lie groups $\rho_m : \mathrm{GL}(m, \mathbb{R}) \to \mathbb{R}^*$, $\rho_m(a) = (\det a)^{1/m}$, $\big(a \in \mathrm{GL}(m, \mathbb{R})\big)$; obviously, $L^* \otimes TM$ is an oriented vector bundle. We say that L is *the line bundle of* M^m.

Let c be a conformal structure on M. Then positive sections of L^2 correspond to representatives of c. Moreover, a conformal structure on M corresponds to an injective vector bundle morphism $L^2 \hookrightarrow \odot^2 TM$ such that, at each $x \in M$, the positive bases of L^2_x are mapped into the cone of positive definite symmetric bilinear forms on $T^*_x M$; furthermore, a conformal structure on M also corresponds to a Riemannian structure on the vector bundle $L^* \otimes TM$. If $\dim M$ is odd then local sections of L correspond to oriented local representatives of c (that is, local representatives of c together with an orientation of their domain of definition). Let $\mathscr{H}$ be a distribution on M. Then c induces a conformal structure $c|_{\mathscr{H}}$ on $\mathscr{H}$ and, it follows that we have an isomorphism, which depends of c, between L^2 and $L^2_{\mathscr{H}}$, where $L_{\mathscr{H}}$ is the line bundle of $\mathscr{H}$ (that is, if $\mathscr{H}$ is a distribution of dimension n then $L_{\mathscr{H}}$ is the line bundle associated to the frame bundle of $\mathscr{H}$ through ρ_n).

Definition 1.1. (i) Let (M, c, D) be a Weyl space. A *harmonic function, on* (M, c, D), is a real-valued function f, locally defined on M, such that $\mathrm{trace}_c(D\mathrm{d}f) = 0$.

(ii) Let (M, c_M, D^M) and (N, c_N, D^N) be Weyl spaces. A map $\varphi : M \to N$ is a *harmonic map, from* (M, c_M, D^M) *to* (N, c_N, D^N), if $\mathrm{trace}_c(D\mathrm{d}\varphi) = 0$.

(iii) Let (M, c_M, D^M) and (N, c_N, D^N) be Weyl spaces. A map $\varphi : M \to N$ is a *harmonic morphism, from* (M, c_M, D^M) *to* (N, c_N, D^N), if for any harmonic function

$f : V \to \mathbb{R}$, on (N, c_N, D^N), with V an open set of N such that $\varphi^{-1}(V)$ is nonempty, $f \circ \varphi : \varphi^{-1}(V) \to \mathbb{R}$ is a harmonic function, on (M, c_M, D^M).

Obviously, any harmonic function is a harmonic map and a harmonic morphism, if $\mathbb{R}$ is endowed with its conformal structure and canonical connection.

Definition 1.2. Let (M, c_M) and (N, c_N) be conformal manifolds. A *horizontally (weakly) conformal map, from* (M, c_M) *to* (N, c_N), is a map $\varphi : M \to N$ such that, at each point $x \in M$, either $\mathrm{d}\varphi_x = 0$ or $\mathrm{d}\varphi_x|_{(\ker \mathrm{d}\varphi_x)^\perp}$ is a conformal linear isomorphism from $\big((\ker \mathrm{d}\varphi_x)^\perp, (c_M)_x|_{(\ker \mathrm{d}\varphi_x)^\perp}\big)$ onto $\big(T_{\varphi(x)}N, (c_N)_{\varphi(x)}\big)$.

Let (M, c_M) and (N, c_N) be conformal manifolds and let $\varphi : (M, c_M) \to (N, c_N)$ be a horizontally conformal submersion; denote by L_M and L_N the line bundles of M and N, respectively. Let $\mathscr{H} = (\ker \mathrm{d}\varphi)^\perp$ and denote by $L_{\mathscr{H}}$ its line bundle. As $L_M^2 = L_{\mathscr{H}}^2$, the differential of φ induces a vector bundle isomorphism Λ from L_M^2 to $\varphi^*(L_N^2)$; Λ is called the *square dilation of* φ (cf. [2]). We have $\Lambda = \frac{1}{n}\, c(\mathrm{d}\varphi, \mathrm{d}\varphi)$, where $n = \dim N$ and c is the conformal structure on the bundle $T^*M \otimes \varphi^*(TN)$ induced by c_M, c_N and φ. Therefore, if φ is horizontally weakly conformal, then Λ extends to a (smooth) section of $\mathrm{Hom}\big(L_M^2, \varphi^*(L_N^2)\big)$ which is zero over the set of critical points of φ.

Note that, if $\dim N$ is odd, $\mathrm{d}\varphi$ defines a vector bundle morphism from $(L_{\mathscr{H}}^* \otimes \mathscr{H}, c|_{\mathscr{H}})$ to $(L_N^* \otimes TN, c_N)$ which is an orientation preserving isometry, on each fibre.

We end this section with the following basic result on harmonic morphisms (see [7] , [2] and the references therein).

Theorem 1.3. *A map between Weyl spaces is a harmonic morphism if and only if it is a harmonic map which is horizontally weakly conformal.*

2. F-STRUCTURES

We recall the well-known notion of almost F-structure (cf. [16]).

Definition 2.1. An *almost F-structure*, on a manifold M, is a section F of $\mathrm{End}(TM)$ such that $F^3 + F = 0$.

If F is an almost F-structure on M then we denote by T^0M, $T^{1,0}M$, $T^{0,1}M$ the eigenbundles of $F^{\mathbb{C}} \in \Gamma\big(\mathrm{End}(T^{\mathbb{C}}M)\big)$ corresponding to the eigenvalues 0, i, $-\mathrm{i}$, respectively.

Let (M, c) be a conformal manifold. An almost F-structure F on M is *compatible with* c if $F^* = -F$, where F^* denotes the adjoint of F. An *almost F-structure on* (M, c) is an almost F-structure on M, compatible with c.

Let F be an almost F-structure on M. Then T^0M is the zero distribution if and only if F is an almost complex structure on M.

Let (M, c) be a conformal manifold. An almost F-structure F on M is compatible with c if and only if $(T^0M)^\perp = T^{1,0}M \oplus T^{0,1}M$ and $T^{1,0}M$ is isotropic. Thus, any almost F-structure on (M, c) is determined by its eigenbundle corresponding to i (or $-$i).

Definition 2.2. Let F^M and F^N be almost F-structures on M and N, respectively. We shall say that a map $\varphi : (M, F^M) \to (N, F^N)$ is *holomorphic* if

$$\mathrm{d}\varphi(T^0 M \oplus T^{1,0} M) \subseteq T^0 N \oplus T^{1,0} N$$

(equivalently, $\mathrm{d}\varphi(T^0 M \oplus T^{0,1} M) \subseteq T^0 N \oplus T^{0,1} N$).

Note that, the condition $\varphi : (M, F^M) \to (N, F^N)$ be holomorphic is less restrictive than the condition that $\mathrm{d}\varphi$ intertwine the almost F-structures. However, if F^N is an almost complex structure then φ is holomorphic if and only if $\mathrm{d}\varphi \circ F^M = F^N \circ \mathrm{d}\varphi$.

Definition 2.3. We shall say that an almost F-structure F on M is *integrable* if, for any point $x \in M$, there exists a holomorphic submersion $\varphi : (U, F|_U) \to (N, J)$, from some open neighbourhood U of x, onto a complex manifold (N, J) , such that $\ker \mathrm{d}\varphi = T^0 U$.

An integrable almost F-structure will be called an *F-structure*. We shall say that an F-structure is *simple* if there exists a holomorphic submersion $\varphi : (M, F) \to (N, J)$, with connected fibres, onto a complex manifold (N, J) , such that $\ker \mathrm{d}\varphi = T^0 M$.

Remark 2.4. 1) The complex structure of any complex manifold is a simple F-structure.

2) Let F be an almost F-structure on M. Then F is integrable if and only if $T^0 M \oplus T^{1,0} M$ is (formally) integrable.

The *torsion* (cf. [6]) of F is the tensor field defined by

$$N_F(X, Y) = [FX, FY] - F[X, FY] - F[FX, Y] + F^2[X, Y]$$

for any vector fields X , Y on M. Then $N_F = 0$ if and only if $T^0 M$, $T^{1,0} M$, $T^0 M \oplus T^{1,0} M$ and $T^{1,0} M \oplus T^{0,1} M$ are integrable. Hence, if $N_F = 0$ then F is integrable, but the converse does not hold. For example, let S^3 be endowed with its canonical conformal structure and let F be the almost F-structure on it with respect to which the Hopf fibration $\varphi : S^3 \to \mathbb{C}P^1$ is holomorphic and $T^0 S^3 = \ker \mathrm{d}\varphi$. Obviously, F is integrable but, as $T^{1,0} S^3 \oplus T^{0,1} S^3$ is nowhere integrable, N_F is nowhere zero.

3) Let (M, F^M) and (N, F^N) be manifolds endowed with simple F-structures and let $\psi_M : (M, F^M) \to (P_M, J^M)$ and $\psi_N : (N, F^N) \to (P_N, J^N)$, respectively, be the corresponding holomorphic submersions.

A map $\Phi : (M, F^M) \to (N, F^N)$ is holomorphic if and only if there exists a holomorphic map $\varphi : (P_M, J^M) \to (P_N, J^N)$ such that $\varphi \circ \psi_M = \psi_N \circ \Phi$.

3. Twistorial structures

For the remainder of the paper, we refer the reader, for further details, to [7] , [8] , [12] and to the references therein.

The following definition reformulates, from the perspective of [12] , a definition of [13] .

Definition 3.1. An *almost twistorial structure*, on a manifold M, is a quadruple $\tau = (P, M, \pi, F)$ where $\pi : P \to M$ is a locally trivial fibre space and F is an almost F-structure on P such that $\ker \mathrm{d}\pi$ is preserved by F . The almost twistorial structure τ is *integrable* if F is integrable. Assuming F a simple F-structure, let $\pi_Z:(P,F)\to(Z,J)$ be the corresponding holomorphic submersion. Then, the complex manifold (Z,J) is the *twistor space* of τ .

A *twistorial structure* is an integrable almost twistorial structure. A twistorial structure $\tau=(P,M,\pi,F)$ is *simple* if F is a simple F-structure and each leaf of T^0P intersects each fibre of π at most once.

See [8] for a more general definition of the notion of almost twistorial structure.

Next, we formulate the examples of almost twistorial structures with which we shall work.

Example 3.2. If F is an almost F-structure on M then (M,M,Id_M,F) is an almost twistorial structure.

In particular, let (M^2,c) be a two-dimensional oriented conformal manifold. Then, as the identity component of $\mathrm{CO}(2,\mathbb{R})$ is isomorphic to $\mathbb{C}^*$, there exists a unique (almost) Hermitian structure J on (M^2,c) such that if $X\in TM$ then (X,JX) is a positively oriented frame on M^2; then (M,M,Id,J) is a twistorial structure. Similarly, any oriented vector bundle of (real) rank two, endowed with a conformal structure, is a complex line bundle.

Example 3.3 ([5] , [13])**.** Let (M^3,c,D) be a three-dimensional Weyl space. Let $\pi: P\to M$ be the bundle of nonzero skew-adjoint F-structures on (M^3,c) . Obviously, P is also the bundle of nonzero skew-adjoint F-structures on the oriented Riemannian bundle $(L^*\otimes TM,c)$. Therefore, P is isomorphic to the sphere bundle of $(L^*\otimes TM,c)$ (any element p , of the sphere bundle of $(L^*\otimes TM,c)$, corresponds to the linear F-structure F_p , on $\big((L^*\otimes TM)_{\pi(p)},c_{\pi(p)}\big)$, defined by $F_p(q)=p\times q$, $q\in(L^*\otimes TM)_{\pi(p)}$). In particular, the typical fibre and the structural group of P are $\mathbb{C}P^1$ and $\mathrm{PGL}(\mathbb{C},1)$, respectively.

The bundle P is also isomorphic with the bundle of oriented two-dimensional spaces tangent to M^3. It follows that there exists a bijective correspondence between one-dimensional foliations on M^3, with oriented orthogonal complement, and almost F-structures on (M^3,c) . Furthermore, under this bijection, conformal one-dimensional foliations correspond to (integrable) F-structures.

Let $\mathscr{H}\subseteq TP$ be the connection induced by D on P. We denote by $\mathscr{H}^0$, $\mathscr{H}^{1,0}$ the subbundles of $\mathscr{H}^{\mathbb{C}}$ such that, at each $p\in P$, the subspaces $\mathscr{H}^0_p$, $\mathscr{H}^{1,0}_p\subseteq\mathscr{H}^{\mathbb{C}}_p$ are the horizontal lifts of the eigenspaces of $p^{\mathbb{C}}\in\mathrm{End}(T^{\mathbb{C}}_{\pi(p)}M)$ corresponding to the eigenvalues 0 , i , respectively.

We define the almost F-structure $\mathcal{F}$ on P with respect to which $T^0P=\mathscr{H}^0$ and $T^{1,0}P=(\ker\mathrm{d}\pi)^{1,0}\oplus\mathscr{H}^{1,0}$. Then $(P,M,\pi,\mathcal{F})$ is an almost twistorial structure on M.

Example 3.4 ([1])**.** Let (M^4,c,D) be an oriented four-dimensional Weyl space. Let $\pi:P\to M$ be the bundle of positive orthogonal complex structures on (M^4,c) . Obviously, P is also the bundle of positive orthogonal complex structures on the oriented Riemannian bundle $(L^*\otimes TM,c)$. Let E be the adjoint bundle of $(L^*\otimes TM,c)$ and let $*_c$ be the involution of E induced by the Hodge star-operator of $(L^*\otimes M^4,c)$, under the isomorphism $E=\Lambda^2(L\otimes T^*M)$. Then $E=E_+\oplus E_-$ where $E_\pm$ is the vector bundle, of rank three, formed of the eigenvectors of $*_c$ corresponding to the eigenvalue ± 1. There exists a unique oriented Riemannian structure $< \cdot, \cdot >_\pm$ on $E_\pm$ with respect to which $AB = - < A, B >_\pm \mathrm{Id}_{TM} \pm A \times B$ for any $A, B \in E_\pm$. It follows that P is the sphere bundle of $(E_+, < \cdot, \cdot >_+)$.

Similarly to Example 3.3, there exists a bijective correspondence between two-dimensional distributions $\mathscr{F}$ on M^4, with oriented orthogonal complement, and pairs (J, K) of almost Hermitian structures on (M^4, c), with J positive and K negative, such that $J|_{\mathscr{F}^\perp} = K|_{\mathscr{F}^\perp}$ is the complex structure of the oriented conformal bundle of rank two $(\mathscr{F}^\perp, c|_{\mathscr{F}^\perp})$.

Let $\mathscr{H} \subseteq TP$ be the connection induced by D on P. We denote by $\mathscr{H}^{1,0}$ the subbundle of $\mathscr{H}^{\mathbb{C}}$ such that, at each $p \in P$, the subspace $\mathscr{H}^{1,0}_p \subseteq \mathscr{H}^{\mathbb{C}}_p$ is the horizontal lift of the eigenspace of $p^{\mathbb{C}} \in \mathrm{End}(T^{\mathbb{C}}_{\pi(p)}M)$ corresponding to the eigenvalue i. We define the almost complex structure $\mathcal{J}$ on P with respect to which $T^{1,0}P = (\ker \mathrm{d}\pi)^{1,0} \oplus \mathscr{H}^{1,0}$. Then $(P, M, \pi, \mathcal{J})$ is an almost twistorial structure on M.

Next, we recall the conditions under which the almost twistorial structures of Examples 3.3 and 3.4 are integrable.

Theorem 3.5 ([5]). *Let (M^3, c, D) be a three-dimensional Weyl space and let $\tau = (P, M, \pi, \mathcal{F})$ be the almost twistorial structure of Example 3.3.*

Then τ is integrable if and only if (M^3, c, D) is Einstein–Weyl.

Theorem 3.6 ([1]). *Let (M^4, c, D) be an oriented four-dimensional Weyl space and let $\tau = (P, M, \pi, \mathcal{J})$ be the almost twistorial structure of Example 3.4.*

Then τ depends only of (M^4, c). Moreover, τ is integrable if and only if (M^4, c) is anti-self-dual.

We end this section with a proof of Theorems 3.5 and 3.6 (cf. [9]).

Proof of Theorems 3.5 and 3.6. Let (M^m, c, D) be a Weyl space, $\dim M = m \in \{3, 4\}$; if $m = 4$ assume M^4 oriented. If $m = 3$ let $G = \mathrm{CO}(3, \mathbb{C})$ and if $m = 4$ let G be the connected component of the identity of $\mathrm{CO}(4, \mathbb{C})$. Thus, M^m is endowed with a (complex) G-structure $(G(M), M, G)$ which is equipped with the principal connection $\mathscr{H} \subseteq T\big(G(M)\big)$ corresponding to D.

For any $\xi \in \mathbb{C}^m$, let $B(\xi)$ be the section of $T^{\mathbb{C}}\big(G(M)\big)$ which, at each $u \in G(M)$, is the horizontal lift of $u(\xi) \in T^{\mathbb{C}}_{\pi(u)}M$, where $\pi : G(M) \to M$ is the canonical projection.

Note that, $(\ker \mathrm{d}\pi)^{\mathbb{C}}$ is the trivial complex vector bundle with fibre $\mathfrak{g} \oplus \overline{\mathfrak{g}}$, where $\mathfrak{g}$ is the Lie algebra of G. Furthermore, for any $A \in \mathfrak{g}$, $A' \in \overline{\mathfrak{g}}$ and $\xi, \eta \in \mathbb{C}^m$ the following relations hold (cf. [6]):

$$
\begin{aligned}
[A, A'] &= 0 , \\
[A, B(\xi)] &= B(A\xi) , \\
[A', B(\xi)] &= 0 , \\
[B(\xi), B(\eta)] &= -\,\Omega(B(\xi), B(\eta)) ,
\end{aligned}
\tag{3.1}
$$

where Ω is the curvature form of $\mathscr{H}$ and where we have denoted by the same symbols elements of $\mathfrak{g}^{\mathbb{C}}$ and the corresponding fundamental (complex) vector fields on $G(M)$.

Let F_0 be a nonzero skew-adjoint F-structure on $\mathbb{R}^m$; if $m=4$ assume F_0 to be a positive orthogonal complex structure. Denote by p_0 the corresponding subspace $T^0\mathbb{R}^m\oplus T^{0,1}\mathbb{R}^m\subseteq\mathbb{C}^m$ and let $H\subseteq G$ be the subgroup fixing p_0. Note that, if $m=3$ then $G(M)/H$ is the bundle of degenerate two-dimensional spaces on (M^3,c) whilst if $m=4$ then $G(M)/H$ is the bundle of self-dual spaces on (M^4,c).

Let $\mathscr{F}_0$ be the subbundle of $\mathscr{H}^{\mathbb{C}}$ spanned by all $B(\xi)$, $\xi\in p_0$, and let $\mathfrak{h}$ be the Lie algebra of H. Then the almost twistorial structure of Example 3.3 or 3.4, associated to (M^m,c,D), according to $m=3$ or $m=4$, is integrable if and only if the subbundle $\mathscr{F}=\mathscr{F}_0\oplus\big(G(M)\times\mathfrak{h}\big)\oplus\big(G(M)\times\overline{\mathfrak{g}}\big)$ of $T^{\mathbb{C}}\big(G(M)\big)$ is integrable.

From (3.1) it follows that $\mathscr{F}$ is integrable if and only if $\Omega(B(\xi),B(\eta))(p_0)\subseteq p_0$ for all $\xi,\eta\in p_0$. Let L be the line bundle of M^m and let R be the curvature form of the connection induced by D on $L^*\otimes TM$. Then $\mathscr{F}$ is integrable if and only if $c(R(X,Y)X,Y)=0$ for all $X,Y\in T^{\mathbb{C}}M$ spanning an element of $G(M)/H$.

If $m=3$ let $\Pi\subseteq\mathrm{G}_2(\mathbb{C}^3)$ be the space of degenerate two-dimensional subspaces of $\mathbb{C}^3$ and if $m=4$ let $\Pi\subseteq\mathrm{G}_2(\mathbb{C}^4)$ be the space of self-dual subspaces of $\mathbb{C}^4$. Then the Plücker embedding $\mathrm{G}_2(\mathbb{C}^m)\to P\big(\Lambda^2(\mathbb{C}^m)\big)$ maps Π onto the conic of null directions in $\Lambda^2(\mathbb{C}^3)$ or $\Lambda^2_+(\mathbb{C}^4)$, according to $m=3$ or $m=4$, where $\Lambda^2_+(\mathbb{C}^4)$ is the space of self-dual forms on $\mathbb{C}^4$.

Let $\mathcal{R}$ be the restriction to $\Lambda^2(TM)$ or $\Lambda^2_+(TM)$, according to $m=3$ or $m=4$, of the L^2-valued quadratic form defined by $X\wedge Y\mapsto c(R(X,Y)X,Y)$, $X,Y\in TM$. It follows that $\mathscr{F}$ is integrable if and only if there exists a section μ of L^{*2} such that $\mathcal{R}=\mu\,c$. The proof follows. $\square$

4. Twistorial maps

We start this section, by formulating the definition of twistorial maps.

Definition 4.1 (cf. [13], [12]). Let $\tau_M=(P_M,M,\pi_M,F^M)$ and $\tau_N=(P_N,N,\pi_N,F^N)$ be almost twistorial structures and let $\varphi:M\to N$ be a map. Suppose that φ is endowed with a map $\Phi:Q_M\to Q_N$, where Q_M and Q_N are submanifolds of P_M and P_N, respectively, such that:

1) $\pi_M|_{Q_M}:Q_M\to M$ and $\pi_N|_{Q_N}:Q_N\to N$ are locally trivial fibres spaces.
2) TQ_M and TQ_N are preserved by F^M and F^N, respectively.
3) $\varphi\circ\pi_M|_{Q_M}=\pi_N|_{Q_N}\circ\Phi$.

Then $\varphi:(M,\tau_M)\to(N,\tau_N)$ is a *twistorial map (with respect to Φ)* if the map $\Phi:(Q_M,F^M|_{TQ_M})\to(Q_N,F^N|_{TQ_N})$ is holomorphic. If, further, $(Q_M,F^M|_{TQ_M})$ and $(Q_N,F^N|_{TQ_N})$ are simple F-structures, with $\psi_M:(Q_M,F^M|_{TQ_M})\to Z_{M,\Phi}$ and $\psi_N:(Q_N,F^N|_{TQ_N})\to Z_{N,\Phi}$, respectively, the corresponding holomorphic submersions, then Φ induces a holomorphic map $Z_{M,\Phi}\to Z_{N,\Phi}$, which is called the *twistorial representation of φ (with respect to Φ)*.

Next, we give the examples of twistorial maps with which we shall work.

Example 4.2. Let (M^2, c_M) and (N^2, c_N) be two-dimensional oriented conformal manifolds. Let $\tau_M = (M, M, \mathrm{Id}_M, J^M)$ and $\tau_N = (N, N, \mathrm{Id}_N, J^N)$ be the twistorial structures of Example 3.2 associated to (M^2, c_M) and (N^2, c_N), respectively.

Let $\varphi : M^2 \to N^2$ be a map. Obviously, $\varphi : (M^2, \tau_M) \to (N^2, \tau_N)$ is twistorial (with respect to φ) if and only if $\varphi : (M^2, J^M) \to (N^2, J^N)$ is holomorphic.

The following example is, essentially, due to [5].

Example 4.3. Let (M^3, c_M, D) be a three-dimensional Weyl space and let (N^2, c_N) be a two-dimensional oriented conformal manifold. Let $\tau_M = (P, M, \pi, \mathcal{F})$ be the almost twistorial structure of Example 3.3, associated to (M^3, c_M, D), and let $\tau_N = (N, N, \mathrm{Id}_N, J)$ be the twistorial structure of Example 3.2, associated to (N^2, c_N).

Let $\varphi : M^3 \to N^2$ be a submersion and let F^φ be the almost F-structure on (M^3, c_M) determined by $\ker \mathrm{d}\varphi$ and the orientation of N^2. Define $\varPhi = \varphi \circ {p_\varphi}^{-1} : p_\varphi(M) \to N$ where p_φ is the section of P corresponding to F^φ.

The following assertions are equivalent:

(i) $\varphi : (M^3, \tau_M) \to (N^2, \tau_N)$ is twistorial (with respect to $\varPhi$).

(ii) $p_\varphi : (M^3, F^\varphi) \to (P^5, \mathcal{F})$ is holomorphic and $\varphi : (M^3, c_M) \to (N^2, c_N)$ is horizontally conformal.

(iii) $\varphi : (M^3, c_M) \to (N^2, c_N)$ is horizontally conformal and the fibres of $\varPhi$ are tangent to the connection induced by D on P^5.

(iv) $\varphi : (M^3, c_M, D) \to (N^2, c_N)$ is a horizontally conformal submersion with geodesic fibres.

Example 4.4 ([14]). Let (M^4, c_M) and (N^2, c_N) be oriented conformal manifolds of dimensions four and two, respectively. Let $\tau_M = (P, M, \pi, \mathcal{J})$ be the almost twistorial structure of Example 3.4, associated to (M^4, c_M), and let $\tau_N = (N, N, \mathrm{Id}_N, J)$ be the twistorial structure of Example 3.2, associated to (N^2, c_N).

Let $\varphi : M^4 \to N^2$ be a submersion and let J^φ be the almost Hermitian structure on (M^4, c_M) determined by $\ker \mathrm{d}\varphi$ and the orientation of N^2. Define $\varPhi = \varphi \circ {p_\varphi}^{-1}$: $p_\varphi(M) \to N$ where p_φ is the section of P corresponding to J^φ.

The following assertions are equivalent:

(i) $\varphi : (M^4, \tau_M) \to (N^2, \tau_N)$ is twistorial (with respect to $\varPhi$).

(ii) $p_\varphi : (M^4, J^\varphi) \to (P^6, \mathcal{J})$ is holomorphic and $\varphi : (M^4, c_M) \to (N^2, c_N)$ is horizontally conformal.

(iii) J^φ is integrable and $\varphi : (M^4, c_M) \to (N^2, c_N)$ is horizontally conformal.

Example 4.5 ([5] , [3]). Let (M^4, c_M) be a four-dimensional oriented conformal manifold and let (N^3, c_N, D^N) be a three-dimensional Weyl space. Let $\tau_M = (P_M, M, \pi_M, \mathcal{J})$ be the almost twistorial structure of Example 3.4, associated to (M^4, c_M), and let $\tau_N = (P_N, N, \pi_N, \mathcal{F})$ be the almost twistorial structure of Example 3.3, associated to (N^3, c_N, D^N).

Let $\varphi : M^4 \to N^3$ be a submersion. Let $\mathscr{V} = \ker \mathrm{d}\varphi$ and $\mathscr{H} = \mathscr{V}^\perp$. Then the orientation of M^4 corresponds to an isomorphism between $\mathscr{V}$ and the line bundle of $\mathscr{H}$. Therefore $(\mathscr{V}^* \otimes \mathscr{H}, c|_{\mathscr{H}})$ is an oriented Riemannian vector bundle. We define

$\Phi : P_M \to P_N$ by

$$\Phi(p) = \tfrac{1}{\|\,\mathrm{d}\varphi(V^* \otimes p(V))\|}\,\mathrm{d}\varphi(V^* \otimes p(V))\,,$$

where $\{V\}$ is any basis of $\mathscr{V}_{\pi_M(p)}$ and $\{V^*\}$ its dual basis, $(p \in P_M)$.

Let $I^{\mathscr{H}}$ be the $\mathscr{V}$-valued two-form on $\mathscr{H}$ defined by $I^{\mathscr{H}}(X,Y) = -\mathscr{V}[X,Y]$, for any sections X and Y of $\mathscr{H}$. Then $*_{\mathscr{H}} I^{\mathscr{H}}$ is a horizontal one-form on M^4, where $*_{\mathscr{H}}$ is the Hodge star-operator of $(\mathscr{V}^* \otimes \mathscr{H}, c|_{\mathscr{H}})$. Denote by D_+ the Weyl connection on (M^4, c_M) defined by $D_+ = D + *_{\mathscr{H}} I^{\mathscr{H}}$ where D is the Weyl connection of $(M^4, c_M, \mathscr{V})$.

The following assertions are equivalent:

(i) $\varphi : (M^4, \tau_M) \to (N^3, \tau_N)$ is twistorial (with respect to Φ).

(ii) $\varphi : (M^4, c_M) \to (N^3, c_N)$ is horizontally conformal and $\varphi^*(D^N) = \mathscr{H} D_+$ as partial connections on $\mathscr{H}$, over $\mathscr{H}$.

5. Harmonic morphisms and twistorial maps

In this section, we discuss the relations between harmonic morphisms and the twistorial maps of Examples 4.2, …, 4.5. Firstly, any map between two-dimensional orientable conformal manifolds is a harmonic morphism if and only if, with respect to suitable orientations, it is a twistorial map.

Proposition 5.1 ([7]). *Let (M^m, c_M, D^M) and (N^n, c_N, D^N) be Weyl spaces of dimensions m and n, respectively, where $(m,n) \in \{(3,2),(4,2),(4,3)\}$. If m (resp. n) is even then M^m (resp. N^n) is assumed to be oriented. Endow M^m and N^n, according to their dimensions, with the almost twistorial structures τ_M and τ_N, respectively, of Examples 3.2, 3.3 or 3.4. Let $\varphi : M^m \to N^n$ be a submersion.*

(i) *If $(m,n) = (3,2)$ then the following assertions are equivalent:*

(i1) *$\varphi : (M^3, c_M, D^M) \to (N^2, c_N)$ is a harmonic morphism.*

(i2) *$\varphi : (M^3, \tau_M) \to (N^2, \tau_N)$ is a twistorial map.*

(ii) *If $(m,n) = (4,2)$ then the following assertions are equivalent:*

(ii1) *$\varphi : (M^4, c_M, D^M) \to (N^2, c_N)$ is a harmonic morphism and $\varphi : (M^4, \tau_M) \to (N^2, \tau_N)$ is a twistorial map.*

(ii2) *$\varphi : (M^4, c_M) \to (N^2, c_N)$ is horizontally conformal and the fibres of Φ are tangent to the connection induced by D^M on P_M.*

(iii) *If $(m,n) = (4,3)$ then any two of the following assertions imply the third:*

(iii1) *$\varphi : (M^4, c_M, D^M) \to (N^3, c_N, D^N)$ is a harmonic morphism.*

(iii2) *$\varphi : (M^4, \tau_M) \to (N^3, \tau_N)$ is a twistorial map.*

(iii3) *The fibres of Φ are tangent to the connection induced by D^M on P_M.*

Proof. Assertion (i) is an immediate consequence of the fundamental equation for horizontally conformal submersions between Weyl spaces [7].

Let $\mathscr{V} = \ker \mathrm{d}\varphi$ and let $\mathscr{H} = \mathscr{V}^{\perp}$.

To prove assertion (ii), let $T^{1,0}M$ and $T^{0,1}M$ be the eigenbundles of J^{φ} corresponding to i and $-\mathrm{i}$, respectively. If φ is horizontally conformal, then $T^{0,1}M$ is parallel along $T^{0,1}M \cap \mathscr{H}^{\mathbb{C}}$, with respect to D^M. It follows that $\varphi : (M^4, \tau_M) \to (N^2, \tau_N)$ is a twistorial map if and only if it is horizontally conformal and $T^{0,1}M$ is parallel along $T^{0,1}M \cap \mathscr{V}^{\mathbb{C}}$. Also, by the fundamental equation, $\varphi : (M^4, c_M, D^M) \to (N^2, c_N)$

is a harmonic morphism if and only if it is horizontally conformal and its fibres are minimal, with respect to D^M; equivalently, φ is horizontally conformal and $T^{0,1}M$ is parallel along $T^{1,0}M \cap \mathscr{V}^{\mathbb{C}}$. Thus, assertion (ii1) holds if and only if φ is horizontally conformal and $T^{0,1}M$ is parallel along $\mathscr{V}$ which, clearly, is equivalent to (ii2).

Assertion (iii) follows from the fundamental equation and the fact that if φ is horizontally conformal then (iii3) is equivalent to $\mathscr{H}D^M = \mathscr{H}D + \frac{1}{2} *_{\mathscr{H}} I^{\mathscr{H}}$, where D is the Weyl connection of $(M^4, c_M, \mathscr{V})$. □

Note that, statement (ii) of Proposition 5.1, applied to the particular case when D^M is the Levi-Civita connection of a representative of c_M, reformulates a result of [14].

Remark 5.2. Let (M^4, c_M) be a four-dimensional oriented conformal manifold and let (N^3, c_N, D^N) be a three-dimensional Weyl space. Let τ_M and τ_N be the almost twistorial structures, of Examples 3.4 and 3.3, associated to (M^4, c_M) and (N^3, c_N, D^N), respectively.

Suppose that there exist three submersive harmonic morphisms ψ_j, $j = 1, 2, 3$, with one-dimensional fibres, on (N^3, c_N, D^N), such that the distributions orthogonal to the fibres of ψ_j, $j = 1, 2, 3$, are orientable, and $\ker \mathrm{d}\psi_j$ is orthogonal to $\ker \mathrm{d}\psi_k$, if $j \neq k$.

Let $\varphi : (M^4, \tau_M) \to (N^3, \tau_N)$ be a twistorial map. Then, as $\psi_j \circ \varphi$ is twistorial, $J^{\psi_j \circ \varphi}$ is integrable, $(j = 1, 2, 3)$. Moreover, as $\ker \mathrm{d}\psi_j$ is orthogonal to $\ker \mathrm{d}\psi_k$, if $j \neq k$, the Hermitian structures $J^{\psi_j \circ \varphi}$, $j = 1, 2, 3$, determine a hyper-Hermitian structure on (M^4, c_M). Let D^M be the Obata connection of this hyper-Hermitian structure.

Then $\varphi : (M^4, c_M, D^M) \to (N^3, c_N, D^N)$ is a harmonic morphism [7].

In the following results, as in Proposition 5.1, the given Weyl spaces are endowed, according to their dimensions, with the almost twistorial structures of Examples 3.2, 3.3 or 3.4.

Theorem 5.3 ([7], cf. [14], [10]). *Let (M^4, c_M, D^M) and (N^n, c_N, D^N) be Einstein–Weyl spaces of dimensions 4 and n, respectively, where $n \in \{2, 3\}$; assume M^4 orientable and, if $n = 2$, assume N^2 oriented.*

Then a submersion $\varphi : (M^4, c_N, D^N) \to (N^n, c_N, D^N)$ is a harmonic morphism if and only if φ is twistorial, with respect to a suitable orientation of M^4, and the fibres of Φ are tangent to the connection induced by D^M on P_M.

For harmonic morphisms between Einstein–Weyl spaces of dimensions four and three, the result of Theorem 5.3 takes a more precise form, as follows.

Theorem 5.4 ([7]). *Let (M^4, c_M, D^M) and (N^3, c_N, D^N) be Einstein–Weyl spaces of dimensions four and three, respectively; assume M^4 orientable.*

Then a submersion $\varphi : (M^4, c_N, D^N) \to (N^3, c_N, D^N)$ is a harmonic morphism if and only if, with respect to a suitable orientation, (M^4, c_M) is anti-self-dual, φ is twistorial and, locally, either

(i) *D^M is the Obata connection of a hyper-Hermitian structure, on (M^4, c_M), constructed as in Remark 5.2*, or

(ii) *D^M is the Levi-Civita connection of an Einstein representative g of c_M, with* *nonzero scalar curvature, and the fibres of the twistorial representation of* φ *are tangent to the holomorphic distribution, on the twistor space of* (M^4, c_M) *, determined by* g *.*

Example 5.5. The harmonic morphisms given by the Gibbons-Hawking and the Beltrami fields constructions (see [11] , [15]) satisfy assertion (i) of Theorem 5.4 .

The harmonic morphisms of warped-product type (see [2] , [15]) with one-dimensional fibres from an oriented four-dimensional Riemannian manifold with nonzero constant sectional curvature satisfy assertion (ii) of Theorem 5.4 .

Finally, we present the following result.

Theorem 5.6 ([7])**.** *Let* (M^4, c_M, D^M) *be an orientable Einstein–Weyl space of dimension four.*

Then, locally, there can be defined on (M^4, c_M, D^M) *at least* five *distinct foliations of dimension two which produce harmonic morphisms if and only if one of the following two assertions holds:*

(i) D^M *is the Weyl connection of a Hermitian structure locally defined on* (M^4, c_M)*;*

(ii) (M^4, c_M) *is anti-self-dual, with respect to a suitable orientation, and* D^M *is the Levi-Civita connection of a local Einstein representative of* c_M *.*

The interested reader may consult [7] for the proofs of Theorems 5.3 , 5.4 and 5.6 , and for further facts on harmonic morphisms between Weyl spaces.

References

[1] M.F. Atiyah, N.J. Hitchin, I.M. Singer, Self-duality in four-dimensional Riemannian geometry, *Proc. Roy. Soc. London Ser. A*, **362** (1978) 425–461.

[2] P. Baird, J.C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Math. Soc. Monogr. (N.S.), no. 29, Oxford Univ. Press, Oxford, 2003.

[3] D.M.J. Calderbank, Selfdual Einstein metrics and conformal submersions, Preprint, Edinburgh University, 2000 (available from `http://www-users.york.ac.uk/~dc511/mpapers.html` , math.DG/0001041).

[4] J. Eells, S. Salamon, Twistorial construction of harmonic maps of surfaces into four-manifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **12** (1985) 589–640.

[5] N.J. Hitchin, Complex manifolds and Einstein's equations, *Twistor geometry and nonlinear systems (Primorsko, 1980)*, 73–99, Lecture Notes in Math., 970, Springer, Berlin, 1982.

[6] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, I, II, Wiley Classics Library (reprint of the 1963, 1969 original), Wiley-Interscience Publ., Wiley, New-York, 1996.

[7] E. Loubeau, R. Pantilie, Harmonic morphisms between Weyl spaces and twistorial maps, Preprint, The University of Brest, 2004, I.M.A.R., 2005 (available from `http://arxiv.org/abs/math.DG/0405327`).

[8] E Loubeau, R. Pantilie, Harmonic morphisms between Weyl spaces and twistorial maps II, (in preparation).

[9] N.R. O'Brian, J.H. Rawnsley, Twistor spaces, *Ann. Global Anal. Geom.*, **3** (1985) 29–58.

[10] R. Pantilie, Harmonic morphisms with 1-dimensional fibres on 4-dimensional Einstein manifolds, *Comm. Anal. Geom.*, **10** (2002) 779–814.

[11] R. Pantilie, J.C. Wood, A new construction of Einstein self-dual manifolds, *Asian J. Math.*, **6** (2002) 337–348.

[12] R. Pantilie, J.C. Wood, Twistorial harmonic morphisms with one-dimensional fibres on self-dual four-manifolds, *Q. J. Math*, (to appear).

[13] J.H. Rawnsley, f-structures, f-twistor spaces and harmonic maps, *Geometry seminar "Luigi Bianchi" II — 1984*, 85–159, Lecture Notes in Math., 1164, Springer, Berlin, 1985.
[14] J.C. Wood, Harmonic morphisms and Hermitian structures on Einstein 4-manifolds. *Internat. J. Math.*, **3** (1992) 415–439.
[15] J.C. Wood, Harmonic morphisms between Riemannian manifolds, These proceedings.
[16] K. Yano, On a structure defined by a tensor field f of type $(1,1)$ satisfying $f^3+f=0$, *Tensor (N.S.)*, **14** (1963) 99–109.

E-mail address: Radu.Pantilie@imar.ro

R. Pantilie, Institutul de Matematică "Simion Stoilow" al Academiei Române, C.P. 1-764, 014700, Bucureşti, România