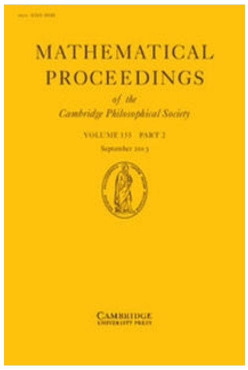

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Conformal actions and harmonic morphisms

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Abstract

We give necessary and sufficient conditions for a conformal foliation locally generated by conformal vector fields to produce harmonic morphisms. Natural constructions of harmonic maps and morphisms are thus obtained. Also we obtain reducibility results for harmonic morphisms induced by (infinitesimal) conformal actions on Einstein manifolds.

0. *Introduction*

Harmonic morphisms are smooth maps between Riemannian manifolds which pull back germs of harmonic functions to germs of harmonic functions. By a basic result of Fuglede and Ishihara [**8**, **11**] harmonic morphisms are harmonic maps which are horizontally weakly conformal.

In order to capture all the harmonic morphisms which can be defined (maybe just locally) on a given Riemannian manifold it is natural to consider the problem of finding foliations whose leaves can be given locally as fibres of (submersive) harmonic morphisms [**22**], i.e. *foliations which produce harmonic morphisms*. By the above-mentioned basic result, these must be conformal foliations. For example on a two-dimensional Riemannian manifold a foliation produces harmonic morphisms if and only if it is locally generated by conformal vector fields. But this is not true in higher dimensions. In this paper we give necessary and sufficient conditions for a foliation locally generated by conformal vector fields to produce harmonic morphisms.

In Section 1 we review some basic facts concerning foliations which produce harmonic morphisms. Then in Section 2 we exploit well-known relations between the mean curvature form and the adapted Bott connection of a distribution on a Riemannian manifold to obtain the formulae needed to prove, in Section 3, the characterisation of the (infinitesimal) conformal actions which produce harmonic morphisms. This (Theorem 3·2) generalizes the result which we have previously obtained for isometric actions [**15**].

In Section 4 we consider homothetic foliations locally generated by conformal vector fields. Homothetic foliations are ones whose leaves are locally the fibres of

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horizontally homothetic maps; they are important since a foliation with minimal leaves produces harmonic morphisms if and only if it is a homothetic foliation [**14**]. The result of Theorem 3·2 when applied to a homothetic foliation shows that the condition of producing harmonic morphisms depends, in this case, just on the corresponding smooth (infinitesimal) action and on the integrability tensor of the orthogonal complement of the foliation. It follows that most of the natural constructions of Riemannian foliations locally generated by Killing fields and which produce harmonic morphisms, which we have obtained in [**15**], can be generalized to homothetic foliations locally generated by conformal vector fields.

In Section 5 we give the geometric characterization of homothetic (infinitesimal) actions which induce homothetic foliations and their relations with harmonic morphisms. Some examples of such actions are also given.

The consideration of foliations locally generated by homothetic vector fields is motivated by the fact that on a Ricci-flat manifold any foliation $\mathscr{V}$ locally generated by conformal vector fields and which produces harmonic morphisms is either locally generated by homothetic vector fields or any harmonic morphism produced by $\mathscr{V}$ can be locally decomposed into a harmonic morphism with geodesic fibres, constant dilation and integrable horizontal distribution followed by another harmonic morphism. This is shown in Section 6 (Theorem 6·1(iii)) in which are obtained reducibility results for foliations locally generated by conformal vector fields and which produce harmonic morphisms on Einstein manifolds.

1. *Foliations which produce harmonic morphisms*

Foliations whose leaves are locally fibres of (submersive) harmonic morphisms were introduced in [**22**]. We recall the following definition.

Definition 1·1. Let (M, g) be a (connected) Riemannian manifold and let $\mathscr{V}$ be (the tangent bundle of) a foliation on it.

We shall say that $\mathscr{V}$ *produces harmonic morphisms on* (M, g) if each point of M has an open neighbourhood U which is the domain of a submersive harmonic morphism $\varphi: (U, g|_U) \to (N, h)$ whose fibres are open subsets of the leaves of $\mathscr{V}$.

By the well-known result of Fuglede and Ishihara [**8**, **11**] harmonic morphisms are harmonic maps which are horizontally weakly conformal. It follows that any foliation which produces harmonic morphisms is a conformal foliation.

Definition 1·2. (1) Let $\mathscr{V}$ be a conformal foliation on the Riemannian manifold (M, g). A smooth positive function $\lambda: U \to \mathbb{R}$ on an open subset U of M will be called a *local dilation* of $\mathscr{V}$ if $\mathscr{V}|_U$ is a Riemannian foliation on $(U, \lambda^2 g|_U)$. If $U = M$ then λ is called a (*global*) *dilation* of $\mathscr{V}$.

(2) Let $\mathscr{V}$ be a foliation which produces harmonic morphisms on the Riemannian manifold (M, g). Let λ be a local dilation of $\mathscr{V}$ which restricts to give dilations of harmonic morphisms which locally define $\mathscr{V}$. Then $\rho = \lambda^{2-n}$ is called a *local density* of $\mathscr{V}$. If λ is globally defined on M then ρ is called a (*global*) *density*.

The terminology of Definition 1·2(2) is motivated by the following fact.

Remark 1·3. Let $\mathscr{V}$ be a foliation which produces harmonic morphisms on (M, g). Let ω be a local volume form for $\mathscr{V}$ and $\mathscr{H} = \mathscr{V}^{\perp}$. A positive smooth function ρ is a local density for $\mathscr{V}$ if and only if $\rho\omega$ is invariant under the parallel displacement determined by $\mathscr{H}$ [**6**, **14**].

A foliation of codimension two produces harmonic morphisms if and only if it is conformal and has minimal leaves [**22**]. This follows from the corresponding result for harmonic morphisms due to Baird and Eells [**2**]. For foliations of codimension not equal to 2 we have the following reformulation of a result of Bryant [**6**].

PROPOSITION 1·4. *Let $\mathscr{V}$ be a conformal foliation on (M, g) of* codim $\mathscr{V} \neq 2$ *and $\mathscr{H}$ its orthogonal complement. Let ${}^{\mathscr{V}}B$ and ${}^{\mathscr{H}}B$ be the second fundamental forms of $\mathscr{V}$ and $\mathscr{H}$, respectively.*

Then, $\mathscr{V}$ produces harmonic morphisms on (M, g) if and only if the one-form

$$(n-2)\,\mathrm{trace}\,({}^{\mathscr{H}}B)^{\flat} - n\,\mathrm{trace}\,({}^{\mathscr{V}}B)^{\flat} \tag{1·1}$$

is closed. (*Here* $\mathrm{trace}\,({}^{\mathscr{H}}B) = \sum_a {}^{\mathscr{H}}B(X_a, X_a)$, $\mathrm{trace}\,({}^{\mathscr{V}}B) = \sum_r {}^{\mathscr{V}}B(V_r, V_r)$ *for local orthonormal frames $\{X_a\}$ and $\{V_r\}$ of $\mathscr{H}$ and $\mathscr{V}$, respectively and where ${}^{\flat}: TM \to T^*M$ is the 'musical' isomorphism induced by g.*)

For a proof of Proposition 1·4 the reader can consult [**4**], [**6**] or [**14**].

Remark 1·5. (1) If $\mathscr{V}$ produces harmonic morphisms then any local density ρ of it is characterized by

$$n\,\mathrm{grad}\,(\log\rho) = -(n-2)\,\mathrm{trace}\,({}^{\mathscr{H}}B) + n\,\mathrm{trace}\,({}^{\mathscr{V}}B).$$

(2) The vertical component of the exterior derivative of the one-form given by (1·1) is always zero (see Proposition 2·18(*a*) below).

PROPOSITION 1·6. *Let $\mathscr{V}$ be a foliation which produces harmonic morphisms on (M, g). Then there exists a Riemannian regular covering $\xi: (\widetilde{M}, \widetilde{g}) \to (M, g)$ with the following properties*:

(i) *$\xi^*(\mathscr{V})$ admits a global density*;

(ii) *if $\eta: (P, k) \to (M, g)$ is any Riemannian regular covering such that $\eta^*(\mathscr{V})$ admits a global density then there exists a unique Riemannian regular covering $\sigma: (P, k) \to (\widetilde{M}, \widetilde{g})$ such that $\eta = \xi \circ \sigma$.*

Moreover, ξ is the unique Riemannian regular covering satisfying (i) *and* (ii).

Proof. Let $[a] \in H^1(M, \mathbb{R})$ be the cohomology class defined by the differentials of the logarithms of the local densities of $\mathscr{V}$ and let $\xi: \widetilde{M} \to M$ be the regular covering corresponding to it.

Let $\widetilde{g} = \xi^*(g)$. It is obvious that $\xi^*(\mathscr{V})$ produces harmonic morphisms on $(\widetilde{M}, \widetilde{g})$.

Also $\xi^*[a] = 0 \in H^1(\widetilde{M}, \mathbb{R})$. Hence, there exists a positive smooth function $\rho: \widetilde{M} \to (0, \infty)$ such that $\xi^*(a) = \mathrm{d}\log\rho$.

Then ρ is a global density of $\xi^*(\mathscr{V})$.

Let $\eta: (P, k) \to (M, g)$ be any other Riemannian regular covering such that $\eta^*(\mathscr{V})$ admits a global density. If H and K are the (Abelian) groups of ξ and η, respectively, then ξ and η correspond to surjective group morphisms $\pi_1(M) \to H$ and $\pi_1(M) \to K$, respectively, where $\pi_1(M)$ is the fundamental group of M (see [**19**, part I, section 14·6]).

Now, $\eta^*(\mathscr{V})$ admits a global density if and only if $\eta^*(\xi)$ is a trivial covering and this happens if and only if the image of the injective group morphism $\pi_1(P) \to \pi_1(M)$ is contained in the kernel of the group morphism $\pi_1(M) \to H$. But the image of $\pi_1(P) \to \pi_1(M)$ is equal to the kernel of $\pi_1(M) \to K$ and hence the latter can be factorized $\pi_1(M) \to H \to K$. The surjective group morphism $H \to K$ induces a Riemannian regular covering $\sigma\colon (P, k) \to (\widetilde{M}, \widetilde{g})$ having the required properties.

The uniqueness of ξ is obvious.

The following simple lemma will be used later on (cf. [**18**, chapter IV, example 4·10]).

Lemma 1·7. *Let $\mathscr{V}$ be a foliation on (M, g) and let $V \in \Gamma(\mathscr{V})$ be a conformal vector field.*

Then $[V, X] = 0$ for any basic vector field X.

Proof. Let $\mathscr{H} = \mathscr{V}^{\perp}$ and $X \in \Gamma(\mathscr{H})$ a basic vector field. Then $[V, X] \in \Gamma(\mathscr{V})$.

But V is conformal and hence we can write

$$0 = (\mathscr{L}_V g)(W, X) = -g(W, [V, X])$$

for any $W \in \Gamma(\mathscr{V})$. Hence $[V, X] = 0$.

The result of Lemma 1·7 is equivalent to the fact that any conformal vector field tangent to a foliation $\mathscr{V}$ is an infinitesimal automorphism of the orthogonal complement of $\mathscr{V}$.

2. *Mean curvature forms and adapted Bott connections*

In this section we recall [**17**] that the exterior derivative of the mean curvature form of a distribution is the curvature form of the connection induced on the determinant bundle of the distribution by its adapted Bott connection. By using this, or by direct calculation we obtain formulae for the exterior derivative of the mean curvature form of a distribution. Some of these formulae apply to prove that if a conformal foliation $\mathscr{V}$ has integrable orthogonal complement $\mathscr{H}$ and if $\mathscr{V}$ and $\mathscr{H}$ both have basic mean curvature forms then $\mathscr{V}$ produces harmonic morphisms.

Let $\mathscr{V}$ and $\mathscr{H}$ be two complementary orthogonal distributions (not necessarily integrable) on (M, g). We shall denote by the same symbols $\mathscr{V}$ and $\mathscr{H}$ the induced projections on $\mathscr{V}$ and $\mathscr{H}$, respectively.

Definition 2·1 (see [**18**, **20**]). The adapted Bott connection $\overset{\mathscr{H}}{\nabla}$ on $\mathscr{H}$ is defined by

$$\overset{\mathscr{H}}{\nabla}_E X = \mathscr{H}[\mathscr{V}E, X] + \mathscr{H}(\nabla_{\mathscr{H}E} X)$$

for $E \in \Gamma(TM)$, $X \in \Gamma(\mathscr{H})$ where ∇ is the Levi–Civita connection of (M, g).

The adapted Bott connection $\overset{\mathscr{V}}{\nabla}$ on $\mathscr{V}$ is defined similarly by reversing the roles of $\mathscr{V}$ and $\mathscr{H}$.

Remark 2·2 (see [**18**, **20**]). It is easy to see that $\overset{\mathscr{H}}{\nabla}$ is compatible with the metric induced by g on $\mathscr{H}$ if and only if $\mathscr{H}$ is totally geodesic. Nevertheless, since $\overset{\mathscr{H}}{\nabla}_X = \mathscr{H}\nabla_X$ for any $X \in \Gamma(\mathscr{H})$ we have that $\overset{\mathscr{H}}{\nabla}_X(g|_{\mathscr{H}}) = 0$.

Let ${}^{\mathscr{H}}I$ be the integrability tensor of $\mathscr{H}$ which is the $\mathscr{V}$-valued horizontal two-form defined by ${}^{\mathscr{H}}I(X, Y) = -\mathscr{V}[X, Y]$ for X, $Y \in \Gamma(\mathscr{H})$.

Proposition 2·3. *Let $\mathscr{H}$ be a distribution on (M, g). Then*

$$\mathrm{d}(\mathrm{trace}\,({}^{\mathscr{H}}\!B)^{\flat})(X, Y) = g(\mathrm{trace}\,({}^{\mathscr{H}}\!B), {}^{\mathscr{H}}\!I(X, Y))\ , \tag{2·1}$$

$$\mathrm{d}(\mathrm{trace}\,({}^{\mathscr{H}}\!B)^{\flat})(X, V) = \overset{\mathscr{V}}{\nabla}_X(\mathrm{trace}\,({}^{\mathscr{H}}\!B)^{\flat})(V) \tag{2·2}$$

for any horizontal vectors X, Y and vertical vector V.

Proof. This is a straightforward calculation using the fact that $\mathrm{trace}\,({}^{\mathscr{H}}\!B)$ is a vertical vector field.

Let $n = \dim \mathscr{H}$ and let $\bigwedge^n \mathscr{H}$ be the determinant line bundle of $\mathscr{H}$.

Let ${}^{\mathscr{H}}\!R \in \Gamma(\mathrm{End}(\mathscr{H}) \otimes \bigwedge^2(T^*M))$ be the curvature form of $\overset{\mathscr{H}}{\nabla}$. Then the curvature form of the connection induced by $\overset{\mathscr{H}}{\nabla}$ on $\bigwedge^n \mathscr{H}$ is $\mathrm{trace}\,({}^{\mathscr{H}}\!R) \in \Gamma(\bigwedge^2(T^*M))$.

Proposition 2·4 (see [**17**]). *Let $\mathscr{H}$ be a distribution on (M, g). Then*

$$\mathrm{trace}\,({}^{\mathscr{H}}\!R) = \mathrm{d}(\mathrm{trace}\,({}^{\mathscr{H}}\!B)^{\flat}).$$

Proof. Let $\overset{\mathscr{H}}{\omega}$ be a local volume form of $\mathscr{H}$ considered with respect to the metric induced by g.

Recall that $\mathrm{trace}\,({}^{\mathscr{H}}\!R) = \mathrm{d}A$ where A is any local connection form of the connection induced by $\overset{\mathscr{H}}{\nabla}$ on $\bigwedge^n \mathscr{H}$. Thus it suffices to show that

$$\overset{\mathscr{H}}{\nabla}_E \overset{\mathscr{H}}{\omega} = -g(E, \mathrm{trace}\,({}^{\mathscr{H}}\!B))\, \overset{\mathscr{H}}{\omega} \tag{2·3}$$

for any $E \in TM$.

If $E \in \mathscr{H}$ then the right-hand side of (2·3) is zero. Also, the left-hand side is zero because if $E \in \mathscr{H}$ then $\overset{\mathscr{H}}{\nabla}_E(g|_{\mathscr{H}}) = 0$ (see Remark 2·2).

If $E \in \mathscr{V}$ then $\overset{\mathscr{H}}{\nabla}_E \overset{\mathscr{H}}{\omega} = \mathscr{H}^*(\mathscr{L}_E \overset{\mathscr{H}}{\omega})$. Thus, if $E \in \mathscr{V}$ then (2·3) reduces to a well-known formula (see [**22**]).

Proposition 2·5 (see [**18**, **20**]). *Let $X \in \Gamma(\mathscr{H})$ be a horizontal vector field. Then the following assertions are equivalent*:

(i) $\overset{\mathscr{H}}{\nabla}_V X = 0$ *for any* $V \in \Gamma(\mathscr{V})$;
(ii) $\mathscr{H}[X, V] = 0$ *for any* $V \in \Gamma(\mathscr{V})$;
(iii) $\mathscr{L}_X(\Gamma(\mathscr{V})) \subseteq \Gamma(\mathscr{V})$.

If $\mathscr{H}$ is integrable then the following assertions can be added:

(iv) $\mathscr{L}_X \circ \mathscr{V} = \mathscr{V} \circ \mathscr{L}_X$;
(v) $\mathscr{L}_X \circ \mathscr{H} = \mathscr{H} \circ \mathscr{L}_X$;
(vi) *X is an infinitesimal automorphism of $\mathscr{V}$ (i.e. $(\xi_t)_*(\mathscr{V}) \subseteq \mathscr{V}$ where (ξ_t) is the local flow of X).*

Proof. The equivalences (i) $\Longleftrightarrow$ (ii) and (ii) $\Longleftrightarrow$ (iii) are trivial.

It is easy to see that if $\mathscr{H}$ is integrable then (iv) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (v). Also, (ii) $\Longleftrightarrow$ (vi) follows easily from [**12**, chapter 1, corollary 1·10].

Example 2·6 (see [**18**, **20**]). (1) Suppose that $\mathscr{V}$ is integrable. Then any basic vector field $X \in \Gamma(\mathscr{H})$ for $\mathscr{V}$ with respect to $\mathscr{H}$ is an infinitesimal automorphism of $\mathscr{V}$. In fact, if $\mathscr{V}$ is integrable a horizontal vector field $X \in \Gamma(\mathscr{H})$ is basic if and only if any of the assertions (i), (ii), (iii) or (vi) holds.

(2) If $\mathscr{H}$ is integrable then, by Lemma 1·7, any conformal vector field X which is horizontal is an infinitesimal automorphism of $\mathscr{V}$.

We recall (cf. [**20**, 4·34]) the following definition which does not require any assumption on the distribution $\mathscr{H}$.

Definition 2·7. Let $E \in \Gamma(TM)$. The *horizontal divergence* $\mathrm{div}_{\mathscr{H}} E$ *of* E is defined by

$$\mathscr{H}^*(\mathscr{L}_E \overset{\mathscr{H}}{\omega}) = \mathrm{div}_{\mathscr{H}} E \, \overset{\mathscr{H}}{\omega},$$

where $\overset{\mathscr{H}}{\omega}$ is any local volume form of $\mathscr{H}$ (considered with the metric induced by g).

(The vertical divergence $\mathrm{div}_{\mathscr{V}}$ is defined similarly, note that

$$\mathrm{div}\, E = \mathrm{div}_{\mathscr{H}} E + \mathrm{div}_{\mathscr{V}} E$$

for any $E \in \Gamma(TM)$.)

It is not difficult to see that $\mathrm{div}_{\mathscr{H}} E$ is globally well-defined (i.e. it does not depend on $\overset{\mathscr{H}}{\omega}$). In fact, a standard calculation gives the following proposition.

PROPOSITION 2·8. *Let* $E \in \Gamma(TM)$. *Then* $\mathrm{div}_{\mathscr{H}} E$ *is the* (*pointwise*) *trace of the linear endomorphism* $\mathscr{H} \longrightarrow \mathscr{H}$ *defined by* $Y \longmapsto \mathscr{H}(\nabla_Y E)$.

Remark 2·9. (1) Obviously, if $\mathscr{H}$ is integrable and $X \in \Gamma(\mathscr{H})$ then the restriction of $\mathrm{div}_{\mathscr{H}} X$ to each leaf L of $\mathscr{H}$ is equal to the divergence of the restriction of X to $(L, g|_L)$.

(2) If $V \in \Gamma(\mathscr{V})$ then $\mathrm{div}_{\mathscr{H}} V = -g(\mathrm{trace}\,({}^{\mathscr{H}}\!B), V)$ (see [**22**]).

LEMMA 2·10. (*a*) *Suppose that* $X \in \Gamma(\mathscr{H})$ *satisfies any of the assertions* (i), (ii) *or* (iii) *of* Proposition 2·5. *Then*

$${}^{\mathscr{H}}\!R(V,W)X = \mathscr{H}(\nabla_{{}^{\mathscr{V}}\!I(V,W)}X)$$

for any vertical V *and* W.

(*b*) *Suppose that* $\mathscr{H}$ *is integrable and let* X, $Y \in \Gamma(\mathscr{H})$ *and* $V \in \Gamma(\mathscr{V})$ *be such that* $[V,X] = 0 = [V,Y]$. *Then*

$${}^{\mathscr{H}}\!R(V,X)Y = \overset{\mathscr{H}}{\nabla}_V(\mathscr{H}\nabla_Y X).$$

Proof. (*a*) Let V, $W \in \Gamma(\mathscr{V})$. Then

$$\begin{aligned} {}^{\mathscr{H}}\!R(V,W)X &= [\overset{\mathscr{H}}{\nabla}_V, \overset{\mathscr{H}}{\nabla}_W]X - \overset{\mathscr{H}}{\nabla}_{[V,W]}X \\ &= -\overset{\mathscr{H}}{\nabla}_{\mathscr{V}[V,W]}X - \overset{\mathscr{H}}{\nabla}_{\mathscr{H}[V,W]}X = \mathscr{H}(\nabla_{{}^{\mathscr{V}}\!I(V,W)}X). \end{aligned}$$

(*b*) We have

$$\begin{aligned} {}^{\mathscr{H}}\!R(V,X)Y &= [\overset{\mathscr{H}}{\nabla}_V, \overset{\mathscr{H}}{\nabla}_X]Y - \overset{\mathscr{H}}{\nabla}_{[V,X]}Y \\ &= \overset{\mathscr{H}}{\nabla}_V(\overset{\mathscr{H}}{\nabla}_X Y) = \overset{\mathscr{H}}{\nabla}_V(\mathscr{H}\nabla_X Y) \\ &= \overset{\mathscr{H}}{\nabla}_V(\mathscr{H}[X,Y] + \mathscr{H}\nabla_Y X). \end{aligned}$$

Because $\mathscr{H}$ is integrable we have that $\mathscr{H}[X,Y] = [X,Y]$ and from $[V,X] = 0 = [V,Y]$, by using the Jacobi identity, we obtain that $[V,[X,Y]] = 0$. Hence $\overset{\mathscr{H}}{\nabla}_V(\mathscr{H}[X,Y]) = 0$ and the lemma follows.

Let $\{X_a\}$ be a local frame for $\mathscr{H}$ over the open subset $U \subseteq M$ and let $\{V_r\}$ be a local frame for $\mathscr{V}$ over U. We shall denote 'horizontal' indices by a, b, c and 'vertical' indices by r, s, t.

LEMMA 2·11. (*a*) *Suppose that $\{X_a\}$ are infinitesimal automorphisms of $\mathscr{V}$. Then*

$$\mathrm{d}(\mathrm{trace}\,({}^{\mathscr{H}}\!B)^{\flat})_{rs} = (c^a_{ba} + \mathrm{div}_{\mathscr{H}} X_b)^{\mathscr{V}} I^b_{rs}, \tag{2·4}$$

where $\{c^c_{ab}\}$ are defined by $\mathscr{H}[X_a, X_b] = c^c_{ab} X_c$.

(*b*) *If both $\mathscr{H}$ and $\mathscr{V}$ are locally generated by infinitesimal automorphisms of $\mathscr{V}$ and $\mathscr{H}$, respectively, and $\mathscr{H}$ is integrable then*

$$\mathscr{V}^*(\mathrm{d}(\mathrm{trace}\,({}^{\mathscr{H}}\!B)^{\flat})) = \mathrm{div}_{\mathscr{H}}({}^{\mathscr{V}}I).$$

Proof. (*a*) This follows from Propositions 2·4, 2·8 and Lemma 2·10(*a*).

(*b*) This follows from (*a*) and the fact that $\mathscr{V}^*(\mathscr{L}_X({}^{\mathscr{V}}I)) = 0$ for any infinitesimal automorphism $X \in \Gamma(\mathscr{H})$ of $\mathscr{V}$.

(Note that it seems to be impossible to formulate invariantly assertion (*a*) of Lemma 2·11.)

PROPOSITION 2·12 (see [**17**]). *If $\mathscr{V}$ is integrable then* $\mathrm{d}(\mathrm{trace}\,({}^{\mathscr{H}}\!B)^{\flat})(V, W) = 0$ *for any vertical V and W.*

Proof. This follows from Lemma 2·11 because if $\mathscr{V}$ is integrable then any basic vector field for $\mathscr{V}$ with respect to $\mathscr{H}$ is an infinitesimal automorphism of $\mathscr{V}$.

PROPOSITION 2·13. *Suppose that both $\mathscr{V}$ and $\mathscr{H}$ are integrable. Then the following assertions are equivalent.*

(i) *The mean curvature form of $\mathscr{H}$ is closed;*

(ii) *The mean curvature form of $\mathscr{H}$ is basic (for $\mathscr{V}$).*

Proof. This follows from Propositions 2·3 and 2·12.

PROPOSITION 2·14. *Suppose that $\mathscr{H}$ is integrable and locally generated by infinitesimal automorphisms of $\mathscr{V}$ and let $V \in \Gamma(\mathscr{V})$ and $X \in \Gamma(\mathscr{H})$ be infinitesimal automorphisms of $\mathscr{H}$ and $\mathscr{V}$, respectively. Then*

$$\mathrm{d}(\mathrm{trace}\,({}^{\mathscr{H}}\!B)^{\flat})(V, X) = V(\mathrm{div}_{\mathscr{H}} X) = -\overset{\mathscr{V}}{\nabla}_X(\mathrm{trace}\,({}^{\mathscr{H}}\!B)^{\flat})(V). \tag{2·5}$$

Proof. This follows from Propositions 2·3, 2·4, 2·8 and Lemma 2·10(*b*).

By reversing the roles of $\mathscr{V}$ and $\mathscr{H}$ in Proposition 2·3 and Lemma 2·11 and Proposition 2·14 we obtain the corresponding formulae for $\mathrm{d}(\mathrm{trace}\,({}^{\mathscr{V}}\!B)^{\flat})$.

The following lemma holds for any complementary orthogonal distributions $\mathscr{H}$ and $\mathscr{V}$.

LEMMA 2·15. *Let f be any smooth function on M. Then*

$$\overset{\mathscr{H}}{\nabla}_V(\mathscr{H}^*(\mathrm{d}f))(X) = \overset{\mathscr{V}}{\nabla}_X(\mathscr{V}^*(\mathrm{d}f))(V) \tag{2·6}$$

for any vertical V and horizontal X.

Proof. Let X and V be vector fields which are horizontal and vertical, respectively. The following relation is trivial

$$V(X(f)) - X(V(f)) - [V,X](f) = 0. \tag{2·7}$$

But (2·7) is equivalent to the following

$$V(X(f)) - \mathscr{H}[V,X](f) = X(V(f)) - \mathscr{V}[X,V](f)$$

which is obviously equivalent to (2·6).

From some of the above results we obtain the following.

Proposition 2·16. *Let $\mathscr{V}$ be a foliation of* $\operatorname{codim}\mathscr{V} = n \neq 2$ *which produces harmonic morphisms on* (M,g). *Then the following assertions are equivalent.*

(i) *The mean curvature form of $\mathscr{V}$ is basic*;

(ii) *The mean curvature form of $\mathscr{H}$ is invariant under the parallel displacement determined by $\mathscr{H}$ (i.e.* $\overset{\mathscr{V}}{\nabla}_X(\operatorname{trace}(^{\mathscr{H}}B)^\flat)(V) = 0$).

Proof. Recall that $\operatorname{trace}(^{\mathscr{H}}B)^\flat = n\mathscr{V}^*(\mathrm{d}\log\lambda)$ (see [**4**]) for any local dilation λ of $\mathscr{V}$.

Also, recall that from the fundamental equation of Baird and Eells ([**2**], see also [**4**]) it follows that the assertion (i) is equivalent to the fact that $\operatorname{trace}(^{\mathscr{V}}B)^\flat = -(n-2)\mathscr{H}^*(\mathrm{d}\log\lambda)$ for any local density λ^{2-n} of $\mathscr{V}$.

Now the equivalence (i) $\Longleftrightarrow$ (ii) follows from Lemma 2·15.

Theorem 2·17. *Let $\mathscr{V}$ be a conformal foliation on (M,g) of* $\operatorname{codim}\mathscr{V} \neq 2$. *Suppose that the orthogonal complement $\mathscr{H}$ of $\mathscr{V}$ is integrable.*

Then any two of the following assertions imply the remaining assertion.

(i) *$\mathscr{V}$ produces harmonic morphisms*;

(ii) *The mean curvature form of $\mathscr{V}$ is basic* (*for $\mathscr{H}$*);

(iii) *The mean curvature form of $\mathscr{H}$ is basic* (*for $\mathscr{V}$*).

Moreover, if any two of (i), (ii) *or* (iii) *hold then both $\mathscr{V}$ and $\mathscr{H}$ have closed mean curvature forms.*

Proof. If (i) holds then the equivalence (ii) $\Longleftrightarrow$ (iii) follows from Proposition 2·16.

Suppose that both the assertions (ii) and (iii) hold. Then (i) follows from Propositions 2·13 and 1·4.

Proposition 2·18. (*a*) *If $\mathscr{V}$ is integrable then*

$$\mathrm{d}(\operatorname{trace}(^{\mathscr{H}}B)^\flat)(V,W) = 0 = \mathrm{d}(\operatorname{trace}(^{\mathscr{V}}B)^\flat)(V,W) \tag{2·8}$$

for any vertical V and W.

(*b*) *Let $\mathscr{V}$ be a conformal foliation on (M,g). Then the following assertions are equivalent.*

(i) *For any local dilation λ of $\mathscr{V}$ the one-form* $\operatorname{trace}(^{\mathscr{V}}B)^\flat + (n-2)\mathscr{H}^*(\mathrm{d}\log\lambda)$ *is basic* ($n = \dim\mathscr{H}$);

(ii) *For any horizontal X and vertical V we have*

$$\mathrm{d}((n-2)\operatorname{trace}(^{\mathscr{H}}B)^\flat - n\operatorname{trace}(^{\mathscr{V}}B)^\flat)(X,V) = 0. \tag{2·9}$$

Proof. (*a*) The first equality of (2·8) follows from Proposition 2·12.

The second equality of (2·8) follows from (2·1) of Proposition 2·3 by reversing the roles of $\mathscr{H}$ and $\mathscr{V}$.

(*b*) Let λ be a local dilation of $\mathscr{V}$ and recall that $\operatorname{trace}({}^{\mathscr{H}}\!B) = n\mathscr{V}(\operatorname{grad}(\log\lambda))$. Hence, by applying (2·2) of Proposition 2·3 and Lemma 2·15 we obtain that

$$\begin{aligned}((n-2)\mathrm{d}(\operatorname{trace}({}^{\mathscr{H}}\!B)^\flat) - n\mathrm{d}(\operatorname{trace}({}^{\mathscr{V}}\!B)^\flat))(X,V)\\ &= (n-2)\overset{\mathscr{V}}{\nabla}_X(\operatorname{trace}({}^{\mathscr{H}}\!B)^\flat)(V) + n\overset{\mathscr{H}}{\nabla}_V(\operatorname{trace}({}^{\mathscr{V}}\!B)^\flat)(X)\\ &= n(n-2)\overset{\mathscr{V}}{\nabla}_X(\mathscr{V}^*(\mathrm{d}\log\lambda))(V) + n\overset{\mathscr{H}}{\nabla}_V(\operatorname{trace}({}^{\mathscr{V}}\!B)^\flat)(X)\\ &= n(n-2)\overset{\mathscr{H}}{\nabla}_V(\mathscr{H}^*(\mathrm{d}\log\lambda))(X) + n\overset{\mathscr{H}}{\nabla}_V(\operatorname{trace}({}^{\mathscr{V}}\!B)^\flat)(X)\end{aligned}$$

for any horizontal X and vertical V and the proof of (i) $\Longleftrightarrow$ (ii) follows from the fact that the basic vector fields (for $\mathscr{V}$) are precisely those horizontal vector fields which are infinitesimal automorphisms of $\mathscr{V}$ (see Example 2·6(1)).

3. *The characterization of the conformal actions which produce harmonic morphisms*

On a two-dimensional Riemannian manifold a foliation (of dimension one) produces harmonic morphisms if and only if it is locally generated by conformal vector fields. This follows from the fact that a harmonic morphism to a one-dimensional Riemannian manifold is essentially a harmonic function and, if the domain is two-dimensional, then any harmonic function is locally the real part of a conformal map. If the manifold has dimension greater than two then it is not true that any foliation locally generated by conformal vector fields produces harmonic morphisms. In this section we shall give necessary and sufficient conditions for a foliation locally generated by conformal vector fields to produce harmonic morphisms (Theorem 3·2). To state this result we need the following:

PROPOSITION 3·1. *Let $\mathscr{V}$ be a foliation on (M,g) locally generated by conformal vector fields* (*i.e. in the neighbourhood of each point a local frame for $\mathscr{V}$ made up of conformal vector fields can be found*). *Let $\mathscr{H}$ be the orthogonal complement of $\mathscr{V}$ and let ${}^{\mathscr{H}}\!I$ be its integrability tensor.*

Let $\{V_r\}$ be a local frame of $\mathscr{V}$ made up of conformal vector fields. Define $\operatorname{trace}(\operatorname{ad}({}^{\mathscr{H}}\!I))$ *by* $\operatorname{trace}(\operatorname{ad}({}^{\mathscr{H}}\!I)) = c^s_{rs}\,{}^{\mathscr{H}}\!I^r$ *where ${}^{\mathscr{H}}\!I = V_r \otimes {}^{\mathscr{H}}\!I^r$ and $[V_r, V_s] = c^t_{rs}V_t$. Then this is a well-defined horizontal two-form which is independent of the frame $\{V_r\}$.*

Proof. Let U be the domain of a local dilation λ. Then $\mathscr{V}|_U$, viewed as a foliation on $(U, \lambda^2 g|_U)$, is Riemannian.

Let $\{V_r\}$ be a local frame of $\mathscr{V}$, over a subset of U, made up of conformal vector fields of (M,g). Then V_r are Killing fields with respect to $\lambda^2 g|_U$. From [**15**, proposition 1·10] it follows that $\operatorname{trace}(\operatorname{ad}({}^{\mathscr{H}}\!I)) = c^s_{rs}\,{}^{\mathscr{H}}\!I^r$ is a well-defined horizontal two-form which is independent of the frame $\{V_r\}$.

We now state the main result of this section.

THEOREM 3·2. *Let $\mathscr{V}$ be a conformal foliation of* $\operatorname{codim}\mathscr{V} = n \neq 2$ *on (M^m, g), $m \geqslant 3$. Suppose that $\mathscr{V}$ is locally generated by conformal vector fields and let $\mathscr{H}$ be its orthogonal complement.*

Then the following assertions are equivalent:

(i) *$\mathscr{V}$ produces harmonic morphisms*;

(ii) *The mean curvature form of $\mathscr{V}$ is basic and the following relation holds*:

$$\operatorname{trace}(\operatorname{ad}({}^{\mathscr{H}}I)) = \frac{m-2}{n}\, g(\operatorname{trace}({}^{\mathscr{H}}B), {}^{\mathscr{H}}I). \tag{3·1}$$

Proof. By Proposition 1·4, $\mathscr{V}$ produces harmonic morphisms if and only if

$$(n-2)\mathrm{d}(\operatorname{trace}({}^{\mathscr{H}}B)^{\flat}) - n\mathrm{d}(\operatorname{trace}({}^{\mathscr{V}}B)^{\flat}) = 0. \tag{3·2}$$

By Proposition 2·18(a) the left-hand side of (3·2) is automatically zero when evaluated on a pair of vertical vectors.

Let λ be a local dilation of $\mathscr{V}$.

Let $V \in \Gamma(\mathscr{V})$ be a conformal vector field on (M, g). It is obvious that V when restricted to any leaf L of $\mathscr{V}$ is a conformal vector field on $(L, g|_L)$. Using this it is easy to see that

$$\operatorname{div}_{\mathscr{V}} V = -(m-n)\, V(\log\lambda). \tag{3·3}$$

Let $X \in \Gamma(\mathscr{H})$ be a basic vector field. Then

$$\begin{aligned}\overset{\mathscr{H}}{\nabla}_V(\mathscr{H}^*(\mathrm{d}\log\lambda))(X) &= V(X(\log\lambda)) = X(V(\log\lambda))\\ &= -\frac{1}{m-n}X(\operatorname{div}_{\mathscr{V}} V) = \frac{1}{m-n}\overset{\mathscr{H}}{\nabla}_V(\operatorname{trace}({}^{\mathscr{V}}B)^{\flat})(X),\end{aligned} \tag{3·4}$$

where we have also applied Lemma 1·7 and Proposition 2·14 (reversing the roles of $\mathscr{H}$ and $\mathscr{V}$ in the latter). From Proposition 2·18(b) and (3·4) it follows that the left-hand side of (3·2) is zero when evaluated on a pair made up of a vertical vector and a horizontal vector if and only if $\mathscr{V}$ has basic mean curvature form.

Now, using Lemma 2·11(a) (with the roles of $\mathscr{H}$ and $\mathscr{V}$ reversed) and (3·3) it is easy to see that

$$\mathrm{d}(\operatorname{trace}({}^{\mathscr{V}}B)^{\flat})(X,Y) = \operatorname{trace}(\operatorname{ad}({}^{\mathscr{H}}I))(X,Y) - \frac{m-n}{n}g(\operatorname{trace}({}^{\mathscr{H}}B), {}^{\mathscr{H}}I(X,Y)) \tag{3·5}$$

for any horizontal X and Y. By combining (3·5) and Proposition 2·3 we obtain that the left-hand side of (3·2) is zero when evaluated on a pair of horizontal vectors if and only if (3·1) holds. The theorem is proved.

Remark 3·3. By Proposition 2·16 the first condition of Theorem 3·2(ii) (i.e. $\mathscr{V}$ has basic mean curvature form) can be replaced with the fact that the mean curvature form of $\mathscr{H}$ is invariant under the parallel displacement determined by $\mathscr{H}$.

Corollary 3·4. *Let $\mathscr{V}$ be a foliation on (M^m, g), $m \geqslant 3$, which is locally generated by conformal vector fields and has integrable orthogonal complement. Then the following assertions are equivalent.*

(i) *$\mathscr{V}$ produces harmonic morphisms*;

(ii) *$\mathscr{V}$ has basic mean curvature form.*

Moreover, if either assertion (i) *or* (ii) *holds then both $\mathscr{V}$ and its orthogonal complement have closed mean curvature forms.*

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) is an immediate consequence of Theorem 3·2. The last assertion follows from Proposition 2·3, Lemma 2·11, Proposition 2·14 and Theorem 3·2.

The following result of [**15**, theorem 1·12] can be viewed as a consequence of Theorem 3·2.

COROLLARY 3·5. *Let* $\mathscr{V}$ *be a Riemannian foliation of* $\operatorname{codim}\mathscr{V} \neq 2$ *on* (M^m, g), $m \geqslant 3$, *and let* ${}^{\mathscr{H}}I$ *be the integrability tensor field of its orthogonal complement. Suppose that* $\mathscr{V}$ *is locally generated by Killing fields.*

Then the following assertions are equivalent:

(i) $\mathscr{V}$ *produces harmonic morphisms*;

(ii) $\operatorname{trace}(\operatorname{ad}({}^{\mathscr{H}}I)) = 0$.

Proof. It is obvious that $\mathscr{V}$ has basic mean curvature. Also, $\operatorname{trace}({}^{\mathscr{H}}B) = 0$ and the proof follows from Theorem 3·2.

4. *Homothetic foliations locally generated by conformal vector fields*

A one-dimensional foliation of codimension greater than two which is locally generated by conformal vector fields produces harmonic morphisms if and only if it is a homothetic foliation. This was proved in [**14**] (see also Corollary 4·6 below) where the notion of homothetic foliation was introduced. From there we recall the following:

Definition 4·1. Let $\mathscr{V}$ be a distribution on the Riemannian manifold (M, g). We shall say that $\mathscr{V}$ is *homothetic* if it is conformal and the mean curvature form of its orthogonal complement is closed.

Then, a conformal foliation is homothetic if and only if in a neighbourhood of each point a local dilation can be defined which is constant along horizontal curves. (Equivalently, any local dilation can be locally decomposed as the product of a function constant along horizontal curves and a function constant along vertical curves.) From this it follows that the homothetic foliations are characterised by the property that their leaves are locally fibres of horizontally homothetic submersions. Also, note that by Proposition 1·4 a foliation of codimension not equal to 2 which has minimal leaves produces harmonic morphisms if and only if it is a homothetic foliation. (See [**1**] for other relations between harmonic maps and minimal submanifolds.)

From results of Sections 2 and 3 we obtain necessary and sufficient conditions for a foliation to be homothetic.

COROLLARY 4·2. *Let* $\mathscr{V}$ *be a conformal foliation on* (M, g) *with integrable orthogonal complement* $\mathscr{H}$. *If both* $\mathscr{V}$ *and* $\mathscr{H}$ *have basic mean curvature forms then* $\mathscr{V}$ *is a homothetic foliation. If further* $\operatorname{codim}\mathscr{V} \neq 2$ *and* $\dim M \geqslant 3$ *then* $\mathscr{V}$ *produces harmonic morphisms.*

Proof. This is an immediate consequence of Theorem 2·17.

PROPOSITION 4·3. *Let* $\mathscr{V}$ *be a foliation of* $\operatorname{codim}\mathscr{V} = n$ *on* (M^m, g) *which is locally generated by conformal vector fields and let* $\mathscr{H}$ *be its orthogonal complement.*

If $\mathscr{V}$ *is homothetic then* $g(\operatorname{trace}({}^{\mathscr{H}}B), {}^{\mathscr{H}}I) = 0$.

Conversely, if $g(\operatorname{trace}({}^{\mathscr{H}}B), {}^{\mathscr{H}}I) = 0$ *then the following assertions are equivalent*:

(i) $\mathscr{V}$ *is a homothetic foliation*;

(ii) *the mean curvature form of* $\mathscr{V}$ *is basic*;

(iii) *the mean curvature form of* $\mathscr{H}$ *is invariant under the parallel displacement determined by* $\mathscr{H}$;

(iv) *in the neighbourhood of each point of* M *there a exists a local dilation* λ *of* $\mathscr{V}$ *such that for any horizontal vector* X *and conformal vector field* V *tangent to* $\mathscr{V}$ *we have* $X(V(\log\lambda)) = 0$.

Proof. The first assertion is an immediate consequence of formula (2·1) from Proposition 2·3.

Suppose now that $g(\mathrm{trace}\,({}^{\mathscr{H}}\!B), {}^{\mathscr{H}}\!I) = 0$.

The equivalence (i) $\Longleftrightarrow$ (iii) follows from Proposition 2·3.

From the proof of Theorem 3·2 it follows that

$$\frac{1}{m-n}\overset{\mathscr{V}}{\nabla}_V(\mathrm{trace}\,({}^{\mathscr{V}}\!B)^\flat)(X) = X(V(\log\lambda)) = \overset{\mathscr{H}}{\nabla}_V(\mathscr{H}^*(\mathrm{d}\log\lambda))(X) \qquad (4{\cdot}1)$$

for any $X \in \Gamma(\mathscr{H})$, any conformal vector field $V \in \Gamma(\mathscr{V})$ and any local dilation λ of $\mathscr{V}$. The first equality of (4·1) implies that (ii) $\Longleftrightarrow$ (iv).

From the second equality of (4·1) and (2·2) of Proposition 2·3 we obtain that (iii) $\Longleftrightarrow$ (iv) and the proposition is proved.

Theorem 4·4. *Let* $\mathscr{V}$ *be a foliation of* $\mathrm{codim}\,\mathscr{V} \neq 2$ *on* (M^m, g), $m \geqslant 3$, *which is locally generated by conformal vector fields. Then any two of the following assertions imply the remaining assertion.*

(i) $\mathscr{V}$ *produces harmonic morphisms*;

(ii) $\mathscr{V}$ *is homothetic*;

(iii) $\mathrm{trace}\,(\mathrm{ad}({}^{\mathscr{H}}\!I)) = 0$.

Proof. This is a consequence of Proposition 4·3 and of Theorem 3·2.

Remark 4·5. (1) There is another way to prove that in Theorem 4·4 if (ii) holds then (i) $\Longleftrightarrow$ (iii). To see this let λ be a local dilation of $\mathscr{V}$ which is constant along horizontal curves. Then, with respect to $\lambda^2 g$, $\mathscr{V}$ is a Riemannian foliation locally generated by Killing fields. Thus, condition (ii) of Theorem 3·2 says the same thing when written for a local frame made up of fields which are conformal with respect to g and when written for the same frame but made up of Killing fields with respect to $\lambda^2 g$. Moreover, since λ is constant along horizontal curves, $\mathscr{V}$ produces harmonic morphisms with respect to g if and only if it produces harmonic morphisms with respect to $\lambda^2 g$ [**14**, corollary 1·9]. The proof now follows from Corollary 3·5.

(2) It is not difficult to see using Theorem 4·4 that the following classes of foliations of codimension not equal to two produce harmonic morphisms:

(i) Homothetic foliations locally generated by conformal fields and with integrable orthogonal complement;

(ii) Homothetic foliations generated by the local action of an Abelian Lie group of conformal transformations;

(iii) Homothetic foliations generated by the action of a unimodular closed subgroup of the group of conformal transformations;

(iv) Homothetic foliations formed by the fibres of principal bundles for which the total space is endowed with a metric such that the structural group acts by conformal transformations and the connection induced on the determinant bundle of the adjoint bundle is flat.

From Theorem 4·4 we obtain the following.

COROLLARY 4·6. *Let $\mathscr{V}$ be a foliation of* $\operatorname{codim}\mathscr{V} \neq 2$ *on* (M^m, g), $m \geqslant 3$ *which is locally generated by conformal vector fields. Suppose that the orthogonal complement $\mathscr{H}$ of $\mathscr{V}$ is integrable. Then the following assertions are equivalent.*

(i) *$\mathscr{V}$ produces harmonic morphisms;*

(ii) *$\mathscr{V}$ is homothetic.*

Remark 4·7. If $\dim \mathscr{V} = 1$ then (i) and (ii) of Corollary 4·6 imply that $\mathscr{V}$ is locally generated by conformal vector fields [**14**, proposition 2·5].

Next we give a construction of a foliation which produce harmonic morphisms has basic mean curvature form but is nowhere homothetic.

Example 4·8. Let $\varphi\colon (M^{n+1}, h) \to (N^n, \bar{h})$, $n \geqslant 1$, be a Riemannian submersion with geodesic fibres and let $\mathscr{V}$ be the foliation formed by the fibres of φ.

Suppose that V is a local vertical field such that $h(V, V) = 1$. Because φ has geodesic leaves we have that $[V, X] = 0$ for any basic X.

Let $\theta = V^\flat$ and $\Omega = \mathrm{d}\theta$. It is easy to see that $\Omega = 0$ if and only if the horizontal distribution $\mathscr{H}$ is integrable. Also, Ω is basic and since $\mathrm{d}\Omega = \mathrm{d}\mathrm{d}\theta = 0$ at least locally we can find a basic one-form A such that $\Omega = -\mathrm{d}A$.

Thus $\mathrm{d}\theta = \Omega = -\mathrm{d}A$ and hence $\mathrm{d}(A + \theta) = 0$. It follows that at least locally we can write $A + \theta = \mathrm{d}\sigma$ for some smooth local function σ on M. Note that the horizontal component of $\mathrm{d}\sigma$ is basic, being equal to A.

Supposing that σ is defined on the whole M, let g^σ be the Riemannian metric on M defined by

$$g^\sigma = e^{-2\sigma}\,\varphi^*(\bar{h}) + e^{(2n-4)\sigma}\,\theta^2.$$

Then $\varphi\colon (M, g^\sigma) \to (N, \bar{h})$ is a harmonic morphism [**6**]. Moreover, the mean curvature form of $\mathscr{V}$ with respect to g^σ is $(2-n)A$ and therefore is basic. However, the connected components of the fibres of φ form a homothetic foliation with respect to g^σ only over the set of points where the horizontal distribution is integrable. Thus if $\mathscr{H}$ is nowhere integrable then $\mathscr{V}$ is nowhere homothetic with respect to g^σ.

Let ρ be any function which has the same properties as σ (i.e. $\varphi\colon (M, g^\rho) \to (N, \bar{h})$ is a harmonic morphism, the mean curvature form of $\mathscr{V}$ with respect to g^ρ is basic and $\mathscr{V}$ is nowhere homothetic on (M, g^ρ)). Then, there exists a unique constant $c \in \mathbb{R}$ such that $\rho - c\sigma$ is, at least locally, a basic function.

To see this note that because the induced foliation is nowhere homothetic then we must have:

(i) $M = \{x \in M \,|\, V(V(\rho)) = 0\}$. (Otherwise on some open subset of M the level hypersurfaces of $V(\rho)$ would be integral submanifolds of the horizontal distribution.)

(ii) The interior of the set $\{x \in M \,|\, V(\rho) = 0\}$ is empty. (Otherwise the restriction of $\mathscr{V}$ to some open subset of M would be Riemannian.)

Thus we have $V(\rho) = c$, for some constant $c \neq 0$. Hence $\mathrm{d}\rho = c\theta + B$.

Then B must be basic (because $X(V(\rho)) = 0$ for any horizontal X) and hence $0 = c\mathrm{d}\theta + \mathrm{d}B$ which is equivalent to $\mathrm{d}B = -c\Omega$.

It follows that $\mathrm{d}(\rho - c\sigma) = \mathrm{d}\rho - c\mathrm{d}\sigma = c\theta + B - c\theta - cA = B - cA$.

Because $B - cA$ is a closed basic one-form, at least locally, we can find a basic function whose differential is equal to $\mathrm{d}(\rho - c\,\sigma)$ and hence $\rho - c\,\sigma$ is, at least locally, a basic function.

5. *Homothetic actions and harmonic morphisms*

Recall that a vector field V on a Riemannian manifold (M, g) is homothetic if $\mathscr{L}_V g = a\,g$ for some constant $a \in \mathbb{R}$ (see [**23**]).

The first thing to note about a foliation locally generated by homothetic vector fields is the following.

Proposition 5·1. *Let $\mathscr{V}$ be a foliation on (M, g) locally generated by homothetic vector fields. Then either $\mathscr{V}$ is Riemannian and locally generated by Killing vector fields or $\mathscr{V}$ is nowhere Riemannian.*

Proof. Let $P = \{\, x \in M \mid \mathscr{V}$ is Riemannian at $x\,\}$.

It is obvious that P is closed. Also let $\{V_r\}$ be a local frame of $\mathscr{V}$, over the connected open subset U, made up of homothetic vector fields. It is obvious that if $U \cap P \neq \varnothing$ then V_r are Killing fields and thus $U \subseteq P$. Hence P is also open and since M is connected either $P = M$ or $P = \varnothing$.

From Proposition 4·3 we obtain the following.

Corollary 5·2. *Let $\mathscr{V}$ be a foliation locally generated by homothetic vector fields on the Riemannian manifold (M, g).*

Then $\mathscr{V}$ is a homothetic foliation if and only if $g(\mathrm{trace}\,({}^{\mathscr{H}}\!B), {}^{\mathscr{H}}\!I) = 0$.

In particular, a foliation locally generated by homothetic vector fields and with integrable orthogonal complement is a homothetic foliation.

Proof. Let V be a homothetic vector field on (M, g) which is tangent to the foliation. Then it is easy to see that $\mathscr{L}_V g = -2V(\log\lambda) g$ where λ is any local dilation of the foliation. But V is homothetic and hence $V(\log\lambda)$ is a constant function. The proof now follows from (i) $\Longleftrightarrow$ (iv) of Proposition 4·3.

Remark 5·3. Let $\mathscr{V}$ be a conformal foliation on (M, g) and define the vertical one-form μ by the relation $(\mathscr{L}_V g)(X, Y) = \mu(V)\, g(X, Y)$ where V is vertical and X, Y are horizontal [**21**]. Then $\mu = -2\,\mathscr{V}^*(\mathrm{d}(\log\lambda))$ where λ is a local dilation of $\mathscr{V}$ and because $\mathrm{trace}\,({}^{\mathscr{H}}\!B) = n\,\mathscr{V}(\mathrm{grad}\,(\log\lambda))$ we have $\mu = -(2/n)\,\mathrm{trace}\,({}^{\mathscr{H}}\!B)^\flat$ (see [**4**]).

By Corollary 5·2, if $\mathscr{V}$ is locally generated by homothetic vector fields then $\mathscr{V}$ is a homothetic foliation if and only if $\mu({}^{\mathscr{H}}\!I) = 0$.

Proposition 5·4. *Let $\mathscr{V}$ be a one-dimensional foliation of* $\mathrm{codim}\,\mathscr{V} \neq 2$ *on (M^m, g), $m \geqslant 3$, which is not a Riemannian foliation and is locally generated by homothetic vector fields. Then the following assertions are equivalent*:

(i) *$\mathscr{V}$ produces harmonic morphisms*;
(ii) *$\mathscr{V}$ is a homothetic foliation*;
(iii) *$\mathscr{H}$ is integrable.*

Proof. It is easy to see that because $\mathscr{V}$ is locally generated by homothetic vector fields its mean curvature form is basic.

The proof now follows from Theorem 3·2 and Corollary 5·2.

Note that in Proposition 5·4 the equivalence (ii) $\Longleftrightarrow$ (iii) holds also when codim $\mathscr{V} = 2 = \dim M$.

PROPOSITION 5·5 (cf. [**21**, proposition 2·83]). *Let $\mathscr{V}$ be a foliation on (M^m, g) locally generated by homothetic vector fields. Then there exists a Riemannian foliation $\mathscr{W} \subseteq \mathscr{V}$ locally generated by Killing fields. Moreover, if $\mathscr{V}$ is not Riemannian then* $\dim \mathscr{V} = \dim \mathscr{W} + 1$.

Proof. Suppose that $\mathscr{V}$ is not Riemannian. Then by Proposition 5·1 the foliation $\mathscr{V}$ is nowhere Riemannian. Since $\mathscr{V}$ is conformal we can find local dilations of it in the neighbourhood of each point. Let λ be a local dilation of $\mathscr{V}$ defined on the open subset $U \subseteq M$. For $x \in U$ let

$$\mathscr{W}_x = \{\, V \in \mathscr{V}_x \mid V(\log\lambda) = 0 \,\} = \mathscr{V}_x \cap \operatorname{grad}(\log\lambda)^{\perp}_x .$$

Since any two local dilations of $\mathscr{V}$ differ by a factor which is constant along the leaves it follows that $\mathscr{W}_x$ does not depend on λ. Because $\mathscr{V}$ is nowhere Riemannian, $\mathscr{W}_x \neq \mathscr{V}_x$. Also $\operatorname{grad}(\log\lambda)$ is nonvanishing and hence $\dim(\operatorname{grad}(\log\lambda)^{\perp}_x) = m - 1$ where $m = \dim M$. We have

$$\left.\begin{aligned} \dim \mathscr{W}_x &= \dim(\mathscr{V}_x \cap \operatorname{grad}(\log\lambda)^{\perp}_x) \\ &= \dim \mathscr{V}_x + \dim(\operatorname{grad}(\log\lambda)^{\perp}_x) - \dim(\mathscr{V}_x + \operatorname{grad}(\log\lambda)^{\perp}_x). \end{aligned}\right\} \qquad (5·1)$$

It follows that the minimal value of $\dim \mathscr{W}_x$ occurs if and only if $\mathscr{V}_x + \operatorname{grad}(\log\lambda)^{\perp}_x = T_xM$. If this is the case then $\dim \mathscr{W}_x = \dim \mathscr{V}_x + (m-1) - m = \dim \mathscr{V}_x - 1$. Since $\mathscr{W}_x \subset \mathscr{V}_x$, $\mathscr{W}_x \neq \mathscr{V}_x$ it follows that $\dim \mathscr{W}_x = \dim \mathscr{V}_x - 1$. Thus $\mathscr{W} = (\mathscr{W}_x)_{x\in M}$ defines a distribution on M. Then $\mathscr{W}$ is integrable because it is the intersection of two transversal foliations.

Let $V \in \Gamma(\mathscr{V})$ be a homothetic vector field. It is easy to see that if $V_x \in \mathscr{W}_x$ then $V \in \Gamma(\mathscr{W})$. Since $\mathscr{V}$ is locally generated by homothetic vector fields it follows that $\mathscr{W}$ is locally generated by Killing fields. (This also implies that $\mathscr{W}$ is integrable since any Killing field which is tangent to $\mathscr{V}$ must be tangent to $\mathscr{W}$ and the bracket of any two Killing fields is a Killing field.)

We can now characterise geometrically the homothetic (infinitesimal) actions which induce homothetic foliations and their relations with harmonic morphisms.

THEOREM 5·6. *Let $\mathscr{V}$ be a foliation locally generated by homothetic vector fields and let $\mathscr{H}$ be its orthogonal complement.*

Then the following assertions are equivalent:

(*a*) *$\mathscr{V}$ is a homothetic foliation;*

(*b*) *Either $\mathscr{V}$ is Riemannian and locally generated by Killing fields or there exists a Riemannian foliation $\mathscr{W} \subseteq \mathscr{V}$ locally generated by Killing fields such that* $\dim \mathscr{V} = \dim \mathscr{W} + 1$ *and the distribution $\mathscr{F} = \mathscr{W} \oplus \mathscr{H}$ is integrable.*

Moreover, if (*a*) *or* (*b*) *hold and* $\dim \mathscr{V} \geqslant 2$, codim $\mathscr{V} \geqslant 3$ *then the following assertions are equivalent*:

(i) *$\mathscr{V}$ produces harmonic morphisms;*

(ii) *the restriction of $\mathscr{W}$ to any leaf of $\mathscr{F}$ produces harmonic morphisms.*

Proof. The equivalence (*a*) $\Longleftrightarrow$ (*b*) follows from Corollary 5·2 and Proposition 5·5.

Let $V \in \Gamma(\mathscr{V})$ be a homothetic vector field which is not Killing. (Such a vector field can be found in the neighbourhood of each point of M because $\mathscr{V}$ is locally generated by homothetic vector fields and $\mathscr{W} \pitchfork \mathscr{V}$.) Note that for any Killing field $W \in \Gamma(\mathscr{W})$ we have that $[V, W]$ is also Killing and hence $[V, W] \in \Gamma(\mathscr{W})$. Using this fact together with Theorem 3·2 and Corollary 5·2 it is not difficult to see that the assertions (i), and (ii) are equivalent.

Remark 5·7. (1) If $\mathscr{V}$ is homothetic then the leaves of $\mathscr{F}$ are level hypersurfaces of the local dilations of $\mathscr{V}$ which are constant along horizontal curves.

(2) In Theorem 5·6 if (a) or (b) hold and $\operatorname{codim} \mathscr{V} = 1$ then (i) $\Longleftrightarrow$ (ii).

Let G be a Lie group which acts to the right by homotheties on (M, g) and for $a \in G$ let $\rho(a) \in (0, \infty)$ be the conformal factor of the homothetic transformation induced by $a \in G$ on (M, g). Then it is easy to see that $\rho: G \to (0, \infty)$ is a morphism of Lie groups (hence, if ρ is nonconstant, G is isomorphic to a semi-direct product of $\ker\rho$ and $((0, \infty), \cdot)$). In particular, if G is compact then ρ is constant. Nevertheless, if G is compact then there might exist *local* morphisms of Lie groups $G \to (0, \infty)$ (see Example 5·8(3), below) which can be used to construct homothetic *local* actions.

Here are a few examples of morphisms of Lie groups $\rho: G \to (0, \infty)$.

Example 5·8. (1) For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ define $\rho: Gl_n(\mathbb{K}) \to (0, \infty)$ by $\rho(a) = |\det a|$.

(2) For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ define $\rho: CO_n(\mathbb{K}) \to (0, \infty)$ by $|Au| = \rho(A)|u|$ for $u \in \mathbb{K}^n$ and $A \in CO_n(\mathbb{K})$.

(3) The canonical morphisms $U_n \to S^1$ and $\mathrm{Spin}^{\mathfrak{c}}_n \to S^1$ when composed with the exponential of $\arg: S^1 \setminus \{-1\} \to (-\pi, \pi)$ induce *local* morphisms of Lie groups $U_n \to (0, \infty)$ and $\mathrm{Spin}^{\mathfrak{c}}_n \to (0, \infty)$, respectively.

From now on we shall suppose that G acts *freely* on M. In this case there exists a natural isomorphism of vector bundles $\mathscr{V} = M \times \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of G.

Hence ${}^{\mathscr{H}}I$ can be viewed as a $\mathfrak{g}$-valued two form on M which has properties similar to the properties of the curvature form of a principal connection (in particular, $R_a^*({}^{\mathscr{H}}I) = \operatorname{Ad} a^{-1} \cdot {}^{\mathscr{H}}I$ where R_a is the transformation induced by $a \in G$ on M).

Also ρ_* can be viewed as a vertical one form on M. Moreover, we have that $\rho_* = \mu$ (see Remark 5·3 for the definition of μ).

It follows from Corollary 5·2 that the foliation induced by the free action of G on (M, g) is homothetic if and only if $\rho_*({}^{\mathscr{H}}I) = 0$.

By identifying G with an orbit we can induce on it a metric which we shall denote by γ. Then it is easy to see that $\rho^{-2}\gamma$ is right invariant.

Suppose that ρ is nonconstant and let $\mathscr{V}$ be the foliation on G formed by the connected components of the fibres of ρ. This is generated by the action of the normal subgroup $H = \ker\rho$. Then it is obvious that H acts by isometries on (G, γ) and hence $\mathscr{V}$ is a Riemannian foliation on it.

Also, $\mathscr{H}$ $(= \mathscr{V}^{\perp})$ is a (one-dimensional) homothetic foliation with geodesic leaves for which ρ^{-1} is a global dilation.

Thus both $\mathscr{V}$ and $\mathscr{H}$ produce harmonic morphisms and, in particular, ρ induces a harmonic function on (G, γ) (which gives another argument for the fact that if G is compact then ρ cannot be globally defined unless is constant).

Example 5·9. Let G and ρ be as in Example 5·8(1) or (2) and let $h = |\mathrm{d}x \cdot x^{-1}|^2$. Then $g = \rho^2\, h$ has all the above properties.

Next we show that the results of Theorem 5·6 takes a more concrete form in the case of homothetic free actions.

PROPOSITION 5·10. *Let G be a Lie group which acts freely by* homotheties *on (M, g) and let $\mathscr{V}^G$ be the induced foliation. Let $\rho\colon G \to (0, \infty)$ be the corresponding morphism of Lie groups and let $H = \ker \rho$.*

Then the following assertions are equivalent:

(i) *$\mathscr{V}^G$ is a homothetic foliation;*

(ii) *there exists a hypersurface N of M such that H acts by* isometries *on $(N, g|_N)$ to generate a Riemannian foliation $\mathscr{V}^H$ and such that $M = N \times_H G$.*

Further, if (i), *or* (ii), *holds and $2 \leqslant \dim G \leqslant \dim M - 3$ then $\mathscr{V}^G$ produces harmonic morphisms if and only if $\mathscr{V}^H$ produces harmonic morphisms.*

Proof. Let $\mathscr{H}$ be the orthogonal complement of $\mathscr{V}^G$ and let $\mathscr{W}$ be the foliation induced by the *isometric* (free) action of H on M. Then, assertion (i) is equivalent to the fact that $\rho_*({}^{\mathscr{H}}I) = 0$ which, since ${}^{\mathscr{H}}I$ is the integrability tensor of $\mathscr{H}$, is equivalent to the fact that the distribution $\mathscr{F} = \mathscr{W} \oplus \mathscr{H}$ is integrable.

Suppose that (i) holds and let N be a leaf of $\mathscr{F}$. Then $\mathscr{V}^H = \mathscr{W}|_N$ and the implication (i)$\Rightarrow$(ii) follows.

The implication (i)$\Leftarrow$(ii) is now obvious.

The last assertion follows from the fact that if (i) holds then ${}^{\mathscr{H}}I$ is $\mathfrak{h}$-valued and $R_a^*({}^{\mathscr{H}}I) = \mathrm{Ad}a^{-1} \cdot {}^{\mathscr{H}}I$.

Remark 5·11. In Proposition 5·10 we also have that if (i), or (ii), holds then g is determined by ρ and the induced metric h on N.

To see this recall that we identified the metric γ induced on G with an orbit. Suppose the identification and N were chosen so that the identity element of G is contained in N. Then H acts by isometries on $(N \times G, \pi_N^*(h) + \pi_G^*(\gamma))$ (where $\pi_N\colon N \times G \to N$ and $\pi_G\colon N \times G \to G$ are the canonical projections) and (M, g) is the induced isometric quotient.

Example 5·12. Let $\rho\colon G \to (0, \infty)$ be as in Example 5·8 and let $H = \ker \rho$. Then $\mathrm{trace}\,(\mathrm{ad}\, \mathfrak{h}) = 0$ (here, as above, $\mathfrak{h}$ is the Lie algebra of H).

Let (Q, M, H) be a principal bundle whose total space is endowed with a Riemannian metric h such that H acts by isometries on (Q, h). (Note that any such h corresponds to a triple $(\gamma, \mathscr{H}, k)$ where γ is a Riemannian metric on the vector bundle $\mathrm{Ad}\, Q \to M$, $\mathscr{H}$ is a principal connection on (Q, M, H) and k is a Riemannian metric on M.)

From Remark 5·11 it follows that a metric (and just a local metric, for ρ from Example 5·8(3)) can be found on $P = Q \times_H G$ with respect to which the foliation induced by G is homothetic (but not Riemannian) and produces harmonic morphisms.

6. *Conformal actions and harmonic morphisms on Einstein manifolds*

In this section we study foliations which are locally generated by conformal vector fields and produce harmonic morphisms on Einstein manifolds. Note that, as before, no compactness or completeness assumptions are made. The main results of this section are the following:

Theorem 6·1. *Let* (M^m, g), $m \geqslant 3$ *be an Einstein manifold* (Ricci $= c\,g$, $c \in \mathbb{R}$). *Let* $\mathscr{V}$ *be a foliation of* codim $\mathscr{V} \neq 2$ *locally generated by conformal vector fields.*

Suppose that $\mathscr{V}$ *produces harmonic morphisms on* (M^m, g). *Then either* $\mathscr{V}$ *is Riemannian and locally generated by Killing fields or the set of points where* $\mathscr{V}$ *is Riemannian has empty interior. Moreover, we have the following*:

(i) *if* $c > 0$ *then either* $\mathscr{V}$ *is Riemannian and locally generated by Killing vector fields or any harmonic morphism produced by* $\mathscr{V}$ *can be locally decomposed into a harmonic morphism with geodesic fibres and integrable horizontal distribution followed by another harmonic morphism*;

(ii) *if* $c < 0$ *then at least outside the points where* $\mathscr{V}$ *is Riemannian any harmonic morphism produced by* $\mathscr{V}$ *can be locally decomposed into a harmonic morphism with geodesic fibres and integrable horizontal distribution followed by another harmonic morphism*;

(iii) *if* $c = 0$ *then either* $\mathscr{V}$ *is locally generated by homothetic vector fields or any harmonic morphism produced by* $\mathscr{V}$ *can be locally decomposed into a harmonic morphism with geodesic fibres, constant dilation and integrable horizontal distribution followed by another harmonic morphism.*

Corollary 6·2. *Let* (M^m, g), $m \geqslant 3$ *be an Einstein manifold* (Ricci $= c\,g$) *and let* $\mathscr{V}$ *be a foliation on it of* codim $\mathscr{V} \neq 2$ *and locally generated by conformal vector fields.*

Suppose that $\mathscr{V}$ *produces harmonic morphisms on* (M^m, g).

(i) *If* $c > 0$ *then any harmonic morphism produced by* $\mathscr{V}$ *can be locally decomposed into two harmonic morphisms in which the first one either has geodesic fibres and integrable horizontal distribution or is induced by an isometric quotient.*

(ii) *If* $c < 0$ *then at least outside a set with empty interior any harmonic morphism produced by* $\mathscr{V}$ *can be locally decomposed into two harmonic morphisms in which the first one either has geodesic fibres and integrable horizontal distribution or is induced by an isometric quotient.*

Note that Corollary 6·2 is an extension of corollary 5·9 from [**14**].

Theorem 6·3. *Let* (M^m, g) *be a Ricci-flat Riemannian manifold and let* $\mathscr{V}$ *be a homothetic foliation on it of* dim $\mathscr{V} \geqslant 2$, codim $\mathscr{V} \geqslant 3$ *and locally generated by conformal vector fields.*

Suppose that $\mathscr{V}$ *produces harmonic morphisms on* (M^m, g). *Then one of the following assertions hold.*

(*a*) $\mathscr{V}$ *is Riemannian and locally generated by Killing fields.*

(*b*) *There exists a Riemannian foliation* $\mathscr{W} \subseteq \mathscr{V}$, dim $\mathscr{W}$ = dim $\mathscr{V} - 1$, *locally generated by Killing fields and such that* $\mathscr{F} = \mathscr{W} \oplus \mathscr{H}$ *is integrable and the restriction of* $\mathscr{W}$ *to any leaf of* $\mathscr{F}$ *produces harmonic morphisms.*

(*c*) *Any harmonic morphism produced by* $\mathscr{V}$ *can be locally decomposed into two harmonic morphisms in which the first one has geodesic fibres, constant dilation and integrable horizontal distribution.*

Corollary 6·4. *Let* (M^m, g) *be a Ricci-flat Riemannian manifold and let* $\mathscr{V}$ *be a foliation on it, locally generated by conformal vector fields and with integrable orthogonal complement,* dim $\mathscr{V} \geqslant 2$, codim $\mathscr{V} \geqslant 3$.

Suppose that $\mathscr{V}$ *produces harmonic morphisms on* (M^m, g). *Then one of the following assertions hold.*

(*a*) *$\mathscr{V}$ is Riemannian and locally generated by Killing fields.*

(*b*) *There exists a Riemannian foliation $\mathscr{W} \subseteq \mathscr{V}$,* $\dim \mathscr{W} = \dim \mathscr{V} - 1$, *locally generated by Killing fields and such that $\mathscr{F} = \mathscr{W} \oplus \mathscr{H}$ is integrable and the restriction of $\mathscr{W}$ to any leaf of $\mathscr{F}$ produces harmonic morphisms.*

(*c*) *Any harmonic morphism produced by $\mathscr{V}$ can be locally decomposed into two harmonic morphisms in which the first one has geodesic fibres, constant dilation and integrable horizontal distribution.*

The proofs of the above results are based on results obtained in the previous sections. We also need a few lemmas some of which are well-known.

Lemma 6·5 (cf. [**13**]). *Let $\mathscr{V}$ be a one-dimensional foliation on (M, g). Then the following assertions are equivalent.*

(i) *$\mathscr{V}$ is a homothetic foliation with geodesic leaves and integrable orthogonal complement;*

(ii) *$\mathscr{V}$ is locally generated by* (*nowhere zero*) *conformal vector fields $V \in \Gamma(\mathscr{V})$ such that* $\mathrm{d}V^\flat = 0$.

Proof. Conformal vector fields $V \in \Gamma(TM)$, $\mathscr{L}_V g = 2\,\mu\, g$, such that $\mathrm{d}V^\flat = 0$ are characterized by the relation $\nabla V = \mu \mathrm{Id}_{TM}$. Now it is obvious that any such V which is nowhere zero generates a conformal foliation with geodesic leaves and integrable orthogonal complement. Moreover, $e^{-|V|}$ is a local dilation for it whose gradient is tangent to the leaves and thus V generates a homothetic foliation.

Conversely, if $\mathscr{V}$ satisfies (i) then it produces harmonic morphisms. But $\mathscr{V}$ is homothetic and hence $\mathscr{V}$ is locally generated by conformal vector fields [**14**, proposition 2·5]. Now, it is easy to see that a conformal vector field V which generates a foliation with geodesic leaves and integrable orthogonal complement satisfies $\mathrm{d}V^\flat = 0$.

Lemma 6·6 (see [**23**]). *Let V be a conformal vector field on the Einstein manifold (M^m, g), $\mathscr{L}_V g = 2\,\sigma g$,* Ricci $= c\,g$. *Then*

$$\nabla \mathrm{d}\sigma = -\frac{c}{m-1}\,\sigma g \tag{6·1}$$

and, in particular,

$$\Delta\sigma = \frac{c\,m}{m-1}\,\sigma. \tag{6·2}$$

Proof. Formula (6·1) follows after a straightforward but tedious computation (see [**23**]).

Lemma 6·7. *Let f be a smooth function on (M, g) such that $\nabla \mathrm{d}f = -kfg$ for some constant $k \in \mathbb{R}$. Then*

$$kf^2 + |\mathrm{d}f|^2 = \text{constant}.$$

Proof. This is obvious.

Lemma 6·8. *Let $\mathscr{V}$ be a foliation* ($\operatorname{codim} \mathscr{V} > 0$) *on (M, g) and let $V \in \Gamma(\mathscr{V})$ be such that $\nabla V^\flat = \mu\, g$ for some smooth function μ on M.*

If for some $x \in M$ we have that $V_x = 0$ then $\mu(x) = 0$.

Proof. Let $X \in \Gamma(\mathscr{V}^\perp)$ be a basic vector field. Then $[V, X] \in \Gamma(\mathscr{V})$. But

$$[V, X] = \nabla_V X - \nabla_X V = \nabla_V X - \mu X.$$

Thus if $V_x = 0$ then $[V, X]_x = -\mu(x) X_x \in \mathscr{V}_x$ and hence $\mu(x) = 0$.

Lemma 6·9. *Let* $\mathscr{V}$ *be a foliation on* (M, g) *and let* $\operatorname{grad} f \in \Gamma(\mathscr{V})$ *be such that* $\nabla \mathrm{d} f = -kfg$ *for some nonnegative constant* $k \geqslant 0$.

If for some $x \in M$ *we have that* $(\operatorname{grad} f)_x = 0$ *then* $\operatorname{grad} f = 0$. *Moreover, if* $k > 0$ *then* $f = 0$.

Proof. If $k = 0$ then $|\operatorname{grad} f| = \text{constant}$.

If $k > 0$ and for some $x \in M$ we have that $(\operatorname{grad} f)_x = 0$ then by Lemma 6·8 we have $f(x) = 0$. The proof now follows from Lemma 6·7.

Proof of Theorem 6·1. Let $\mathscr{H}$ be the orthogonal complement of $\mathscr{V}$. Let $V \in \Gamma(\mathscr{V})$ be a conformal vector field. Then at least locally we can write $\mathscr{L}_V g = -2V(\log \lambda) g$ for some local dilation λ of $\mathscr{V}$. By Lemma 1·7 we have that V is an infinitesimal automorphism of $\mathscr{H}$.

Because $\mathscr{V}$ produces harmonic morphisms from Theorem 3·2 it follows that the mean curvature form of $\mathscr{V}$ is basic. Applying Proposition 2·14 with the roles of $\mathscr{V}$ and $\mathscr{H}$ reversed we obtain that for any basic vector field $X \in \Gamma(\mathscr{H})$ we have that $X(V(\log \lambda)) = 0$. It follows that $\operatorname{grad}(V(\log \lambda)) \in \Gamma(\mathscr{V})$.

Now recall that (M, g) is an Einstein manifold and thus it is an analytic manifold (see [**5**]). From the regularity of solutions for elliptic operators and (6·2) it follows that $V(\log \lambda)$ is an analytic function. Hence either $\mathscr{V}$ is Riemannian or the interior of the set where $\mathscr{V}$ is Riemannian is empty.

From Lemmas 6·5 and 6·6 it follows that if $\operatorname{grad}(V(\log \lambda))$ is nowhere zero then it generates a one-dimensional homothetic foliation $\mathscr{F}$ with geodesic leaves and integrable orthogonal complement. Moreover $\mathscr{F} \subseteq \mathscr{V}$. Also, note that if $c = 0$ then $\operatorname{grad}(V(\log \lambda))$ is a parallel vector field.

Let $x \in M$ and suppose that for any conformal vector field $V \in \Gamma(\mathscr{V})$ we have that $\operatorname{grad}(V(\log \lambda)) = 0$ at x.

If $c \neq 0$ then from Lemma 6·8 it follows that $\mathscr{V}$ is Riemannian at x. This establishes assertion (ii). Further, if $c > 0$ then from Lemma 6·9 it follows that $\mathscr{V}$ is Riemannian in a neighbourhood of x and this establishes assertion (i).

If $c = 0$ then from Lemma 6·9 it follows that in a neighbourhood of x we have that $\mathscr{V}$ is generated by homothetic vector fields.

Proof of Corollary 6·2. This follows from assertions (i) and (ii) of Theorem 6·1.

Proof of Theorem 6·3. This follows from Theorem 5·6 and Theorem 6·1.

Proof of Corollary 6·4. This follows from Proposition 4·3 and Theorem 6·3.

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