HARMONIC MORPHISMS WITH ONE-DIMENSIONAL FIBRES

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Abstract

We study harmonic morphisms by placing them into the context of conformal foliations. Most of the results we obtain hold for fibres of dimension one and codomains of dimension not equal to two. We consider *foliations which produce harmonic morphisms* on both compact and noncompact Riemannian manifolds.

By using integral formulae, we prove an extension to one-dimensional foliations which produce harmonic morphisms of the well-known result of S. Bochner concerning Killing fields on compact Riemannian manifolds with nonpositive Ricci curvature.

From the noncompact case, we improve a result of R. L. Bryant [9] regarding harmonic morphisms with one-dimensional fibres defined on Riemannian manifolds of dimension at least four with constant sectional curvature. Our method gives an entirely new and geometrical proof of Bryant's result.

The concept of *homothetic foliation* (or, more generally, *homothetic distribution*) which we introduce, appears as a useful tool both in proofs and in providing new examples of harmonic morphisms, with fibres of any dimension.

Introduction

Harmonic morphisms are maps, between Riemannian manifolds, which pull-back (local) harmonic functions to (local) harmonic functions. By the well-known result, independently proved by B. Fuglede [13] and T. Ishihara [20], harmonic morphisms are harmonic maps which are horizontally (weakly) conformal.

One of the problems in the theory of harmonic morphisms was suggested to me by J. C. Wood: *Given a Riemannian manifold, find and classify all harmonic morphisms which can be defined on it.*

As was pointed out by P. Baird and J. Eells in [2], there is a significant difference between the case when the codomain of a harmonic morphism is of dimension two and the case when it is not. In [2], P. Baird and J. Eells proved that a horizontally conformal map which takes values in a two-dimensional Riemannian manifold is a harmonic morphism if and only if its fibres are minimal. Also, it is well-known that, in the case of a codomain of dimension two, the property of being a harmonic morphism is preserved under conformal changes of the metric

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on the codomain. When the codomain is not of dimension two, a harmonic morphism is characterised by the property that it is horizontally conformal and the parallel displacement defined by the horizontal distribution (i.e. the distribution orthogonal to the fibres), preserves the *mass* of the fibres—here the fibres are given the mass *density* λ^{2-n} where λ denotes the dilation (see Proposition 1.6 , below). This generalizes the well-known fact that a Riemannian submersion is harmonic if and only if its fibres are minimal, and this holds if and only if the parallel displacement defined by the horizontal distribution preserves volumes. From this it follows that the (nonconstant) map and the metric on the domain of a harmonic morphism determine, up to a homothety, the metric on the codomain (see Proposition 1.12 , below).

Harmonic morphisms with (regular) fibres of dimension one defined on three-dimensional Riemannian manifolds, have been deeply investigated in several papers by P. Baird and J. C. Wood, see the Bibliography of Harmonic Morphisms [18] , maintained by S. Gudmundsson.

In higher dimensions, if $\varphi : (M^{n+1}, g) \to (N^n, h)$ is a nonconstant harmonic morphism, then by a basic result, due to P. Baird [1] , if $n \geq 4$ then φ is submersive, and if $n = 3$ then φ can have only isolated critical points. Moreover in [1] there is constructed a harmonic morphism defined on a (deformed) four-dimensional sphere, whose (regular) fibres have dimension one and which has two (isolated) critical points. Other examples of harmonic morphisms $S^4 \to S^3$, which are homotopic to the suspension of the Hopf fibration $S^3 \to S^2$ are given by P. Baird and A. Ratto in [3] . In [28] , Y. L. Ou and J. C. Wood proved that up to homotheties the only quadratic polynomial $\mathbb{R}^4 \to \mathbb{R}^3$ which is a harmonic morphism is the one which restricts to the Hopf fibration $S^3 \to S^2$. Orthogonal projection $\mathbb{R}^4 \to \mathbb{R}^3$ is another harmonic morphism. This illustrates one of the main results from R. L. Bryant's paper [9] , that if $\varphi : (M^{n+1}, g) \to (N^n, h)$ is a submersive harmonic morphism, $n \geq 3$ and (M, g) is simply-connected with constant sectional curvature, then φ is one of the following two types: either *(i)* the foliation induced on (M, g) is Riemannian with leaves generated by a Killing field or *(ii)* φ is horizontally homothetic and has geodesic fibres orthogonal to a foliation formed of umbilical hypersurfaces.

A very fruitful idea, due to J. C. Wood [37] , was to place the study of harmonic morphisms into the framework of conformal foliations. An illustration of this is the paper [4] of P. Baird and J. C. Wood. The origins of many of our results can be found in these papers. To formalize these ideas, say that a foliation *produces harmonic morphisms* if its leaves can be locally given as fibres of submersive harmonic morphisms. In [9] , R. L. Bryant pointed out that any conformal foliation of codimension not equal to two produces harmonic morphisms if and only if a certain one-form is closed (see Proposition 1.16 , below). In particular, if the mean curvature forms of a conformal foliation $\mathcal{V}$ and of its orthogonal complement $\mathcal{H}$ are closed then $\mathcal{V}$ produces harmonic morphisms. It turns out that a conformal foliation $\mathcal{V}$ for which its orthogonal complement $\mathcal{H}$ has closed mean curvature form has the property that it can be locally defined by horizontally homothetic submersions (Proposition 1.18). Following a suggestion of J. C. Wood , we shall call such foliations *homothetic*. Some of the properties of homothetic foliations, together with new examples of harmonic morphisms given by them, are presented in Section 1 . For instance, there we prove that a foliation with minimal leaves of codimension not equal to two produces harmonic morphisms if and only if it is a homothetic foliation (Corollary 1.24). Also, we prove that if a foliation $\mathcal{V}$ of codimension not equal to two, is generated by the action of a Lie group of isometries which admits a bi-invariant metric, and is such that the canonical representation of an isotropy group is irreducible, then $\mathcal{V}$ produces harmonic morphisms (Theorem 1.27). This result can be viewed as an extension of Bryant's observation [9] that nonvanishing Killing fields, defined on Riemannian manifolds of dimension not equal to three, generate foliations which produce harmonic morphisms.

In Section 2 we present some particular results which appear when the fibres of a harmonic morphism are one-dimensional. We prove the converse of the above mentioned observation of R. L. Bryant thus showing that a Riemannian one-dimensional foliation of codimension not equal to two produces harmonic morphisms if and only if is locally generated by Killing fields (Proposition 2.3). This can be generalized to homothetic foliations to prove that a homothetic one-dimensional foliation of codimension not equal to two produces harmonic morphisms if and only if is locally generated by conformal vector fields (Proposition 2.5). Theorem 2.9 is a global version of a result of R. L. Bryant [9, Theorem 1] which gives a local normal form for the metric of the total space of a harmonic morphism with one-dimensional fibres. On the way a simpler proof for Bryant's local result is obtained. Section 2 also contains a few technical results which will be often used in the sequel.

Sections 3 and 4 contain results on foliations which produce harmonic morphisms on compact Riemannian manifolds. In Section 3 , the results are related to the mixed curvature of the almost product Riemannian structure induced by the foliation. For a one-dimensional foliation $\mathcal{V}$ this is just the Ricci curvature restricted to $\mathcal{V}$. We prove that on a compact Riemannian manifold with negative Ricci curvature there exists no one-dimensional foliation which produces harmonic morphisms and admits a global density (Theorem 3.4). This is an extension of the well-known result of S. Bochner on Killing fields. Theorem 4.12(i) is an extension of a result of M. Berger, concerning Killing fields on compact even-dimensional Riemannian manifolds with positive sectional curvature, to homothetic one-dimensional foliations which produce harmonic morphisms. In Section 4 other results on harmonic morphisms with fibres of dimension one are obtained from integral formulae involving the mass of the fibre and the scalar curvatures of the domain and codomain. In Section 3 it is also proved that if a pair of complementary orthogonal foliations have mean curvature vectors which are gradient vector fields, then the mixed curvature is nonpositive (in particular, the sectional curvature cannot be positive) (Proposition 3.7). From this, some consequences to homothetic foliations which produce harmonic morphisms are derived.

In Section 5 we prove (Theorem 5.7) that, on an Einstein manifold with dimension at least four, any one-dimensional foliation which produces harmonic morphisms and has integrable orthogonal complement is either Riemannian and locally generated by Killing fields or is homothetic and has geodesic leaves. In Corollary 5.9 we prove that this still holds if we replace the integrability assumption on the orthogonal complement with the condition that the foliation be homothetic. As with the other results in this section this does not require the compactness or the completeness of the manifold. Section 5 , also contains other similar (but local) results, proved under the hypothesis that certain components of the curvature tensor are basic or zero. All these conditions are satisfied by manifolds with constant sectional curvature and, in this way, we obtain a completely new proof for the above mentioned main result from [9]. In fact, we improve Bryant's result by showing that, instead of a simple foliation on a simply-connected manifold, we can consider an orientable foliation which admits a global density (Theorem 5.14). In Section 5 we also prove a Kaluza-Klein type result (Proposition 5.4).

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1. Foliations which produce harmonic morphisms

Let (M, g) be a Riemannian manifold and $\mathcal{V}$ (the tangent bundle of) a foliation or, more generally, a distribution on it. We shall denote the orthogonal complement of $\mathcal{V}$ by $\mathcal{H}$. Then, $\mathcal{H}$ and $\mathcal{V}$ will be called the *horizontal* and *vertical* distributions, respectively. The corresponding projections will be denoted by the same letters $\mathcal{H}$ and $\mathcal{V}$. We shall denote by X, Y horizontal vector fields, i.e. sections of $\mathcal{H}$ and by U, V vertical vector fields, i.e. sections of $\mathcal{V}$.

Recall that, given a submersive harmonic morphism $\varphi : (M, g) \to (N, h)$, the connected components of its fibres form a conformal foliation. Conversely, we make the following definition (cf. [37]):

Definition 1.1. Let (M, g) be a Riemannian manifold (it will always be assumed that M is connected) and let $\mathcal{V}$ be (the tangent bundle of) a foliation on it.

We will say that $\mathcal{V}$ *produces harmonic morphisms on* (M, g) if each point of M has an open neighbourhood O which is the domain of a submersive harmonic morphism $\varphi : (O, g|_O) \to (N, h)$ whose fibres are open subsets of the leaves of $\mathcal{V}$.

Remark 1.2. When $\operatorname{codim}\mathcal{V} = 2$, $\mathcal{V}$ produces harmonic morphisms if and only if it is conformal and its leaves are minimal [37] ; in this case any local submersion φ on M whose fibres are open subsets of the leaves can be made into a harmonic morphism. Indeed, it suffices to choose a metric on the codomain such that φ is horizontally conformal.

We will see (as an immediate consequence of Corollary 1.14) that, when $\mathrm{codim}\mathcal{V} \neq 2$ and $\mathcal{V}$ produces harmonic morphisms, then each local submersion $\varphi : O \to N$ on M for which the first Betti number of the total space O is zero and whose fibres are open subsets of the leaves of $\mathcal{V}$ can be made to be a harmonic morphism (i.e. there exists a Riemannian metric h on N such that $\varphi : (O, g|_O) \to (N, h)$ is a harmonic morphism).

Definition 1.3. Let $\mathcal{V}$ be a conformal foliation on the Riemannian manifold (M, g). A smooth positive function $\lambda : O \to \mathbb{R}$ on an open subset O of M will be called a *local dilation* of $\mathcal{V}$ if $\mathcal{V}|_O$ is a Riemannian foliation on $(O, \lambda^2 g|_O)$. If $O = M$ then we shall call λ a *(global) dilation of* $\mathcal{V}$.

Remark 1.4. 1) It is obvious that local dilations for a conformal foliation $\mathcal{V}$ can be found in the neighbourhood of each point; in fact, this is equivalent to the definition of its conformality. If $\mathcal{V}$ is *simple*, i.e. its leaves are the fibres of a (horizontally conformal) submersion φ, then it admits a (global) dilation, for example, the dilation of φ.

2) A smooth positive function λ is a local dilation for $\mathcal{V}$ if and only if

$$\big(\mathcal{L}_U(\lambda^2 g)\big)(X, Y) = 0$$

for any vertical vector field U and any horizontal vector fields X, Y, where $\mathcal{L}$ denotes Lie differentiation. Hence, if we multiply a local dilation of a conformal foliation by a smooth positive function which is constant along the leaves then we obtain another local dilation of the foliation. Conversely, if two local dilations λ_j, $j = 1, 2$, of a conformal foliation $\mathcal{V}$ have the same domain then $\lambda_2 = \lambda_1 \rho$ where the factor ρ is a smooth positive function, constant along the leaves of $\mathcal{V}$.

3) Let $\mathcal{V}$ be a foliation, not necessary conformal. Let $\mathcal{H}$ denote its orthogonal complement. Recall that its second fundamental form ${}^{\mathcal{H}}B$ is the horizontal $\mathcal{V}$-valued tensor field defined by

$$(1.1) \qquad {}^{\mathcal{H}}B(X, Y) = \frac{1}{2}\mathcal{V}(\nabla_X Y + \nabla_Y X) ,$$

where X, Y are horizontal vector fields (see [32, Ch.IV, 3.16]). A simple calculation (see [5]) gives the following formula

$$(\mathcal{L}_U\, g)(X, Y) = -2g\big({}^{\mathcal{H}}B(X, Y), U\big) ,$$

where, U, X, Y are as above.

It follows quickly that any local dilation λ of a conformal foliation $\mathcal{V}$ is characterised by the relation

$$(1.2) \qquad \mathrm{trace}({}^{\mathcal{H}}B) = n\mathcal{V}(\mathrm{grad}(\log\lambda)) ,$$

where $n = \mathrm{codim}\mathcal{V}$.

Note that formula (1.1) shows that $\mathcal{V}$ is a conformal foliation if and only if $\mathcal{H}$

is an umbilical distribution, i.e. ${}^{\mathcal{H}}B(X,X)$ is independent of X for $g(X,X)=1$, if $\mathcal{H}$ is integrable this condition says that its integral submanifolds are umbilical (see [32]).

Let $\varphi:(M^m,g)\to(N^n,h)$ be a horizontally conformal submersion with dilation λ. Let τ denote the tension field of φ and ${}^{\mathcal{V}}B$ the second fundamental of the foliation $\mathcal{V}$ induced by the fibres, then we have the *fundamental equation* [2] (see [5] for a different proof):

$$\tau+\mathrm{trace}({}^{\mathcal{V}}B)+(n-2)\mathcal{H}(\mathrm{grad}(\log\lambda))=0\,. \tag{1.3}$$

From this, P. Baird and J. Eells concluded:

Proposition 1.5 ([2])**.** (a) *When $n=2$, φ is a harmonic morphism if and only if its fibres are minimal.*

(b) *When $n\neq 2$ any two of the following assertions imply the remaining assertion:*

(i) *φ is a harmonic morphism,*
(ii) *φ has minimal fibres,*
(iii) *φ is horizontally homothetic (i.e. λ is constant along horizontal curves).*

Note that in above proposition it is unnecessary for φ to be submersive (see [5]).

Let ω denote a local volume form of $\mathcal{V}$. It can easily be seen [29] that, the fundamental equation (1.3) is equivalent to

$$\mathcal{V}^*\big(\mathcal{L}_X(\lambda^{2-n}\,\omega)\big)=\lambda^{2-n}\,g(X,\tau)\,\omega\,, \tag{1.4}$$

for any horizontal vector field X. Thus, we have the following:

Proposition 1.6 ([29])**.** *A horizontally conformal submersion with dilation λ is a harmonic morphism if and only if the parallel displacement defined by the horizontal distribution preserves the* mass *of the fibres, where the fibres are given the mass* density λ^{2-n}.

Definition 1.7. Let $\mathcal{V}$ be a foliation of codimension n, which produces harmonic morphisms on (M,g). Let λ be a local dilation of $\mathcal{V}$ which restricts to give dilations of harmonic morphisms which locally define $\mathcal{V}$. Then $\rho=\lambda^{2-n}$ is called a *local density* of $\mathcal{V}$. If λ is globally defined on M then ρ is called a *(global) density.*

We next discuss how much the metric of M can be changed preserving the property of producing harmonic morphisms.

Proposition 1.8. *Let $\mathcal{V}$ be a foliation on (M,g), with* $\dim\mathcal{V}=p$ *and* $\mathrm{codim}\mathcal{V}=n$. *Let r and s be smooth positive functions on M. Let $g^{\mathcal{H}}$ and $g^{\mathcal{V}}$ denote the horizontal and the vertical components of g, and set $\tilde{g}=s^2\,g^{\mathcal{H}}+r^2\,g^{\mathcal{V}}$.*

(a) *If $n\neq 2$, then, any two of the following assertions imply the remaining assertion:*

(i) *$\mathcal{V}$ produces harmonic morphisms on (M, g) ,*

(ii) *$\mathcal{V}$ produces harmonic morphisms on $(M, \tilde{g})$,*

(iii) *$r^p\, s^{n-2}$ is locally the product of a function constant on horizontal curves and a function constant on vertical curves.*

(b) *If $n = 2$, then the same implications are true after replacing* (iii) *with:*

(iii$'$) *r is constant along horizontal curves.*

Proof. Suppose that $\mathcal{V}$ is conformal and let λ and $\tilde{\lambda}$ be local dilations of $\mathcal{V}$ with respect to g and $\tilde{g}$, respectively. Then $\tilde{\lambda} = a\, s^{-1}\lambda$, where a is a smooth positive function which is constant along vertical curves.

Let ω and $\tilde{\omega}$ be local volume forms of $\mathcal{V}$ with respect to g and $\tilde{g}$, respectively. Then $\tilde{\omega} = r^p\, \omega$.

It follows that

$$\tilde{\lambda}^{2-n}\, \tilde{\omega} = a^{2-n}(s^{n-2}\, r^p)(\lambda^{2-n}\omega) \,. \tag{1.5}$$

To prove that (i),(ii)$\Rightarrow$(iii) note that (1.5) and Proposition 1.6 implies that if λ^{2-n} and $\tilde{\lambda}^{2-n}$ are local densities of $\mathcal{V}$ with respect to g and $\tilde{g}$, respectively, then

$$s^{n-2}\, r^p = a^{n-2}\, b \,, \tag{1.6}$$

where b is a smooth positive function constant along horizontal curves.

To prove that (i),(iii)$\Rightarrow$(ii) , suppose that λ^{2-n} is a local density of $\mathcal{V}$ with respect to g and choose smooth positive functions a and b which satisfy (1.6) and such that a is constant along vertical curves and b is constant along horizontal curves. Now, (1.5) implies that $\tilde{\lambda} = a\, s^{-1}\lambda$ corresponds to a local density of $\mathcal{V}$ with respect to $\tilde{g}$.

The proof of (ii),(iii)$\Rightarrow$(i) is similar. □

Corollary 1.9. *Let $\mathcal{V}$ be a foliation with* $\operatorname{codim}\mathcal{V} \neq 2$ *on (M, g) . Let a and b be smooth positive functions on M such that a is constant along vertical curves and b is constant along horizontal curves. Then the following assertions are equivalent:*

(i) *$\mathcal{V}$ produces harmonic morphisms on (M, g) ,*

(ii) *$\mathcal{V}$ produces harmonic morphisms on $(M,\, a^2\, b^2\, g)$.*

If $\operatorname{codim}\mathcal{V} = 2$ *then* (i) $\Longleftrightarrow$ (ii) *if and only if the function a is constant on M.*

Proof. This is an immediate consequence of Proposition 1.8 . □

Proposition 1.8 suggests the following:

Definition 1.10. Let $\mathcal{V}$ be a distribution of dimension p and codimension n on the Riemannian manifold (M, g). For a positive smooth function σ on M we define the metric ${}^{\sigma}g$ by

$${}^{\sigma}g = \sigma^2\, g^{\mathcal{H}} + \sigma^{\frac{4-2n}{p}}\, g^{\mathcal{V}} \,,$$

where, $g^{\mathcal{H}}$ and $g^{\mathcal{V}}$ are the horizontal and the vertical components of g , respectively.

Proposition 1.11. *Let $\mathcal{V}$ be a conformal foliation on (M, g) and let σ be a positive smooth function on M. Then,*

(i) *$\mathcal{V}$ is also a conformal foliation on $(M, {}^{\sigma}g)$. Furthermore, λ is a local dilation of $\mathcal{V}$ with respect to g if and only if $\lambda\,\sigma^{-1}$ is a local dilation with respect to ${}^{\sigma}g$.*

(ii) *$\mathcal{V}$ produces harmonic morphisms on (M, g) if and only if it produces harmonic morphisms on $(M, {}^{\sigma}g)$.*

(iii) *If $\mathcal{V}$ produces harmonic morphisms and admits a global dilation λ such that λ^{2-n} is a density for $\mathcal{V}$ with respect to g then, $\mathcal{V}$ is a Riemannian foliation with minimal leaves on $(M, {}^{\lambda}g)$.*

Proof. Statement (i) follows from Remark 1.4(2) whilst (ii) follows from Proposition 1.8.

If $\operatorname{codim}\mathcal{V} = 2$ assertion (iii) is obvious. If $\operatorname{codim}\mathcal{V} \neq 2$ first note that if λ is a local dilation of $\mathcal{V}$ with respect to g then $\mathcal{V}$ is a Riemannian foliation on $(M, {}^{\lambda}g)$. Now the proof of Proposition 1.8 shows that if λ^{2-n} is a density for $\mathcal{V}$ then the constant function $\tilde{\lambda} = 1$ is a dilation which corresponds to a density for $\mathcal{V}$ with respect to ${}^{\lambda}g$. Thus, by (1.3) the leaves of $\mathcal{V}$ are minimal submanifolds of $(M, {}^{\lambda}g)$. □

The next result shows that the metric on the codomain is much more rigid.

Proposition 1.12. *Let $\varphi_j : (M, g) \to (N, h_j)$, $j = 1, 2$, be nonconstant harmonic morphisms having the same fibres. Suppose that N is connected and* $\dim N \neq 2$.

Then, h_1 and h_2 are homothetic.

Proof. Let λ_j be the dilation of φ_j $(j = 1, 2)$. Then $\lambda_2 = \lambda_1\,\sigma$ where $\sigma : M \to \mathbb{R}$ is a smooth positive function, constant along the fibres.

Recall from Proposition 1.6 that the property that *φ_j is a harmonic morphism* is equivalent to the property that *the parallel displacement defined by the horizontal distribution preserves $\lambda_j^{2-n}\,\omega$*, where $n = \dim N$ and ω is a local volume form for the vertical distribution. Hence, σ is also constant along horizontal curves.

It follows that σ is constant on M, and the proposition is proved. □

An immediate consequence of Proposition 1.12 is the following:

Corollary 1.13. *A foliation of codimension $q \neq 2$ which produces harmonic morphisms is given by a Haefliger structure* [19] *with values in the groupoid of germs of homothetic diffeomorphisms of the sheaf of germs of Riemannian metrics on $\mathbb{R}^q$.*

Corollary 1.14. *Let (M, g) be a Riemannian manifold with zero first Betti number. Let $\mathcal{V}$ be a foliation of codimension not equal to two which produces harmonic morphisms on (M, g).*

Then, $\mathcal{V}$ admits a global density λ^{2-n}. Furthermore, $\mathcal{V}$ is a Riemannian foliation with minimal leaves on $(M, {}^{\lambda}g)$.

Proof. Let $\varphi_j : (O_j, g|_{O_j}) \to (N_j, h_j)$ be submersive harmonic morphisms whose fibres are open subsets of leaves of $\mathcal{V}$ and $M = \bigcup_j O_j$.

If λ_j is the dilation of φ_j then for any indices j, k we can write $\lambda_k = \lambda_j \, \rho_{jk}$, where in view of Proposition 1.12, ρ_{jk} is constant on each component of $O_j \cap O_k$.

It is obvious that $\{O_j, \log(\rho_{jk})\}$ determines an element of $H^1(M;\mathbb{R})$. Since this group is trivial, for each j we can find a_j constant on each component of O_j such that $\rho_{jk} = a_j \, a_k^{-1}$ on $O_j \cap O_k$, for all j, k.

It follows that for each j, k we have $\lambda_j \, a_j = \lambda_k \, a_k$ on $O_j \cap O_k$ and hence we can define $\lambda : M \to \mathbb{R}$ such that $\lambda = \lambda_j \, a_j$, on O_j. It is obvious that λ^{2-n} is a global density for $\mathcal{V}$.

The last assertion follows from Proposition 1.11(iii). □

In the following theorem we attach assertion (iv) to a well-known list of equivalent assertions (see [25, Appendix B]).

Theorem 1.15. *Let M be a compact manifold with zero first Betti number.*

For a foliation $\mathcal{V}$ on M with compact leaves the following assertions are equivalent:

(i) *the holonomy group of each leaf of $\mathcal{V}$ is finite,*

(ii) *there exists a metric g on M such that $\mathcal{V}$ is a Riemannian foliation on (M, g),*

(iii) *there exists a metric g on M such that the leaves of $\mathcal{V}$ are minimal submanifolds of (M, g),*

(iv) *there exists a metric g on M such that $\mathcal{V}$ produces harmonic morphisms on (M, g).*

Moreover, if $\operatorname{codim}\mathcal{V} = 2$, *is not necessary to assume that the first Betti number of M is zero.*

Proof. It is well-known that the assertions (i), (ii) and (iii) are equivalent. Moreover, if any of these properties holds then there exists a metric g on M such that $\mathcal{V}$ is a Riemannian foliation with minimal leaves on (M, g). (To see this let h be a metric on M with respect to which $\mathcal{V}$ has minimal leaves and let k be a metric on M with respect to which $\mathcal{V}$ is Riemannian. If $\mathcal{H}$ is the orthogonal complement of $\mathcal{V}$ with respect to h let $g = h^{\mathcal{V}} + k^{\mathcal{H}}$ where, $h^{\mathcal{V}}$ is the vertical component of h and $k^{\mathcal{H}}$ is the horizontal component of k.) But any Riemannian foliation with minimal leaves produces harmonic morphisms.

Conversely, suppose that $\mathcal{V}$ produces harmonic morphisms on (M, g). If $\operatorname{codim}\mathcal{V} = 2$ then by Proposition 1.5, $\mathcal{V}$ has minimal leaves (see [37]). If $\operatorname{codim}\mathcal{V} \neq 2$ then by Corollary 1.14, there exists a global density λ^{2-n} of $\mathcal{V}$, and $\mathcal{V}$ is a Riemannian foliation with minimal leaves on $(M, {}^{\lambda}g)$. □

We now characterise conformal foliations which produce harmonic morphisms. Recall that a conformal foliation $\mathcal{V}$ of $\operatorname{codim}\mathcal{V} = 2$ produces harmonic morphisms if and only if its leaves are minimal. For $\operatorname{codim}\mathcal{V} \neq 2$, the situation is more complicated and we have the following reformulation of a result of R. L. Bryant [9] (see [5] for another treatment).

Proposition 1.16. *Let $\mathcal{V}$ be a conformal foliation on (M,g) of* $\operatorname{codim}\mathcal{V}\neq 2$ *and $\mathcal{H}$ its orthogonal complement. Let ${}^{\mathcal{V}}\!B$ and ${}^{\mathcal{H}}\!B$ be the second fundamental forms of $\mathcal{V}$ and $\mathcal{H}$, respectively.*

Then, $\mathcal{V}$ produces harmonic morphisms on (M,g) if and only if the vector field

$$(n-2)\operatorname{trace}({}^{\mathcal{H}}\!B)-n\operatorname{trace}({}^{\mathcal{V}}\!B)$$

is locally a gradient vector field. (Here $\operatorname{trace}({}^{\mathcal{H}}\!B)=\sum_j {}^{\mathcal{H}}\!B(X_j,X_j)$, $\operatorname{trace}({}^{\mathcal{V}}\!B)=\sum_\alpha {}^{\mathcal{V}}\!B(U_\alpha,U_\alpha)$ *for local orthonormal frames $\{X_j\}$ and $\{U_\alpha\}$ of $\mathcal{H}$ and $\mathcal{V}$, respectively.)*

Proof. Note that the following relation holds

$$(n-2)\operatorname{trace}({}^{\mathcal{H}}\!B)-n\operatorname{trace}({}^{\mathcal{V}}\!B)=n(n-2)\operatorname{grad}(\log\lambda)\,, \tag{1.7}$$

if and only if:

$$\operatorname{trace}({}^{\mathcal{H}}\!B)=n\mathcal{V}(\operatorname{grad}(\log\lambda))\,, \tag{1.7a}$$

and

$$\operatorname{trace}({}^{\mathcal{V}}\!B)=-(n-2)\mathcal{H}(\operatorname{grad}(\log\lambda))\,. \tag{1.7b}$$

By Remark 1.4(3), (1.7a) holds if and only λ is a local dilation of $\mathcal{V}$. This together with the fundamental equation (1.3), imply that (1.7a) and (1.7b) hold if and only if $\mathcal{V}$, restricted to the domain of λ, produces harmonic morphisms and λ^{2-n} is a density of it. □

Note that Corollary 1.14 can be proved using (1.7).

Let $p=\dim\mathcal{V}$. By the *mean curvature form* of $\mathcal{V}$ we mean the one-form $\frac{1}{p}(\operatorname{trace}({}^{\mathcal{V}}\!B))^\flat$ obtained by applying the musical isomorphism $\flat : TM\to T^*M$ with similar terminology for $\mathcal{H}$. Then we have.

Corollary 1.17. *Let $\mathcal{V}$ be a conformal foliation with* $\operatorname{codim}\mathcal{V}\neq 2$ *and let $\mathcal{H}$ be its orthogonal complement. Then any two of the following assertions imply the remaining assertion:*

(i) *$\mathcal{V}$ produces harmonic morphisms,*
(ii) *$\mathcal{V}$ has closed mean curvature form,*
(iii) *$\mathcal{H}$ has closed mean curvature form.*

Proof. This is an immediate consequence of Proposition 1.16 and the fact that $(\operatorname{trace}({}^{\mathcal{V}}\!B))^\flat$ is closed if and only if $\operatorname{trace}({}^{\mathcal{V}}\!B)$ is locally a gradient and similar for $\mathcal{H}$. □

We now introduce a new sort of foliations midway between conformal and Riemannian foliations.

Proposition 1.18. *For a conformal foliation $\mathcal{V}$ on the Riemannian manifold (M, g) the following assertions are equivalent:*

(i) *the leaves of $\mathcal{V}$ can be, locally, given as fibres of horizontally homothetic submersions;*

(ii) *each point of M has an open neighbourhood on which there can be defined a local dilation of $\mathcal{V}$ which is constant along horizontal curves;*

(iii) *any local dilation λ of $\mathcal{V}$, defined on an open subset O with zero first Betti number, is a product $\lambda = a\,b$, where a and b are positive smooth functions such that a is constant along vertical curves and b is constant along horizontal curves;*

(iv) *the mean curvature of the orthogonal complement of $\mathcal{V}$ is locally a gradient vector field.*

Proof. The equivalence (i) $\iff$ (ii) is obvious. Also, by Remark 1.4(2) it follows that (ii)$\Leftarrow$(iii).

By the same remark, if (ii) holds then, any local dilation $\lambda : O \to \mathbb{R}$ of $\mathcal{V}$ is, locally, a product $\lambda = a\,b$ as in (iv). If on the same open subset of O, we also have $\lambda = a_1\,b_1$ then, $a^{-1}\,a_1 = b\,b_1^{-1} =$const.. Hence the differentials of the logarithms of the factors a, b from the local decompositions of λ define closed one-forms on O. If the first Betti number of O is zero then these one-forms are exact and the implication (ii)$\Rightarrow$(iii) is proved.

The equivalence (ii) $\iff$ (iv) follows from Remark 1.4(3). □

Remark 1.19. 1) If a conformal foliation satisfies one of the properties from the above proposition then the holonomy groupoid of each leaf is formed of germs of homothetic diffeomorphisms.

2) In (iv) above instead of $H^1(O;\mathbb{R}) = 0$ we could ask that the first basic cohomology group (see [34]) of $\mathcal{V}|_O$ be zero. This follows from the fact that the set of differentials $\{\mathrm{d}\,a\}$ define a closed *basic* one-form on O.

3) Alternatively, we could ask that the orthogonal complement $\mathcal{H}|_O$ of $\mathcal{V}|_O$ to be an Ehresmann connection [8] with trivial holonomy (in particular, $\mathcal{H}|_O$ is integrable). To see this, note that the set of differentials $\{\mathrm{d}\,b\}$ define a closed one-form which, when restricted to a leaf L, is exact (because it coincide with $\mathrm{d}(\lambda|_L)$). When $\mathcal{H}$ is an Ehresmann connection with trivial holonomy these exact forms can be matched together to define an exact form on O.

The notion of *local dilations* for conformal foliations can be generalized to conformal *distributions*, although, in this case, these might not exist. Nevertheless, if a conformal distribution admits local dilations then these share the same properties (Remark 1.4(2) and (3)) as the local dilations of a conformal foliation. Moreover, assertions (ii), (iii) and (iv) from Proposition 1.18 remain equivalent for conformal distributions which admit local dilations in a neighbourhood of each point.

Proposition 1.18 suggests the following definition.

Definition 1.20. Let $\mathcal{V}$ be a distribution on the Riemannian manifold (M, g). We will say that $\mathcal{V}$ is *homothetic* if it is conformal and the mean curvature of its orthogonal complement is locally a gradient vector field.

Remark 1.21. Note that, unlike conformal distributions, homothetic distributions always admit local dilations, even if nonintegrable. Indeed, if $\mathcal{V}$ is a homothetic distribution with $\operatorname{codim}\mathcal{V} = n$ and $\mathcal{H}$ is its orthogonal complement, then any local smooth positive function λ on M which has the property

$$n \operatorname{grad}(\log \lambda) = \operatorname{trace}({}^{\mathcal{H}}B)\,, \tag{1.9}$$

is a local dilation of $\mathcal{V}$.

Moreover, if $\mathcal{V}$ is a homothetic distribution and λ is a local dilation of it defined on an open set O such that $\lambda = a\,b$ as in (iii) from Proposition 1.18, then any other local dilation defined on O is of this form.

Proposition 1.22. *Let $\mathcal{V}$ be a foliation on (M, g) with orthogonal complement $\mathcal{H}$.*

If $\mathcal{H}$ is a homothetic distribution then the parallel displacement defined by it is formed of (local) homothetic diffeomorphisms between leaves of $\mathcal{V}$.

Conversely, if $\mathcal{H}$ is conformal, integrable and the parallel displacement defined by it is formed of (local) homothetic diffeomorphisms between leaves of $\mathcal{V}$, then it is a homothetic distribution.

Proof. If $\mathcal{H}$ is conformal and integrable then it admits local dilations. Let λ be a local dilation of $\mathcal{H}$. Then for any horizontal vector field X invariant under the holonomy of $\mathcal{V}$ (i.e. for any basic vector field X) we have:

$$\mathcal{L}_X(\lambda^2\,(g|_{\mathcal{V}})) = 0\,. \tag{1.10}$$

If (ψ_t) is the local flow of X, then (1.10) is equivalent to the fact that for any t we have

$$(\psi_t)^*(\lambda^2(g|_{\mathcal{V}})) = \lambda^2(g|_{\mathcal{V}})\,. \tag{1.11}$$

The proof follows from (1.11) by using the fact that, if $\mathcal{H}$ is conformal, then it is homothetic if and only if in the neighbourhood of each point a local dilation can be found, which is constant along the leaves of $\mathcal{V}$. □

Proposition 1.23. *Let $\mathcal{V}$ be a homothetic foliation of codimension not equal to two on (M, g). Then, the following assertions are equivalent:*

(i) *$\mathcal{V}$ produces harmonic morphisms;*

(ii) *the mean curvature of $\mathcal{V}$ is locally a gradient vector field.*

In particular, a homothetic foliation whose orthogonal complement is a homothetic distribution produces harmonic morphisms with umbilical fibres.

Proof. Since $\mathcal{V}$ is homothetic we have that $\operatorname{trace}({}^{\mathcal{H}}B)$ is, locally, a gradient vector field. From Proposition 1.16 it follows that $\mathcal{V}$ produces harmonic morphisms if and only if $\operatorname{trace}({}^{\mathcal{V}}B)$ is, locally, a gradient vector field.

The last assertion follows from the fact that (see Remark 1.4(3)) $\mathcal{V}$ is umbilical if and only if $\mathcal{H}$ is a conformal distribution. $\square$

Corollary 1.24. *Let $\mathcal{V}$ be a foliation with minimal leaves and* $\mathrm{codim}\mathcal{V} \neq 2$ *. Then, $\mathcal{V}$ produces harmonic morphisms if and only if it is homothetic.*

Remark 1.25. 1) In Proposition 1.23 condition (ii) is a bit more general than saying $\mathcal{H}$ is homothetic since it is not assumed to be conformal.

2) Any Riemannian foliation is homothetic.

3) Given a horizontally homothetic submersion the connected components of its fibres form a homothetic foliation. However, note that if $\varphi : (M, g) \to (N, h)$ is a horizontally conformal submersion such that its fibres form a homothetic foliation then it is not true, in general, that φ is horizontally homothetic. If the first Betti number of M is zero then φ can be factorised into a horizontally homothetic submersion followed by a conformal diffeomorphism. (By Remark 1.19(2)(3) this factorization can also be done when the first Betti number of N is zero or when the horizontal distribution $\mathcal{H}$ is an Ehresmann connection with trivial holonomy, in which case $\mathcal{H}$ is integrable and M is diffeomorphic with the product of N and the fibre.)

Example 1.26. 1) *Double-warped-products* (see [33], and the references therein). If (M^p, g) and (N^q, h) are Riemannian manifolds and $r : M \to \mathbb{R}$ and $s : N \to \mathbb{R}$ are positive smooth functions then the double-warped-product of (M^p, g) and (N^q, h) is defined to be:

$$M_s \times {}_r N = (M \times N, s^2\, \pi_M^* g + r^2\, \pi_N^* h) \,,$$

where, π_M and π_N are the projections onto M and N , respectively.

The projections $\pi_M : M_s \times {}_r N \to (M, g)$ and $\pi_N : M_s \times {}_r N \to (N, h)$ are horizontally homothetic so their fibres define a pair of complementary orthogonal homothetic foliations. Conversely, any Riemannian manifold endowed with a pair of complementary orthogonal homothetic foliations is canonically locally isometric to a double-warped-product. Hence, by Proposition 1.23 , when $p, q \neq 2$ both of the foliations induced by the factors of a double-warped-product produce harmonic morphisms. More precisely, if $p \neq 2$ the following projection is a harmonic morphism with umbilical fibres:

$$\pi_M : M_s \times {}_r N \longrightarrow (M, r^{\frac{2q}{p-2}}\, g) \,. \tag{1.12}$$

The fact that the above projection is a harmonic morphism also follows from Proposition 1.11(iii) .

A concrete example of a warped product is provided by the open subsets in spheres $(S^{p+q} \backslash S_0^{p-1}, g_{p+q})$, where $S_0^{p-1} = \mathbb{R}^p \cap S^{p+q}$ and $\mathbb{R}^p \equiv \{x \in \mathbb{R}^{p+q+1} \,|\, x^{p+1} = \ldots = x^{p+q+1} = 0\}$ and g_{p+q} is the restriction of the canonical metric on S^{p+q} .

The warped-product is the one induced by the following diffeomorphism:

$$\Phi : S^{p+q} \setminus S_0^{p-1} \longrightarrow S_+^p \times S^q$$
$$\Phi(x,y) = \big((x,|y|), \frac{1}{|y|}y\big) .$$

where, $S_+^p = \big\{x \in S^{p+1} \,|\, x^{p+1} > 0\big\}$. To make Φ an isometry we must give $S_+^p \times S^q$ the warped-product structure $S_+^p \times_r S^q$ where, $r(x^1,\dots,x^{p+1}) = x^{p+1}$.

Thus, if $p \neq 2$, (1.12) particularises to give the following harmonic morphism with umbilical fibres:

$$\varphi : (S^{p+q} \setminus S_0^{p-1}, g_{p+q}) \longrightarrow (S_+^p, r^{\frac{2q}{p-2}}\, g_p)$$
$$\varphi(x^1,\dots,x^{p+q+1}) = \big(\, x^1,\dots,x^p, \sqrt{(x^{p+1})^2 + \ldots + (x^{p+q+1})^2}\,\big) .$$

By Proposition 1.12 any other metric on S_+^p with respect to which φ above is a harmonic morphism is homothetic to the one considered. Also, note that, although φ can be extended to a continuous map on S^{p+q}, the considered metric on the codomain cannot be extended over the points where the extension of φ takes values.

The last example can be viewed as the projection of the action of a group of isometries. In Theorem 1.27 below we shall give a generalization of this idea.

2) *Double-twisted-products*(see [33], and the references therein). These are defined in the same way as double-warped-products, but now $r, s : M \times N \to \mathbb{R}$. It is easy to see that a Riemannian manifold endowed with a pair of complementary orthogonal foliations which are both umbilical (equivalently, both conformal) is, canonically, locally isometric to a double-twisted-product.

If $p \neq 2$, it follows from Proposition 1.8 that the foliation $\mathcal{V}$ induced by the second factor of the double-twisted-product $M_s \times {}_r N$ produces harmonic morphism if and only if the function $r^q \cdot s^{p-2}$ is, locally, the product of a function constant on M and a function constant on N.

If $p = 2$, then $\mathcal{V}$ produces harmonic morphisms if and only if r is a function defined on N. In this case, $\mathcal{V}$ has totally geodesic leaves and $M_s \times {}_r N$ is isometric to the twisted product $M_s \times \widetilde{N}$ where $\widetilde{N} = (N, r^2\, h)$.

It follows that a pair of complementary orthogonal umbilical foliations both of codimension not equal to two are both homothetic if and only if each of them produces harmonic morphisms.

The following theorem can be viewed as an extension of a result of R. L. Bryant [9] that any Killing field, on a Riemannian manifold of dimension not equal to three, generates a foliation which produces harmonic morphisms.

Theorem 1.27. *Let G be a Lie group which acts as an isometry group on the Riemannian manifold (M,g). Suppose that the following conditions are satisfied:*

(i) *The orbits of the action of G on M have the same codimension not equal to two.*

(ii) *There exists on G a bi-invariant Riemannian metric.*
(iii) *The canonical representation of an isotropy group is irreducible.*
Then, the connected components of the orbits form a Riemannian foliation with umbilical leaves which produces harmonic morphisms.

Proof. The fact that (i) implies that the connected components of the orbits form a Riemannian foliation is well-known (see [32, Ch.IV, 4.10]). Let $\mathcal{V}$ be this foliation.

Let U be a vector field on M which is induced by an element of $\mathfrak{g}$, the Lie algebra of G, via the action. Then U is a Killing field on (M,g) and hence, for any vertical field V and basic field X, we have:

$$0=(\mathcal{L}_U\,g)(V,X)=-g(V,[U,X])\,.$$

Since $[U,X]$ is vertical (X being basic), we get that $[U,X]=0$. This means that the action commutes with the (local) flows of the basic fields. Hence the (local) flow of a basic field is formed of (local) automorphisms of the action of G on M, and, in particular, any two orbits have conjugate isotropy groups.

Let γ be a metric on $\mathfrak{g}$ which is invariant under the action of the adjoint group $\mathrm{Ad}G$ (its existence is assured by (ii)). If H is the isotropy group at $x\in M$ of an orbit F, and $\mathfrak{h}$ its Lie algebra, let $\mathfrak{l}$ be its orthogonal complement with respect to γ. Then, as is well-known, there exists a natural isomorphism between T_xF and $\mathfrak{l}$, under which the canonical representation of H on T_xF corresponds to the restriction, to $\mathfrak{l}$, of the adjoint representation of H on $\mathfrak{g}$. Now, (iii) implies that the metrics on T_xF induced by g and γ must be homothetic (see [22, vol.I, Appendix 5]).

Thus, there exists a smooth positive function ρ on M, such that, up to the isomorphisms described above, we have $\rho(x)^{-2}\,\gamma=g_x|_{T_xF}$, for each $x\in M$. Moreover, since G acts by isometries, ρ is constant along leaves of $\mathcal{V}$.

Since the basic vector fields are infinitesimal automorphisms of the action of G on M, their flows are formed of isometries between orbits, considered with the metrics induced by $\rho^2\,g$. Thus the orthogonal complement $\mathcal{H}$ of $\mathcal{V}$ is a conformal distribution with dilation ρ. Moreover, since ρ is constant along the leaves of $\mathcal{V}$ we see that $\mathcal{H}$ is homothetic. By Proposition 1.23, $\mathcal{V}$ produces harmonic morphisms with umbilical fibres. □

Example 1.28. The special orthogonal group $SO(q+1)$ acts on S^{p+q} to give the same Riemannian foliation which produces harmonic morphisms as in Example 1.26(1). (Recall that $SO(q+1)$ admits a bi-invariant Riemannian metric and that the isotropy group is, in this case, $SO(q)$ which acts irreducible on $\mathbb{R}^q$.)

Similarly, let Q^* be formed of the regular points of the quadric Q, defined by:

$$Q=\bigl\{x\in\mathbb{R}^{p+q+1}\,|\,\sum_{j=1}^{p}(x^j)^2+\sum_{j=p+1}^{p+q+1}c_j(x^j)^2=0\,\bigr\}\,.$$

Then, the theorem above implies that the principal orbits of the action of $SO(p)$ on Q^* form a Riemannian foliation with umbilical leaves which produces harmonic morphisms.

More general examples of this type can be obtained by considering other hypersurfaces of $\mathbb{R}^{p+q+1}$ which are invariant under the action of $SO(p)$.

2. Some basic facts on harmonic morphisms with one-dimensional fibres

In this section we present, for later use, a few facts about one-dimensional foliations which produce harmonic morphisms. Here, $\mathcal{V}$ will always denote a one-dimensional foliation.

The following lemma will be used several times in this paper. The case $n = 2$ was used by P. Baird and J. C. Wood in [4, §3].

Lemma 2.1. *Let $\mathcal{V}$ be a conformal one dimensional foliation on (M^{n+1}, g). Then, the following assertions are equivalent.*

(i) *$\mathcal{V}$ produces harmonic morphisms;*

(ii) *each point has a neighbourhood on which a local dilation λ of $\mathcal{V}$, can be found, such that, if V is a vertical field with $g(V, V) = \lambda^{2n-4}$, then $[V, X] = 0$ for any basic field X.*

Proof. From (1.4) it follows that assertion (i) is equivalent to the possibility of finding in the neighbourhood of each point a local dilation λ of $\mathcal{V}$ such that

$$\mathcal{V}^*(\mathcal{L}_X(\lambda^{2-n}\,\omega)) = 0 \tag{2.1}$$

for any basic vector field X and where ω is a local volume of $\mathcal{V}$.

If V is as in (ii) and θ is its dual vertical one-form (i.e. θ is the unique vertical one-form such that $\theta(V) = 1$) then $\lambda^{n-2}\,\theta$ is a local volume form of $\mathcal{V}$. Hence (2.1) is equivalent to $(\mathcal{L}_X\,\theta)(V) = 0$ which is equivalent to $[V, X] = 0$. □

Remark 2.2. 1) From the proof above we see that (ii) is a characterisation for those local dilations which restrict to give dilations of harmonic morphisms which locally define the foliation.

2) If V is as above, let θ be its dual vertical one-form. Using the fact that $[V, X] = 0$ for any basic vector field X, it follows that the two-form $\Omega = \mathrm{d}\,\theta$ is basic. In fact, θ and Ω are, respectively, the connection form and the curvature form of a principal (local) connection (see Theorem 2.9).

The implication (iii)⇒(i) from the following proposition is due to R. L. Bryant [9].

Proposition 2.3. *For $n \neq 2$, let $\mathcal{V}$ be a one-dimensional Riemannian foliation on (M^{n+1}, g) and let $\mathcal{H}$ be its orthogonal complement. Then, the following assertions are equivalent:*

(i) *$\mathcal{V}$ produces harmonic morphisms,*

(ii) *$\mathcal{H}$ is a homothetic distribution,*

(iii) *$\mathcal{V}$ is locally generated by Killing fields.*

Furthermore, if $\mathcal{V}$ is orientable and the first Betti number of M is zero then (iii) *above can be replaced by*

(iii$'$) *$\mathcal{V}$ is globally generated by a Killing field.*

Proof. (i) $\iff$ (ii) This follows from Proposition 1.23 , since, being Riemannian, $\mathcal{V}$ is homothetic and, being of codimension one, $\mathcal{H}$ is conformal.

(ii)$\Rightarrow$(iii) Let ρ be a local dilation of $\mathcal{H}$ which is constant along the leaves of $\mathcal{V}$ and let V be a local vertical field such that $g(V,V) = \rho^{-2}$.

Because ρ is constant along the leaves of $\mathcal{V}$ we have

$$(\mathcal{L}_V\, g)(V,V) = 0\,. \tag{2.2}$$

Because ρ is a local dilation of $\mathcal{H}$ we have

$$(\mathcal{L}_X(\rho^2\, g))(V,V) = 0 \tag{2.3}$$

for any horizontal vector field X. It is easy to see that (2.3) is equivalent to $g([X,V],V) = 0$. This implies that for any horizontal vector field we have

$$(\mathcal{L}_V\, g)(V,X) = 0\,. \tag{2.4}$$

Since $\mathcal{V}$ is Riemannian we have

$$(\mathcal{L}_V\, g)(X,Y) = 0\,, \tag{2.5}$$

for any horizontal vector fields X and Y.

Equations (2.2),(2.4) and (2.5) show that V is a Killing field.

(iii)$\Rightarrow$(ii) Since $\dim \mathcal{V} = 1$, the orthogonal complement $\mathcal{H}$ of $\mathcal{V}$ is a conformal distribution.

As in the proof of Theorem 1.27 , if V is a (local) nonvanishing Killing field, which (locally) generates $\mathcal{V}$, and $|V|$ its norm then $|V|^{-1}$ is a local dilation for the horizontal distribution $\mathcal{H}$. Moreover, $\mathcal{H}$ is homothetic, since $|V|$ is constant along the leaves of $\mathcal{V}$.

The last assertion follows from the fact that when the first Betti number of M is zero and $\mathcal{V}$ is orientable we can find a global density λ^{2-n} of $\mathcal{V}$ (which is also a local dilation for $\mathcal{H}$) and a vertical vector field V defined on M, such that $g(V,V) = \lambda^{2n-4}$. □

Remark 2.4. 1) Note that if $n = 2$ then (i)$\Rightarrow$(ii) $\iff$ (iii). In fact, in this case, a one-dimensional foliation $\mathcal{V}$ produces harmonic morphisms on (M^3,g) if and only if its leaves are geodesics (see [4]). Thus, being of codimension one, $\mathcal{H}$ is a Riemannian distribution. However, if $n = 2$ then (ii)$\Rightarrow$(i) fails, as simple examples show.

2) If in the above proposition we further assume that $\mathcal{H}$ is integrable then we can add the assertion

(iv) $\mathcal{V}$ induces, locally, a warped product structure on (M,g).

One might guess that a similar proposition, to the one above, holds, in general, for any conformal one-dimensional foliation, just by replacing 'Killing fields', with 'conformal fields'. It is not difficult to see that this is not true, the actual situation being described by the following:

Proposition 2.5. *For $n \geq 3$, let $\mathcal{V}$ be a one-dimensional foliation on (M^{n+1}, g). Then, any two of the following assertions imply the remaining assertion.*

(i) *$\mathcal{V}$ produces harmonic morphisms,*

(ii) *$\mathcal{V}$ (or $\mathcal{H}$) is homothetic,*

(iii) *$\mathcal{V}$ is locally generated by conformal vector fields.*

Furthermore, if $\mathcal{V}$ is orientable and the first Betti number of M is zero then (iii) *above can be replaced by*

(iii$'$) *$\mathcal{V}$ is (globally) generated by a conformal field.*

Proof. (i),(ii)$\Rightarrow$(iii) Let λ^{2-n} be a local density for $\mathcal{V}$. By Proposition 1.18, we can suppose that $\lambda = a\,b$, where a is constant along leaves and b is constant along horizontal curves.

Let W be a local vertical vector field such that $g(W, W) = a^{2n-4}\,b^{-2}$. It is a straightforward calculation to check that W is a local conformal vector field on (M, g).

(ii),(iii)$\Rightarrow$(i) Since $\mathcal{V}$ is homothetic, by Proposition 1.18, we can find a local dilation b of $\mathcal{V}$ which is constant along horizontal curves.

Let W be a local conformal vector field which (locally) generates $\mathcal{V}$. We can suppose that b and W are defined on the same open subset of M. It is easy to see that, since W is conformal, we have that $b^2\,g(W, W)$ is constant along leaves.

We can choose a smooth positive local function a on M such that $g(W, W) = a^{2n-4}\,b^{-2}$. Hence a is constant along the leaves and thus $\lambda = a\,b$ is a local dilation of $\mathcal{V}$.

If V is a local field, tangent to the leaves and such that $g(V, V) = \lambda^{2n-4}$ then, from the fact that W is conformal it follows that $[V, X] = 0$ for any basic X. Hence, by Lemma 2.1, $\mathcal{V}$ is a foliation which produces harmonic morphisms.

(iii),(i)$\Rightarrow$(ii) Let λ^{2-n} be a local density for $\mathcal{V}$. Let V be a local vector field, tangent to the leaves and such that $g(V, V) = \lambda^{2n-4}$, and let W be a local conformal vector field tangent to the leaves. (We can suppose that V and W are defined on the same open set.)

Since W is conformal, for any basic X we have $(\mathcal{L}_W\, g)(W, X) = 0$, and hence, $[W, X] = 0$. But by Lemma 2.1 we also have $[V, X] = 0$, for any basic X. Hence if b is such that $W = bV$, then b is constant on horizontal curves.

Since λ is a local dilation of the conformal foliation $\mathcal{V}$, from Remark 1.4, we see that

$$(\mathcal{L}_W\, g)(V, V) = W(\log(\lambda^{-2}))g(V, V)\,. \tag{2.6}$$

Relation (2.6) together with $W = bV$ implies after a straightforward calculation that $\lambda^{n-1}\,b$ is constant along leaves. Thus, we can write $\lambda = r\,s$ where r, s are positive smooth functions on M such that r is constant along the leaves and s is constant along horizontal curves. From Proposition 1.18 , we get that $\mathcal{V}$ is a homothetic foliation. □

Remark 2.6. Note that, if in Proposition 2.5 we have $n = 2$, the implication (ii),(iii)⇒(i) fails, the other implications still holding.

If $n = 1$, then (i) $\iff$ (iii) but they do not imply (ii).

The following lemmas will be used, mainly in the last section (cf. [4, Remark 5.3]).

Lemma 2.7. *Let $\mathcal{V}$ be a one-dimensional homothetic foliation on (M, g). Then, at least away of the points where $\mathcal{V}$ is Riemannian, its orthogonal complement is integrable.*

Proof. By Proposition 1.18 , $\mathcal{V}$ admits a local dilation λ whose gradient is vertical. The points $x \in M$, where $\mathcal{V}$ is not Riemannian are characterised by $(\operatorname{grad}\lambda)_x \neq 0$. Hence, in a neighbourhood of such a point, the level hypersurfaces of λ are integral submanifolds of the horizontal distribution. □

Lemma 2.8. *Let $\varphi : (M^{n+1}, g) \to (N^n, h)$ be a harmonic morphism with one-dimensional fibres. Let λ denote the dilation of φ let V be a (local) vertical vector field on M such that $g(V, V) = \lambda^{2n-4}$.*

Then, the following assertions are equivalent:

(i) *the fibres of φ form a homothetic foliation at least away of the interior of the set $\{x \in M \mid \mathrm{d}(V(\sigma))(x) = 0 \neq (V(\sigma))(x)\}$.*

(ii) *for any basic field X , we have $V(X(\log\lambda)) = 0$.*

Proof. Let μ be the vertical one-form on M such that for any horizontal fields X, Y we have $(\mathcal{L}_V\, g)(X, Y) = \mu(V)g(X, Y)$. Hence, by the definition of λ we have $\mu(V) = -2V(\log\lambda)$.

Let $\mathcal{H}$ be the horizontal distribution and ${}^{\mathcal{H}}\!B$ its second fundamental form. Using (1.2) we obtain the following relation:

$$\mu = -\frac{2}{n}(\operatorname{trace}({}^{\mathcal{H}}\!B))^{\flat} \,.$$

Hence, $\mathcal{V}$ is homothetic if and only if μ is closed.

By Lemma 2.1 , for any basic X we have $[V, X] = 0$, and hence:

$$\begin{aligned}(\mathrm{d}\,\mu)(V, X) &= -V(\mu(X)) - X(\mu(V)) - \mu[V, X] = -X(\mu(V))\\ &= 2X(V(\log\lambda)) = 2V(X(\log\lambda))\,.\end{aligned}$$

The lemma follows. □

In [4, Proposition 3.5] , P. Baird and J. C. Wood gave a *global* description of the metric of a Riemannian manifold of dimension three, on which a harmonic morphism can be defined. In [9, Theorem 1] , R. L. Bryant gave a *local* description of the metric of the total space of a submersive harmonic morphism with one-dimensional fibres (with no restriction on the dimension of the total space). The following theorem explains how this result can be globalized, giving also a simpler proof of Bryant's local result.

Theorem 2.9. *Let* $\varphi : (M^{n+1}, g) \to (N^n, h)\,, n \geq 1\,,$ *be a submersive harmonic morphism with connected one-dimensional fibres of the same homotopy type. Let* λ *be the dilation of* φ *and suppose that* $\mathcal{V}\,(= \ker\varphi_*)$ *is orientable.*

Then, there exists:

(i) *a principal bundle* $\pi : P \to N$ *with group* $G = (\mathbb{R}, +)$ *or* $G = (S^1, \cdot)\,,$

(ii) *a principal connection* $\theta \in \Gamma(T^*P)$ *on* $\pi\,,$

(iii) *a diffeomorphic embedding* $\iota : M \to P$

such that:

1) $\pi \circ \iota = \varphi\,,$

2) $g = \lambda^{-2}(\varphi^* h) + \lambda^{2n-4}(\iota^*\theta)^2\,.$

Furthermore, if the fibres are all diffeomorphic to circles, or are all complete with respect to the metric induced by ${}^{\lambda}g$ *, then* ι *is onto, and hence,* φ *itself is a principal bundle and the horizontal distribution is a principal connection on it.*

Note that by the result of P. Baird [9] we know that φ is automatically submersive except when $n \leq 3$.

Proof. Let V be a vertical field such that $g(V, V) = \lambda^{2n-4}$. By Lemma 2.1, the horizontal distribution $\mathcal{H}$ is invariant under the local flow of V. Thus, the integral curves of V are the fibres of a local principal bundle, and $\mathcal{H}$ is a principal connection on it. If θ is the vertical one-form dual to V then, it is obvious that ${}^{\lambda}g = \varphi^* h + \theta^2$. (This establishes [9, Theorem 1].)

To end the proof we shall prove the following assertions:

(a) If the fibres are diffeomorphic to circles then φ is a principal bundle with group $(S^1, \cdot)$.

(b) If the fibres are diffeomorphic to $\mathbb{R}$ then, there exists a diffeomorphic embedding $\iota : M \to N \times \mathbb{R}$, such that $\pi_1 \circ \iota = \varphi$, and a principal connection on the trivial principal bundle $\pi_1 : N \times \mathbb{R} \to N$, with group $(\mathbb{R}, +)$, such that $\mathcal{H}$ is the restriction to M of it.

From now on all the considerations which will be made in this proof, will be done with respect to the metric ${}^{\lambda}g$ on M.

For $x \in M$, let $I_x \subseteq \mathbb{R}$ be the open interval which is the domain of the (maximal) geodesic with velocity V_x. Let $Q = \big\{(x, r) \in M \times \mathbb{R} \,|\, r \in I_x\big\}$, and define $\Psi : Q \to M$, by $\Psi(x, r) = \exp rV_x$.

If the fibres are all circles then $Q = M \times \mathbb{R}$. Since θ is relatively closed, by Stokes theorem the fibres have the same length. Hence Ψ descends to a map $M \times S^1 \to M$ which is a free action of $(S^1, \cdot)$ on M. Thus assertion (a) is proved.

Suppose now that the fibres are diffeomorphic to $\mathbb{R}$. If they are all complete with respect to the metric induced from ${}^{\lambda}g$ then, $Q = M \times \mathbb{R}$ and Ψ represents a free action of $(\mathbb{R},+)$ on M , and thus the proof of the theorem is finished. Otherwise, since φ is a submersion, we can find local sections of it in the neighbourhood of each point of N. Let $\mathcal{S}$ be a family of such sections whose domains form an open covering $\{O_s\}_{s\in\mathcal{S}}$ of N.

Let $s\,,\,t\in\mathcal{S}$. For $x\in O_s\cap O_t$, let $a_{s\,t}(x)$ be the (unique) real number such that $t(x)=\Psi\big(s(x),a_{s\,t}(x)\big)$.

It is obvious that $\{a_{s\,t}\}_{s,t\in\mathcal{S}}$ is a cocycle with values in $(\mathbb{R},+)$, which induces a principal bundle. This bundle is trivial because $\mathbb{R}$ is contractible.

Moreover, set $A_s=s^*\theta$; then the family of one-forms $\{A_s\}_{s\in\mathcal{S}}$, defines a principal connection on this bundle.

The total space $N\times\mathbb{R}$ of this bundle, can be retrieved, as usual, from the cocycle $\{a_{s\,t}\}_{s,t\in\mathcal{S}}$ as the space of equivalence classes $[x,r]$, under the identifications $[x,r]\equiv[x,a_{s\,t}(x)r]$, $x\in O_s\cap O_t$.

For $x\in M$, let $s\in\mathcal{S}$ be such that $\varphi(x)\in O_s$ and $r_x\in I_{s(\varphi(x))}$ be the real number which satisfies $x=\Psi(s(\varphi(x)),r_x)$. (Note that r_x depends just on x and s.). We can define $\iota:M\to N\times\mathbb{R}$, by $\iota(x)=[\varphi(x),r_x]$, $x\in M$, and the theorem follows. □

Remark 2.10. The proof of above theorem, can be simplified considerably when $\mathcal{H}$ is an Ehresmann connection (see [8] for the definition of Ehresmann connection).

It is not difficult to prove that a sufficient condition for $\mathcal{H}$ to be an Ehresmann connection is that V be a complete vector field.

3. Mixed curvature and harmonic morphisms

To prove the results in this section we shall use a formula which relates the curvature of a Riemannian manifold to the geometric properties of a pair of complementary orthogonal distributions on it. Equivalent forms of it are already known (see [33] and the references therein), a fact which we discovered whilst writing this paper. However, we believe our proof is more direct.

Let (M,g) be a Riemannian manifold and $\mathcal{H}$, $\mathcal{V}$ a pair of complementary orthogonal distributions on it, with $\dim\mathcal{H}=p$ and $\dim\mathcal{V}=q$. As before, $\mathcal{H}$ and $\mathcal{V}$ will be called the *horizontal* and the *vertical* distribution, respectively. Also, the corresponding projections will be denoted with the same letters $\mathcal{H}$ and $\mathcal{V}$ and we shall denote by X, Y horizontal vector fields and by U, V vertical vector fields. Let ${}^{\mathcal{V}}\!B$ and ${}^{\mathcal{V}}\!I$ be the second fundamental form and integrability tensor of $\mathcal{V}$, respectively. Recall that they are the unique $\mathcal{H}$-valued vertical tensor fields which satisfy

$$\begin{aligned}{}^{\mathcal{V}}\!B(U,V)&=\frac{1}{2}\mathcal{H}(\nabla_UV+\nabla_VU)\,,\\{}^{\mathcal{V}}\!I(U,V)&=-\mathcal{H}[U,V]\,.\end{aligned}$$

Note the minus sign above.

Recall that trace$({}^{\mathcal{V}}B)$ is q times the mean curvature of $\mathcal{V}$, i.e. if $\{U_\alpha\}$ is a local orthonormal frame for $\mathcal{V}$ then trace$({}^{\mathcal{V}}B)=\sum_\alpha {}^{\mathcal{V}}B(U_\alpha,U_\alpha)$. We shall also consider the trace free part ${}^{\mathcal{V}}B_0$ defined by

$$ {}^{\mathcal{V}}B_0 = {}^{\mathcal{V}}B - \frac{1}{q}\,\mathrm{trace}({}^{\mathcal{V}}B)\otimes g^{\mathcal{V}} $$

where $g^{\mathcal{V}}$ is the vertical component of g. Also, ${}^{\mathcal{H}}B$, ${}^{\mathcal{H}}I$, ${}^{\mathcal{H}}B_0$ are defined similarly by reversing the roles of $\mathcal{V}$ and $\mathcal{H}$.

As in [33] we denote by s_{mix} the mixed scalar curvature which is the sum of the sectional curvatures of all planes spanned by a horizontal and a vertical vector from an orthonormal frame adapted to the decomposition $TM=\mathcal{H}\oplus\mathcal{V}$. This invariant was introduced in [10].

Proposition 3.1. *With the notations above we have*

$$ \begin{aligned} &\mathrm{div}(\mathrm{trace}({}^{\mathcal{H}}B)) + \mathrm{div}(\mathrm{trace}({}^{\mathcal{V}}B)) + \frac{p-1}{p}\big|\,\mathrm{trace}({}^{\mathcal{H}}B)\big|^2 + \frac{q-1}{q}\big|\,\mathrm{trace}({}^{\mathcal{V}}B)\big|^2 \\ &+\frac{1}{4}\big|{}^{\mathcal{H}}I\big|^2 + \frac{1}{4}\big|{}^{\mathcal{V}}I\big|^2 = \big|{}^{\mathcal{H}}B_0\big|^2 + \big|{}^{\mathcal{V}}B_0\big|^2 + s_{mix}\,. \end{aligned} \tag{3.1} $$

To prove this we first establish a preliminary lemma.

Let $\{E_a\}=\{X_j,U_\alpha\}$ be a local orthonormal frame on (M,g) adapted to the decomposition $TM=\mathcal{H}\oplus\mathcal{V}$. We shall always denote by j, k 'horizontal' indices, by α, β 'vertical' indices whilst a, b, c will be any kind of indices. Let ∇ be the Levi-Civita connection of (M,g) and let (Γ^a_{bc}) be the Christoffel symbols of (M,g) with respect to the chosen local frame (i.e. $\nabla_{E_a}E_b=\Gamma^c_{ba}E_c$, note the order of the indices). Recall that $\Gamma^a_{bc}=-\Gamma^b_{ac}$; we shall frequently use this without comment. Then it is easy to see that we have the following relations:

$$ \begin{aligned} \Gamma^j_{\alpha\beta} &= {}^{\mathcal{V}}B^j_{\alpha\beta} + \frac{1}{2}\,{}^{\mathcal{V}}I^j_{\alpha\beta}\,, \\ \Gamma^\alpha_{jk} &= {}^{\mathcal{H}}B^\alpha_{jk} + \frac{1}{2}\,{}^{\mathcal{H}}I^\alpha_{jk}\,. \end{aligned} \tag{3.2} $$

Lemma 3.2. *With the notations above the following relations hold:*

$$ \begin{aligned} \mathrm{div}(\mathrm{trace}({}^{\mathcal{H}}B)) + |\,\mathrm{trace}({}^{\mathcal{H}}B)|^2 &= \sum_{j,\alpha,\beta}\big(U_\alpha(\Gamma^\alpha_{jj}) - \Gamma^\beta_{\alpha\alpha}\Gamma^\beta_{jj}\big)\,, \\ \mathrm{div}(\mathrm{trace}({}^{\mathcal{V}}B)) + |\,\mathrm{trace}({}^{\mathcal{V}}B)|^2 &= \sum_{j,k,\alpha}\big(X_j(\Gamma^j_{\alpha\alpha}) - \Gamma^k_{jj}\Gamma^k_{\alpha\alpha}\big)\,. \end{aligned} $$

Proof. We will prove just the first formula, the proof of the second one then follows by reversing roles of $\mathcal{V}$ and $\mathcal{H}$.

We have

$$ \mathrm{trace}({}^{\mathcal{H}}B) = \sum_{j,\alpha} {}^{\mathcal{H}}B^\alpha_{jj}\cdot U_\alpha = \sum_{j,\alpha}\Gamma^\alpha_{jj}\cdot U_\alpha\,. $$

For a vector field A, define $A^b_{;a}$ by $\nabla_{E_a} A = A^b_{;a}\, E_b$, then, we have

$$\begin{aligned}
\operatorname{div}(\operatorname{trace}({}^{\mathcal{H}}B)) &= (\operatorname{trace}({}^{\mathcal{H}}B))^a_{;a} = (\operatorname{trace}({}^{\mathcal{H}}B))^j_{;j} + (\operatorname{trace}({}^{\mathcal{H}}B))^\alpha_{;\alpha} \\
&= X_j((\operatorname{trace}({}^{\mathcal{H}}B))^j) \ + \ \Gamma^j_{aj}(\operatorname{trace}({}^{\mathcal{H}}B))^a \\
&+ \ U_\alpha((\operatorname{trace}({}^{\mathcal{H}}B))^\alpha) \ + \ \Gamma^\alpha_{a\alpha}(\operatorname{trace}({}^{\mathcal{H}}B))^a \,.
\end{aligned}$$

Now the first term from the right-hand side is zero and hence we can write

$$\begin{aligned}
\operatorname{div}(\operatorname{trace}({}^{\mathcal{H}}B)) &= \sum_{j,k,\alpha,\beta} (\Gamma^j_{\alpha j}\Gamma^\alpha_{kk} + U_\alpha(\Gamma^\alpha_{kk}) + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{kk}) \\
&= \sum_{j,k,\alpha,\beta} (-\Gamma^\alpha_{jj}\Gamma^\alpha_{kk} + U_\alpha(\Gamma^\alpha_{kk}) - \Gamma^\beta_{\alpha\alpha}\Gamma^\beta_{kk}) \\
&= -|\operatorname{trace}({}^{\mathcal{H}}B)|^2 + \sum_{k,\alpha,\beta} (U_\alpha(\Gamma^\alpha_{kk}) - \Gamma^\beta_{\alpha\alpha}\Gamma^\beta_{kk}) \,.
\end{aligned}$$

The lemma is proved. □

Proof of Proposition 3.1. Let R denote the curvature tensor of (M, g) and $\{\Gamma^a_b\}$ the components of the local connection form of ∇ with respect to $\{E_a\}$ given by $\Gamma^c_b(E_a) = \Gamma^c_{ba}$, then,

$$\begin{aligned}
\sum_{j,\alpha} R_{j\alpha j\alpha} &= \sum_{j,\alpha} R^j_{\alpha j\alpha} = \sum_{j,\alpha,a} (d\Gamma^j_\alpha + \Gamma^j_a \wedge \Gamma^a_\alpha)_{j\alpha} \\
&= \sum_{j,\alpha,a} \{X_j(\Gamma^j_{\alpha\alpha}) - U_\alpha(\Gamma^j_{\alpha j}) - \Gamma^j_\alpha([X_j, U_\alpha]) + \Gamma^j_{aj}\Gamma^a_{\alpha\alpha} - \Gamma^j_{a\alpha}\Gamma^a_{\alpha j}\}
\end{aligned}$$

$$\begin{aligned}
= \sum_{j,k,\alpha,\beta,a} \{X_j(\Gamma^j_{\alpha\alpha}) + U_\alpha(\Gamma^\alpha_{jj}) \quad &- \quad \Gamma^j_{\alpha k} g(X_k, [X_j, U_\alpha]) \\
&- \quad \Gamma^j_{\alpha\beta} g(U_\beta, [X_j, U_\alpha]) - \Gamma^a_{jj}\Gamma^a_{\alpha\alpha} + \Gamma^a_{j\alpha}\Gamma^a_{\alpha j}\}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j,\alpha,a} \{X_j(\Gamma^j_{\alpha\alpha}) + U_\alpha(\Gamma^\alpha_{jj}) - \Gamma^a_{jj}\Gamma^a_{\alpha\alpha}\} \\
&+ \sum_{j,k,\alpha,\beta,a} \{-\Gamma^j_{\alpha k} g(X_k, \nabla_{X_j} U_\alpha - \nabla_{U_\alpha} X_j) - \Gamma^j_{\alpha\beta} g(U_\beta, \nabla_{X_j} U_\alpha - \nabla_{U_\alpha} X_j) + \Gamma^a_{jj}\Gamma^a_{\alpha\alpha}\} \\
&= \operatorname{div}(\operatorname{trace}({}^{\mathcal{H}}B)) + \operatorname{div}(\operatorname{trace}({}^{\mathcal{V}}B)) + |\operatorname{trace}({}^{\mathcal{H}}B)|^2 + |\operatorname{trace}({}^{\mathcal{V}}B)|^2 \\
&+ \sum_{j,k,\alpha,\beta,a} (-\Gamma^j_{\alpha k}\Gamma^k_{\alpha j} + \Gamma^j_{\alpha k}\Gamma^k_{j\alpha} - \Gamma^j_{\alpha\beta}\Gamma^\beta_{\alpha j} + \Gamma^j_{\alpha\beta}\Gamma^\beta_{j\alpha} + \Gamma^a_{j\alpha}\Gamma^a_{\alpha j}) \\
&= \operatorname{div}(\operatorname{trace}({}^{\mathcal{H}}B)) + \operatorname{div}(\operatorname{trace}({}^{\mathcal{V}}B)) + |\operatorname{trace}({}^{\mathcal{H}}B)|^2 + |\operatorname{trace}({}^{\mathcal{V}}B)|^2 \\
&- \sum_{j,k,\alpha,\beta} (\Gamma^\alpha_{jk}\Gamma^\alpha_{kj} + \Gamma^j_{\alpha\beta}\Gamma^j_{\beta\alpha}) + \sum_{j,k,\alpha,\beta,a} (-\Gamma^j_{\alpha k}\Gamma^j_{k\alpha} - \Gamma^\alpha_{j\beta}\Gamma^\alpha_{\beta j} + \Gamma^a_{j\alpha}\Gamma^a_{\alpha j}) \,.
\end{aligned}$$

Since the last sum is zero, using relations (3.2) we obtain the left-hand side of the formula. □

Application. When $\dim M = 2$, the formula says that, if on a surface we have a pair of orthogonal foliations by curves, then the sum of the divergences of the curvature vectors of the curves at their intersection point is equal to the Gauss curvature of the surface at that point. Hence, if a closed surface admits a one-dimensional foliation, then, since its orthogonal complement is another one-dimensional foliaton, by the divergence theorem the total curvature of any metric on it is zero. This gives a particular case of the well-known Hopf theorem, that the Euler number of a closed manifold which admits a nonvanishing tangent vector field is zero.

Suppose for the rest of this section that $\dim M \geq 3$. The following Proposition is a generalization to conformal one-dimensional foliations of the corresponding results for Riemannian one-dimensional foliations from [31].

Proposition 3.3. *Let (M, g) be compact.*

(i) *If (M, g) has nonpositive Ricci curvature, then any conformal one-dimensional foliation is Riemannian and its orthogonal complement is a totally geodesic foliation. Further,* $\mathrm{Ricci}(U, U) = 0$ *for any U tangent to the foliation.*

(ii) *If (M, g) has negative Ricci curvature then there exists no one-dimensional conformal foliation on it.*

Proof. By passing to a finite covering, if necessary, we can suppose that both the foliation $\mathcal{V}$ and the manifold M are oriented.

Since $\mathcal{V}$ is conformal we have ${}^{\mathcal{H}}B_0 = 0$. But, as for any codimension one foliation, $\mathcal{H}$ is also conformal. Hence ${}^{\mathcal{V}}B_0 = 0$.

Next, note that, because $\mathcal{V}$ is one-dimensional, the mixed curvature is equal to the Ricci curvature restricted to $\mathcal{V}$.

Thus integrating (3.1) gives

$$\int_M \mathrm{Ricci}(U, U)\, v_g = \int_M \left\{ \frac{p-1}{p} |\,\mathrm{trace}({}^{\mathcal{H}}B)|^2 + \frac{1}{4} |{}^{\mathcal{H}}I|^2 \right\} v_g$$

where U is a unit vector field tangent to $\mathcal{V}$ and v_g is the volume element of (M, g).

The proposition follows. $\square$

By a well-known result of S. Bochner (see [21, Ch.II, Theorem 4.3]) any Killing field on a compact Riemannian manifold with nonpositive Ricci tensor is parallel. The following theorem can be viewed as an extension of that result.

Theorem 3.4. *On a compact Riemannian manifold with nonpositive Ricci curvature any one-dimensional foliation which produces harmonic morphisms and admits a global density is locally generated by parallel vector fields. In particular, it is Riemannian, has geodesic leaves and its orthogonal complement is a totally geodesic Riemannian foliation. Hence the foliation corresponds to a local Riemannian product structure of the manifold. In particular, the universal cover of (M, g) is a Riemannian product. If M is simply-connected or the foliation is* *simple and the base space is simply-connected, then the foliation corresponds to a Riemannian product structure on* (M, g) .

Proof. If the dimension of the manifold is three then the leaves are geodesics. This together with Proposition 3.3 , gives the result.

Assume that the manifold has dimension greater than three. As before, by passing to a finite covering, if necessary, we can suppose that both the foliation $\mathcal{V}$ and the manifold M are oriented.

By Proposition 3.3 the foliation is Riemannian; hence, by the proof of Proposition 2.3 , it is globally generated by a Killing field, namely $\rho^{-1}U$ where ρ is a global density for $\mathcal{V}$ and U is a unit vertical vector field. Now, Bochner's result mentioned above implies that the foliation is generated by parallel vector fields. Hence $\mathcal{V}$ is a Riemannian foliation by geodesics and its orthogonal complement is a totally geodesic Riemannian foliation.

The fact that $\mathcal{V}$ induces on the universal cover of (M, g) a Riemannian product structure follows from the de Rham decomposition theorem. If the foliation is simple then the leaves are compact and hence any curve in the base space admits (global) horizontal lifts, which induce an isometry between the fibres over the endpoints of the curve. Since the horizontal distribution is integrable, this isometry depends only on the homotopy class of the curve. It follows that when the base space is simply-connected, this isometry depends only on the two fibres and the theorem is proved. □

Remark 3.5. 1) From the proof of Corollary 1.14 it follows that any local density of a foliation which produces harmonic morphisms corresponds to a local section of a regular covering $\xi \in H^1(M;\mathbb{R})$. Thus in the hypothesis of Theorem 3.4 we can replace the condition of the existence of a global density with the condition that the holonomy group of ξ to be finite.

2) In Proposition 3.3(i) and Theorem 3.4 we can replace the condition on the Ricci curvature by the condition: $\int_M \mathrm{Ricci}(U, U)\, v_g \leq 0$ for any vector field U tangent to the foliation.

3) The above mentioned result of S. Bochner can be proved by using (3.1) . In fact, if V is a Killing field on (M, g) which generates the (possibly singular) foliation $\mathcal{V}$, and $\sigma = |V|$ then σ^{-1} is a dilation for the homothetic distribution $\mathcal{H}$ (see the proof of Proposition 2.3). By (1.9) we have $\mathrm{trace}({}^{\mathcal{V}}B) = \mathrm{grad}(\log(\sigma^{-1}))$. It is easy to see that, in this case, (3.1) gives (sign convention for the Laplacian as in [7])

$$\sigma\,\Delta\sigma + |\,\mathrm{grad}\,\sigma|^2 + \frac{1}{4}\,\sigma^2\,\left|{}^{\mathcal{H}}I\right|^2 = \mathrm{Ricci}(V, V)\ . \tag{3.3}$$

If σ attains a maximum at a point where V is not zero then the left hand side of (3.3) is nonnegative from which the result follows.

4) Recall that, by another well-known result of S. Bochner, on a compact

Riemannian manifold with positive Ricci curvature there exists no harmonic one-forms (in particular, the first Betti number of such a manifold is zero). As is well-known (see [7]) this can be proved by using the Weitzenböck formula for the Hodge Laplacian acting on exterior forms. Also formula (3.1) can be obtained from the Weitzenböck formula applied to a local volume form of one of the two distributions.

By Corollary 1.14, any foliation which produces harmonic morphisms on a simply-connected manifold admits a global density and hence in this case the hypotheses of the above theorem can be weakened. Also, we have the following:

Corollary 3.6. *Any nonconstant submersive harmonic morphism with fibres of dimension one which is defined on a compact Riemannian manifold such that the Ricci curvature* $\mathrm{Ricci}(U,U)$ *is nonpositive when U is tangent to the fibres is totally geodesic (up to a conformal transformation of the codomain if this is two-dimensional). Hence, if the total space or the base space is simply-connected, up to a homothety of the codomain (up to a conformal transformation of the codomain if this is two-dimensional), it is a projection of a Riemannian product.*

In order to apply it to *nonnegative* curvature, note that formula (3.1) can also be written

$$\begin{aligned}&\mathrm{div}(\mathrm{trace}({}^{\mathcal{H}}\!B)) + \mathrm{div}(\mathrm{trace}({}^{\mathcal{V}}\!B)) + \big|\,\mathrm{trace}({}^{\mathcal{H}}\!B)\big|^2 + \big|\,\mathrm{trace}({}^{\mathcal{V}}\!B)\big|^2\\ &\qquad + \frac{1}{4}\big|{}^{\mathcal{H}}\!I\big|^2 + \frac{1}{4}\big|{}^{\mathcal{V}}\!I\big|^2 = \big|{}^{\mathcal{H}}\!B\big|^2 + \big|{}^{\mathcal{V}}\!B\big|^2 + s_{mix}\,.\end{aligned} \tag{3.4}$$

The next result applies to arbitrary foliations, not necessary conformal.

Proposition 3.7. *Let (M,g) be a compact Riemannian manifold.*

(i) *Let $\mathcal{V}$ and $\mathcal{H}$ be two complementary orthogonal foliations whose mean curvatures are (globally) gradient vector fields. If the mixed curvature is nonnegative then $\mathcal{V}$ and $\mathcal{H}$ are totally geodesic and hence they induce on (M,g) a local Riemannian product structure. Thus, the universal cover of (M,g) is globally a Riemannian product.*

(ii) *If the mixed curvature is positive then there exists no pair of complementary orthogonal foliations on (M,g) for which the mean curvatures are gradient vector fields.*

Proof. If $\mathrm{trace}({}^{\mathcal{H}}\!B) = \mathrm{grad}(\log u)$ and $\mathrm{trace}({}^{\mathcal{V}}\!B) = \mathrm{grad}(\log v)$ for some smooth positive functions u and v on M, then, (3.4) gives the following:

$$-\Delta(\log u) - \Delta(\log v) + |\,\mathrm{grad}(\log u)|^2 + |\,\mathrm{grad}(\log u)|^2 = |{}^{\mathcal{H}}\!B|^2 + |{}^{\mathcal{V}}\!B|^2 + s_{mix}\,. \tag{3.5}$$

Equation (3.5) can be written as follows:

$$-u^{-1}\Delta u - v^{-1}\Delta v = |{}^{\mathcal{H}}\!B|^2 + |{}^{\mathcal{V}}\!B|^2 + s_{mix}\,. \tag{3.6}$$

Since $\operatorname{grad} u$ and $\operatorname{grad} v$ are orthogonal (the former being vertical whilst the latter is horizontal) relation (3.6) can be written as follows:

$$-u^{-1}v^{-1}\Delta(uv) = |{}^{\mathcal{H}}B|^2 + |{}^{\mathcal{V}}B|^2 + s_{mix}\,.$$

The proof follows by multiplying by $u\,v$ and integrating over M. □

Corollary 3.8. *Let (M,g) be compact and with zero first Betti number. Let $\mathcal{V}$ be a homothetic foliation with* $\operatorname{codim}\mathcal{V} \neq 2$ *which produces harmonic morphisms on (M,g) and has integrable orthogonal complement.*

(i) *Then the total mixed curvature $\int_M s_{mix}\, v_g$ is nonpositive.*

(ii) *If the mixed curvature is nonnegative then it is identically zero and $\mathcal{V}$ and $\mathcal{H}$ are totally geodesic. Hence, the universal cover of (M,g) is globally a Riemannian product.*

On a compact Riemannian manifold with positive sectional curvature there exists no homothetic foliation which produces harmonic morphisms and has integrable orthogonal complement.

In Theorem 4.12(i) we shall prove that for one-dimensional foliations the last assertion from above proposition is true, without the integrability assumption on $\mathcal{H}$, when $\dim M$ is even and greater than two.

Proof. Since $\mathcal{V}$ is homothetic the mean curvature form of $\mathcal{H}$ is closed. But $\mathcal{V}$ produces harmonic morphisms and hence, by Corollary 1.17, the mean curvature form of $\mathcal{V}$ is also closed. Since the first Betti number of M is zero both mean curvatures are globally gradient vector fields. The proof follows from Proposition 3.7. □

Remark 3.9. 1) Recall (Remark 1.25) that Riemannian foliations are homothetic, as are the foliations with minimal leaves of codimension not equal to two and which produce harmonic morphisms.

2) If $\operatorname{codim}\mathcal{V} = 1$ then the integrability assumption on the orthogonal complement of $\mathcal{V}$, made above, can be removed. Further, the mixed curvature is equal to the restriction of the Ricci curvature to its orthogonal complement.

3) Corollary 3.8 admits further consequences in a similar way to Corollary 3.6.

Theorem 3.10. *Let (M,g) be a compact Riemannian manifold of dimension at least four, with zero first Betti number and with Ricci curvature of constant sign.*

Then, there exists no orientable one-dimensional homothetic foliation which produces harmonic morphisms on (M,g) and which has integrable orthogonal complement.

Note that, by Lemma 2.7, the integrability of the orthogonal complement above is automatic except on the set where $\mathcal{V}$ is Riemannian.

Proof. Suppose that there exists a foliation $\mathcal{V}$ with the stated properties. By Corollary 3.8 the Ricci curvature of (M,g) is nonpositive. Since M has zero Betti number, $\mathcal{V}$ admits a global density. From Theorem 3.4 it follows that $\mathcal{V}$

is locally generated by parallel vector fields. Moreover, being orientable and admiting a global density, as in the proof of Theorem 3.4, $\mathcal{V}$ must be *globally* generated by a parallel vector field. Hence, the first Betti number of M is nonzero. Contradiction! $\square$

Corollary 3.11. *On a compact Riemannian manifold with positive Ricci curvature, there exists no nonvanishing Killing field with integrable orthogonal complement.*

The following immediate consequence of Proposition 3.1 slightly improves Proposition 5.9 and Proposition 5.10 from [35].

Corollary 3.12. *If $\mathcal{V}$ is a Riemannian foliation on (M,g), and $s_{mix}<0$ at least at one point of M, then $\mathcal{V}$ cannot be totally geodesic.*

4. Harmonic morphisms with one-dimensional fibres defined on compact manifolds

Throughout this section $\varphi:(M^{n+1},g)\to(N^n,h)\,,n\geq1$, will denote a nonconstant harmonic morphism defined on a compact Riemannian manifold. Recall that, by a result of P. Baird [1], φ is automatically submersive if $n\geq4$. Since closed, all the fibres of φ are compact. As is well known [13] if φ is nonconstant then it is open, hence it is surjective and N is also compact. Let λ denote the dilation of φ; we shall denote by the same letter h the metric on N and the metric ${}^{\lambda}g$ on M of Definition 1.10. This metric should be seen just as an auxiliary tool and thus, whenever we denote a geometric object on the total space of φ without mentioning a metric then it should be understood that the metric considered is g.

Definition 4.1. We define the *mass of a (regular) fibre* of φ to be the positive number

$$\mathfrak{m}=\int_{\mathrm{fibre}}\lambda^{2-n}\,v_{\mathrm{fibre}}\,.$$

Where v_{fibre} is the volume measure of the fibre induced by the metric (see [23]).

By Proposition 1.6 the mass is independent of the (regular) fibre and it can be defined without any restriction on the dimensions.

Theorem 4.2. *Let $\varphi:(M^{n+1},g)\to(N^n,h)\,,(n\geq1)$ be a submersive harmonic morphism and let $\mathcal{S}^M=\int_M s^M\,v_g$, $\mathcal{S}^N=\int_N s^N\,v_h$ be the total scalar curvature of (M,g) and (N,h), respectively. Then*

$$\begin{aligned}\mathcal{S}^M-\mathfrak{m}\,\mathcal{S}^N&=n(n-1)\|\mathcal{V}(\mathrm{grad}_g(\log\lambda))\|^2\\&\quad-(n-1)(n-2)\|\mathcal{H}(\mathrm{grad}_g(\log\lambda))\|^2-\frac{1}{4}\|I\|^2\,.\end{aligned}\tag{4.1}$$

Let $n \neq 2$. Then

$$
\begin{aligned}
(4.2)\qquad \int_M \lambda^2 (s^M - \lambda^2\, s^N)\, v_g = n(n-5) \int_M \lambda^2 |\mathcal{V}(\mathrm{grad}_g(\log\lambda))|^2\, v_g \\
-(n^2-3n+6)\int_M \lambda^2 |\mathcal{H}(\mathrm{grad}_g(\log\lambda))|^2\, v_g - \frac{1}{4}\int_M \lambda^2 |I|^2\, v_g\,.
\end{aligned}
$$

Here, $I = {}^{\mathcal{H}}I$ is the integrability tensor of the horizontal distribution $\mathcal{H}$.

Before proving this we need some preparations. The following lemma can be obtained by a straightforward computation.

Lemma 4.3. *Let $\sigma = \log\lambda$, and (as before) let V be a local vertical field such that $g(V,V) = e^{(2n-4)\sigma}$, and θ its dual vertical 1-form. Then, for any basic X and Y we have:*

$$
\begin{aligned}
\mathcal{H}({}^g\nabla_X Y) =& \mathcal{H}({}^h\nabla_X Y) - X(\sigma)Y - Y(\sigma)X + \mathcal{H}(\mathrm{grad}_h\,\sigma)h(X,Y)\,,\\
\mathcal{V}({}^g\nabla_X Y) =& \big\{e^{(-2n+2)\sigma}\, V(\sigma)h(X,Y) - \frac{1}{2}\,\Omega(X,Y)\big\}V\,,\\
\mathcal{H}({}^g\nabla_V X) =& -V(\sigma)X + \frac{1}{2}\,e^{(2n-2)\sigma}\,(i_X\Omega)^{\#_h}\,,\\
\mathcal{V}({}^g\nabla_V X) =& (n-2)X(\sigma)V\,,\\
\mathcal{H}({}^g\nabla_V V) =& -(n-2)e^{(2n-2)\sigma}\,\mathcal{H}(\mathrm{grad}_h\,\sigma)\,,\\
\mathcal{V}({}^g\nabla_V V) =& (n-2)V(\sigma)V\,.
\end{aligned}
$$

Here, ${}^g\nabla$ and ${}^h\nabla$ are, respectively, the Levi-Civita connections of (M,g) and (M,h), and $\Omega = \mathrm{d}\,\theta$.

We shall also use the *horizontal Laplacian* of the associated Riemannian submersion with geodesic fibres. This was introduced in [6] and it can be defined as follows:

Definition 4.4. If $\varphi : M \to (N,h)$ is a submersion endowed with a distribution $\mathcal{H}$ which is complementary to $\ker\varphi_*$, then the *horizontal Laplacian* $\varphi^*\Delta^N$ is the second-order differential operator which on a local function f defined in the neighbourhood of the point $x \in M$ acts as follows:

$$
(\varphi^*\Delta^N)(f) = -\sum_j \big\{X_j(X_j(f)) - ((\varphi^*\nabla^N)_{X_j}X_j)(f)\big\}\,;
$$

here, $\{X_j\}$ is a local orthonormal frame of $\mathcal{H}$ (considered with the metric induced by h) formed of basic vector fields (i.e. sections of $\mathcal{H}$ which are projectable by φ to vector fields on N), and ∇^N is the Levi-Civita connection of (N,h).

Remark 4.5. Note that $(\varphi^*\Delta^N)(f\circ\varphi) = (\Delta^N f)\circ\varphi$ for any local smooth function f on N.

Lemma 4.6. *Let $\varphi : (M^{n+1}, g) \to (N^n, h)$ be a submersive harmonic morphism and let Δ^M and Δ^N be the Laplace operators on (M, g) and (N, h), respectively. Then*

$$\Delta^M f = e^{2\sigma}\,(\varphi^*\Delta^N)(f) - e^{(-2n+4)\sigma}\left\{V(V(f)) - 2(n-1)V(f)V(\sigma)\right\}.$$

Remark 4.7. Note that $V(V(f))$ is just minus the 'vertical' Laplacian [6] applied to f, of the Riemannian submersion with geodesic fibres associated to $\varphi : (M, g) \to (N, h)$. More generally, the 'vertical' Laplacian of $\varphi : (M, g) \to (N, h)$ is defined by $(\Delta^{\text{fibre}} f)(x) = (\Delta^{\varphi^{-1}(\varphi(x))}(f|_{\varphi^{-1}(\varphi(x))}))(x)$ where $\Delta^{\varphi^{-1}(\varphi(x))}$ is the Laplacian of the fibre through x endowed with the metric induced by g. If φ is a Riemannian submersion with totally geodesic fibres, then the sum of the horizontal and the vertical Laplacians is equal to the Laplacian of the total space.

Proof of Theorem 4.2. Recall from the previous section that s_{mix} is the sum of the sectional curvatures of all planes on (M, g) spanned by a horizontal and a vertical vector from an orthonormal frame adapted to the decomposition $TM = \mathcal{H} \oplus \mathcal{V}$. Let $s^{\mathcal{H}}$ denote twice the sum of the sectional curvatures of all planes on (M, g) spanned by the horizontal vectors of a frame as above.

Using the previous two lemmas and the fact that $I = V \otimes \Omega$, after a straightforward computation the following relation can be obtained. (Another way to obtain it is to use the previous two lemmas together with Corollary 2.2.4, from [17].)

$$\begin{aligned} s^{\mathcal{H}} - e^{2\sigma} s^N &= -2(n-1)\Delta^M\sigma - (n-1)(n-2)e^{2\sigma}|\mathcal{H}(\operatorname{grad}_h\sigma)|^2 \\ &\quad -2(n-1)e^{(-2n+4)\sigma}\,V(V(\sigma)) + (3n-4)(n-1)e^{(-2n+4)\sigma}\,V(\sigma)^2 - \frac{3}{4}|I|^2 . \end{aligned} \tag{4.3}$$

Using (1.2), (1.3) together with Lemma 4.6 and (3.1), we obtain:

$$\begin{aligned} s_{mix} &= (n-2)\Delta^M\sigma + 2(n-1)e^{(-2n+4)\sigma}\,V(V(\sigma)) \\ &\qquad -(3n-4)(n-1)e^{(-2n+4)\sigma}\,V(\sigma)^2 + \frac{1}{4}|I|^2 . \end{aligned} \tag{4.4}$$

But it is obvious that $s^M = s^{\mathcal{H}} + 2s_{mix}$ and hence from (4.3) and (4.4) we obtain

$$\begin{aligned} s^M - e^{2\sigma} s^N &= -2\Delta^M\sigma + \frac{2(n-1)}{n}e^{n\sigma}\Delta^{\text{fibre}}(e^{-n\sigma}) \\ &\quad + n(n-1)|\mathcal{V}(\operatorname{grad}_g\sigma)|^2 - (n-1)(n-2)|\mathcal{H}(\operatorname{grad}_g\sigma)|^2 - \frac{1}{4}|I|^2 . \end{aligned} \tag{4.5}$$

Integrating (4.5) gives (4.1). Relation (4.5) can also be written as follows:

$$\begin{aligned} e^{2\sigma} s^M - e^{4\sigma} s^N &= -\Delta^M(e^{2\sigma}) + \frac{2(n-1)}{n-2}e^{n\sigma}\Delta^{\text{fibre}}(e^{(-n+2)\sigma}) \\ &\quad + n(n-5)e^{2\sigma}|\mathcal{V}(\operatorname{grad}_g\sigma)|^2 - (n^2-3n+6)e^{2\sigma}|\mathcal{H}(\operatorname{grad}_g\sigma)|^2 - \frac{1}{4}e^{2\sigma}|I|^2 . \end{aligned} \tag{4.6}$$

Integrating (4.6) gives (4.2), since

$$\int_M e^{n\sigma}\Delta^{\mathrm{fibre}}(e^{(-n+2)\sigma})\,v_g=\int_N v_h\int_{\mathrm{fibre}}\Delta^{\mathrm{fibre}}(e^{(-n+2)\sigma})\,v_{\mathrm{fibre}}=0$$

□

Remark 4.8. 1) Suppose that $n=1$, i.e. $\varphi:(M^2,g)\to\mathbb{R}$ is a harmonic function defined on the surface (M^2,g). Then, equation (4.4) above reads:

$$K=-\Delta(\log|\mathrm{d}\,\varphi|)\,,$$

where, K is the Gauss curvature of (M,g). As is well-known this can be proved, also, by using the local isothermal coordinates induced by φ.

2) Computing $\lambda^2(s^{\mathcal{H}}+s_{mix})$, by adding (4.3) and (4.4) from the above proof, there can be obtained formula (2.2) from [26] applied to harmonic morphisms with fibres of dimension one.

Proposition 4.9. *Let $\varphi:(M^{n+1},g)\to(N^n,h)$, $n\geq 2$ be a submersive harmonic morphism. If U is a unit vector field tangent to the fibres of φ then*

$$\mathcal{S}^M\leq\mathfrak{m}\,\mathcal{S}^N+\int_M \mathrm{Ricci}(U,U)\,v_g$$

and equality holds if and only if φ has geodesic fibres and $\mathcal{H}$ is integrable.

Note that since, $\mathrm{Ricci}(U,U)$ is quadratic in U, we do not need $\mathcal{V}$ to be orientable.

Proof. First recall that $\mathrm{Ricci}(U,U)=s_{mix}$, then take the sum of (4.3) and (4.4) and use the definition of $\mathfrak{m}$. □

Corollary 4.10. *Let $\varphi:(M^{n+1},g)\to(N^n,h)$, $n\geq 2$, be a submersive harmonic morphism.*

(i) *If φ induces a Riemannian foliation on (M,g) then $\mathcal{S}^M\leq\mathfrak{m}\,\mathcal{S}^N$ and equality holds if and only if φ is totally geodesic (up to a conformal transformation of the codomain if $n=2$).*

(ii) *If φ has geodesic fibres and $\mathcal{H}$ is integrable then $\mathcal{S}^M\geq\mathfrak{m}\,\mathcal{S}^N$ and equality holds if and only if φ is totally geodesic (up to a conformal transformation of the codomain if $n=2$).*

From Lemma 2.7 it follows that when the set of the points where $\mathcal{V}$ is Riemannian has measure zero then the integrability assumption on $\mathcal{H}$ in (ii) is superfluous.

Proof. (i) This is a trivial consequence of formula (4.1) from Theorem 4.2.
(ii) If $n=2$, then (4.1) from Theorem 4.2 gives the result. If $n\neq 2$, by Proposition 1.5(b) we have $\mathcal{H}(\mathrm{grad}_g\,\lambda)=0$. Now apply formula (4.1). □

Corollary 4.11. *If $n\in\{3,4,5\}$ then $\int_M\lambda^2(s^M-\lambda^2\,s^N)\,v_g\leq 0$. For $n\in\{3,4\}$ equality holds if and only if φ is totally geodesic and for $n=5$ equality holds if and only if φ has geodesic fibres and $\mathcal{H}$ is integrable.*

Therefore, for $n \in \{3,4,5\}$ *if* (M^{n+1}, g) , (N^n, h) *are compact with* $s^M \geq 0$, $s^N \leq 0$ *and at least one of these inequalities is strict then there exists no nonconstant submersive harmonic morphism* $\varphi : (M^{n+1}, g) \to (N^n, h)$.

This improves Theorem 2.5 from [26] for the dimensions considered.

Proof. This is an immediate consequence of formula (4.2) from Theorem 4.2. □

To end this section we prove two results on homothetic one-dimensional foliations which produce harmonic morphisms on compact manifolds, the first of them being a generalization (refered to in Section 3) of a well-known result of M. Berger (see [21, Ch.II, Corollary 5.7]) concerning Killing fields. To prove the first of these two results we shall use a computation (Lemma 5.1) from the next section.

Theorem 4.12. *Let* M *be compact with dimension at least four.*

(i) *If* $\dim M$ *is even and* (M, g) *has positive sectional curvature then there exists no homothetic one-dimensional foliation which produces harmonic morphisms on* (M, g).

(ii) *If* (M, g, J) *is Kähler and has zero first Betti number then any homothetic one-dimensional foliation which produces harmonic morphisms on* (M, g) *is Riemannian and locally generated by Killing fields.*

Proof. (i) Since (M, g) has positive sectional curvature it has, in particular, positive Ricci curvature. Thus from Bochner's result (see Remark 3.5(4)) it follows that the first Betti number of M is zero .

Suppose that there exists a homothetic foliation $\mathcal{V}$ which produces harmonic morphisms on (M, g) . Because the first Betti number of M is zero, by Corollary 1.14 , $\mathcal{V}$ admits a global density λ^{2-n} , where $n + 1 = \dim M$. We shall denote by $h = {}^{\lambda}g$ the associated Riemannian metric on M which makes $\mathcal{V}$ a Riemannian foliation with geodesic leaves.

Since $\mathcal{V}$ is homothetic, by Proposition 1.18 , λ is of the form e^{a+b} with $(\mathrm{d}\, a)^{\mathcal{V}} = (\mathrm{d}\, b)^{\mathcal{H}} = 0$. At a point $x \in M$ where $a - b$ attains a minimum we have

$$\begin{aligned} 0 &\leq ({}^{h}\nabla \,\mathrm{d}\, a)(V, V) - ({}^{h}\nabla \,\mathrm{d}\, b)(V, V) = -({}^{h}\nabla \,\mathrm{d}\, b)(V, V) \, , \\ 0 &\leq ({}^{h}\nabla \,\mathrm{d}\, a)(X, X) - ({}^{h}\nabla \,\mathrm{d}\, b)(X, X) = ({}^{h}\nabla \,\mathrm{d}\, a)(X, X) \, , \end{aligned} \tag{4.7}$$

where, V is as before and X is any horizontal vector at x .

Now, evaluated at x , the first formula from Lemma 5.1 gives

$$\begin{aligned} R(X, V, X, V) &= -(n-2) e^{(2n-4)(a+b)} \, ({}^{h}\nabla \,\mathrm{d}\, a)(X, X) \\ &\quad + e^{-2(a+b)} \, ({}^{h}\nabla \,\mathrm{d}\, b)(V, V) \, h(X, X) + \frac{1}{4} e^{(4n-6)(a+b)} \, h(i_X \Omega, i_X \Omega) \, , \end{aligned} \tag{4.8}$$

for any horizontal vector X.

Since Ω is skew-symmetric it must have even rank at each point. But $\dim M$ is even and $i_V \Omega = 0$ and hence there must be a horizontal vector X at x such

that $i_X\Omega = 0$. By (4.7) and (4.8), the sectional curvature of the plane spanned by X and V would be nonpositive, contradicting the hypothesis.
(ii) Since the first Betti number of M is zero the foliation $\mathcal{V}$ must admit a global density. By passing to a two-fold covering if necessary, we can suppose that $\mathcal{V}$ is oriented. Hence, by Proposition 2.5, it must be generated by a conformal vector field. But by a well-known result of A. Lichnerowicz (see [7, 2.125(ii)]) on a compact Kähler manifold of complex dimension greater than two any conformal vector field is Killing.

The theorem is proved. □

Remark 4.13. 1) Note that statement (i) from Theorem 4.12 fails if $\dim M$ is *odd*, for example the Hopf maps $S^{2n+1} \to \mathbb{C}P^n$ are harmonic Riemannian submersions.

2) In Theorem 4.12(i), if the sectional curvature K of (M^{n+1}, g) satisfies $K \geq a^2 > 0$ on M for some positive constant a, then the compactness assumption on M can be replaced by the weaker condition that (M, g) is complete. This follows from a well-known result of S. B. Myers (see [7, 6.51]), because in this case the Ricci curvature is $\geq na^2\, g$. (A similar remark can be made for Corollary 3.11.)

Corollary 4.14. *Let (M, g) be a compact Riemannian manifold of even dimension at least four, with zero first Betti number and whose sectional curvature has constant sign.*

Then, there exists no orientable one-dimensional homothetic foliation which produces harmonic morphisms on (M, g).

Proof. This follows from Theorem 3.4 and Theorem 4.12(i). □

5. Some local and global results for one-dimensional foliations which produce harmonic morphisms

All of the main results of this section hold for Riemannian manifolds of dimension at least four. None of the results of this section requires the compactness or the completeness of the manifold.

If the manifold has dimension at least four then by Proposition 2.3 a one-dimensional Riemannian foliation produces harmonic morphisms on it if and only if it is locally generated by Killing fields and from Corollary 1.24 it follows that a foliation by geodesics produces harmonic morphisms if and only if it is homothetic.

The first main result of this section concerns a one-dimensional foliation $\mathcal{V}$ which has integrable orthogonal complement and which produces harmonic morphisms on an Einstein manifold. We prove that $\mathcal{V}$ is one of these two types. Also, we prove that this still holds if we replace the integrability assumption on the orthogonal complement by the condition that the foliation be homothetic.

In [9, Theorem 3] R. L. Bryant considered a submersive harmonic morphism φ

with connected one-dimensional fibres defined on a simply-connected Riemannian manifold of dimension at least four and with constant sectional curvature. Bryant's result is that φ is one of the following two types: either there exists a nonvanishing Killing field tangent to the fibres of φ or the fibres are geodesics orthogonal to an umbilical foliation by hypersurfaces. We improve this result by showing that, on a Riemannian manifold with constant sectional curvature any orientable one-dimensional foliation which produces harmonic morphisms and admits a global density is either Riemannian and its leaves are generated by a nonvanishing Killing field or is homothetic and the leaves are geodesics orthogonal to an umbilical foliation by hypersurfaces. In this way an entirely new proof for Bryant's result is obtained.

In this section $\mathcal{V}$ will always denote a one-dimensional foliation which produces harmonic morphisms on (M^{n+1}, g) $(n \geq 1)$ and $\rho = e^{(2-n)\sigma}$ will denote a local density of it. As before, $h = {}^{e^{\sigma}}g$ will denote the associated (local) metric on M with respect to which $\mathcal{V}$ is Riemannian and has geodesic leaves and $\mathcal{H}$ will denote the orthogonal complement of $\mathcal{V}$.

Using Lemma 4.3, the following lemma can be obtained by a straightforward computation. Another way to obtain it is to use Lemma 4.3 and S. Gudmundsson's fundamental equations of a horizontally conformal submersion [17].

Lemma 5.1. *Let X, Y, Z be horizontal and V vertical vectors on M, such that $g(V,V) = e^{(2n-4)\sigma}$; then the curvature tensor of (M,g) has the following components*

$$\begin{aligned} R(X,V,Y,V) = &-\frac{1}{2}(n-2)e^{(2n-4)\sigma}(\mathcal{L}_{\mathcal{H}(\mathrm{grad}_h \sigma)}\, h)(X,Y) \\ &-(n-2)e^{(2n-4)\sigma}\{nX(\sigma)Y(\sigma) - |\mathcal{H}(\mathrm{grad}_h \sigma)|_h^2\, h(X,Y)\} \\ &+e^{-2\sigma}\{V(V(\sigma)) - (n-1)V(\sigma)^2\}h(X,Y) \\ &+\frac{1}{4}\,e^{(4n-6)\sigma}\, h(i_X\Omega, i_Y\Omega)\,, \end{aligned} \tag{5.1}$$

$$\begin{aligned} R(X,Y,Z,V) = &-\frac{1}{2}e^{(2n-4)\sigma}({}^h\nabla\Omega)(X,Y,Z) \\ &+\frac{1}{2}(n-1)e^{(2n-4)\sigma}\{X(\sigma)\Omega(Y,Z) + Y(\sigma)\Omega(Z,X) - 2Z(\sigma)\Omega(X,Y)\} \\ &-e^{-2\sigma}\{X(V(\sigma)) - (n-2)X(\sigma)V(\sigma)\}h(Y,Z) \\ &+e^{-2\sigma}\{Y(V(\sigma)) - (n-2)Y(\sigma)V(\sigma)\}h(X,Z) \\ &+\frac{1}{2}e^{(2n-4)\sigma}\{\Omega(X, \mathrm{grad}_h \sigma)h(Y,Z) - \Omega(Y, \mathrm{grad}_h \sigma)h(X,Z)\}\,. \end{aligned} \tag{5.2}$$

Here, as before, ${}^h\nabla$ denote the Levi-Civita connection of (M,h).

A similar, but longer, formula can be given for the case $R(X,Y,Z,H)$ when X, Y, Z, H are horizontal [30].

The first formula of the following lemma follows after a straightforward computation using (5.1) and Gudmundsson's formula given in [17, Theorem 2.2.3(5)]. The second formula follows from (5.2).

Lemma 5.2. *Suppose that $\mathcal{V}$ restricted to the domain of the local density $\rho = e^{(2-n)\sigma}$ is a simple foliation (i.e. the leaves are the fibres of a submersion) and let $\varphi : (O, g|_O) \to (N, \bar{h})$ be the induced harmonic morphism. If* Ricci *denotes the Ricci tensor of (M, g) and* ^{N}Ricci *denotes the Ricci tensor of $(N, \bar{h})$, then,*

$$\begin{aligned}\mathrm{Ricci}(X,Y) &= ({}^N\mathrm{Ricci})(\varphi_*X,\varphi_*Y) - \frac{1}{2}\,e^{(2n-2)\sigma}\,h(i_X\Omega, i_Y\Omega)\\ &\quad - e^{-2\sigma}\,(\Delta^M\sigma)\,h(X,Y) - (n-1)(n-2)X(\sigma)Y(\sigma)\,,\end{aligned}\tag{5.3}$$

$$\begin{aligned}\mathrm{Ricci}(X,V) &= \frac{1}{2}\,e^{(2n-2)\sigma}\,({}^h\mathrm{d}^*\,\Omega)(X) + (n-1)e^{(2n-2)\sigma}\,\Omega(X,\mathrm{grad}_h\,\sigma)\\ &\quad + (n-1)X(V(\sigma)) - (n-1)(n-2)X(\sigma)V(\sigma)\,.\end{aligned}\tag{5.4}$$

where $^h\,\mathrm{d}^*$ is the codifferential on (M, h).

Remark 5.3. If in above formulae we put $n = 2$ we obtain particular cases of formulae of P. Baird and J. C. Wood [4, Proposition 4.2].

A consequence of Lemma 5.2 is the following Kaluza-Klein type result:

Proposition 5.4. *For $n \geq 3$, let $\varphi : (M^{n+1}, g) \to (N^n, \bar{h})$ be a harmonic morphism with one-dimensional geodesic fibres.*

(a) *If $\mathcal{H}$ is integrable then the following assertions are equivalent:*

(i) *(M, g) is Einstein,*

(ii) *$(N, \bar{h})$ is Einstein and the following relation holds*

$$(n-1)V(V(\sigma)) - (n-1)(n-2)(V(\sigma))^2 = e^{(2n-2)\sigma}\,c^N - \frac{1}{2}\,e^{(4n-4)\sigma}\,\left|\Omega\right|_h^2$$

where c^N is the Einstein constant of $(N^n, \bar{h})$.

(b) *When $n = 4$ and M and N are oriented consider, also, the following assertion:*

(iii) *Ω is the pull-back of a (anti-)self-dual form on $(N, \bar{h})$.*

Then, any two of the assertions (i), (ii) *and* (iii) *imply the remaining assertion.*

Proof. (a) By Proposition 1.5, we have that $X(\sigma) = 0$ for any horizontal X. Lemma 2.1 implies that $[V, X] = 0$ and hence $X(V(\sigma)) = V(X(\sigma)) = 0$. By hypothesis, $\Omega = 0$ so, from (5.4) for any horizontal X we have $\mathrm{Ricci}(X, V) = 0$. Similarly, from (5.3) we get:

$$\mathrm{Ricci}(X,Y) = ({}^N\mathrm{Ricci})(\varphi_*X,\varphi_*Y) - e^{-2\sigma}\,(\Delta^M\sigma)\,h(X,Y)\,.\tag{5.5}$$

It follows that (M, g) is Einstein if and only if $(N, \bar{h})$ is.

(b) Let $\bar{\Omega}$ be the two-form on N such that $\varphi^*(\bar{\Omega}) = \Omega$. Note that $({}^h\mathrm{d}^*\,\Omega)|_{\mathcal{H}} = 0$ if and only $\bar{\Omega}$ is coclosed on $(N, \bar{h})$.

If (iii) holds the equivalence (i) $\iff$ (ii) can be proved in a similar way to

(a) , using the fact that any closed (anti-)self-dual form is coclosed and that, for any two-form ω on a four-dimensional oriented Euclidean space $(E^4, <,>)$ and $u, v \in E$, we have:

$$< i_u\omega, i_v\omega >= \frac{1}{2}|\omega|^2 < u, v > +2 < i_u\omega_+, i_v\omega_- > \tag{5.6}$$

where, ω_+ and ω_- are, respectively, the self-dual and the anti-self-dual components of ω.

To prove (i),(ii)$\Rightarrow$(iii) first note that, by (5.4) , $\bar{\Omega}$ is coclosed.

Now, recall that (5.6) gives the decomposition of the symmetric bilinear map $(u, v) \mapsto < i_u\omega, i_v\omega >$ into its 'spherical' part and its 'trace-free' part. Also, the bilinear map $(u, v) \mapsto < i_u\alpha, i_v\beta >$ induces a natural isomorphism between the space of 'trace-free' symmetric bilinear maps and $\Lambda^2_+(E) \otimes \Lambda^2_-(E)$. Using these facts it is easy to see that at each point $\bar{\Omega}$ is either self-dual or anti-self-dual.

If $N_\pm = \{\, y \in N \,|\, (\bar{\Omega}_\pm)_y = 0 \,\}$ then by the Baire category theorem at least one of the two sets N_+ and N_- has nonempty interior. If N_+ has nonempty interior then by Aronszajn's unique continuation theorem (see [11] and recall that $\bar{\Omega}$, and hence also $\bar{\Omega}_+$, is closed and coclosed) $\bar{\Omega}_+ = 0$ and hence $\bar{\Omega}$ is anti-self-dual. □

Remark 5.5. 1) From Lemma 2.7 we see that if the foliation given by the fibres of φ is nowhere Riemannian then $\mathcal{H}$ is automatically integrable.

2) Since the decomposition of two-forms into self-dual and anti-self-dual forms is conformally invariant, the condition that Ω be the pull-back of a (anti-)self-dual form is equivalent to the condition that Ω restricted to the horizontal distribution be (anti-)self-dual.

The following elementary algebraic lemma will be used several times in this section.

Lemma 5.6. *Let E be an Euclidean linear space of dimension at least two and α a linear function on it such that for any pair of orthogonal vectors $\{u, v\}$ we have $\alpha(u)\alpha(v) = 0$.*

Then $\alpha = 0$.

Proof. Let u, $v \in E$ be orthogonal and such that $|u| = |v|$. Since $u + v$, $u - v$ are also orthogonal we get that $0 = \alpha(u + v)\alpha(u - v) = \alpha(u)^2 - \alpha(v)^2$.

Thus $\alpha(u) = \pm\alpha(v)$ and since by hypothesis at least one of must be zero they are both zero. The lemma is proved. □

Recall that on a Riemannian manifold of dimension at least four a Riemannian foliation with one-dimensional leaves produces harmonic morphisms if and only if it is locally generated by Killing fields (Proposition 2.3) and a foliation by geodesics produces harmonic morphisms if and only if it is homothetic (Corollary 1.24). The next few results give conditions under which these are the only possible types of one-dimensional foliations which produces harmonic morphisms. They require a technical lemma which will be proved at the end of the section (Lemma 5.17).

Theorem 5.7. *Let* (M^{n+1}, g) *be an Einstein manifold of dimension* $n+1 \geq 4$*, and let* $\mathcal{V}$ *be a one-dimensional foliation with integrable orthogonal complement.*

Then $\mathcal{V}$ *produces harmonic morphisms if and only if* either

(i) $\mathcal{V}$ *is Riemannian and locally generated by Killing fields* or

(ii) $\mathcal{V}$ *is homothetic and its leaves are geodesics.*

Moreover, if both (i) *and* (ii) *occur then* (M^{n+1}, g) *is Ricci flat.*

Proof. By passing to a Riemannian covering if necessary, we can suppose that $\mathcal{V}$ admits a global density.

By the remarks above we need to prove just the 'only if' part.

If $\mathcal{H}$ is integrable, then from (5.3), for any orthogonal pair $\{X, Y\}$ formed of basic vector fields we have:

$$ {}^N\operatorname{Ricci}(\varphi_* X, \varphi_* Y) = (n-1)(n-2)X(\sigma)Y(\sigma)\,. \tag{5.7} $$

Since $n \geq 3$ and the left-hand side of (5.7) is a basic function we get that $X(\sigma)Y(\sigma)$ is a basic function.

Also, from (5.4) we obtain that $V(X(\sigma)) = X(V(\sigma)) = (n-2)V(\sigma)X(\sigma)$.

Hence:

$$ \begin{aligned} 0 =& V(X(\sigma)Y(\sigma)) = V(X(\sigma))Y(\sigma) + X(\sigma)V(Y(\sigma)) \\ =& 2(n-2)V(\sigma)X(\sigma)Y(\sigma)\,. \end{aligned} $$

If, at a point x we have that $V(\sigma)(x) \neq 0$, then this holds in an open neighbourhood O of x. It follows that $X(\sigma)Y(\sigma) = 0$ on O.

Using Lemma 5.6 we see that $\operatorname{grad}\sigma$ restricted to O is vertical and hence, from Proposition 1.5, it follows that $\mathcal{V}$ restricted to O has geodesic leaves. Thus, we have proved that at each point either $\mathcal{V}$ is Riemannian or geodesic. Using Lemma 5.17 we get that locally either (i) or (ii) holds.

To prove that this alternative holds globally first recall that, being Einstein, (M, g) is analytic (see [7, 5.26]).

On an open subset where $\mathcal{V}$ has geodesic leaves, by (3.1), σ satisfies the equation

$$ -n\Delta\sigma + n(n-1)|\operatorname{grad}_g \sigma|^2 = c \tag{5.8} $$

where, c is the Einstein constant of (M, g) and Δ is the Laplacian of (M, g). Hence, by the regularity of solutions for elliptic equations (see [7, page 467]) we get that σ is analytic on any open set where $\mathcal{V}$ has geodesic leaves.

Also, σ is analytic on an open set where $\mathcal{V}$ is Riemannian since, by (3.1), on such open sets σ satisfies the equation

$$ \Delta\sigma = \frac{1}{n-2}\,c\,. \tag{5.9} $$

Because the alternative (i) or (ii) holds locally, σ is analytic on M.

Now suppose that locally both (i) and (ii) occur. In this case, being analytic, σ satisfies both (5.8) and (5.9) on M. It is easy to see that this implies that $|\operatorname{grad}_g \sigma|^2 = 2c/n(n-2)$ is constant on M. Also, if locally both (i) and (ii) occur, there must be at least one point $x \in M$ where $(\operatorname{grad}\sigma)_x = 0$. Thus, if locally both situations (i) and (ii) occur then σ is constant on M and (M, g) is Ricci flat. □

Remark 5.8. In the above proof the fact that σ is analytic on an open subset of M where $\mathcal{V}$ has geodesic leaves follows, also, from the fact that if (M, g) is Einstein and φ a local harmonic morphism with geodesic fibres produced by $\mathcal{V}$ then by Proposition 5.4 the codomain of φ is also Einstein and hence is analytic in harmonic coordinates (see [7, 5.26]). Thus, φ is an analytic horizontally conformal (actually, homothetic) submersion between analytic Riemannian manifolds. Hence, the dilation of φ is analytic.

Next we prove that Theorem 5.7 still holds if we replace the integrability assumption on $\mathcal{H}$ with the condition that $\mathcal{V}$ be a homothetic foliation.

Corollary 5.9. *Let (M, g) be an Einstein manifold of dimension at least four endowed with a one-dimensional homothetic foliation $\mathcal{V}$.*

Then $\mathcal{V}$ produces harmonic morphisms if and only if either

(i) *$\mathcal{V}$ is Riemannian and locally generated by Killing fields* or

(ii) *$\mathcal{V}$ has geodesic leaves and integrable orthogonal complement.*

Moreover, if both (i) *and* (ii) *occur then (M, g) is Ricci flat.*

Proof. Again we need to prove just the 'only if' part.

Let

$$F_1 = \{x \in M \mid \mathcal{V} \text{ is Riemannian at } x\},$$
$$F_2 = \{x \in M \mid \mathcal{V} \text{ is geodesic at } x\}.$$

By Lemma 2.7 we have that $\mathcal{H}$, the orthogonal complement of $\mathcal{V}$, is integrable at least on $M \setminus F_1$. From Theorem 5.7 it follows that on $M \setminus F_1$ either $\mathcal{V}$ is Riemannian or geodesic. Using Lemma 5.17 we get that locally either $\mathcal{V}$ is Riemannian or geodesic, i.e. $M = \overset{\circ}{F_1} \cup \overset{\circ}{F_2}$, where $\overset{\circ}{F}$ means the interior of F. If $\overset{\circ}{F_1} \cap \overset{\circ}{F_2} \neq \emptyset$ then, by the proof of Theorem 5.7, $\overset{\circ}{F_2} \subseteq \overset{\circ}{F_1}$. Hence, $\mathcal{V}$ is either Riemannian or geodesic, on M. If $\mathcal{V}$ is not Riemannian then $\mathcal{H}$ is integrable, by Lemma 2.7.

The last assertion follows from Theorem 5.7. □

Remark 5.10. For the alternative (i) or (ii) from the above corollary to hold locally it is enough to suppose that the Ricci tensor take the value zero when evaluated on a pair formed of a horizontal and a vertical vector.

In the next two propositions we shall assume curvature conditions which are automatically satisfied when (M, g) has constant sectional curvature.

Proposition 5.11. *For $n \geq 3$, let $\mathcal{V}$ be a one-dimensional foliation on (M^{n+1}, g). Suppose that the orthogonal complement of $\mathcal{V}$ is integrable, that for any horizontal vector X we have $\mathrm{Ricci}(X, V) = 0$ and that for any pair $\{X, Y\}$ of orthogonal basic vector fields the function $e^{(-2n+4)\sigma} R(X, V, Y, V)$ is basic.*

Then $\mathcal{V}$ produces harmonic morphisms if and only if locally either

(i) *$\mathcal{V}$ is Riemannian and generated by a Killing field* or

(ii) *$\mathcal{V}$ is homothetic and its leaves are geodesic.*

Proof. Again we need to prove just the 'only if' part.

By Lemma 5.17 and Proposition 2.3, it is sufficient to prove that at each point $\mathcal{V}$ is either Riemannian or geodesic.

Let $\{X, Y\}$ be a pair of basic vector fields orthonormal with respect to h. Then the hypothesis together with (5.1) implies that

(5.10)
$$e^{(-2n+4)\sigma} R(X, V, Y, V) = -\frac{1}{2}(n-2)(\mathcal{L}_{\mathcal{H}(\mathrm{grad}_h \sigma)} h)(X, Y) - n(n-2)X(\sigma)Y(\sigma).$$

Also, with the given hypotheses, from (5.4) we see that

$$V(X(\sigma)) = X(V(\sigma)) = (n-2)X(\sigma)V(\sigma)\,. \tag{5.11}$$

Since the left-hand side of (5.10) is basic by hypothesis, we can write

$$0 = \frac{1}{2}V((\mathcal{L}_{\mathcal{H}(\mathrm{grad}_h \sigma)} h)(X, Y)) + nV(X(\sigma)Y(\sigma))\,.$$

Hence using the properties of the Lie derivative together with (5.11) we get

$$\begin{aligned} 0 \ &= \ \frac{1}{2}(\mathcal{L}_V(\mathcal{L}_{\mathcal{H}(\mathrm{grad}_h \sigma)} h))(X, Y) + \frac{1}{2}(\mathcal{L}_{\mathcal{H}(\mathrm{grad}_h \sigma)} h)([V, X], Y) \\ &+ \ \frac{1}{2}(\mathcal{L}_{\mathcal{H}(\mathrm{grad}_h \sigma)} h)(X, [V, Y]) + 2n(n-2)V(\sigma)X(\sigma)Y(\sigma)\,. \end{aligned}$$

Since $[\mathcal{L}_A, \mathcal{L}_B] = \mathcal{L}_{[A,B]}$, for any vector fields A, B, and $[V, X] = 0$ for any basic field X, we get

$$\begin{aligned} 0 =& \frac{1}{2}(\mathcal{L}_{[V,\mathcal{H}(\mathrm{grad}_h \sigma)]} h)(X, Y) + \frac{1}{2}(\mathcal{L}_{\mathcal{H}(\mathrm{grad}_h \sigma)}(\mathcal{L}_V h))(X, Y) \\ &+ 2n(n-2)V(\sigma)X(\sigma)Y(\sigma)\,. \end{aligned}$$

Since $\mathcal{V}$ is a Riemannian foliation with respect to h, the last relation becomes

$$0 = \frac{1}{2}(\mathcal{L}_{[V,\mathcal{H}(\mathrm{grad}_h \sigma)]} h)(X, Y) + 2n(n-2)V(\sigma)X(\sigma)Y(\sigma)\,.$$

Now, from (5.11) one can easily obtain

$$[V, \mathcal{H}(\mathrm{grad}_h \sigma)] = (n-2)V(\sigma)\mathcal{H}(\mathrm{grad}_h \sigma)\,.$$

From the last two relations we obtain

$$\begin{aligned} 0 &= \frac{1}{2}(n-2)V(\sigma)(\mathcal{L}_{\mathcal{H}(grad_h\sigma)}\,h)(X,Y) + \frac{1}{2}(n-2)X(V(\sigma))Y(\sigma) \\ &+ \frac{1}{2}(n-2)Y(V(\sigma))X(\sigma) + 2n(n-2)V(\sigma)X(\sigma)Y(\sigma)\,. \end{aligned}$$

The last relation together with (5.10) and (5.11) gives

$$e^{(-2n+4)\sigma}\,V(\sigma)R(X,V,Y,V) = 2(n-1)(n-2)V(\sigma)X(\sigma)Y(\sigma)\,. \tag{5.12}$$

If at a point x we have that $\mathcal{V}$ is not Riemannian (i.e. $V(\sigma)(x) \neq 0$) then (5.12) together with the fact that $e^{(-2n+4)\sigma}\,R(X,V,Y,V)$ is basic gives us that in a neighbourhood O of x we have $V(X(\sigma)Y(\sigma)) = 0$. Using (5.11), this implies that $X(\sigma)Y(\sigma) = 0$. Using Lemma 5.6, we get that in O, $\mathrm{grad}_h\,\sigma$ is vertical. Now apply Lemma 5.17. □

Proposition 5.12. *For $n \geq 4$, let $\mathcal{V}$ be a one-dimensional foliation on the Riemannian manifold (M^{n+1}, g) and let R be its Riemannian curvature tensor. Suppose that $\mathcal{V}$ produces harmonic morphisms and that for any triple $\{X,Y,Z\}$ of orthogonal basic vector fields the function $e^{(-2n+4)\sigma}\,R(X,Y,Z,V)$ is basic.*

Then except, possibly, at the points where the orthogonal complement of $\mathcal{V}$ is integrable, we have that $X(V(\sigma)) = 0$ for any horizontal X .

If $n = 3$ and, together with the above curvature assumption, we have that $\mathrm{Ricci}(X,Y)$ *is basic for any pair of orthogonal basic vector fields $\{X,Y\}$ then $X(V(\sigma)) = 0$ for any horizontal X .*

Proof. Let $\{X,Y,Z\}$ be orthogonal basic fields. Put them into (5.2) and multiply by $e^{(-2n+4)\sigma}$. Then, the first term from the right hand side is basic by a simple calculation. The terms in the last three lines are zero and we conclude that the second term from the right hand side of (5.2) is basic.

Hence, for any triple $\{X,Y,Z\}$ of orthogonal basic fields the following relation holds:

$$V(X(\sigma))\Omega(Y,Z) \;+\; V(Y(\sigma))\Omega(Z,X) \;=\; 2V(Z(\sigma))\Omega(X,Y)\,.$$

Rewriting this after a circular permutation of the vectors in the frame and then subtracting the second relation from the first gives the following:

$$V(X(\sigma))\Omega(Y,Z) = V(Z(\sigma))\Omega(X,Y)\,. \tag{5.13}$$

Hence, if $\{T,X,Y,Z\}$ are orthogonal basic fields we have the following:

$$\begin{aligned} V(T(\sigma))V(X(\sigma))\Omega(Y,Z) &= V(T(\sigma))V(Z(\sigma))\Omega(X,Y) \\ = -V(Z(\sigma))V(T(\sigma))\Omega(Y,X) &= -V(Z(\sigma))V(X(\sigma))\Omega(T,Y) \\ = -V(X(\sigma))V(Z(\sigma))\Omega(T,Y) &= -V(X(\sigma))V(T(\sigma))\Omega(Y,Z)\,. \end{aligned}$$

And hence

$$V(T(\sigma))V(X(\sigma))\Omega(Y,Z) = 0\,.$$

It follows that on $S\big(= \big\{ x \in M \,|\, \Omega_x \neq 0 \big\}\big)$ for any pair $\{X, Y\}$ of orthogonal basic fields we have $V(X(\sigma))V(Y(\sigma)) = 0$. By Lemma 5.6 this implies that on S we have $V(X(\sigma)) = 0$ for any basic vector field X .

Suppose now that $\dim M = 4$ and let S be defined as above. Then, it suffices to prove that $\mathcal{V}$ is homothetic on S and on the interior of $M \setminus S$.

Let $\{X, Y\}$ be a pair of orthogonal basic vector fields defined on the interior of $M \setminus S$. Then from (5.3) we obtain that $X(\sigma)Y(\sigma)$ is a basic function. If X and Y have the same length with respect to h then this applies also to the pair $\{X+Y, X-Y\}$ and hence $X(\sigma)^2 - Y(\sigma)^2$ is a basic function. Hence $X(\sigma)^2$ and $Y(\sigma)^2$ are basic and thus $X(\sigma)$ and $Y(\sigma)$ are basic.

Since $\dim M = 4$ and $i_V\Omega = 0$ (here, as before, i denotes the interior product) each point from S has an open neighbourhood on which a unique basic vector field Z can be defined such that $i_Z\Omega = 0$ and $h(Z, Z) = 1$.

We can choose basic vector fields X and Y such that $\{X, Y, Z\}$ is an orthonormal local frame for $\mathcal{H}$ with respect to h . Since, on S , $\Omega \neq 0$ we have that $\Omega(X, Y) \neq 0$. By (5.13) and the way Z was chosen we have:

$$V(Z(\sigma))\Omega(X, Y) = V(X(\sigma))\Omega(Y, Z) = 0\,.$$

Hence $V(Z(\sigma)) = 0$.

Now, a simple calculation gives the following relations:

$$\begin{aligned} &h(i_X\Omega, i_Y\Omega) = 0\,, \\ &h(i_X\Omega, i_X\Omega) = h(i_Y\Omega, i_Y\Omega)\,. \end{aligned}$$

The first of the above relations together with (5.3) applied to $\{X, Y\}$ gives that the function $X(\sigma)Y(\sigma)$ is basic. The second of the above relations together with (5.3) applied to $\{X+Y, X-Y\}$ gives that the function $X(\sigma)^2 - Y(\sigma)^2$ is basic. Hence $X(\sigma)$ and $Y(\sigma)$ are basic. □

Proposition 5.13. *Let $\mathcal{V}$ be a one-dimensional foliation which produces harmonic morphisms on (M^{n+1}, g) . Suppose that the following conditions are satisfied, for any horizontal X :*

(i) $X(V(\sigma)) = 0$,

(ii) $\mathrm{Ricci}(X, V) = 0$.

Then $\mathcal{V}$ is homothetic.

Proof. By Lemma 2.8 it is sufficient to prove that $\mathcal{V}$ is homothetic on the interior S of the set $\{x \in M \,|\, \mathrm{d}(V(\sigma))(x) = 0 \neq (V(\sigma))(x)\}$.

By (5.4) on S we have

$$e^{(2n-2)\sigma}\Big\{\frac{1}{2}({}^h\mathrm{d}^*\,\Omega)(X) + (n-1)\Omega(X, \mathrm{grad}_h\,\sigma)\Big\} = (n-1)(n-2)X(\sigma)V(\sigma), \tag{5.14}$$

for any basic vector field X .

By hypothesis the right hand side above is a basic function on S . Also, the second factor from the left hand side of (5.14) is basic and thus, if this second factor is nonzero, then $e^{(2n-2)\sigma}$ is a basic function. This implies that $\mathcal{V}$ is Riemannian and, in particular, homothetic.

If the second factor from the left hand side of (5.14) is zero on an open subset S_0 of S then the right hand is also zero and hence $\mathcal{V}$ has geodesic fibres on S_0. From Corollary 1.24 it follows that $\mathcal{V}$ is homothetic on S_0. $\square$

It is obvious that a space with constant sectional curvature satisfies the curvature assumptions of all of the previous theorems of this section. In fact, in this case, we have the following:

Theorem 5.14. *For $n \geq 3$ let (M^{n+1}, g) be a Riemannian manifold with constant sectional curvature and $\mathcal{V}$ a one-dimensional foliation on (M^{n+1}, g).*

Then, $\mathcal{V}$ produces harmonic morphisms if and only if either

(i) *$\mathcal{V}$ is Riemannian and locally generated by Killing fields* or

(ii) *$\mathcal{V}$ is a homothetic foliation by geodesics with integrable orthogonal complement.*

Moreover, both (i) *and* (ii) *can occur only if (M^{n+1}, g) is flat.*

Furthermore, if $\mathcal{V}$ is orientable and admits a global density then (i) *can be replaced by*

(i′) *$\mathcal{V}$ is (globally) generated by a nonvanishing Killing field.*

Proof. Again we need to prove just the 'only if' part.

From Theorem 5.7, Proposition 5.12 and Proposition 5.13 we obtain that at each point either $\mathcal{V}$ is Riemannian or geodesic.

Using Lemma 5.17 we get that locally either $\mathcal{V}$ is Riemannian or geodesic. In particular, $\mathcal{V}$ is homothetic. Now the proof of the alternative (i) or (ii) follows from Corollary 5.9.

If both (i) and (ii) hold then, from formula (5.1), it follows that (M, g) is flat.

The fact that (i) can be replaced by (i′) when $\mathcal{V}$ is orientable and admits a global density follows from the proof of Proposition 2.3. $\square$

Corollary 5.15. *For $n \geq 3$ let (M^{n+1}, g) be a Riemannian manifold with constant sectional curvature and let $\varphi : (M^{n+1}, g) \to (N^n, h)$ be a submersive harmonic morphism with orientable vertical distribution.*

Then, either

(i) *the fibres of φ form a Riemannian foliation generated by a Killing field* or

(ii) *φ is horizontally homothetic and has geodesic fibres orthogonal to an umbilical foliation by hypersurfaces.*

Lemma 5.16. *Let M^m be a manifold and $F_1, F_2 \subseteq M$, closed subsets such that $M = F_1 \cup F_2$.*

Then, $M = \mathrm{cl}(\mathring{F}_1) \cup \mathrm{cl}(\mathring{F}_2)$.

Proof. Suppose that there exists $x \notin \mathrm{cl}(\mathring{F}_1) \cup \mathrm{cl}(\mathring{F}_2)$. Then there exists an open neighbourhood O of x, which is homeomorphic with $\mathbb{R}^m$ and such that $O \cap \mathrm{cl}(\mathring{F}_j) = \emptyset$, $j = 1, 2$.

For $j = 1, 2$, let $G_j = O \cap F_j$. Since F_j is closed, G_j is closed in O. Also, since O is open we have that the interior of G_j in O is the same as its interior in M, which is empty by the way O was chosen.

Hence, G_j is nowhere dense in O. From $M = F_1 \cup F_2$ we get that $\mathbb{R}^m \approx O = G_1 \cup G_2$. But this is impossible, by the Baire category theorem. □

Lemma 5.17. *Let $\mathcal{V}$ be a one-dimensional foliation with geodesic leaves on (M, g) and let $\mathcal{H}$ be its orthogonal complement. Let $f : M \to \mathbb{R}$ be a smooth function such that at each point $x \in M$ either $\mathcal{V}(\operatorname{grad} f)_x = 0$ or $\mathcal{H}(\operatorname{grad} f)_x = 0$.*

Then each point of M has an open neighbourhood O such that either $\mathcal{V}(\operatorname{grad} f) = 0$ on O or $\mathcal{H}(\operatorname{grad} f) = 0$ on O.

Proof. Let

$$\begin{aligned} F_1 =& \{ x \in M \mid \mathcal{V}(\operatorname{grad} f)_x = 0 \} , \\ F_2 =& \{ x \in M \mid \mathcal{H}(\operatorname{grad} f)_x = 0 \} . \end{aligned}$$

By hypothesis, $M = F_1 \cup F_2$ and hence, by Lemma 5.16, we have that

$$M = \operatorname{cl}(\overset{\circ}{F_1}) \cup \operatorname{cl}(\overset{\circ}{F_2}) . \tag{5.15}$$

Let V be a local unit vertical vector field, then, $[V, X] = 0$ for any basic vector field X. Also, using (5.15) we see that for any basic vector field X, we have:

$$V(X(f)) = X(V(f)) = 0 . \tag{5.16}$$

Note that $\mathcal{H}$ is integrable on $M \setminus F_1$, because on this set the level hypersurfaces of f are integral manifolds for $\mathcal{H}$. Using this fact together with (5.16) and the hypothesis it is easy to see that $\mathcal{H}(\operatorname{grad} f)$ and $\mathcal{V}(\operatorname{grad} f)$ are, locally, gradient vector fields. Thus, we can find local smooth functions a and b such that:

$$\begin{gathered} f = a + b \qquad \text{and,} \\ \mathcal{V}(\operatorname{grad} a) = 0 = \mathcal{H}(\operatorname{grad} b) \end{gathered}$$

It follows that F_1 is the set of critical points of b, and F_2 is the set of critical points of a. Since locally, M is diffeomorphic to the product between an open subset of a leaf and a local base, the lemma quickly follows. □

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