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TWISTOR THEORY FOR CO-CR QUATERNIONIC MANIFOLDS AND RELATED STRUCTURES

BY

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ABSTRACT

In a general and non-metrical framework, we introduce the class of co-CR quaternionic manifolds, which contains the class of quaternionic manifolds, whilst in dimension three it particularizes to give the Einstein–Weyl spaces. We show that these manifolds have a rich natural Twistor Theory and, along the way, we obtain a heaven space construction for quaternionic-Kähler manifolds.

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Introduction

Over any three-dimensional conformal manifold M, endowed with a conformal connection, there is a sphere bundle Z endowed with a natural CR structure [14]. Furthermore, if M is real analytic, then [13] the CR structure of Z is induced by a germ unique embedding of Z into a three-dimensional complex manifold $\widetilde{Z}$ which is the twistor space of an anti-self-dual manifold $\widetilde{M}$; accordingly, M is a hypersurface in $\widetilde{M}$, and the latter is called the **heaven space** (due to [18]; cf. [13]) of M (endowed with the given conformal connection).

In [17] (see Section 2), we obtained the higher dimensional versions of these constructions by introducing the notion of **CR quaternionic manifold**. Thus, the generic submanifolds of codimensions at most $2k-1$, of a quaternionic manifold of dimension $4k$, are endowed with natural CR quaternionic structures. Moreover, assuming real-analyticity, any CR quaternionic manifold is obtained this way through a germ unique embedding into a quaternionic manifold [17].

Returning to the three-dimensional case, by [8], if the inclusion of M into $\widetilde{M}$ admits a retraction which is twistorial (that is, its fibres correspond to a (one-dimensional) holomorphic foliation on $\widetilde{Z}$), then the connection used to construct the CR structure on Z may be assumed to be a Weyl connection; moreover, there is a natural correspondence between such retractions and Einstein–Weyl connections on M. Furthermore, (locally) any Einstein–Weyl connection ∇ on M determines a complex surface Z_∇ and a holomorphic submersion from $\widetilde{Z}$ onto it; then Z_∇ is the twistor space of (M,∇) [8].

Furthermore, the correspondence between Einstein–Weyl spaces and their twistor spaces is similar to the correspondence between anti-self-dual manifolds and their twistor spaces (see also [16]). Furthermore, from the point of view of Twistor Theory, the anti-self-dual manifolds are just four-dimensional quaternionic manifolds (see [9]).

This raises the obvious question: *is there a natural class of manifolds, endowed with twistorial structures, which contains both the quaternionic manifolds and the three-dimensional Einstein–Weyl spaces*?

In this paper, where the adopted point of view is essentially non-metrical, we answer this question in the affirmative by introducing, in a general framework, the notion of **co-CR quaternionic manifolds** and we initiate the study of their twistorial properties. This notion is based on the **(co-)CR quaternionic vector spaces** which were introduced and classified in [17] (see Section 1, and also Appendix A for an alternative definition) and, up to the integrability, it is dual to the notion of **CR quaternionic manifolds**.

An interesting situation to consider is when a manifold may be endowed with both a CR quaternionic and a co-CR quaternionic structure which are **compatible**. This gives the notion of ***f*-quaternionic** manifold, which has two twistor spaces. The simplest example is provided by the three-dimensional Einstein–Weyl spaces, endowed with the twistorial structures of [14] and [8], respectively; furthermore, the above-mentioned twistorial retraction admits a natural generalization to the f-quaternionic manifolds (Corollary 4.5). Also, the quaternionic manifolds may be characterised as f-quaternionic manifolds for which the two twistor spaces coincide.

Other examples of f-quaternionic manifolds are the Grassmannian $\mathrm{Gr}_3^+(l+3,\mathbb{R})$ of oriented three-dimensional vector subspaces of $\mathbb{R}^{l+3}$ and the flag manifold $\mathrm{Gr}_2^0(2n+2,\mathbb{C})$ of two-dimensional complex vector subspaces of $\mathbb{C}^{2n+2}(=\mathbb{H}^{n+1})$ which are isotropic with respect to the underlying complex symplectic structure of $\mathbb{C}^{2n+2}$, $(l,n\geq 1)$. The twistor spaces of their underlying co-CR quaternionic structures are the hyperquadric Q_{l+1} of isotropic one-dimensional complex vector subspaces of $\mathbb{C}^{l+3}$ and $\mathrm{Gr}_2^0(2n+2,\mathbb{C})$ itself, respectively. Also, their heaven spaces are the Wolf spaces $\mathrm{Gr}_4^+(l+4,\mathbb{R})$ and $\mathrm{Gr}_2(2n+2,\mathbb{C})$, respectively (see Examples 4.6 and 4.7 for details). Another natural class of f-quaternionic manifolds is described in Example 4.8.

The notion of almost f-quaternionic manifold appears also in a different form, in [10]. However, there any adequate integrability condition is not considered. Also, in [5], [1] and [4], particular classes of almost f-quaternionic manifolds are considered, under particular dimensional assumptions and/or in a metrical framework.

Let N be the heaven space of a real analytic f-quaternionic manifold M, with $\dim N = \dim M + 1$. If the connection of the f-quaternionic structure on M is induced by a torsion-free connection on M, then the twistor space of N is endowed with a natural holomorphic distribution of codimension one which is transversal to the twistor lines corresponding to the points of $N \setminus M$. Furthermore, this construction also works if, more generally, M is a real analytic CR quaternionic manifold which is a ***q*-umbilical** hypersurface of its heaven space N. Then, under a non-degeneracy condition, this distribution defines a holomorphic contact structure on the twistor space of N. Therefore, according to [15] , it determines a quaternionic-Kähler structure on $N \setminus M$ (cf. [5], [7]).

It is well known (see, for example, [20] and the references therein) that the three-dimensional Einstein–Weyl spaces are one of the basic ingredients in constructions of anti-self-dual (Einstein) manifolds. One of the aims of this paper is to give a first indication that the study of co-CR quaternionic manifolds will lead to a better understanding of quaternionic(-Kähler) manifolds.

1. Brief review of (co-)CR quaternionic vector spaces

The group of automorphisms of the (unital) associative algebra of quaternions $\mathbb{H}$ is $\mathrm{SO}(3)$ acting trivially on $\mathbb{R}\,(\subseteq \mathbb{H}\,)$ and canonically on $\mathrm{Im}\mathbb{H}$.

A **linear hypercomplex structure** on a (real) vector space E is a morphism of associative algebras $\sigma : \mathbb{H} \to \mathrm{End}(E)$. A **linear quaternionic structure** on E is an equivalence class of linear hypercomplex structures, where two linear hypercomplex structures $\sigma_1, \sigma_2 : \mathbb{H} \to \mathrm{End}(E)$ are **equivalent** if there exists $a \in \mathrm{SO}(3)$ such that $\sigma_2 = \sigma_1 \circ a$. A **hypercomplex/quaternionic vector space** is a vector space endowed with a linear hypercomplex/quaternionic structure (see [2], [9]).

If $\sigma : \mathbb{H} \to \mathrm{End}(E)$ is a linear hypercomplex structure on a vector space E, then the unit sphere Z in $\sigma(\mathrm{Im}\mathbb{H}\,) \subseteq \mathrm{End}(E)$ is the corresponding space of **admissible linear complex structures**. Obviously, Z depends only on the linear quaternionic structure determined by σ.

Let E and E' be quaternionic vector spaces and let Z and Z' be the corresponding spaces of admissible linear complex structures. A linear map $t : E \to E'$ is **quaternionic**, with respect to some function $T : Z \to Z'$, if $t \circ J = T(J) \circ t$, for any $J \in Z$ (see [2]). If, further, $t \neq 0$, then T is unique and an orientation preserving isometry (see [9]).

The basic example of a quaternionic vector space is $\mathbb{H}^k$ endowed with the linear quaternionic structure given by its canonical (left) $\mathbb{H}$-module structure. Moreover, for any quaternionic vector space of dimension $4k$ there exists a quaternionic linear isomorphism from it onto $\mathbb{H}^k$. The group of quaternionic linear automorphisms of $\mathbb{H}^k$ is $\mathrm{Sp}(1) \cdot \mathrm{GL}(k, \mathbb{H}\,)$ acting on it by $\big(\pm(a, A), x\big) \mapsto axA^{-1}$, for any $\pm(a, A) \in \mathrm{Sp}(1) \cdot \mathrm{GL}(k, \mathbb{H}\,)$ and $x \in \mathbb{H}^k$. If we restrict this action to $\mathrm{GL}(k, \mathbb{H}\,)$, then we obtain the group of hypercomplex linear automorphisms of $\mathbb{H}^k$.

If $\sigma : \mathbb{H} \to \mathrm{End}(E)$ is a linear hypercomplex structure then $\sigma^* : \mathbb{H} \to \mathrm{End}(E^*)$, where $\sigma^*(q)$ is the transpose of $\sigma(\overline{q})$, $(q \in \mathbb{H})$, is the **dual linear hypercomplex structure**. Accordingly, we define the dual of a linear quaternionic structure.

Definition 1.1 ([17]): A **linear co-CR quaternionic structure** on a vector space U is a pair (E, ρ), where E is a quaternionic vector space and $\rho : E \to U$ is a surjective linear map such that $\ker \rho \cap J(\ker \rho) = \{0\}$, for any admissible linear complex structure J on E.

A **co-CR quaternionic vector space** is a vector space endowed with a linear co-CR quaternionic structure.

Dually, a **CR quaternionic vector space** is a triple (U, E, ι), where E is a quaternionic vector space and $\iota : U \to E$ is an injective linear map such that $\mathrm{im}\,\iota + J(\mathrm{im}\,\iota) = E$, for any admissible linear complex structure J on E.

A map $t : (U, E, \rho) \to (U', E', \rho')$ between co-CR quaternionic vector spaces is **co-CR quaternionic linear** (with respect to some map $T : Z \to Z'$) if there exists a map $\widetilde{t} : E \to E'$ which is quaternionic linear (with respect to T) such that $t \circ \rho = \rho' \circ \widetilde{t}$.

By duality, we also have the notion of **CR quaternionic linear map**.

Note that if (U, E, ι) is a CR quaternionic vector space, then the inclusion $\iota : U \to E$ is CR quaternionic linear. Dually, if (U, E, ρ) is a co-CR quaternionic vector space, then the projection $\rho : E \to U$ is co-CR quaternionic linear.

By working with pairs (U, E), where E is a quaternionic vector space and $U \subseteq E$ is a real vector subspace, we call $(\mathrm{Ann}\, U, E^*)$ **the dual pair** of (U, E), where the annihilator $\mathrm{Ann}\, U$ is formed of those $\alpha \in E^*$ such that $\alpha|_U = 0$.

Any CR quaternionic vector space (U, E, ι) corresponds to the pair $(\mathrm{im}\,\iota, E)$, whilst any co-CR quaternionic vector space (U, E, ρ) corresponds to the pair $(\ker \rho, E)$. These associations define functors in the obvious way.

To any pair (U, E) we associate a (coherent analytic) sheaf over Z as follows. Let $E^{0,1}$ be the holomorphic vector bundle over Z whose fibre over any $J \in Z$ is the $-\mathrm{i}$ eigenspace of J. Let $u : E^{0,1} \to Z \times (E/U)^{\mathbb{C}}$ be the composition of the inclusion $E^{0,1} \to Z \times E^{\mathbb{C}}$ followed by the projection $Z \times E^{\mathbb{C}} \to Z \times (E/U)^{\mathbb{C}}$.

Definition 1.2 ([19]): $\mathcal{U} = \mathcal{U}_- \oplus \mathcal{U}_+$ is **the sheaf of** (U, E), where $\mathcal{U}_- = \ker u$ and $\mathcal{U}_+ = \mathrm{coker}\, u$.

If (U,E) corresponds to a (co-)CR quaternionic vector space, then $\mathcal{U}$ is its holomorphic vector bundle, introduced in [17]. In fact, (U,E) corresponds to a co-CR quaternionic vector space if and only if $\mathcal{U}$ is a holomorphic vector bundle and $\mathcal{U}=\mathcal{U}_+$. Dually, (U,E) corresponds to a CR quaternionic vector space if and only if $\mathcal{U}=\mathcal{U}_-$ (note that $\mathcal{U}_-$ is a holomorphic vector bundle for any pair). See [19] for more information on the functor $(U,E)\mapsto\mathcal{U}$.

Here are the basic examples of (co-)CR quaternionic vector spaces.

Example 1.3 (cf. [17]): (1) Let V_k, $(k\geq 1)$ be the vector subspace of $\mathbb{H}^k$ formed of all vectors of the form $(z_1\,,\overline{z_1}+z_2\,\mathrm{j}\,,z_3-\overline{z_2}\,\mathrm{j}\,,\ldots)$, where $z_1,\ldots,z_k$ are complex numbers and $\overline{z_k}=(-1)^k z_k$. Then $(V_k,\mathbb{H}^k)$ corresponds to a co-CR quaternionic vector space and its holomorphic vector bundle is $\mathcal{O}(2k)$. Hence, the dual pair is a CR quaternionic vector space and its holomorphic vector bundle is $\mathcal{O}(-2k)$.

(2) Let $V_0'=\{0\}$ and, for $k\geq 1$, let V_k' be the vector subspace of $\mathbb{H}^{2k+1}$ formed of all vectors of the form $(z_1,\overline{z_1}+z_2\,\mathrm{j},z_3-\overline{z_2}\,\mathrm{j},\ldots,\overline{z_{2k-1}}+z_{2k}\,\mathrm{j},-\overline{z_{2k}}\,\mathrm{j})$, where $z_1,\ldots,z_{2k}$ are complex numbers. Then $(V_k',\mathbb{H}^{2k+1})$ corresponds to a co-CR quaternionic vector space and its holomorphic vector bundle is $2\mathcal{O}(2k+1)$. Hence, the dual pair is a CR quaternionic vector space and its holomorphic vector bundle is $2\mathcal{O}(-2k-1)$.

Also, by [17], any (co-)CR quaternionic vector space is isomorphic to a product, unique up to the order of factors, in which each factor is given by Example 1.3(1) or (2).

Definition 1.4: A **linear f-quaternionic structure** on a vector space U is a pair (E,V), where E is a quaternionic vector space such that $U,V\subseteq E$, $E=U\oplus V$ and $J(V)\subseteq U$, for any $J\in Z$.

An **f-quaternionic vector space** is a vector space endowed with a linear f-quaternionic structure.

Let (U,E,V) be an f-quaternionic vector space; denote by $\iota:U\to E$ the inclusion and by $\rho:E\to U$ the projection determined by the decomposition $E=U\oplus V$.

Then (E,ι) and (E,ρ) are linear CR-quaternionic and co-CR-quaternionic structures, respectively, which are **compatible**.

The **f-quaternionic linear maps** are defined, accordingly, by using the compatible linear CR and co-CR quaternionic structures determining a linear f-quaternionic structure.

From any f-quaternionic vector space (U,E,V), with $\dim E = 4k$, $\dim V = l$, there exists an f-quaternionic linear isomorphism onto $(\mathrm{Im}\mathbb{H})^l \times \mathbb{H}^{4k-l}$ (this follows, for example, from the classification of (co-)CR quaternionic vector spaces [17]).

We end this section with the description of the Lie group G of f-quaternionic linear isomorphisms of $(\mathrm{Im}\mathbb{H})^l \times \mathbb{H}^m$. For this, let $\rho_k : \mathrm{Sp}(1)\cdot\mathrm{GL}(k,\mathbb{H}) \to \mathrm{SO}(3)$ be the Lie group morphism defined by $\rho_k(q\cdot A) = \pm q$, for any $q\cdot A \in \mathrm{Sp}(1)\cdot \mathrm{GL}(k,\mathbb{H})$, $(k \geq 1)$. Denote

$$H = \big\{ (A,A') \in \big(\mathrm{Sp}(1)\cdot\mathrm{GL}(l,\mathbb{H})\big) \times \big(\mathrm{Sp}(1)\cdot\mathrm{GL}(m,\mathbb{H})\big) \,|\, \rho_l(A) = \rho_m(A') \big\}.$$

Then H is a closed subgroup of $\mathrm{Sp}(1)\cdot\mathrm{GL}(l+m,\mathbb{H})$ and G is the closed subgroup of H formed of those elements $(A,A') \in H$ such that A preserves $\mathbb{R}^l \subseteq \mathbb{H}^l$. This follows from the fact that there are no nontrivial f-quaternionic linear maps from $\mathrm{Im}\mathbb{H}$ to $\mathbb{H}$ (and from $\mathbb{H}$ to $\mathrm{Im}\mathbb{H}$). Now, the canonical basis of $\mathrm{Im}\mathbb{H}$ induces a linear isomorphism $(\mathrm{Im}\mathbb{H})^l = \big(\mathbb{R}^l\big)^3$ and, therefore, an effective action σ of $\mathrm{GL}(l,\mathbb{R})$ on $(\mathrm{Im}\mathbb{H})^l$. We define an effective action of $\mathrm{GL}(l,\mathbb{R}) \times \big(\mathrm{Sp}(1)\cdot\mathrm{GL}(m,\mathbb{H})\big)$ on $(\mathrm{Im}\mathbb{H})^l \times \mathbb{H}^m$ by

$$(A, q\cdot B)(X,Y) = \big(q\big(\sigma(A)(X)\big)q^{-1}, qYB^{-1}\big),$$

for any $A \in \mathrm{GL}(l,\mathbb{R})$, $q\cdot B \in \mathrm{Sp}(1)\cdot\mathrm{GL}(m,\mathbb{H})$, $X \in (\mathrm{Im}\mathbb{H})^l$ and $Y \in \mathbb{H}^m$.

Proposition 1.5: *There exists an isomorphism of Lie groups*

$$G = \mathrm{GL}(l,\mathbb{R}) \times \big(\mathrm{Sp}(1)\cdot\mathrm{GL}(m,\mathbb{H})\big),$$

given by $(A,A') \mapsto (A|_{\mathbb{R}^l}, A')$, *for any* $(A,A') \in G$.

In particular, the group of f-quaternionic linear isomorphisms of $(\mathrm{Im}\mathbb{H})^l$ *is isomorphic to* $\mathrm{GL}(l,\mathbb{R}) \times \mathrm{SO}(3)$.

Note that the group of f-quaternionic linear isomorphisms of $\mathrm{Im}\mathbb{H}$ is $\mathrm{CO}(3)$.

2. A few basic facts on CR quaternionic manifolds

In this section we recall, for the reader's convenience, a few basic facts on CR quaternionic manifolds (we refer to [17] for further details).

A (smooth) **bundle of associative algebras** is a vector bundle whose typical fibre is a (finite-dimensional) associative algebra and whose structural group is the group of automorphisms of the typical fibre. Let A and B be bundles of associative algebras. A morphism of vector bundles $\rho : A \to B$ is called a

morphism of bundles of associative algebras if ρ restricted to each fibre is a morphism of associative algebras.

Recall that a **quaternionic vector bundle** over a manifold M is a real vector bundle E over M endowed with a pair (A,ρ) where A is a bundle of associative algebras, over M, with typical fibre $\mathbb{H}$ and $\rho : A \to \mathrm{End}(E)$ is a morphism of bundles of associative algebras; we say that (A,ρ) is a **linear quaternionic structure on** E (see [6]). Standard arguments (see [9]) apply to show that a quaternionic vector bundle of (real) rank $4k$ is just a (real) vector bundle endowed with a reduction of its structural group to $\mathrm{Sp}(1)\cdot\mathrm{GL}(k,\mathbb{H})$.

If (A,ρ) defines a linear quaternionic structure on a vector bundle E, then we denote $Q=\rho(\mathrm{Im}A)$, and by Z the sphere bundle of Q.

Recall [22] (see [9]) that a manifold is **almost quaternionic** if and only if its tangent bundle is endowed with a linear quaternionic structure.

Definition 2.1: Let E be a quaternionic vector bundle on a manifold M and let $\iota : TM \to E$ be an injective morphism of vector bundles. We say that (E,ι) is an **almost CR quaternionic structure** on M if (E_x,ι_x) is a linear CR quaternionic structure on T_xM, for any $x\in M$.

An **almost CR quaternionic manifold** is a manifold endowed with an almost CR quaternionic structure.

On any almost CR quaternionic manifold (M,E,ι) for which E is endowed with a connection ∇, compatible with its linear quaternionic structure, there can be defined a natural almost twistorial structure, as follows. For any $J\in Z$, let $\mathcal{B}_J\subseteq T^{\mathbb{C}}_JZ$ be the horizontal lift, with respect to ∇, of $\iota^{-1}\big(E^J\big)$, where $E^J\subseteq E^{\mathbb{C}}_{\pi(J)}$ is the eigenspace of J corresponding to $-\mathrm{i}$. Define $\mathcal{C}_J=\mathcal{B}_J\oplus(\ker \mathrm{d}\pi)^{0,1}_J$, $(J\in Z)$. Then $\mathcal{C}$ is an almost CR structure on Z and $(Z,M,\pi,\mathcal{C})$ is **the almost twistorial structure of** (M,E,ι,∇).

Definition 2.2: An **(integrable almost) CR quaternionic structure** on M is a triple (E,ι,∇), where (E,ι) is an almost CR quaternionic structure on M and ∇ is an almost quaternionic connection of (M,E,ι) such that the almost twistorial structure of (M,E,ι,∇) is integrable (that is, $\mathcal{C}$ is integrable). Then (M,E,ι,∇) is a **CR quaternionic manifold** and the CR manifold $(Z,\mathcal{C})$ is its **twistor space**.

A main source of CR quaternionic manifolds is provided by the submanifolds of quaternionic manifolds.

Definition 2.3: Let (M,E,ι,∇) be a CR quaternionic manifold and let $(Z,\mathcal{C})$ be its twistor space. We say that (M,E,ι,∇) is **realizable** if M is an embedded submanifold of a quaternionic manifold N such that $E=TN|_M$, as quaternionic vector bundles, and $\mathcal{C}=T^{\mathbb{C}}Z\cap(T^{0,1}Z_N)|_M$, where Z_N is the twistor space of N.

Then N is **the heaven space** of (M,E,ι,∇).

By [17, Corollary 5.4], any real-analytic CR quaternionic manifold is realizable.

3. Co-CR quaternionic manifolds

An **almost co-CR structure** on a manifold M is a complex vector subbundle $\mathcal{C}$ of $T^{\mathbb{C}}M$ such that $\mathcal{C}+\overline{\mathcal{C}}=T^{\mathbb{C}}M$. An **(integrable almost) co-CR structure** is an almost co-CR structure whose space of sections is closed under the bracket.

Note that if $\varphi:M\to(N,J)$ is a submersion onto a complex manifold, then $(\mathrm{d}\varphi)^{-1}\big(T^{0,1}N\big)$ is a co-CR structure on M; moreover, any co-CR structure is, locally, of this form.

Definition 3.1: Let E be a quaternionic vector bundle on a manifold M and let $\rho:E\to TM$ be a surjective morphism of vector bundles. Then (E,ρ) is called an **almost co-CR quaternionic structure**, on M, if (E_x,ρ_x) is a linear co-CR quaternionic structure on T_xM, for any $x\in M$. If, further, E is a hypercomplex vector bundle, then (E,ρ) is called an **almost hyper-co-CR structure** on M. An **almost co-CR quaternionic manifold** (**almost hyper-co-CR manifold**) is a manifold endowed with an almost co-CR quaternionic structure (almost hyper-co-CR structure).

Any almost co-CR quaternionic (hyper-co-CR) structure (E,ρ) for which ρ is an isomorphism is an almost quaternionic (hypercomplex) structure.

Example 3.2: Let (M,c) be a three-dimensional conformal manifold and let $L=\big(\Lambda^3TM\big)^{1/3}$ be the line bundle of M. Then, $E=L\oplus TM$ is an oriented vector bundle of rank four endowed with a (linear) conformal structure such that $L=(TM)^{\perp}$. Therefore, E is a quaternionic vector bundle and (M,E,ρ) is an almost co-CR quaternionic manifold, where $\rho:E\to TM$ is the projection. Moreover, any three-dimensional almost co-CR quaternionic manifold is obtained this way.

Next, we are going to introduce a natural almost twistorial structure (see [16] for the definition of almost twistorial structures) on any almost co-CR quaternionic manifold (M,E,ρ) for which E is endowed with a connection ∇ compatible with its linear quaternionic structure.

For any $J\in Z$, let $\mathcal{C}_J\subseteq T^{\mathbb{C}}_J Z$ be the direct sum of $(\ker \mathrm{d}\pi)^{0,1}_J$ and the horizontal lift, with respect to ∇, of $\rho(E^J)$, where E^J is the eigenspace of J corresponding to $-\mathrm{i}$. Then $\mathcal{C}$ is an almost co-CR structure on Z and $(Z,M,\pi,\mathcal{C})$ is **the almost twistorial structure of** (M,E,ρ,∇).

The following definition is motivated by [9, Remark 2.10(2)].

Definition 3.3: A **co-CR quaternionic manifold** is an almost co-CR quaternionic manifold (M,E,ρ) endowed with a compatible connection ∇ on E such that the associated almost twistorial structure $(Z,M,\pi,\mathcal{C})$ is integrable (that is, $\mathcal{C}$ is integrable). If, further, E is a hypercomplex vector bundle and the connection induced by ∇ on Z is trivial, then (M,E,ρ,∇) is a **hyper-co-CR manifold**.

Example 3.4: Let (M,c) be a three-dimensional conformal manifold and let (E,ρ) be the corresponding almost co-CR structure, where $E=L\oplus TM$ with L the line bundle of M. Let D be a Weyl connection on (M,c) and let $\nabla = D^L\oplus D$, where D^L is the connection induced by D on L. It follows that (M,E,ρ,∇) is co-CR quaternionic if and only if (M,c,D) is Einstein–Weyl (that is, the trace-free symmetric part of the Ricci tensor of D is zero).

Furthermore, let μ be a section of L^* such that the connection defined by

$$D^{\mu}_X Y = D_X Y + \mu\, X\times_c Y,$$

for any vector fields X and Y on M, induces a flat connection on $L^*\otimes TM$. Then (M,E,ι,∇^{μ}) is, locally, a hyper-co-CR manifold, where $\nabla^{\mu}=(D^{\mu})^L\oplus D^{\mu}$, with $(D^{\mu})^L$ the connection induced by D^{μ} on L (this follows from well-known results; see [16] and the references therein).

Let $\tau=(Z,M,\pi,\mathcal{C})$ be the twistorial structure of a co-CR quaternionic manifold (M,E,ρ,∇). Recall [16] that τ is **simple** if and only if $\mathcal{C}\cap\overline{\mathcal{C}}$ is a simple foliation (that is, its leaves are the fibres of a submersion) whose leaves intersect each fibre of π at most once. Then $\big(T,\mathrm{d}\varphi(\mathcal{C})\big)$ is the **twistor space** of τ, where $\varphi:Z\to T$ is the submersion whose fibres are the leaves of $\mathcal{C}\cap\overline{\mathcal{C}}$.

Example 3.5: Any co-CR quaternionic vector space is a co-CR quaternionic manifold, in an obvious way; moreover, the associated twistorial structure is simple and its twistor space is just its holomorphic vector bundle.

THEOREM 3.6: *Let (M, E, ρ, ∇) be a co-CR quaternionic manifold,* $\operatorname{rank} E = 4k$, $\operatorname{rank}(\ker \rho) = l$. *If the twistorial structure of (M, E, ρ, ∇) is simple, then it is real analytic and its twistor space is a complex manifold of dimension $2k-l+1$ endowed with a locally complete family of complex projective lines $\{Z_x\}_{x \in M^{\mathbb{C}}}$. Furthermore, for any $x \in M$, the normal bundle of the corresponding twistor line Z_x is the holomorphic vector bundle of (T_xM, E_x, ρ_x).*

Proof. Let $(Z, M, \pi, \mathcal{C})$ be the twistorial structure of (M, E, ρ, ∇). Let $\varphi : Z \to T$ be the submersion whose fibres are the leaves of $\mathcal{C} \cap \overline{\mathcal{C}}$. Obviously, $\mathrm{d}\varphi(\mathcal{C})$ defines a complex structure on T of dimension $2k - l + 1$. Furthermore, if for any $x \in M$ we denote $Z_x = \varphi(\pi^{-1}(x))$, then Z_x is a complex submanifold of T whose normal bundle is the holomorphic vector bundle of (T_xM, E_x, ρ_x). The proof follows from [12] and [21, Proposition 2.5]. ∎

PROPOSITION 3.7: *Let (M, E, ρ, ∇) be a co-CR quaternionic manifold whose twistorial structure is simple; denote by $\varphi : Z \to T$ the corresponding holomorphic submersion onto its twistor space. Then (M, E, ρ, ∇) is hyper-co-CR if and only if there exists a surjective holomorphic submersion $\psi : T \to \mathbb{C}P^1$ such that the fibres of $\psi \circ \varphi$ are integral manifolds of the connection induced by ∇ on Z.*

Proof. Denote by $\mathscr{H}$ the connection induced by ∇ on Z. Then $\mathscr{H}$ is integrable if and only if $\mathrm{d}\varphi(\mathscr{H})$ is a holomorphic foliation on T; furthermore, this foliation is simple if and only if E is hypercomplex and $\mathscr{H}$ is the trivial connection on Z. ∎

4. f-Quaternionic manifolds

Let F be an almost f-structure on a manifold M; that is, F is a field of endomorphisms of TM such that $F^3 + F = 0$. Denote by $\mathcal{C}$ the eigenspace of F with respect to $-\mathrm{i}$ and let $\mathcal{D} = \mathcal{C} \oplus \ker F$. Then $\mathcal{C}$ and $\mathcal{D}$ are **compatible** almost CR and almost co-CR structures, respectively. An (**integrable almost**) **f-structure** is an almost f-structure for which the corresponding almost CR and almost co-CR structures are integrable.

Definition 4.1: An **almost f-quaternionic structure** on a manifold M is a pair (E,V), where E is a quaternionic vector bundle on M, and TM and V are vector subbundles of E such that $E = TM \oplus V$ and $J(V) \subseteq TM$, for any $J \in Z$. An **almost hyper-f-structure** on a manifold M is an almost f-quaternionic structure (E,V) on M such that E is a hypercomplex vector bundle. An **almost f-quaternionic manifold** (**almost hyper-f-manifold**) is a manifold endowed with an almost f-quaternionic structure (almost hyper-f-structure).

With the same notations as in Definition 4.1 , an almost f-quaternionic structure (almost hyper-f-structure) for which V is the zero bundle is an almost quaternionic structure (almost hypercomplex structure).

Let k and l be positive integers, $k \geq l$, and denote by $G_{k,l}$ the group of f-quaternionic linear isomorphisms of $(\mathrm{Im}\mathbb{H}\,)^l \times \mathbb{H}^{\,k-l}$. The next result is an immediate consequence of the description of $G_{k,l}$ given in Section 1.

Proposition 4.2: *Let M be a manifold of dimension $4k-l$. Then any almost f-quaternionic structure (E,V) on M, with $\operatorname{rank} E = 4k$ and $\operatorname{rank} V = l$, corresponds to a reduction of the frame bundle of M to $G_{k,l}$.*

Furthermore, if $(P,M,G_{k,l})$ is the reduction of the frame bundle of M, corresponding to (E,V), then V is the vector bundle associated to P through the canonical morphism of Lie groups $G_{k,l} \to \mathrm{GL}(l,\mathbb{R})$.

Example 4.3: (1) A three-dimensional almost f-quaternionic manifold is just a (three-dimensional) conformal manifold.

(2) Let N be an almost quaternionic manifold endowed with a Hermitian metric and let M be a hypersurface in N. Then $\left(TN|_M,(TM)^{\perp}\right)$ is an almost f-quaternionic structure on M.

Obviously, any almost f-quaternionic structure (E,V) on a manifold M corresponds to a pair (E,ι) and (E,ρ) of compatible almost CR quaternionic and co-CR quaternionic structures on M, where $\iota : TM \to E$ and $\rho : E \to TM$ are the inclusion and projection, respectively.

Definition 4.4: Let (M,E,V) be an almost f-quaternionic manifold. Let (E,ι) and (E,ρ) be the almost CR quaternionic and co-CR quaternionic structures, respectively, corresponding to (E,V). Let ∇ be a connection on E compatible with its linear quaternionic structure, and let τ and τ_c be the almost twistorial structures of (M,E,ι,∇) and (M,E,ρ,∇), respectively. We say that (M,E,V,∇) is an **f-quaternionic manifold** if the almost twistorial structures τ and τ_c are integrable. If, further, E is hypercomplex and ∇ induces the trivial flat connection on Z, then (M,E,V,∇) is an (**integrable almost**) **hyper-f-manifold**.

Let (M,E,V,∇) be an f-quaternionic manifold, and let Z and Z_c be the twistor spaces of τ and τ_c, respectively (we assume, for simplicity, that τ_c is simple). Then Z is called the **CR twistor space** and Z_c is called the **twistor space** of (M,E,V,∇).

Let (M,E,V) be an almost f-quaternionic manifold and let ∇ be a connection on E compatible with its linear quaternionic structure. Let $\mathcal{C}$ and $\mathcal{D}$ be the almost CR and almost co-CR structures on Z determined by ∇ and the underlying almost CR quaternionic and almost co-CR quaternionic structures of (M,E,V), respectively. Then $\mathcal{C}$ and $\mathcal{D}$ are compatible; therefore (M,E,V,∇) is f-quaternionic if and only if the corresponding almost f-structure on Z is integrable.

Let (M,E,V) be an almost f-quaternionic manifold, $\operatorname{rank} E = 4k$, $\operatorname{rank} V = l$, and D some compatible connection on M (equivalently, D is a linear connection on M which corresponds to a principal connection on the reduction to $G_{k,l}$, of the frame bundle of M, corresponding to (E,V)). Then D induces a connection D^V on V. Moreover, $\nabla = D^V \oplus D$ is compatible with the linear quaternionic structure on E.

Corollary 4.5: *Let (M,E,V,∇) be an f-quaternionic manifold,* $\operatorname{rank} E = 4k$, $\operatorname{rank} V = l$, *where $\nabla = D^V \oplus D$ for some compatible connection D on M. Denote by τ and τ_c the associated twistorial structures. Then, locally, the twistor space of (M,τ_c) is a complex manifold, of complex dimension $2k-l+1$, endowed with a locally complete family of complex projective lines each of which has normal bundle $2(k-l)\mathcal{O}(1)\oplus l\,\mathcal{O}(2)$.*

Furthermore, if (M,E,V,∇) is real analytic then, locally, there exists a twistorial map from the corresponding heaven space N, endowed with its twistorial structure, to (M,τ_c) which is a retraction of the inclusion $M\subseteq N$.

Proof. By passing to a convex open set of D, if necessary, we may suppose that τ_c is simple. Thus, the first assertion is a consequence of Theorem 3.6. The second statement follows from the fact that there exists a holomorphic submersion from the twistor space of N, endowed with its twistorial structure, to the twistor space of (M,τ_c), which maps diffeomorphically twistor lines onto twistor lines. ∎

Note that if $\dim M = 3$, then Corollary 4.5 gives results of [13] and [8].

Example 4.6: Let $M^{3l} = \mathrm{Gr}_3^+(l+3,\mathbb{R})$ be the Grassmann manifold of oriented vector subspaces of dimension 3 of $\mathbb{R}^{l+3}$, $(l \geq 1)$. Alternatively, M^{3l} can be defined as the Riemannian symmetric space $\mathrm{SO}(l+3)/\big(\mathrm{SO}(l)\times\mathrm{SO}(3)\big)$. As the structural group of the frame bundle of M^{3l} is $\mathrm{SO}(l)\times\mathrm{SO}(3)$, from Proposition 4.2 we obtain that M^{3l} is canonically endowed with an almost f-quaternionic structure. Moreover, if we endow M^{3l} with its Levi-Civita connection, then we obtain an f-quaternionic manifold. Its twistor space is the hyperquadric Q_{l+1} of isotropic one-dimensional complex vector subspaces of $\mathbb{C}^{l+3}$, considered as the complexification of the (real) Euclidean space of dimension $l+3$. Further, the CR twistor space Z of M^{3l} can be described as the closed submanifold of $Q_{l+1}\times M^{3l}$ formed of those pairs (ℓ,p) such that $\ell\subseteq p^{\mathbb{C}}$. Under the orthogonal decomposition $\mathbb{R}^{l+4}=\mathbb{R}\oplus\mathbb{R}^{l+3}$, we can embed M^{3l} as a totally geodesic submanifold of the quaternionic manifold $\widetilde{M}^{4l}=\mathrm{Gr}_4^+(l+4,\mathbb{R})$ as follows: $p\mapsto\mathbb{R}\oplus p$, $(p\in M^{3l})$. Recall (see [15]) that the twistor space of $\widetilde{M}^{4l}$ is the manifold $\widetilde{Z}=\mathrm{Gr}_2^0(l+4,\mathbb{C})$ of isotropic complex vector subspaces of dimension 2 of $\mathbb{C}^{l+4}$, where the projection $\widetilde{Z}\to\widetilde{M}$ is given by $q\mapsto p$, with q a self-dual subspace of $p^{\mathbb{C}}$ (in particular, $p^{\mathbb{C}}=q\oplus\overline{q}$). Consequently, the CR twistor space Z of M^{3l} can be embedded in $\widetilde{Z}$ as follows: $(\ell,p)\mapsto q$, where q is the unique self-dual subspace of $(\mathbb{R}\oplus p)^{\mathbb{C}}$ which intersects $p^{\mathbb{C}}$ along ℓ.

In the particular case $l=1$ we obtain the well-known fact (see [3]) that the twistor space of S^3 is $Q_2\ \big(=\mathbb{C}P^1\times\mathbb{C}P^1\big)$. Also, the CR twistor space of S^3 can be identified with the sphere bundle of $\mathcal{O}(1)\oplus\mathcal{O}(1)$. Similarly, the dual of M^{3l} is, canonically, an f-quaternionic manifold whose twistor space is an open set of Q_{l+1}.

Example 4.7: Let $\mathrm{Gr}_2^0(2n+2,\mathbb{C})$ be the complex hypersurface of the Grassmannian $\mathrm{Gr}_2(2n+2,\mathbb{C})$ of two-dimensional complex vector subspaces of $\mathbb{C}^{2n+2}$ $\big(=\mathbb{H}^{n+1}\big)$ formed of those $q\in\mathrm{Gr}_2(2n+2,\mathbb{C})$ which are isotropic with respect to the underlying complex symplectic structure ω of $\mathbb{C}^{2n+2}$; note that

$$\mathrm{Gr}_2^0(2n+2,\mathbb{C})=\mathrm{Sp}(n+1)/\big(\mathrm{U}(2)\times\mathrm{Sp}(n-1)\big).$$

Then $\mathrm{Gr}_2^0(2n+2,\mathbb{C})$ is a real-analytic f-quaternionic manifold and its heaven space is $\mathrm{Gr}_2(2n+2,\mathbb{C})$. Its twistor space is $\mathrm{Gr}_2^0(2n+2,\mathbb{C})$ itself, considered as a complex manifold.

To describe the CR twistor space of $\mathrm{Gr}_2^0(2n+2,\mathbb{C})$, firstly, recall that the twistor space of $\mathrm{Gr}_2(2n+2,\mathbb{C})$ is the flag manifold $\mathrm{F}_{1,2n+1}(2n+2,\mathbb{C})$ formed of the pairs (ℓ,p) with ℓ and p complex vector subspaces of $\mathbb{C}^{2n+2}$ of dimensions 1 and $2n+1$, respectively, such that $\ell\subseteq p$.

Now, let $Z\subseteq \mathrm{Gr}_2^0(2n+2,\mathbb{C})\times\mathrm{Gr}_2^0(2n+2,\mathbb{C})$ be formed of the pairs (p,q) such that $p\cap q$ and $p\cap q^{\perp}$ are nontrivial and the latter is contained by the kernel of $\omega|_{q^{\perp}}$, where the orthogonal complement is taken with respect to the underlying Hermitian metric of $\mathbb{C}^{2n+2}$. Then the embedding $Z\to\mathrm{F}_{1,2n+1}(2n+2,\mathbb{C})$, $(p,q)\mapsto(p\cap q, q^{\perp}+p\cap q)$ induces a CR structure with respect to which Z is the CR twistor space of $\mathrm{Gr}_2^0(2n+2,\mathbb{C})$.

Note that if $n=1$ we obtain the f-quaternionic manifold of Example 4.6 with $l=2$.

The next example is related to a construction of [23] (see also [9, Example 4.4]).

Example 4.8: Let M be a quaternionic manifold, ∇ a quaternionic connection on it and Z its twistor space.

Then Z is the sphere bundle of an oriented Riemannian vector bundle of rank three Q. By extending the structural group of the frame bundle $\big(\mathrm{SO}(Q),M,\mathrm{SO}(3,\mathbb{R})\big)$ of Q we obtain a principal bundle $\big(H,M,\mathbb{H}^*/\mathbb{Z}_2\big)$.

Let $q\in S^2$ ($\subseteq\mathrm{Im}\mathbb{H}$). The morphism of Lie groups $\mathbb{C}^*\to\mathbb{H}^*$, $a+b\,\mathrm{i}\mapsto a-bq$ induces an action of $\mathbb{C}^*$ on H whose quotient space is Z (considered with its underlying smooth structure); denote by $\psi_q:H\to Z$ the projection. Moreover, $(H,Z,\mathbb{C}^*)$ is a principal bundle on which ∇ induces a principal connection for which the $(0,2)$ component of its curvature form is zero. Therefore, the complex structures of Z and of the fibres of H induce, through this connection, a complex structure J_q on H.

We thus obtain a hypercomplex manifold $(H,J_{\mathrm{i}},J_{\mathrm{j}},J_{\mathrm{k}})$ which is the heaven space of an f-quaternionic structure on $\mathrm{SO}(Q)$ (in fact, a hyper-f structure). Note that the twistor space of $\mathrm{SO}(Q)$ is $\mathbb{C}P^1\times Z$ and the corresponding projection from $S^2 \times \mathrm{SO}(Q)$ onto $\mathbb{C}P^1 \times Z$ is given by $(q,u) \mapsto \big(q, \psi_q(u)\big)$, for any $(q,u) \in S^2 \times \mathrm{SO}(Q)$.

If $M = \mathbb{H}P^k$, then the factorisation through $\mathbb{Z}_2$ is unnecessary and we obtain an f-quaternionic structure on S^{4k+3} with heaven space $\mathbb{H}^{k+1} \setminus \{0\}$ and twistor space $\mathbb{C}P^1 \times \mathbb{C}P^{2k+1}$.

Let (M,E,V) be an almost f-quaternionic manifold, with $\operatorname{rank} V = l$, and $(P,M,G_{k,l})$ the corresponding reduction of the frame bundle of M, where $\operatorname{rank} E = 4k$. Then $TM = (V \otimes Q) \oplus W$, where W is the quaternionic vector bundle associated to P through the canonical morphism of Lie groups $G_{k,l} \longrightarrow \mathrm{Sp}(1) \cdot \mathrm{GL}(k-l, \mathbb{H})$. Note that W is the largest quaternionic vector subbundle of E contained by TM.

Theorem 4.9: *Let (M,E,V) be an almost f-quaternionic manifold and let D be a compatible torsion-free connection,* $\operatorname{rank} E = 4k$, $\operatorname{rank} V = l$*; suppose that $(k,l) \neq (2,2), (1,0)$. Then (M,E,V,∇) is f-quaternionic, where $\nabla = D^V \oplus D$. Moreover, W is integrable if and only if it is geodesic, with respect to D (equivalently, $D_X Y$ is a section of W, for any sections X and Y of W).*

Proof. Let $\iota : TM \to E$ be the inclusion and $\rho : E \to TM$ the projection. It quickly follows that we may apply [17, Theorem 4.6] to obtain that (M,E,ι,∇) is CR quaternionic. To prove that (M,E,ρ,∇) is co-CR quaternionic we apply [17, Theorem A.3] to D. Thus, we obtain that it is sufficient to show that for any $J \in Z$ and any $X,Y,Z \in E^J$ we have $R^D(\rho(X),\rho(Y))(\rho(Z)) \in \rho(E^J)$, where E^J is the eigenspace of J, with respect to $-\mathrm{i}$, and R^D is the curvature form of D; equivalently, for any $J \in Z$ and any $X,Y,Z \in E^J$ we have $R^\nabla(\rho(X),\rho(Y))Z \in E^J$, where R^∇ is the curvature form of ∇. The proof of the fact that (M,E,V,∇) is f-quaternionic follows, similarly to the proof of [17, Theorem 4.6]. The last statement follows quickly from the fact that $(\nabla_X J)(Y)$ is a section of W, for any section J of Z and X, Y of W. ∎

From the proof of Theorem 4.9 we immediately obtain the following.

Corollary 4.10: *Let (M,E,V) be an almost f-quaternionic manifold and let D be a compatible torsion-free connection,* $\operatorname{rank} E \geq 8$*. Then (M,E,ρ,∇) is co-CR quaternionic, where $\rho : E \to TM$ is the projection and $\nabla = D^V \oplus D$.*

Next, we prove two realizability results for f-quaternionic manifolds.

Proposition 4.11: *Let* (M, E, V, ∇) *be an* f*-quaternionic manifold,* $\operatorname{rank} V = 1$*, where* $\nabla = D^V \oplus D$ *for some compatible connection* D *on* M*. Then* (M, E, ι, ∇) *is realizable, where* $\iota : TM \to E$ *is the inclusion.*

Proof. By passing to a convex open set of D, if necessary, we may suppose that the twistorial structure $(Z, M, \pi, \mathcal{D})$ of the co-CR quaternionic manifold (M, E, ρ) is simple, where $\rho : E \to TM$ is the projection. Thus, by Theorem 3.6 , we have that $(Z, M, \pi, \mathcal{D})$ is real analytic. It follows that $Q^{\mathbb{C}}$ is real analytic which, together with the relation $TM = (V \otimes Q) \oplus W$, quickly gives that the twistorial structure $(Z, M, \pi, \mathcal{C})$ of (M, E, ι) is real analytic. By [17, Corollary 5.4] the proof is complete. ∎

The next result is an immediate consequence of Theorem 4.9 and Proposition 4.11 .

Corollary 4.12: *Let* (M, E, V) *be an almost* f*-quaternionic manifold, with* $\operatorname{rank} V = 1$, $\operatorname{rank} E \geq 8$ *, and let* ∇ *be a torsion-free connection on* E *compatible with its linear quaternionic structure, and induced by a connection on* M*. Then* (M, E, ι, ∇) *is realizable, where* $\iota : TM \to E$ *is the inclusion.*

We end this section with the following result.

Proposition 4.13: *Let* (M, E, V, ∇) *be a real analytic* f*-quaternionic manifold, with* $\operatorname{rank} V = 1$ *, where* $\nabla = D^V \oplus D$ *for some torsion-free compatible connection* D *on* M*. Let* N *be the heaven space of* (M, E, ι, ∇)*, where* $\iota : TM \to E$ *is the inclusion, and denote by* Z_N *its twistor space. Then* Z_N *is endowed with a nonintegrable holomorphic distribution* $\mathscr{H}$ *of codimension one, transversal to the twistor lines corresponding to the points of* $N \setminus M$*.*

Proof. By passing to a complexification, we may assume all the objects complex analytic. Furthermore, excepting Z, we shall denote by the same symbols the corresponding complexifications. As for Z, this will denote the bundle of isotropic directions of Q. Then any $p \in Z$ corresponds to a vector subspace E^p of E. Let $\mathscr{F}$ be the distribution on Z such that $\mathscr{F}_p$ is the horizontal lift, with respect to ∇, of $\iota^{-1}(E^p)$, $(p \in Z)$. As (M, E, V, ∇) is (complex) f-quaternionic, $\mathscr{F}$ is integrable. Moreover, locally, we may suppose that its leaf space is Z_N . Let $\mathscr{G}$ be the distribution on Z such that, at each $p \in Z$, we have that $\mathscr{G}_p$ is the horizontal lift of $(V_x \otimes p^{\perp}) \oplus W_x$, where $x = \pi(p)$. Define $\mathscr{K} = \mathscr{G} \oplus \ker \mathrm{d}\pi$. Then the complex analytic versions of Cartan's structural equations and [11, Proposition III.2.3] , straightforwardly show that $\mathscr{K}$ is projectable with respect to $\mathscr{F}$. Thus, $\mathscr{K}$ projects to a distribution $\mathscr{H}$ on Z_N of codimension one. Furthermore, by using again [11, Proposition III.2.3], we obtain that $\mathscr{H}$ is nonintegrable. ∎

5. Quaternionic-Kähler manifolds as heaven spaces

A **quaternionic-Kähler** manifold is a quaternionic manifold endowed with a (semi-Riemannian) Hermitian metric whose Levi-Civita connection is quaternionic and whose scalar curvature is assumed nonzero.

Let (M, E, ι, ∇) be a CR quaternionic manifold with $\operatorname{rank} E = \dim M + 1$. Let W be the largest quaternionic vector subbundle of E contained by TM and denote by $\mathcal{I}$ the (Frobenius) integrability tensor of W. From the integrability of the almost twistorial structure of (M, E, ι, ∇) it follows that, for any $J \in Z$, the two-form $\mathcal{I}|_{E^J}$ takes values in $E^J/(E^J \cap W^{\mathbb{C}})$; as this is one-dimensional the condition $\mathcal{I}|_{E^J}$ nondegenerate has an obvious meaning.

Definition 5.1: A CR quaternionic manifold (M, E, ι, ∇), with $\operatorname{rank} E = \dim M + 1$, is **nondegenerate** if $\mathcal{I}|_{E^J}$ is nondegenerate, for any $J \in Z$.

Let M be a submanifold of a quaternionic manifold N and Z the twistor space of N.

Denote by B the second fundamental form of M with respect to some quaternionic connection ∇ on N; that is, B is the (symmetric) bilinear form on M, with values in $(TN|_M)/TM$, characterised by $B(X, Y) = \sigma(\nabla_X Y)$, for any vector fields X, Y on M, where $\sigma : TN|_M \to (TN|_M)/TM$ is the projection.

Definition 5.2: We say that M is **q-umbilical** in N if for any $J \in Z|_M$ the second fundamental form of M vanishes along the eigenvectors of J which are tangent to M.

From [9, Propositions 1.8(ii) and 2.8] it quickly follows that the notion of q-umbilical submanifold, of a quaternionic manifold, does not depend of the quaternionic connection used to define the second fundamental form.

Note that if $\dim N = 4$, then we retrieve the usual notion of umbilical submanifold. Also, if a quaternionic manifold is endowed with a Hermitian metric, then any umbilical submanifold of it is q-umbilical.

The notion of q-umbilical submanifold of a quaternionic manifold can be easily extended to CR quaternionic manifolds. Indeed, just define the second fundamental form B of (M,E,ι,∇) by $B(X,Y)=\frac{1}{2}\,\sigma(\nabla_XY+\nabla_YX)$, for any vector fields X and Y on M, where $\sigma:E\to E/TM$ is the projection.

Theorem 5.3: *Let N be the heaven space of a real analytic CR quaternionic manifold (M,E,ι,∇), with $\operatorname{rank}E=\dim M+1$. If M is q-umbilical in N, then the twistor space Z_N of N is endowed with a nonintegrable holomorphic distribution $\mathscr{H}$ of codimension one, transversal to the twistor lines corresponding to the points of $N\setminus M$. Furthermore, the following assertions are equivalent:*

(i) *$\mathscr{H}$ is a holomorphic contact structure on Z_N.*
(ii) *(M,E,ι,∇) is nondegenerate.*

Proof. By passing to a complexification, we may assume all the objects complex analytic. Also, we may assume ∇ torsion free. Furthermore, excepting Z, which will be soon described, below, we shall denote by the same symbols the corresponding complexifications.

Let $\dim N=4k$. As the complexification of $\mathrm{Sp}(1)\cdot\mathrm{GL}(k,H)$ is $\mathrm{SL}(2,\mathbb{C})\cdot\mathrm{GL}(2k,\mathbb{C})$, we may assume that, locally, $TN=H\otimes F$ where H and F are (complex analytic) vector bundles of rank 2 and $2k$, respectively. Also, H is endowed with a nowhere zero section ε of Λ^2H^* and $\nabla=\nabla^H\otimes\nabla^F$, for some connections ∇^H and ∇^F on H and F, respectively, with $\nabla^H\varepsilon=0$.

Then, by restricting to a convex neighbourhood of ∇, if necessary, Z_N is the leaf space of the foliation $\mathscr{F}_N$ on PH which, at each $[u]\in PH$, is given by the horizontal lift, with respect to ∇^H of $[u]\otimes F_{\pi_H(u)}$, where $\pi_H:H\to N$ is the projection. Let $Z=PH|_M$ and let $\mathscr{F}$ be the foliation induced by $\mathscr{F}_N$ on Z. Note that the leaf space of $\mathscr{F}$ is Z_N.

Let $PH+PF^*$ be the restriction to N of $PH\times PF^*$. Then

$$([u],[\alpha])\mapsto[u]\otimes\ker\alpha$$

defines an embedding of $PH+PF^*$ into the Grassmann bundle P of $(2k-1)$-dimensional vector spaces tangent to N. As $\nabla=\nabla^H\otimes\nabla^F$, this embedding preserves the connections induced by ∇^H, ∇^F and ∇ on $PH+PF^*$ and P. Let $\mathscr{F}_P$ be the distribution on P which, at each $p\in P$, is the horizontal lift, with respect to ∇, of $p\subseteq T_{\pi_P(p)}N$, where $\pi_P:P\to N$ is the projection. Then the restriction of $\mathscr{F}_P$ to $PH+PF^*$ is a distribution $\mathscr{F}'$ on $PH+PF^*$.

The map $Z \to P$, $[u] \mapsto TM \cap \big([u] \otimes F_{\pi_H(u)}\big)$, is an embedding whose image is contained by $PH + PF^*$. Moreover, the fact that M is q-umbilical in N is equivalent to the fact that $\mathscr{F}$ is the restriction of $\mathscr{F}_P$ to Z.

If for any $([u],[\alpha]) \in PH + PF^*$ we take the preimage of $\ker(\varepsilon(u) \otimes \alpha)$ through the projection of $PH + PF^*$, we obtain a distribution of codimension one $\mathscr{G}'$ on $PH + PF^*$ which contains $\mathscr{F}'$. Furthermore, $\mathscr{G} = TZ \cap \mathscr{G}'$ is a codimension one distribution on Z which contains $\mathscr{F}$.

To prove that $\mathscr{G}$ is projectable with respect to $\mathscr{F}$, firstly, observe that this is equivalent to the fact that the integrability tensor of $\mathscr{G}$ is zero when evaluated on the pairs in which one of the vectors is from $\mathscr{F}$. Thus, as $\mathscr{F}$ is integrable, $\mathscr{F} = \mathscr{F}'|_Z$ and $\mathscr{G} = TZ \cap \mathscr{G}'$, it is sufficient to prove that, at each $p \in PH + PF^*$, the integrability tensor of $\mathscr{G}'$ is zero when evaluated on the pairs formed of a vector from a basis of $\mathscr{F}'_p$ and a vector from a basis of a space complementary to $\mathscr{F}'_p$.

Let $\mathrm{SL}(H)$ and $\mathrm{GL}(F)$ be the frame bundles of H and F, respectively, and let $\mathrm{SL}(H) + \mathrm{GL}(F)$ be the restriction to N of $\mathrm{SL}(H) \times \mathrm{GL}(F)$. Then the kernel of the differential of the projection of $\mathrm{SL}(H) + \mathrm{GL}(F)$ is the trivial vector bundle over $\mathrm{SL}(H) + \mathrm{GL}(F)$ with fibre $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{gl}(2k,\mathbb{C})$. Also, note that, for any $(u,v) \in \mathrm{SL}(H) + \mathrm{GL}(F)$, we have that $u \otimes v$ is a (complex-quaternionic) frame on N.

Let G be the closed subgroup of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{GL}(2k,\mathbb{C})$ which preserves some fixed pair $\big([x_0],[\alpha_0]\big) \in \mathbb{C}P^1 \times P\big((\mathbb{C}^{2k})^*\big)$. Then

$$PH + PF^* = \big(\mathrm{SL}(H) + \mathrm{GL}(F)\big)/G$$

and we denote $\mathscr{F}'' = (\mathrm{d}\mu)^{-1}(\mathscr{F}')$ and $\mathscr{G}'' = (\mathrm{d}\mu)^{-1}(\mathscr{G}')$, where μ is the projection from $\mathrm{SL}(H) + \mathrm{GL}(F)$ onto $PH + PF^*$.

For any $\xi \in \mathbb{C}^2 \otimes \mathbb{C}^{2k}$ we define a horizontal vector field $B(\xi)$ which, at any $(u,v) \in \mathrm{SL}(H) + \mathrm{GL}(F)$, is the horizontal lift of $(u \otimes v)(\xi)$. Then $\mathscr{F}''$ is generated by the Lie algebra of G and all $B(x_0 \otimes y)$ with $\alpha_0(y) = 0$. Also, $\mathscr{G}''$ is generated by $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{gl}(2k,\mathbb{C})$ and all $B(\xi)$ with $\big(\varepsilon_0(x_0) \otimes \alpha_0\big)(\xi) = 0$, where ε_0 is the volume form on $\mathbb{C}^2$.

Further, similarly to [11, Proposition III.2.3], we have $\big[A_1 \oplus A_2, B(x_1 \otimes x_2)\big] = B(A_1x_1 \otimes x_2 + x_1 \otimes A_2x_2)$, for any $A_1 \in \mathfrak{sl}(2,\mathbb{C})$, $A_2 \in \mathfrak{gl}(2k,\mathbb{C})$, $x_1 \in \mathbb{C}^2$ and $x_2 \in \mathbb{C}^{2k}$. Also, because ∇ is torsion-free we have that, for any $\xi, \eta \in \mathbb{C}^2 \otimes \mathbb{C}^{2k}$, the horizontal component of $\big[B(\xi), B(\eta)\big]$ is zero. These facts quickly show that, at each $(u,v) \in \mathrm{SL}(H) + \mathrm{GL}(F)$, the integrability tensor of $\mathscr{G}''$ is zero when evaluated on the pairs formed of a vector from a basis of $\mathscr{F}''_{(u,v)}$ and a vector from a basis of a space complementary to $\mathscr{F}''_{(u,v)}$. Consequently, $\mathscr{G}$ is projectable with respect to $\mathscr{F}$.

Next, we shall prove that $\mathscr{G}$ is nonintegrable. For this, firstly, observe that those (u,v) in $\big(\mathrm{SL}(H)+\mathrm{GL}(F)\big)|_M$ for which $u\otimes v$ preserves the corresponding tangent space to M form a principal bundle, which we shall call 'the bundle of adapted frames', whose structural group K can be described as follows. We may write $\mathbb{C}^2\otimes\mathbb{C}^{2k}=\mathfrak{gl}(2,\mathbb{C})\oplus\big(\mathbb{C}^2\otimes\mathbb{C}^{2k-2}\big)$ so that K is the closed subgroup of $\mathrm{SL}(2,\mathbb{C})\times\mathrm{GL}(2k,\mathbb{C})$ which preserves $\mathrm{Id}_{\mathbb{C}^2}$. Thus, K contains $\mathrm{SL}(2,\mathbb{C})$ acting on $\mathfrak{gl}(2,\mathbb{C})\oplus\big(\mathbb{C}^2\otimes\mathbb{C}^{2k-2}\big)$ by $(a,(\xi,\eta))\mapsto(a\xi a^{-1},\eta)$, for any $a\in\mathrm{SL}(2,\mathbb{C})$, $\xi\in\mathfrak{gl}(2,\mathbb{C})$ and $\eta\in\mathbb{C}^2\otimes\mathbb{C}^{2k-2}$.

Note that TM is the bundle associated to the bundle of adapted frames through the action of K on $\mathfrak{sl}(2,\mathbb{C})\oplus\big(\mathbb{C}^2\otimes\mathbb{C}^{2k-2}\big)$. Also, Z $(\subseteq P)$ is the quotient of the bundle of adapted frames through the closed subgroup of K preserving $\mathbb{C}\xi_0\oplus\big(\ker\xi_0\otimes\mathbb{C}^{2k-2}\big)$, for some fixed $\xi_0\in\mathfrak{sl}(2,\mathbb{C})\setminus\{0\}$ with $\det\xi_0=0$.

If we, locally, consider a principal connection on the bundle of adapted frames, then we can define, similarly to above, the corresponding 'standard horizontal vector fields' $B(\xi)$, for any $\xi\in\mathfrak{sl}(2,\mathbb{C})\oplus\big(\mathbb{C}^2\otimes\mathbb{C}^{2k-2}\big)$, so that $\mathscr{G}$ corresponds to the distribution generated by the Lie algebra of K and $\mathscr{F}_1$, where $\mathscr{F}_1$ is formed of all $B(\xi)$ with $\xi\in\mathbb{C}^2\otimes\mathbb{C}^{2k-2}$ or $\xi\in\mathfrak{sl}(2,\mathbb{C})$ such that $\xi(\ker\xi_0)\subseteq\ker\xi_0$. Thus, if we take $\xi\in\mathfrak{sl}(2,\mathbb{C})$ with $\xi(\ker\xi_0)\subseteq\ker\xi_0$ and $A\in\mathfrak{sl}(2,\mathbb{C})$ such that $[A,\xi](\ker\xi_0)\nsubseteq\ker\xi_0$, then A and $B(\xi)$ determine sections of $\mathscr{G}$ whose bracket is not a section of $\mathscr{G}$.

Finally, the equivalence of the assertions (i) and (ii) is a straightforward consequence of the fact that if we denote by W the largest complex-quaternionic subbundle of $TN|_M$ contained by TM, then $\mathscr{F}_1+(\mathrm{d}\pi)^{-1}(W)=\mathscr{G}$, where $\pi:Z\to M$ is the projection. ∎

The next result follows from [15] and Theorem 5.3.

Corollary 5.4: *The following assertions are equivalent, for a real analytic hypersurface M embedded in a quaternionic manifold N:*

(i) *M is nondegenerate and q-umbilical.*

(ii) *By passing, if necessary, to an open neighbourhood of M, there exists a metric g on $N\setminus M$ such that $(N\setminus M,g)$ is quaternionic-Kähler and the twistor lines determined by the points of M are tangent to the contact distribution, on the twistor space of N, corresponding to g.*

If $\dim M = 3$, then Corollary 5.4 and [17, Corollary 5.5] give the main result of [13] . Also, the 'quaternionic contact' manifolds of [5] (see [7]) are nondegenerate q-umbilical CR quaternionic manifolds.

Appendix A. The intrinsic description of linear (co-)CR quaternionic structures

A **conjugation**, on a quaternionic vector space, is an involutive quaternionic automorphism (not equal to the identity); in particular, the corresponding orientation-preserving isometry on the space of admissible complex structures is a symmetry in a line.

Example A.1 ([6]): Let $U^{\mathbb{H}} = \mathbb{H} \otimes U$ be the **quaternionification** of a vector space U (the tensor product is taken over $\mathbb{R}$), endowed with the linear quaternionic structure induced by the multiplication to the left.

If $q \in S^2$, then the association $q' \otimes u \mapsto -qq'q \otimes u$, for any $q' \in \mathbb{H}$ and $u \in U$, defines a conjugation on $U^{\mathbb{H}}$.

In fact, more can be proved.

Proposition A.2: *Any pair of distinct commuting conjugations τ_1 and τ_2 on a quaternionic vector space E determines a quaternionic linear isomorphism $E = U^{\mathbb{H}}$, for some vector space U, so that τ_1 and τ_2 are defined, as in Example A.1, by two orthogonal imaginary unit quaternions.*

Proof. Let $T_1, T_2 : Z \to Z$ be the orientation-preserving isometries corresponding to τ_1 τ_2, respectively, where Z is the space of admissible linear complex structures on E.

As T_1 and T_2 are commuting symmetries in lines ℓ_1 and ℓ_2, respectively, it follows that either $\ell_1 = \ell_2$ or $\ell_1 \perp \ell_2$. In the former case, we would have $T_1T_2 = \mathrm{Id}_Z$ which, together with the fact that τ_1 and τ_2 are commuting involutions, implies $\tau_1 = \tau_2$, a contradiction. Thus, if ℓ_1 and ℓ_2 are generated by I and J, respectively, then $IJ = -IJ$; denote $K = IJ$.

Now, $E = U^+ \oplus U^-$, where $U^{\pm} = \ker\big(\tau_1 \mp \mathrm{Id}_E\big)$. Furthermore, as $\tau_1\tau_2 = \tau_2\tau_1$, we have $U^+ = V^+ \oplus V^-$ and $U^- = W^+ \oplus W^-$, where $V^{\pm} = \ker\big(\tau_2|_{U^+} \mp \mathrm{Id}_{U^+}\big)$ and $W^{\pm} = \ker\big(\tau_2|_{U^-} \mp \mathrm{Id}_{U^-}\big)$.

A straightforward argument shows that $IV^+ = V^-$, $JV^+ = W^+$ and $KV^+ = W^-$. Thus, if we denote $U = V^+$, then $E = U \oplus IU \oplus JU \oplus KU$ and the association $q\otimes u \mapsto q_0u+q_1Iu+q_2Ju+q_3Ku$, for any $q = q_0+q_1\mathrm{i}+q_2\mathrm{j}+q_3\mathrm{k} \in \mathbb{H}$ and $u \in U$, defines a quaternionic linear isomorphism from $U^{\mathbb{H}}$ onto E which is as required. ∎

The quaternionification of a linear map is defined in the obvious way. Then a quaternionic linear map between the quaternionifications of two vector spaces is the quaternionification of a linear map if and only if it intertwines two distinct commuting conjugations.

Let U be a vector space and let Λ be the space of conjugations on $U^{\mathbb{H}}$.

The next proposition reformulates a result of [6].

Proposition A.3: *There exist natural correspondences between the following:*

(i) *linear quaternionic structures on U;*

(ii) *quaternionic vector subspaces $B \subseteq U^{\mathbb{H}}$ such that $U^{\mathbb{H}} = B\oplus\sum_{\tau\in\Lambda}\tau(B)$;*

(iii) *quaternionic vector subspaces $C \subseteq U^{\mathbb{H}}$ such that $U^{\mathbb{H}} = C\oplus\bigcap_{\tau\in\Lambda}\tau(C)$.*

Furthermore, the correspondences are such that $C = \sum_{\tau\in\Lambda}\tau(B)$ and $B = \bigcap_{\tau\in\Lambda}\tau(C)$.

We can now give the intrinsic description of linear CR quaternionic structures.

Proposition A.4: *There exists a natural correspondence between the following:*

(i) *linear CR quaternionic structures on U;*

(ii) *quaternionic vector subspaces $C \subseteq U^{\mathbb{H}}$ such that*

 (ii1) $C\cap\bigcap_{\tau\in\Lambda}\tau(C) = 0$,

 (ii2) $C+\sigma(C) = U^{\mathbb{H}}$, *for any $\sigma \in \Lambda$.*

Proof. If (E,ι) is a linear CR quaternionic structure on U, then $C = \left(\iota^{\mathbb{H}}\right)^{-1}(C_E)$ satisfies assertion (ii), where C_E is the quaternionic vector subspace of $E^{\mathbb{H}}$ given by assertion (iii) of Proposition A.3.

Conversely, if C is as in (ii), then on defining $E = U^{\mathbb{H}}/C$ and ι to be the composition of the inclusion of U into $U^{\mathbb{H}}$ followed by the projection from the latter onto E we obtain the corresponding linear CR quaternionic structure. ∎

Finally, by duality, we also have

Proposition A.5: *There exists a natural correspondence between the following:*

(i) *linear co-CR quaternionic structures on U;*
(ii) *quaternionic vector subspaces $B \subseteq U^{\mathbb{H}}$ such that*
 (ii1) $U^{\mathbb{H}} = B + \sum_{\tau \in \Lambda} \tau(B)$,
 (ii2) $B \cap \sigma(B) = 0$, *for any $\sigma \in \Lambda$.*

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