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ON THE CLASSIFICATION OF THE REAL VECTOR SUBSPACES OF A QUATERNIONIC VECTOR SPACE

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Abstract We prove the classification of the real vector subspaces of a quaternionic vector space by using a covariant functor which associates, to any pair formed of a quaternionic vector space and a real subspace, a coherent sheaf over the sphere.

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1. Introduction

Let X_E be the space of (real) vector subspaces of a vector space E. Then, X_E is a disjoint union of Grassmannians and $\mathrm{GL}(E)$ acts transitively on each of its components.

If E is endowed with a linear geometric structure, corresponding to the Lie subgroup $G \subseteq \mathrm{GL}(E)$, then it is natural to ask whether or not the action induced by G on X_E is still transitive on each component and, if not, to find explicit representatives for each orbit.

For example, if E is a Euclidean vector space and, accordingly, G is the orthogonal group, then the orthonormalization process shows that G acts transitively on each component of X_E.

Suppose, instead, that E is endowed with a linear complex structure J; equivalently, $E = \mathbb{C}^k$ and $G = \mathrm{GL}(k, \mathbb{C})$. Then, for any vector subspace U of E we have a decomposition $U = F \times V$, where F is a complex vector subspace of E and V is totally real (that is, $V \cap JV = 0$); obviously, the filtration $0 \subseteq F \subseteq U$ is canonical. Consequently, the subspaces $\mathbb{C}^m \times \mathbb{R}^l$, where $2m + l \leqslant 2k$, are representatives for each of the orbits of $\mathrm{GL}(k, \mathbb{C})$ on $X_{\mathbb{C}^k}$.

The corresponding decomposition for the real subspaces of a hypercomplex vector space (that is, $E = \mathbb{H}^k$ and $G = \mathrm{GL}(k, \mathbb{H})$) was obtained in [**2**].

By using a different method, we obtain the decomposition and the canonical filtration for the real subspaces of a quaternionic vector space; that is, $E = \mathbb{H}^k$ and $G = \mathrm{Sp}(1) \cdot \mathrm{GL}(k, \mathbb{H})$. This involves a covariant functor from the category of pairs (U, E), where E is a quaternionic vector space and $U \subseteq E$ is a real vector subspace (with the obvious morphisms induced by the linear quaternionic maps), to the category of coherent sheaves on the Riemann sphere. We mention that a similar functor appeared in [**7**] (see [**8**]).

2. Complex and (co-)CR vector spaces

A *linear complex structure* on a (real) vector space U is a linear map $J\colon U \to U$ such that $J^2 = -\operatorname{Id}_U$. Then, on associating to any linear complex structure the $-\mathrm{i}$ eigenspace of its complexification, we obtain a (bijective) correspondence between the space of linear complex structures on U and the space of complex vector subspaces C of $U^{\mathbb{C}}$ such that $C \oplus \bar{C} = U^{\mathbb{C}}$.

This suggests that we consider the following two less restrictive conditions for a complex vector subspace C of $U^{\mathbb{C}}$:

(1) $C \cap \bar{C} = 0$,

(2) $C + \bar{C} = U^{\mathbb{C}}$.

Furthermore, conditions (1) and (2) are dual to each other. That is, $C \subseteq U^{\mathbb{C}}$ satisfies (1) if and only if $\operatorname{Ann} C \subseteq (U^{\mathbb{C}})^*$ satisfies (2), where $\operatorname{Ann} C = \{\alpha \in (U^{\mathbb{C}})^* \mid \alpha|_C = 0\}$ is the *annihilator* of C.

Now, it is a standard fact that if $C \subseteq U^{\mathbb{C}}$ satisfies (1), then it is called a *linear CR structure* on U.

Therefore, a complex vector subspace C of $U^{\mathbb{C}}$ satisfying $C + \bar{C} = U^{\mathbb{C}}$ is called a *linear co-CR structure* on U [**6**].

Thus, a complex vector subspace of $U^{\mathbb{C}}$ is a linear co-CR structure on U if and only if its annihilator is a linear CR structure on U^*.

A vector space endowed with a linear (co-)CR structure is a *(co-)CR vector space.*

If U is a vector subspace of a vector space E, endowed with a linear complex structure J, then $C = U^{\mathbb{C}} \cap E^J$ is a linear CR structure on U, where E^J is the $-\mathrm{i}$ eigenspace of J. Moreover, if we further assume that $U + JU = E$, then (E, J) is, up to complex linear isomorphisms, the unique complex vector space, containing U, such that $C = U^{\mathbb{C}} \cap E^J$.

Thus, we have the following fact.

Proposition 2.1 (see [6]). *Any CR vector space corresponds to a pair (U, E), where (E, J) is a complex vector space and U is a vector subspace of E such that $U + JU = E$.*

We also have the following dual fact.

Proposition 2.2 (see [6]). *Any co-CR vector space corresponds to a pair (V, E), where (E, J) is a complex vector space and V is a vector subspace of E such that $V \cap JV = 0$.*

Proof. Let (E, J) be a complex vector space and let $V \subseteq E$ be totally real; that is, $V \cap JV = 0$. Let $U = E/V$ and let $\pi\colon E \to U$ be the projection. Then, $\pi(E^J)$ is a linear co-CR structure on U and the proof follows quickly. □

Let (E, J) be a complex vector space and let U be a vector subspace of E. Then, obviously, $F = U \cap JU$ is invariant under J, and therefore $(F, J|_F)$ is a complex vector subspace of (E, J). Moreover, $(F, J|_F)$ is the biggest complex vector subspace of (E, J) contained by U. Consequently, if V is a complement of F in U, then V is totally real in E.

Thus, we have a decomposition $U = F \oplus V$; moreover, the filtration $0 \subseteq F \subseteq U$ is canonical.

As already suggested, it is useful to consider pairs (U, E), with E a complex vector space and U a vector subspace of E. A morphism $t\colon (U, E) \to (U', E')$, between two such pairs, is a complex linear map $t\colon E \to E'$ such that $t(U) \subseteq U'$. Also, there is an obvious notion of product: $(U, E) \times (U', E') = (U \times U', E \times E')$.

Proposition 2.3 (see [2]). *Any pair formed of a complex vector space and a real vector subspace admits a decomposition, unique up to the order of factors, as a (finite) product, in which each factor is either* $(\mathbb{C}, \mathbb{C})$, $(\mathbb{R}, \mathbb{C})$ *or* $(0, \mathbb{C})$.

Proof. Let (E, J) be a complex vector space and let U be a vector subspace of E. We have seen that $U = F \times V$, where $F = U \cap JU$ and V is a complement of F in U. From the fact that $V \cap JV = 0$, it follows that $F \cap (V + JV) = 0$.

Let $E' \subseteq E$ be a complex vector subspace complementary to $F \oplus (V + JV)$. We obviously have that (U, E) is isomorphic to $(F, F) \times (V, V + JV) \times (0, E')$.

To complete the proof, just note that (F, F), $(V, V + JV)$ and $(0, E')$ decompose as products, in which each factor is of the form $(\mathbb{C}, \mathbb{C})$, $(\mathbb{R}, \mathbb{C})$ and $(0, \mathbb{C})$, respectively. $\square$

If we apply Proposition 2.3 to the pair corresponding to a (co-)CR vector space, then we obtain the following facts, dual to each other.

(1) The pair corresponding to a CR vector space admits a decomposition, unique up to the order of factors, as a product, in which each factor is either $(\mathbb{C}, \mathbb{C})$ or $(\mathbb{R}, \mathbb{C})$.

(2) The pair corresponding to a co-CR vector space admits a decomposition, unique up to the order of factors, as a product, in which each factor is either $(\mathbb{R}, \mathbb{C})$ or $(0, \mathbb{C})$.

Thus, we have the following result.

Corollary 2.4. *Any pair formed of a complex vector space and a real vector subspace admits a decomposition as a product of the pair corresponding to a CR vector space and the pair corresponding to a co-CR vector space.*

3. Quaternionic vector spaces

The automorphism group of the (unital) associative algebra of quaternions is $\mathrm{SO}(3, \mathbb{R})$, acting trivially on $\mathbb{R}$ and canonically on $\mathrm{Im}\,\mathbb{H} (= \mathbb{R}^3)$. Thus, if E is a vector space, then there exists a natural action of $\mathrm{SO}(3, \mathbb{R})$ on the space of morphisms of associative algebras from $\mathbb{H}$ to $\mathrm{End}(E)$; that is, on the space of *linear hypercomplex structures* on E. The (non-empty) orbits of this action are the *linear quaternionic structures* on E.

A *quaternionic (hypercomplex) vector space* is a vector space endowed with a linear quaternionic (hypercomplex) structure (see [**1**, **4**]).

Let E be a quaternionic vector space and let $\rho\colon \mathbb{H} \to \mathrm{End}(E)$ be a representative of its linear quaternionic structure. Then, obviously, the space $Z = \rho(S^2)$ of *admissible linear complex structures* on E depends only on the linear quaternionic structure of E. We denote by E^J the $-\mathrm{i}$ eigenspace of $J \in Z$.

The linear quaternionic structure on E corresponds to a linear quaternionic structure on its dual E^* given by the morphism of associative algebras from $\mathbb{H}$ to $\mathrm{End}(E^*)$, which maps any $q \in \mathbb{H}$ to the transpose of $\rho(\bar{q})$. Thus, any admissible linear complex structure J on E corresponds to the admissible linear complex structure J^*, which is the opposite of the transpose of J; note that, $(E^*)^{J^*}$ is the annihilator of E^J.

Let E and E' be quaternionic vector spaces and let Z and Z' be the corresponding spaces of admissible linear complex structures, respectively. A *linear quaternionic map* from E to E' is a linear map $t\colon E \to E'$ such that, for some function $T\colon Z \to Z'$, we have that $t \circ J = T(J) \circ t$ for any $J \in Z$; consequently, if $t \neq 0$, then T is unique and an orientation preserving isometry (see [**4**]).

The (left) $\mathbb{H}$-module structure on $\mathbb{H}^k$ determines a linear quaternionic structure on it. Moreover, for any quaternionic vector space E, with $\dim E = 4k$, there exists a linear quaternionic isomorphism from E onto $\mathbb{H}^k$. The group of linear quaternionic automorphisms of $\mathbb{H}^k$ is $\mathrm{Sp}(1) \cdot \mathrm{GL}(k, \mathbb{H})$, acting on $\mathbb{H}^k$ by $(\pm(a, A), q) \mapsto aqA^{-1}$, for any $\pm(a, A) \in \mathrm{Sp}(1) \cdot \mathrm{GL}(k, \mathbb{H})$ and $q \in \mathbb{H}^k$ (see [**4**]).

We end this section by showing how to define the product of two quaternionic vector spaces E and E'. Let $T\colon Z \to Z'$ be an orientation preserving isometry between the spaces of admissible linear complex structures on E and E'.

If $\rho\colon \mathbb{H} \to \mathrm{End}(E)$ represents the linear quaternionic structure of E, then T is the restriction of a unique linear map $\tilde{T}\colon \rho(\mathbb{H}) \to \mathrm{End}(E')$ such that $\tilde{T} \circ \rho$ determines the linear quaternionic structure on E'.

Then, $q \mapsto (\rho(q), \tilde{T}(\rho(q)))$, $q \in \mathbb{H}$, defines the *product linear quaternionic structure* on $E \times E'$ (with respect to T).

Note that, although the product of two quaternionic vector spaces is well defined (that is, it does not depend on the particular isometry T), it does not make the category of quaternionic vector spaces abelian. Nevertheless, it is obvious that the category of hypercomplex vector spaces is abelian.

4. Pairs formed of a quaternionic vector space and a real vector subspace

The category of quaternionic vector spaces is a full subcategory of the category whose objects are pairs (U, E), where E is a quaternionic vector space and $U \subseteq E$ is a real vector subspace. The morphisms between two such pairs (U, E) and (U', E') are the linear quaternionic maps $t\colon E \to E'$ such that $t(U) \subseteq U'$ (see [**2**]).

If U is a real vector subspace of a quaternionic vector space E, we call $(\mathrm{Ann}\, U, E^*)$ the *dual* of (U, E).

We shall see that there are three basic subcategories of the category of pairs formed of a quaternionic vector space and a real vector subspace, two of which are related to the Twistor Theory (see [**6**]).

Definition 4.1. Let E be a quaternionic vector space and let Z be its space of admissible linear complex structures.

If $\iota\colon U \to E$ is an injective linear map, then (E,ι) is a *linear CR quaternionic structure* on U if $\operatorname{im}\iota + J(\operatorname{im}\iota) = E$ for any $J \in Z$.

A *CR quaternionic vector space* is a vector space endowed with a linear CR quaternionic structure.

By duality, we obtain the notion of a *co-CR quaternionic vector space*.

To any co-CR quaternionic vector space (U,E,ρ) we associate the pair $(\ker\rho, E)$. Thus, the category of co-CR quaternionic vector spaces is a full subcategory of the category of pairs formed of a quaternionic vector space and a real vector subspace; by duality, the latter also includes the category of CR quaternionic vector spaces.

See [**6**] for further information on (co-)CR quaternionic vector spaces.

Remark 4.2.

(1) Let U be a real vector subspace of a quaternionic vector space E. Then, (U,E) is given by a CR quaternionic vector space if and only if its dual is given by a co-CR quaternionic vector space.

(2) Any quaternionic vector space E is both CR and co-CR quaternionic. When we consider E a CR quaternionic vector space, the associated pair is (E,E), while when we consider E a co-CR quaternionic vector space, the associated pair is $(0,E)$.

We shall construct a covariant functor from the category of pairs, formed of a quaternionic vector space and a real vector subspace, to the category of coherent analytic sheaves, over the sphere, endowed with a conjugation covering the antipodal map (see [**3**] for the basic properties of coherent analytic sheaves and [**7**] for coherent analytic sheaves, over the sphere, endowed with a conjugation covering the antipodal map, briefly called 'σ-sheaves').

For this, firstly note that if E is a quaternionic vector space, with $Z(=S^2)$ the space of admissible linear complex structures, then

$$E^{0,1} = \bigcup_{J\in Z} \{J\} \times E^J$$

is a holomorphic vector subbundle of $Z \times E^{\mathbb{C}}$. Now, if $U \subseteq E$ is a real vector subspace, then the projection $E \to E/U$ induces, by restriction, a morphism of holomorphic vector bundles $E^{0,1} \to Z \times (E/U)^{\mathbb{C}}$. Let $\mathcal{U}_-$ and $\mathcal{U}_+$ be the kernel and cokernel, respectively, of this morphism of holomorphic vector bundles.

Definition 4.3. We call $\mathcal{U} = \mathcal{U}_- \oplus \mathcal{U}_+$ the *(coherent analytic) sheaf of* (U,E).

The proof of the following proposition is straightforward.

Proposition 4.4. *The association $(U,E)\mapsto\mathcal{U}$ defines a covariant functor $\mathcal{F}$ from the category of pairs, formed of a quaternionic vector space E and a real vector subspace $U\subseteq E$, to the category of coherent sheaves, on the sphere, endowed with a conjugation covering the antipodal map. Furthermore, $\mathcal{F}$ has the following properties.*

(i) *For any morphism $t\colon (U,E)\to(U',E')$, we have that $\mathcal{F}(t)$ maps $\mathcal{F}(U,E)_\pm$ to $\mathcal{F}(U',E')_\pm$.*

(ii) *If (U,E) is given by a (co-)CR quaternionic vector space, then $\mathcal{F}(U,E)$ is its holomorphic vector bundle.*

With the same notation as in Proposition 4.4, if $\mathcal{U}=\mathcal{U}_+$, then E/U is the space of (global) sections of $\mathcal{U}$ intertwining the antipodal map and the conjugation.

We now give the basic examples of pairs whose sheaves are torsion free (cf. [**2**, **7**]; see also [**6**, **8**]).

Example 4.5. Let $q_1,\dots,q_{k+1}\in S^2$, $k\geqslant 1$, be such that $q_i\neq\pm q_j$ if $i\neq j$. For $j=1,\dots,k$, let

$$e_j=(\underbrace{0,\dots,0}_{j-1},q_j,q_{j+1},\underbrace{0,\dots,0}_{k-j}).$$

Define $U_0=\mathbb{R}$ and, for $k\geqslant 1$, let $U_k=\mathbb{R}^{k+1}+\mathbb{R}e_1+\cdots+\mathbb{R}e_k$.

Then, the sheaf of $(U_k,\mathbb{H}^{k+1})$ is $\mathcal{O}(2k+2)$ for any $k\in\mathbb{N}$. Note that the projection $\mathbb{H}^{k+1}\to\mathbb{H}^{k+1}/U_k$ defines a co-CR quaternionic vector space and the sheaf of the dual of $(U_k,\mathbb{H}^{k+1})$ is $\mathcal{O}(-2k-2)$ for any $k\in\mathbb{N}$.

Example 4.6. Let $V_0=\{0\}$ and, for $k\geqslant 1$, let V_k be the vector subspace of $\mathbb{H}^{2k+1}$ formed of all vectors of the form

$$(z_1,\overline{z_1}+z_2\mathrm{j},z_3-\overline{z_2}\mathrm{j},\dots,\bar{z}_{2k-1}+z_{2k}\mathrm{j},-\overline{z_{2k}}\mathrm{j}),$$

where $z_1,\dots,z_{2k}$ are complex numbers.

Then, the sheaf of $(V_k,\mathbb{H}^{2k+1})$ is $2\mathcal{O}(2k+1)$ for any $k\in\mathbb{N}$. Note that the projection $\mathbb{H}^{2k+1}\to\mathbb{H}^{2k+1}/V_k$ defines a co-CR quaternionic vector space and the sheaf of the dual of $(V_k,\mathbb{H}^{2k+1})$ is $2\mathcal{O}(-2k-1)$ for any $k\in\mathbb{N}$.

The next class of pairs is taken from [**2**].

Example 4.7. For $k\geqslant 1$ and $q\in S^2$, let $W_{k,q}$ be the real vector subspace of $\mathbb{H}^k$ formed of all vectors of the form

$$(a_1+b_1q+b_2\mathrm{i},a_2+b_2q+b_3\mathrm{i},\dots,a_{k-1}+b_{k-1}q+b_k\mathrm{i},a_k+b_kq),\tag{4.1}$$

where $a_1,b_1,\dots,a_k,b_k$ are real numbers, and we have assumed that $q\neq\pm\mathrm{i}$; if $q=\pm\mathrm{i}$, then we replace i by j in (4.1).

Then, for any $p \in S^2 \setminus \{\pm q\}$ we have that $W_{k,q} \cap pW_{k,q} = 0$, while $W_{k,q} \cap qW_{k,q}$ has dimension two. Together with [**7**, Proposition 3.1], this implies that the sheaf of $(W_{k,q}, \mathbb{H}^k)$ is the indecomposable torsion sheaf with conjugation, supported at $\pm q$, and of Chern number $2k$.

5. The main results

Now, we can prove the following.

Theorem 5.1 (cf. [2]). *Any pair formed of a quaternionic vector space and a real vector subspace admits a decomposition, unique up to the order of factors, as a (finite) product, in which each factor is given by one of the Examples 4.5, 4.6 or 4.7, or is the dual of one of the Examples 4.5 or 4.6.*

Proof. Let $\mathcal{U}$ be the sheaf of (U, E). By the dual of [**6**, Proposition 4.7], we have that $\mathcal{U}_-$ is the holomorphic vector bundle of a CR quaternionic vector space (U_-, E_-). Furthermore, from the diagram of the proof of [**7**, Theorem 4.8] (adapted to the case of sheaves with conjugations), we obtain that there exists an injective morphism $t\colon (U_-, E_-) \to (U, E)$, which induces an injective linear map $E_-/U_- \to E/U$; equivalently, $U_- = E_- \cap t^{-1}(U)$. Therefore, t admits a cokernel (U_+, E_+) whose sheaf is, obviously, $\mathcal{U}_+$.

Thus, we may assume that $\mathcal{U} = \mathcal{U}_+$ and, consequently, we have an exact sequence

$$0 \to E^{0,1} \to Z \times (E/U)^{\mathbb{C}} \to \mathcal{U} \to 0. \tag{5.1}$$

Then, the cohomology exact sequence of (5.1) gives a canonical isomorphism (which intertwines the conjugations) $(E/U)^{\mathbb{C}} = H^0(Z, \mathcal{U})$.

Furthermore, the morphism $(0, E) \to (U, E)$ determines a surjective sheaf morphism $\mathcal{E} \to \mathcal{U}$ whose kernel is $Z \times U^{\mathbb{C}}$ (with the corresponding morphism to $\mathcal{E}$ given by the inclusion $Z \times U^{\mathbb{C}} \to Z \times E^{\mathbb{C}}$ followed by the projection $Z \times E^{\mathbb{C}} \to \mathcal{E}$). Thus, we also have that

$$0 \to Z \times U^{\mathbb{C}} \to \mathcal{E} \to \mathcal{U} \to 0. \tag{5.2}$$

The cohomology exact sequence of (5.2), together with the isomorphisms $(E/U)^{\mathbb{C}} = H^0(Z, \mathcal{U})$ and $E^{\mathbb{C}} = H^0(Z, \mathcal{E})$, shows that the inclusion $U \to E$ is determined by $\mathcal{U}$.

Now, tensorising (5.2) with $\mathcal{O}(-2)$, where $\mathcal{O}(-1)$ is the tautological line bundle over $Z(= \mathbb{C}P^1)$, and by passing to cohomology, we deduce that $H^1(Z, \mathcal{U} \otimes \mathcal{O}(-2)) = 0$; equivalently, in the Birkhoff–Grothendieck decomposition of $\mathcal{U}$ there are no trivial terms. Together with Examples 4.5–4.7, this completes the proof. □

Let (U, E) be a pair formed of a quaternionic vector space and a real vector subspace.

Then, (U, E) is a *torsion pair* if it corresponds to a torsion sheaf; equivalently, (U, E) is a product of pairs as in Example 4.7.

The pair (U, E) is *torsion free* if its sheaf is torsion free; equivalently, it is a holomorphic vector bundle.

Corollary 5.2.

(i) *Let (U,E) and (U',E') be pairs formed of a quaternionic vector space and a real vector subspace. Suppose that either (U,E) or (U',E') are torsion free. Then, $(U\times U',E\times E')$ does not depend on the particular isometry used to define $E\times E'$.*

(ii) *Any pair (U,E) formed of a quaternionic vector space and a real vector subspace decomposes uniquely as the product of a torsion pair and (the pairs given by) a CR quaternionic vector space and a co-CR quaternionic vector space; moreover, the filtration $(0,0)\subseteq(U_-,E_-)\subseteq(U_-,E_-)\times(U_t,E_t)\subseteq(U,E)$ is canonical, where (U_t,E_t) is the torsion pair and (U_-,E_-) is the CR quaternionic vector space.*

Proof. If $\mathcal{U}$ is a holomorphic vector bundle over S^2 and $T\colon S^2\to S^2$ is a holomorphic diffeomorphism, then $T^{-1}(\mathcal{U})$ is isomorphic to $\mathcal{U}$ and, furthermore, the same holds for bundles, endowed with a conjugation covering the antipodal map, and their pull-backs through orientation preserving isometries. Assertion (i) follows quickly.

Assertion (ii) follows from (i) and the proof of Theorem 5.1. □

Finally, note that the 'augmented (strengthened) $\mathbb{H}$-modules' of [**5**] (respectively, [**7**]) are just pairs whose decompositions contain no terms of the form $(\mathbb{H},\mathbb{H})$ (respectively, $(0,\mathbb{H})$); equivalently, in the decompositions of their sheaves there are no terms of Chern number -1 (respectively, 1).

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