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On holomorphic maps and Generalized Complex Geometry

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ABSTRACT

We introduce a natural notion of *holomorphic map* between generalized complex manifolds and we prove some related results on Dirac structures and generalized Kähler manifolds.

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0. Introduction

The *generalized complex structures* [1,2] contain, as particular cases, the complex and symplectic structures. Although for the latter structures there exist well known definitions which give the corresponding morphisms (holomorphic maps and Poisson morphisms, respectively), it still lacks a suitable notion of *holomorphic map* with respect to which the class of generalized complex manifolds to become a category.

In this paper we introduce such a notion (Definition 4.1) based on the following considerations. Firstly, holomorphic maps between generalized complex manifolds should be invariant under *B-field transformations*. This is imposed by the fact that the group of (orthogonal) automorphisms of the Courant bracket (which defines the integrability in Generalized Complex Geometry) on a manifold is the semidirect product of the group of diffeomorphisms and the additive group of closed two-forms on the manifold [1]. Secondly, by [1], underlying any linear generalized complex structure there are:

- a linear Poisson structure (that is, a constant Poisson structure on the vector space; see Section 1), and
- a linear co-CR structure (that is, a linear CR structure on the dual vector space; see Section 3),

both of which are preserved under linear *B*-field transformations. Moreover, these two structures determine, up to a (non-unique) linear *B*-field transformation, the given generalized linear complex structure.

A *generalized complex linear map* is a co-CR linear Poisson morphism (Definition 3.1). It follows quickly that a linear map is generalized complex if and only if, up to linear *B*-fields transformations, it is the product of a (classical) complex linear map, between complex vector spaces, and a linear Poisson morphism, between symplectic vector spaces (Proposition 3.2).

A *holomorphic map* between generalized (almost) complex manifolds is a map whose differential is generalized complex (Definition 4.1). Then, essentially, all of the above mentioned (linear) facts hold, locally, in the setting of generalized complex manifolds (Theorem 4.4 and Proposition 4.5).

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The first examples are the classical holomorphic maps, the Poisson morphisms between symplectic manifolds and their products (Example 4.6).

Other large classes of natural examples can be obtained by working with compact or nilpotent Lie groups (Examples 4.7 and 4.8).

Further motivation for our notion of holomorphicity comes from generalized Kähler geometry. For example, if (g, b, J_+, J_-) is the bi-Hermitian structure corresponding to a generalized Kähler manifold (M, L_1, L_2) then the holomorphic functions of (M, L_1) and (M, L_2) are the bi-holomorphic functions of (M, J_+, J_-) and $(M, J_+, -J_-)$, respectively (Remark 5.1). Other natural properties of the holomorphic maps between generalized Kähler manifolds are obtained in Sections 5 and 6 (Remark 5.6(2) and Corollaries 6.7 and 6.8).

Along the way, we obtain results on generalized Kähler manifolds, such as the factorization result Theorem 6.10; see, also, Corollaries 5.7, 6.3 and 6.4, the first of which is a significant improvement of [3, Theorem A].

The paper is organized as follows. In Section 1, after recalling [4] some basic facts on linear Dirac structures, we give explicit descriptions (Proposition 1.3) for the pull-back and push-forward of a linear Dirac structure, which we then use to show that any linear Dirac structure is, in a natural way, the pull-back of a linear Poisson structure (Corollary 1.5; cf. [5, 6]), which we call the *canonical (linear) Poisson quotient* (cf. [6]), of the given linear Dirac structure. The smooth version (Theorem 2.3; cf. [4–6]) of this result is proved in Section 2 together with some other results on Dirac structures. For example, there we show (Corollary 2.5) that, locally, any regular Dirac structure is, up to a B-field transformation, of the form $\mathcal{V} \oplus \mathrm{Ann}(\mathcal{V})$, where $\mathcal{V}$ is (the tangent bundle of) a foliation.

In Section 3, we introduce the notion of generalized complex linear map, along the above mentioned lines. It follows that two generalized linear complex structures L_1 and L_2, on a vector space V, can be identified if and only if L_2 is the linear B-field transform of the push-forward of L_1, through a linear isomorphism of V (Corollary 3.3). Also, we explain (Remark 3.4) why another definition of the notion of generalized complex linear map is, in our opinion, inadequate.

In Section 4, we review some basic facts on generalized complex manifolds and we introduce the corresponding notion of holomorphic map. It follows that if a real analytic map φ, between real analytic regular generalized complex manifolds, is holomorphic then, locally, up to the complexification of a real analytic B-field transformation, the complexification of φ descends to a complex analytic Poisson morphism between canonical Poisson quotients (Proposition 4.10). Also, we show that the pseudo-horizontally conformal submersions with minimal two-dimensional fibers, from Riemannian manifolds, provide natural constructions of generalized complex structures (Example 4.11).

In Section 5, we prove (Theorem 5.3) that if (g, b, J_+, J_-) is the bi-Hermitian structure corresponding to a generalized Kähler structure and we denote $\mathcal{H}^\pm = \ker(J_+ \mp J_-)$ then the following conditions are equivalent:

- $\mathcal{H}^\pm$ integrable;
- $\mathcal{H}^\pm$ geodesic;

It follows that, under natural conditions, the holomorphic maps between generalized Kähler manifolds descend to holomorphic maps between Kähler manifolds (Remark 5.6). Also, we classify the generalized Kähler manifolds M for which $TM = \mathcal{H}^+ \oplus \mathcal{H}^-$ (Corollary 5.7).

In Section 6, we describe, in terms of *tamed symplectic manifolds* (see Definition 6.1) the generalized Kähler manifolds for which either $\mathcal{H}_+$ or $\mathcal{H}_-$ is zero; the obtained result (Theorem 6.2) also appears, in a different form, in [7]. Also, in Corollary 6.3, we prove a factorization result for generalized Kähler manifolds with $\mathcal{H}^+$ a holomorphic foliation, with respect to J_+ and J_-, and $\mathcal{H}^- = 0$ (or $\mathcal{H}^+ = 0$ and $\mathcal{H}^-$ a holomorphic foliation, with respect to J_+ and J_-); see, also, Corollary 6.4 for a similar result and Theorem 6.10 for a generalization.

Furthermore, we explain how the associated holomorphic Poisson structures of [8] fit into our approach (Theorem 6.5, Remark 6.6), we deduce some consequences for holomorphic diffeomorphisms (Corollary 6.7), and we show that, under natural conditions, the holomorphic maps between generalized Kähler manifolds are holomorphic Poisson morphisms (Corollary 6.8).

1. Linear Dirac structures

In this section we recall ([4]; see [5,6,1]) some basic facts on linear Dirac structures.

Let V be a (real or complex, finite dimensional) vector space. The symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $V \oplus V^*$ defined by

$$\langle u + \alpha, v + \beta \rangle = \frac{1}{2}\big(\alpha(v) + \beta(u)\big),$$

for any $u + \alpha, v + \beta \in V \oplus V^*$, corresponds, up to the factor $\frac{1}{2}$, to the canonical isomorphism $V \oplus V^* \xrightarrow{\sim} (V \oplus V^*)^*$ defined by $u + \alpha \longmapsto \alpha + u$, for any $u + \alpha \in V \oplus V^*$. In particular, $\langle \cdot, \cdot \rangle$ is nondegenerate and, if V is real, its index is $\dim V$. Thus, the dimension of the maximal isotropic subspaces of $V \oplus V^*$ (endowed with $\langle \cdot, \cdot \rangle$) is equal to $\dim V$.

Definition 1.1 ([4]). A *linear Dirac structure* on V is a maximal isotropic subspace of $V \oplus V^*$.

If b is a bilinear form on V then we shall denote by the same letter the corresponding linear map from V to V^*; thus, $b(u)(v) = b(u, v)$, for any $u, v \in V$.

Let $E \subseteq V$ be a vector subspace and let $\varepsilon \in \Lambda^2 E^*$; denote

$$L(E, \varepsilon) = \{u + \alpha \mid u \in E, \alpha|_E = \varepsilon(u)\}.$$

From the fact that ε is skew-symmetric it follows easily that $L(E, \varepsilon)$ is isotropic. Also, $L(E, 0) = E \oplus \mathrm{Ann}(E)$, where $\mathrm{Ann}(E) = \{\alpha \in V^* \mid \alpha|_E = 0\}$.

We shall denote by π and $^*\pi$ the projections from $V \oplus V^*$ onto V and V^*, respectively. Also, if $L \subseteq V \oplus V^*$ then $L^\perp$ denotes the 'orthogonal complement' of L with respect to $\langle \cdot, \cdot \rangle$.

Proposition 1.2 *([4])*. *Let L be an isotropic subspace of $V \oplus V^*$ and let $E = \pi(L)$.*
Then there exists a unique $\varepsilon \in \Lambda^2 E^*$ such that $L \subseteq L(E, \varepsilon)$. In particular, if L is a linear Dirac structure then $L = L(E, \varepsilon)$. Furthermore, $V \cap L = \ker \varepsilon$ and $^*\pi(L) = \mathrm{Ann}(V \cap L)$.

Let L be a linear Dirac structure on V. If $^*\pi(L) = V^*$ then L is called a *linear Poisson structure* (see [4]). By Proposition 1.2, if L is a linear Poisson structure then $L = L(V^*, \eta)$ for some *bivector* $\eta \in \Lambda^2 V$ (cf. [9]).

Let V and W be vector spaces endowed with linear Dirac structures L_V and L_W, respectively, and let $f : V \to W$ be a linear map. Denote

$$f_*(L_V) = \{f(X) + \eta \mid X + f^*(\eta) \in L_V\},$$

$$f^*(L_W) = \{X + f^*(\eta) \mid f(X) + \eta \in L_W\}.$$

Proposition 1.3. *Let $f : V \to W$ be a linear map. Let $L(E, \varepsilon)$ and $L(F, \eta)$ be linear Dirac structures on V and W, respectively. Then*

$$f_*\big(L(E, \varepsilon)\big) = L\big(f\big((E \cap \ker f)^{\perp_\varepsilon}\big), \check{\varepsilon}\big),$$

$$f^*\big(L(F, \eta)\big) = L\big(f^{-1}(F), f^*(\eta)\big),$$

where $\check{\varepsilon}$ is characterized by $f^*(\check{\varepsilon}) = \varepsilon$ on $(E \cap \ker f)^{\perp_\varepsilon}$.

Proof. It is easy to prove that $f_*(L_V)$ and $f^*(L_W)$ are isotropic subspaces of $W \oplus W^*$ and $V \oplus V^*$, respectively.

Next, we show that there exists a unique two-form $\check{\varepsilon}$ on $f\big((E \cap \ker f)^{\perp_\varepsilon}\big)$ such that $f^*(\check{\varepsilon}) = \varepsilon$ on $(E \cap \ker f)^{\perp_\varepsilon}$. For this, it is sufficient to prove that if $X_1, X_2 \in (E \cap \ker f)^{\perp_\varepsilon}$ are such that $f(X_1) = f(X_2)$ then $\varepsilon(X_1, Y) = \varepsilon(X_2, Y)$, for any $Y \in (E \cap \ker f)^{\perp_\varepsilon}$. Now, if $X_1, X_2 \in (E \cap \ker f)^{\perp_\varepsilon}$, then $X_1, X_2 \in E$ and, as $X_1 - X_2 \in \ker f$, we have $\varepsilon(X_1 - X_2, Y) = 0$, for any $Y \in (E \cap \ker f)^{\perp_\varepsilon}$.

Thus, to complete the proof it is sufficient to show that

$$f_*\big(L(E, \varepsilon)\big) \supseteq L\big(f\big((E \cap \ker f)^{\perp_\varepsilon}\big), \check{\varepsilon}\big),$$
$$f^*\big(L(F, \eta)\big) \supseteq L\big(f^{-1}(F), f^*(\eta)\big). \tag{1.1}$$

Let $Y + \xi \in L\big(f\big((E \cap \ker f)^{\perp_\varepsilon}\big), \check{\varepsilon}\big)$; equivalently, there exists $X \in (E \cap \ker f)^{\perp_\varepsilon}$ such that $f(X) = Y$ and $\xi(f(X')) = \check{\varepsilon}(f(X), f(X'))$, for any $X' \in (E \cap \ker f)^{\perp_\varepsilon}$.

We claim that $Y + \xi \in f_*\big(L(E, \varepsilon)\big)$; equivalently, there exists $X \in (E \cap \ker f)^{\perp_\varepsilon}$ such that $f(X) = Y$ and $\xi(f(X')) = \varepsilon(X, X')$, for any $X' \in E$.

It is easy to prove that, if $X \in (E \cap \ker f)^{\perp_\varepsilon}$ is such that $f(X) = Y$, then $\xi(f(X')) = \varepsilon(X, X')$, for any $X' \in (E \cap \ker f) \cup (E \cap \ker f)^{\perp_\varepsilon}$.

It follows that, for any $X \in (E \cap \ker f)^{\perp_\varepsilon}$ with $f(X) = Y$, there exists $X_1 \in \ker(\varepsilon|_{E \cap \ker f})$ such that $\xi(f(X')) = \varepsilon(X + X_1, X')$, for any $X' \in E$; as, then, we also have $X_1 \in (E \cap \ker f)^{\perp_\varepsilon}$ and $f(X_1) = 0$, this shows that $Y + \xi = f(X + X_1) + \xi \in f_*(L_V)$.

To prove the second relation of (1.1), let $X + \xi \in L\big(f^{-1}(F), f^*(\eta)\big)$; equivalently, $f(X) \in F$ and $\xi(X') = \eta(f(X), f(X'))$ for any $X' \in f^{-1}(F)$. As $f^{-1}(F) \supseteq \ker f$, there exists $\check{\xi}$ in the dual of $f(V)$ such that $\xi = f^*(\check{\xi})$. Obviously, we can extend $\check{\xi}$ to a one-form on W, which we shall denote by the same symbol $\check{\xi}$, such that $\check{\xi}(Y) = \eta(f(X), Y)$, for any $Y \in F$; equivalently, $f(X) + \check{\xi} \in L(F, \eta)$. Therefore $X + \xi = X + f^*(\check{\xi}) \in f^*\big(L(F, \eta)\big)$.
The proof is complete. $\quad \square$

Definition 1.4 (*See [5,6,1]*). Let V and W be vector spaces endowed with linear Dirac structures L_V and L_W, respectively, and let $f : V \to W$ be a linear map.
Then $f_*(L_V)$ and $f^*(L_W)$ are called the *push-forward* and *pull-back, by f, of L_V and L_W*, respectively.

Note that, if $f : (V, L_V) \to (W, L_W)$ is a linear map between vector spaces endowed with linear Poisson structures then the following assertions are equivalent (see [5,6]):

(i) f is a *linear Poisson morphism* (that is, $f(\eta_V) = \eta_W$, where η_V and η_W are the bivectors defining L_V and L_W, respectively; see [10]).
(ii) $f_*(L_V) = L_W$.

From Proposition 1.3, we easily obtain the following result.

Corollary 1.5 (*Cf. [5,6]*)**.** *Let V be a vector space endowed with a linear Dirac structure $L = L(E, \varepsilon)$. Let $W = \ker \varepsilon$ and denote by $\varphi : V \to V/W$ the projection.*
Then $L = \varphi^*(\varphi_*(L))$ and $\varphi_*(L)$ is a linear Poisson structure on V/W.

2. Dirac structures

In this section, we shall work in the smooth and (real or complex) analytic categories. All the notations of Section 1 will be applied to tangent bundles of manifolds and to (differentials of) maps between manifolds.

Definition 2.1 (*[4]*)**.** An *almost Dirac structure* on a manifold M is a maximal isotropic subbundle of $TM \oplus T^*M$.
An almost Dirac structure is *integrable* if its space of sections is closed under the *Courant bracket* defined by

$$[X + \alpha, Y + \beta] = [X, Y] + \frac{1}{2}\mathrm{d}(\iota_X\beta - \iota_Y\alpha) + \iota_X\mathrm{d}\beta - \iota_Y\mathrm{d}\alpha,$$

for any sections $X + \alpha$ and $Y + \beta$ of $TM \oplus \Lambda(T^*M)$, where ι denotes the interior product.
A *Dirac structure* is an integrable almost Dirac structure.

Let L be a Dirac structure on M. If $\pi(L) = TM$ then L is a *presymplectic structure* whilst if $^*\pi(L) = T^*M$ then L is a *Poisson structure* [4] (cf. [9]).
Recall [4, Section 4] that a point of a manifold endowed with an almost Dirac structure L is called *regular* if, in some open neighborhood of it, $\pi(L)$ and $^*\pi(L)$ are bundles.
The following result follows from the fact that it is sufficient to be proved for maps of constant rank between manifolds endowed with regular almost Dirac structures.

Proposition 2.2. *Let M and N be manifolds endowed with the almost Dirac structures L_M and L_N, respectively. Let $\varphi : M \to N$ be a map which maps regular points of L_M to regular points of L_N.*

 (i) *If L_M is integrable and $\varphi_*(L_M) = L_N$ then L_N is integrable.*
(ii) *If L_N is integrable and $\varphi^*(L_N) = L_M$ then L_M is integrable.*

Next, we prove the following result.

Theorem 2.3 (*Cf. [4–6]*)**.** *Let L be a Dirac structure on M such that $^*\pi(L)$ is a subbundle of T^*M. Then, locally, there exist submersions φ on M such that $\varphi_*(L)$ is a Poisson structure and $L = \varphi^*(\varphi_*(L))$; moreover, these submersions are (germ) unique, up to Poisson diffeomorphisms of their codomains.*

Proof. By hypothesis, $TM \cap L$ is a subbundle of TM. Furthermore, as L is integrable, $TM \cap L$ is (the tangent bundle to) a foliation.
Let $F = {}^*\pi(L)$ and let η be the section of $\Lambda^2 F^*$ such that $L = L(F, \eta)$. Note that, $F(=\mathrm{Ann}(TM \cap L))$ is locally spanned by the differentials of functions which are basic with respect to $TM \cap L$.
Let f and g be functions, locally defined on M, such that $\mathrm{d}f$ and $\mathrm{d}g$ are sections of F. Then there exist vector fields X and Y, locally defined on M, such that $X + \mathrm{d}f$ and $Y + \mathrm{d}g$ are local sections of L; in particular, we have $\eta(\mathrm{d}f, \mathrm{d}g) = X(g) = -Y(f)$. Hence $[X + \mathrm{d}f, Y + \mathrm{d}g] = [X, Y] + \mathrm{d}(\eta(\mathrm{d}f, \mathrm{d}g))$ and we deduce that $\eta(\mathrm{d}f, \mathrm{d}g)$ is basic with respect to $TM \cap L$.
The proof follows quickly from Corollary 1.5 and Proposition 2.2. □

Under the same hypotheses, as in Theorem 2.3, we call $\varphi_*(L)$ the *canonical (local) Poisson quotient* of L.
Next, we prove the following (cf. [4, Proposition 4.1.2]).

Proposition 2.4. *Let $L = L(E, \varepsilon)$ be a Dirac structure on M and let $x \in M$ be a regular point of L; denote by P the leaf of E through x.*
Then for any submanifold Q of M transversal to E, such that $x \in Q$ and $\dim Q = \dim M - \dim P$, there exists a submersion ρ from some open neighborhood U of x in M onto some open neighborhood V of x in P such that $\rho_*(L|_U) = L(TV, \varepsilon|_V)$ and the fiber of ρ through x is an open set of Q.

Proof. From Theorem 2.3 it follows that we may assume L a Poisson structure.
If we ignore the fact that the fiber of ρ through x is fixed then the proposition is a consequence of [9, Corollary 2.3] and Proposition 1.3. To complete the proof just note that in the proof of [9, Theorem 2.1] (and, consequently, of [9, Corollary 2.3], as well), at each step, the two functions involved may be assumed constant along Q. □

Recall (see [1,5]) that any closed two-form B on M corresponds to a *B-field transformation* which is the automorphism of $TM \oplus T^*M$, preserving the Courant bracket, defined by

$$\exp(B)(X + \alpha) = X + B(X) + \alpha$$

for any $X + \alpha \in TM \oplus T^*M$, where, as before, we have identified B with the corresponding section of $\mathrm{Hom}(TM, T^*M)$. It is easy to prove that if $L = L(E, \varepsilon)$ is an almost Dirac structure on M then $\exp(B)(L) = L(E, \varepsilon + B|_E)$.

Corollary 2.5. *Let L be a regular Dirac structure on M; denote $E = \pi(L)$. Then, locally, there exist two-forms B on M such that* $\exp(B)(L) = E \oplus \mathrm{Ann}E.$

Proof. By Proposition 2.4, locally, there exist submersions $\rho : M \to P$ onto presymplectic manifolds $\big(P, L(TP, \omega)\big)$ such that $\rho_*(L) = L(TP, \omega)$.

Then $B = -\rho^*(\omega)$ is as required. $\quad\square$

We end this section with the following result which will be used later on.

Proposition 2.6. *Let $\varphi : (M, L_M) \to (N, L_N)$ be a Poisson morphism, of constant rank, between regular Poisson manifolds such that* $\mathrm{d}\varphi(E_M) \subseteq E_N$, *where E_M and E_N are the (symplectic) foliations determined by L_M and L_N, respectively.*

Then, locally, there exist submersions $\rho : M \to (P, \omega)$ and $\sigma : N \to (Q, \eta)$ onto symplectic manifolds, and a Poisson morphism $\psi : (P, \omega) \to (Q, \eta)$ such that:

  (i) *$TM = E_M \oplus \ker \mathrm{d}\rho$ and $\rho_*(L_M) = L(TP, \omega)$;*
 (ii) *$TN = E_N \oplus \ker \mathrm{d}\sigma$ and $\sigma_*(L_N) = L(TQ, \eta)$;*
(iii) *$\sigma \circ \varphi = \psi \circ \rho$.*

Proof. From Proposition 1.3 we obtain that $\mathrm{d}\varphi(E_M) = E_N$. As, locally, φ is the composition of a submersion followed by an immersion, it follows that we may assume that φ is a surjective submersion.

By Proposition 2.4, locally, there exists a submersion $\sigma : M \to (Q, \eta)$ onto a symplectic manifold such that assertion (ii) is satisfied.

Let $\mathscr{V}$ be the distribution on M generated by all of the Hamiltonian vector fields determined by $u \circ \sigma \circ \varphi$, with u a function on Q; obviously, $\mathscr{V} \subseteq E_M$. Then arguments similar to the inductive step of the proof of [9, Theorem 2.1] show that:

(a) *$\mathscr{V}$ is a foliation mapped by $\sigma \circ \varphi$ onto TQ;*
(b) *$\mathscr{V}$ and $E_M \cap \ker \mathrm{d}\varphi$ are nondegenerate and complementary orthogonal with respect to the symplectic structure ω_M of E_M;*
(c) *ω_M restricted to $\mathscr{V}$ is projectable (onto η) with respect to $\sigma \circ \varphi$;*
(d) *ω_M restricted to $E_M \cap \ker \mathrm{d}\varphi$ is projectable with respect to $\mathscr{V}$.*

Consequently, (E_M, ω_M) induces on any fiber M' of $\sigma \circ \varphi$ a Poisson structure L' such that, locally, (M, L_M) is the product of (M', L') and $\big(Q, L(TQ, \eta)\big)$.

By Proposition 2.4, locally, there exists a submersion $\rho' : M' \to (P', \omega')$ such that $\ker \mathrm{d}\rho' \oplus (E_M \cap TM') = TM'$ and $\rho'_*(L') = L(TP', \omega')$.

If we define $(P, \omega) = (P', \omega') \times (Q, \eta)$, $\rho = \rho' \times \sigma$ and $\psi : P \to Q$ the projection then it is easy to see that ρ, σ and ψ are as required. $\quad\square$

3. Generalized complex linear maps

A *linear generalized complex structure* on a vector space V is a maximal isotropic subspace $L = L(E, \varepsilon)$ of $V^{\mathbb{C}} \oplus \big(V^{\mathbb{C}}\big)^*$ such that $L \cap \overline{L} = \{0\}$ [1,2]; equivalently, $E + \overline{E} = V^{\mathbb{C}}$ and $\mathrm{Im}\big(\varepsilon|_{E \cap \overline{E}}\big)$ is nondegenerate [1].

The condition $E + \overline{E} = V^{\mathbb{C}}$ means that E is a *linear co-CR structure* on V [11]; equivalently, the annihilator E^0 of E is a linear CR structure on V (that is, $E^0 \cap \overline{E^0} = \{0\}$).

On the other hand, as $\mathrm{Im}\big(\varepsilon|_{E \cap \overline{E}}\big)$ is nondegenerate, $L\big(E \cap \overline{E}, \mathrm{Im}(\varepsilon|_{E \cap \overline{E}})\big)$ is a linear Poisson structure on V.

If $L = L(E, \varepsilon)$ is a linear generalized complex structure then we call E and $L\big(E \cap \overline{E}, \mathrm{Im}(\varepsilon|_{E \cap \overline{E}})\big)$ *the associated linear co-CR and Poisson structures*, respectively.

A map $f : (V, E_V) \to (W, E_W)$ between vector spaces endowed with linear (co-)CR structures is a *(co-)CR linear map* if it is linear and $f(E_V) \subseteq E_W$.

Definition 3.1. A linear map between vector spaces endowed with linear generalized complex structures is *generalized complex linear* if it is a co-CR linear Poisson morphism, with respect to the associated linear co-CR and Poisson structures.

Note that, Definition 3.1 is invariant under linear B-field transformations. Also, the composition of two generalized complex linear maps is a generalized complex linear map.

Proposition 3.2. *Let $f : V \to W$ be a linear map between vector spaces endowed with linear generalized complex structures L_V and L_W, respectively.*

Then the following assertions are equivalent:

 (i) *f is generalized complex linear.*
(ii) *Up to linear B-field transformations, f is the direct sum of a a complex linear map, between complex vector spaces, and a linear Poisson morphism, between symplectic vector spaces.*

Proof. Suppose that (i) holds and let E_V and E_W be the linear co-CR structures associated to L_V and L_W, respectively.

As $f : (V, E_V) \to (W, E_W)$ is co-CR linear, we obtain $f(E_V \cap \overline{E_V}) \subseteq E_W \cap \overline{E_W}$. But f is, also, a linear Poisson morphism, with respect to the linear Poisson structures associated to L_V and L_W, respectively. From Proposition 1.3 we obtain that $f(E_V \cap \overline{E_V}) = E_W \cap \overline{E_W}$. Moreover, f restricts to give a linear Poisson morphism between $E_V \cap \overline{E_V}$ and $E_W \cap \overline{E_W}$, endowed with the linear symplectic structures corresponding to the linear Poisson structures associated to L_V and L_W, respectively.

We, also, obtain $f^{-1}(E_W \cap \overline{E_W}) = \ker f + (E_V \cap \overline{E_V})$ and, consequently, there exist complementary vector spaces V' and W' of $E_V \cap \overline{E_V}$ and $E_W \cap \overline{E_W}$ in V and W, respectively, such that $f(V') \subseteq W'$.

It is obvious that E_V and E_W induce linear complex structures on V' and W', respectively. Moreover, f restricts to give a complex linear map between these two complex vector spaces.

Now, (i) $\Longrightarrow$ (ii) follows quickly from [1, Theorem 4.13], whilst (ii) $\Longrightarrow$ (i) is trivial. $\square$

The next result is an immediate consequence of Proposition 3.2.

Corollary 3.3. *Let $f : V \to W$ be a linear isomorphism between vector spaces endowed with linear generalized complex structures L_V and L_W, respectively.*

Then the following assertions are equivalent:

(i) *f is generalized complex linear.*
(ii) *$f_*(L_V) = L_W$, up to linear B-field transformations.*

We end this section with the following:

Remark 3.4. It has been proposed another definition for the notion of generalized complex linear map by imposing that the product of the graphs of the map and of its transpose be invariant under the product of the (endomorphisms corresponding to the) generalized linear complex structures, of the domain and codomain [12] (see [13]).

However this notion is not invariant under linear B-field transformations as we shall now explain.

Let (V, J) be a complex vector space and let b be a two-form on V; denote by L_J the linear generalized complex structure corresponding to J. Then the map $\mathrm{Id}_V : (V, L_J) \to (V, L_{(\exp b)(L_J)})$ satisfies the above mentioned condition if and only if b is of type $(1, 1)$, with respect to J.

Certainly, this inconvenience would be removed if we take this definition up to linear B-field transformations. However, a straightforward calculation shows that there are no such maps between symplectic vector spaces U and V with $\dim U - \dim V = 2$, a rather unnatural restriction.

4. Holomorphic maps between generalized complex manifolds

From now on, unless otherwise stated, all the manifolds are assumed connected and smooth and all the maps are assumed smooth.

A *generalized almost complex structure* on M is a complex vector subbundle L of $T^{\mathbb{C}}M \oplus (T^{\mathbb{C}}M)^*$ such that L_x is a linear generalized complex structure on $T_x M$, for any $x \in M$. An *integrable* generalized complex structure is a generalized almost complex structure whose space of sections is closed under the (complexification of the) Courant bracket; a *generalized (almost) complex manifold* is a manifold endowed with a generalized (almost) complex structure [1,2].

Definition 4.1. A map between generalized almost complex manifolds is *holomorphic* if, at each point, its differential is generalized complex linear.

A point x of a generalized almost complex manifold (M, L) is *regular* if it is regular for the associated almost Poisson structure; equivalently, in some open neighborhood of x, $\pi(L)$ is a complex vector subbundle of $T^{\mathbb{C}}M$.

An *almost (co-)CR structure* on a manifold M is a complex vector subbundle $\mathcal{C}$ of $T^{\mathbb{C}}M$ such that $\mathcal{C}_x$ is a linear (co-)CR structure on $T_x M$, for any $x \in M$. An *integrable* almost (co-)CR structure is an almost (co-)CR structure whose space of sections is closed under the (Lie) bracket; a *(co-)CR structure* is an integrable almost (co-)CR structure (see [11]).

Note that, the eigenbundles of a complex structure are both CR and co-CR structures. Also, a generalized almost complex structure L on M is regular (at each point) if and only if $\pi(L)$ is an almost co-CR structure on M.

Let $\varphi : M \to N$ be a submersion onto a complex manifold (N, J); denote by $T^{1,0}N$ the eigenbundle of J corresponding to i. Then $\mathrm{d}\varphi^{-1}(T^{1,0}N)$ is a co-CR structure on M. Conversely, any co-CR structure is, locally, obtained this way.

A map between manifolds endowed with almost (co-)CR structures is *(co-)CR holomorphic* if, at each point, its differential is a (co-)CR linear map.

An *almost f-structure* is a $(1, 1)$-tensor field F such that $F^3 + F = 0$. Any almost f-structure on M corresponds to a pair formed of an almost CR structure $\mathcal{C}$ and an almost co-CR structure $\mathcal{D}$, which are compatible [11]; these are given by $\mathcal{C} = T^{1,0}M$ and $\mathcal{D} = T^0 M \oplus T^{1,0}M$, where $T^0 M$ and $T^{1,0}M$ are the eigenbundles of F corresponding to 0 and i, respectively.

An almost f-structure is *(co-)CR integrable* if the associated almost (co-)CR structure is integrable. An *(integrable almost) f-structure* is an almost f-structure which is both CR and co-CR integrable [11].

An almost f-structure F and a two-form ω on M are *compatible* if ω is nondegenerate on $T^0 M$ and $\iota_X \omega = 0$, for any $X \in T^{1,0}M \oplus T^{0,1}M$.

A generalized (almost) complex structure L on M is in *normal form* if $L = L(T^0 M \oplus T^{1,0}M, i\omega)$ for some compatible almost f-structure and two-form ω on M. Note that, a generalized almost complex structure in normal form is regular.

Proposition 4.2. *Let $L = L(T^0M \oplus T^{1,0}M, i\omega)$ be the generalized almost complex structure in normal form, corresponding to the compatible almost f-structure F and two-form ω on M.*
Then the following assertions are equivalent:

(i) *L is integrable.*
(ii) *F is integrable, $L(T^0M, \omega)$ is a Poisson structure and ω is invariant under the parallel displacement of $T^{1,0}M \oplus T^{0,1}M$.*

Proof. From [1, Proposition 4.19] it follows quickly that assertion (i) is equivalent to the fact that F is co-CR integrable and $(d\omega)|_{T^0M \oplus T^{1,0}M} = 0$. Assuming F co-CR integrable, the latter condition is equivalent to the fact that $L(T^0M, \omega)$ is a Poisson structure, F is CR integrable and $(\mathcal{L}_X\omega)|_{T^0M} = 0$ for any vector field X tangent to $T^{1,0}M \oplus T^{0,1}M$, where $\mathcal{L}$ denotes the Lie derivative. □

All of the examples of generalized complex structures of [14] are in normal form. Similarly, we have the following example, due to [15].

Example 4.3. Let G be a compact Lie group of even rank assumed, for simplicity, semisimple. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{k}$ be the Lie algebra of a maximal torus in G.
Let $\mathfrak{c}$ be a Borel subalgebra of $\mathfrak{g}^{\mathbb{C}}$ containing $\mathfrak{k}^{\mathbb{C}}$. Any such Borel subalgebra is obtained by choosing a base for the root system of $\mathfrak{g}^{\mathbb{C}}$ corresponding to $\mathfrak{k}^{\mathbb{C}}$ (see [16]): $\mathfrak{c} = \mathfrak{k}^{\mathbb{C}} \oplus \bigoplus_{\alpha \succ 0} \mathfrak{g}^{\alpha}$, where $\mathfrak{g}^{\alpha}$ is the root space of $\mathfrak{g}^{\mathbb{C}}$ corresponding to the root α.
As $\overline{\mathfrak{g}^{\alpha}} = \mathfrak{g}^{-\alpha}$ (see [17]), we have $\mathfrak{c} + \bar{\mathfrak{c}} = \mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{c} \cap \bar{\mathfrak{c}} = \mathfrak{k}^{\mathbb{C}}$. Consequently, $\mathfrak{c}$ corresponds to a left invariant co-CR structure $\mathcal{C}$ on G (for any $a \in G$, we have that $\mathcal{C}_a$ is the left translation of $\mathfrak{c}$, at a).
Let ω be a linear symplectic form on $\mathfrak{k}$ (dim $\mathfrak{k}$ = rank G is even), extended to $\mathfrak{g}$ such that $\iota_X\omega = 0$ for any $X \in \bigoplus_{\alpha \succ 0} \mathfrak{g}^{\alpha}$. We shall denote by the same letter ω the left invariant two-form on G, determined by ω.
Then $L(\mathcal{C}, i\omega)$ is a generalized complex structure on G in normal form.

The next result follows from the proof of [1, Theorem 4.35].

Theorem 4.4. *Let L be a regular generalized almost complex structure on M and let L' be the associated almost Poisson structure. Then the following assertions are equivalent:*

(i) *L is integrable.*
(ii) *$\pi(L)$ and L' are integrable and, locally, for any submersion $\rho : M \to P$, with dim P = rank$(\pi(L'))$ and $\rho_*(L')$ a symplectic structure on P, we have that, up to a B-field transformation, L is in normal form with respect to the f-structure on M determined by $\pi(L)$ and $\pi(L) \cap \ker d\rho$.*

We can, now, give the smooth version of Proposition 3.2.

Proposition 4.5. *Let $\varphi : (M, L_M) \to (N, L_N)$ be a map between generalized complex manifolds. Then the following assertions are equivalent:*

(i) *φ is holomorphic.*
(ii) *On an open neighborhood of each regular point of L_M on which φ has constant rank, up to B-field transformations, φ is the product of a Poisson morphism between symplectic manifolds and a holomorphic map between complex manifolds.*

Proof. This follows quickly from Proposition 2.6, [1, Theorem 4.35] and Theorem 4.4. □

Next, we give examples of holomorphic maps between generalized complex manifolds.

Example 4.6. The classical holomorphic maps, the Poisson morphisms between symplectic manifolds, and their products are, obviously, holomorphic maps between generalized complex manifolds.
Moreover, by Proposition 4.5, any holomorphic map $\varphi : (M, L_M) \to (N, L_N)$ between generalized complex manifolds is of this form, up to B-field transformations, on an open neighborhood of each regular point of L_M on which φ has constant rank.

Example 4.7. Let G be a compact Lie group endowed with the generalized complex structures $L = L(\mathcal{C}, i\omega)$ of Example 4.3.
Let K be the maximal torus of G whose Lie algebra is used to define $\mathcal{C}$. Obviously, $d\varphi(\mathcal{C})$ defines a left invariant complex structure on G/K, where $\varphi : G \to G/K$ is the projection.
Then $\varphi : (G, L) \to (G/K, d\varphi(\mathcal{C}))$ is a holomorphic map.

Example 4.8. Let G/H be a compact inner symmetric space (see [17, page 23] for the definition and [17, page 38] for a table of examples) with rank G(= rank H) even; denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively.
Endow G with the generalized complex structures $L(\mathcal{C}, i\omega)$ of Example 4.3, determined by a Borel subalgebra $\mathfrak{c}$ of $\mathfrak{g}^{\mathbb{C}}$ containing the Lie algebra of a maximal torus of H (also a maximal torus of G, as G/H is inner).
It follows that $\mathfrak{d} = \mathfrak{c} \cap \mathfrak{h}^{\mathbb{C}}$ is a Borel subalgebra of $\mathfrak{h}^{\mathbb{C}}$. Let $\mathcal{D}$ be the left invariant co-CR structure induced by $\mathfrak{d}$, on H, and let $\eta = \omega|_H$.
Then the inclusion map from $(H, L(\mathcal{D}, i\eta))$ to $(G, L(\mathcal{C}, i\omega))$ is holomorphic.
Fairly similar examples can be obtained by working with nilpotent Lie groups endowed with the generalized complex structures of [14].

The following facts are immediate consequences of the definitions.

Remark 4.9. (1) A map between regular generalized almost complex manifolds is holomorphic if and only if it is a co-CR Poisson morphism, with respect to the associated almost co-CR and Poisson structures.

(2) Let $\varphi : (M, L_M) \to (N, L_N)$ be a diffeomorphism between generalized complex manifolds. Then φ is holomorphic if and only if, in an open neighborhood of each regular point of M, we have $\varphi_*(L_M) = L_N$, up to B-field transformations.

(3) The composition of two holomorphic maps, between generalized (almost) complex manifolds is holomorphic.

(4) Let (M, L) be a generalized complex manifold. The $\bar{\partial}$ operator on functions [1] (see [18], and, also, [19,20]) is defined as follows. If f is a complex-valued function on M then $\bar{\partial}f$ is the L-component of df with respect to the decomposition $T^{\mathbb{C}}M \oplus (T^{\mathbb{C}}M)^* = L \oplus \bar{L}$. Then f is holomorphic if and only if $\bar{\partial}f = 0$. Note that, if $L = L(E, \varepsilon)$ is regular then the holomorphic (local) functions on (M, L) are just the co-CR holomorphic functions on (M, E). Equivalently, if E is locally defined by the submersion $\varphi : M \to (N, J)$ onto the complex manifold (N, J) (that is, $E = \mathrm{d}\varphi^{-1}(T^{1,0}N)$) then, locally, any holomorphic function on (M, L) is the composition of φ followed by a holomorphic function on (N, J).

(5) Let (M, J_M, L_M) and (N, J_N, L_N) be complex manifolds endowed with holomorphic Poisson structures (see [21] for the notion of holomorphic Poisson structure, and [22] for a generalization); denote by $\widetilde{L}_M$ and $\widetilde{L}_N$ the generalized complex structures associated to L_M and L_N, respectively (see [8]). For any map $\varphi : M \to N$, any two of the following assertions imply the third:

  (i) $\varphi : (M, J_M, L_M) \to (N, J_N, L_N)$ is a holomorphic Poisson morphism;

  (ii) $\varphi : (M, \widetilde{L}_M) \to (N, \widetilde{L}_N)$ is holomorphic;

  (iii) $\varphi : (M, J_M) \to (N, J_N)$ is holomorphic and it maps the leaves of the holomorphic symplectic foliation associated to L_M into leaves of the holomorphic symplectic foliation associated to L_N.

From Theorem 2.3 we obtain the following result.

Proposition 4.10. *Let (M, L_M) and (N, L_N) be regular real analytic generalized complex manifolds and let $\varphi : M \to N$ be a real analytic map.*

If φ is holomorphic then, locally, up to the complexification of a real analytic B-field transformation, the complexification of φ descends to a complex analytic Poisson morphism between canonical Poisson quotients.

Let $L(E, \mathrm{i}\varepsilon)$ be a generalized complex structure in normal form on a Riemannian manifold (M, g).

Then E is coisotropic (that is, $E^{\perp}$ is isotropic), with respect to g, if and only if $E \cap \bar{E}$ is locally defined by pseudo-horizontally conformal submersions onto complex manifolds (a map from a Riemannian manifold to an almost complex manifold is *pseudo-horizontally conformal* if it pulls back $(1, 0)$-forms to isotropic forms).

Also, if ε^k has constant norm, with respect to g, where $\dim(E \cap \bar{E}) = 2k$, then the leaves of $E \cap \bar{E}$ are minimal submanifolds of (M, g).

Conversely, we have the following:

Example 4.11. Let $\varphi : (M, g) \to (N, J)$ be a pseudo-horizontally conformal submersion from a Riemannian manifold onto an almost complex manifold, with $\dim M = \dim N + 2$.

Denote $\mathscr{V} = \ker \mathrm{d}\varphi$, $\mathscr{H} = \mathscr{V}^{\perp}$ and let ω be the volume form of $\mathscr{V}$. Also, let F be the unique skew-adjoint almost f-structure on M such that $\ker F = \mathscr{V}$ and, with respect to which, φ is co-CR holomorphic. Obviously, F and ω are compatible; denote by L the corresponding generalized almost complex structure in normal form.

From Proposition 4.2 it follows that L is integrable if and only if J is integrable, the fibers of φ are minimal and the integrability tensor of $\mathscr{H}$ is of type $(1, 1)$; note that, if $\dim M = 4$ then this is equivalent to the condition that φ is a harmonic morphism (see [23]), where N is endowed with the conformal structure with respect to which J is a Hermitian structure.

Moreover, any generalized complex structure, in normal form, on a Riemannian manifold such that the corresponding f-structure is skew-adjoint, the associated Poisson structure has rank two and its symplectic form that has norm 1 is, locally, obtained this way.

The pseudo-horizontally conformal submersions with totally geodesic fibers onto complex manifolds, for which the integrability tensor of the horizontal distribution is of type $(1, 1)$, admit a twistorial description from which it follows that they abound on Riemannian manifolds of constant curvature [24] (cf. [23]).

Also, see [25] for a study of the harmonic pseudo-horizontally conformal submersions with minimal fibers and [23] for twistorial constructions of harmonic morphisms with two-dimensional fibers on four-dimensional Riemannian manifolds.

5. Generalized Kähler manifolds

We start this section by recalling from [1] a few facts on generalized Kähler manifolds.

A *generalized (almost) Kähler manifold* is a manifold M endowed with two generalized (almost) complex structures such that the corresponding sections $\mathscr{J}_1$ and $\mathscr{J}_2$ of $\mathrm{End}(TM \oplus T^*M)$ commute and $\mathscr{J}_1\mathscr{J}_2$ is negative definite.

Any generalized almost Kähler structure (L_1, L_2) on a manifold M corresponds to a quadruple (g, b, J_+, J_-) where g is a Riemannian metric, b is a two-form and $J_{\pm}$ are almost Hermitian structures on (M, g). The (bijective) correspondence is given by $L_1 = L^+ \oplus L^-$, $L_2 = L^+ \oplus \overline{L^-}$, where

$$L^{\pm} = \left\{ X + (b \pm g)(X) \mid X \in V^{\pm} \right\}$$

with $V^{\pm}$ the eigenbundles of $J_{\pm}$ corresponding to i.

According to [1, Theorem 6.28], the following assertions are equivalent:

(i) L_1 and L_2 are integrable.
(ii) L_+ and L_- are integrable.
(iii) $J_\pm$ are integrable and parallel with respect to $\nabla^\pm = \nabla^g \pm \frac{1}{2}g^{-1}h$, where ∇^g is the Levi-Civita connection of g and $h = \mathrm{d}b$ (equivalently, $J_\pm$ are integrable and $\mathrm{d}_\pm^c \omega_\pm = \mp h$, where $\omega_\pm$ are the Kähler forms of $J_\pm$).

Now, if we (pointwisely) denote $E_j = \pi(L_j)$, $(j = 1, 2)$, then $E_1 = V^+ + V^-$ and $E_2 = V^+ + \overline{V^-}$. Hence, $E_1^\perp = V^+ \cap V^-$, $E_2^\perp = V^+ \cap \overline{V^-}$ and, therefore, E_1 and E_2 are coisotropic.

Remark 5.1. Let (M, L_1, L_2) be a generalized Kähler manifold.

(1) The (skew-adjoint) almost f-structures F_j determined by E_j and $E_j^\perp$ are integrable; we call F_j *the f-structures of L_j*, $(j = 1, 2)$.
(2) The holomorphic functions of (M, L_1) and (M, L_2) are the bi-holomorphic functions of (M, J_+, J_-) and $(M, J_+, -J_-)$, respectively.

Let $\mathscr{H}^\pm = \ker(J_+ \mp J_-)$. Then $\mathscr{H}^+$ and $\mathscr{H}^-$ are orthogonal; this follows from $\mathscr{H}^+ = (V^+ \cap V^-) \oplus \overline{(V^+ \cap V^-)}$ and $\mathscr{H}^- = (V^+ \cap \overline{V^-}) \oplus \overline{(V^+ \cap \overline{V^-})}$. Denote $\mathscr{V} = (\mathscr{H}^+ \oplus \mathscr{H}^-)^\perp$.

Note that, $\mathscr{H}^+$, $\mathscr{H}^-$ and $\mathscr{V}$ are invariant under J_+ and J_-. Also, $J_+ - J_-$ and $J_+ + J_-$ are invertible on $\mathscr{V}$.

Proposition 5.2. *The following assertions are equivalent:*

(i) L_1 and L_2 are regular.
(ii) $\mathscr{H}^+$ and $\mathscr{H}^-$ are distributions on M.
(iii) $\mathscr{V}$ is a distribution on M.

Proof. The obvious relations

$$E_1 = (V^+ \cap V^-)^\perp = (V^+ \cap V^-) \oplus \mathscr{H}^- \oplus \mathscr{V},$$
$$E_2 = (V^+ \cap \overline{V^-})^\perp = (V^+ \cap \overline{V^-}) \oplus \mathscr{H}^+ \oplus \mathscr{V}$$

imply

$$E_1 \cap \overline{E_1} = \mathscr{H}^- \oplus \mathscr{V} = (\mathscr{H}^+)^\perp,$$
$$E_2 \cap \overline{E_2} = \mathscr{H}^+ \oplus \mathscr{V} = (\mathscr{H}^-)^\perp$$

which show that (i) $\Longleftrightarrow$ (ii).

Also, as the dimensions of $\mathscr{H}^+$ and $\mathscr{H}^-$ are upper semicontinuous functions on M, assertion (ii) holds if and only if $\mathscr{H}^+ \oplus \mathscr{H}^- (= \mathscr{V}^\perp)$ is a distribution on M. □

Next, we prove the following result.

Theorem 5.3. *Let (M, L_1, L_2) be a generalized Kähler manifold with L_1 regular.*
Then the following assertions are equivalent:

(i) $\mathscr{H}^+$ is integrable.
(ii) $\mathscr{H}^+$ is geodesic.

Furthermore, if (i) or (ii) holds then the leaves of $\mathscr{H}^+$, endowed with $(g, J_\pm)$, are Kähler manifolds. Also, if $\mathscr{H}^+$ is holomorphic, with respect to J_+ or J_-, then both (i) and (ii) hold.

To prove Theorem 5.3 we need some preparations.

Let $\mathscr{H}$ be a distribution on a Riemannian manifold (M, g) endowed with a linear connection ∇; denote $\mathscr{V} = \mathscr{H}^\perp$.

The *second fundamental form* of $\mathscr{H}$, with respect to ∇, is the $\mathscr{V}$-valued symmetric two-form $B^{\mathscr{H}}$ on $\mathscr{H}$ defined by $B^{\mathscr{H}}(X, Y) = \frac{1}{2}\mathscr{V}(\nabla_X Y + \nabla_Y X)$; then $\mathscr{H}$ is geodesic, with respect to ∇, if and only if $B^{\mathscr{H}} = 0$ (cf. [23]).

The next result follows from a straightforward calculation.

Lemma 5.4 (Cf. [26]). *Let (M, g, J) be a Hermitian manifold endowed with a distribution $\mathscr{H}$ and a conformal connection ∇ such that $\nabla J = 0$.*
If $\mathscr{V}$ is integrable and J-invariant then the following relation holds:

$$2g\big(B^{\mathscr{H}}(JX, Y), V\big) + g\big(I^{\mathscr{H}}(X, Y), JV\big) = g\big(T(V, JX), Y\big) + g\big(T(V, X), JY\big),$$

for any $X, Y \in \mathscr{H}$ and $V \in \mathscr{V}$, where T is the torsion of ∇ and $I^{\mathscr{H}}$ is the integrability tensor of $\mathscr{H}$, defined by $I^{\mathscr{H}}(X, Y) = -\mathscr{V}[X, Y]$, for any sections X and Y of $\mathscr{H}$.

To prove Theorem 5.3 we also need the following lemma.

Lemma 5.5. *Let (M, J) be a complex manifold and let $\mathscr{H}$ be a holomorphic distribution on (M, J). The following assertions are equivalent:*

(i) $\mathscr{H}$ *is integrable.*
(ii) $\mathscr{H}^{1,0}$ *is a CR structure.*

Proof. This is obvious.　□

Proof of Theorem 5.3. We may assume that, also, L_2 is regular.

Obviously, the second fundamental form of $\mathscr{H}^+$, with respect to ∇^g, is equal to the second fundamental forms of $\mathscr{H}^+$, with respect to $\nabla^\pm$.

As L_1 and L_2 are integrable we have that E_1 and E_2 are integrable and, consequently, $\mathscr{H}^+ \oplus \mathscr{V}$ and $\mathscr{H}^- \oplus \mathscr{V}$ are integrable; in particular, the integrability tensor of $\mathscr{H}^+$ takes values in $\mathscr{V}$. Furthermore, $\mathscr{H}^+ \oplus \mathscr{V}$ and $\mathscr{H}^- \oplus \mathscr{V}$ are holomorphic with respect to both J_+ and J_-.

Now, by applying Lemma 5.4 to $\mathscr{H} = \mathscr{H}^+$ twice, with respect to ∇^+ and ∇^-, we quickly obtain

$$4g\big(B^{\mathscr{H}^+}(J_\pm X, Y), V\big) = -g\big(I^{\mathscr{H}^+}(X, Y), (J_+ + J_-)(V)\big),$$

for any $X, Y \in \mathscr{H}^+$ and $V \in \mathscr{H}^- \oplus \mathscr{V}$. As $J_+ + J_-$ is invertible on $\mathscr{V}$, we obtain that (i) $\Longleftrightarrow$ (ii).

If $\mathscr{H}^+$ is holomorphic with respect to J_+ or J_- then $\mathscr{H}^+$ is integrable by Lemma 5.5 and the fact that the eigenbundles of $J_\pm|_{\mathscr{H}^+}$ corresponding to i are equal to $V^+ \cap V^-$ which is integrable.

To complete the proof just note that if $\mathscr{H}^+$ is integrable then (g, b, J_+, J_-) induces, by restriction, a generalized Kähler structure on each leaf L of $\mathscr{H}^+$ and $J_+ = J_-$ on L.　□

Remark 5.6. (1) Let (M, L_1, L_2) be a generalized Kähler manifold with L_1 regular. If $\mathscr{H}^+$ is integrable then, by Theorem 5.3, the co-CR structure associated to L_1 (that is, E_1) is, locally, given by holomorphic Riemannian submersions from $(M, g, J_\pm)$ onto Kähler manifolds (P, h, J); in particular, the leaves of $\mathscr{H}^+$, endowed with $(g, J_\pm)$ can be, locally, identified with (P, h, J).

(2) If (M, L_1^M, L_2^M) and (N, L_1^N, L_2^N) are generalized Kähler manifolds with $\mathscr{H}_M^+$ and $\mathscr{H}_N^+$ integrable distributions then any holomorphic map $\varphi : (M, L_1^M) \to (N, L_1^N)$ descends, locally (with respect to the Riemannian submersions of Remark 5.6(1)), to a holomorphic map between Kähler manifolds.

Let (M_j, g_j, J_j) be Kähler manifolds, $(j = 1, 2)$. Then on $M_1 \times M_2$ there are two nonequivalent natural generalized Kähler structures: the *first product* is just the Kähler product structure whilst the *second product* is given by $L_1 = L(T^{1,0}M_1 \times TM_2, i\omega_2)$ and $L_2 = L(T^{1,0}M_2 \times TM_1, i\omega_1)$, where ω_j are the Kähler forms of J_j, $(j = 1, 2)$; see Section 6, for the corresponding definitions in a more general setting. Note that, both L_1 and L_2 are in normal form; moreover, the corresponding almost f-structures are skew-adjoint (and, thus, unique with this property).

We end this section with the following consequence of Theorem 5.3 (cf. [3, Theorem A]).

Corollary 5.7. *Any generalized Kähler manifold with $\mathscr{V} = 0$ is, up to a unique B-field transformation, locally given by the second product of two Kähler manifolds.*

Proof. Let (M, L_1, L_2) be a generalized Kähler manifold with $\mathscr{V} = 0$. Then, Proposition 5.2 implies that $\mathscr{H}^\pm$ are complementary orthogonal distributions on M.

As L_1 and L_2 are integrable, we have $\mathscr{H}^\pm$ integrable. Furthermore, by (the proof of) Theorem 5.3, we have that $\mathscr{H}^\pm$ are geodesic foliations which are holomorphic with respect to both $J_\pm$; moreover, $(g, J_\pm)$ induce, by restriction, Kähler structures on their leaves.

If $L_2 = L(E_2, \varepsilon_2)$ then, from the definitions it follows that $\varepsilon_2 = (b - i\eta)|_{E_2}$, where η is the two-form on M characterized by $\iota_X \eta = 0$ if $X \in \mathscr{H}_-$ and $\eta|_{\mathscr{H}^+}$ is the Kähler form of $J_+|_{\mathscr{H}^+}$. As $(\mathcal{L}_X \eta)(Y, Z) = 0$ for any sections X of $\mathscr{H}^-$ and Y, Z of $\mathscr{H}^+$, and $(d\varepsilon_2)(X, Y, Z) = 0$ for any $X, Y, Z \in E_2$, we obtain that $(db)(X, Y, \bar{Z}) = 0$ for any $X \in V^+ \cap \mathscr{H}^- (=E_2 \cap \mathscr{H}^-)$ and $Y, Z \in V^+ \cap \mathscr{H}^+$. Furthermore, from Lemma 5.4, applied to $\mathscr{H} = \mathscr{H}^+$ with $J = J_+$ and $\nabla = \nabla^+$, we obtain $(db)(X, Y, Z) = 0$ for any $X \in \mathscr{H}^-$ and $Y, Z \in V^+ \cap \mathscr{H}^+$.

It follows that $db = 0$ and the proof is complete.　□

6. Tamed symplectic and generalized Kähler manifolds

The following definition is fairly standard.

Definition 6.1. A *tamed almost symplectic manifold* is a manifold M endowed with a nondegenerate two-form ε and an almost complex structure J such that $\varepsilon(JX, X) > 0$ for any nonzero $X \in TM$.

A *tamed symplectic manifold* is a tamed almost symplectic manifold (M, ε, J) such that J and $\varepsilon^{-1}J^*\varepsilon$ are integrable and $d\varepsilon = 0$.

Obviously, (M, ε, J) is a tamed symplectic manifold if and only if ε is a symplectic form, $T^{1,0}M$ and $(T^{1,0}M)^{\perp_\varepsilon}$ are integrable, and $\varepsilon(JX, X) > 0$, for any nonzero $X \in TM$.

The next result also appears, in a different form, in [7].

Theorem 6.2. *Let M be a manifold endowed with a nondegenerate two-form ε and an almost complex structure J; denote $J_+ = J$ and $J_- = -\varepsilon^{-1}J^*\varepsilon$. Let g and b be the symmetric and skew-symmetric parts, respectively, of εJ.*

Then the following assertions are equivalent:

 (i) *(M, ε, J) is a tamed symplectic manifold.*
(ii) *(g, b, J_+, J_-) defines a generalized Kähler structure such that $J_+ + J_-$ is invertible.*

Moreover, up to a unique B-field transformation, any generalized Kähler structure, on M, with $J_+ + J_-$ invertible is obtained this way from a tamed symplectic structure.

Proof. Firstly, note that $\varepsilon(J_+X, Y) = -\varepsilon(X, J_-Y)$, for any $X, Y \in TM$. This implies that

$$g(X, Y) = \frac{1}{2}\varepsilon\big((J_+ + J_-)(X), Y\big),$$
$$b(X, Y) = \frac{1}{2}\varepsilon\big((J_+ - J_-)(X), Y\big),$$
(6.1)

for any $X, Y \in TM$.

Therefore (M, ε, J) is a tamed almost symplectic manifold if and only if the quadruple (g, b, J_+, J_-) defines a generalized almost Kähler manifold with $J_+ + J_-$ invertible.

Now, with respect to $J_\pm$, we have $\omega_\pm = -\varepsilon^{1,1}$, $b^{1,1} = 0$ and $b^{2,0} = \pm i\varepsilon^{2,0}$. It quickly follows that if $J_\pm$ are integrable then $d\varepsilon = 0$ if and only if $d^c_\pm \omega_\pm = \mp db$.

We have thus proved that (i) $\Longleftrightarrow$ (ii).

Suppose that (g, b, J_+, J_-) corresponds to the generalized Kähler structure (L_1, L_2) on M. Then $J_+ + J_-$ is invertible if and only if $\pi(L_2) = TM$. Hence, if $J_+ + J_-$ is invertible then, up to a unique B-field transformation, we have $L_2 = L(TM, i\varepsilon)$ for some symplectic form ε on M and, consequently,

$$i\varepsilon(X - iJ_+X, Y) = (b + g)(X - iJ_+X, Y),$$
$$i\varepsilon(X + iJ_-X, Y) = (b - g)(X + iJ_-X, Y),$$
(6.2)

for any $X, Y \in TM$. By using the fact that $J_+ + J_-$ is invertible, from (6.2) we quickly obtain that g and b satisfy (6.1). Together with the fact that g and b are symmetric and skew-symmetric, respectively, this shows that $J_- = -\varepsilon^{-1}J_+^*\varepsilon$ and the proof follows. $\square$

It is easy to rephrase Theorem 6.2 so that to obtain the description of generalized Kähler manifolds with $J_+ - J_-$ invertible.

Let (M, L_1^M, L_2^M) and (N, L_1^N, L_2^N) be generalized Kähler manifolds corresponding to the tamed symplectic manifolds (M, ε_M, J_M) and (N, ε_N, J_N), respectively. Then $(M \times N, L_1^M \times L_1^N, L_2^M \times L_2^N)$ and $(M \times N, L_1^M \times L_2^N, L_2^M \times L_1^N)$ are called the *first* and *second product* of (M, L_1^M, L_2^M) and (N, L_1^N, L_2^N), respectively; note that, the first product is the generalized Kähler manifold corresponding to $(M \times N, \varepsilon_M + \varepsilon_N, J_M \times J_N)$.

Corollary 6.3. *Any generalized Kähler manifold with $\mathscr{H}^+$ a holomorphic foliation, with respect to J_+ and J_-, and $\mathscr{H}^- = 0$ is, up to a unique B-field transformation, locally given by the first product of a Kähler manifold and a generalized Kähler manifold for which both $J_+ + J_-$ and $J_+ - J_-$ are invertible.*

Proof. Let (M, L_1, L_2) be a generalized Kähler manifold with $\mathscr{H}^+$ a distribution and $\mathscr{H}_- = 0$. Then, by Theorem 6.2, up to a unique B-field transformation, we have that (M, L_1, L_2) corresponds to the tamed symplectic manifold (M, ε, J).

Thus, by (6.1), we have $\iota_X b = 0$ for any $X \in \mathscr{H}^+$ and $\varepsilon = \eta + \varepsilon'$ where η and ε' are the two-forms on M characterized by $\iota_X \eta = 0$, $(X \in \mathscr{V})$, $\iota_X \varepsilon' = 0$, $(X \in \mathscr{H}^+)$, $\eta = \omega_+$ on $\mathscr{H}^+$, and $\varepsilon' = \varepsilon$ on $\mathscr{V}$.

If, further, $\mathscr{H}^+$ is holomorphic, with respect to J_+ and J_-, then, by Theorem 5.3, it is also integrable, geodesic and its leaves endowed with (g, J) are Kähler manifolds; in particular, $d\eta = 0$ on $\mathscr{H}^+$. As, also, $\mathscr{V}$ is a holomorphic foliation, it quickly follows that $(\mathscr{L}_X \eta)(Y, Z) = 0$ for any sections X of $\mathscr{V}$ and Y, Z of $\mathscr{H}^+$; consequently, $d\eta = 0$.

We have thus obtained $d\varepsilon' = 0$ which implies $(\mathscr{L}_X \varepsilon')(Y, Z) = 0$ for any sections X of $\mathscr{H}^+$ and Y, Z of $\mathscr{V}$. Together with (6.1), this gives $(\mathscr{L}_X b)(Y, Z) = 0$ and $(\mathscr{L}_X g)(Y, Z) = 0$ for any sections X of $\mathscr{H}^+$ and Y, Z of $\mathscr{V}$; in particular, this shows that $\mathscr{V}$ is geodesic. The proof follows. $\square$

Obviously, a result similar to Corollary 6.3 holds for any generalized Kähler manifold with $\mathscr{H}^+ = 0$ and $\mathscr{H}^-$ an integrable distribution.

Corollary 6.4. *Let (M, L_1, L_2) be a generalized Kähler manifold such that L_2 is in normal form with respect to its f-structure and the two-form ε on M.*

Then, in a neighborhood of each regular point of L_1, we have that (M, L_1, L_2) is the second product of a Kähler manifold and a generalized Kähler manifold determined by a tamed symplectic manifold.

Proof. Assume that L_1 regular. Define $\varepsilon_\pm$ to be the (complex linear) two-forms on $T_+^{1,0}M + T_-^{1,0}M$ such that $\varepsilon_\pm = \varepsilon$ on $T_\pm^{1,0}M$ and $\iota_X \varepsilon_\pm = 0$ if $X \in T_\mp^{1,0}M$.

Obviously, $d\varepsilon_\pm = 0$ on $T_\pm^{1,0}M$. Also, from the fact that $\iota_X \varepsilon_\pm = 0$ if $X \in T_\mp^{1,0}M$ it quickly follows that if $X_\pm, Y_\pm, Z_\pm \in T_\pm^{1,0}M$ then $d\varepsilon_\pm(X_\mp, Y_\mp, Z_\pm) = 0$; together with the fact that $\varepsilon = \varepsilon_+ + \varepsilon_-$ on $T_+^{1,0}M + T_-^{1,0}M$, this implies that $d\varepsilon_\pm(X_\pm, Y_\pm, Z_\mp) = 0$. Thus, we have proved that $d\varepsilon_\pm = 0$ on $T_+^{1,0}M + T_-^{1,0}M$.

Therefore $\ker \varepsilon_\pm = T_\mp^{1,0} \oplus (T_\mp^{0,1} \cap \mathscr{H}^-)$ is integrable which implies that $\mathscr{H}^-$ is an antiholomorphic distribution on $(M, J_\mp)$. Hence, by Lemma 5.5, we have that $\mathscr{H}^-$ is integrable and the proof follows from Theorem 5.3 and the fact that $\ker \varepsilon = \mathscr{H}^-$. $\square$

Let (M, ε, J) be a tamed almost symplectic manifold. With the same notations as in Corollary 6.4, if (M, L_1, L_2) is the generalized Kähler manifold determined by (M, ε, J) then, from (6.1), it follows that $L_1 = L(T_+^{1,0}M + T_-^{1,0}M, i\varepsilon_+ - i\varepsilon_-)$.

Theorem 6.5 (*Cf. [8]*). *Let (M, ε, J) be a tamed almost symplectic manifold and let (M, L_1, L_2) be the corresponding generalized almost Kähler manifold; denote by $\rho^\pm : T^{\mathbb{C}}M \to T_\pm^{1,0}M$ the projections.*

If (M, L_1, L_2) is generalized Kähler then $J_\pm$ are integrable and $\rho_*^\pm(L_2)$ are holomorphic Poisson structures on $(M, J_\pm)$, respectively. Furthermore, the converse holds if also $J_+ - J_-$ is invertible; moreover, in this case, if (M, L_1, L_2) is generalized Kähler then $\rho_*^\pm(L_2)$ are holomorphic symplectic structures on $(M, J_\pm)$, respectively.

Proof. Assume, for simplicity, that (M, ε, J) is real analytic. Also, we may assume L_1 regular. If (M, L_1, L_2) is generalized Kähler then, by passing to the complexification of (M, ε, J), from Proposition 1.3 and the proof of Corollary 6.4 we obtain that $\rho_*^\pm(L_2)$ are the canonical Poisson quotients of $L(T_+^{0,1}M + T_-^{0,1}M, i\overline{\varepsilon_\mp})$.

If $J_+ \pm J_-$ are invertible and $J_\pm$ are integrable then $\rho_*^\pm(L_2)$ are holomorphic Poisson structures on $(M, J_\pm)$ if and only if $d\varepsilon_\pm = 0$. $\square$

We call the $\rho_*^\pm(L_2)$ of Theorem 6.5 the *holomorphic Poisson structures associated to* (M, L_1, L_2).

Remark 6.6 (*Cf. [8,7]*).

(1) Let (M, L_1, L_2) be a generalized Kähler manifold with $J_+ + J_-$ invertible. Denote by $\eta_\pm$ the (real) bivectors on M which determine the holomorphic Poisson structures on $(M, J_\pm)$, respectively, associated to (M, L_1, L_2); that is, with respect to $J_\pm$, we have $\eta_\pm^{1,1} = 0$ and the holomorphic bivectors corresponding to $\rho_*^\pm(L_2)$ are $\eta_\pm^{2,0}$, respectively.

It quickly follows that

$$\eta_- = -\eta_+ = \frac{1}{2}(J\varepsilon^{-1} + \varepsilon^{-1}J^*) = \frac{1}{2}(J_+ - J_-)\varepsilon^{-1} = \frac{1}{4}[J_+, J_-]g^{-1},$$

where (M, ε, J) is the tamed symplectic manifold associated to (M, L_1, L_2).

Hence, the symplectic foliation associated to η_+ is given by $\mathscr{V}(=\mathrm{im}(J_+ - J_-))$.

(2) If the generalized almost Kähler structure (L_1, L_2) on M corresponds to the quadruple (g, b, J_+, J_-) then (L_2, L_1) corresponds to $(g, b, J_+, -J_-)$. Assume that (M, L_1, L_2) is a generalized Kähler manifold with $J_+ + J_-$ and $J_+ - J_-$ invertible and let η_+ and η'_+ be the bivectors which determine, as in (1), the holomorphic symplectic structures associated to (M, L_1, L_2) and (M, L_2, L_1), respectively. Then (6.3) implies that $\eta'_+ = -\eta_+$.

Next, we prove some results on holomorphic maps between generalized Kähler manifolds.

Corollary 6.7. *Let (M, L_1, L_2) be a generalized almost Kähler manifold with $J_+ + J_-$ and $J_+ - J_-$ invertible.*

If $\varphi : M \to M$ is a diffeomorphism then any two of the following assertions imply the third:

  (i) $\varphi : (M, L_1) \to (M, L_1)$ *is holomorphic.*

 (ii) $\varphi : (M, L_2) \to (M, L_2)$ *is holomorphic.*

(iii) $[d\varphi, J_+ J_-] = 0$.

Proof. Let $L = L(T_+^{1,0}M + T_-^{1,0}M, \varepsilon_1)$. By using the first relation of (6.1), we obtain

$$(\mathrm{Im}\,\varepsilon_1)(J_+ - J_-) = \varepsilon(J_+ + J_-), \tag{6.3}$$

which, firstly, shows that if (iii) holds then (i) $\Longleftrightarrow$ (ii).

Furthermore, (6.3) implies that $\varepsilon^{-1}(\mathrm{Im}\,\varepsilon_1)$ is skew-adjoint, with respect to g, and, consequently, $\varepsilon - \mathrm{Im}\,\varepsilon_1$ is invertible. This fact together with (6.3) proves that (i), (ii) $\Longrightarrow$ (iii). $\square$

Corollary 6.8. *Let (M, L_1^M, L_2^M) and (N, L_1^N, L_2^N) be generalized Kähler manifolds, with $J_+^M + J_-^M$ and $J_+^N + J_-^N$ invertible, and let $\varphi : M \to N$ be a map.*

 (i) *If $\varphi : (M, L_1^M) \to (N, L_1^N)$ and $\varphi : (M, J_\pm^M) \to (N, J_\pm^N)$ are holomorphic then φ is a holomorphic Poisson morphism between the corresponding associated holomorphic Poisson manifolds; moreover, the converse holds if φ is an immersion.*

(ii) *If $\varphi : (M, L_2^M) \to (N, L_2^N)$ and, either, $\varphi : (M, J_+^M) \to (N, J_+^N)$ or $\varphi : (M, J_-^M) \to (N, J_-^N)$ are holomorphic maps then φ is a holomorphic Poisson morphism between the associated holomorphic Poisson structures.*

Proof. Assertion (i) follows from Proposition 1.3 and the proof of Theorem 6.5.

To prove (ii), note that if $\varphi : (M, L_2^M) \to (N, L_2^N)$ is holomorphic then $\varphi : (M, J_+^M) \to (N, J_+^N)$ is holomorphic if and only if $\varphi : (M, J_-^M) \to (N, J_-^N)$ is holomorphic. The proof quickly follows from Remark 6.6(1). □

If $(g, J_\pm)$ are Kähler structures on M then $(g, 0, J_+, J_-)$ corresponds to a generalized Kähler structure (L_1, L_2) on M; furthermore, if b is a closed two-form on M then (g, b, J_+, J_-) corresponds to $\big((\exp b)(L_1), (\exp b)(L_2)\big)$.

Example 6.9 (*Cf. [8]*)**.** Let (M, g, I, J, K) be a hyper-Kähler manifold. Denote by $\omega_I, \omega_J, \omega_K$ the Kähler forms of I, J, K, respectively, and let $\varepsilon = -(\omega_J + \omega_K)$.

Then (M, ε, J) is a tamed symplectic manifold. The corresponding generalized Kähler structure (L_1, L_2) is given by (g, b, J_+, J_-), where $b = \omega_I, J_+ = J$ and $J_- = K$. Also, $L_1 = L\big(T^{\mathbb{C}}M, 2\omega_I - \mathrm{i}(\omega_J - \omega_K)\big)$, $L_2 = L\big(T^{\mathbb{C}}M, -\mathrm{i}(\omega_J + \omega_K)\big)$ and $\varepsilon_+ = -\mathrm{i}(\omega_I - \mathrm{i}\omega_J)$, $\varepsilon_- = -(\omega_K - \mathrm{i}\omega_I)$.

We end with a generalization of Corollaries 5.7 and 6.3.

Theorem 6.10. *Let (M, L_1, L_2) be a generalized Kähler manifold. Then the following assertions are equivalent:*

(i) *$\mathscr{H}^+ \oplus \mathscr{H}^-$ is a holomorphic foliation with respect to J_+ and J_-.*
(ii) *Locally, up to a B-field transformation, (M, L_1, L_2) is the first product of a generalized Kähler manifold for which $J_+ \pm J_-$ are invertible and the second product of two Kähler manifolds.*

Proof. By applying Lemma 5.4 to $\mathscr{H} = \mathscr{H}^+ \oplus \mathscr{H}^-$ twice, with respect to ∇^+ and ∇^-, we obtain

$$
\begin{aligned}
2g\big(B^{\mathscr{H}^+ \oplus \mathscr{H}^-}(J_+X_+, X_-), V\big) + g\big(I^{\mathscr{H}^+ \oplus \mathscr{H}^-}(X_+, X_-), J_+V\big) &= (\mathrm{d}b)(V, J_+X_+, X_-) + (\mathrm{d}b)(V, X_+, J_+X_-), \\
2g\big(B^{\mathscr{H}^+ \oplus \mathscr{H}^-}(J_+X_+, X_-), V\big) + g\big(I^{\mathscr{H}^+ \oplus \mathscr{H}^-}(X_+, X_-), J_-V\big) &= -(\mathrm{d}b)(V, J_+X_+, X_-) + (\mathrm{d}b)(V, X_+, J_+X_-),
\end{aligned}
\tag{6.4}
$$

for any $X_\pm \in \mathscr{H}^\pm$ and $V \in \mathscr{V}$. Consequently, we, also, have

$$
g\big(I^{\mathscr{H}^+ \oplus \mathscr{H}^-}(X_+, X_-), (J_+ - J_-)(V)\big) = 2\mathrm{d}b(V, J_+X_+, X_-),
\tag{6.5}
$$

for any $X_\pm \in \mathscr{H}^\pm$ and $V \in \mathscr{V}$.

Suppose that (i) holds. Then, by (6.5), we have $\mathrm{d}b(V, X_+, X_-) = 0$, for any $X_\pm \in \mathscr{H}^\pm$ and $V \in \mathscr{V}$. Moreover, from Corollaries 5.7 and 6.3 it follows that $\mathrm{d}b(X, Y, Z) = 0$ if $X, Y, Z \in \mathscr{H}^+ \oplus \mathscr{H}^-$ or $X \in \mathscr{H}^\pm$ and $Y, Z \in \mathscr{V} \oplus \mathscr{H}^\pm$.

As $\mathrm{d}(\mathrm{d}b) = 0$, this shows that $\mathrm{d}b$ is basic with respect to $\mathscr{H}^+ \oplus \mathscr{H}^-$. Hence, locally, there exists a two-form b', basic with respect to $\mathscr{H}^+ \oplus \mathscr{H}^-$, such that $\mathrm{d}b = \mathrm{d}b'$.

Furthermore, from (6.4) and (6.5) we obtain $B^{\mathscr{H}^+ \oplus \mathscr{H}^-}(X_+, X_-) = 0$, for any $X_\pm \in \mathscr{H}^\pm$. Together with Theorem 5.3 and Corollary 6.3, this shows that $\mathscr{V}$ and $\mathscr{H}^+ \oplus \mathscr{H}^-$ are geodesic foliations on (M, g).

Thus, we have proved that (M, L_1, L_2) is the first product of a generalized Kähler manifold with $\mathscr{H}^+ = 0 = \mathscr{H}^-$ and a generalized Kähler manifold with $\mathscr{V} = 0$. Hence, by Corollary 5.7, assertion (ii) holds.

The implication (ii) $\Longrightarrow$ (i) is trivial. □

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