Math. Proc. Camb. Phil. Soc. (2008), **145**, 141
doi:10.1017/S0305004108001060 *Printed in the United Kingdom*
First published online 27 March 2008

Harmonic morphisms with one-dimensional fibres on conformally-flat Riemannian manifolds

BY RADU PANTILIE†

Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, *RO*-014700 *Bucharest, Romania.*
e-mail: Radu.Pantilie@imar.ro

(*Received* 15 *January* 2007; *revised* 2 *June* 2007)

Abstract

We classify the harmonic morphisms with one-dimensional fibres (1) from real-analytic conformally-flat Riemannian manifolds of dimension at least four (Theorem 3·1), and (2) between conformally-flat Riemannian manifolds of dimensions at least three (Corollaries 3·4 and 3·6).

Also, we prove (Proposition 2·5) an integrability result for any real-analytic submersion, from a constant curvature Riemannian manifold of dimension $n + 2$ to a Riemannian manifold of dimension 2, which can be factorised as an n-harmonic morphism with two-dimensional fibres, to a conformally-flat Riemannian manifold, followed by a horizontally conformal submersion, ($n \geqslant 4$).

Introduction

Harmonic morphisms between Riemannian manifolds are maps which pull-back (local) harmonic functions to harmonic functions. By a basic result, a map is a harmonic morphism if and only if it is a harmonic map which is horizontally weakly conformal (see [**1**]).

There are, now, several classification results for harmonic morphisms with one-dimensional fibres. In [**2**], it was proved that there are just two types of such harmonic morphisms from Riemannian manifolds, with constant curvature, of dimension at least four. This result was generalized, in [**12**], to Einstein manifolds of dimension at least five; in dimension four, the situation is different, there appears a third type of harmonic morphism [**10**] (see [**13**]). Also, in [**14**], are classified the 'twistorial' harmonic morphisms with one-dimensional fibres from self-dual four-manifolds.

In this paper, we classify the harmonic morphisms with one-dimensional fibres from conformally-flat Riemannian manifolds of dimension at least four. We prove that, assuming real-analyticity, there are just two types of such harmonic morphisms (Theorem 3·1), one of which (the 'Killing type'), also, appears in the above mentioned results, whilst the second type is an extension of the 'warped product type', involved in [**2**, **10**, **12**, **14**].

Furthermore, we classify the (smooth) harmonic morphisms with one-dimensional fibres between conformally-flat Riemannian manifolds (Corollary 3·4). It follows that *the Hopf*

† The author gratefully acknowledges that this work was partially supported by a CEx Grant no. 2-CEx 06-11-22/25.07.2006.

polynomial map $\mathbb{R}^4 \to \mathbb{R}^3$, $(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2z_1\overline{z_2})$, *is, up to local conformal diffeomorphisms, the only harmonic morphism with one-dimensional fibres and nonintegrable horizontal distribution between conformally-flat Riemannian manifolds, of dimensions at least three* (Corollary 3·6).

Let $\varphi\colon (M^4, g) \to (P^2, k)$ be a submersive harmonic morphism between (oriented) Riemannian manifolds of dimensions four and two, respectively. In [**15**], it is proved that if (M^4, g) is Einstein then one of the two almost Hermitian structures on (M^4, g), with respect to which φ is holomorphic, is integrable and parallel along the fibres of φ. This result was generalized in [**6**] to Einstein–Weyl spaces. Here (Proposition 2·5), we give a higher dimensional version of this result which applies to pairs formed of a submersive n-harmonic morphism $\varphi\colon (M^{n+2}, g) \to (N^n, h)$, from a constant curvature Riemannian manifold to a conformally-flat Riemannian manifold, and a horizontally-conformal submersion $\psi\colon (N^n, h) \to (P^2, k)$ which is real-analytic in flat local coordinates, $(n \geqslant 4)$.

1. *Harmonic morphisms with one-dimensional fibres*

In this section we recall a few facts on harmonic morphisms with one-dimensional fibres.

Unless otherwise stated, all the manifolds are assumed to be connected and smooth and all the maps are assumed to be smooth.

Definition 1·1. A *harmonic morphism* (*between Riemannian manifolds*) is a map $\varphi : (M^m, g) \to (N^n, h)$ such that if U is an open set of N, with $\varphi^{-1}(U) \neq \emptyset$, and f is a harmonic function on $(U, h|_U)$ then $f \circ \varphi$ is a harmonic function on $(\varphi^{-1}(U), g|_{\varphi^{-1}(U)})$.

Definition 1·2. A map between Riemannian manifolds $\varphi\colon (M^m, g) \to (N^n, h)$ is *horizontally weakly conformal* if, for any $x \in M$, either $d\varphi_x = 0$ or, $d\varphi_x$ is surjective and, for any $X, Y \in (\ker d\varphi_x)^{\perp}$, we have $h(d\varphi(X), d\varphi(Y)) = \lambda(x)^2 g(X, Y)$ for some positive number $\lambda(x)$.

The function λ, extended to be zero over the set of points $x \in M$ where $d\varphi_x = 0$, is called the *dilation* of φ. (Note that the dilation λ is continuous on M^m whilst the *square dilation* λ^2 is smooth on M^m.)

Let $\varphi\colon (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion; denote by λ its dilation. Then $\lambda = 1$ if and only if φ is a *Riemannian submersion*.

If $m = n$ then a bijective map $\varphi\colon (M^m, g) \to (N^m, h)$ is a horizontally conformal submersion if and only if it is a *conformal diffeomorphism*. The dilation of a conformal diffeomorphism φ is called the *conformality factor* of φ.

Next, we recall the following basic result (see [**1**]).

THEOREM 1·3. *A map is a harmonic morphism if and only if it is a harmonic map which is horizontally weakly conformal.*

As usual, if $\varphi\colon (M^m, g) \to N^n$ is a submersion we denote by $\mathscr{V} = \ker d\varphi$ the *vertical distribution* and by $\mathscr{H} = \mathscr{V}^{\perp}$ the *horizontal distribution*.

For horizontally conformal submersions with one-dimensional fibres the condition of harmonicity can be expressed as follows.

PROPOSITION 1·4 ([**2**], see [**1, 8**]). *Let* $\varphi\colon (M^{n+1}, g) \to (N^n, h)$ *be a horizontally conformal submersion with one-dimensional fibres,* $(n \geqslant 1)$. *Let* λ *be the dilation of* φ *and let* V *be the vertical vector field (well-defined up to sign) such that* $g(V, V) = \lambda^{2n-4}$.

The following assertions are equivalent:

(i) $\varphi : (M^{n+1}, g) \to (N^n, h)$ *is a harmonic morphism,*

(ii) $[V, X] = 0$ *for any basic (horizontal) vector field* X.

Furthermore, if (i) *or* (ii) *holds then* $\Omega = d\theta$ *is basic, where* θ *is the vertical dual of* V, *characterised by* $\theta(V) = 1$ *and* $\theta|_{\mathscr{H}} = 0$.

Let $\varphi\colon (M^{n+1}) \to (N^n, h)$ be a harmonic morphism with one-dimensional fibres. With the same notations as in Proposition 1·4, the vector field V is called the *fundamental vector field*. It is easy to prove that $\Omega = 0$ if and only if $\mathscr{H}$ is integrable. Also, we have $g = \lambda^{-2}\varphi^*(h) + \lambda^{2n-4}\theta^2$ and, hence, V is a Killing vector field if and only if $\mathscr{V}$ is a Riemannian foliation; equivalently, $V(\lambda) = 0$ ([**2**], see [**1**, **8**]). If V is Killing then φ is called of *Killing type*.

LEMMA 1·5 ([**9**], cf. [**1**]). *Let* $\varphi : (M^{n+1}, g) \to (N^n, h)$ *be a harmonic morphism. Let* λ *be its dilation and let* V *be the fundamental vector field of* φ*; we shall denote by* $\sigma = \log\lambda$.

Then we have the following relations for the curvature tensors R^M *and* R^N *of* (M^{n+1}, g) *and* (N^n, h)*, respectively*:

$$\begin{aligned} R^M(X, V, Y, V) = &-\frac{1}{2}(n-2)e^{(2n-4)\sigma}(\mathcal{L}_{\mathscr{H}(\mathrm{grad}_h\sigma)}h)(X, Y) \\ &-(n-2)e^{(2n-4)\sigma}\{nX(\sigma)Y(\sigma) - |\mathscr{H}(\mathrm{grad}_h\sigma)|_h^2\, h(X, Y)\} \\ &+e^{-2\sigma}\{V(V(\sigma)) - (n-1)V(\sigma)^2\}h(X, Y) \\ &+\frac{1}{4}\, e^{(4n-6)\sigma}\, h(i_X\Omega, i_Y\Omega), \end{aligned} \tag{1·1}$$

$$\begin{aligned} R^M(X, Y, Z, V) = &-\frac{1}{2}e^{(2n-4)\sigma}({}^h\nabla\Omega)(X, Y, Z) \\ &+\frac{1}{2}(n-1)e^{(2n-4)\sigma}\{X(\sigma)\Omega(Y, Z) + Y(\sigma)\Omega(Z, X) \\ &-2Z(\sigma)\Omega(X, Y)\} - e^{-2\sigma}\{X(V(\sigma)) - (n-2)X(\sigma)V(\sigma)\}h(Y, Z) \\ &+e^{-2\sigma}\{Y(V(\sigma)) - (n-2)Y(\sigma)V(\sigma)\}h(X, Z) \\ &+\frac{1}{2}e^{(2n-4)\sigma}\{\Omega(X, \mathrm{grad_h}\sigma)\mathrm{h(Y, Z)} - \Omega(\mathrm{Y}, \mathrm{grad_h}\sigma)\mathrm{h(X, Z)}\}, \end{aligned} \tag{1·2}$$

$$\begin{aligned} &R^M(X, Y, Z, H) = e^{-2\sigma}\varphi^*(R^N)(X, Y, Z, H) \\ &-\frac{1}{4}\, e^{(2n-4)\sigma}\{\Omega(H, X)\Omega(Y, Z) + \Omega(H, Y)\Omega(Z, X) - 2\Omega(H, Z)\Omega(X, Y)\} \\ &-\frac{1}{2}\, e^{-2\sigma}V(\sigma)\,\{-\Omega(Y, H)h(X, Z) + \Omega(X, H)h(Y, Z) \\ &-\Omega(X, Z)h(Y, H) + \Omega(Y, Z)h(X, H)\} - e^{-2\sigma}\{X(\sigma)H(\sigma)h(Y, Z) \\ &-X(\sigma)Z(\sigma)h(Y, H) - Y(\sigma)H(\sigma)h(X, Z) + Y(\sigma)Z(\sigma)h(X, H)\} \\ &+e^{-2\sigma}\{h(X, Z)h({}^h\nabla_Y(\mathscr{H}(\mathrm{grad}_h\sigma)), H) - h(Y, Z)h({}^h\nabla_X(\mathscr{H}(\mathrm{grad}_h\sigma)), H) \\ &+h(Y, H)h({}^h\nabla_X(\mathscr{H}(\mathrm{grad}_h\sigma)), Z) - h(X, H)h({}^h\nabla_Y(\mathscr{H}(\mathrm{grad}_h\sigma)), Z)\} \\ &-e^{-2\sigma}\{h(X, Z)h(Y, H) - h(X, H)h(Y, Z)\}\{e^{(-2n+2)\sigma}\, V(\sigma)^2 + |\mathscr{H}(\mathrm{grad}_h\sigma|_h^2\}, \end{aligned} \tag{1·3}$$

where X, Y, Z, H *are horizontal and* ${}^h\nabla$ *denotes the Levi–Civita connection of* (M, h).

Remark 1·6. See [**1**] and the references therein for more information on harmonic morphisms between Riemannian manifolds and, in particular, for the notion of p-harmonic morphism. Also, see [**6**, **7**, **11**] for harmonic morphisms in the more general setting of Weyl geometry.

2. *Conformally-flat Riemannian manifolds*

Firstly, we recall (see [**4**]) the definition of the Weyl tensor of a Riemannian manifold.

Let (M^m, g) be a Riemannian manifold. For h and k sections of $\odot^2(T^*M)$ (that is, h and k are symmetric covariant tensor fields of degree two on M^m), we shall denote by $h \owedge k$ the section of $\odot^2(\Lambda^2(T^*M))$ defined by

$$\begin{aligned}(h \owedge k)(T, X, Y, Z) =& h(T, Y)k(X, Z) + h(X, Z)k(T, Y)\\ &- h(T, Z)k(X, Y) - h(X, Y)k(T, Z),\end{aligned}$$

for any $T, X, Y, Z \in TM$.

If S is a (1,3)-tensor field on (M, g) then we shall denote by the same symbol S the (0,4)-tensor field defined by $S(T, X, Y, Z) = -g(S(T, X, Y), Z)$, for any $T, X, Y, Z \in TM$.

The *Weyl (curvature) tensor* of (M^m, g) is the (1,3)-tensor field W characterised by the following two conditions:

(1) trace(X $\mapsto$ W(X, Y)Z) $= 0$, for any $Y, Z \in TM$,

(2) $R = g \owedge r + W$ for some (necessarily unique) section r of $\odot^2(T^*M)$, where R is the curvature tensor of (M^m, g).

The Weyl tensor is conformally invariant; that is, if we denote by W^g the Weyl tensor of (M^m, g) then $W^{\lambda^2 g} = W^g$, for any positive function λ on M^m.

The Riemannian manifold (M^m, g) is called *(locally) conformally-flat* if for each point of M^m there exists an open neighbourhood U and a conformal diffeomorphism φ from U onto some open set of $\mathbb{R}^m$ (endowed with its canonical Riemannian metric); the local coordinates on U induced by φ are called *flat*.

From Liouville's theorem on local conformal diffeomorphisms between Euclidean spaces (see [**1**]), it follows easily that if (M^m, g) is conformally-flat then M^m is real-analytic in flat local coordinates ($m \geqslant 2$).

The following theorem is due to H. Weyl (see [**4**]).

THEOREM 2·1. *A Riemannian manifold, of dimension at least four, is conformally-flat if and only if its Weyl tensor is zero.*

(See [**4**] for the case when the dimension is less than four.)

We do not imagine that the following result is new.

PROPOSITION 2·2. *Let* (M^m, g) *be a Riemannian manifold,* ($m \geqslant 4$). *The following assertions are equivalent*:

(i) (M^m, g) *is conformally-flat*,

(ii) $R(X, Y, X, Y) = 0$ *for any* $X, Y \in TM$ *spanning an isotropic space on* (M, g), *where* R *is the curvature tensor of* (M, g), *and* TM *now denotes the complexified tangent bundle.*

Proof. Clearly, assertion (ii) is equivalent to $W(X, Y, X, Y) = 0$ for any $X, Y \in TM$ spanning an isotropic space on (M, g), where W is the Weyl tensor of (M, g). Therefore, by Theorem 2·1, we have (i) $\Rightarrow$ (ii).

Suppose that (ii) holds and let $(X_1, \ldots, X_m)$ be an orthonormal frame on (M^m, g). Then for any distinct $i, j, k, l \in \{1, \ldots, m\}$ we have

$$W(X_i \pm \mathrm{i}X_j, X_k + \mathrm{i}X_l, X_i \pm \mathrm{i}X_j, X_k + \mathrm{i}X_l) = 0 .$$

This is equivalent to the following two relations

$$W(X_i, X_k + \mathrm{i}X_l, X_i, X_k + \mathrm{i}X_l) = W(X_j, X_k + \mathrm{i}X_l, X_j, X_k + \mathrm{i}X_l), \tag{2·1}$$

$$W(X_i, X_k + \mathrm{i}X_l, X_j, X_k + \mathrm{i}X_l) = 0 . \tag{2·2}$$

Also, by applying condition (2) of the definition of the Weyl tensor, we obtain

$$\sum_{r=1}^{m} W(X_r, X_k + \mathrm{i}X_l, X_r, X_k + \mathrm{i}X_l) = 0 . \tag{2·3}$$

From (2·1) and (2·3), it follows that $W(X_j, X_k + \mathrm{i}X_l, X_j, X_k + \mathrm{i}X_l) = 0$ and, hence, $W_{jkjk} = W_{jljl}$, for any distinct $j, k, l \in \{1, \ldots, m\}$. Therefore, for any distinct $i, j \in \{1, \ldots, m\}$, we have

$$(m-1)W_{ijij} = \sum_{r=1}^{m} W_{irir} = 0 .$$

From (2·2) we obtain that, for any distinct $i, j, k, l \in \{1, \ldots, m\}$, we have

$$\begin{aligned} W_{ikjk} &= W_{iljl}, \\ W_{ikjl} &= -W_{iljk}. \end{aligned} \tag{2·4}$$

The first relation of (2·4) implies $W_{ikjk} = 0$, whilst from the second and the algebraic Bianchi identity it follows quickly that $W_{ijkl} = 0$, for any distinct $i, j, k, l \in \{1, \ldots, m\}$.

Thus, if (ii) holds then $W = 0$ which, by Theorem 2·1, is equivalent to (i).

If $\mathscr{H}$ is a distribution on a Riemannian manifold (M^m, g) we shall denote by $I^{\mathscr{H}}$ the integrability tensor of $\mathscr{H}$, which is the $\mathscr{V}$-valued horizontal two-form on M^m defined by $I^{\mathscr{H}}(X, Y) = -\mathscr{V}[X, Y]$, for any horizontal vector fields X and Y, where $\mathscr{V} = \mathscr{H}^{\perp}$.

Next, we prove the following:

PROPOSITION 2·3. *Let $\varphi: (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion between conformally-flat Riemannian manifolds,* $(m \geqslant n \geqslant 4)$.

Then $g\big(I^{\mathscr{H}}(X, Y), I^{\mathscr{H}}(X, Y)\big) = 0$, for any horizontal vectors X and Y spanning an isotropic space on (M^m, g).

Proof. As both the hypothesis and the conclusion are conformally-invariant, we may suppose that $\varphi : (M^m, g) \to (N^n, h)$ is a Riemannian submersion. Then the proof follows easily from Proposition 2·2 and the following well-known relation of B. O'Neill (see [**1**]):

$$R^M(X, Y, X, Y) = \varphi^*(R^N)(X, Y, X, Y) - \frac{3}{4}\, g\big(\mathscr{V}[X, Y], \mathscr{V}[X, Y]\big),$$

for any horizontal vector fields X and Y.

COROLLARY 2·4. *Any horizontally conformal submersion, with fibres of dimension at most two, between conformally-flat Riemannian manifolds has integrable horizontal distribution, if the codomain has dimension at least four.*

Proof. Let $\varphi\colon (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion between conformally-flat Riemannian manifolds, $m \geqslant n \geqslant 4$.

Let $x \in M$ and let $E \subseteq T_x M$ be an oriented four-dimensional subspace. From Proposition 2·3, it follows that $I^{\mathscr{H}}_x\colon \Lambda^2_+ E \to \mathscr{V}_x$ is conformal, where $\Lambda^2_+ E$ is the space of self-dual bivectors on $(E, g|_E)$ (by definition, a bivector $v \in \Lambda^2 E$ is self-dual if $*_E v = v$ where $*_E$ is the Hodge $*$-operator of $(E, g|_E)$). As $\Lambda^2_+ E$ is three-dimensional, we obtain that either $I^{\mathscr{H}}_x = 0$ or $\dim(\mathscr{V}_x) \geqslant 3$.

We end this section with an application of Corollary 2·4.

An *almost CR-structure*, on a manifold M^m, is a section J of $\mathrm{End}(\mathscr{H})$ such that $J^2 = -\mathrm{Id}_{\mathscr{H}}$, where $\mathscr{H}$ is some distribution on M^m. Obviously, J is determined by its eigenbundle corresponding to $-\mathrm{i}$ (or i). Furthermore, a subbundle $\mathscr{F}$ of the complexified tangent bundle of M^m is the eigenbundle corresponding to $-\mathrm{i}$ of an almost CR-structure on M^m if and only if $\mathscr{F} \cap \overline{\mathscr{F}} = \{0\}$.

Let J be an almost CR-structure on M^m and let $\mathscr{F}$ be its eigenbundle corresponding to $-\mathrm{i}$; J is called *integrable* if for any $X,\ Y \in \Gamma(\mathscr{F})$ we have $[X, Y] \in \Gamma(\mathscr{F})$. A *CR-structure* is an integrable almost CR-structure (see [**7**]). If M^m is endowed with a Riemannian metric g then $\mathscr{F}$ is isotropic, with respect to g, if and only if J is orthogonal, with respect to (the Riemannian metric induced on $\mathscr{H}$ by) g.

For example, any oriented two-dimensional distribution $\mathscr{V}$, on a Riemannian manifold (M^m, g), determines two orthogonal CR-structures on (M^m, g); at each point $x \in M$, these are given by the rotations of angles $\pm\pi/2$ on $\mathscr{V}_x$ (cf. [**15**]).

Let $\varphi\colon (M^{n+2}, g) \to (N^n, h)$ and $\psi : (N^n, h) \to (P^2, k)$ be horizontally conformal submersions, $n \geqslant 2$. Let $\mathscr{V} = \ker d\varphi$, $\mathscr{H} = \mathscr{V}^\perp$ and let $\mathscr{K} \subseteq \mathscr{H}$ be the horizontal lift of $(\ker d\psi)^\perp$. Assume $\mathscr{V}$ and P^2 oriented and orient $\mathscr{K}$ such that the isomorphism $\mathscr{K} = (\psi \circ \varphi)^*(TP)$ to be orientation preserving.

Then the positive/negative orthogonal CR-structures determined by $\mathscr{V}$ and the positive orthogonal CR-structure determined by $\mathscr{K}$ sum up to give orthogonal almost CR-structures $J^{\varphi,\psi}_\pm$ on (M^{n+2}, g). Obviously, if we endow (P^2, k) with its positive Hermitian structure J^P then $\psi \circ \varphi : (M^{n+2}, J^{\varphi,\psi}_\pm) \to (P^2, J^P)$ is holomorphic; that is, the differential of $\psi \circ \varphi$ intertwines $J^{\varphi,\psi}_\pm$ and J^P.

We call $J^{\varphi,\psi}_\pm$ the *positive/negative almost CR-structures associated to* φ *and* ψ.

PROPOSITION 2·5. *Let* $\varphi\colon (M^{n+2}, g) \to (N^n, h)$ *be a submersive n-harmonic morphism from a Riemannian manifold of constant curvature to a conformally-flat Riemannian manifold, and let* $\psi\colon (N^n, h) \to (P^2, k)$ *be a horizontally conformal submersion which is real-analytic in flat local coordinates,* $(n \geqslant 4)$*; assume* $\mathscr{V}$ $(= \ker d\varphi)$ *and* P^2 *oriented. Denote by* $J^{\varphi,\psi}_\pm$ *the almost CR-structures associated to* φ *and* ψ.

Then either $J^{\varphi,\psi}_+$ *or* $J^{\varphi,\psi}_-$ *is integrable and parallel along the fibres of* φ.

Proof. As the n-Laplacian on n-dimensional Riemannian manifolds is conformally invariant, we may suppose (N^n, h) real-analytic, in flat local coordinates. Therefore, also, φ is real-analytic.

Note that φ has minimal fibres [**5**]. Also, by Corollary 2·4, the distribution $\mathscr{H}$ is integrable.

Let $\mathscr{F}_\pm$ be the eigenbundles of $J^{\varphi,\psi}_\pm$ corresponding to $-\mathrm{i}$. Let Y be a basic vector field which locally generates $\mathscr{F}_+ \cap \mathscr{F}_-$. Then Y is isotropic. Moreover, from the fact that φ

and ψ are horizontally conformal, it follows that $\nabla_Y Y$ is proportional to Y, where ∇ is the Levi-Civita connection of (M^{n+2}, g).

There exists an isotropic vertical vector field U such that $\mathscr{F}_+$ and $\mathscr{F}_-$ are, locally, generated by $\{U, Y\}$ and $\{\overline{U}, Y\}$, respectively; we may suppose that $g(U, \overline{U}) = 1$. As Y is basic, $[U, Y]$ and $[\overline{U}, Y]$ are vertical. Thus, $\mathscr{F}_+$ and $\mathscr{F}_-$ are integrable if and only if $g([U, Y], U) = 0$ and $g([\overline{U}, Y], \overline{U}) = 0$, respectively.

As (M^{n+2}, g) has constant curvature, $R^M(U, Y, Y, \overline{U}) = 0$. On the other hand, a straightforward calculation shows that (cf. [**6**])

$$R^M(U, Y, Y, \overline{U}) = g([U, Y], U)g([\overline{U}, Y], \overline{U}) \,. \tag{2·5}$$

The proof follows.

Remark 2·6. (1) If $n = 2$ then the conclusion of Proposition 2·5 holds under the assumption that (M^4, g) is Einstein [**15**] (see [**6**] for a generalization of this result to Einstein–Weyl spaces).

If $n = 3$ then the proof of Proposition 2·5 works under the further assumption that the horizontal distribution of φ is integrable.

(2) Proposition 2·5, also, holds under the assumption that φ is a real-analytic horizontally conformal submersions such that the mean curvature of $\mathscr{V}$ takes values in $(\mathscr{V} \oplus \mathscr{K})^{\perp}$. Also, note that, in the proof, we have not use the fact that $d\varphi(\mathscr{K})^{\perp}$ $\big(= \ker d\psi\big)$ is integrable.

3. *The main results*

This section is devoted to the following result and its consequences.

THEOREM 3·1. *Let $\varphi\colon (M^{n+1}, g) \to (N^n, h)$ be a harmonic morphism between Riemannian manifolds, $(n \geqslant 3)$; denote by λ the dilation of φ.*

If (M^{n+1}, g) is real-analytic and conformally-flat then either:

(i) *φ is of Killing type;* or

(ii) *the horizontal distribution of φ is integrable and its leaves endowed with the metrics induced by $\lambda^{-2n+4}g$ have constant curvature.*

Proof. By a result of [**12**], at least away of the critical points (which may occur only if $n = 3$, see [**1**]), we have $\varphi : (M^{n+1}, g) \to (N^n, h)$ real-analytic.

As the dimension of the intersection of (the complexification of) $\mathscr{H}$ with any isotropic two-dimensional space, on (M^{n+1}, g), is at least 1, Proposition 2·2 implies that (M^{n+1}, g) is conformally-flat if and only if, for any $U \in \Gamma(\mathscr{V})$ and $X, Y \in \Gamma(\mathscr{H})$ with $g(U, U) = g(X, X)$, $g(X, Y) = 0$, $g(Y, Y) = 0$, we have $R^M(U \pm \mathrm{i}X, Y, U \pm \mathrm{i}X, Y) = 0$; equivalently,

$$\begin{aligned} &R^M(U, Y, U, Y) = R^M(X, Y, X, Y) \\ &R^M(U, Y, X, Y) = 0 \,. \end{aligned} \tag{3·1}$$

From (1·2), it follows quickly that the second relation of (3·1) is equivalent to

$$({}^h\nabla_Y \Omega)(X, Y) + 3(n-1)Y(\sigma)\Omega(X, Y) = 0 \,. \tag{3·2}$$

Thus, by assuming X and Y basic and using Proposition 1·4, we obtain

$$Y(V(\sigma))\Omega(X, Y) = 0, \tag{3·3}$$

where V is the fundamental vector field of φ.

Next, we shall use the first relation of (3·1). For this, we assume X and Y basic with $g(X,X) = e^{-2\sigma}$ (equivalently, $h(X,X) = 1$), and $U = e^{-(n-1)\sigma}V$ (so that, $g(U,U) = g(X,X)$). Thus, the first relation of (3·1) becomes

$$e^{-(2n-2)\sigma}R^M(V,Y,V,Y) = R^M(X,Y,X,Y)$$

which, by applying (1·1) and (1·3), is equivalent to

$$\begin{aligned} R^N(X,Y,X,Y) = &- (n-1)h({}^h\nabla_Y(\mathscr{H}(\mathrm{grad_h}\sigma)), \mathrm{Y}) - (\mathrm{n}-1)^2\mathrm{Y}(\sigma)^2 \\ &+ \frac{1}{4}e^{(2n-2)\sigma}\big\{h(i_Y\Omega, i_Y\Omega) + 3\Omega(X,Y)^2\big\}, \end{aligned} \tag{3·4}$$

where we have denoted by the same symbol R^N and its pull-back by φ to M^{n+1}.

We may assume that Y is the horizontal lift of an isotropic geodesic (local) vector field on (the complexification of) (N^n, h); equivalently, ${}^h\nabla_Y Y = 0$. Then (3·4) becomes

$$\begin{aligned} R^N(X,Y,X,Y) = &- (n-1)Y(Y(\sigma)) - (n-1)^2Y(\sigma)^2 \\ &+ \frac{1}{4}e^{(2n-2)\sigma}\big\{h(i_Y\Omega, i_Y\Omega) + 3\Omega(X,Y)^2\big\}\,. \end{aligned} \tag{3·5}$$

As $R^N(X,Y,X,Y)$ is basic, from (3·3) and (3·5) it easily follows that either $\Omega = 0$ or

$$V(\sigma)\big\{h(i_Y\Omega, i_Y\Omega) + 3\Omega(X,Y)^2\big\} = 0\,. \tag{3·6}$$

Now, from $\Omega \neq 0$ it follows that there exist $Y \in \mathscr{H}$ isotropic and $X \in Y^\perp \cap \mathscr{H}$ such that the second factor of the left-hand side of (3·6) is not zero. Thus, we have proved that either $\Omega = 0$ (equivalently, $\mathscr{H}$ is integrable) or $V(\sigma) = 0$ (equivalently, φ is of Killing type).

Next, we study the case $\Omega = 0$. Then (3·2) (and hence, also, the second relation of (3·1)) is automatically satisfied, whilst (3·4) is equivalent to

$$R^N(X,Y,X,Y) = {}^h\nabla(d^{\mathscr{H}}u)(Y,Y) - (d^{\mathscr{H}}u)(Y)^2, \tag{3·7}$$

where $u = -(n-1)\sigma$ and, recall that, X and Y are basic with $h(X,X) = 1$, $h(X,Y) = 0$ and $h(Y,Y) = 0$.

Let $h_1 = e^{2u}h|_{\mathscr{H}} = e^{(-2n+4)\sigma}g|_{\mathscr{H}}$.

We have proved that, if $\mathscr{H}$ is integrable, (3·1) is equivalent to the fact that the curvature tensor R^P of any leaf P of $\mathscr{H}$, endowed with the metric induced by h_1, satisfies $R^P(X,Y,X,Y) = 0$.

It follows that if $\mathscr{H}$ is integrable then h_1 induces a conformally-flat Einstein metric on each leaf of $\mathscr{H}$; equivalently, h_1 induces a metric of constant curvature on each leaf of $\mathscr{H}$. The proof is complete.

Example 3·2. Let (N^n, h) be $\mathbb{R}^n$, endowed with the canonical metric, and let

$$M^{n+1} = \big\{(t,x) \in \mathbb{R}\times\mathbb{R}^n \,\big|\, |tx| < 1\big\},$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$.

Define $\lambda : M^{n+1} \to (0,\infty)$ by $\lambda(t,x) = (1-|tx|^2)^{\frac{1}{n-1}}$, $(t,x) \in M^{n+1}$, and let $g = \lambda^{-2}h + \lambda^{2n-4}dt^2$.

Then φ: $(M^{n+1}, g) \to (N^n, h)$, $(t,x) \mapsto x$, is a harmonic morphism which satisfies assertion (ii) of Theorem 3·1; in particular, (M^{n+1}, g) is conformally-flat, $(n \geqslant 3)$. Furthermore, φ is neither of Killing type nor are its fibres geodesics.

Remark 3·3. If $n = 3$ then Theorem 3·1 holds, also, in the complex-analytic category. Indeed, the only point in the proof of Theorem 3·1 where it is essential for φ to be 'real' is when we deduce from $\Omega \neq 0$ that there exist $Y \in \mathscr{H}$ isotropic and $X \in Y^{\perp} \cap \mathscr{H}$ such that the second factor of the left-hand side of (3·6) is not zero. But, if $n = 3$ and $h(X, X) = 1$ then

$$h(i_Y\Omega, i_Y\Omega) + 3\Omega(X, Y)^2 = 4\Omega(X, Y)^2,$$

which, also, in the complex-analytic category, is not zero, for suitable choices of X and Y, if $\Omega \neq 0$.

Next, we discuss the case when both the domain and codomain, of a harmonic morphism with one-dimensional fibres, are conformally-flat; the notations are as in Section 1.

COROLLARY 3·4. *Let* $\varphi\colon (M^{n+1}, g) \to (N^n, h)$ *be a submersive harmonic morphism with connected one-dimensional fibres,* ($n \geqslant 3$).

The following assertions are equivalent:

(i) (M^{n+1}, g) *and* (N^n, h) *are conformally-flat.*

(ii) *one of the following assertions holds:*

(iia) φ *is of Killing type,* $n = 3$, *and, up to homotheties,* Ω *is the volume form of a Riemannian foliation by geodesic surfaces, of sectional curvature* 1, *on* $(N^3, \lambda^{-4}h)$,

(iib) *the horizontal distribution of* φ *is integrable and its leaves endowed with the metrics induced by* $\lambda^{-2n+4}g$ *have constant curvature.*

Proof. If $n \geqslant 4$ this follows from Corollary 2·4 and the proof of Theorem 3·1.

Assume $n = 3$. Then by the proof of Theorem 3·1, if (M^4, g) is conformally-flat, on each connected component of a dense open subset of M^4, either φ is of Killing type or (iib) holds.

If φ is of Killing type then there exist Weyl connections $D_{\pm}$ on $(N^3, [h])$ such that $\varphi : (M^4, [g]) \to (N^3, [h], D_{\pm})$ is $\pm$twistorial, in the sense of [**7**, example 4·8] (cf. [**14**]). Furthermore, (M^4, g) is conformally-flat if and only if both $D_{\pm}$ are Einstein–Weyl (see [**14**] and the references therein). Also, if (N^3, h) is conformally-flat then $D_{\pm}$ are Einstein–Weyl if and only if, locally, $D_{\pm}$ are the Levi–Civita connections of constant curvature representatives $h_{\pm}$ of $[h]$ (see [**3**]).

We claim that if $n = 3$ and φ is of Killing type then, with the same notations as above, the following assertions are equivalent:

(a) up to homotheties, Ω is the volume form of a Riemannian foliation by geodesic surfaces, of sectional curvature 1, on $(N^3, \lambda^{-4}h)$;

(a′) $D_{\pm}$ are, locally, the Levi-Civita connections of constant curvature representatives $h_{\pm}$ of $[h]$, and $D_+ \neq D_-$.

Indeed, if φ is of Killing type then, by replacing g and h with $\lambda^{-2}g$ and $\lambda^{-4}h$, respectively, we may suppose that φ is a Riemannian submersion with geodesic fibres. Then the Lee forms $\alpha_{\pm}$ of $D_{\pm}$, with respect to h, are given by $\alpha_{\pm} = \pm *_h \Omega$ (see [**7**, **14**]), where $*_h$ is the Hodge $*$-operator of h, with respect to some local orientation, and we have denoted by the same symbol Ω and the two-form on N^3 whose pull-back by φ is Ω. Hence, if (a′) holds then $h_{\pm} = e^{\pm 2u}h$ where u is characterised by $du = *_h\Omega$; in particular, u is a harmonic (local) function on (N^3, h).

It follows that (a′) is equivalent to the following assertion:

(a″) locally, there exists a nonconstant function u on N^3 such that

$$du = *_h\Omega,\ \ (\nabla^h du)(Y, Y) = 0,\ \ \mathrm{Ric}^h(Y, Y) = -du(Y)^2,$$

for any isotropic vector Y on (N^3, h), where ∇^h is the Levi-Civita connection of (N^3, h) and Ric^h is the Ricci tensor of (N^3, h).

Now, by applying, for example, Lemma 1·5, we obtain that assertion (a″) is equivalent to the following:

(a‴) locally, there exists a nonconstant function u on N^3 such that

$$du = *_h\Omega,\ \nabla^h du = 0$$

and the level surfaces of u have sectional curvature equal to $|du|^2$.

The proof of (a) ⇔ (a′) follows.

We have thus proved that (ii) ⇒ (i), and if (i) holds then, also, (ii) holds on each connected component of a dense open subset of M^4.

To complete the prof of (i) ⇒ (ii), define a connection ∇ on $\mathscr{H}$ by

$$\nabla_E X = \mathscr{H}\nabla^{\lambda^{-2}g}_{\mathscr{H}E} X + \mathscr{H}[\mathscr{V}E, X]$$

for any vector field E and horizontal vector field X, where $\nabla^{\lambda^{-2}g}$ is the Levi–Civita connection of $(M^4, \lambda^{-2}g)$.

If we assume (i) then, from the fact that (ii) holds on each connected component of a dense open subset of M^4, it follows quickly that $\nabla\Omega = 0$. Therefore either Ω is nowhere zero or $\Omega = 0$ on M^4. The proof is complete.

Example 3·5. Let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ be the *Hopf polynomial map* defined by $\pi(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1\overline{z_2})$, for any $(z_1, z_2) \in \mathbb{R}^4\ (= \mathbb{C}^2)$.

Then $\pi|_{\mathbb{R}^4\setminus\{0\}}$ satisfies assertion (iia) of Corollary 3·4.

We end with the following consequence of Corollary 3·4.

COROLLARY 3·6. *The Hopf polynomial map $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ is, up to local conformal diffeomorphisms with basic conformality factors, the only harmonic morphism with one-dimensional fibres and nonintegrable horizontal distribution between conformally-flat Riemannian manifolds, of dimensions at least three.*

Proof. This follows from the fact that any harmonic morphism which satisfies assertion (iia) of Corollary 3·4 is, locally, uniquely determined, up to conformal diffeomorphisms with basic conformality factors.

Acknowledgements. I am grateful to John C. Wood for useful comments.

REFERENCES

[**1**] P. BAIRD AND J. C. WOOD. *Harmonic morphisms between Riemannian manifolds.* London Math. Soc. Monogr. (N.S.), no. 29 (Oxford University Press, 2003).

[**2**] R. L. BRYANT. Harmonic morphisms with fibres of dimension one. *Comm. Anal. Geom.* **8** (2000), 219–265.

[**3**] D. M. J. CALDERBANK. The Faraday 2-form in Einstein–Weyl geometry. *Math. Scand.* **89** (2001), 97–116.

[**4**] J. LAFONTAINE. Conformal geometry from the Riemannian viewpoint. *Conformal geometry* (Bonn, 1985/1986), *Aspects Math.* E12, (Vieweg, Braunschweig, 1988), 65–92.

[**5**] E. LOUBEAU. The Fuglede–Ishihara and Baird–Eells theorems for $p > 1$. *Contemp. Math.* **288** (2001), 376–380.

[**6**] E. LOUBEAU and R. PANTILIE. Harmonic morphisms between Weyl spaces and twistorial maps, *Comm. Anal. Geom.* **14** (2006), 847881.

[**7**] E. LOUBEAU and R. PANTILIE. Harmonic morphisms between Weyl spaces and twistorial maps II. Preprint, IMAR, Bucharest (2006), (math.DG/0610676).

[**8**] R. PANTILIE. Harmonic morphisms with one-dimensional fibres. *Internat. J. Math.* **10** (1999), 457–501.

[**9**] R. PANTILIE. *Submersive harmonic maps and morphisms*. PhD Thesis (University of Leeds, 2000).

[**10**] R. PANTILIE. Harmonic morphisms with 1-dimensional fibres on 4-dimensional Einstein manifolds. *Comm. Anal. Geom.* **10** (2002), 779–814.

[**11**] R. PANTILIE. Harmonic morphisms between Weyl spaces. *Modern Trends in Geometry and Topology*. Proceedings of the Seventh International Workshop on Differential Geometry and Its Applications. (Deva, Romania, 5-11 September, 2005), 321–332.

[**12**] R. PANTILIE and J. C. WOOD. Harmonic morphisms with one-dimensional fibres on Einstein manifolds. *Trans. Amer. Math. Soc.* **354** (2002), 4229–4243.

[**13**] R. PANTILIE and J. C. WOOD. A new construction of Einstein self-dual manifolds. *Asian J. Math.* **6** (2002) 337–348.

[**14**] R. PANTILIE and J. C. WOOD. Twistorial harmonic morphisms with one-dimensional fibres on self-dual four-manifolds. *Quart. J. Math.* **57** (2006), 105–132.

[**15**] J. C. WOOD. Harmonic morphisms and Hermitian structures on Einstein 4-manifolds. *Internat. J. Math.* **3** (1992), 415–439.