Radu Pantilie · John C. Wood

Topological restrictions for circle actions and harmonic morphisms

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Abstract. Let M^m be a compact oriented smooth manifold admitting a smooth circle action with isolated fixed points which are isolated as singularities as well. Then all the Pontryagin numbers of M^m are zero and its Euler number is nonnegative and even. In particular, M^m has signature zero. We apply this to obtain non-existence of harmonic morphisms with one-dimensional fibres from various domains, and a classification of harmonic morphisms from certain 4-manifolds.

Introduction

It is well known that, if a compact oriented smooth manifold M admits a smooth *free* circle action, then its Euler number and all its Pontryagin numbers are zero. This follows from the fact that the tangent bundle of M is the Whitney sum of a trivial real line bundle and the pull back of the tangent bundle of the orbit space.

In this paper we generalize this by proving that *if M is a compact oriented smooth manifold which admits a smooth circle action with isolated fixed points which are isolated as singularities as well then (i) all the Pontryagin numbers of M are zero (in particular, the signature of M is zero), (ii) the Euler number of M is even and is equal to the number of fixed points* (Theorem 1.1). We obtain this by using a well known formula of R. Bott [9] (see also [21]). Also, we apply an idea of J.D.S. Jones to prove (Theorem 2.8) that *the signature of a compact oriented* 4*-manifold endowed with a non-trivial circle action for which each fixed point has equal exponents is equal to the Euler number of the normal bundle of the components of dimension* 2 *of the fixed point set.*

Harmonic morphisms are smooth maps $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds which preserve Laplace's equation. They are characterised as harmonic maps which are *horizontally weakly conformal* [16], [19], i.e., for each point $x \in M$, either $\mathrm{d}\varphi_x = 0$ or $\mathrm{d}\varphi_x$ is surjective and maps $\mathcal{H}_x = (\ker \mathrm{d}\varphi_x)^\perp$ conformally onto $T_{\varphi(x)}N$. Classification results for harmonic morphisms with one-dimensional fibres appear in [4], [5], [6], [10], [27], [28], [29], [30]. In [1] it is proved that any non-constant harmonic morphism with one-dimensional fibres defined on a Riemannian manifold of dimension at least five is submersive whilst,

R. Pantilie, J.C. Wood: Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, UK. e-mail: r.pantilie@leeds.ac.uk; j.c.wood@leeds.ac.uk.

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for domains of dimension four, only isolated critical points can occur. Moreover, in [1] it is proved that any non-constant harmonic morphism $(M^4, g) \to (N^3, h)$ induces a locally smooth circle action on M^4. We show in fact that, at least outside the critical set, the action is smooth and free (a consequence of Proposition 3.1). It follows that the result of Theorem 1.1 can be applied to obtain topological restrictions on the total space of a harmonic morphism with one-dimensional fibres. These are obtained in Theorem 3.3 from which it immediately follows that $S^{2n} (n \geq 3)$, $\mathbb{C}P^n$ $(n \geq 1)$, $\mathbb{H}P^n$ $(n \geq 2)$, $S^{2n} \times P_g$ $(n \geq 2, g \neq 1$ or $n = 1, g \geq 2)$, the complex surfaces of degree $d \neq 2$ in $\mathbb{C}P^3$ can never be the total space of a non-constant harmonic morphism with one-dimensional fibres whatever metrics we put on them. The result regarding $\mathbb{C}P^2$ answers a question formulated by P. Baird in a conversation.

We apply Theorem 3.3 to prove that, *up to homotheties and Riemannian coverings, the canonical projection $T^4 \to T^3$ between flat tori is the only harmonic morphism with one-dimensional fibres which is defined on a compact half-conformally flat Riemannian* 4*-manifold which is scalar-flat* (Proposition 3.7). Finally, in Proposition 3.9 we show that, if $\varphi : (M^4, g) \to (N^3, h)$ is a non-constant harmonic morphism between orientable Einstein manifolds, then (M^4, g) is half-conformally flat.

1. A topological restriction for circle actions

By a *singularity* of a group action we mean a point at which the isotropy group is non-trivial. A *fixed point* is a singularity where the isotropy group is the entire group.

Let M^m be a compact oriented smooth manifold of dimension $m \geq 1$ endowed with a smooth circle action. Let F denote its fixed point set and V its infinitesimal generator. Let g be a Riemannian metric on M with respect to which V is a Killing vector field. Such a metric can be obtained by averaging an arbitrary Riemannian metric over the action. Let ∇ denote the Levi-Civita connection on (M^m, g) with curvature form R.

Obviously F is the zero set of V and thus, its connected components are totally-geodesic submanifolds of (M, g) of even codimension (see [21]).

Let $x \in F$ and suppose that the connected component N of x in F has codimension $2r$. Because $(\nabla V)_x$ is a skew-symmetric endomorphism of $(T_x M, g_x)$, with respect to a suitably chosen orthonormal frame, $(\nabla V)_x$ is represented by the direct sum of the zero square matrix of dimension $m - 2r$ and $\oplus_{j=1}^r \left(\begin{smallmatrix} 0 & -m_j \\ m_j & 0 \end{smallmatrix}\right)$ where $m_j > 0$. In fact, since V integrates to give an S^1 action on M^m, it follows from [22, Chapter I, Proposition 1.9] that, $m_j \in \mathbb{Z}$. Indeed, since V is Killing, its flow commutes with the exponential map; hence, via the exponential map at x, the linear flow induced by $(\nabla V)_x$ on $T_x M$ is locally equivalent to the flow of V. Following [20] we shall call the (positive) integers m_j the *exponents* of the action at the fixed point x. This is, of course, motivated by the fact that the exponential map of (M^m, g) at x induces a local equivalence between the given S^1 action and the following S^1 action on $\mathbb{R}^m = \mathbb{R}^{m-2r} \oplus \mathbb{C}^r$:

$$z \cdot (x_1, \ldots, x_{m-2r}, z_1, \ldots, z_r) = (x_1, \ldots, x_{m-2r}, z^{m_1} z_1, \ldots, z^{m_r} z_r).$$

In particular, this shows that the exponents $(m_1, \ldots, m_r)$ do not depend on the metric g. This also follows from the fact that, at each $x \in F$, we can write $(\nabla V)_x = -\mathcal{L}_V|_{T_x M}$. Furthermore, if $x \in F$ is an isolated fixed point (equivalently $m = 2r$), then the orientation induced by the corresponding orthonormal frame on $T_x M$ does not depend of g. Let $\epsilon(x)$ be $+1$ or -1 according to whether or not this orientation agrees with the orientation of $T_x M$ (cf. [20]).

The main result of this section is the the following.

Theorem 1.1. *Let M^m be a compact oriented smooth manifold which admits a smooth circle action whose fixed points are isolated singularities.*

Then (i) *all the Pontryagin numbers of M^m are zero,* (ii) *the Euler number of M^m is even and is equal to the number of fixed points. In particular, the signature of M^m is zero.*

Proof. Let $x \in F$. Because x is an isolated singularity the exponents at x are all equal to 1. Equivalently, there exists an orthonormal basis of $(T_x M, g_x)$ with respect to which the matrix of $(\nabla V)_x$ is the direct sum of n copies of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Thus, if $F \neq \emptyset$, then $\dim M = 2n$ is even. Let f be an $\mathrm{Ad}(\mathrm{SO}(2n))$-invariant symmetric polynomial of degree $p \leq n$. Then, by a result of R. Bott [9, Theorem 2] (see also [21, Theorem II.6.1]) we have

$$\sum_{x \in F} \frac{f((\nabla V)_x)}{\chi_n(\frac{1}{2\pi}(\nabla V)_x)} = \int_M f(R) \tag{1.1}$$

where χ_n is the Pfaffian (see [21, p. 68] or [26, p. 309]) and $f(R)$ is the closed $2p$-form on M which represents the cohomology class induced by the Chern–Weil morphism applied to f via the Levi-Civita connection ∇ of (M^{2n}, g) (see [22, Chapter XII] or [26, Appendix C]). Note that the right hand side of (1.1) is zero if $p < n$.

It is easy to prove that

$$\chi_n(\tfrac{1}{2\pi}(\nabla V)_x) = \frac{(-1)^n \, \epsilon(x)}{(2\pi)^n} \ . \tag{1.2}$$

By taking $f = 1$, from (1.1) and (1.2), we obtain

$$\sum_{x \in F} \epsilon(x) = 0 \ . \tag{1.3}$$

By taking $f = \frac{1}{(2\pi)^n} \chi_n$, from (1.1) and the Gauss-Bonnet theorem we obtain that the Euler number of M^{2n} is equal to the cardinal of F (this also follows from the Poincaré-Hopf theorem or from [21, Theorem II.5.5]). But, by (1.3), the cardinal of F must be even and hence the Euler number of M^{2n} is even.

By definition, if $\dim M$ is not divisible by four then all the Pontryagin numbers of M are zero.

Suppose that $n = 2p$ and let $i_1, \dots, i_r$ be a partition of p. Denote by p_{i_k} the $\mathrm{Ad}(\mathrm{SO}(2m))$-invariant symmetric polynomial of degree $2i_k$ such that $p_{i_k}(\frac{1}{2\pi} R)$ represents the i_k'th Pontryagin class of M.

Let $x \in F$ and recall that $(\nabla V)_x$ is the direct sum of n copies of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then it is obvious that for any $\mathrm{Ad}(\mathrm{SO}(2n))$-invariant symmetric polynomial f we have

$$f((\nabla V)_x) = c(f, n) \tag{1.4}$$

where $c(f, n)$ is a constant which depends just on f and n but not on $x \in F$.

By taking $f = p_{i_1} \cdots p_{i_r}$ in (1.1) it follows from (1.2), (1.3), (1.4) that all the Pontryagin numbers of M^{2n} are zero. The fact that M^{2n} has zero signature follows from Hirzebruch's Signature Theorem (see [26]). □

Remark 1.2. With the same hypotheses as in Theorem 1.1, it follows from [11] that, if the set of fixed points is nonempty and the orbit space is a manifold, then $\dim M = 4$ and the number of fixed points is even and equal to the Euler number of M. Then (1.3) corresponds to an equality in the proof of [11, Proposition 3.5(b)].

2. Blowing-up isolated fixed points of circle actions

For $j = 1, \dots, r$ let m_j be positive integers. We consider the following action on $\mathbb{C}^r$:

$$z \cdot (z_1, \dots, z_r) = (z^{m_1} z_1, z^{m_2} z_2, \dots, z^{m_r} z_r) \ . \tag{2.1}$$

Obviously, the fixed point set of this action is $\{0\}$.

In what follows we need the notion of 'equivariant connected sum' of two manifolds endowed with circle actions. The idea of the following construction comes from [11] (we use $-\mathbb{C}^r$ to denote $\mathbb{C}^r$ considered with the orientation opposite to its usual one).

Definition 2.1. *Let M and N be manifolds,* $\dim M = \dim N = 2r$, *both endowed with non-trivial circle actions.*

Let $x \in M$ and $y \in N$ be isolated fixed points of these actions having the same set of exponents $\{m_1, \dots, m_r\}$, $m_1 \leq \dots \leq m_r$. Further, assume that $\epsilon(x) = -\epsilon(y)$.

Then suitably chosen neighbourhoods U_x and U_y about x and y in M and N, respectively, are equivariantly diffeomorphic to open balls of radius three about $0 \in \epsilon(x)\mathbb{C}^r$ and $0 \in \epsilon(y)\mathbb{C}^r$, respectively, where $\mathbb{C}^r$ is endowed with the circle action of (2.1).

The equivariant connected sum of M and N (about $x \in M$ and $y \in N$) *is the quotient $(M \setminus V_x \sqcup N \setminus V_y)/\!\sim$ where V_x and V_y correspond, via the above equivariant diffeomorphisms, to the open balls $\epsilon(x)B(1)$ and $\epsilon(y)B(1)$, of radius* 1, *whilst $\sim$ is induced by the identification $(2+t)u \sim (2-t)u$ with $u \in \partial B(1)$ and $t \in [-1, 1]$.*

Thus the actions glue together to give a non-trivial circle action on the connected sum $M \# N$.

Example 2.2. Let $\mathbb{C}^r$ and $-\mathbb{C}^r$ be endowed with the action given by (2.1). It is easy to see that the connected sum $\mathbb{C}^r \# - \mathbb{C}^r$, suitably constructed about $0 \in \mathbb{C}^r$ in each term, inherits in a canonical way a circle action. Moreover this action is without fixed points.

The following definition is based on an idea of J.D.S. Jones which arose in a private conversation. We formulate it only for isolated fixed points although it can be given for any connected component of the fixed point set.

Definition 2.3. *Let $x \in F$ be an isolated fixed point with exponents $(m_1, \dots, m_r)$. The* blow-up *of M (considered with the given action) at x is the equivariant connected sum of M and $-\epsilon(x)\mathbb{C}P^r$, about $x \in M$ and $[1, 0, \dots, 0] \in \mathbb{C}P^r$, where $\mathbb{C}P^r$ is considered with the S^1 action*

$$z \cdot [z_0, z_1, \dots, z_r] = [z_0, z^{m_1} z_1, \dots, z^{m_r} z_r] \,.$$

In what follows, the following obvious lemma will play an important role.

Lemma 2.4. *Let M be a manifold endowed with a circle action and let F be its fixed point set.*

Let x be an isolated fixed point whose exponents are equal: $m_1 = \dots = m_r$.

Let $\widehat{F}$ be the fixed point set of the induced action on the blow-up of M at x. Then

$$\widehat{F} = (F \setminus \{x\}) \cup \mathbb{C}P^{r-1}$$

where $\mathbb{C}P^{r-1} = \{\, [z_0, \dots, z_r] \in \mathbb{C}P^r \mid z_0 = 0 \,\}$.

Remark 2.5. From the above lemma it follows that if, besides isolated fixed points, a circle action has only components of codimension two, then after blowing up all the isolated fixed points we obtain a manifold endowed with a circle action whose fixed point set is of codimension two.

In particular, if the starting manifold is of dimension four then after blowing-up the isolated fixed points we obtain a manifold endowed with a circle action whose fixed point set is of dimension two.

For the next lemma recall the $\mathrm{Ad}(\mathrm{SO}(4))$-invariant polynomial p_1 on $so(4)$ given by $p_1(A) = \sum_{i<j} (a^i_j)^2$ where $A = (a^i_j)_{i,j=1,\dots,4}$. As is usual, we shall also denote by p_1 the corresponding $\mathrm{Ad}(\mathrm{SO}(4))$-invariant symmetric bilinear form on $so(4)$.

If (M^4, g) is a Riemannian 4-manifold then, if R denotes the curvature form of the Levi-Civita connection of (M^4, g), by the Chern–Weil theorem, $p_1(\frac{1}{2\pi} R)$ represents the first Pontryagin class of M^4 (see [22], [26]).

Also we need the following definition (see [21, p. 69]).

Definition 2.6. *Let (M^{2n}, g) be a Riemannian manifold and let V be a Killing vector field on it. Let N be a component of codimension $2r$ of the zero set of V.*

For an Ad(SO(2n))-invariant polynomial f of degree n on $so(2n)$ the residue of V over N^{2n-2r} is given by

$$\mathrm{Res}_f(N)\, t^{n-r} = \int_N \frac{f(\frac{1}{2\pi}(tR + \nabla V))}{\chi_r(\frac{1}{2\pi}(tR^{\perp} + (\nabla V)^{\perp}))} \tag{2.2}$$

where ∇ is the Levi-Civita connection of (M^{2n}, g), R its curvature form, $^{\perp}$ denotes the components in $End(TN^{\perp})$ and χ_r is the Pfaffian. (Here we expand the right hand side of (2.2) *as a power series in t.)*

Lemma 2.7. *Let (M^4, g) be an oriented 4-manifold and let V be a Killing vector field on it. Suppose that the zero set of V has a component N of dimension two.*

Then the residue $\mathrm{Res}_{p_1}(N)$ is given by

$$\mathrm{Res}_{p_1}(N) = \chi(TN^{\perp})[N]$$

where $\chi(TN^{\perp})[N]$ is the Euler number of the normal bundle $TN^{\perp}$ of N.

Proof. From Definition 2.6 it follows that we can write

$$\mathrm{Res}_{p_1}(N)\, t = \int_N \frac{p_1(\frac{1}{2\pi}(tR + \nabla V))}{\frac{1}{2\pi}(tR + \nabla V)^{\perp}} \tag{2.3}$$

where $\left(\begin{smallmatrix} 0 & a^{\perp} \\ a^{\perp} & 0 \end{smallmatrix}\right)$ denotes the 'normal' component of a section a of $End(TN) \oplus End(TN^{\perp})$ and recall that $(\nabla V)|_N$ is a section of $End(TN) \oplus End(TN^{\perp})$ (see [21, Chapter II, Theorem 5.3]). In fact $(\nabla V)^{\perp}|_N : N \to \mathbb{R}$ is the nowhere zero function on N characterised by

$$(\nabla V)|_N = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & (\nabla V)^{\perp} \\ -(\nabla V)^{\perp} & 0 \end{smallmatrix}\right),$$

then (2.3) becomes

$$\begin{aligned} \mathrm{Res}_{p_1}(N)\, t &= \frac{1}{2\pi} \int_N \big(t^2 p_1(R, R) + 2t p_1(R, \nabla V) + p_1(\nabla V, \nabla V)\big) \\ &\quad \times \Big(\frac{1}{(\nabla V)^{\perp}} - t\, \frac{R^{\perp}}{((\nabla V)^{\perp})^2} + t^2 \frac{(R^{\perp})^2}{((\nabla V)^{\perp})^3} - \cdots \Big) \\ &= \frac{t}{2\pi} \int_N \big(2p_1(R, U) - R^{\perp}\big) \end{aligned} \tag{2.4}$$

where $U = \frac{1}{(\nabla V)^{\perp}} \nabla V|_N$ - note that with respect to a suitably chosen adapted orthonormal frame we have $U = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Also note that $J = U|_N$ is an almost complex structure on the normal bundle $TN^{\perp}$ (see [21] for the general case).

Because N is a totally geodesic submanifold of (M, g) we have that $p_1(R, U) = R^{\perp}$; hence, from (2.4) it follows that

$$\mathrm{Res}_{p_1}(N) = \frac{1}{2\pi} \int_N R^{\perp}\, . \tag{2.5}$$

By the same reason we also have that $J \otimes R^{\perp} = \left(\begin{smallmatrix} 0 & R^{\perp} \\ -R^{\perp} & 0 \end{smallmatrix}\right)$ is the curvature form of the connection induced by ∇ on $TN^{\perp}$.

The proof now follows from the Chern–Weil theorem. □

We now state the main result of this section.

Theorem 2.8. *Let M^4 be a compact oriented 4-manifold endowed with a non-trivial circle action.*

Let $F = F_0 \cup F_2$ be the fixed point set, where F_0 is the set of isolated fixed points and F_2 is the union of the components of dimension 2 of the fixed point set. Suppose that the exponents of each isolated fixed point $x \in F_0$ are equal.

Then the signature $\sigma[M]$ of M is given by

$$\sigma[M] = \sum_{x \in F_0} \epsilon(x) = \chi(T F_2^{\perp})[F_2] \, . \tag{2.6}$$

In particular, the signature of M is given by the Euler number of the normal bundle of the components of dimension 2 of the zero set of V.

Proof. Let $\widehat{M}$ be the manifold (endowed with a circle action) obtained by blowing-up the isolated fixed points, i.e. the points of F_0.

Then, since signatures add when taking connected sums, the signature $\sigma[\widehat{M}]$ of $\widehat{M}$ is given by

$$\sigma[\widehat{M}] = \sigma[M] - \sum_{x \in F_0} \epsilon(x) \; . \tag{2.7}$$

Because the exponents of each isolated fixed point are equal, by hypothesis, the induced circle action on $\widehat{M}$ has *no* isolated fixed points. This follows from Lemma 2.4 (see also Remark 2.5).

Thus we can apply [20, Theorem 4.2] to obtain that $\widehat{M}$ has signature zero. Combining this with (2.7) gives

$$\sigma[M] = \sum_{x \in F_0} \epsilon(x) \; , \tag{2.8}$$

i.e. the first equality of (2.6).

Now, take a metric on M with respect to which S^1 acts by isometries. Then by applying the Bott formula (see [21, Chapter II, Theorem 6.1]) to the infinitesimal generator of this action we obtain

$$p_1[M] = \mathrm{Res}_{p_1}(F_0) + \mathrm{Res}_{p_1}(F_2) \; . \tag{2.9}$$

Since $p_1(A) = \pm 2\chi_2(A)$ for $A \in so(4)_{\pm}$ (because $A = (a^i_j) \in so(4)_{\pm}$ if and only if $a^1_2 = \pm a^3_4$, $a^1_3 = \mp a^2_4$ and $a^1_4 = \pm a^2_3$), we have $\mathrm{Res}_{p_1}(F_0) = 2 \sum_{x \in F_0} \epsilon(x)$. By using this fact and Lemma 2.7 the equation (2.9) becomes

$$p_1[M] = 2 \sum_{x \in F_0} \epsilon(x) + \chi(T F_2^{\perp})[F_2] \; . \tag{2.10}$$

But by Hirzebruch's theorem, $p_1[M] = 3\sigma[M]$, which together with (2.8) and (2.10) gives

$$3 \sum_{x \in F_0} \epsilon(x) = 2 \sum_{x \in F_0} \epsilon(x) + \chi(T F_2^{\perp})[F_2] \; ; \tag{2.11}$$

this immediately yields the second equality of (2.6). □

Remark 2.9. 1) Obviously, Theorem 2.8 generalizes the result of [20, Theorem 4.2]. However, note that our proof of Theorem 2.8 uses that result.

Also, Theorem 2.8 generalizes the result of Theorem 1.1 for dimension four.

2) By applying a calculation similar to the one in the proof Lemma 2.7 we can prove the following result: *If the fixed point set of a circle action on a compact oriented manifold M has codimension two and trivial normal bundle then all the Pontryagin numbers of M are zero.*

3) See [23], [20] for some related results on circle actions.

3. Applications to harmonic morphisms

Let $\varphi : (M^{n+1}, g) \to (N^n, h)$, $n \geq 1$, be a non-constant harmonic morphism between compact oriented Riemannian manifolds. Then, by a result of P. Baird [1], the set Σ of critical points of φ is empty if $\dim M \geq 5$ and is discrete if $\dim M = 4$. For $x \in M \setminus \Sigma$ set $\mathcal{V}_x = \ker \mathrm{d}\varphi_x$ and let $\mathcal{H}_x = \mathcal{V}_x^{\perp}$. The resulting distributions $\mathcal{V}$ and $\mathcal{H}$ on $M \setminus \Sigma$ shall be called, as usual, the *vertical* distribution and *horizontal* distribution, respectively.

Proposition 3.1. *Let $\varphi : (M^{n+1}, g) \to (N^n, h)$ $(n \geq 1)$ be a non-constant harmonic morphism between compact oriented Riemannian manifolds. Let Σ be the set of critical points of φ.*

Then $\varphi|_{M\setminus\varphi^{-1}(\varphi(\Sigma))}$ can be factorised as $\xi \circ \psi$ where $\psi : M \setminus \varphi^{-1}(\varphi(\Sigma)) \to P$ is the projection of an S^1 principal bundle (for which $\mathcal{H}|_{M\setminus\varphi^{-1}(\varphi(\Sigma))}$ is a principal connection) and $\xi : P \to N \setminus \varphi(\Sigma)$ is a smooth covering projection. Moreover, for $n = 3$, the smooth free S^1 action on $M \setminus \varphi^{-1}(\varphi(\Sigma))$ extends to a continuous S^1 action on M which is smooth over $M \setminus \Sigma$ and whose fixed point set is Σ.

Furthermore, if k is the unique metric on P with respect to which the mapping $\xi : (P, k) \to (N \setminus \varphi(\Sigma), h|_{N\setminus\varphi(\Sigma)})$ is a Riemannian covering then the mapping $\psi : (M \setminus \varphi^{-1}(\varphi(\Sigma)), g|_{M\setminus\varphi^{-1}(\varphi(\Sigma))}) \to (P, k)$ is a submersive harmonic morphism with connected fibres.

Proof. Because M and N are oriented, $\mathcal{V}$ is orientable. Thus we can choose $V \in \Gamma(\mathcal{V}|_{M\setminus\Sigma})$ such that $g(V, V) = \lambda^{2n-4}$. Clearly, V is smooth on $M \setminus \Sigma$. Furthermore, when $n = 3$, since $|V| \to 0$ as we approach a critical point, V extends to a continuous vector field on M whose zero set is Σ and the flow of V extends to a continuous flow on M whose fixed point set is Σ.

For any $n \geq 1$, since φ is proper, it quickly follows that $\varphi|_{M\setminus\varphi^{-1}(\varphi(\Sigma))} \to N \setminus \varphi(\Sigma)$ is a proper map. By a simple extension of a result of C. Ehresmann [15], φ restricted to $M \setminus \varphi^{-1}(\varphi(\Sigma))$ is the projection of a locally trivial fibre bundle. In particular, the orbit space P of $V|_{M\setminus\varphi^{-1}(\varphi(\Sigma))}$ is a smooth manifold. Thus, $\varphi|_{M\setminus\varphi^{-1}\varphi(\Sigma))}$ can be factorised as $\xi \circ \psi$ where $\psi : M \setminus \varphi^{-1}(\varphi(\Sigma)) \to P$ has connected fibres and $\xi : P \to N \setminus \varphi(\Sigma)$ is a covering projection.

Let $k = \xi^*(h)$ be the unique metric on P with respect to which $\xi : (P, k) \to (N \setminus \varphi(\Sigma), h|_{N\setminus\varphi(\Sigma)})$ becomes a Riemannian covering. It is obvious that $\psi : (M \setminus \varphi^{-1}(\varphi(\Sigma)), g|_{M\setminus\varphi^{-1}(\varphi(\Sigma))}) \to (P, k)$ is a submersive harmonic morphism with compact connected fibres. From [27, Theorem 2.9] it follows that ψ is the projection of a circle bundle where the action on the total space $M \setminus \varphi^{-1}(\varphi(\Sigma))$ is induced by the flow of $V|_{M\setminus\varphi^{-1}(\varphi(\Sigma))}$. □

Remark 3.2. 1) In [32, Proposition 4.2] are given similar decompositions of maps between (topological) compact manifolds. Also, from [32, Lemma 2.6, Proposition 4.2] it follows that if $\varphi : M^4 \to N^3$ is a map with isolated critical points between compact oriented manifolds then the number of critical points of φ is equal to the Euler number of M^4 (cf. Remark 1.2).

2) Recall ([16], [19], [7]) that, for $n = 1$, a harmonic morphism from (M^2, g) to (N^1, h) is, essentially, a harmonic function.

When $n = 2$, given a non-constant harmonic morphism $\varphi : (M^3, g) \to (N^2, h)$ with M^3 compact the S^1 action extends *smoothly* over the set of critical points and induces on M^3 a structure of a smooth Seifert fibre space [6] and the factorisation in Proposition 3.1 extends smoothly to M^3.

When $n = 3$, the S^1 action extends smoothly if (M^4, g) is Einstein [28] and, in this case, the factorisation again extends smoothly to M^4.

When $n \geq 4$, then any non-constant harmonic morphism $\varphi : (M^{n+1}, g) \to (N^n, h)$ is submersive [1]. Thus, if M^{n+1} is compact (more generally, if $\mathcal{H}$ is an Ehresmann connection for φ) then φ can be factorised as a harmonic morphism with connected fibres followed by a Riemannian covering (over $\varphi(M)$).

From Theorem 1.1 and Proposition 3.1 we obtain the following.

Theorem 3.3. *Let $\varphi : (M^{n+1}, g) \to (N^n, h)$ $(n \geq 3)$ be a non-constant harmonic morphism between compact oriented Riemannian manifolds.*

Then all the Pontryagin numbers of M^{n+1} are zero. In particular, the signature of M^{n+1} is zero.

Further, if $n \geq 4$ then the Euler number of M^{n+1} is zero. If $n = 3$, then the Euler number of M^4 is even and is equal to the number of critical points of φ.

Proof. If $n \geq 4$ then, by Proposition 3.1, there exists a free S^1 action on M^{n+1} whose orbits are connected components of the fibres of φ. Hence, by the Hopf theorem, the Euler number of M^{n+1} is zero. Also, as is well-known (immediate consequence of (1.1)), all the Pontryagin numbers of M^{n+1} are zero.

Suppose that $n = 3$. If the set of critical points Σ is empty, then the same argument as above implies that the Euler number and the Pontryagin number of M^4 are zero.

Suppose that $\Sigma \neq \emptyset$ and let $x \in \Sigma$ and $y = \varphi(x)$. By Proposition 3.1 we can assume that $\varphi|_{M\setminus\varphi^{-1}(\varphi(\Sigma))}$ has connected fibres. Let $B^3 \subseteq N^3$ be a neighbourhood of $y \in N^3$ such that $B^3 \cap \varphi(\Sigma) = \{y\}$ and which is diffeomorphic to the closed ball of radius two centred at zero in $\mathbb{R}^3$. Then $\varphi^{-1}(B^3)$ is a four-dimensional submanifold-with-boundary of M^4 such that $\varphi^{-1}(B^3) \cap \Sigma = \{x\}$. Furthermore, by Proposition 3.1, $\varphi|_{\varphi^{-1}(B^3)\setminus\{x\}}$ is the projection of an S^1-bundle over $B^3 \setminus \{y\}$. Because B^3 is the cone over S^2, $\varphi^{-1}(B^3)$ is the cone over its boundary. It easily follows (cf. [1]) that the boundary of $\varphi^{-1}(B^3)$ must be simply-connected.

Consider the embedding $S^2 = \partial B^3 \subseteq N^3 \setminus \Sigma$. Let $k \in \mathbb{Z}$ be the Chern number of the S^1 bundle $(\varphi^{-1}(S^2),\ S^2,\ S^1)$. Then, if $k \neq 0$, $\varphi^{-1}(S^2) \cong S^3/\mathbb{Z}_{|k|}$

and, in particular, the fundamental group of $\varphi^{-1}(S^2)$ is $\mathbb{Z}_{|k|}$. (If $k = 0$ the bundle $(\varphi^{-1}(S^2), S^2, S^1)$ is trivial.) But, $\varphi^{-1}(S^2)$ is diffeomorphic to the boundary of $\varphi^{-1}(B^3)$ which we have seen is simply-connected. It follows that $k = \pm 1$ and, thus, we can suppose that $\varphi|_{\varphi^{-1}(B^3)\setminus\{x\}}$ is smoothly equivalent to the projection of the cylinder of the Hopf bundle (S^3, S^2, S^1).

Thus, by taking, if necessary, the equivariant connected sum of M and $-k\mathbb{C}^2$, about $x \in M$ and $0 \in \mathbb{C}^2$, where $\mathbb{C}^2$ is considered with its canonical circle action, we can suppose that on $\varphi^{-1}(B^3)$ we have a smooth circle action having x as a fixed point outside of which the action is free. By repeating this procedure about each point of Σ we obtain on M^4 a smooth circle action whose fixed point set is Σ, outside which the action is free.

By Theorem 1.1, the Pontryagin number of M^4 is zero and its Euler number is even and equal to the cardinal of Σ. $\square$

Remark 3.4. With the same notations as in the proofs of Theorem 1.1 and Theorem 3.3 we have that $\epsilon(x) = k$ for each $x \in \Sigma$.

Also, (1.3), applied in the context of the proof of Theorem 3.3 with φ having connected fibres, can be proved by applying the Stokes' theorem and the Chern–Weil theorem; that is, by assuming, if necessary, that $V|_{M\setminus\Sigma}$ has periodicity 2π then

$$0 = \frac{1}{2\pi}\int_{N\setminus \cup_{x\in\Sigma}\overset{\circ}{B}_x} \mathrm{d}F = \sum_{x\in\Sigma}\Big(-\frac{1}{2\pi}\int_{\partial B_x} F\Big) = \sum_{x\in\Sigma}\epsilon(x)$$

where $F \in \Gamma(\Lambda^2(T^*(N\setminus\varphi(\Sigma)))$ is the curvature form of any principal connection on the principal bundle $\varphi|_{M\setminus\Sigma}$ and $B_x \subseteq N$ is a closed ball about each $\varphi(x)$, $x \in \Sigma$, such that $B_{x_1} \cap B_{x_2} = \emptyset$ for $x_1 \neq x_2$.

Let P_g be the closed oriented surface of genus $g \geq 0$. For $d \geq 1$, let

$$S_d = \left\{ [z] \in \mathbb{C}P^3 \mid {z_1}^d + \cdots + {z_4}^d = 0 \right\}$$

be a complex surface of degree d. Since the Euler number of $\mathbb{C}P^n$ is $n+1$, of $S^{2n} \times P_g$ is $4(1-g)$ and the first Pontryagin number of S_d (see [13, page 12]) is $d(4-d^2)$, we have the following immediate consequence of Theorem 3.3.

Corollary 3.5. S^{2n} $(n \geq 3)$, $\mathbb{C}P^n$ $(n \geq 1)$, $\mathbb{H}P^n$ $(n \geq 2)$, $S^{2n} \times P_g$ $(n \geq 2,\ g \neq 1$ *or* $n = 1,\ g \geq 2)$, S_d $(d \neq 2)$ *can never be the domain of a non-constant harmonic morphism with one-dimensional fibres whatever metrics we put on them.*

Remark 3.6. 1) There exists a harmonic morphism from S^4 with suitable metric to S^3 [3] (see also [7]).

2) The projections $S^{2n+1} \times P_g \to \mathbb{C}P^n \times P_g$ induced by the Hopf fibrations $S^{2n+1} \to \mathbb{C}P^n$ are Riemannian submersions with totally-geodesic fibres, with respect to suitable multiples of their standard metrics, and are thus harmonic morphisms with one-dimensional fibres.

3) Other constructions of even-dimensional compact manifolds which cannot be the total space of a harmonic morphism with one-dimensional fibres whatever

metrics we put on them can be easily obtained by using products and/or connected sums of manifolds.

4) We do not know if $S_2 = S^2 \times S^2$ with suitable metric, can be the domain of a non-constant harmonic morphism to some Riemannian 3-manifold. By [29, Theorem 3.6], there exists no non-constant harmonic morphism from $S^2 \times S^2$, endowed with the canonical metric, to some Riemannian 3-manifold.

Next, by applying Theorem 3.3, we prove the following result.

Proposition 3.7. *Let (M^4, g) be a compact orientable half-conformally flat Riemannian four-manifold with zero scalar curvature and let $\varphi : (M^4, g) \to (N^3, h)$ be a non-constant harmonic morphism.*

Then, up to homotheties and Riemannian coverings, φ is the canonical projection $T^4 \to T^3$ between flat tori.

Proof. Choose one of the orientations of M^4 and let ω_g be the corresponding volume form with respect to g.

Let $p_1[M]$ be the Pontryagin number of M^4. By the Chern–Weil theorem we have

$$p_1[M] = \frac{1}{4\pi^2} \int_M \left(|W^+|^2 - |W^-|^2 \right) \omega_g \ , \tag{3.1}$$

where W is the Weyl tensor of (M^4, g) and W^+, W^- are its self-dual and anti-self-dual components, respectively (see [8, 13.8]).

By Theorem 3.3, $p_1[M] = 0$ and hence $W^{\pm} = 0 \iff W^{\mp} = 0$. Thus, if (M^4, g) is half-conformally flat then it is conformally flat.

Now, recall that, by the Gauss-Bonnet theorem, the Euler number of M^4 is given by (see [8], [25]):

$$\chi[M] = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} - \frac{|\,{}^M\mathrm{Ricci}_0\,|^2}{2} + |W^+|^2 + |W^-|^2 \right) \omega_g \ , \tag{3.2}$$

where s is the scalar curvature of (M^4, g) and ${}^M\mathrm{Ricci}_0$ is the trace-free part of the Ricci tensor of (M^4, g).

If (M^4, g) is half-conformal flat and its scalar curvature is zero then (3.2) becomes

$$\chi[M] = -\frac{1}{16\pi^2} \int_M |\,{}^M\mathrm{Ricci}_0\,|^2 \, \omega_g \ .$$

But, by Theorem 3.3, $\chi[M] \geq 0$ and hence (M^4, g) must be Einstein.

If (M^4, g) is Einstein and conformally flat then (M^4, g) has constant sectional curvature k^M (see [8]). By [27, Proposition 3.3(ii)] and [28, Proposition 3.6] we cannot have $k^M < 0$. If $k^M > 0$ then, up to homotheties, the universal cover of (M^4, g) is S^4, a situation which cannot occur (see [10, Section 3] or [7]). Hence (M^4, g) must be flat.

The proof now follows from [27, Theorem 3.4] and an argument as in the proof of [28, Theorem 3.8]. $\square$

From the proof of Proposition 3.7 we obtain the following.

Proposition 3.8. *Let $\varphi : (M^4, g) \to (N^3, h)$ be a non-constant harmonic morphism defined on an orientable compact Riemannian four-manifold.*

Then (M^4, g) is half-conformally flat if and only if it is conformally flat.

We end with a sufficient condition for the domain of a harmonic morphism $\varphi : (M^4, g) \to (N^3, h)$ to be half-conformally flat. Note that this does not require any compactness or completeness assumption.

Proposition 3.9. *Let (M^4, g) be an orientable Einstein four-manifold.*

Suppose that there exists a non-constant harmonic morphism $\varphi : (M^4, g) \to (N^3, h)$ to an orientable Einstein three-manifold (N^3, h).

Then (M^4, g) is half-conformally flat.

Proof. Let $x \in M$ be a regular point of φ and $y = \varphi(x)$. Let $Y_0 \in T_y N$ be a unit vector. Because (N^3, h) is of constant curvature, by [5], there exists an open neighbourhood U of y and a submersive harmonic morphism $\psi_{Y_0} : (U, h|_U) \to P^2$ with values in some Riemann surface P^2 such that its fibre through y is tangent to Y_0. Then for any other unit vector $Y \in T_y N$ we can compose ψ_{Y_0} with an isometry to obtain a submersive harmonic morphism ψ_Y whose fibre through y is tangent to Y.

Then, $\psi_Y \circ \varphi : (\varphi^{-1}(U), g|_{\varphi^{-1}(U)}) \to P^2$ is a submersive harmonic morphism from an orientable Einstein four-manifold to a Riemann surface. By [34, Theorem 1.1], there exists an (integrable) Hermitian structure J_Y on $(\varphi^{-1}(U), g|_{\varphi^{-1}(U)})$ with respect to which $\psi_Y \circ \varphi : (\varphi^{-1}(U), J_Y) \to P^2$ is holomorphic. We can suppose that all the J_Y induce the same orientation σ on $\varphi^{-1}(U)$. Thus any orthogonal complex structure on $(T_x M, g_x)$, that is positive with respect to σ_x, can be extended to a Hermitian structure defined on some neighbourhood of x. It follows that (M^4, g) is half-conformally flat. □

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