# Submersive Harmonic Maps and Morphisms 

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## Introduction

This book is based on my PhD Thesis [49. From the later work I have included here only the classification of harmonic morphisms with onedimensional fibres on Einstein manifolds of dimension at least five [55] and on conformally-flat Riemannian manifolds of dimension at least four [53]. Elsewhere, I shall collect some of my other subsequent work, such as twistorial maps and the related theory of harmonic morphisms between Weyl spaces (see, for example, [56] , 42], [6]).

Harmonic morphisms between Riemannian manifolds are smooth maps which pull back harmonic functions to harmonic functions.

By the fundamental theorem of B. Fuglede [21] and T. Ishihara [33], harmonic morphisms were characterised as harmonic maps which are horizontally weakly conformal.

The next important step towards a better understanding of harmonic morphisms was done by P. Baird and J. Eells [5] who found the necessary and sufficient condition for a harmonic morphism to have minimal (regular) fibres (see Proposition 1.1.10, below). In the particular case of a map $\varphi:\left(M^{m}, g\right) \rightarrow\left(N^{2}, h\right)$ to a Riemannian manifold $\left(N^{2}, h\right)$ of dimension two the result of P. Baird and J. Eells states that $\varphi$ is a harmonic morphism if and only if $\varphi$ is horizontally weakly conformal and its regular fibres are minimal submanifolds of $\left(M^{m}, g\right)$. This result was then elegantly formulated in the language of conformal foliations by J.C. Wood 68].

The first classification results for harmonic morphisms were due to P. Baird and J.C. Wood (see [7, [9]) who completely classified harmonic morphisms
with one-dimensional fibres from three-dimensional Riemannian manifolds with constant curvature.

Later, R.L. Bryant 14 (see Corollary 3.8.5) proved that from a Riemannian manifold $\left(M^{n+1}, g\right), n \geq 3$, with constant sectional curvature there are just two types of submersive harmonic morphisms $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right)$ with one-dimensional fibres, namely either
(i) there exists a nowhere zero Killing vector field tangent to the fibres of $\varphi$ or
(ii) $\varphi$ has geodesic fibres orthogonal to an umbilical foliation by hypersurfaces.

Moreover, the type (i) was new (whilst type (ii) was known to P. Baird and J. Eells (5).

We study harmonic morphisms by placing them into the framework of conformal foliations, an idea due to J.C. Wood [68]. Unless otherwise stated, we consider only foliations of codimension greater than or equal to three.

In Chapter 1] we present some basic facts on foliations which produce harmonic morphisms (i.e. foliations which can be locally defined by submersive harmonic morphisms). We recall Bryant's result [14] which states that a conformal foliation produces harmonic morphisms if and only if a certain one-form is closed (see Proposition 1.3.1). In particular, if the mean curvature forms of a conformal foliation $\mathscr{V}$ and of its orthogonal complement $\mathscr{H}$ are closed then $\mathscr{V}$ produces harmonic morphisms. Following a suggestion of J.C. Wood we call a foliation homothetic if it can be locally defined by horizontally homothetic submersions. This is equivalent to the condition that the foliation is conformal and its orthogonal complement has closed mean curvature form.

Note also that, although a Riemannian one-dimensional foliation produces harmonic morphisms if and only if it is locally generated by Killing vector fields [14], a one-dimensional foliation locally generated by conformal vector fields produces harmonic morphisms if and only if it is a homothetic foliation (a consequence of Proposition 3.1.5). This starts to show the importance of homothetic foliations. Another equivalent condition for a conformal
foliation $\mathscr{V}$ to be homothetic is that any local dilation (Definition 1.1.8) of it can be locally written as the product of a function constant along leaves and a function constant along horizontal curves (in particular, a conformal foliation $\mathscr{V}$ is homothetic if and only if, in a neighbourhood of each point, a local dilation for $\mathscr{V}$ can be found which is constant along horizontal curves). This latter property is also satisfied by any positive smooth function $f$ defined on the Riemannian manifold $(M, g)$ and which has the property that $\mathscr{V}$ produces harmonic morphisms on both $(M, g)$ and $\left(M, f^{2} g\right)$.

For a foliation $\mathscr{V}$ on $M$ there exists a metric $g$ on $M$ with respect to which $\mathscr{V}$ is a homothetic foliation if and only if $\mathscr{V}$ is a foliation of type (A) in the sense of I. Vaisman [66]. However, all the above geometrical properties of the homothetic foliations appear to be new.

In terms of foliations which produce harmonic morphisms the types (i) and (ii) above correspond to:
(i) Riemannian one-dimensional foliations locally generated by Killing vector fields,
(ii) homothetic foliations with geodesic leaves orthogonal to an umbilical foliation by hypersurfaces.

The starting point of the results of Chapter 2 was to try to generalise the type (i) above to foliations of higher dimensions. In Theorem 2.2 .6 we give necessary and sufficient conditions for a conformal foliation locally generated by conformal vector fields to produce harmonic morphisms. In the particular case when the foliation is Riemannian and locally generated by Killing vector fields this condition depends only on the induced infinitesimal action and on the integrability tensor of the horizontal distribution. Many natural constructions of Riemannian foliations which produce harmonic morphisms can be thus obtained. Moreover, most of the constructions can be generalised to homothetic foliations locally generated by conformal vector fields.

In Chapter 2 we also study foliations locally generated by homothetic vector fields and their relations to homothetic foliations and harmonic morphisms. This is motivated by the fact that, on a Ricci-flat manifold, given any
foliation $\mathscr{V}$ locally generated by conformal vector fields and which produces harmonic morphisms, either $\mathscr{V}$ is locally generated by homothetic vector fields or any harmonic morphism produced by $\mathscr{V}$ can be locally decomposed into a totally geodesic harmonic morphism followed by another harmonic morphism. This is shown in Section 2.6 where other similar factorisation results are obtained.

In Chapter 3 we present [48], [52] , [55], [53] the classification of harmonic morphisms with one-dimensional fibres on Einstein manifolds and on conformally-flat Riemannian manifolds. More precisely, we prove the following for a one-dimensional foliation which produces harmonic morphisms on a Riemannian manifold ( $M, g$ ):

- If $(M, g)$ is Einstein, $\operatorname{dim} M \geq 5$, then $\mathscr{V}$ is of type (i) or type (ii) (Theorem 3.4.1).
- If $(M, g)$ is Einstein, $\operatorname{dim} M=4$, then $\mathscr{V}$ is of one of the types (i), (ii) or (iii) (Theorem $\sqrt{3.5 .4}$ ) where types (i) and (ii) are as before and type (iii) is the following: $\left(M^{4}, g\right)$ is Ricci-flat and, up to homotheties, any harmonic morphism $\varphi:\left(U,\left.g\right|_{U}\right) \rightarrow\left(N^{3}, h\right)$, with dilation $\lambda$, produced by $\mathscr{V}$ such that $\left.\mathscr{V}\right|_{U}$ and $N^{3}$ are orientable is (locally) described by:
(a) $\left(N^{3}, h\right)$ has constant sectional curvature equal to one;
(b) $\frac{1}{2} \mathrm{~d}\left(\lambda^{-2}\right)$ is a (flat) principal connection for $\mathscr{V}$ with respect to suitably chosen $V \in \Gamma(\mathscr{V})$ such that $g(V, V)=\lambda^{2}$ (Definition 3.5.1);
(c) the local connection form $A$ of $\mathscr{H}$ with respect to $\frac{1}{2} \mathrm{~d}\left(\lambda^{-2}\right)$ satisfies the equation $\mathrm{d} A+2 * A=0$ on $\left(N^{3}, h\right)$ where $*$ is the Hodge star-operator of $\left(N^{3}, h\right)$ with respect to some orientation of $N^{3}$.
- If $(M, g)$ is analytic and conformally-flat, $\operatorname{dim} M \geq 4$, then $\mathscr{V}$ is of type (i) or type (ii') (Theorem 3.8.1), where type (i) is as before and type (ii') is the following: the orthogonal complement of $\mathscr{V}$ is integrable and its leaves endowed with the metrics induced by $\rho^{2} g$ have constant sectional curvature, where $\rho$ is any local density of $\mathscr{V}$.

Note that, any foliation of type (ii) on a constant curvature Riemannian manifold satisfies the condition (ii'), above.

On the other hand, a foliation $\mathscr{V}$ of type (iii) can never be of type (i), whilst its orthogonal complement is integrable if and only if $\mathscr{V}$ is of type (ii). Therefore the above mentioned classification result of R.L. Bryant is an immediate consequence of Theorems 3.4.1, 3.5.4 and 3.8.1.

Examples of harmonic morphisms of type (iii) are given in Section 3.6. There we prove that these are always submersive (Proposition 3.6.1). Also, we show that any surjective harmonic morphism of type (iii) with connected fibres and complete codomain is, up to homotheties and Riemannian coverings, the restriction of the radial projection $\left(\mathbb{R}^{4} \backslash\{0\}, g_{a}\right) \rightarrow S^{3}$ where $g_{a}$ is the EguchiHanson II metric [19] (when $a=0, g_{a}$ is the restriction of the canonical metric on $\mathbb{R}^{4}$ and $\varphi_{0}$ is a well-known harmonic morphism simultaneously of type (ii) and (iii) ). In particular, there exists no surjective harmonic morphism of type (iii) whose domain and codomain are both complete.

See [54, [56] , 42], [43] for more relations to Einstein and self-dual metric constructions and for relations to Twistor Theory of the harmonic morphisms of type (iii).

In Section 3.7 we study surjective harmonic morphisms $\varphi:\left(M^{4}, g\right) \rightarrow$ $\left(N^{3}, h\right)$ between complete Einstein manifolds of dimensions four and three, respectively. If $M^{4}$ and $N^{3}$ are simply-connected we prove the following:

- If $\varphi$ is submersive then, up to homotheties, it is one of the following projections $\mathbb{R}^{4} \rightarrow \mathbb{R}^{3}, H^{4} \rightarrow \mathbb{R}^{3}, H^{4} \rightarrow H^{3}$ induced by the following canonical warped-product decompositions $\mathbb{R}^{4}=\mathbb{R}^{1} \times \mathbb{R}^{3}, H^{4}=H^{1} \times_{r} \mathbb{R}^{3}$, $H^{4}=H^{1} \times{ }_{s} H^{3}$ where $H^{k}$ is the hyperbolic space of dimension $k$ (Theorem 3.7.1).
- If $\varphi$ has exactly one critical point then there exists $a \geq 0$ such that, up to homotheties, $\varphi:\left(\mathbb{R}^{4}, g_{a}\right) \rightarrow\left(\mathbb{R}^{3}, h_{0}\right)$ is the Hopf polynomial with $g_{a}$ the Gibbons-Hawking Taub-NUT metric $(a>0)$ and $g_{0}, h_{0}$ the canonical metrics on $\mathbb{R}^{4}, \mathbb{R}^{3}$, respectively (Theorem 3.7.10).

In Chapter 4 we present some results on harmonic morphisms defined on compact Riemannian manifolds. In Section 4.1 these are based on a known formula of P. Walczak [67] which we recall in Appendix A.1. The results of

Section 4.2 are based on two integral formulae which are proved in Theorem 4.2.2. From Chapter 4 we mention here the following results:

- On a compact Riemannian manifold with nonpositive Ricci curvature any one-dimensional foliation which produces harmonic morphisms and admits a global density is locally generated by parallel vector fields (Theorem 4.1.2).
- On a compact Riemannian manifold with positive sectional curvature there exists no homothetic foliation which produces harmonic morphisms and has integrable orthogonal complement (Corollary 4.1.6).
- For $n \in\{3,4,5\}$ if $\left(M^{n+1}, g\right),\left(N^{n}, h\right)$ are compact with scalar curvatures $s^{M} \geq 0, s^{N} \leq 0$ and, if at least one of these inequalities is strict, then there exists no nonconstant submersive harmonic morphism $\varphi:\left(M^{n+1}, g\right) \rightarrow$ $\left(N^{n}, h\right)$ (Corollary 4.2.10).
- If $\left(M^{4}, g\right)$ is a compact four-dimensional Einstein manifold and $\varphi$ : $\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ is a non-constant harmonic morphism then, up to homotheties and Riemannian coverings, $\varphi$ is the canonical projection $T^{4} \rightarrow T^{3}$ between flat tori (Theorem 4.3.2).


## CHAPTER 1

## Foliations which produce harmonic morphisms

### 1.1. Foliations and harmonic morphisms

We recall some basic definitions and results on harmonic morphisms and conformal foliations.

Definition 1.1.1 (21], 33]). A harmonic morphism between Riemannian manifolds is a smooth map $\varphi:\left(M^{m}, g\right) \rightarrow\left(N^{n}, h\right)$ such that for any harmonic function $f:\left(U,\left.h\right|_{U}\right) \rightarrow \mathbb{R}$, defined on some open Riemannian submanifold $\left(U,\left.h\right|_{U}\right)$ of $(N, h)$ with $\varphi^{-1}(U) \neq \emptyset$, the pull back $f \circ \varphi:\left(\varphi^{-1}(U),\left.g\right|_{\varphi^{-1}(U)}\right) \rightarrow \mathbb{R}$ is a harmonic function.

By the fundamental result of B. Fuglede 21] and T. Ishihara [33] the harmonic morphisms form a special class of harmonic maps which we now describe.

Definition 1.1.2 ([21], [33]). A smooth $\operatorname{map} \varphi:\left(M^{m}, g\right) \rightarrow\left(N^{n}, h\right)$ between Riemannian manifolds is called horizontally weakly conformal if at each point $x \in M$ either $\varphi$ is submersive (i.e. $\varphi_{*, x}: T_{x} M \rightarrow T_{\varphi(x)} N$ is surjective) and $\left.\varphi_{*}\right|_{\mathscr{H}_{x}}$ is conformal, where $\mathscr{H}_{x}=\left(\operatorname{ker} \varphi_{*, x}\right)^{\perp}$, or $\varphi_{*, x}=0$.

Let $\lambda: M \rightarrow \mathbb{R}$ be the nonnegative function such that $\lambda(x)$ is the conformal factor of $\left.\varphi_{*}\right|_{\mathscr{H}_{x}}$ if $x \in M$ is a regular point of $\varphi$ and $\lambda(x)=0$ if $x$ is a critical point of $\varphi$. Then $\lambda$ is called the dilation of $\varphi$.

Remark 1.1.3 ([21], [33]). Let $\lambda$ be the dilation of the horizontally weakly conformal map $\varphi:\left(M^{m}, g\right) \rightarrow\left(N^{n}, h\right)$. Then, because $\lambda^{2}=\frac{1}{n}\left|\varphi_{*}\right|^{2}, \lambda$ is continuous on $M$ and smooth outside the set of critical points; $\lambda^{2}$ is smooth on $M$.

Theorem 1.1.4 ([21, [33]). A map $\varphi:(M, g) \rightarrow(N, h)$ is a harmonic morphism if and only if $\varphi$ is a harmonic map which is horizontally weakly conformal.

Let ( $M, g$ ) be a Riemannian manifold (it will always be assumed that $M$ is paracompact and connected) and $\mathscr{V}$ (the tangent bundle of) a foliation or, more generally, a distribution on it. The orthogonal complement of $\mathscr{V}$ will de denoted by $\mathscr{H}$. Then, $\mathscr{H}$ and $\mathscr{V}$ will be called the horizontal and vertical distributions, respectively. Following [11, we shall denote the corresponding projections by the same letters $\mathscr{H}$ and $\mathscr{V}$; we shall denote by $X, Y$ horizontal vector fields, i.e. sections of $\mathscr{H}$ and by $U, V$ vertical vector fields, i.e. sections of $\mathscr{V}$.

Definition 1.1.5 (see [68]). A foliation $\mathscr{V}$ on $(M, g)$ is called conformal if for any vertical vector field $U \in \Gamma(\mathscr{V})$ and horizontal vector fields $X, Y \in \Gamma(\mathscr{H})$ we have

$$
\left(\mathcal{L}_{U} g\right)(X, Y)=\mu(U) g(X, Y)
$$

for some vertical one-form $\mu \in \Gamma(T M)$ and where $\mathcal{L}$ denotes the Lie differentiation.

By Theorem 1.1 .4 the connected components of a submersive harmonic morphism $\varphi:(M, g) \rightarrow(N, h)$ form a conformal foliation. Conversely, we make the following definition:

Definition 1.1.6 (cf. [68]). Let $(M, g)$ be a Riemannian manifold and let $\mathscr{V}$ be a foliation on it.

We shall say that $\mathscr{V}$ produces harmonic morphisms on $(M, g)$ if $\mathscr{V}$ is locally defined by submersive harmonic morphisms (i.e. each point of $M$ has an open neighbourhood $U$ which is the domain of a submersive harmonic morphism $\varphi:\left(U,\left.g\right|_{U}\right) \rightarrow(N, h)$ whose fibres are open subsets of the leaves of $\mathscr{V}$.

Remark 1.1.7. When $\operatorname{codim} \mathscr{V}=2, \mathscr{V}$ produces harmonic morphisms if and only if it is conformal and its leaves are minimal [68] ; in this case any local submersion $\varphi$ on $M$ whose fibres are open subsets of the leaves can be made into a harmonic morphism. Indeed, it suffices to choose a metric on the
codomain such that $\varphi$ is horizontally conformal.
We shall see (as an immediate consequence of Corollary 1.1.14) that, when $\operatorname{codim} \mathscr{V} \neq 2$ and $\mathscr{V}$ produces harmonic morphisms, then each submersion $\varphi: U \rightarrow N$ defined on an open subset $U \subseteq M$ with zero first Betti number and which locally defines $\mathscr{V}$ can be made into a harmonic morphism (i.e. there exists a Riemannian metric $h$ on $N$ such that $\varphi:\left(O,\left.g\right|_{O}\right) \rightarrow(N, h)$ is a harmonic morphism).

Definition 1.1.8 (cf. [66, (1.4)]). Let $\mathscr{V}$ be a conformal foliation on the Riemannian manifold $(M, g)$. A smooth positive function $\lambda: O \rightarrow \mathbb{R}$ on an open subset $O$ of $M$ will be called a local dilation of $\mathscr{V}$ if $\left.\mathscr{V}\right|_{O}$ is a Riemannian foliation on $\left(O,\left.\lambda^{2} g\right|_{O}\right)$. If $O=M$ then we shall call $\lambda a$ (global) dilation of $\mathscr{V}$.

Remark 1.1.9. 1) It is obvious that local dilations for a conformal foliation $\mathscr{V}$ can be found in the neighbourhood of each point; in fact, this is equivalent to the definition of its conformality. If $\mathscr{V}$ is simple, i.e. its leaves are the fibres of a (horizontally conformal) submersion $\varphi$, then it admits a (global) dilation, for example, the dilation of $\varphi$.
2) A smooth positive function $\lambda$ is a local dilation for $\mathscr{V}$ if and only if

$$
\left(\mathcal{L}_{U}\left(\lambda^{2} g\right)\right)(X, Y)=0
$$

for any vertical vector field $U$ and any horizontal vector fields $X, Y$. Hence, if we multiply a local dilation of a conformal foliation by a smooth positive function which is constant along the leaves then we obtain another local dilation of the foliation. Conversely, if two local dilations $\lambda_{j}, j=1,2$, of a conformal foliation $\mathscr{V}$ have the same domain then $\lambda_{2}=\lambda_{1} \rho$ where the factor $\rho$ is a smooth positive function, constant along the leaves of $\mathscr{V}$.
3) Let $\mathscr{V}$ be a foliation, not necessarily conformal. Let $\mathscr{H}$ denote its orthogonal complement. Recall that its second fundamental form $\mathscr{H}_{B}$ is the horizontal $\mathscr{V}$-valued tensor field defined by

$$
\begin{equation*}
\mathscr{H}_{B}(X, Y)=\frac{1}{2} \mathscr{V}\left(\nabla_{X} Y+\nabla_{Y} X\right), \tag{1.1.1}
\end{equation*}
$$

where $X, Y$ are horizontal vector fields (see [60, Ch. IV, 3.16]). A simple calculation (see [9]) gives the following formula

$$
\begin{equation*}
\left(\mathcal{L}_{U} g\right)(X, Y)=-2 g\left({ }^{\mathscr{H}} B(X, Y), U\right) \tag{1.1.2}
\end{equation*}
$$

where $U, X, Y$ are as above.
It follows quickly (see [9]) that any local dilation $\lambda$ of a conformal foliation $\mathscr{V}$ is characterised by the relation

$$
\begin{equation*}
\operatorname{trace}\left({ }^{\mathscr{H}} B\right)=n \mathscr{V}(\operatorname{grad}(\log \lambda)) \tag{1.1.3}
\end{equation*}
$$

where $n=\operatorname{codim} \mathscr{V}$.
Note that formula $\sqrt{1.1 .2}$ shows that $\mathscr{V}$ is a conformal foliation if and only if $\mathscr{H}$ is an umbilical distribution, i.e. ${ }^{\mathscr{H}} B(X, X)$ is independent of $X$ for $g(X, X)=1$, if $\mathscr{H}$ is integrable this condition says that its integral submanifolds are umbilical (see [60]).

Let $\varphi:\left(M^{m}, g\right) \rightarrow\left(N^{n}, h\right)$ be a horizontally conformal submersion with dilation $\lambda$. Let $\tau$ denote the tension field of $\varphi$ and ${ }^{\mathscr{V}} B$ the second fundamental of the foliation $\mathscr{V}$ induced by the fibres, then we have the fundamental equation [5] (see [9] for a different proof):

$$
\begin{equation*}
\tau+\operatorname{trace}\left({ }^{\mathscr{V}} B\right)+(n-2) \mathscr{H}(\operatorname{grad}(\log \lambda))=0 \tag{1.1.4}
\end{equation*}
$$

From this, P. Baird and J. Eells concluded:
Proposition 1.1.10 ([5]). (a) When $n=2, \varphi$ is a harmonic morphism if and only if its fibres are minimal.
(b) When $n \neq 2$ any two of the following assertions imply the remaining assertion:
(i) $\varphi$ is a harmonic morphism,
(ii) $\varphi$ has minimal fibres,
(iii) $\varphi$ is horizontally homothetic (i.e. $\lambda$ is constant along horizontal curves).

Note that in the above proposition it is unnecessary for $\varphi$ to be submersive (see [9]).

Let $\omega$ denote a local volume form of $\mathscr{V}$. It can easily be seen that, the fundamental equation (1.1.4) is equivalent to

$$
\begin{equation*}
\mathscr{V}^{*}\left(\mathcal{L}_{X}\left(\lambda^{2-n} \omega\right)\right)=\lambda^{2-n} g(X, \tau) \omega, \tag{1.1.5}
\end{equation*}
$$

for any horizontal vector field $X$. Thus, we have the following:
Proposition 1.1.11 ([14], [47]). A horizontally conformal submersion with dilation $\lambda$ is a harmonic morphism if and only if the parallel displacement defined by the horizontal distribution preserves the mass of the fibres, where the fibres are given the mass density $\lambda^{2-n}$.

Definition 1.1.12. Let $\mathscr{V}$ be a foliation of codimension $n$, which produces harmonic morphisms on $(M, g)$. Let $\lambda$ be a local dilation of $\mathscr{V}$ which restricts to give dilations of harmonic morphisms which locally define $\mathscr{V}$. Then $\rho=$ $\lambda^{2-n}$ is called a local density of $\mathscr{V}$. If $\lambda$ is globally defined on $M$ then $\rho$ is called $a$ (global) density.

Proposition 1.1.13. Let $\mathscr{V}$ be a foliation which produces harmonic morphisms on $(M, g)$. Then there exists a Riemannian regular covering $\xi$ : $(\widetilde{M}, \widetilde{g}) \rightarrow(M, g)$ with the following properties:
(i) $\xi^{*}(\mathscr{V})$ admits a global density.
(ii) If $\eta:(P, k) \rightarrow(M, g)$ is any Riemannian regular covering such that $\eta^{*}(\mathscr{V})$ admits a global density then there exists a unique Riemannian regular covering $\sigma:(P, k) \rightarrow(\widetilde{M}, \widetilde{g})$ such that $\eta=\xi \circ \sigma$.

Moreover, $\xi$ is the unique Riemannian regular covering satisfying (i) and (ii).

Proof. Let $[a] \in H^{1}(M, \mathbb{R})$ be the cohomology class defined by the differentials of the logarithms of the local densities of $\mathscr{V}$ and let $\xi: \widetilde{M} \rightarrow M$ be the regular covering corresponding to it ( $\widetilde{M}$ is connected).

Let $\widetilde{g}=\xi^{*}(g)$. It is obvious that $\xi^{*}(\mathscr{V})$ produces harmonic morphisms on $(\widetilde{M}, \widetilde{g})$.

Also $\xi^{*}[a]=0 \in H^{1}(\widetilde{M}, \mathbb{R})$. Hence, there exists a positive smooth function $\rho: \widetilde{M} \rightarrow(0, \infty)$ such that $\xi^{*}(a)=\mathrm{d} \log \rho$.

Then $\rho$ is a global density of $\xi^{*}(\mathscr{V})$.

Let $\eta:(P, k) \rightarrow(M, g)$ be any other Riemannian regular covering such that $\eta^{*}(\mathscr{V})$ admits a global density. If $H$ and $K$ are the groups of $\xi$ and $\eta$, respectively, then $\xi$ and $\eta$ correspond to surjective group morphisms $\pi_{1}(M) \rightarrow H$ and $\pi_{1}(M) \rightarrow K$, respectively, where $\pi_{1}(M)$ is the fundamental group of $M$ (see [61, Part I, §14.6]).

Now, $\eta^{*}(\mathscr{V})$ admits a global density if and only if $\eta^{*}(\xi)$ is a trivial covering and this happens if and only if the image of the injective group morphism $\pi_{1}(P) \rightarrow \pi_{1}(M)$ is contained in the kernel of the group morphism $\pi_{1}(M) \rightarrow H$. But the image of $\pi_{1}(P) \rightarrow \pi_{1}(M)$ is equal to the kernel of $\pi_{1}(M) \rightarrow K$ and hence the surjective group morphism $\pi_{1}(M) \rightarrow H$ can be factorised $\pi_{1}(M) \rightarrow K \rightarrow H$. The surjective group morphism $K \rightarrow H$ induces a Riemannian regular covering $\sigma:(P, k) \rightarrow(\widetilde{M}, \widetilde{g})$ having the required properties.

The uniqueness of $\xi$ is obvious.

Corollary 1.1.14. Let $(M, g)$ be a Riemannian manifold with zero first Betti number. Let $\mathscr{V}$ be a foliation of codimension not equal to two which produces harmonic morphisms on $(M, g)$.

Then, $\mathscr{V}$ admits a global density $\lambda^{2-n}$.

### 1.2. Metric deformations

We next discuss how much the metric of $M$ can be changed preserving the property of producing harmonic morphisms (cf. [3, [44).

Proposition 1.2.1 (cf. [44, Theorem 5.1]). Let $\mathscr{V}$ be a foliation on ( $M, g$ ), with $\operatorname{dim} \mathscr{V}=p$ and $\operatorname{codim} \mathscr{V}=n$. Let $r$ and $s$ be smooth positive functions on $M$. Let $g^{\mathscr{H}}$ and $g^{\mathscr{V}}$ denote the horizontal and the vertical components of $g$, and set $\tilde{g}=s^{2} g^{\mathscr{H}}+r^{2} g^{\mathscr{V}}$.
(a) If $n \neq 2$, then, any two of the following assertions imply the remaining assertion:
(i) $\mathscr{V}$ produces harmonic morphisms on $(M, g)$,
(ii) $\mathscr{V}$ produces harmonic morphisms on $(M, \tilde{g})$,
(iii) $r^{p} s^{n-2}$ is locally the product of a function constant on horizontal curves and a function constant on vertical curves.
(b) If $n=2$, then the same implications are true after replacing (iii) with:
(iii') $r$ is constant along horizontal curves.
Proof. Suppose that $\mathscr{V}$ is conformal and let $\lambda$ and $\tilde{\lambda}$ be local dilations of $\mathscr{V}$ with respect to $g$ and $\tilde{g}$, respectively. Then $\tilde{\lambda}=a s^{-1} \lambda$, where $a$ is a smooth positive function which is constant along vertical curves.

Let $\omega$ and $\tilde{\omega}$ be local volume forms of $\mathscr{V}$ with respect to $g$ and $\tilde{g}$, respectively. Then $\tilde{\omega}=r^{p} \omega$.

It follows that

$$
\begin{equation*}
\tilde{\lambda}^{2-n} \tilde{\omega}=a^{2-n}\left(s^{n-2} r^{p}\right)\left(\lambda^{2-n} \omega\right) . \tag{1.2.1}
\end{equation*}
$$

To prove that (i), (ii) $\Rightarrow$ (iii) note that 1.2 .1 ) and Proposition 1.1.11 implies that if $\lambda^{2-n}$ and $\tilde{\lambda}^{2-n}$ are local densities of $\mathscr{V}$ with respect to $g$ and $\tilde{g}$, respectively, then

$$
\begin{equation*}
s^{n-2} r^{p}=a^{n-2} b \tag{1.2.2}
\end{equation*}
$$

where $b$ is a smooth positive function constant along horizontal curves.
To prove that (i), (iii) $\Rightarrow$ (ii), suppose that $\lambda^{2-n}$ is a local density of $\mathscr{V}$ with respect to $g$ and choose smooth positive functions $a$ and $b$ which satisfy (1.2.2) and such that $a$ is constant along vertical curves and $b$ is constant along horizontal curves. Now, (1.2.1) implies that $\tilde{\lambda}=a s^{-1} \lambda$ corresponds to a local density of $\mathscr{V}$ with respect to $\tilde{g}$.

The proof of (ii), (iii) $\Rightarrow$ (i) is similar.
Corollary 1.2.2 (cf. [44). Let $\mathscr{V}$ be a foliation with $\operatorname{codim} \mathscr{V} \neq 2$ on $(M, g)$. Let $a$ and $b$ be smooth positive functions on $M$ such that $a$ is constant along vertical curves and $b$ is constant along horizontal curves. Then the following assertions are equivalent:
(i) $\mathscr{V}$ produces harmonic morphisms on $(M, g)$,
(ii) $\mathscr{V}$ produces harmonic morphisms on $\left(M, a^{2} b^{2} g\right)$.

If $\operatorname{codim} \mathscr{V}=2$ then (i) $\Longleftrightarrow$ (ii) if and only if the function $a$ is constant on $M$.

Proof. This is an immediate consequence of Proposition 1.2.1.
Propositions 1.1 .11 and 1.2 .1 suggests the following:
Definition 1.2.3 ([44], cf. [47]). Let $\mathscr{V}$ be a distribution of dimension $p$ and codimension $n$ on the Riemannian manifold $(M, g)$. For a positive smooth function $\sigma$ on $M$ we define the metric ${ }^{\sigma} g$ by

$$
\sigma_{g}=\sigma^{2} g^{\mathscr{H}}+\sigma^{\frac{4-2 n}{p}} g^{\mathscr{V}},
$$

where $g^{\mathscr{H}}$ and $g^{\mathscr{V}}$ are the horizontal and the vertical components of $g$, respectively.

Proposition 1.2.4 (cf. [44]). Let $\mathscr{V}$ be a conformal foliation on ( $M, g$ ) and let $\sigma$ be a positive smooth function on $M$. Then,
(i) $\mathscr{V}$ is also a conformal foliation on $\left(M,{ }^{\sigma}\right)$ ). Furthermore, $\lambda$ is a local dilation of $\mathscr{V}$ with respect to $g$ if and only if $\lambda \sigma^{-1}$ is a local dilation with respect to ${ }^{\sigma} g$.
(ii) $\mathscr{V}$ produces harmonic morphisms on $(M, g)$ if and only if it produces harmonic morphisms on $\left(M,{ }^{\sigma}\right)$ ).
(iii) If $\mathscr{V}$ produces harmonic morphisms and admits a global dilation $\lambda$ such that $\lambda^{2-n}$ is a density for $\mathscr{V}$ with respect to $g$ then, $\mathscr{V}$ is a Riemannian foliation with minimal leaves on $\left(M,{ }^{\lambda} g\right)$.

Proof. Statement (i) follows from Remark 1.1 .9 (2) whilst (ii) follows from Proposition 1.2.1

If codim $\mathscr{V}=2$ assertion (iii) is obvious. If $\operatorname{codim} \mathscr{V} \neq 2$ first note that if $\lambda$ is a local dilation of $\mathscr{V}$ with respect to $g$ then $\mathscr{V}$ is a Riemannian foliation on $\left(M,{ }^{\lambda} g\right)$. Now the proof of Proposition 1.2 .1 shows that if $\lambda^{2-n}$ is a density for $\mathscr{V}$ then the constant function $\tilde{\lambda}=1$ is a dilation which corresponds to a density for $\mathscr{V}$ with respect to ${ }^{\lambda} g$. Thus, by (1.1.4) the leaves of $\mathscr{V}$ are minimal submanifolds of $\left(M,{ }^{\lambda} g\right)$.

The next result shows that the metric on the codomain is much more rigid.

Proposition 1.2.5 ([14]). Let $\varphi_{j}:(M, g) \rightarrow\left(N, h_{j}\right), j=1,2$, be nonconstant harmonic morphisms having the same fibres. Suppose that $N$ is connected and $\operatorname{dim} N \neq 2$.

Then, $h_{1}$ and $h_{2}$ are homothetic.
Proof. Let $\lambda_{j}$ be the dilation of $\varphi_{j}(j=1,2)$. Then $\lambda_{2}=\lambda_{1} \sigma$ where $\sigma: M \rightarrow \mathbb{R}$ is a smooth positive function, constant along the fibres.

Recall from Proposition 1.1.11 that the property that $\varphi_{j}$ is a harmonic morphism is equivalent to the property that the parallel displacement defined by the horizontal distribution preserves $\lambda_{j}^{2-n} \omega$, where $n=\operatorname{dim} N$ and $\omega$ is a local volume form for the vertical distribution. Hence, $\sigma$ is also constant along horizontal curves.

It follows that $\sigma$ is constant on $M$, and the proposition is proved.
An immediate consequence of Proposition 1.2.5 is the following:
Corollary 1.2.6. A foliation of codimension $q \neq 2$ which produces harmonic morphisms is given by a Haefliger structure [30] with values in the groupoid of germs of homothetic diffeomorphisms of the sheaf of germs of Riemannian metrics on $\mathbb{R}^{q}$.

In the following theorem we attach assertion (iv) to a well-known list of equivalent assertions (see [45, Appendix B]).

Theorem 1.2.7. Let $M$ be a compact manifold with zero first Betti number.
For a foliation $\mathscr{V}$ on $M$ with compact leaves the following assertions are equivalent:
(i) the holonomy group of each leaf of $\mathscr{V}$ is finite,
(ii) there exists a metric $g$ on $M$ such that $\mathscr{V}$ is a Riemannian foliation on ( $M, g$ ),
(iii) there exists a metric $g$ on $M$ such that the leaves of $\mathscr{V}$ are minimal submanifolds of $(M, g)$,
(iv) there exists a metric $g$ on $M$ such that $\mathscr{V}$ produces harmonic morphisms on $(M, g)$.

Moreover, if $\operatorname{codim} \mathscr{V}=2$, is not necessary to assume that the first Betti number of $M$ is zero.

Proof. It is well-known that the assertions (i), (ii) and (iii) are equivalent. Moreover, if any of these properties holds then there exists a metric $g$ on $M$ such that $\mathscr{V}$ is a Riemannian foliation with minimal leaves on $(M, g)$. (To see this let $h$ be a metric on $M$ with respect to which $\mathscr{V}$ has minimal leaves and let $k$ be a metric on $M$ with respect to which $\mathscr{V}$ is Riemannian. If $\mathscr{H}$ is the orthogonal complement of $\mathscr{V}$ with respect to $h$ and $\mathcal{K}$ is the orthogonal complement of $\mathscr{V}$ with respect to $k$ let $g=h^{\mathscr{V}}+k^{\mathscr{H}}$ where, $h^{\mathscr{V}}$ is the vertical component of $h$ and $k^{\mathscr{H}}$ is the metric on $\mathscr{H}$ induced by the restriction of $k$ to $\mathcal{K}$ via the canonical isomorphisms of vector bundles $\mathscr{H} \rightarrow T M / \mathscr{V} \rightarrow \mathcal{K}$.) But any Riemannian foliation with minimal leaves produces harmonic morphisms.

Conversely, suppose that $\mathscr{V}$ produces harmonic morphisms on $(M, g)$. If $\operatorname{codim} \mathscr{V}=2$ then by Proposition 1.1.10, $\mathscr{V}$ has minimal leaves (see 68]). If codim $\mathscr{V} \neq 2$ then by Corollary 1.1.14, there exists a global density $\lambda^{2-n}$ of $\mathscr{V}$, and $\mathscr{V}$ is a Riemannian foliation with minimal leaves on $\left(M,{ }^{\lambda} g\right)$.

### 1.3. Characterisation of the conformal foliations which produce harmonic morphisms

We now characterise conformal foliations which produce harmonic morphisms. Recall [68] that a conformal foliation $\mathscr{V}$ of $\operatorname{codim} \mathscr{V}=2$ produces harmonic morphisms if and only if its leaves are minimal. For $\operatorname{codim} \mathscr{V} \neq 2$, the situation is more complicated and we have the following reformulation of a result of R.L. Bryant [14] (see [9] for another treatment).

Proposition 1.3.1 ([14]). Let $\mathscr{V}$ be a conformal foliation of $\operatorname{codim} \mathscr{V} \neq 2$ on $(M, g)$ and let $\mathscr{H}$ be its orthogonal complement. Let ${ }^{\mathscr{V}} B$ and ${ }^{\mathscr{H}} B$ be the second fundamental forms of $\mathscr{V}$ and $\mathscr{H}$, respectively.

Then, $\mathscr{V}$ produces harmonic morphisms on $(M, g)$ if and only if the vector
field

$$
(n-2) \operatorname{trace}\left({ }^{\mathscr{H}} B\right)-n \operatorname{trace}\left({ }^{\mathscr{V}} B\right)
$$

is locally a gradient vector field. (Here trace $\left({ }^{\mathscr{H}} B\right)=\sum_{j} \mathscr{H}_{B}\left(X_{j}, X_{j}\right)$ and trace $\left({ }^{(/} B\right)=\sum_{\alpha}{ }^{\mathscr{V}} B\left(U_{\alpha}, U_{\alpha}\right)$ for local orthonormal frames $\left\{X_{j}\right\}$ and $\left\{U_{\alpha}\right\}$ of $\mathscr{H}$ and $\mathscr{V}$, respectively.)

Proof. Note that the following relation holds

$$
\begin{equation*}
(n-2) \operatorname{trace}\left({ }^{\mathscr{H}} B\right)-n \operatorname{trace}\left({ }^{\mathscr{V}} B\right)=n(n-2) \operatorname{grad}(\log \lambda), \tag{1.3.1}
\end{equation*}
$$

if and only if:

$$
\begin{equation*}
\operatorname{trace}\left({ }^{\mathscr{H}} B\right)=n \mathscr{V}(\operatorname{grad}(\log \lambda)), \tag{1.3.1}
\end{equation*}
$$

and

$$
\operatorname{trace}\left({ }^{\mathscr{V}} B\right)=-(n-2) \mathscr{H}(\operatorname{grad}(\log \lambda)) .
$$

By Remark 1.1.9(3), 1.3.1 ${ }^{\text {a }}$ ) holds if and only $\lambda$ is a local dilation of $\mathscr{V}$. This together with the fundamental equation (1.1.4 , imply that 1.3.1 and (1.3.1b hold if and only if $\mathscr{V}$, restricted to the domain of $\lambda$, produces harmonic morphisms and $\lambda^{2-n}$ is a density of it.

Note that Corollary 1.1.14 can be proved using 1.3.1).
Proposition 1.3.1 provides one of the essential ingredients necessary to obtain the following result (see [9] for a proof).

Proposition 1.3.2. Let $(M, g)$ be a real-analytic Riemannian manifold, and let $\mathscr{V}$ be a foliation which produces harmonic morphisms on $(M, g)$.

Then $\mathscr{V}$ is a real-analytic foliation. Moreover, if $\operatorname{codim} \mathscr{V} \neq 2$, then any harmonic morphism produced by $\mathscr{V}$ is a real-analytic map onto a realanalytic Riemannian manifold. If $\operatorname{codim} \mathscr{V}=2$, then $\mathscr{V}$ is locally defined by real-analytic submersive harmonic morphisms onto real-analytic Riemannian two-manifolds.

Corollary 1.3.3. Let $\varphi:(M, g) \rightarrow(N, h)$ be a submersive harmonic morphism from a real-analytic manifold onto a smooth manifold.

If $\operatorname{dim} N \neq 2$ then there exists a real-analytic structure on $N$ with respect to which $\varphi$ and $h$ are real-analytic. If $\operatorname{dim} N=2$ then $\varphi$ is real-analytic with respect to the real-analytic structure on $N$ induced by the conformal structure of $h$.

Remark 1.3.4. In Corollary 1.3.3, if $\operatorname{dim} N=2$, then we can remove the hypothesis that $\varphi$ is submersive; note, however, that the metric of $N$ may always be chosen such that not to be real-analytic (just apply a suitable conformal deformation).

If $\operatorname{dim} N \geq 3$, we do not know if the hypothesis that $\varphi$ is submersive can be removed.

### 1.4. Homothetic foliations

By the mean curvature form of $\mathscr{V}$ we mean the one-form $\left(\operatorname{trace}\left({ }^{(V} B\right)\right)^{b}$ obtained by applying the musical isomorphism (see [11]) ${ }^{b}: T M \rightarrow T^{*} M$ with similar terminology for $\mathscr{H}$ (see [64). Then we have.

Corollary 1.4.1. Let $\mathscr{V}$ be a conformal foliation with codim $\mathscr{V} \neq 2$ and let $\mathscr{H}$ be its orthogonal complement. Then any two of the following assertions imply the other one:
(i) $\mathscr{V}$ produces harmonic morphisms,
(ii) $\mathscr{V}$ has closed mean curvature form,
(iii) $\mathscr{H}$ has closed mean curvature form.

Proof. This is an immediate consequence of Proposition 1.3 .1 and the fact that $\left(\operatorname{trace}\left({ }^{(/} B\right)\right)^{b}$ is closed if and only if $\operatorname{trace}\left({ }^{(/} B\right)$ is locally a gradient and similarly for $\mathscr{H}$.

We now introduce a new sort of foliation midway between conformal and Riemannian foliations.

Proposition 1.4.2. For a conformal foliation $\mathscr{V}$ on the Riemannian manifold $(M, g)$ the following assertions are equivalent:
(i) the leaves of $\mathscr{V}$ can be, locally, given as fibres of horizontally homothetic submersion;
(ii) each point of $M$ has an open neighbourhood on which there can be defined a local dilation of $\mathscr{V}$ which is constant along horizontal curves;
(iii) any local dilation $\lambda$ of $\mathscr{V}$, defined on an open subset $O$ with zero first Betti number, is a product $\lambda=a b$, where $a$ and $b$ are positive smooth functions such that $a$ is constant along vertical curves and $b$ is constant along horizontal curves;
(iv) the mean curvature of the orthogonal complement of $\mathscr{V}$ is locally a gradient vector field.

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) is obvious. Also, by Remark 1.1.9(2) it follows that $($ ii $) \Leftarrow$ (iii).

By the same remark, if (ii) holds then, any local dilation $\lambda: O \rightarrow \mathbb{R}$ of $\mathscr{V}$ is, locally, a product $\lambda=a b$ as in (iii). If on the same open subset of $O$, we also have $\lambda=a_{1} b_{1}$ then, $a^{-1} a_{1}=b b_{1}^{-1}=$ const. Hence the differentials of the logarithms of the factors $a, b$ from the local decompositions of $\lambda$ define closed one-forms on $O$. If the first Betti number of $O$ is zero then these one-forms are exact and the implication (ii) $\Rightarrow$ (iii) is proved.

The equivalence (ii) $\Longleftrightarrow$ (iv) follows from Remark 1.1.9(3).

Remark 1.4.3. 1) If a conformal foliation satisfies one of the properties from the above proposition then the holonomy groupoid of each leaf is formed of germs of homothetic diffeomorphisms.
2) In (iii) above instead of $H^{1}(O ; \mathbb{R})=0$ we could ask that the first basic cohomology group (see [64]) of $\left.\mathscr{V}\right|_{O}$ be zero. This follows from the fact that the set of differentials $\{\mathrm{d} a\}$ define a closed basic one-form on $O$.
3) Alternatively, it is sufficient to ask that the orthogonal complement $\left.\mathscr{H}\right|_{O}$ of $\left.\mathscr{V}\right|_{O}$ is an Ehresmann connection [12] with trivial holonomy (in particular, $\left.\mathscr{H}\right|_{O}$ is integrable). To see this, note that the set of differentials $\{\mathrm{d}(\log b)\}$ define a closed one-form which, when restricted to a leaf $L$, is exact (because it coincide with $\left.\mathrm{d}\left(\left.(\log \lambda)\right|_{L}\right)\right)$. When $\mathscr{H}$ is an Ehresmann connection with trivial holonomy these exact forms can be matched together to define an exact form on $O$.

The notion of local dilations for conformal foliations can be generalized to conformal distributions, although, in this case, these might not exist. Nevertheless, if a conformal distribution admits local dilations then these share the same properties (Remark $1.1 .9(2)$ and (3)) as the local dilations of a conformal foliation. Moreover, assertions (ii), (iii) and (iv) from Proposition 1.4 .2 remain equivalent for conformal distributions which admit local dilations in a neighbourhood of each point.

Proposition 1.4.2 suggests the following definition.
Definition 1.4.4 (cf. [66, §2]). Let $\mathscr{V}$ be a distribution on the Riemannian manifold $(M, g)$. We shall say that $\mathscr{V}$ is homothetic if it is conformal and the mean curvature of its orthogonal complement is locally a gradient vector field.

Remark 1.4.5. 1) Let $\mathscr{V}$ be a foliation on $M$. Then, there exists a metric $g$ on $M$ such that $\mathscr{V}$ is a homothetic foliation on $(M, g)$ if and only if $\mathscr{V}$ is a foliation of type (A) in the sense of I. Vaisman 66].
2) Note that, unlike conformal distributions, homothetic distributions always admit local dilations, even if nonintegrable. Indeed, if $\mathscr{V}$ is a homothetic distribution with codim $\mathscr{V}=n$ and $\mathscr{H}$ is its orthogonal complement, then any local smooth positive function $\lambda$ on $M$ which has the property

$$
\begin{equation*}
n \operatorname{grad}(\log \lambda)=\operatorname{trace}\left({ }^{\mathscr{H}} B\right), \tag{1.4.1}
\end{equation*}
$$

is a local dilation of $\mathscr{V}$.
Moreover, if $\mathscr{V}$ is a homothetic distribution and $\lambda$ is a local dilation of it defined on an open set $O$ such that $\lambda=a b$ as in (iii) from Proposition 1.4.2, then any other local dilation defined on $O$ is of this form.

Proposition 1.4.6. Let $\mathscr{V}$ be a foliation on ( $M, g$ ) with orthogonal complement $\mathscr{H}$.

If $\mathscr{H}$ is a homothetic distribution then the parallel displacement defined by it consists of (local) homothetic diffeomorphisms between leaves of $\mathscr{V}$.

Conversely, if $\mathscr{H}$ is conformal, integrable and the parallel displacement defined by it is formed of (local) homothetic diffeomorphisms between leaves of $\mathscr{V}$, then it is a homothetic distribution.

Proof. If $\mathscr{H}$ is conformal and integrable then it admits local dilations. Let $\lambda$ be a local dilation of $\mathscr{H}$. Then for any horizontal vector field $X$ invariant under the holonomy of $\mathscr{V}$ (i.e. for any basic vector field $X$ ) we have:

$$
\begin{equation*}
\mathcal{L}_{X}\left(\lambda^{2}\left(\left.g\right|_{\mathscr{V}}\right)\right)=0 . \tag{1.4.2}
\end{equation*}
$$

If $\left(\psi_{t}\right)$ is the local flow of $X$, then (1.4.2) is equivalent to the fact that for any $t$ we have

$$
\begin{equation*}
\left(\psi_{t}\right)^{*}\left(\lambda^{2}\left(\left.g\right|_{\mathscr{V}}\right)\right)=\lambda^{2}\left(\left.g\right|_{\mathscr{V}}\right) . \tag{1.4.3}
\end{equation*}
$$

The proof follows from 1.4 .3 ) by using the fact that, if $\mathscr{H}$ is conformal, then it is homothetic if and only if in the neighbourhood of each point a local dilation can be found, which is constant along the leaves of $\mathscr{V}$.

Proposition 1.4.7. Let $\mathscr{V}$ be a homothetic foliation of codimension not equal to two on $(M, g)$. Then, the following assertions are equivalent:
(i) $\mathscr{V}$ produces harmonic morphisms;
(ii) the mean curvature of $\mathscr{V}$ is locally a gradient vector field.

In particular, a homothetic foliation whose orthogonal complement is a homothetic distribution produces harmonic morphisms with umbilical fibres.

Proof. Since $\mathscr{V}$ is homothetic we have that trace $\left({ }^{\mathscr{H}} B\right)$ is, locally, a gradient vector field. From Proposition 1.3 .1 it follows that $\mathscr{V}$ produces harmonic morphisms if and only if trace $\left({ }^{2} B\right)$ is, locally, a gradient vector field.

The last assertion follows from the fact that (see Remark 1 1.1.9(3)) $\mathscr{V}$ is umbilical if and only if $\mathscr{H}$ is a conformal distribution.

Corollary 1.4.8. Let $\mathscr{V}$ be a foliation with minimal leaves and $\operatorname{codim} \mathscr{V} \neq 2$. Then, $\mathscr{V}$ produces harmonic morphisms if and only if it is homothetic.

See [1] for other relations between harmonic maps and minimal submanifolds.

Remark 1.4.9. 1) In Proposition 1.4 .7 condition (ii) is a bit more general than saying $\mathscr{H}$ is homothetic since it is not assumed to be conformal.
2) Any Riemannian foliation is homothetic.
3) Given a horizontally homothetic submersion the connected components of its fibres form a homothetic foliation. Conversely, note that if $\varphi$ : $(M, g) \rightarrow(N, h)$ is a horizontally conformal submersion such that its fibres form a homothetic foliation then it is not true, in general, that $\varphi$ is horizontally homothetic. If the first Betti number of $M$ is zero then $\varphi$ can be factorised into a horizontally homothetic submersion followed by a conformal diffeomorphism. (By Remark $1.4 .3(2)$, (3) this factorization can also be done when the first Betti number of $N$ is zero or when the horizontal distribution $\mathscr{H}$ is an Ehresmann connection with trivial holonomy, in which case $\mathscr{H}$ is integrable and $M$ is diffeomorphic to the product of $N$ and the fibre.)

Example 1.4.10. 1) Doubly-warped-products (see [57], and the references therein). If $\left(M^{p}, g\right)$ and $\left(N^{q}, h\right)$ are Riemannian manifolds and $r: M \rightarrow \mathbb{R}$ and $s: N \rightarrow \mathbb{R}$ are positive smooth functions then the doubly-warped-product of $\left(M^{p}, g\right)$ and $\left(N^{q}, h\right)$ is defined to be:

$$
M_{s} \times{ }_{r} N=\left(M \times N, s^{2} \pi_{M}^{*}(g)+r^{2} \pi_{N}^{*}(h)\right),
$$

where $\pi_{M}$ and $\pi_{N}$ are the projections onto $M$ and $N$, respectively.
The projections $\pi_{M}: M_{s} \times{ }_{r} N \rightarrow(M, g)$ and $\pi_{N}: M_{s} \times{ }_{r} N \rightarrow(N, h)$ are horizontally homothetic so their fibres define a pair of complementary orthogonal homothetic foliations. Conversely, any Riemannian manifold endowed with a pair of complementary orthogonal homothetic foliations is canonically locally isometric to a doubly-warped-product. Hence, by Proposition 1.4.7, when $p, q \neq 2$ both of the foliations induced by the factors of a doubly-warpedproduct produce harmonic morphisms. More precisely, if $p \neq 2$ the following projection is a harmonic morphism with umbilical fibres:

$$
\begin{equation*}
\pi_{M}: M_{s} \times{ }_{r} N \longrightarrow\left(M, r^{\frac{2 q}{p-2}} g\right) . \tag{1.4.4}
\end{equation*}
$$

The fact that the above projection is a harmonic morphism also follows from Proposition 1.2.4(iii).

A concrete example of a warped-product is provided by the open subsets in spheres $\left(S^{p+q} \backslash S_{0}^{p-1}, g_{p+q}\right)$, where $S_{0}^{p-1}=\mathbb{R}^{p} \cap S^{p+q}$ and $\mathbb{R}^{p} \equiv\{x \in$
$\left.\mathbb{R}^{p+q+1} \mid x^{p+1}=\ldots=x^{p+q+1}=0\right\}$ and $g_{p+q}$ is the restriction of the canonical metric on $S^{p+q}$. The warped-product is the one induced by the following diffeomorphism:

$$
\begin{aligned}
& \Phi: S^{p+q} \backslash S_{0}^{p-1} \longrightarrow S_{+}^{p} \times S^{q} \\
& \Phi(x, y)=\left((x,|y|), \frac{1}{|y|} y\right) .
\end{aligned}
$$

where $S_{+}^{p}=\left\{x \in S^{p+1} \mid x^{p+1}>0\right\}$. To make $\Phi$ an isometry we must give $S_{+}^{p} \times S^{q}$ the warped-product structure $S_{+}^{p} \times_{r} S^{q}$ where $r\left(x^{1}, \ldots, x^{p+1}\right)=x^{p+1}$.

Thus, if $p \neq 2$, (1.4.4) particularises to give the following harmonic morphism with umbilical fibres:

$$
\begin{aligned}
& \varphi:\left(S^{p+q} \backslash S_{0}^{p-1}, g_{p+q}\right) \longrightarrow\left(S_{+}^{p}, r^{\frac{2 q}{p-2}} g_{p}\right) \\
& \varphi\left(x^{1}, \ldots, x^{p+q+1}\right)=\left(x^{1}, \ldots, x^{p}, \sqrt{\left(x^{p+1}\right)^{2}+\ldots+\left(x^{p+q+1}\right)^{2}}\right)
\end{aligned}
$$

By Proposition 1.2 .5 any other metric on $S_{+}^{p}$ with respect to which $\varphi$ above is a harmonic morphism is homothetic to the one considered. Also, note that, although $\varphi$ can be extended to a continuous map $\widetilde{\varphi}$ on $S^{p+q}$, the considered metric on the codomain cannot be extended to the range of $\widetilde{\varphi}$.
2) Doubly-twisted-products (see [57], and the references therein). These are defined in the same way as doubly-warped-products, but now $r, s: M \times$ $N \rightarrow \mathbb{R}$. It is easy to see that a Riemannian manifold endowed with a pair of complementary orthogonal foliations which are both umbilical (equivalently, both conformal) is, canonically, locally isometric to a doubly-twisted-product.

If $p \neq 2$, it follows from Proposition 1.2.1 that the foliation $\mathscr{V}$ induced by the second factor of the doubly-twisted-product $M_{s} \times{ }_{r} N$ produces harmonic morphisms if and only if the function $r^{q} s^{p-2}$ is, locally, the product of a function constant on $M$ and a function constant on $N$.

If $p=2$, then $\mathscr{V}$ produces harmonic morphisms if and only if $r$ is a function defined on $N$. In this case, $\mathscr{V}$ has totally geodesic leaves and $M_{s} \times{ }_{r} N$ is isometric to the twisted product $M_{s} \times \widetilde{N}$ where $\widetilde{N}=\left(N, r^{2} h\right)$.

It follows that a pair of complementary orthogonal umbilical foliations both of codimension not equal to two are both homothetic if and only if each of them produces harmonic morphisms.

## CHAPTER 2

## Group actions and harmonic morphisms

### 2.1. Mean curvature forms and adapted Bott connections

In this section we shall establish the fact that the exterior derivative of the mean curvature form of a distribution is the curvature form of the connection induced on the determinant bundle of the distribution by its adapted Bott connection (see 59). By using this or by direct calculation we obtain formulae for the exterior derivative of the mean curvature form of a distribution. (We do not imagine that these formulae are new, but we could not find them in the literature.) Some of these formulae apply to prove that if a conformal foliation $\mathscr{V}$ has integrable orthogonal complement $\mathscr{H}$ and if $\mathscr{V}$ and $\mathscr{H}$ both have basic mean curvature forms then $\mathscr{V}$ produces harmonic morphisms.

The following simple lemma will be used later on.

Lemma 2.1.1 (cf. [60, Chapter IV, Example 4.10]). Let $\mathscr{V}$ be a foliation on $(M, g)$ and let $V \in \Gamma(\mathscr{V})$ be a conformal vector field.

Then $[V, X]=0$ for any basic vector field $X$.

Proof. Let $\mathscr{H}=\mathscr{V}^{\perp}$ and $X \in \Gamma(\mathscr{H})$ a basic vector field. Then $[V, X] \in$ $\Gamma(\mathscr{V})$.

But $V$ is conformal and hence we can write

$$
0=\left(\mathcal{L}_{V} g\right)(W, X)=-g(W,[V, X])
$$

for any $W \in \Gamma(\mathscr{V})$. Hence $[V, X]=0$.

The result of Lemma 2.1.1 is equivalent to the fact that any conformal vector field tangent to a foliation $\mathscr{V}$ is an infinitesimal automorphism of the
orthogonal complement of $\mathscr{V}$ (see [60] or 35] for the definition of the infinitesimal automorphism of a distribution).

Let $\mathscr{V}$ and $\mathscr{H}$ be two complementary orthogonal distributions (not necessarily integrable) on $(M, g)$.

Definition 2.1.2 (see [64). The adapted Bott connection $\nabla$ on $\mathscr{H}$ is defined by

$$
\stackrel{\mathscr{H}}{E} X=\mathscr{H}[\mathscr{V} E, X]+\mathscr{H}\left(\nabla \mathscr{H}_{E} X\right)
$$

for $E \in \Gamma(T M), X \in \Gamma(\mathscr{H})$ where $\nabla$ is the Levi-Civita connection of $(M, g)$.
The adapted Bott connection $\nabla$ on $\mathscr{V}$ is defined similarly by reversing the roles of $\mathscr{V}$ and $\mathscr{H}$.

Remark 2.1.3 (see [64]). It is easy to see that $\nabla \not \mathscr{}$ is compatible with the metric induced by $g$ on $\mathscr{H}$ if and only if $\mathscr{H}$ is totally geodesic. Nevertheless, since $\stackrel{\mathscr{H}}{X}^{\nabla_{X}} \mathscr{H} \nabla_{X}$ for any $X \in \Gamma(\mathscr{H})$ we have that $\nabla_{X}\left(\left.g\right|_{\mathscr{H}}\right)=0$.

Let ${ }^{\mathscr{H}} I$ be the integrability tensor of $\mathscr{H}$ which is the $\mathscr{V}$-valued horizontal two-form defined by ${ }^{\mathscr{H}} I(X, Y)=-\mathscr{V}[X, Y]$ for $X, Y \in \Gamma(\mathscr{H})$.

Proposition 2.1.4. Let $\mathscr{H}$ be a distribution on $(M, g)$. Then

$$
\begin{gather*}
\mathrm{d}\left(\operatorname{trace}\left(\mathscr{\mathscr { H }}_{B}\right)^{b}\right)(X, Y)=g\left(\operatorname{trace}\left(\mathscr{\mathscr { H }}_{B}\right),{ }^{\mathscr{H}} I(X, Y)\right),  \tag{2.1.1}\\
\mathrm{d}\left(\operatorname{trace}\left(\mathscr{H}^{\mathscr{H}} B\right)^{b}\right)(X, V)=\nabla_{X}\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right)^{b}\right)(V) \tag{2.1.2}
\end{gather*}
$$

for any horizontal vectors $X, Y$ and vertical vector $V$.
Proof. This is a straightforward calculation using the fact that trace $\left({ }^{\mathscr{H}} B\right)$ is a vertical vector field.

Let $n=\operatorname{dim} \mathscr{H}$ and let $\bigwedge^{n} \mathscr{H}$ be the determinant line bundle of $\mathscr{H}$.
Let $\mathscr{H}_{R} \in \Gamma\left(\operatorname{End}(\mathscr{H}) \otimes \Lambda^{2}\left(T^{*} M\right)\right)$ be the curvature form of ${ }^{\mathscr{H}}$. Then the curvature form of the connection induced by $\nabla^{\mathscr{H}}$ on $\bigwedge^{n} \mathscr{H}$ is trace $\left({ }^{\mathscr{H}} R\right) \in$ $\Gamma\left(\bigwedge^{2}\left(T^{*} M\right)\right)$.

Proposition 2.1.5 (see [59]). Let $\mathscr{H}$ be a distribution on $(M, g)$. Then

$$
\operatorname{trace}\left({ }^{\mathscr{H}} R\right)=\mathrm{d}\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right)^{b}\right) .
$$

Proof. Let $\stackrel{\mathscr{H}}{\omega}$ be a local volume form of $\mathscr{H}$ considered with respect to the metric induced by $g$.

Recall that trace $\left({ }^{\mathscr{H}} R\right)=\mathrm{d} A$ where $A$ is any local connection form of the connection induced by $\stackrel{\mathscr{H}}{\nabla}$ on $\bigwedge^{n} \mathscr{H}$. Thus it suffices to show that

$$
\begin{equation*}
\stackrel{\mathscr{H}}{E}_{E}^{\mathscr{H}}=-g\left(E, \operatorname{trace}\left({ }^{\mathscr{H}} B\right)\right) \stackrel{\mathscr{H}}{\omega} \tag{2.1.3}
\end{equation*}
$$

for any $E \in T M$.
If $E \in \mathscr{H}$ then the right hand side of $(2.1 .3$ is zero. Also, the left hand side is zero because if $E \in \mathscr{H}$ then $\stackrel{\mathscr{H}}{ }^{\nabla}\left(\left.g\right|_{\mathscr{H}}\right)=0$ (see Remark 2.1.3.).
 to a well-known formula (see [68]).

Proposition 2.1.6 (see [60], 64]). Let $X \in \Gamma(\mathscr{H})$ be a horizontal vector field. Then the following assertions are equivalent.
(i) $\stackrel{\mathscr{H}}{ }^{\nabla} X=0$ for any $V \in \Gamma(\mathscr{V})$;
(ii) $\mathscr{H}[X, V]=0$ for any $V \in \Gamma(\mathscr{V})$;
(iii) $\mathcal{L}_{X}(\Gamma(\mathscr{V})) \subseteq \Gamma(\mathscr{V})$;
(iv) $X$ is an infinitesimal automorphism of $\mathscr{V}$.

Proof. The equivalences (i) $\Longleftrightarrow$ (ii) and (ii) $\Longleftrightarrow$ (iii) are trivial. Also, (ii) $\Longleftrightarrow$ (iv) follows easily from [36, Chapter 1, Corollary 1.10].

Remark 2.1.7. If in Proposition 2.1.6 we further assume that $\mathscr{H}$ is integrable then the following assertions can be added.
(v) $\mathcal{L}_{X} \circ \mathscr{V}=\mathscr{V} \circ \mathcal{L}_{X}$;
(vi) $\mathcal{L}_{X} \circ \mathscr{H}=\mathscr{H} \circ \mathcal{L}_{X}$.

Example 2.1.8 (see [64]). Suppose that $\mathscr{V}$ is integrable. Then any basic vector field $X \in \Gamma(\mathscr{H})$ for $\mathscr{V}$ with respect to $\mathscr{H}$ is an infinitesimal automorphism of $\mathscr{V}$; in fact, if $\mathscr{V}$ is integrable a horizontal vector field $X \in \Gamma(\mathscr{H})$ is basic if and only if any of the assertions (i), (ii), (iii) or (iv) holds.

The following definition does not require any assumption on the distribution $\mathscr{H}$.

Definition 2.1.9 (see [64, (4.34)]). Let $E \in \Gamma(T M)$. The horizontal divergence $\operatorname{div}_{\mathscr{H}} E$ of $E$ is defined by

$$
\mathscr{H}^{*}\left(\mathcal{L}_{E} \stackrel{\mathscr{H}}{\omega}\right)=\left(\operatorname{div}_{\mathscr{H}} E\right) \stackrel{\mathscr{H}}{\omega},
$$

where $\mathscr{H}_{\omega}^{\mathscr{\omega}}$ is any local volume form of $\mathscr{H}$ (endowed with the metric induced by g).

The vertical divergence $\operatorname{div}_{\mathscr{V}}$ is defined similarly (note that

$$
\operatorname{div} E=\operatorname{div}_{\mathscr{H}} E+\operatorname{div}_{\mathscr{V}} E
$$

for any $E \in \Gamma(T M))$.
A standard calculation gives the following proposition.
Proposition 2.1.10. Let $E \in \Gamma(T M)$. Then $\operatorname{div}_{\mathscr{H}} E$ is the (pointwise) trace of the linear endomorphism $\mathscr{H} \longrightarrow \mathscr{H}$ defined by $Y \longmapsto \mathscr{H}\left(\nabla_{Y} E\right)$.

Hence, $\operatorname{div}_{\mathscr{H}} E$ is globally well-defined (i.e. it does not depend on $\stackrel{\mathscr{H}}{\omega}$ ).
Remark 2.1.11. 1) Obviously, if $\mathscr{H}$ is integrable and $X \in \Gamma(\mathscr{H})$ then the restriction of $\operatorname{div}_{\mathscr{H}} X$ to each leaf $L$ of $\mathscr{H}$ is equal to the divergence of the restriction of $X$ to $\left(L,\left.g\right|_{L}\right)$.
2) If $V \in \Gamma(\mathscr{V})$ then $\operatorname{div}_{\mathscr{H}} V=-g\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right), V\right)$.

Lemma 2.1.12. (a) Suppose that $X \in \Gamma(\mathscr{H})$ is an infinitesimal automorphism of $\mathscr{V}$. Then

$$
\mathscr{H}_{R(V, W) X}=\mathscr{H}\left(\nabla_{\mathscr{V}_{I(V, W)}} X\right)
$$

for any vertical $V$ and $W$.
(b) Suppose that $\mathscr{H}$ is integrable and let $X, Y \in \Gamma(\mathscr{H})$ and $V \in \Gamma(\mathscr{V})$ be such that $[V, X]=0=[V, Y]$. Then

$$
\mathscr{H}_{R}(V, X) Y=\nabla_{V}\left(\mathscr{H} \nabla_{Y} X\right) .
$$

Proof. (a) Let $V, W \in \Gamma(\mathscr{V})$. Then

$$
\begin{aligned}
\mathscr{H}_{R}(V, W) X & =\left[\stackrel{\mathscr{H}}{V}, \stackrel{\mathscr{H}}{W}^{\nabla_{W}}\right] X-\stackrel{\mathscr{H}}{[V, W]} \\
& =-\stackrel{\mathscr{H}}{\mathscr{V}}[V, W] X-\stackrel{\mathscr{H}}{\mathscr{H}}[V, W] X=\mathscr{H}\left(\nabla_{\mathscr{V}_{I}(V, W)} X\right) .
\end{aligned}
$$

(b) We have

$$
\begin{aligned}
\mathscr{H}_{R(V, X) Y}= & {\left[\stackrel{\mathscr{H}}{V}, \stackrel{\mathscr{H}}{X}^{\nabla^{2}}\right] Y-\stackrel{\mathscr{H}}{[V, X]} } \\
= & \nabla_{V}\left(\nabla_{X} Y\right)=\nabla_{V}\left(\mathscr{H} \nabla_{X} Y\right) \\
= & \nabla_{V}\left(\mathscr{H}[X, Y]+\mathscr{H} \nabla_{Y} X\right) .
\end{aligned}
$$

Because $\mathscr{H}$ is integrable we have that $\mathscr{H}[X, Y]=[X, Y]$ and from $[V, X]=0=[V, Y]$, by using the Jacobi identity, we obtain that $[V,[X, Y]]=$ 0 . Hence $\stackrel{\mathscr{H}}{V}^{\nabla}(\mathscr{H}[X, Y])=0$ and the lemma follows.

Let $\left\{X_{a}\right\}$ be a local frame for $\mathscr{H}$ over the open subset $U \subseteq M$ and let $\left\{V_{r}\right\}$ be a local frame for $\mathscr{V}$ over $U$. We shall denote 'horizontal' indices by $a, b, c$ and 'vertical' indices by $r, s, t$.

Lemma 2.1.13. (a) Suppose that the $X_{a}$ are infinitesimal automorphisms of $\mathscr{V}$.

Then

$$
\begin{equation*}
\mathrm{d}\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right)^{b}\right)_{r s}=\left(c_{b a}^{a}+\operatorname{div}_{\mathscr{H}} X_{b}\right)^{\mathscr{V}} I_{r s}^{b} \tag{2.1.4}
\end{equation*}
$$

where $\left\{c_{a b}^{c}\right\}$ are defined by $\mathscr{H}\left[X_{a}, X_{b}\right]=c_{a b}^{c} X_{c}$.
(b) If both $\mathscr{H}$ and $\mathscr{V}$ are locally generated by infinitesimal automorphisms of $\mathscr{V}$ and $\mathscr{H}$, respectively, and $\mathscr{H}$ is integrable then

$$
\mathscr{V}^{*}\left(\mathrm{~d}\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right)^{b}\right)\right)=\operatorname{div}_{\mathscr{H}}\left({ }^{\mathscr{V}} I\right) .
$$

Proof. (a) This follows from Proposition 2.1.5, Proposition 2.1.10 and Lemma 2.1.12(a) .
(b) This follows from (a) and the fact that $\mathscr{V}^{*}\left(\mathcal{L}_{X}\left({ }^{\mathscr{V}} I\right)\right)=0$ for any infinitesimal automorphism $X \in \Gamma(\mathscr{H})$ of $\mathscr{V}$.
(Note that it seems to be impossible to formulate invariantly assertion (a) of Lemma 2.1.13.)

Proposition 2.1.14 (see [59]). If $\mathscr{V}$ is integrable then $\mathrm{d}\left(\operatorname{trace}\left(\mathscr{H}_{B}\right)^{b}\right)(V, W)=$ 0 for any vertical $V$ and $W$.

Proof. This follows from Lemma 2.1.13 because if $\mathscr{V}$ is integrable then any basic vector field for $\mathscr{V}$ is an infinitesimal automorphism of $\mathscr{V}$.

Proposition 2.1.15. Suppose that both $\mathscr{V}$ and $\mathscr{H}$ are integrable. Then the following assertions are equivalent.
(i) The mean curvature form of $\mathscr{H}$ is closed;
(ii) The mean curvature form of $\mathscr{H}$ is basic (for $\mathscr{V}$ ).

Proof. This follows from Proposition 2.1.4 and Proposition 2.1.14.
Proposition 2.1.16. Suppose that $\mathscr{H}$ is integrable and locally generated by infinitesimal automorphisms of $\mathscr{V}$. Let $V \in \Gamma(\mathscr{V})$ and let $X \in \Gamma(\mathscr{H})$ be an infinitesimal automorphism of $\mathscr{V}$. Then

$$
\begin{equation*}
\mathrm{d}\left(\operatorname{trace}\left(\mathscr{H}^{\mathscr{H}} B\right)^{b}\right)(V, X)=V\left(\operatorname{div}_{\mathscr{H}} X\right)=-\nabla_{X}\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right)^{b}\right)(V) . \tag{2.1.5}
\end{equation*}
$$

Proof. Note that 2.1 .5 is tensorial in $V$ and thus we can suppose that $V$ is basic for $\mathscr{H}$. Then the proof follows from Proposition 2.1.4, Proposition 2.1.5, Proposition 2.1.10 and Lemma 2.1.12(b) .

By reversing the roles of $\mathscr{V}$ and $\mathscr{H}$ in Proposition 2.1.4 and Lemma 2.1.13 and Proposition 2.1.16 we obtain the corresponding formulae for $\mathrm{d}\left(\operatorname{trace}\left({ }^{\mathscr{V}} B\right)^{b}\right)$.

The following simple lemma holds for any complementary orthogonal distributions $\mathscr{H}$ and $\mathscr{V}$.

Lemma 2.1.17. Let $f$ be any smooth function on $M$. Then

$$
\begin{equation*}
\stackrel{\mathscr{H}}{\nabla}_{V}\left(\mathscr{H}^{*}(\mathrm{~d} f)\right)(X)=\stackrel{\mathscr{V}}{X}\left(\mathscr{V}^{*}(\mathrm{~d} f)\right)(V) \tag{2.1.6}
\end{equation*}
$$

for any vertical $V$ and horizontal $X$.

Proof. Let $X$ and $V$ be vector fields which are horizontal and vertical, respectively. The following relation is trivial

$$
\begin{equation*}
V(X(f))-X(V(f))-[V, X](f)=0 . \tag{2.1.7}
\end{equation*}
$$

But 2.1.7) is equivalent to the following

$$
V(X(f))-\mathscr{H}[V, X](f)=X(V(f))-\mathscr{V}[X, V](f)
$$

which is obviously equivalent to 2.1.6).
Proposition 2.1.18. Let $\mathscr{V}$ be a foliation of $\operatorname{codim} \mathscr{V}=n \neq 2$ which produces harmonic morphisms on $(M, g)$. Then the following assertions are equivalent.
(i) The mean curvature form of $\mathscr{V}$ is basic;
(ii) The mean curvature form of $\mathscr{H}$ is invariant under the parallel displacement determined by $\mathscr{H}$ (i.e. $\left.\nabla_{X}^{\mathscr{V}}\left(\operatorname{trace}\left(\mathscr{H}_{B}\right)^{b}\right)(V)=0\right)$.

Proof. Recall that $\operatorname{trace}\left({ }^{\mathscr{H}} B\right)^{b}=n \mathscr{V}^{*}(\mathrm{~d} \log \lambda)$ (see [9]) for any local dilation $\lambda$ of $\mathscr{V}$.

Also, from the fundamental equation (1.1.4) of P. Baird and J. Eells it follows that $\operatorname{trace}\left({ }^{\mathscr{V}} B\right)^{b}=-(n-2) \mathscr{H}^{*}(\mathrm{~d} \log \lambda)$ for any local density $\lambda^{2-n}$ of $\mathscr{V}$.

Now the equivalence (i) $\Longleftrightarrow$ (ii) follows from Lemma 2.1.17.
Theorem 2.1.19. Let $\mathscr{V}$ be a conformal foliation on $(M, g)$ of $\operatorname{codim} \mathscr{V} \neq 2$. Suppose that the orthogonal complement $\mathscr{H}$ of $\mathscr{V}$ is integrable.

Then any two of the following assertions imply the remaining assertion.
(i) $\mathscr{V}$ produces harmonic morphisms;
(ii) The mean curvature form of $\mathscr{V}$ is basic (for $\mathscr{H}$ );
(iii) The mean curvature form of $\mathscr{H}$ is basic (for $\mathscr{V})$.

Moreover, if any two of (i), (ii) or (iii) hold then both $\mathscr{V}$ and $\mathscr{H}$ have closed mean curvature forms.

Proof. If (i) holds then the equivalence (ii) $\Longleftrightarrow$ (iii) follows from Proposition 2.1.18.

Suppose that both the assertions (ii) and (iii) hold. Then (i) follows from Proposition 2.1.15 and Proposition 1.3.1.

Proposition 2.1.20. (a) If $\mathscr{V}$ is integrable then

$$
\begin{equation*}
\mathrm{d}\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right)^{b}\right)(V, W)=0=\mathrm{d}\left(\operatorname{trace}\left({ }^{\mathscr{V}} B\right)^{b}\right)(V, W) \tag{2.1.8}
\end{equation*}
$$

for any vertical $V$ and $W$.
(b) Let $\mathscr{V}$ be a conformal foliation on ( $M, g$ ). Then the following assertions are equivalent.
(i) For any local dilation $\lambda$ of $\mathscr{V}$ the one-form

$$
\operatorname{trace}\left({ }^{\mathscr{V}} B\right)^{b}+(n-2) \mathscr{H}^{*}(\mathrm{~d} \log \lambda)
$$

is basic $(n=\operatorname{dim} \mathscr{H})$;
(ii) For any horizontal $X$ and vertical $V$ we have

$$
\begin{equation*}
\mathrm{d}\left((n-2) \operatorname{trace}\left({ }^{\mathscr{H}} B\right)^{\mathrm{b}}-n \operatorname{trace}\left({ }^{\mathscr{V}} B\right)^{\mathrm{b}}\right)(X, V)=0 . \tag{2.1.9}
\end{equation*}
$$

Proof. (a) The first equality of (2.1.8) follows from Proposition 2.1.14.
The second equality of (2.1.8) follows from 2.1.1) of Proposition 2.1.4 by reversing the roles of $\mathscr{H}$ and $\mathscr{V}$.
(b) Let $\lambda$ be a local dilation of $\mathscr{V}$ and recall that, according to (1.1.3), we have $\operatorname{trace}\left({ }^{\mathscr{H}} B\right)=n \mathscr{V}(\operatorname{grad}(\log \lambda))$. Hence, by applying (2.1.2) of Proposition 2.1.4 and Lemma 2.1.17 we obtain that

$$
\begin{aligned}
& \left((n-2) \mathrm{d}\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right)^{b}\right)-n \mathrm{~d}\left(\operatorname{trace}\left({ }^{\mathscr{V}} B\right)^{b}\right)\right)(X, V) \\
= & (n-2){ }_{\nabla}^{\mathscr{V}} \\
= & \left.n(n-2) \operatorname{Hace}^{\mathscr{V}}\left({ }^{\mathscr{H}} B\right)^{b}\right)(V)+n \mathscr{V}_{V}\left(\operatorname{trace}\left({ }^{\mathscr{V}} B\right)^{b}\right)(X) \\
= & n(n-2) \mathscr{\mathscr { H }}_{V}\left(\mathscr{H}^{*}(\mathrm{~d} \log \lambda)\right)(V)+n \stackrel{\mathscr{F}}{V}\left(\operatorname{trace}\left({ }^{\mathscr{V}} B\right)^{b}\right)(X)+n \mathscr{\mathscr { H }}_{V}\left(\operatorname{trace}\left({ }^{\mathscr{V}} B\right)^{b}\right)(X)
\end{aligned}
$$

for any horizontal $X$ and vertical $V$ and the proof of (i) $\Longleftrightarrow$ (ii) follows from the fact that the basic vector fields (for $\mathscr{V}$ ) are precisely those horizontal vector fields which are infinitesimal automorphisms of $\mathscr{V}$ (see Example 2.1.8(1)).

### 2.2. Characterisation of the conformal actions which produce harmonic morphisms

On a two-dimensional Riemannian manifold a foliation (of dimension one) produces harmonic morphisms if and only if it is locally generated by
conformal vector fields. This follows from the well-known fact that a harmonic morphism to a one-dimensional Riemannian manifold is essentially a harmonic function and, if the domain is two-dimensional, this is locally the real part of a conformal map. If the manifold has dimension greater than two then it is not true that any foliation locally generated by conformal vector fields produces harmonic morphisms. In this section we shall give necessary and sufficient conditions for a foliation locally generated by conformal vector fields to produce harmonic morphisms (Theorem 2.2.6). To state this result we need some preparations.

Firstly, we recall the following definition.
Definition 2.2 .1 (cf. [63]). Let $\mathscr{V}$ be a foliation on the Riemannian manifold ( $M, g$ ).

Then $\mathscr{V}$ is locally generated by conformal (resp. Killing) vector fields if in the neighbourhood of each point a local frame for $\mathscr{V}$ can be found which is formed of conformal (resp. Killing) vector fields.

We also need the following:
Definition 2.2.2. Let $\mathscr{V}$ be an orientable foliation of dimension $p$ on a smooth manifold $M$. Let $\mathscr{H}$ be a complementary distribution (i.e. $\mathscr{V} \oplus \mathscr{H}=T M$ ). Let ${ }^{\mathscr{H}}$ I be its integrability tensor. Let $\stackrel{\mathscr{\omega}}{\omega}$ be a volume form on $\mathscr{V}$ (i.e. a vertical nonvanishing p-form).

Suppose that $\mathscr{V}$ is locally generated by local frames $\left\{V_{r}\right\}$ such that
(1) $\mathscr{V}^{*}\left(\mathcal{L}_{V_{r}} \stackrel{\mathscr{}}{\omega}\right)=0$,
(2) $V_{r}$ is an infinitesimal automorphism of $\mathscr{H}$, for any $r$.

We define the two-form $\operatorname{trace}\left(\operatorname{ad}\left({ }^{\mathscr{K}} I\right)\right)$ on $M$ by

$$
\operatorname{trace}\left(\operatorname{ad}\left(\mathscr{H}^{\prime}\right)\right)=c_{r s}^{s} \mathscr{H}^{r} I^{r},
$$

where $\mathscr{H}_{I}=V_{r} \otimes \mathscr{H}_{I}^{r}$ and $\left[V_{r}, V_{s}\right]=c_{r s}^{t} V_{t}$.
Example 2.2.3. Let $\mathscr{V}$ be an orientable foliation on $(M, g), \mathscr{H}=\mathscr{V}^{\perp}$ and $\mathscr{\omega}$ a volume form for $\mathscr{V}$ with respect to $g$.

Then, (i) if $\left\{V_{r}\right\}$ is a local frame for $\mathscr{V}$ made up of Killing fields then they
satisfy (1) and (2) of Definition 2.2.2; (ii) more generally, the $\left\{V_{r}\right\}$ satisfy (1) of Definition 2.2 .2 if and only if their restrictions to each leaf are divergencefree.

It can be shown, directly, that the above definition is independent of the local frame $\left\{V_{r}\right\}$ of $\mathscr{V}$ such that (1) and (2) above, hold. This also follows from the following proposition.

Proposition 2.2.4. Let $\mathscr{V}, \mathscr{H}, \mathscr{H}_{I}, \stackrel{\mathscr{\omega}}{\omega},\left\{V_{r}\right\}$ be as in Definition 2.2.2 and let ${ }^{2} R$ be the curvature form of $\nabla$.

Then,
(a) $\operatorname{trace}\left(\operatorname{ad}\left({ }^{\mathscr{H}} I\right)\right)=\operatorname{trace}\left({ }^{\mathscr{V}} R\right)$;
(b) $\mathscr{V}^{*}\left(\mathcal{L}_{\mathscr{H}_{I(X, Y)}}{ }^{\mathscr{W}}\right)=\operatorname{trace}\left(\operatorname{ad}\left(\mathscr{H}_{I}\right)\right)(X, Y) \stackrel{\mathscr{\omega}}{\omega}$ for any basic vector fields $X$ and $Y$.

Further, the following assertions are equivalent:
(i) $\operatorname{trace}\left(\operatorname{ad}\left({ }^{\mathscr{H}} I\right)\right)=0$,
(ii) At least locally there can be defined smooth positive basic functions $\rho$ such that $\mathscr{V}^{*}\left(\mathcal{L}_{X}(\rho \stackrel{\mathscr{W}}{\omega})\right)=0$ for any horizontal field $X$,
(iii) $\mathscr{V}^{*}\left(\mathcal{L}_{\mathscr{H}_{I(X, Y)}} \stackrel{\mathscr{V}}{\omega}\right)=0$ for any basic vector fields $X$ and $Y$.

In particular, if the first Betti number of $M$ is zero and (i) holds then $\mathscr{V}$ is taut (i.e. there exists a Riemannian metric on $M$ with respect to which the leaves of $\mathscr{V}$ are minimal).

Proof. Let $g$ be a Riemannian metric on $M$ such that $\mathscr{H}=\mathscr{V}^{\perp}$ and $\stackrel{\mathscr{\omega}}{ }$ is equal to the induced volume form on $\mathscr{V}$, and let $\stackrel{\mathscr{V}}{\nabla}$ be the adapted Bott connection on $\mathscr{V}$ corresponding to $g$.

Note that (1) of Definition 2.2 .2 is equivalent to the fact that $\operatorname{div}_{\mathscr{V}} V_{\alpha}=0$.
The proof of (a) and (b) follows from Proposition 2.1.5, Lemma 2.1.13, Proposition 2.1.14 and Propostion 2.1.16 reversing the roles of $\mathscr{H}$ and $\mathscr{V}$.

Assertion (a) is equivalent to the fact that $\operatorname{trace}\left(\operatorname{ad}\left({ }^{\mathscr{H}} I\right)\right)$ is the curvature form of the connection induced by $\stackrel{\mathscr{V}}{ }$ on $\Lambda^{r}(\mathscr{V})$. It is easy to see that this connection is flat if and only if assertion (ii) holds. The equivalence (i) $\Longleftrightarrow$ (ii)
now follows from (a).
The equivalence (i) $\Longleftrightarrow$ (iii) follows from (b).

Remark 2.2.5. Note that trace $\left(\operatorname{ad}\left({ }^{\mathscr{H}} I\right)\right)$ is well-defined also for nonorientable foliations (in which case $\stackrel{\mathscr{V}}{\omega}$ is defined just up to the sign). Also, note that $\operatorname{trace}\left(\operatorname{ad}\left({ }^{\mathscr{H}} I\right)\right)$ is well-defined for foliations locally generated by conformal vector fields.

We now state the main result of this section.

Theorem 2.2.6. Let $\mathscr{V}$ be a conformal foliation of $\operatorname{codim} \mathscr{V}=n \neq 2$ on $\left(M^{m}, g\right), m \geq 3$. Suppose that $\mathscr{V}$ is locally generated by conformal vector fields and let $\mathscr{H}$ denote its orthogonal complement.

Then the following assertions are equivalent:
(i) $\mathscr{V}$ produces harmonic morphisms.
(ii) The mean curvature form of $\mathscr{V}$ is basic and the following relation holds:

$$
\begin{equation*}
\operatorname{trace}\left(\operatorname{ad}\left({ }^{\mathscr{H}} I\right)\right)=\frac{m-2}{n} g\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right),{ }^{\mathscr{H}} I\right) . \tag{2.2.1}
\end{equation*}
$$

Proof. By Proposition 1.3.1, $\mathscr{V}$ produces harmonic morphisms if and only if

$$
\begin{equation*}
(n-2) \mathrm{d}\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right)^{b}\right)-n \mathrm{~d}\left(\operatorname{trace}\left({ }^{\mathscr{V}} B\right)^{b}\right)=0 . \tag{2.2.2}
\end{equation*}
$$

By Proposition 2.1.20(a) the left hand side of (2.2.2) is automatically zero when evaluated on a pair of vertical vectors.

Let $\lambda$ be a local dilation of $\mathscr{V}$.
Let $V \in \Gamma(\mathscr{V})$ be a conformal vector field on $(M, g)$. It is obvious that $V$ restricted to any leaf $L$ of $\mathscr{V}$ is a conformal vector field on ( $L,\left.g\right|_{L}$ ). Using this it is easy to see that

$$
\begin{equation*}
\operatorname{div}_{\mathscr{V}} V=-(m-n) V(\log \lambda) . \tag{2.2.3}
\end{equation*}
$$

Let $X \in \Gamma(\mathscr{H})$ be a basic vector field. Then

$$
\begin{align*}
\ddot{\nabla}_{V}\left(\mathscr{H}^{*}(\mathrm{~d} \log \lambda)\right)(X) & =V(X(\log \lambda))=X(V(\log \lambda)) \\
& =-\frac{1}{m-n} X(\operatorname{div} \mathscr{\mathscr { V }} V)=\frac{1}{m-n} \stackrel{\mathscr{H}}{V}\left(\operatorname{trace}\left({ }^{(/} B\right)^{\mathrm{b}}\right)(X), \tag{2.2.4}
\end{align*}
$$

where we have also applied Lemma 2.1.1 and Proposition 2.1.16 (reversing the roles of $\mathscr{H}$ and $\mathscr{V}$ in the latter). From Proposition 2.1.20(b) and (2.2.4) it follows that the left hand side of $(2.2 .2)$ is zero when evaluated on a pair made up of a vertical vector and a horizontal vector if and only if $\mathscr{V}$ has basic mean curvature form.

Now, using Lemma 2.1.13(a) (with the roles of $\mathscr{H}$ and $\mathscr{V}$ reversed) and (2.2.3) it is easy to see that

$$
\begin{equation*}
\mathrm{d}\left(\operatorname{trace}\left({ }^{\mathscr{V}} B\right)^{\mathrm{b}}\right)(X, Y)=\operatorname{trace}\left(\operatorname{ad}\left({ }^{\mathscr{H}} I\right)\right)(X, Y)-\frac{m-n}{n} g\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right),{ }^{\mathscr{H}} I(X, Y)\right) \tag{2.2.5}
\end{equation*}
$$

for any horizontal $X$ and $Y$. By combining 2.2.5 and Proposition 2.1.4 we obtain that the left hand side of $(2.2 .2)$ is zero when evaluated on a pair of horizontal vectors if and only if (2.2.1) holds. The theorem is proved.

Remark 2.2.7. The first condition of Theorem 2.2.6(ii) (i.e. $\mathscr{V}$ has basic mean curvature form) can be replaced with the fact that the mean curvature form of $\mathscr{H}$ is invariant under the parallel displacement determined by $\mathscr{H}$.

Corollary 2.2.8. Let $\mathscr{V}$ be a foliation on $\left(M^{m}, g\right), m \geq 3$, which is locally generated by conformal vector fields and has integrable orthogonal complement. Then the following assertions are equivalent.
(i) $\mathscr{V}$ produces harmonic morphisms;
(ii) $\mathscr{V}$ has basic mean curvature form.

Moreover, if either assertion (i) or (ii) holds then both $\mathscr{V}$ and its orthogonal complement have closed mean curvature forms.

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) is an immediate consequence of Theorem 2.2.6.

The last assertion follows from Proposition 2.1.4, Lemma 2.1.13, Proposition 2.1.16 and Theorem 2.2.6.

### 2.3. Isometric actions and harmonic morphisms

From Theorem 2.2.6 we obtain the following.
Corollary 2.3.1. Let $\mathscr{V}$ be a Riemannian foliation of $\operatorname{codim} \mathscr{V} \neq 2$ on $\left(M^{m}, g\right)$, $m \geq 3$, and let $\mathscr{H}_{I}$ be the integrability tensor field of its orthogonal complement. Suppose that $\mathscr{V}$ is locally generated by Killing fields.

Then the following assertions are equivalent:
(i) $\mathscr{V}$ produces harmonic morphisms,
(ii) $\operatorname{trace}\left(\operatorname{ad}\left({ }^{\mathscr{H}} I\right)\right)=0$.

Proof. It is obvious that $\mathscr{V}$ has basic mean curvature. Also, $\operatorname{trace}\left({ }^{\mathscr{H}} B\right)=$ 0 and the proof follows from Theorem 2.2.6.

Remark 2.3.2. It is easy to see that Corollary 2.3 .1 holds more generally for Riemannian foliations locally generated by infinitesimal automorphisms of the horizontal distribution which when restricted to any leaf are divergence-free.

When the foliation is simple the result of Corollary 2.3.1 takes a more concrete form.

Corollary 2.3.3. Let $\varphi:(M, g) \rightarrow(N, h), \operatorname{dim} N \neq 2$, be a Riemannian submersion whose fibres are connected and locally generated by Killing fields and let ${ }^{\mathscr{H}}$ I be the integrability tensor of the horizontal distribution. Then the following assertions are equivalent.
(i) $\varphi$ lifts to a harmonic morphism $\widetilde{\varphi}:(\widetilde{M}, \widetilde{g}) \rightarrow(\widetilde{N}, \widetilde{h})$ where $(\widetilde{M}, \widetilde{g}) \rightarrow$ $(M, g)$ is a Riemannian regular covering and $\widetilde{N} \rightarrow N$ is a regular covering such that $\widetilde{h}$ and the pull-back of $h$ to $\widetilde{N}$ are conformally equivalent.
(ii) $\operatorname{trace}\left(\operatorname{ad}\left({ }^{\mathscr{H}} I\right)\right)=0$.

Proof. From Corollary 2.3.1 it follows that it is sufficient to prove that if $\mathscr{V}\left(=\operatorname{ker} \varphi_{*}\right)$ produces harmonic morphisms then (i) holds.

Let $\widetilde{M} \rightarrow M$ be the regular covering which corresponds to the cohomology
class $[a] \in H^{1}(M ; \mathbb{R})$ induced by the differentials of the logarithms of the dilations of the (local) harmonic morphisms produced by $\mathscr{V}\left(=\operatorname{ker} \varphi_{*}\right)$. (From the fundamental equation it follows that $a$ can be also defined as the oneform obtained by applying the musical isomorphism $b$ to $\frac{-1}{n-2} \operatorname{trace}\left({ }^{V} B\right)$.) It is obvious that the pull-back of $[a]$ to $\widetilde{M}$ is zero; let $\lambda$ be a smooth positive function on $\widetilde{M}$ such that $\mathrm{d}(\log \lambda)$ is equal to the pull-back of $a$ to $\widetilde{M}$.

Since $a$ is basic (see Remark 2.3.2(1)) there exists a regular covering $\widetilde{N} \rightarrow N$ whose pull-back by $\varphi$ is $\widetilde{M} \rightarrow M$. It is obvious that $\varphi$ lifts to a smooth map $\widetilde{\varphi}: \widetilde{M} \rightarrow \widetilde{N}$. Note that we can suppose that $\lambda$ is constant along the fibres of $\widetilde{\varphi}$ and hence there exists a positive smooth function $\bar{\lambda}$ on $\widetilde{N}$ such that $\widetilde{\varphi}^{*}(\bar{\lambda})=\lambda$.

Let $\widetilde{g}$ be the pull-back of $g$ to $\widetilde{M}$ and $\bar{h}$ be the pull-back of $h$ to $\widetilde{N}$. Then $\widetilde{\varphi}:(\widetilde{M}, \widetilde{g}) \rightarrow\left(\widetilde{N}, \bar{\lambda}^{2} \bar{h}\right)$ is a harmonic morphism and the corollary is proved.

Remark 2.3.4. If in Corollary 2.3.3 we have that $H^{1}(M ; \mathbb{R})=0$ or $H^{1}(N ; \mathbb{R})=$ 0 then assertion (i) can be replaced by the following stronger assertion:
(i') There exists a Riemannian metric $h_{1}$ on $N$ which is conformally equivalent to $h$ such that $\varphi:(M, g) \rightarrow\left(N, h_{1}\right)$ is a harmonic morphisms.

The same improvement can be made if the foliation formed by the fibres is generated by a commuting family of Killing fields $\left\{V_{1}, \ldots, V_{r}\right\}$ (in particular, if the foliation is generated by an Abelian Lie group of isometries). To see this define $\lambda$ by $g\left(V_{1} \wedge \ldots \wedge V_{r}, V_{1} \wedge \ldots \wedge V_{r}\right)=\lambda^{2 n-4}, n=\operatorname{dim} N$. Then $\lambda$ is the dilation of the induced harmonic morphism.

In some cases trace $\left(\operatorname{ad}\left({ }^{\mathscr{H}} I\right)\right)$ can be defined in a different way:
Lemma 2.3.5. Let $\mathscr{V}$ be a Riemannian foliation on $(M, g)$ generated by the action of a closed subgroup $G$ of the isometry group of $(M, g)$. Let $\left\{V_{r}\right\}$ be a local frame of $\mathscr{V}$ made up of Killing vector fields induced by the action of $G$ and let $\left\{c_{r s}^{t}\right\}$ be defined by $\left[V_{r}, V_{s}\right]=c_{r s}^{t} V_{t}$.

Then there exists a well-defined vertical one-form trace $\circ \mathrm{ad} \in \Gamma\left(\mathscr{V}^{*}\right)$ such that

$$
(\text { trace } \circ \operatorname{ad})\left(V_{r}\right)=c_{r s}^{s}
$$

Moreover, we have trace $\left(\operatorname{ad}\left({ }^{\mathscr{H}} I\right)\right)=(\operatorname{trace} \circ \mathrm{ad})\left({ }^{\mathscr{H}} I\right)$.
Proof. Let $x \in M$ and let $\mathfrak{h}_{x}$ be the Lie algebra of the isotropy group $H_{x}$ at $x$ of the action of $G$ on $M$. Because $G$ is a closed subgroup of the isometry group, the isotropy groups are compact and hence we can find a subspace $\mathfrak{m}_{x}$ of the Lie algebra $\mathfrak{g}$ of $G$ such that $\mathfrak{g}=\mathfrak{h}_{x} \oplus \mathfrak{m}_{x}$ and $\left[\mathfrak{h}_{x}, \mathfrak{m}_{x}\right] \subseteq \mathfrak{m}_{x}$ (for example, take $\mathfrak{m}_{x}$ to be the orthogonal complement of $\mathfrak{h}_{x}$ in $\mathfrak{g}$ with respect to an $\operatorname{Ad} H_{x}$-invariant metric on $\mathfrak{g}$ ).

Then, by identifying, as usual, $\mathfrak{m}_{x}=\mathscr{V}_{x},(\text { trace } \circ \mathrm{ad})_{x}$ is the restriction to $\mathfrak{m}_{x}$ of the trace of the adjoint representation of $\mathfrak{g}$.

If $y=x a, a \in G$, is another point on the same leaf, then we can take $\mathfrak{m}_{y}=\left(\operatorname{Ad} a^{-1}\right)\left(\mathfrak{m}_{x}\right)$.

If $y \in M$ can be joined to $x$ by a horizontal curve then from Lemma 2.1.1 it follows that $H_{y}=H_{x}$.

Hence trace oad is well-defined.
The last assertion of the proposition is obvious.
Let $\mathscr{V}$ be a foliation on $(M, g)$ locally generated by Killing fields. Although traceoad is not always well-defined, to simplify the exposition, we shall write trace $\circ \mathrm{ad}=0$ to mean that in the neighbourhood of each point a local frame $\left\{V_{r}\right\}$ for $\mathscr{V}$ can be found which is made up of Killing fields and is such that $c_{r s}^{s}=0$ where $c_{r s}^{t}$ are defined by $\left[V_{r}, V_{s}\right]=c_{r s}^{t} V_{t}$.

The following two corollaries follows immediately from Corollary 2.3.1.
Corollary 2.3.6. A foliation of codimension not equal to two which is locally generated by Killing fields and which has integrable orthogonal complement produces harmonic morphisms.

Corollary 2.3.7. A foliation of codimension not equal to two which is locally generated by Killing fields and for which trace $\mathrm{oad}=0$ produces harmonic morphisms.

The above result admits the following partial converse.
Proposition 2.3.8. Let $\mathscr{V}$ be a foliation of codimension not equal to two which produces harmonic morphisms on $(M, g)$ and is locally generated by

Killing fields. Let ${ }^{\mathscr{H}}$ I be the integrability tensor of the orthogonal complement $\mathscr{H}$ of $\mathscr{V}$.

Suppose that on each leaf $L$ of $\mathscr{V}$ a point $x \in L$ can be found such that $\mathscr{V}_{x}$ is spanned by $\left\{\mathscr{H}^{\prime}(X, Y) \mid X, Y \in \mathscr{H}_{x}\right\}$.

Then trace $\circ \mathrm{ad}=0$.
Proof. From Corollary 2.3 .1 it follows that it is sufficient to prove that $\mathscr{V}_{x}$ is spanned by $\left\{\mathscr{H}^{\prime} I(X, Y) \mid X, Y \in \mathscr{H}_{x}\right\}$ at each point $x \in M$. From Lemma 2.1.1 it follows that $\left[V,{ }^{\mathscr{H}} I(X, Y)\right]=0$ for any Killing field $V \in \Gamma(\mathscr{V})$ and any basic vector fields $X, Y \in \Gamma(\mathscr{H})$.

The proof follows from the fact that any two points of a leaf can be joined by a curve which is piecewisely an integral curve of a Killing vector field $V \in \Gamma(\mathscr{V})$ (see [60, Chapter I, Theorem 4.4]).

Next we give an example of a Riemannian foliation locally generated by Killing fields for which $\operatorname{trace}\left(\operatorname{ad}\left({ }^{\mathscr{H}} I\right)\right)=0$ but ${ }^{\mathscr{K}} I \neq 0$ and trace $\circ \operatorname{ad} \neq 0$.

Example 2.3.9. Let $\mathcal{F}$ be a Riemannian foliation locally generated by Killing fields and which produces harmonic morphisms on $(M, g)$. Suppose that the orthogonal complement of $\mathcal{F}$ is not integrable (see examples, below).

Let $G$ be the Lie group defined by

$$
G=\left\{\left.\left(\begin{array}{cc}
a & b \\
0 & a^{-1}
\end{array}\right) \right\rvert\, a>0, b \in \mathbb{R}\right\} .
$$

Endow $G$ with a right invariant metric $\gamma$ and consider the Riemannian product manifold ( $M \times G, \pi_{M}^{*}(g)+\pi_{G}^{*}(\gamma)$ ) where $\pi_{M}$ and $\pi_{G}$ are the projections onto $M$ and $G$, respectively. Let $\mathscr{V}=\mathcal{F} \times T G$. It is obvious that $\mathscr{V}$ is a foliation locally generated by Killing fields and which produces harmonic morphisms on $\left(M \times G, \pi_{M}^{*}(g)+\pi_{G}^{*}(\gamma)\right)$.

Notice, however, that the orthogonal complement $\mathscr{H}$ of $\mathscr{V}$ is nonintegrable so $\mathscr{H} I \neq 0$, and also trace $\circ \mathrm{ad} \neq 0$.

For the next application we recall the following definition (cf. [11, 7.84]).
Definition 2.3.10. Let $\left(L^{p}, h\right)$ be a locally homogeneous Riemannian manifold (i.e. a Riemannian manifold whose tangent bundle admits, in a neighbourhood
of each point, local frames made up of Killing fields).
Then $\left(L^{p}, h\right)$ is called naturally reductive if each point $x \in L$ has an open neighbourhood on which a local frame $\left\{V_{r}\right\}_{r=1, \ldots, p}$ made up of Killing fields can be found such that $h\left(\left[V_{r}, V_{s}\right], V_{t}\right)+h\left(V_{s},\left[V_{r}, V_{t}\right]\right)=0$ at $x$.

The following well-known lemma follows from the fact that any skewsymmetric endomorphism is trace-free.

Lemma 2.3.11. Let $(L, h)$ be a naturally reductive locally-homogeneous Riemannian manifold. Then trace $\mathrm{oad}=0$.

The following result follows from Corollary 2.3.7 and Lemma 2.3.11.
Proposition 2.3.12. A foliation of codimension not equal to two which is locally generated by Killing fields and whose leaves are naturally reductive produces harmonic morphisms.

Theorem 2.3.13. Let $G$ be a Lie group which acts as an isometry group on the Riemannian manifold $(M, g)$.

Suppose that the following conditions are satisfied:
(i) The orbits of the action of $G$ on $M$ have the same codimension not equal to two.
(ii) There exists on $G$ a bi-invariant Riemannian metric.
(iii) The canonical representation of an isotropy group is irreducible.

Then, the connected components of the orbits form a Riemannian foliation with umbilical leaves which produces harmonic morphisms.

Proof. The fact that (i) implies that the connected components of the orbits form a Riemannian foliation is well-known (see [60, Chapter IV, Example 4.10]). Let $\mathscr{V}$ be this foliation.

By chosing an $\operatorname{Ad} G$ invariant metric on the Lie algebra of $G$ and restricting it to the orthogonal complement of the Lie algebra of the isotropy group at $x \in M$ we can induce a metric $\bar{h}_{x}$ on $\mathscr{V}_{x}$ which by (iii) must be homothetic to $\left.g_{x}\right|_{V_{x}}$ (see [36, vol. I, Appendix 5]). Then $\bar{h}$ is a metric on $\mathscr{V}$ which can be extended to a metric $h$ on $M$ such that $\left.h\right|_{\mathscr{H}}=\left.g\right|_{\mathscr{H}}$ where $\mathscr{H}$ is the orthogonal
complement of $\mathscr{V}$.
Since $\left.h\right|_{\mathscr{V}}$ is induced by an $\operatorname{Ad} G$ invariant metric, $\mathscr{V}$ has naturally reductive leaves with respect to $h$. But $g$ and $h$ are homothetic when restricted to a leaf and hence the leaves of $\mathscr{V}$ are also naturally reductive with respect to $g$. Moreover, from the fact that $\mathscr{V}$ has totally geodesic leaves with respect to $h$, and $g$ and $h$ are conformal when restricted to $\mathscr{V}$ and equal when restricted to $\mathscr{H}$ it follows that the leaves of $\mathscr{V}$ are umbilical with respect to $g$.

Remark 2.3.14. Note that the same argument as above can be applied to show that the Ricci tensor of each leaf is proportional to the induced metric (see [11, 7.44]); hence, in the above theorem, each leaf is an Einstein manifold.

Theorem 2.3.15. Let $G$ be a closed subgroup of the isometry group of $(M, g)$ which generates a foliation $\mathscr{V}$ of codimension not equal to two.
(i) Suppose that the Lie algebra $\mathfrak{g}$ of $G$ satisfies $\operatorname{trace}(\operatorname{ad} \mathfrak{g})=0$. Then, $\mathscr{V}$ produces harmonic morphisms.
(ii) Conversely, if $\mathscr{V}$ produces harmonic morphisms and on each orbit $Q$ a point $x \in Q$ can be found such that $\mathscr{V}_{x}$ is spanned by $\left\{{ }^{\mathscr{H}} I(X, Y) \mid X, Y \in \mathscr{H}_{x}\right\}$ where ${ }^{\mathscr{H}} I$ is the integrability tensor of the orthogonal complement of $\mathscr{V}$, then $\operatorname{trace}(\mathrm{ad} \mathfrak{g})=0$.

Proof. (i) It is sufficient to prove that trace $\circ \mathrm{ad}=0$.
Let $H$ be the isotropy group of $G$ at $x \in M$. Since $G$ is a closed subgroup of the isometry group we have that $H$ is compact and hence we can find $\mathfrak{m} \subseteq \mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{h} \oplus \mathfrak{m}$ and $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$.

Let $\left\{A_{1}, \ldots, A_{r}\right\}$ be a basis of $\mathfrak{h}$ and $\left\{A_{r+1}, \ldots, A_{s}\right\}$ a basis of $\mathfrak{m}$. Let $\left\{c_{\alpha \beta}^{\gamma}\right\}$ be the corresponding structural constants of $\mathfrak{g}$.

Because trace $(\operatorname{ad} \mathfrak{g})=0$ we have that $\sum_{\alpha=1}^{s} c_{\alpha \beta}^{\alpha}=0$ for $\beta=1, \ldots, s$.
Also, $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ implies that $c_{\alpha \beta}^{\gamma}=0$ for $\alpha, \gamma=1, \ldots, r$ and $\beta=r+$ $1, \ldots, s$.

Let $\beta \in\{r+1, \ldots, s\}$. Then $\sum_{\alpha=r+1}^{s} c_{\alpha \beta}^{\alpha}=-\sum_{\alpha=1}^{r} c_{\alpha \beta}^{\alpha}=0$ and it follows that $(\text { trace } \circ \mathrm{ad})_{x}=0$.
(ii) Note that if a point of an orbit has the assumed property then on
each component of that orbit a point can be found with the same property. The proof now follows from Proposition 2.3.8.

Corollary 2.3.16. Let $G$ be a compact Lie group of isometries of $(M, g)$. If the principal orbits of $G$ have codimension not equal to two then their connected components form a Riemannian foliation which produces harmonic morphisms.

In particular, if $(M, g)$ is compact and the principal orbits of the isometry group are of codimension not equal to two then their connected components form a Riemannian foliation which produces harmonic morphisms.

Remark 2.3.17. Let $G$ be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Recall that if $G$ is connected then $\operatorname{trace}(\operatorname{ad} \mathfrak{g})=0$ if and only if $G$ is unimodular, i.e. its left and right invariant Haar measures (which are unique up to multiplicative constants) are equal.

Theorem 2.3.18. Let $\xi=(P, N, G)$ be a principal bundle, $\operatorname{dim} N \neq 2$, whose total space $P$ is endowed with a Riemannian metric $g$ which is invariant under the action of $G$.

Let $\mathscr{H}$ be the induced principal connection on $\xi$ and $h$ the (unique) Riemannian metric on $N$ such that the projection $\pi:(P, g) \rightarrow(N, h)$ is a Riemannian submersion.

Then the following assertions are equivalent:
(i) The connection induced by $\mathscr{H}$ on the determinant bundle of the adjoint bundle $\operatorname{Ad} \xi$ is flat.
(ii) The projection $\pi$ lifts to a harmonic morphism $\widetilde{\pi}:(\widetilde{P}, \widetilde{g}) \rightarrow(\widetilde{N}, \widetilde{h})$ where $(\widetilde{P}, \widetilde{g}) \rightarrow(P, g)$ is a Riemannian regular covering and $\widetilde{N} \rightarrow N$ is a regular covering such that $\widetilde{h}$ and the pull-back of $h$ to $\widetilde{N}$ are conformally equivalent. Moreover, $(\widetilde{P}, \widetilde{N}, G)$ is, in a natural way, a principal bundle.

Proof. Let $\Omega \in \Gamma\left(\mathfrak{g} \otimes \Lambda^{2}\left(T^{*} P\right)\right)$ be the curvature form of $\mathscr{H}$. It is obvious that trace $\left(\operatorname{ad}^{\mathscr{H}} I\right)=(\operatorname{trace} \circ \mathrm{ad})(\Omega)$. But $(\operatorname{trace} \circ \mathrm{ad})(\Omega)$ is the pull back, by $\pi$, of the curvature form of the determinant bundle of $\operatorname{Ad} \xi$. Hence, (i) is equivalent to the fact that the fibres of $\pi$ form a (Riemannian) foliation which
produces harmonic morphisms.
Now, from the Corollary 2.3 .1 it follows that it is sufficient to prove that $(\widetilde{P}, \widetilde{N}, G)$ is, in a natural way, a principal bundle. Using the same notations as in Corollary 2.3.3 (with $P=M$ ) this follows from the fact that $\widetilde{P}$ is the total space of $\xi+\eta \in H^{1}(N, G \times K)$, where $\eta \in H^{1}(N, K)$ is the regular covering corresponding to $[b] \in H^{1}(N, \mathbb{R})$ and $[b]$ is such that $\pi^{*}[b]=[a]$.

Remark 2.3.19. 1) From the holonomy theorem (see [36]) it follows that assertion (i) of Theorem 2.3 .18 is equivalent to the fact that the identity component $H$ of the holonomy group of $\mathscr{H}$ satisfies $\operatorname{det}\left(\operatorname{Ad}_{G} H\right)=1$.
2) Let $\mathscr{V}$ be a foliation on $(M, g)$ generated by the action of a closed subgroup $G$ of the isometry group of $(M, g)$. Then it can be proved, directly by using the mass invariance characteristic property of harmonic morphisms (see Proposition 1.1.11), that $\mathscr{V}$ produces harmonic morphisms if and only if the identity component $H$ of the holonomy group at $x$ of the orthogonal complement of $\mathscr{V}$ satisfies $\operatorname{det}\left(\operatorname{Ad}_{G} H\right)=1$. (The holonomy group of $\mathscr{H}\left(=\mathscr{V}^{\perp}\right)$ at $x \in M$ consists of those $a \in G$ such that $x$ and $x a$ can be joined by a horizontal path, cf. [12].) In this way another proof can be obtained for the result of Corollary 2.3.1 applied to foliations globally generated by closed subgroups of the isometry group.

Corollary 2.3.20. Let $(P, N, G)$ be a principal bundle, $\operatorname{dim} N \geq 3$, whose total space $P$ is endowed with a Riemannian metric $g$ which is invariant under the action of $G$.

Let $H \subseteq G$ be a closed subgroup and suppose that $\operatorname{trace}(\operatorname{ad} \mathfrak{h})=0$ and $\operatorname{trace}(\mathrm{ad} \mathfrak{g})=0$. Let $E=P \times{ }_{G} G / H$ be the total space of the associated bundle.

Then in a neighbourhood of any point, there exist Riemannian metrics on $E$ and $N$ with respect to which the restriction of the natural projection $E \rightarrow N$ to that neighbourhood is a harmonic morphism.

Proof. It is well-known that $(P, E, H)$ is in a natural way a principal bundle (see [36]). From Theorem 2.3 .18 it follows that both of the foliations induced on $P$ by the actions of $G$ and of $H$ produces harmonic morphisms. Thus, at least locally, we can find metrics on $E$ and on $N$ with respect to
which the projections $P \rightarrow E$ and $P \rightarrow N$ are harmonic morphisms. Since the first of these is surjective this implies that the projection $E \rightarrow N$ can also be made, at least locally, a harmonic morphism (see [18, 2.31] and [28, Proposition 1.1]).

It is obvious that Corollary 2.3 .20 still holds if $\operatorname{dim} G-\operatorname{dim} H \neq 1=$ $\operatorname{dim} N$.

Corollary 2.3.21. Let $(G, g)$ be a Lie group endowed with a right (left) invariant Riemannian metric. Let $K \subseteq H \subseteq G$ be closed subgroups such that $\operatorname{trace}(\operatorname{ad} \mathfrak{k})=0, \operatorname{trace}(\operatorname{ad} \mathfrak{h})=0$ and $\operatorname{dim} G-\operatorname{dim} H \geq 3$.

Then in a neighbourhood of any point, there exist Riemannian metrics on $G / H$ and $G / K$ with respect to which the restriction of the natural projection $G / K \rightarrow G / H$ to that neighbourhood is a harmonic morphism.

It is obvious that Corollary 2.3 .21 still holds if $\operatorname{dim} G-\operatorname{dim} H=1 \neq$ $\operatorname{dim} H-\operatorname{dim} K$.

Example 2.3.22. 1) A one-dimensional Riemannian foliation produces harmonic morphisms if and only if it is locally generated by Killing vector fields. This result is due to R.L. Bryant [14] and the 'if' part follows also from Corollary 2.3.7 (see also Proposition 3.1.3, below).

Let $\varphi:\left(M^{n+1}, g\right) \rightarrow N^{n}, n \geq 3$, be a submersion with connected onedimensional fibres and let $\mathscr{V}$ be the foliation formed by the fibres of $\varphi$. Suppose that there exists a nowhere zero Killing vector field $V \in \Gamma(\mathscr{V})$. Let $\bar{g}$ be the unique metric on $N$ such that $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, \bar{g}\right)$ is a Riemannian submersion and let $\lambda$ be the positive smooth function such that $g(V, V)=\lambda^{2 n-4}$. Then, because $V$ is Killing, $\lambda$ is basic, i.e. there exists a positive smooth function $\bar{\lambda}$ on $N^{n}$ such that $\lambda=\bar{\lambda} \circ \varphi$. Then, $\varphi:(M, g) \rightarrow\left(N, \bar{\lambda}^{2} \bar{g}\right)$ is a harmonic morphism [14]. We shall say that $\varphi$ is induced by an (infinitesimal) isometric action.
2) The foliation formed on (an open subset of) a hypersphere $S^{n}$ by the intersections with it of a parallel family of planes in $\mathbb{R}^{n+1}$ of codimension not equal to three produces harmonic morphisms. This follows from Corollary
2.3.6 or from Theorem 2.3.13. This can also be proved by noting that the foliation is induced by one of the projections of a warped-product (see Example 1.4.10). Similar examples can be obtained on Euclidean spaces and on hyperbolic spaces.
3) Let $\mathbb{K}=\mathbb{R}, \mathbb{C}, \mathbb{H}$ and consider on $G l_{n}(\mathbb{K}), n \geq 2$, the following wellknown right invariant Riemannian metric

$$
g=\sum_{i, j=1}^{n}\left|\mathrm{~d} x_{k}^{i} \cdot\left(x^{-1}\right)_{j}^{k}\right|^{2} .
$$

Let $K \subseteq H \subseteq G \subseteq G l_{n}(\mathbb{K})$ be closed subgroups such that trace $(\operatorname{ad} \mathfrak{k})=0$, $\operatorname{trace}(\operatorname{ad} \mathfrak{h})=0$ and $\operatorname{dim} G-\operatorname{dim} H \neq 2 \neq \operatorname{dim} G-\operatorname{dim} K$. Then, at least locally, a metric can be found on $G / H$ (which is unique up to homotheties) such that the projection $G \rightarrow G / H$ becomes, suitably restricted, a harmonic morphism. (If $G$ or $G / H$ has zero first Betti number then this metric can be defined globally on $G / H$.) Also, at least locally, a metric can be found on the total space of the projection $G / K \rightarrow G / H$ such that the induced foliation produces harmonic morphisms. (If $G / K$ and $G / H$ both have zero first Betti number then there can be defined (global) metrics on them such that the projection $G / K \rightarrow G / H$ becomes a harmonic morphism.)

For example, the foliations formed by the fibres of the following natural maps produce harmonic morphisms:

$$
\begin{aligned}
& G l_{p+q}(\mathbb{K}) \rightarrow G_{p+q, p}(\mathbb{K}) \times G_{p+q, q}(\mathbb{K}), \\
& G l_{p+q}(\mathbb{K}) \rightarrow V_{p+q, p}(\mathbb{K}) \times G_{p+q, q}(\mathbb{K}), \\
& G l_{p+q}(\mathbb{K}) \rightarrow P G l_{p+q}(\mathbb{K})
\end{aligned}
$$

where, for $p, q \geq 1, G_{p+q, p}(\mathbb{K})$ is the Grassmanian manifold of $p$-dimensional subspaces of $\mathbb{K}^{p+q}, V_{p+q, p}(\mathbb{K})$ is the Stiefel manifold of $p$-frames on $\mathbb{K}^{p+q}$ and for the first projection $p+q \geq 3$ if $\mathbb{K}=\mathbb{R}$. If $\mathbb{K}=\mathbb{H}$, or $\mathbb{K}=\mathbb{R}$ and $p+q \geq 3$, then the first Betti number of $G l_{p+q}(\mathbb{K})$ is zero and hence in these cases on the image of each of the above maps a metric can be found such that the induced map becomes a harmonic morphism.

In particular, consider on $G l_{2}^{+}(\mathbb{R})$ the coordinates given by $x=\left(\begin{array}{cc}x_{1} & x_{3} \\ x_{2} & x_{4}\end{array}\right)$.

With respect to these coordinates we have

$$
\begin{aligned}
& g=\frac{1}{\left(x_{1} x_{4}-x_{2} x_{3}\right)^{2}}\left\{\left(x_{3}^{2}+x_{4}^{2}\right)\left(\mathrm{d} x_{1}^{2}+\mathrm{d} x_{2}^{2}\right)+\left(x_{1}^{2}+x_{2}^{2}\right)\left(\mathrm{d} x_{3}^{2}+\mathrm{d} x_{4}^{2}\right)\right. \\
&\left.-2\left(x_{1} x_{3}+x_{2} x_{4}\right)\left(\mathrm{d} x_{1} \mathrm{~d} x_{3}+\mathrm{d} x_{2} \mathrm{~d} x_{4}\right)\right\} .
\end{aligned}
$$

Then on the images of the following maps there exist Riemannian metrics, unique up to homotheties, with respect to which the induced maps are harmonic morphisms.
$\varphi_{1}: G l_{2}^{+}(\mathbb{R}) \rightarrow P G l_{2}^{+}(\mathbb{R})$, given by the natural projection,
$\varphi_{2}: G l_{2}^{+}(\mathbb{R}) \rightarrow \mathbb{R}^{2} \times \mathbb{R} P^{1}$, given by $\varphi_{2}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)=\left(\left(x_{1}, x_{2}\right),\left[x_{3}: x_{4}\right]\right)$.
If $\mathbb{K}=\mathbb{C}$ or $\mathbb{H}$ then we can also consider the foliation induced by the map $G l_{2}(\mathbb{K}) \rightarrow \mathbb{K} P^{1} \times \mathbb{K} P^{1}$.

Other examples can be obtained by considering other linear Lie groups.

### 2.4. Homothetic foliations locally generated by conformal vector fields

From results of Section 2 and 3 we obtain necessary and sufficient conditions for a foliation to be homothetic.

Corollary 2.4.1. Let $\mathscr{V}$ be a conformal foliation on ( $M, g$ ) with integrable orthogonal complement $\mathscr{H}$. If both $\mathscr{V}$ and $\mathscr{H}$ have basic mean curvature forms then $\mathscr{V}$ is a homothetic foliation. If, further, $\operatorname{codim} \mathscr{V} \neq 2$ and $\operatorname{dim} M \geq 3$ then $\mathscr{V}$ produces harmonic morphisms.

Proof. This is an immediate consequence of Theorem 2.1.19.
Proposition 2.4.2. Let $\mathscr{V}$ be a foliation of $\operatorname{codim} \mathscr{V}=n$ on $\left(M^{m}, g\right)$ which is locally generated by conformal vector fields and let $\mathscr{H}$ denote its orthogonal complement.

If $\mathscr{V}$ is homothetic then $g\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right),{ }^{\mathscr{H}} I\right)=0$.
Conversely, if $g\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right),{ }^{\mathscr{H}} I\right)=0$ then the following assertions are equivalent:
(i) $\mathscr{V}$ is a homothetic foliation;
(ii) The mean curvature form of $\mathscr{V}$ is basic;
(iii) The mean curvature form of $\mathscr{H}$ is invariant under the parallel displacement determined by $\mathscr{H}$;
(iv) In the neighbourhood of each point of $M$ there a exists a local dilation $\lambda$ of $\mathscr{V}$ such that for any horizontal vector $X$ and conformal vector field $V$ tangent to $\mathscr{V}$ we have $X(V(\log \lambda))=0$.

Proof. The first assertion is an immediate consequence of formula 2.1.1 from Proposition 2.1.4.

Suppose now that $g\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right),{ }^{\mathscr{L}} I\right)=0$.
The equivalence (i) $\Longleftrightarrow$ (iii) follows from Proposition 2.1.4.
From the proof of Theorem 2.2.6 it follows that

$$
\begin{equation*}
\frac{1}{m-n} \stackrel{V}{\nabla}_{V}\left(\operatorname{trace}\left({ }^{\mathscr{V}} B\right)^{\mathrm{b}}\right)(X)=X(V(\log \lambda))=\stackrel{\mathscr{H}}{\nabla}_{V}\left(\mathscr{H}^{*}(\mathrm{~d} \log \lambda)\right)(X) \tag{2.4.1}
\end{equation*}
$$

for any $X \in \Gamma(\mathscr{H})$, any conformal vector field $V \in \Gamma(\mathscr{V})$ and any local dilation $\lambda$ of $\mathscr{V}$. The first equality of (2.4.1) implies that (ii) $\Longleftrightarrow$ (iv).

From the second equality of (2.4.1) and (2.1.2) of Proposition 2.1.4 we obtain the equivalence (iii) $\Longleftrightarrow$ (iv) and the proposition is proved.

Theorem 2.4.3. Let $\mathscr{V}$ be a foliation of $\operatorname{codim} \mathscr{V} \neq 2$ on $\left(M^{m}, g\right), m \geq 3$, which is locally generated by conformal vector fields. Then any two of the following assertions imply the other one.
(i) $\mathscr{V}$ produces harmonic morphisms;
(ii) $\mathscr{V}$ is homothetic;
(iii) $\operatorname{trace}\left(\operatorname{ad}\left({ }^{\mathscr{H}} I\right)\right)=0$.

Proof. This is a consequence of Proposition 2.4 .2 and of Theorem 2.2.6.

Remark 2.4.4. 1) There is another way to prove that in Theorem 2.4 .3 if (ii) holds then (i) $\Longleftrightarrow$ (iii). To see this let $\lambda$ be a local dilation of $\mathscr{V}$ which is constant along horizontal curves. Then, with respect to $\lambda^{2} g, \mathscr{V}$ is a Riemannian foliation locally generated by Killing fields. Thus, condition (ii) of Theorem 2.2.6 says the same thing when written for a local frame made up of fields which are conformal with respect to $g$ and when written for the same
frame with the metric $\lambda^{2} g$ so that the fields are now Killing fields. Moreover, since $\lambda$ is constant along horizontal curves, $\mathscr{V}$ produces harmonic morphisms with respect to $g$ if and only if it produces harmonic morphisms with respect to $\lambda^{2} g$ (Corollary 1.2.2). The proof of the theorem now follows from Corollary 2.3.1 (which can be proved directly).
2) It is not difficult to see using Theorem 2.4.3 that the following classes of foliations of codimension not equal to two produce harmonic morphisms:

- homothetic foliations locally generated by conformal fields and with integrable orthogonal complement;
- homothetic foliations generated by the local action of an Abelian Lie group of conformal transformations;
- homothetic foliations generated by the action of a unimodular closed subgroup of the group of conformal transformations;
- homothetic foliations formed by the components of the fibres of principal bundles for which the total space is endowed with a metric such that the structural group acts by conformal transformations and the connection induced on the determinant bundle of the adjoint bundle is flat.

From Theorem 2.4.3 we obtain the following.
Corollary 2.4.5. Let $\mathscr{V}$ be a foliation of $\operatorname{codim} \mathscr{V} \neq 2$ on $\left(M^{m}, g\right), m \geq 3$ which is locally generated by conformal vector fields. Suppose that the orthogonal complement $\mathscr{H}$ of $\mathscr{V}$ is integrable. Then the following assertions are equivalent.
(i) $\mathscr{V}$ produces harmonic morphisms;
(ii) $\mathscr{V}$ is homothetic.

Remark 2.4.6. We shall see (Proposition 3.1.5) that if $\operatorname{dim} \mathscr{V}=1$ then (i) and (ii) of Corollary 2.4 .5 imply that $\mathscr{V}$ is locally generated by conformal vector fields.

Next we give a construction of a foliation which produces harmonic morphisms which has basic mean curvature form but is nowhere homothetic.

Example 2.4.7. Let $\varphi:\left(M^{n+1}, h\right) \rightarrow\left(N^{n}, \bar{h}\right), n \geq 1$, be a Riemannian submersion with geodesic fibres and let $\mathscr{V}$ be the foliation formed by the fibres of $\varphi$.

Suppose that $V$ is a local vertical field such that $h(V, V)=1$. Because $\varphi$ has geodesic leaves we have that $[V, X]=0$ for any basic $X$.

Let $\theta=V^{b}$ and $\Omega=\mathrm{d} \theta$. It is easy to see that $\Omega=0$ if and only if the horizontal distribution $\mathscr{H}$ is integrable. Also, $\Omega$ is basic and since $\mathrm{d} \Omega=\mathrm{dd} \theta=0$, at least locally, we can find a basic one-form $A$ such that $\Omega=-\mathrm{d} A$.

Thus $\mathrm{d} \theta=\Omega=-\mathrm{d} A$ and hence $\mathrm{d}(A+\theta)=0$. It follows that, at least locally, we can write $A+\theta=\mathrm{d} \sigma$ for some smooth local function $\sigma$ on $M$. Note that the horizontal component of $\mathrm{d} \sigma$ is basic, being equal to $A$.

Supposing that $\sigma$ is defined on the whole $M$, let $g^{\sigma}$ be the Riemannian metric on $M$ defined by

$$
g^{\sigma}=\mathrm{e}^{-2 \sigma} \varphi^{*}(\bar{h})+\mathrm{e}^{(2 n-4) \sigma} \theta^{2} .
$$

Then $\varphi:\left(M, g^{\sigma}\right) \rightarrow(N, \bar{h})$ is a harmonic morphism [14]. Moreover, the mean curvature form of $\mathscr{V}$ with respect to $g^{\sigma}$ is $(2-n) A$ and therefore is basic. However, from Proposition 2.1.4 and Proposition 2.1.14 it follows that the connected components of the fibres of $\varphi$ form a homothetic foliation with respect to $g^{\sigma}$ only over the set of points where the horizontal distribution is integrable. Thus if $\mathscr{H}$ is nowhere integrable then $\mathscr{V}$ is nowhere homothetic with respect to $g^{\sigma}$.

Let $\rho$ be any other function which has the same properties as $\sigma$ (i.e. $\varphi:\left(M, g^{\rho}\right) \rightarrow(N, \bar{h})$ is a harmonic morphism, the mean curvature form of $\mathscr{V}$ with respect to $g^{\rho}$ is basic and $\mathscr{V}$ is nowhere homothetic on $\left.\left(M, g^{\rho}\right)\right)$. Then, there exists a unique constant $c \in \mathbb{R}$ such that $\rho-c \sigma$ is, at least locally, a basic function.

To see this note that because the induced foliation is nowhere homothetic then we must have:
(i) $M=\{x \in M \mid V(V(\rho))=0\}$ (otherwise on some open subset of $M$ the level hypersurfaces of $V(\rho)$ would be integral submanifolds of the horizontal
distribution);
(ii) the interior of the set $\{x \in M \mid V(\rho)=0\}$ is empty (otherwise the restriction of $\mathscr{V}$ to some open subset of $M$ would be Riemannian).

Thus we have $V(\rho)=c$, for some constant $c \neq 0$. Hence $\mathrm{d} \rho=c \theta+B$.
Then $B$ must be basic (because $X(V(\rho))=0$ for any horizontal $X$ ) and hence $0=c \mathrm{~d} \theta+\mathrm{d} B$ which is equivalent to $\mathrm{d} B=-c \Omega$.

It follows that $\mathrm{d}(\rho-c \sigma)=\mathrm{d} \rho-c \mathrm{~d} \sigma=c \theta+B-c \theta-c A=B-c A$.
Because $B-c A$ is a closed basic one-form, at least locally, we can find a basic function whose differential is equal to $\mathrm{d}(\rho-c \sigma)$ and hence $\rho-c \sigma$ is, at least locally, a basic function.

### 2.5. Homothetic actions and harmonic morphisms

Recall that a vector field $V$ on a Riemannian manifold ( $M, g$ ) is homothetic if $\mathcal{L}_{V} g=a g$ for some constant $a \in \mathbb{R}$ (see [70]).

The first thing to note about a foliation locally generated by homothetic vector fields is the following.

Proposition 2.5.1. Let $\mathscr{V}$ be a foliation on ( $M, g$ ) locally generated by homothetic vector fields. Then either $\mathscr{V}$ is Riemannian and locally generated by Killing vector fields or $\mathscr{V}$ is nowhere Riemannian.

Proof. Let $P=\{x \in M \mid \mathscr{V}$ is Riemannian at $x\}$.
It is obvious that $P$ is closed. Also let $\left\{V_{r}\right\}$ be a local frame of $\mathscr{V}$, defined on a connected open subset $U$ and made up of homothetic vector fields. It is obvious that if $U \cap P \neq \emptyset$ then $V_{r}$ are Killing fields and thus $U \subseteq P$. Hence $P$ is also open and, since $M$ is connected, either $P=M$ or $P=\emptyset$.

From Proposition 2.4.2 we obtain the following.
Corollary 2.5.2. Let $\mathscr{V}$ be a foliation locally generated by homothetic vector fields on the Riemannian manifold $(M, g)$.

Then $\mathscr{V}$ is a homothetic foliation if and only if $g\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right),{ }^{\mathscr{H}} I\right)=0$.
In particular, a foliation locally generated by homothetic vector fields and with integrable orthogonal complement is a homothetic foliation.

Proof. Let $V$ be a homothetic vector field on $(M, g)$ which is tangent to the foliation. Then it is easy to see that $\mathcal{L}_{V} g=-2 V(\log \lambda) g$ where $\lambda$ is any local dilation of the foliation. But $V$ is homothetic and hence $V(\log \lambda)$ is a constant function. The proof now follows from (i) $\Longleftrightarrow$ (iv) of Proposition 2.4.2.

Remark 2.5.3. Let $\mathscr{V}$ be a conformal foliation on $(M, g)$ and define the vertical one-form $\mu$ by the relation $\left(\mathcal{L}_{V} g\right)(X, Y)=\mu(V) g(X, Y)$ where $V$ is vertical and $X, Y$ are horizontal [66] (see [9]). Then $\mu=-2 \mathscr{V}^{*}(\mathrm{~d}(\log \lambda))$ where $\lambda$ is a local dilation of $\mathscr{V}$, and, because trace $\left({ }^{\mathscr{H}} B\right)=n \mathscr{V}(\operatorname{grad}(\log \lambda))$, we have $\mu=-\frac{2}{n} \operatorname{trace}\left({ }^{\mathscr{H}} B\right)^{b}$ (see [9]).

By Corollary 2.5.2, if $\mathscr{V}$ is locally generated by homothetic vector fields then $\mathscr{V}$ is a homothetic foliation if and only if $\mu\left({ }^{\mathscr{H}} I\right)=0$.

Proposition 2.5.4. Let $\mathscr{V}$ be a one-dimensional foliation of $\operatorname{codim} \mathscr{V} \neq 2$ on $\left(M^{m}, g\right), m \geq 3$, which is not a Riemannian foliation and which is locally generated by homothetic vector fields. Then the following assertions are equivalent:
(i) $\mathscr{V}$ produces harmonic morphisms,
(ii) $\mathscr{V}$ is a homothetic foliation,
(iii) $\mathscr{H}$ is integrable.

Proof. It is easy to see that because $\mathscr{V}$ is locally generated by homothetic vector fields its mean curvature form is basic. The proof now follows from Theorem 2.2.6 and Corollary 2.5.2.

Note that in Proposition 2.5 .4 the equivalence (ii) $\Longleftrightarrow$ (iii) holds also when $\operatorname{codim} \mathscr{V}=2$.

Proposition 2.5.5 (cf. [66, Proposition 2.8]). Let $\mathscr{V}$ be a foliation on ( $M^{m}, g$ ) locally generated by homothetic vector fields. Then there exists a Riemannian foliation $\mathscr{W} \subseteq \mathscr{V}$ locally generated by Killing fields. Moreover, if $\mathscr{V}$ is not Riemannian then $\operatorname{dim} \mathscr{V}=\operatorname{dim} \mathscr{W}+1$.

Proof. Suppose that $\mathscr{V}$ is not Riemannian. Then by Proposition 2.5.1 the foliation $\mathscr{V}$ is nowhere Riemannian. Since $\mathscr{V}$ is conformal we can find local
dilations of it in the neighbourhood of each point. Let $\lambda$ be a local dilation of $\mathscr{V}$ defined on the open subset $U \subseteq M$. For $x \in U$ let

$$
\mathscr{W}_{x}=\left\{V \in \mathscr{V}_{x} \mid V(\log \lambda)=0\right\}=\mathscr{V}_{x} \cap\left(\operatorname{grad}(\log \lambda)_{x}\right)^{\perp} .
$$

Since any two local dilations of $\mathscr{V}$ differ locally by a factor which is constant along the leaves it follows that $\mathscr{W}_{x}$ does not depend on $\lambda$. Because $\mathscr{V}$ is nowhere Riemannian, $\mathscr{W}_{x} \neq \mathscr{V}_{x}$. Also $\operatorname{grad}(\log \lambda)$ is nonvanishing and hence $\operatorname{dim}\left(\left(\operatorname{grad}(\log \lambda)_{x}\right)^{\perp}\right)=m-1$ where $m=\operatorname{dim} M$. We have

$$
\begin{aligned}
\operatorname{dim} \mathscr{W}_{x} & =\operatorname{dim}\left(\mathscr{V}_{x} \cap\left(\operatorname{grad}(\log \lambda)_{x}\right)^{\perp}\right) \\
& =\operatorname{dim} \mathscr{V}_{x}+\operatorname{dim}\left(\left(\operatorname{grad}(\log \lambda)_{x}\right)^{\perp}\right)-\operatorname{dim}\left(\mathscr{V}_{x}+\left(\operatorname{grad}(\log \lambda)_{x}\right)^{\perp}\right) .
\end{aligned}
$$

It follows that the minimum value of $\operatorname{dim} \mathscr{W}_{x}$ occurs precisely when $\mathscr{V}_{x}+$ $\left(\operatorname{grad}(\log \lambda)_{x}\right)^{\perp}=T_{x} M$. If this is the case, then $\operatorname{dim} \mathscr{W}_{x}=\operatorname{dim} \mathscr{V}_{x}+(m-1)-$ $m=\operatorname{dim} \mathscr{V}_{x}-1$. Since $\mathscr{W}_{x} \subset \mathscr{V}_{x}, \mathscr{W}_{x} \neq \mathscr{V}_{x}$ it follows that $\operatorname{dim} \mathscr{W}_{x}=\operatorname{dim} \mathscr{V}_{x}-1$. Thus $\mathscr{W}=\left(\mathscr{W}_{x}\right)_{x \in M}$ defines a distribution on $M$. Then $\mathscr{W}$ is integrable because it is the intersection of two transversal foliations.

Let $V \in \Gamma(\mathscr{V})$ be a homothetic vector field. It is easy to see that if $V_{x} \in \mathscr{W}_{x}$ then $V \in \Gamma(\mathscr{W})$. Since $\mathscr{V}$ is locally generated by homothetic vector fields it follows that $\mathscr{W}$ is locally generated by Killing fields. (This also implies that $\mathscr{W}$ is integrable since any Killing field which is tangent to $\mathscr{V}$ must be tangent to $\mathscr{W}$ and the bracket of any two Killing fields is a Killing field.)

We can now characterise geometrically the homothetic (infinitesimal) actions which induce homothetic foliations and their relations with harmonic morphisms.

Theorem 2.5.6. Let $\mathscr{V}$ be a foliation locally generated by homothetic vector fields and let $\mathscr{H}$ be its orthogonal complement.

Then the following assertions are equivalent:
(a) $\mathscr{V}$ is a homothetic foliation;
(b) either $\mathscr{V}$ is Riemannian and locally generated by Killing fields or there exists a Riemannian foliation $\mathscr{W} \subseteq \mathscr{V}$ locally generated by Killing fields such that $\operatorname{dim} \mathscr{V}=\operatorname{dim} \mathscr{W}+1$ and the distribution $\mathscr{F}=\mathscr{W} \oplus \mathscr{H}$ is integrable.

Moreover, if (a) or (b) hold and $\operatorname{dim} \mathscr{V} \geq 2$, codim $\mathscr{V} \geq 3$ then the
following assertions are equivalent:
(i) $\mathscr{V}$ produces harmonic morphisms,
(ii) The restriction of $\mathscr{W}$ to any leaf of $\mathscr{F}$ produces harmonic morphisms.

Proof. The equivalence (a) $\Longleftrightarrow$ (b) follows from Corollary 2.5.2 and Proposition 2.5.5.

Suppose that (a) or (b) hold and $\mathscr{V}$ is not Riemannian. Let $V \in \Gamma(\mathscr{V})$ be a homothetic vector field which is not Killing. (Such a vector field can be found in the neighbourhood of each point of $M$ because $\mathscr{V}$ is locally generated by homothetic vector fields and $\mathscr{W} \neq \mathscr{V}$.) Note that, for any Killing field $W \in \Gamma(\mathscr{W})$ we have that $[V, W]$ is also Killing and hence $[V, W] \in \Gamma(\mathscr{W})$. Using this fact together with Theorem 2.2.6 and Corollary 2.5 .2 it is not difficult to see that the assertions (i) and (ii) are equivalent.

Remark 2.5.7.1) If $\mathscr{V}$ is homothetic then the leaves of $\mathscr{F}$ are level hypersurfaces of the local dilations of $\mathscr{V}$ which are constant along horizontal curves.
2) In Theorem 2.5.6 if (a) or (b) hold and codim $\mathscr{V}=1$ then (i) $\Longleftrightarrow$ (ii).

Let $G$ be a Lie group which acts to the right by homotheties on $(M, g)$ and for $a \in G$ let $\rho(a) \in(0, \infty)$ be the conformal factor of the homothetic transformation induced by $a \in G$ on $(M, g)$. Then it is easy to see that $\rho: G \rightarrow(0, \infty)$ is a morphism of Lie groups (hence, if $\rho$ is nonconstant, $G$ is isomorphic to a semi-direct product of $\operatorname{ker} \rho$ and $((0, \infty), \cdot))$. In particular, if $G$ is compact then $\rho$ is constant. Nevertheless, if $G$ is compact, then there might exist local morphisms of Lie groups $G \rightarrow(0, \infty)$ (see Example 2.5 .8 (3), below) which can be used to construct homothetic local actions.

Here are a few well-known examples of morphisms of Lie groups $\rho: G \rightarrow$ $(0, \infty)$.

Example 2.5.8. 1) For $\mathbb{K}=\mathbb{R}, \mathbb{C}, \mathbb{H}$ define $\rho: G l_{n}(\mathbb{K}) \rightarrow(0, \infty)$ by $\rho(a)=$ $|\operatorname{det} a|$.
2) For $\mathbb{K}=\mathbb{R}, \mathbb{C}, \mathbb{H}$ define $\rho: C O_{n}(\mathbb{K}) \rightarrow(0, \infty)$ by $|A u|=\rho(A)|u|$ for $u \in \mathbb{K}^{n}$ and $A \in C O_{n}(\mathbb{K})$.
3) The canonical morphisms of Lie groups det : $U_{n} \rightarrow S^{1}$ and $S p i n_{n}^{\mathrm{c}}=$
$\operatorname{Spin}_{n} \times_{\mathbb{Z}_{2}} S^{1} \rightarrow S^{1},[a, z] \mapsto z^{2}$, when composed with the exponential of $\arg : S^{1} \backslash\{-1\} \rightarrow(-\pi, \pi)$ induce local morphisms of Lie groups $U_{n} \rightarrow(0, \infty)$ and Spinin $_{n}^{\mathrm{c}} \rightarrow(0, \infty)$, respectively.

From now on we shall suppose that $G$ acts freely on $M$. In this case there exists a natural isomorphism of vector bundles $\mathscr{V}=M \times \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of $G$.

Hence ${ }^{\mathscr{H}} I$ can be viewed as a $\mathfrak{g}$-valued two form on $M$ which has properties similar to the properties of the curvature form of a principal connection (in particular, $R_{a}^{*}\left({ }^{\mathscr{H}} I\right)=\operatorname{Ad} a^{-1} \cdot{ }^{\mathscr{H}} I$ where $R_{a}$ is the transformation induced by $a \in G$ on $M$ ).

Also $\rho_{*}$ can be viewed as a vertical one form on $M$. Moreover, we have that $\rho_{*}=\mu$ (see Remark 2.5.3 for the definition of $\mu$ ).

It follows from Corollary 2.5.2 that the foliation induced by the free action of $G$ on $(M, g)$ is homothetic if and only if $\rho_{*}\left({ }^{\mathscr{H}} I\right)=0$.

By identifying $G$ with an orbit we can induce on it a metric which we shall denote by $\gamma$. Then it is easy to see that $\rho^{-2} \gamma$ is right invariant.

Suppose that $\rho$ is nonconstant and let $\mathscr{V}$ be the foliation on $G$ formed by the components of the fibres of $\rho$. This is generated by the action of the normal subgroup $H=\operatorname{ker} \rho$. Then it is obvious that $H$ acts by isometries on $(G, \gamma)$ and hence $\mathscr{V}$ is a Riemannian foliation on it.

Also, $\mathscr{H}\left(=\mathscr{V}^{\perp}\right)$ is a (one-dimensional) homothetic foliation with geodesic leaves for which $\rho^{-1}$ is a global dilation.

Thus both $\mathscr{V}$ and $\mathscr{H}$ produce harmonic morphisms and, in particular, $\rho$ induces a harmonic function on $(G, \gamma)$ (which gives another argument for the fact that if $G$ is compact then $\rho$ cannot be globally defined unless it is constant).

Example 2.5.9. Let $G$ and $\rho$ be as in Example 2.5.8(1) or (2) and define $h=\left|\mathrm{d} x \cdot x^{-1}\right|^{2}$. Then $g=\rho^{2} h$ has all the above properties.

Next we show that the results of Theorem 2.5.6 takes a more concrete form in the case of homothetic free actions.

Proposition 2.5.10. Let $G$ be a connected Lie group which acts freely to the right by homotheties on $(M, g)$ and let $\mathscr{V}^{G}$ be the induced foliation. Let $\rho$ : $G \rightarrow(0, \infty)$ be the corresponding morphism of Lie groups and let $H=\operatorname{ker} \rho$.

Then the following assertions are equivalent:
(i) $\mathscr{V}^{G}$ is a homothetic foliation;
(ii) there exists a hypersurface $N$ of $M$ such that $H$ acts by isometries on $\left(N,\left.g\right|_{N}\right)$ to generate a Riemannian foliation $\mathscr{V}^{H}$ and such that $M=N \times_{H} G$.

Further, if (i) or (ii) hold and $2 \leq \operatorname{dim} G \leq \operatorname{dim} M-3$, then $\mathscr{V}^{G}$ produces harmonic morphisms if and only if $\mathscr{V}^{H}$ produces harmonic morphisms.

Proof. Let $\mathscr{H}$ be the orthogonal complement of $\mathscr{V}^{G}$ and let $\mathscr{W}$ be the foliation induced by the isometric (free) action of $H$ on $M$. Then, assertion (i) is equivalent to the fact that $\rho_{*}\left({ }^{\mathscr{H}} I\right)=0$ which, since ${ }^{\mathscr{H}} I$ is the integrability tensor of $\mathscr{H}$, is equivalent to the fact that the distribution $\mathscr{F}=\mathscr{W} \oplus \mathscr{H}$ is integrable.

Suppose that (i) holds and let $N$ be a leaf of $\mathscr{F}$. Then $\mathscr{V}^{H}=\left.\mathscr{W}\right|_{N}$ and the implication (i) $\Rightarrow$ (ii) follows.

The implication (i) $\Leftarrow$ (ii) is now obvious.
The last assertion follows from the fact that if (i) holds then $\mathscr{H}_{I}$ is $\mathfrak{h}$ valued and $R_{a}^{*}\left({ }^{\mathscr{H}} I\right)=\operatorname{Ad} a^{-1} \cdot{ }^{\mathscr{H}} I$.

Remark 2.5.11. In Proposition 2.5.10 we also have that if (i), or (ii), holds then $g$ is determined by $\rho$ and the induced metric $h$ on $N$.

To see this recall that we have considered the metric $\gamma$ induced on $G$ by identifying it with an orbit. Suppose that this identification and $N$ were chosen such that the identity element of $G$ is contained in $N$. Then $H$ acts by isometries on $\left(N \times G, \pi_{N}^{*}(h)+\pi_{G}^{*}(\gamma)\right)$ (where $\pi_{N}: N \times G \rightarrow N$ and $\pi_{G}: N \times G \rightarrow G$ are the canonical projections) and ( $M, g$ ) is the induced isometric quotient.

Example 2.5.12. Let $\rho: G \rightarrow(0, \infty)$ be as in Example 2.5 .8 and let $H=$ $\operatorname{ker} \rho$. Then trace $(\operatorname{ad} \mathfrak{h})=0$ (here, as above, $\mathfrak{h}$ is the Lie algebra of $H$ ).

Let $(Q, M, H)$ be a principal bundle whose total space is endowed with a Riemannian metric $h$ such that $H$ acts by isometries on ( $Q, h$ ). (Obviously,
any such $h$ corresponds to a triple $(\gamma, \mathscr{H}, k)$ where $\gamma$ is a Riemannian metric on the vector bundle $\operatorname{Ad} Q \rightarrow M, \mathscr{H}$ is a principal connection on $(Q, M, H)$ and $k$ is a Riemannian metric on $M$.)

From Remark 2.5.11 it follows that a metric (and just a local metric, for $\rho$ from Example 2.5.8(3) ) can be found on $P=Q \times_{H} G$ with respect to which the foliation induced by $G$ is homothetic (but not Riemannian) and produces harmonic morphisms.

### 2.6. Conformal actions and harmonic morphisms on Einstein manifolds

In this section we study foliations which are locally generated by conformal vector fields and produce harmonic morphisms on Einstein manifolds. Note that, as before, no compactness or completeness assumptions are made. The main results of this section are the following:

Theorem 2.6.1. Let $\left(M^{m}, g\right), m \geq 3$ be an Einstein manifold ( ${ }^{M}$ Ricci $=$ $c g, c \in \mathbb{R})$. Let $\mathscr{V}$ be a foliation with $\operatorname{codim} \mathscr{V} \neq 2$ which is locally generated by conformal vector fields.

Suppose that $\mathscr{V}$ produces harmonic morphisms on $\left(M^{m}, g\right)$. Then either $\mathscr{V}$ is Riemannian and locally generated by Killing fields or the set of points where $\mathscr{V}$ is Riemannian has empty interior. Moreover, we have the following:
(i) If $c>0$ then either $\mathscr{V}$ is Riemannian and locally generated by Killing vector fields or any harmonic morphism produced by $\mathscr{V}$ can be locally decomposed into a harmonic morphism with geodesic fibres and integrable horizontal distribution followed by another harmonic morphism.
(ii) If $c<0$ then, at least outside the points where $\mathscr{V}$ is Riemannian, any harmonic morphism produced by $\mathscr{V}$ can be locally decomposed into a harmonic morphism with geodesic fibres and integrable horizontal distribution followed by another harmonic morphism.
(iii) If $c=0$ then either $\mathscr{V}$ is locally generated by homothetic vector fields or any harmonic morphism produced by $\mathscr{V}$ can be locally decomposed into a harmonic morphism with geodesic fibres, constant dilation and integrable horizontal distribution followed by another harmonic morphism.

Corollary 2.6.2. Let $\left(M^{m}, g\right), m \geq 3$ be an Einstein manifold ( ${ }^{M}$ Ricci $\left.=c g\right)$ and let $\mathscr{V}$ be a foliation on it with codim $\mathscr{V} \neq 2$ which is locally generated by conformal vector fields.

Suppose that $\mathscr{V}$ produces harmonic morphisms on $\left(M^{m}, g\right)$.
(i) If $c>0$ then any harmonic morphism produced by $\mathscr{V}$ can be locally decomposed into two harmonic morphisms in which the first one either has geodesic fibres and integrable horizontal distribution or is induced by an isometric quotient.
(ii) If $c<0$ then at least outside a set with empty interior any harmonic morphism produced by $\mathscr{V}$ can be locally decomposed into two harmonic morphisms in which the first one either has geodesic fibres and integrable horizontal distribution or is induced by an isometric quotient.

Remark 2.6.3. We shall see (Proposition 3.2.4) that when $\operatorname{dim} \mathscr{V}=1$ the assertion (i) above holds for $c \in \mathbb{R}$.

Theorem 2.6.4. Let $\left(M^{m}, g\right)$ be a Ricci-flat Riemannian manifold and let $\mathscr{V}$ be a homothetic foliation on it with $\operatorname{dim} \mathscr{V} \geq 2$, codim $\mathscr{V} \geq 3$ which is locally generated by conformal vector fields.

Suppose that $\mathscr{V}$ produces harmonic morphisms on $\left(M^{m}, g\right)$. Then one of the following assertions holds.
(a) $\mathscr{V}$ is Riemannian and locally generated by Killing fields;
(b) there exists a Riemannian foliation $\mathscr{W} \subseteq \mathscr{V}$, $\operatorname{dim} \mathscr{W}=\operatorname{dim} \mathscr{V}-1$, locally generated by Killing fields such that $\mathscr{F}=\mathscr{W} \oplus \mathscr{H}$ is integrable and the restriction of $\mathscr{W}$ to any leaf of $\mathscr{F}$ produces harmonic morphisms;
(c) any harmonic morphism produced by $\mathscr{V}$ can be locally decomposed into two harmonic morphisms in which the first one has geodesic fibres, constant dilation and integrable horizontal distribution.

Corollary 2.6.5. Let $\left(M^{m}, g\right)$ be a Ricci-flat Riemannian manifold and let $\mathscr{V}$ be a foliation on it with $\operatorname{dim} \mathscr{V} \geq 2$, codim $\mathscr{V} \geq 3$ which is locally generated by conformal vector fields and with integrable orthogonal complement, $\operatorname{dim} \mathscr{V} \geq 2$, $\operatorname{codim} \mathscr{V} \geq 3$.

Suppose that $\mathscr{V}$ produces harmonic morphisms on $\left(M^{m}, g\right)$. Then one of
the following assertions holds.
(a) $\mathscr{V}$ is Riemannian and locally generated by Killing fields;
(b) there exists a Riemannian foliation $\mathscr{W} \subseteq \mathscr{V}$, $\operatorname{dim} \mathscr{W}=\operatorname{dim} \mathscr{V}-1$, locally generated by Killing fields such that $\mathscr{F}=\mathscr{W} \oplus \mathscr{H}$ is integrable and the restriction of $\mathscr{W}$ to any leaf of $\mathscr{F}$ produces harmonic morphisms;
(c) any harmonic morphism produced by $\mathscr{V}$ can be locally decomposed into two harmonic morphisms in which the first one has geodesic fibres, constant dilation and integrable horizontal distribution.

The proofs of the above results are based on results obtained in the previous sections. We also need a few lemmas some of which are well-known.

In the proof of the next simple lemma we use a result (Proposition 3.1.5) which we shall prove in Chapter 3. Also, note that Lemma 2.6.6 reformulates, in terms of homothetic foliations, a well-known fact.

Lemma 2.6.6 (cf. [37]). Let $\mathscr{V}$ be a one-dimensional foliation on a Riemannian manifold $(M, g)$. Then the following assertions are equivalent.
(i) $\mathscr{V}$ is a homothetic foliation with geodesic leaves and integrable orthogonal complement;
(ii) $\mathscr{V}$ is locally generated by (nowhere zero) conformal vector fields $V \in$ $\Gamma(\mathscr{V})$ such that $\mathrm{d} V^{b}=0$.

Proof. A vector field $V \in \Gamma(T M)$ is conformal if and only if $\mathcal{L}_{V} g=2 \mu g$ for some function $\mu$; then $\mathrm{d} V^{b}=0$ if and only if $\nabla V=\mu I d_{T M}$. Now it is obvious that any such $V$ which is nowhere zero generates a conformal foliation with geodesic leaves and integrable orthogonal complement. Moreover, $|V|^{-1}$ is a local dilation for it whose gradient is tangent to the leaves and thus $V$ generates a homothetic foliation.

Conversely, if $\mathscr{V}$ satisfies (i) then it produces harmonic morphisms. But $\mathscr{V}$ is homothetic and hence $\mathscr{V}$ is locally generated by conformal vector fields (see Proposition 3.1.5, below). Now, if $V \in \Gamma(\mathscr{V})$ is conformal and nowhere zero and $X \in \Gamma(\mathscr{H})$ is basic then by applying Lemma 2.1.1 we obtain

$$
\begin{equation*}
\left(\mathrm{d} V^{\mathrm{b}}\right)(V, X)=2 g\left(X, \nabla_{V} V\right)=2 g\left(X, \nabla_{U} U\right) g(V, V)=0 \tag{2.6.1}
\end{equation*}
$$

where $U=\frac{1}{|V|} V$. Also, from Proposition 2.1 .4 we obtain

$$
\begin{equation*}
\left(\mathrm{d} V^{\mathrm{b}}\right)(X, Y)=g\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right),{ }^{\mathscr{H}} I(X, Y)\right)=0 \tag{2.6.2}
\end{equation*}
$$

From (2.6.1) and 2.6 .2 it follows that $\mathrm{d} V^{b}=0$ and the lemma is proved.
The following lemma is well-known.

Lemma 2.6.7 (see 70). Let $V$ be a conformal vector field on an Einstein manifold $\left(M^{m}, g\right) ;$ write $\mathcal{L}_{V} g=2 \sigma g,{ }^{M}$ Ricci $=c g$. Then

$$
\begin{equation*}
\nabla \mathrm{d} \sigma=-\frac{c}{m-1} \sigma g \tag{2.6.3}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\Delta \sigma=\frac{c m}{m-1} \sigma . \tag{2.6.4}
\end{equation*}
$$

Proof. Formula (2.6.3 follows after a straightforward but tedious computation (see [70]).

The following simple lemma is well-known.
Lemma 2.6.8. Let $f$ be a smooth function on a Riemannian manifold ( $M, g$ ) such that $\nabla \mathrm{d} f=-k f g$ for some constant $k \in \mathbb{R}$. Then

$$
k f^{2}+|\mathrm{d} f|^{2}=\text { constant }
$$

Proof. Simply compute the differential $\mathrm{d}\left(k f^{2}+|\mathrm{d} f|^{2}\right)$.
The following simple lemma seems to be less well-known.

Lemma 2.6.9. Let $\mathscr{V}$ be a foliation (codim $\mathscr{V}>0)$ on a Riemannian manifold $(M, g)$ and let $V \in \Gamma(\mathscr{V})$ be such that $\nabla V^{b}=\mu g$ for some smooth function $\mu$ on $M$.

If for some $x \in M$ we have that $V_{x}=0$ then $\mu(x)=0$.
Proof. Let $X \in \Gamma\left(\mathscr{V}^{\perp}\right)$ be a basic vector field. Then $[V, X] \in \Gamma(\mathscr{V})$. But

$$
[V, X]=\nabla_{V} X-\nabla_{X} V=\nabla_{V} X-\mu X
$$

Thus if $V_{x}=0$ then $[V, X]_{x}=-\mu(x) X_{x} \in \mathscr{V}_{x}$ and hence $\mu(x)=0$.

The following simple lemma is an immediate consequence of Lemma 2.6.8 and Lemma 2.6.9.

Lemma 2.6.10. Let $\mathscr{V}$ be a foliation on a Riemannian manifold ( $M, g$ ) and let $\operatorname{grad} f \in \Gamma(\mathscr{V})$ be such that $\nabla \mathrm{d} f=-k f g$ for some nonnegative constant $k \geq 0$.

If for some $x \in M$ we have that $(\operatorname{grad} f)_{x}=0$ then $\operatorname{grad} f=0$. Moreover, if $k>0$ then $f=0$.

Proof. If $k=0$ then $|\operatorname{grad} f|=$ constant.
If $k>0$ and for some $x \in M$ we have that $(\operatorname{grad} f)_{x}=0$ then by Lemma 2.6.9 we have $f(x)=0$. The proof now follows from Lemma 2.6.8.

Proof of Theorem 2.6.1. Let $\mathscr{H}$ be the orthogonal complement of $\mathscr{V}$. Let $V \in \Gamma(\mathscr{V})$ be a conformal vector field. Then at least locally we can write $\mathcal{L}_{V} g=-2 V(\log \lambda) g$ for some local dilation $\lambda$ of $\mathscr{V}$. By Lemma 2.1.1 we have that $V$ is an infinitesimal automorphism of $\mathscr{H}$.

Because $\mathscr{V}$ produces harmonic morphisms from Theorem 2.2 .6 it follows that the mean curvature form of $\mathscr{V}$ is basic. Applying Proposition 2.1.16 with the roles of $\mathscr{V}$ and $\mathscr{H}$ reversed we obtain that, for any basic vector field $X \in$ $\Gamma(\mathscr{H})$, we have that $X(V(\log \lambda))=0$. It follows that $\operatorname{grad}(V(\log \lambda)) \in \Gamma(\mathscr{V})$.

Now recall that $(M, g)$ is an Einstein manifold and thus it is an analytic manifold (see [11]). From the regularity of solutions for elliptic operators and 2.6.4 it follows that $V(\log \lambda)$ is an analytic function. Hence either $\mathscr{V}$ is Riemannian or the interior of the set where $\mathscr{V}$ is Riemannian is empty.

From Lemma 2.6.6 and Lemma 2.6.7 it follows that if $\operatorname{grad}(V(\log \lambda))$ is nowhere zero then it generates a one-dimensional homothetic foliation $\mathscr{F}$ with geodesic leaves and integrable orthogonal complement. Moreover $\mathscr{F} \subseteq \mathscr{V}$. Also, note that, if $c=0$, then $\operatorname{grad}(V(\log \lambda))$ is a parallel vector field.

Let $x \in M$ and suppose that for any conformal vector field $V \in \Gamma(\mathscr{V})$ we have $\operatorname{grad}(V(\log \lambda))_{x}=0$.

If $c \neq 0$, then from Lemma 2.6.9 it follows that $\mathscr{V}$ is Riemannian at $x$. This establishes assertion (ii).

If $c>0$ then from Lemma 2.6.10 it follows that $\mathscr{V}$ is Riemannian in a
neighbourhood of $x$ and this establishes assertion (i).
If $c=0$, let $\mathcal{U}$ be a locally finite open covering of $M$ such that each $U \in \mathcal{U}$ is connected and there exists a local frame $\left\{V_{r}^{U}\right\}_{r=1, \ldots, \operatorname{dim} \mathscr{V}}$ for $\mathscr{V}$, over $U$, made up of conformal vector fields: $\mathcal{L}_{V_{r}^{U}} g=\sigma_{r}^{U} g, r=1, \ldots, \operatorname{dim} \mathscr{V}$. Let $F$ be the set of points $x \in M$ at which $\left(\mathrm{d} \sigma_{r}^{U}\right)_{x}=0, r=1, \ldots, \operatorname{dim} \mathscr{V}$, for all $U \in \mathcal{U}$ with $x \in U$. From Lemma 2.6.10 it follows that if $U \in \mathcal{U}$ is such that $F \cap U \neq \emptyset$ then $V_{r}^{U}$ are homothetic vector fields. Since $\mathcal{U}$ is locally finite, this implies that $F$ is open and closed which, because $M$ is connected, establishes assertion (iii).

Proof of Corollary 2.6.2. This follows from assertions (i) and (ii) of Theorem 2.6.1.

Proof of Theorem 2.6.4. This follows from Theorems 2.5.6 and 2.6.1.
Proof of Corollary 2.6.5. This follows from Proposition 2.4 .2 and Theorem 2.6.4.

## CHAPTER 3

## Harmonic morphisms with one-dimensional fibres

### 3.1. Basic facts

In this section we present, for later use, a few facts about one-dimensional foliations which produce harmonic morphisms. Here, $\mathscr{V}$ will always denote a one-dimensional foliation.

The following lemma, due to R.L. Bryant [14], will be used several times in this chapter. The case $n=2$ was used by P. Baird and J.C. Wood in [8, §3].

Lemma 3.1.1. Let $\mathscr{V}$ be a conformal one-dimensional foliation on $\left(M^{n+1}, g\right)$. Then, the following assertions are equivalent.
(i) $\mathscr{V}$ produces harmonic morphisms;
(ii) each point has a neighbourhood on which a local dilation $\lambda$ of $\mathscr{V}$ can be found such that, if $V$ is a vertical field with $g(V, V)=\lambda^{2 n-4}$, then $[V, X]=0$ for any basic field $X$.

Proof. From 1.1.5) it follows that assertion (i) is equivalent to the possibility of finding in the neighbourhood of each point a local dilation $\lambda$ of $\mathscr{V}$ such that

$$
\mathscr{V}^{*}\left(\mathcal{L}_{X}\left(\lambda^{2-n} \omega\right)\right)=0
$$

for any basic vector field $X$ and where $\omega$ is a local volume of $\mathscr{V}$.
If $V$ is as in (ii) and $\theta$ is its dual vertical one-form (i.e. $\theta$ is the unique vertical one-form such that $\theta(V)=1)$ then $\lambda^{n-2} \theta$ is a local volume form of $\mathscr{V}$. Hence (3.1.1) is equivalent to $\left(\mathcal{L}_{X} \theta\right)(V)=0$ which is equivalent to $[V, X]=0$.

Remark 3.1.2. 1) From the proof above we see that (ii) is a characterisation of those local dilations which restrict to give dilations of harmonic morphisms
which locally define the foliation.
2) If $V$ is as above, let $\theta$ be its dual vertical one-form. Using the fact that $[V, X]=0$ for any basic vector field $X$, it follows that the two-form $\Omega=\mathrm{d} \theta$ is basic. (In fact, $\theta$ and $\Omega=\mathrm{d} \theta$ are, respectively, the connection form and the curvature form of a principal (local) connection, see Theorem 3.1.9).

The equivalence (iii) $\Longleftrightarrow$ (i) from the following proposition is due to R.L. Bryant [14].

Proposition 3.1.3. For $n \neq 2$, let $\mathscr{V}$ be a one-dimensional Riemannian foliation on $\left(M^{n+1}, g\right)$ and let $\mathscr{H}$ be its orthogonal complement. Then, the following assertions are equivalent:
(i) $\mathscr{V}$ produces harmonic morphisms,
(ii) $\mathscr{H}$ is a homothetic distribution,
(iii) $\mathscr{V}$ is locally generated by Killing fields.

Furthermore, if $\mathscr{V}$ is orientable and the first Betti number of $M$ is zero then (iii) above can be replaced by
(iii') $\mathscr{V}$ is globally generated by a Killing field.
Proof. (i) $\Longleftrightarrow$ (ii) This follows from Proposition 1.4.7, since, being Riemannian, $\mathscr{V}$ is homothetic and, being of codimension one, $\mathscr{H}$ is conformal.
(ii) $\Rightarrow$ (iii) Let $\rho$ be a local dilation of $\mathscr{H}$ which is constant along the leaves of $\mathscr{V}$ and let $V$ be a local vertical field such that $g(V, V)=\rho^{-2}$.

Because $\rho$ is constant along the leaves of $\mathscr{V}$ we have

$$
\begin{equation*}
\left(\mathcal{L}_{V} g\right)(V, V)=0 . \tag{3.1.2}
\end{equation*}
$$

Because $\rho$ is a local dilation of $\mathscr{H}$ we have

$$
\begin{equation*}
\left(\mathcal{L}_{X}\left(\rho^{2} g\right)\right)(V, V)=0 \tag{3.1.3}
\end{equation*}
$$

for any horizontal vector field $X$. It is easy to see that (3.1.3) is equivalent to $g([X, V], V)=0$. This implies that for any horizontal vector field we have

$$
\begin{equation*}
\left(\mathcal{L}_{V} g\right)(V, X)=0 . \tag{3.1.4}
\end{equation*}
$$

Since $\mathscr{V}$ is Riemannian we have

$$
\begin{equation*}
\left(\mathcal{L}_{V} g\right)(X, Y)=0, \tag{3.1.5}
\end{equation*}
$$

for any horizontal vector fields $X$ and $Y$.
Equations (3.1.2), (3.1.4 and (3.1.5) show that $V$ is a Killing field.
(iii) $\Rightarrow$ (ii) Since $\operatorname{dim} \mathscr{V}=1$, the orthogonal complement $\mathscr{H}$ of $\mathscr{V}$ is a conformal distribution.

If $V$ is a (local) nonvanishing Killing field, which (locally) generates $\mathscr{V}$, and $|V|$ its norm then $|V|^{-1}$ is a local dilation for the horizontal distribution $\mathscr{H}$. Moreover, $\mathscr{H}$ is homothetic, since $|V|$ is constant along the leaves of $\mathscr{V}$.

The last assertion follows from the fact that when the first Betti number of $M$ is zero and $\mathscr{V}$ is orientable we can find a global density $\lambda^{2-n}$ of $\mathscr{V}$ (which is also a local dilation for $\mathscr{H}$ ) and a vertical vector field $V$ defined on $M$, such that $g(V, V)=\lambda^{2 n-4}$.

Remark 3.1.4. 1) Note that if $n=2$ then (i) $\Rightarrow$ (ii) $\Longleftrightarrow$ (iii). In fact, in this case, a one-dimensional foliation $\mathscr{V}$ produces harmonic morphisms on $\left(M^{3}, g\right)$ if and only if its leaves are geodesics (see [8]). Thus, being of codimension one, $\mathscr{H}$ is a Riemannian distribution. However, if $n=2$ then (ii) $\Rightarrow$ (i) fails, as simple examples show.
2) If in the above proposition we further assume that $\mathscr{H}$ is integrable then $\mathscr{V}$ induces, locally, a warped-product structure on $(M, g)$.

One might guess that a similar proposition to the one above holds in general for any conformal one-dimensional foliation, just by replacing 'Killing fields', with 'conformal fields'. It is not difficult to see that this is not true, the actual situation being described by the following:

Proposition 3.1.5. For $n \geq 3$, let $\mathscr{V}$ be a one-dimensional foliation on $\left(M^{n+1}, g\right)$. Then any two of the following assertions imply the remaining assertion.
(i) $\mathscr{V}$ produces harmonic morphisms,
(ii) $\mathscr{V}$ (or $\mathscr{H}$ ) is homothetic,
(iii) $\mathscr{V}$ is locally generated by conformal vector fields.

Furthermore, if $\mathscr{V}$ is orientable and the first Betti number of $M$ is zero then (iii) above can be replaced by
(iii') $\mathscr{V}$ is (globally) generated by a conformal field.
Proof. (i), (ii) $\Rightarrow$ (iii) Let $\lambda^{2-n}$ be a local density for $\mathscr{V}$. By Proposition 1.4.2, we can suppose that $\lambda=a b$, where $a$ is constant along leaves and $b$ is constant along horizontal curves.

Let $W$ be a local vertical vector field such that $g(W, W)=a^{2 n-4} b^{-2}$. It is a straightforward calculation to check that $W$ is a local conformal vector field on $(M, g)$.
(ii), (iii) $\Rightarrow$ (i) Since $\mathscr{V}$ is homothetic, by Proposition 1.4.2, we can find a local dilation $b$ of $\mathscr{V}$ which is constant along horizontal curves.

Let $W$ be a local conformal vector field which (locally) generates $\mathscr{V}$. We can suppose that $b$ and $W$ are defined on the same open subset of $M$. It is easy to see that, since $W$ is conformal, we have that $b^{2} g(W, W)$ is constant along leaves.

We can choose a smooth positive local function $a$ on $M$ such that $g(W, W)=$ $a^{2 n-4} b^{-2}$. Hence $a$ is constant along the leaves and thus $\lambda=a b$ is a local dilation of $\mathscr{V}$.

If $V$ is a local field, tangent to the leaves and such that $g(V, V)=\lambda^{2 n-4}$ then, from the fact that $W$ is conformal it follows that $[V, X]=0$ for any basic $X$. Hence, by Lemma 3.1.1, $\mathscr{V}$ is a foliation which produces harmonic morphisms.
(iii), (i) $\Rightarrow$ (ii) Let $\lambda^{2-n}$ be a local density for $\mathscr{V}$. Let $V$ be a local vector field, tangent to the leaves and such that $g(V, V)=\lambda^{2 n-4}$, and let $W$ be a local conformal vector field tangent to the leaves. We can suppose that $V$ and $W$ are defined on the same open set.

Since $W$ is conformal, for any basic $X$ we have $\left(\mathcal{L}_{W} g\right)(W, X)=0$, and hence, $[W, X]=0$. But, by Lemma 3.1.1 we also have $[V, X]=0$ for any basic $X$. Hence if $b$ is such that $W=b V$, then $b$ is constant on horizontal curves.

Since $\lambda$ is a local dilation of the conformal foliation $\mathscr{V}$, from Remark 1.1.9, we see that

$$
\begin{equation*}
\left(\mathcal{L}_{W} g\right)(V, V)=W\left(\log \left(\lambda^{-2}\right)\right) g(V, V) . \tag{3.1.6}
\end{equation*}
$$

Relation (3.1.6) together with $W=b V$ implies after a straightforward calculation that $\lambda^{n-1} b$ is constant along leaves. Thus, we can write $\lambda=r s$ where $r, s$ are positive smooth functions on $M$ such that $r$ is constant along the leaves and $s$ is constant along horizontal curves. From Proposition 1.4.2, we get that $\mathscr{V}$ is a homothetic foliation.

Remark 3.1.6. Note that, if in Proposition 3.1 .5 we have $n=2$, the implication (ii),(iii) $\Rightarrow$ (i) fails, the other implications still holding.

If $n=1$, then (i) $\Longleftrightarrow$ (iii) but they do not imply (ii).

Lemma 3.1.7 (cf. [8, Remark 5.3]). Let $\mathscr{V}$ be a one-dimensional homothetic foliation on $(M, g)$. Then, at least away of the points where $\mathscr{V}$ is Riemannian, its orthogonal complement is integrable.

Proof. By Proposition $1.4 .2, \mathscr{V}$ admits a local dilation $\lambda$ whose gradient is vertical. The points $x \in M$, where $\mathscr{V}$ is not Riemannian are characterised by $(\operatorname{grad} \lambda)_{x} \neq 0$. Hence, in a neighbourhood of such a point, the level hypersurfaces of $\lambda$ are integral submanifolds of the horizontal distribution.

Lemma 3.1.8. Let $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right)$ be a harmonic morphism with one-dimensional fibres. Let $\lambda$ denote the dilation of $\varphi$ and let $V$ be a (local) vertical vector field on $M$ such that $g(V, V)=\lambda^{2 n-4}$.

Then, the following assertions are equivalent:
(i) the fibres of $\varphi$ form a homothetic foliation at least on the complement of the interior of the set $\{x \in M \mid \mathrm{d}(V(\log \lambda))(x)=0 \neq(V(\log \lambda))(x)\}$;
(ii) for any basic field $X$, we have $V(X(\log \lambda))=0$.

Proof. Let $\mu$ be the vertical one-form on $M$ such that for any horizontal fields $X, Y$ we have $\left(\mathcal{L}_{V} g\right)(X, Y)=\mu(V) g(X, Y)$. Hence, by the definition of $\lambda$ we have $\mu(V)=-2 V(\log \lambda)$.

Let $\mathscr{H}$ be the horizontal distribution and $\mathscr{H}_{B}$ its second fundamental form. Using (1.1.3 we obtain the following relation:

$$
\mu=-\frac{2}{n}\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right)\right)^{b} .
$$

Hence, $\mathscr{V}$ is homothetic if and only if $\mu$ is closed.
By Lemma 3.1.1, for any basic $X$ we have $[V, X]=0$, and hence:

$$
\begin{aligned}
(\mathrm{d} \mu)(V, X) & =-V(\mu(X))-X(\mu(V))-\mu[V, X]=-X(\mu(V)) \\
& =2 X(V(\log \lambda))=2 V(X(\log \lambda)) .
\end{aligned}
$$

The lemma follows.

In [8, Proposition 3.5], P. Baird and J.C. Wood gave a global description of the metric of a Riemannian manifold of dimension three, on which a harmonic morphism can be defined. In [14, Theorem 1], R.L. Bryant gave a local description of the metric of the total space of a submersive harmonic morphism with one-dimensional fibres (with no restriction on the dimension of the total space). The following theorem explains how the latter result can be globalized, giving also a simpler proof of Bryant's local result.

Theorem 3.1.9 (cf. [14, Theorem 1]). Let $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right), n \geq 1$, be a submersive harmonic morphism with connected one-dimensional fibres of the same homotopy type. Let $\lambda$ be the dilation of $\varphi$ and suppose that $\mathscr{V}\left(=\operatorname{ker} \varphi_{*}\right)$ is orientable.

Then, there exists:
(i) a principal bundle $\pi: P \rightarrow N$ with group $G=(\mathbb{R},+)$ or $G=\left(S^{1}, \cdot\right)$,
(ii) a principal connection $\theta \in \Gamma\left(T^{*} P\right)$ on $\pi$,
(iii) a diffeomorphic embedding $\iota: M \rightarrow P$
such that:

1) $\pi \circ \iota=\varphi$,
2) $g=\lambda^{-2}\left(\varphi^{*} h\right)+\lambda^{2 n-4}\left(\iota^{*} \theta\right)^{2}$.

Furthermore, if the fibres are all diffeomorphic to circles, or are all complete with respect to the metric induced by ${ }^{\lambda} g$, then $\iota$ is onto, and hence, $\varphi$ itself is a principal bundle and the horizontal distribution is a principal connection on it.

Note that, by the result of P. Baird [14, we know that $\varphi$ is automatically submersive except when $n \leq 3$.

Proof. Let $V$ be a vertical field such that $g(V, V)=\lambda^{2 n-4}$. By Lemma 3.1.1, the horizontal distribution $\mathscr{H}$ is invariant under the local flow of $V$. Thus, the integral curves of $V$ are the fibres of a local principal bundle, and $\mathscr{H}$ is a principal connection on it. If $\theta$ is the vertical one-form dual to $V$ then, it is obvious that ${ }^{\lambda} g=\varphi^{*} h+\theta^{2}$. (This establishes [14, Theorem 1].)

To end the proof we shall prove the following assertions:
(a) if the fibres are diffeomorphic to circles then $\varphi$ is a principal bundle with group $\left(S^{1}, \cdot\right)$;
(b) if the fibres are diffeomorphic to $\mathbb{R}$ then, there exists a diffeomorphic embedding $\iota: M \rightarrow N \times \mathbb{R}$, such that $\pi_{1} \circ \iota=\varphi$, and a principal connection on the trivial principal bundle $\pi_{1}: N \times \mathbb{R} \rightarrow N$, with group $(\mathbb{R},+)$, such that $\mathscr{H}$ is the restriction to $M$ of it.

From now on, all the considerations which will be made in this proof will be done with respect to the metric ${ }^{\lambda} g$ on $M$.

For $x \in M$, let $I_{x} \subseteq \mathbb{R}$ be the open interval which is the domain of the (maximal) geodesic with velocity $V_{x}$. Let $Q=\left\{(x, r) \in M \times \mathbb{R} \mid r \in I_{x}\right\}$, and define $\Psi: Q \rightarrow M$, by $\Psi(x, r)=\exp r V_{x}$.

If the fibres are all circles then $Q=M \times \mathbb{R}$. Since $\mathrm{d} \theta(V, X)=0$ for any horizontal vector field $X$, by applying the Stokes theorem we obtain that the fibres have the same length. Hence $\Psi$ descends to a map $M \times S^{1} \rightarrow M$ which is a free action of $\left(S^{1}, \cdot\right)$ on $M$. Thus assertion (a) is proved.

Suppose now that the fibres are diffeomorphic to $\mathbb{R}$. If they are all complete with respect to the metric induced from ${ }^{\lambda} g$ then, $Q=M \times \mathbb{R}$ and $\Psi$ represents a free action of $(\mathbb{R},+)$ on $M$, and thus the proof of the theorem is finished. Otherwise, since $\varphi$ is a submersion, we can find local sections of it in the neighbourhood of each point of $N$. Let $\mathcal{S}$ be a family of such sections whose domains form an open covering $\left\{O_{s}\right\}_{s \in \mathcal{S}}$ of $N$.

Let $s, t \in \mathcal{S}$. For $x \in O_{s} \cap O_{t}$, let $a_{s t}(x)$ be the (unique) real number such that $t(x)=\Psi\left(s(x), a_{s t}(x)\right)$.

It is obvious that $\left\{a_{s t}\right\}_{s, t \in \mathcal{S}}$ is a cocycle with values in $(\mathbb{R},+)$, which induces a principal bundle. This bundle is trivial because $\mathbb{R}$ is contractible.

Moreover, set $A_{s}=s^{*} \theta$; then the family of one-forms $\left\{A_{s}\right\}_{s \in \mathcal{S}}$, defines a
principal connection on this bundle.
The total space $N \times \mathbb{R}$ of this bundle, can be retrieved, as usual from the cocycle $\left\{a_{s t}\right\}_{s, t \in \mathcal{S}}$ as the space of equivalence classes $[x, r]$, under the identifications $[x, r] \equiv\left[x, a_{s t}(x) r\right], x \in O_{s} \cap O_{t}$.

For $x \in M$, let $s \in \mathcal{S}$ be such that $\varphi(x) \in O_{s}$ and, $r_{x} \in I_{s(\varphi(x))}$ be the real number which satisfies $x=\Psi\left(s(\varphi(x)), r_{x}\right)$. (Note that $r_{x}$ depends just on $x$ and $s$.) We can define $\iota: M \rightarrow N \times \mathbb{R}$ by $\iota(x)=\left[\varphi(x), r_{x}\right], x \in M$ and the theorem follows.

Remark 3.1.10. The proof of above theorem, can be simplified considerably when $\mathscr{H}$ is an Ehresmann connection (see [12] for the definition of Ehresmann connection). It is not difficult to prove that a sufficient condition for $\mathscr{H}$ to be an Ehresmann connection is that $V$ be a complete vector field.

### 3.2. Towards the classification

All of the main results of this section hold for Riemannian manifolds of dimension at least four. None of the results of this section requires the compactness or the completeness of the manifold.

In this section $\mathscr{V}$ will always denote a one-dimensional foliation which produces harmonic morphisms on $\left(M^{n+1}, g\right)(n \geq 1)$ and $\rho=\mathrm{e}^{(2-n) \sigma}$ will denote a local density of it. As before, $h={ }^{\mathrm{e}^{\sigma}} g$ will denote the associated (local) metric on $M$ with respect to which $\mathscr{V}$ is Riemannian and has geodesic leaves and $\mathscr{H}$ will denote the orthogonal complement of $\mathscr{V}$.

Proposition 3.2.1 (cf. [11, Chapter 9, §J]). For $n \geq 3$, let $\varphi:\left(M^{n+1}, g\right) \rightarrow$ $\left(N^{n}, \bar{h}\right)$ be a harmonic morphism with one-dimensional geodesic fibres.
(a) If $\mathscr{H}$ is integrable then the following assertions are equivalent:
(i) $(M, g)$ is Einstein,
(ii) $(N, \bar{h})$ is Einstein and the following relation holds

$$
\begin{equation*}
\frac{c^{M}}{n} \lambda^{2}-\frac{c^{N}}{n-1} \lambda^{4}+(U(\lambda))^{2}+\frac{3}{4} \lambda^{2 n+2}|\Omega|_{h}^{2}=0, \tag{3.2.1}
\end{equation*}
$$

where $c^{N}$ is the Einstein constant of $\left(N^{n}, \bar{h}\right), U$ is a vertical local vector field such that $g(U, U)=1$ and $c^{M}={ }^{M} \operatorname{Ricci}(U, U)$.

Moreover, if (i) or (ii) holds then

$$
\begin{equation*}
K_{X \wedge Y}^{M}-\frac{c^{M}}{n}=\lambda^{2}\left(K_{\varphi_{*} X \wedge \varphi_{*} Y}^{N}-\frac{c^{N}}{n-1}\right) \tag{3.2.2}
\end{equation*}
$$

where $K^{M}$ and $K^{N}$ are the sectional curvature of $(M, g)$ and $(N, h)$, respectively, and $X, Y$ are horizontal.
(b) When $n=4$ and $M$ and $N$ are oriented consider also the following assertion:
(iii) $\Omega$ is the pull back of a (anti-)self-dual form on $(N, \bar{h})$.

Then, any two of the assertions (i), (ii) and (iii) imply the remaining assertion.

Proof. (a) By Proposition 1.1.10, we have that $X(\sigma)=0$ for any horizontal $X$. Lemma 3.1.1 implies that $[V, X]=0$ and hence $X(V(\sigma))=V(X(\sigma))=$ 0 . By hypothesis, $\Omega=0$ so, from A.2.24 for any horizontal $X$ we have $\operatorname{Ricci}(X, V)=0$. Similarly, from A.2.23 we get:

$$
\begin{equation*}
{ }^{M} \operatorname{Ricci}(X, Y)=\left({ }^{N} \operatorname{Ricci}\right)\left(\varphi_{*} X, \varphi_{*} Y\right)-\mathrm{e}^{-2 \sigma}\left(\Delta^{M} \sigma\right) h(X, Y) \tag{3.2.3}
\end{equation*}
$$

It follows that $(M, g)$ is Einstein if and only if $(N, \bar{h})$ is Einstein and (3.2.1) holds.

If (i) or (ii) holds then (3.2.2 follows from (3.2.1) and the following formula

$$
\lambda^{2} K_{X \wedge Y}^{M}-\lambda^{4} K_{\varphi_{*} X \wedge \varphi_{*} Y}^{N}+(U(\lambda))^{2}=0
$$

which can be obtained directly or as a consequence of a formula of S. Gudmundsson [27].
(b) Let $\bar{\Omega}$ be the two-form on $N$ such that $\varphi^{*}(\bar{\Omega})=\Omega$. Note that $\left.\left({ }^{h} \mathrm{~d}^{*} \Omega\right)\right|_{\mathscr{H}}=$ 0 if and only if $\bar{\Omega}$ is coclosed on $(N, \bar{h})$.

If (iii) holds the equivalence (i) $\Longleftrightarrow$ (ii) can be proved in a similar way to (a) , using the fact that any closed (anti-)self-dual form is coclosed and that, for any two-form $\omega$ on a four-dimensional oriented Euclidean space ( $E^{4},<,>$ ) and $u, v \in E$, we have (see [13]):

$$
\begin{equation*}
<i_{u} \omega, i_{v} \omega>=\frac{1}{2}|\omega|^{2}<u, v>+2<i_{u} \omega_{+}, i_{v} \omega_{-}> \tag{3.2.4}
\end{equation*}
$$

where, $\omega_{+}$and $\omega_{-}$are, respectively, the self-dual and the anti-self-dual components of $\omega$.

The prove (i), (ii) $\Rightarrow$ (iii) we adapt a method of [13].
First note that, by A.2.24), $\bar{\Omega}$ is coclosed.
Now, recall from [13], that (3.2.4) gives the decomposition of the symmetric bilinear map $(u, v) \mapsto<i_{u} \omega, i_{v} \omega>$ into its 'spherical' part and its 'trace-free' part. Also, the bilinear map $(u, v) \mapsto<i_{u} \alpha, i_{v} \beta>$ induces a natural isomorphism between the space of 'trace-free' symmetric bilinear maps and $\Lambda_{+}^{2}(E) \otimes \Lambda_{-}^{2}(E)$ (see [13]). Using these facts it is easy to see that at each point $\bar{\Omega}$ is either self-dual or anti-self-dual.

If $N_{ \pm}=\left\{y \in N \mid\left(\bar{\Omega}_{ \pm}\right)_{y}=0\right\}$ then by the Baire category theorem at least one of the two sets $N_{+}$and $N_{-}$has nonempty interior. If $N_{+}$has nonempty interior then, following [13], we apply Aronszajn's unique continuation theorem (see [17] noting that $\bar{\Omega}$, and hence also $\bar{\Omega}_{+}$, is closed and coclosed) to obtain $\bar{\Omega}_{+}=0$. Hence $\bar{\Omega}$ is anti-self-dual.

Remark 3.2.2. 1) From Lemma 3.1.7 we see that if the foliation given by the fibres of $\varphi$ is nowhere Riemannian then $\mathscr{H}$ is automatically integrable.
2) Since the decomposition of two-forms into self-dual and anti-self-dual parts is conformally invariant, the condition that $\Omega$ be the pull back of a (anti-)self-dual form is equivalent to the condition that $\Omega$ restricted to the horizontal distribution be (anti-)self-dual.

The following elementary algebraic lemma will be useful later on.
Lemma 3.2.3. Let $E$ be an Euclidean linear space of dimension at least two and $\alpha$ a linear function on it such that, for any pair of orthogonal vectors $\{u, v\}$ we have $\alpha(u) \alpha(v)=0$.

Then $\alpha=0$.
Proof. Let $u, v \in E$ be orthogonal and such that $|u|=|v|$. Since $u+v$ and $u-v$ are also orthogonal we get that $0=\alpha(u+v) \alpha(u-v)=\alpha(u)^{2}-\alpha(v)^{2}$.

Thus $\alpha(u)= \pm \alpha(v)$ and since by hypothesis at least one of must be zero they are both zero. The lemma is proven.

Recall that on a Riemannian manifold of dimension at least four a Riemannian foliation with one-dimensional leaves produces harmonic morphisms if and only if it is locally generated by Killing fields (Proposition 3.1.3) and a foliation by geodesics produces harmonic morphisms if and only if it is homothetic (Corollary 1.4.8).

Proposition 3.2.4. Let $\left(M^{n+1}, g\right)$ be an Einstein manifold of dimension $n+1 \geq 4$, and let $\mathscr{V}$ be a one-dimensional foliation which produces harmonic morphisms on $(M, g)$. Suppose that, either, the orthogonal complement $\mathscr{H}$ of $\mathscr{V}$ is integrable, or, $\mathscr{V}$ is a homothetic foliation.

Then either,
(i) $\mathscr{V}$ is a Riemannian foliation locally generated by Killing vector fields, or
(ii) $\mathscr{V}$ is a homothetic foliation by geodesics orthogonal to an umbilical foliation by hypersurfaces.

Proof. By passing to a Riemannian covering if necessary, we can suppose that $\mathscr{V}$ admits a global density.

By the remarks above we need to prove just the 'only if' part.
Suppose that $\mathscr{H}$ is integrable. Then from A.2.23) of Appendix B, for any orthogonal pair $\{X, Y\}$ formed of basic vector fields we have:

$$
\begin{equation*}
{ }^{N} \operatorname{Ricci}\left(\varphi_{*} X, \varphi_{*} Y\right)=(n-1)(n-2) X(\sigma) Y(\sigma) . \tag{3.2.5}
\end{equation*}
$$

Since $n \geq 3$ and the left-hand side of (3.2.5) is a basic function we get that $X(\sigma) Y(\sigma)$ is a basic function.

Also, from A.2.24 we obtain that

$$
V(X(\sigma))=X(V(\sigma))=(n-2) V(\sigma) X(\sigma) .
$$

Hence:

$$
\begin{aligned}
0 & =V(X(\sigma) Y(\sigma))=V(X(\sigma)) Y(\sigma)+X(\sigma) V(Y(\sigma)) \\
& =2(n-2) V(\sigma) X(\sigma) Y(\sigma) .
\end{aligned}
$$

If, at a point $x$ we have that $V(\sigma)(x) \neq 0$, then this holds in an open neighbourhood $O$ of $x$. It follows that $X(\sigma) Y(\sigma)=0$ on $O$.

Using Lemma 3.2.3 we see that grad $\sigma$ restricted to $O$ is vertical and hence, from Proposition 1.1.10, it follows that $\mathscr{V}$ restricted to $O$ has geodesic leaves.

Now, recall that, being Einstein, $(M, g)$ is analytic (see [11, 5.26]). Together with Proposition 1.3 .2 this shows that $\mathscr{V}$ has geodesic leaves.

Thus, we have proved that if $\mathscr{H}$ is integrable then either (i) or (ii) holds on $M$.

Suppose, now, that $\mathscr{V}$ is homothetic and let

$$
F=\{x \in M \mid \mathscr{V} \text { is Riemannian at } x\} .
$$

Suppose that $M \backslash F \neq \emptyset$. Then, by Lemma 3.1.7, we have that $\mathscr{H}$, the orthogonal complement of $\mathscr{V}$, is integrable at least on $M \backslash F$. Therefore, by analyticity, $\mathscr{H}$ is integrable on $M$.

The last assertion follows quickly from A.1.1) and the proof is complete.

Next, we prove the following:
Proposition 3.2.5. Let $\mathscr{V}$ be a one-dimensional foliation which produces harmonic morphisms on $\left(M^{n+1}, g\right)$. Suppose that the following conditions are satisfied, for any horizontal $X$ :
(i) $X(V(\sigma))=0$,
(ii) ${ }^{M} \operatorname{Ricci}(X, V)=0$.

Then $\mathscr{V}$ is homothetic.
Proof. By Lemma 3.1.8 it is sufficient to prove that $\mathscr{V}$ is homothetic on the interior $S$ of the set $\{x \in M \mid \mathrm{d}(V(\sigma))(x)=0 \neq(V(\sigma))(x)\}$.

By A.2.24) on $S$ we have
$\mathrm{e}^{(2 n-2) \sigma}\left\{\frac{1}{2}\left({ }^{h} \mathrm{~d}^{*} \Omega\right)(X)+(n-1) \Omega\left(X, \operatorname{grad}_{h} \sigma\right)\right\}=(n-1)(n-2) X(\sigma) V(\sigma)$,
for any basic vector field $X$.
By hypothesis the right hand side above is a basic function on S. Also, the second factor from the left hand side of (3.2.6) is basic and thus, if this second factor is nonzero, then $\mathrm{e}^{(2 n-2) \sigma}$ is a basic function. This implies that
$\mathscr{V}$ is Riemannian and, in particular, homothetic.
If the second factor from the left hand side of (3.2.6) is zero on an open subset $S_{0}$ of $S$ then the right hand is also zero and hence $\mathscr{V}$ has geodesic fibres on $S_{0}$. From Corollary 1.4 .8 it follows that $\mathscr{V}$ is homothetic on $S_{0}$.

The following proposition will be used later on.
Proposition 3.2.6. Let $(M, g)$ be an Einstein manifold and $\mathscr{V}$ a one-dimensional foliation of codimension not equal to two which produces harmonic morphisms on $(M, g)$.

Then the following assertions are equivalent:
(i) $\mathscr{V}$ has basic mean curvature form;
(ii) $\mathscr{V}$ is a homothetic foliation.

Proof. As above, we may suppose that $\mathscr{V}$ admits a global density $\mathrm{e}^{(2-n) \sigma}$. From the fundamental equation it follows quickly that (i) holds if and only if $X(\sigma)$ is a basic function for any basic vector field $X \in \Gamma(\mathscr{H})$.

Therefore, if $(M, g)$ is Einstein, Proposition 3.2.5 gives that $(\mathrm{i}) \Rightarrow(\mathrm{ii})$.
Conversely, if $\mathscr{V}$ is homothetic then, from Proposition 1.4 .2 it follows quickly that $X(\sigma)$ is a basic function for any basic vector field $X \in \Gamma(\mathscr{H})$.

### 3.3. Constructions of 1 -dim foliations which produce harmonic morphisms on Einstein manifolds

In this section we use well-known results on warped-products and conformal vector fields (see [37, [11, Chapter 9, §J]) to obtain one-dimensional foliations with integrable orthogonal complement and which produce harmonic morphisms on Einstein manifolds which are not of constant curvature.

### 3.3.1. Homothetic foliations with geodesic leaves.

Proposition 3.3.1 (cf. [37, Lemma 13(iv)]). Let $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right)$, $n \geq 3$, be a nonconstant harmonic morphism with geodesic leaves and integrable horizontal distribution.
(i) If $\left(M^{n+1}, g\right)$ has constant curvature then $\left(N^{n}, h\right)$ has constant curvature.
(ii) If $\left(M^{n+1}, g\right)$ is Einstein and $\left(N^{n}, h\right)$ has constant curvature then $\left(M^{n+1}, g\right)$ has constant curvature.

Proof. Assertion (i) is an immediate consequence of (3.2.2).
If $\left(M^{n+1}, g\right)$ is Einstein then from (3.2.1) and from A.2.1) we obtain

$$
\begin{equation*}
K_{X \wedge U}^{M}=\frac{c^{M}}{n}, \tag{3.3.1}
\end{equation*}
$$

where $X$ is any horizontal vector.
The proof of (ii) follows from (3.2.2) and (3.3.1).
Corollary 3.3.2 (cf. [37, Corollary 15]). Let $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ be a harmonic morphism with one-dimensional geodesic leaves and integrable horizontal distribution.

If $\left(M^{4}, g\right)$ is Einstein then both $\left(M^{4}, g\right)$ and $\left(N^{3}, h\right)$ have constant curvature.

Proof. If $\left(M^{4}, g\right)$ is Einstein then by Proposition 3.2.1, $\left(N^{3}, h\right)$ is Einstein. But $N^{3}$ is three-dimensional and thus $\left(N^{3}, h\right)$ has constant curvature. The proof follows from Proposition 3.3.1(ii) .

Corollary 3.3.3 (cf. [37], [11, Chapter 9, §J]). Given any Einstein manifold $\left(N^{n}, h\right)$ of dimension $n$ there exists an Einstein manifold $\left(M^{n+1}, g\right)$ of dimension $n+1$ and a harmonic morphism $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right)$ with geodesic fibres and integrable horizontal distribution.

If $n \geq 4$ and $\left(N^{n}, h\right)$ does not have constant curvature then $\left(M^{n+1}, g\right)$ does not have constant curvature.

Proof. Let $\left(N^{n}, h\right)$ be Einstein and let $\lambda$ be a (local) solution of 3.2.1 (see [11, 9.109]).

Let $M^{n+1}=\mathbb{R} \times N^{n}$ and $g=\mathrm{d} t^{2}+\lambda^{-2} h$. It is obvious that the canonical projection $\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right)$ is a harmonic morphism with geodesic leaves and integrable horizontal distribution. Also $\left(M^{n+1}, g\right)$ is an Einstein manifold by Proposition 3.2.1. Moreover, if $\left(N^{n}, h\right)$ does not have constant curvature then, by Proposition 3.3.1, $\left(M^{n+1}, g\right)$ does not have constant curvature.
3.3.2. Riemannian foliations locally generated by Killing fields. The following results are consequences of [11, Corollary 9.107, 9.108, 9.109]. Proposition 3.3.4 ([11]). Let $\left(N^{n}, h\right)$ be a Riemannian manifold and $\rho$ : $N^{n} \rightarrow(0, \infty)$ a smooth positive function.

Let $M^{n+1}=\mathbb{R} \times N^{n}$ and $g=\rho^{2} \mathrm{~d} t^{2}+h$. Then, the following assertions are equivalent.
(i) $\left(M^{n+1}, g\right)$ is Einstein $\left({ }^{M} \mathrm{Ricci}=c^{M} g\right)$.
(ii) $\left(N^{n}, h\right)$ has constant scalar curvature $s^{N}=(n-1) c^{M}$ and the following relation holds

$$
\begin{equation*}
\stackrel{N}{\nabla} \mathrm{~d} \rho=-\frac{c^{M}}{n} \rho h+\rho Z^{N} \tag{3.3.2}
\end{equation*}
$$

where $\stackrel{N}{\nabla}$ is the Levi-Civita connection on $\left(N^{n}, h\right)$ and $Z^{N}={ }^{N} \operatorname{Ricci}-\left(s^{N} / n\right) h$ is the trace-free part of ${ }^{N}$ Ricci.

Proof. From [11, 9.106a, 9.106c] or, by a straightforward calculation the following equations can be obtained:

$$
\begin{align*}
\left.\left({ }^{M} \mathrm{Ricci}\right)\right|_{T N} & ={ }^{N} \mathrm{Ricci}-\rho^{-1} \stackrel{N}{\nabla} \mathrm{~d} \rho  \tag{3.3.3}\\
\left.\left({ }^{M} \mathrm{Ricci}\right)\right|_{T \mathbb{R}} & =\left.\rho^{-1}\left(\Delta^{N} \rho\right) g\right|_{T \mathbb{R}}
\end{align*}
$$

Also, [11, 9.106b] gives that ${ }^{M} \operatorname{Ricci}(X, \partial / \partial t)=0$ for any $X \in \Gamma(T N)$. From this and 3.3 .3 the proof easily follows.

Corollary 3.3.5. For each $n \geq 5$ there exists Einstein manifolds ( $M^{n+1}, g$ ) not of constant curvature, endowed with a nowhere zero Killing field which has integrable orthogonal complement. Moreover, the construction can be done in such a way that the (locally) induced isometric quotients are also Einstein.

Proof. If the equation $\stackrel{N}{\nabla} \mathrm{~d} \rho=a \rho h$, where $a \in \mathbb{R}$, has solutions then there exists a homothetic one-dimensional foliation with geodesic leaves and integrable orthogonal complement (see Lemma 2.6.6). Recall that $Z^{N}=0$ if and only if $\left(N^{n}, h\right)$ is Einstein (see [11, 1.118]). Hence, [11, 9.109] and Corollary 3.3.3 implies that there exists an Einstein manifold ( $\left.N^{n}, h\right), n \geq 5$, not of constant curvature, on which (3.3.2) has a (local) solution $\rho$ which is
positive. Then, by Proposition 3.3.4, $\left(M^{n+1}, g\right)\left(\right.$ where $M^{n+1}=\mathbb{R} \times N^{n}$ and $\left.g=\rho^{2} \mathrm{~d} t^{2}+h\right)$ is Einstein. Clearly $V=\partial / \partial t$ is a nowhere zero Killing field on ( $M^{n+1}, g$ ).

### 3.4. The classification on Einstein manifolds

In this section we prove the following theorem.
Theorem 3.4.1. Let $(M, g)$ be an Einstein manifold of dimension at least 5 , and let $\mathscr{V}$ be a one-dimensional foliation which produces harmonic morphisms on ( $M, g$ ).

Then either,
(i) $\mathscr{V}$ is a Riemannian foliation locally generated by Killing vector fields, or
(ii) $\mathscr{V}$ is a homothetic foliation by geodesics orthogonal to an umbilical foliation by hypersurfaces.

Proof. By Proposition 1.3.2, if the horizontal distribution $\mathscr{H}$ is integrable on an open subset of $M$, then $\mathscr{H}$ is integrable on $M$. Thus, by Proposition 3.2.4 it is sufficient to prove the case when $\mathscr{H}$ is nowhere integrable. Also, writing $n+1=\operatorname{dim} M$, we can suppose that the leaves of $\mathscr{V}$ are the fibres of a harmonic morphism $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, \bar{h}\right)$, where $\operatorname{dim} N=n$ with $n \geq 4$, and from now on we shall use the notations of Lemma A.2.5.

Let $\Omega^{2} \in \Gamma(\operatorname{End}(\mathscr{H}))$ be the field of self-adjoint negative semi-definite endomorphisms of ( $\mathscr{H},\left.h\right|_{\mathscr{H}}$ ) defined by $h\left(\Omega^{2}(X), Y\right)=-h\left(i_{X} \Omega, i_{Y} \Omega\right)$ for horizontal $X$ and $Y$.

By Lemma A.3.3, $\Omega^{2}$ can be consistently diagonalized on a dense open subset of $M$; let $x_{0} \in M$ be a point of this subset. There is an open neighbourhood $U$ of $x_{0}$ and an orthonormal frame $\left\{X_{1}, \ldots, X_{n}\right\}$ for ( $\mathscr{H},\left.h\right|_{\mathscr{H}}$ ) over $U$ such that $\Omega^{2}\left(X_{i}\right)=-\mu_{i}^{2} X_{i}$ for some continuous functions $\mu_{i}: U \rightarrow[0, \infty)$ with $\mu_{i}^{2}$ smooth. Because $\Omega$ and $\left.h\right|_{\mathscr{H}}$ are basic we also have that $\Omega^{2}$ is basic; hence the $\mu_{i}$ are basic as well. We can thus suppose that the $X_{i}$ are basic.

From A.2.23 we have

$$
\begin{equation*}
{ }^{N} \operatorname{Ricci}\left(\varphi_{*} X_{i}, \varphi_{*} X_{j}\right)=(n-1)(n-2) X_{i}(\sigma) X_{j}(\sigma) \quad(i, j=1, \ldots, n, i \neq j) . \tag{3.4.1}
\end{equation*}
$$

We have the following alternative. Either
(1) there exists $x \in U$ and distinct $j_{1}, j_{2}, j_{3}$ such that $X_{j_{k}}(\sigma)_{x} \neq 0 \quad(k=$ $1,2,3)$, or
(2) for any $x \in U$ there are at most two distinct values of $j$, say $j_{1}, j_{2}$ such that $X_{j_{k}}(\sigma)_{x} \neq 0 \quad(k=1,2)$.

Suppose that (1) holds. By (3.4.1) we have that $X_{i}(\sigma) X_{j}(\sigma)$ is basic for any $i \neq j$. Hence $X_{j_{1}}(\sigma)^{2} X_{j_{2}}(\sigma)^{2} X_{j_{3}}(\sigma)^{2}$ is basic, and, because $X_{j_{k}}(\sigma) \neq 0$ on some open subset of $U$, we have that $X_{j_{k}}(\sigma)$ is basic $(k=1,2,3)$. Thus, if (1) holds, $X_{i}(\sigma)$ is basic for all $i=1, \ldots, n$, on some open subset of $U$. Then, by Proposition 1.3 .2 and Proposition 3.2.6, $\mathscr{V}$ is homothetic on $M$ and the proof follows from Proposition 3.2.4.

Suppose that (2) holds. If $X_{j}(\sigma)=0$ for all $j=1, \ldots, n$, then $\mathscr{V}$ is a homothetic foliation and the proof of the theorem follows from Proposition 3.2.4. Therefore we can suppose that, after renumbering if necessary, we have $X_{1}(\sigma)_{x} \neq 0$ at some point $x \in U$. Then this holds on some open subset of $U$. Then, either $X_{j}(\sigma)=0$ for $j=2, \ldots, n$ on some open subset of $U$, or there exists a point $x \in U$ such that, after renumbering if necessary, $X_{1}(\sigma)_{x} \neq 0$ and $X_{2}(\sigma)_{x} \neq 0$. In the latter case, because (2) holds, we must have that $X_{j}(\sigma)=0(j=3, \ldots, n)$ on some open subset of $U$. It follows that there exists an open subset $U_{1}$ of $U$ such that $X_{j}(\sigma)=0(j \geq 3)$. From now on we shall work on $U_{1}$.

By A.2.23) we have

$$
\begin{align*}
& c^{M} \mathrm{e}^{-2 \sigma}={ }^{N} \operatorname{Ricci}\left(\varphi_{*} X_{i}, \varphi_{*} X_{i}\right)-\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} \mu_{i}^{2} \\
&-\mathrm{e}^{-2 \sigma} \Delta^{M} \sigma-(n-1)(n-2) X_{i}(\sigma)^{2} \quad(i=1, \ldots, n) \tag{3.4.2}
\end{align*}
$$

From (3.4.2) we get

$$
\begin{align*}
& { }^{N} \operatorname{Ricci}\left(\varphi_{*} X_{i}, \varphi_{*} X_{i}\right)-{ }^{N} \operatorname{Ricci}\left(\varphi_{*} X_{j}, \varphi_{*} X_{j}\right)-\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma}\left(\mu_{i}^{2}-\mu_{j}^{2}\right) \\
&  \tag{3.4.3}\\
& \quad-(n-1)(n-2)\left(X_{i}(\sigma)^{2}-X_{j}(\sigma)^{2}\right)=0 \quad(i, j=1, \ldots, n) .
\end{align*}
$$

From (3.4.3) it follows that $\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma}\left(\mu_{i}^{2}-\mu_{j}^{2}\right)$ is basic for $i, j \geq 3$. Thus, if $\mu_{i} \neq \mu_{j}$ at some point for some $i, j \geq 3, i \neq j$, then $\mathrm{e}^{\sigma}$ is basic and so $\mathscr{V}$ is Riemannian on some open subset of $M$; hence, by Proposition 1.3.2, $\mathscr{V}$ is Riemannian on $(M, g)$. It remains to consider the case when $\mu_{3}=\ldots=\mu_{n}=$ $\mu$ for some function $\mu$.

Now, either, $\mu_{1}=\mu_{2}$ on some open subset, or, $\mu_{1} \neq \mu_{2}$ on a dense open subset. In the former case, by (3.4.3), we have that $X_{1}(\sigma)^{2}-X_{2}(\sigma)^{2}$ is basic on some open subset. But, by (3.4.1), $X_{1}(\sigma) X_{2}(\sigma)$ is also basic, and hence $X_{1}(\sigma), X_{2}(\sigma)$ are basic on some open subset. Since $X_{j}(\sigma)=0$ for $j \geq 3, \mathscr{V}$ has basic mean curvature form. Then, by Proposition 3.2.6, $\mathscr{V}$ is homothetic on some open subset and hence, by Proposition 1.3.2, $\mathscr{V}$ is homothetic on $(M, g)$; the proof of the theorem follows from Proposition 3.2.4.

It remains to consider the case when $\mu_{1} \neq \mu_{2}$. Because $\Omega$ is skewsymmetric, at each point $x$, for any $i \in\{1, \ldots, n\}$ with $\mu_{i}(x) \neq 0$ there exists $j \in\{1, \ldots, n\}, j \neq i$, such that $\mu_{i}(x)=\mu_{j}(x)$. Hence, at each point $x$, we have that either $\mu_{1}(x)=\mu(x)$ and $\mu_{2}(x) \neq \mu(x)$ or $\mu_{1}(x) \neq \mu(x)$ and $\mu_{2}(x)=\mu(x)$. Suppose that $\mu_{1}(x) \neq \mu(x)$; then this holds at all points of an open subset, and on that subset we must have $\mu_{2}=\mu$. Moreoever, because $\Omega$ is skew-symmetric, we must have $\mu_{1}=0$ and so $\mu$ is not identically zero; in particular $n-1$ is even, i.e., $n=2 k+1$ for some integer $k \geq 1$.

From (3.4.3) we get

$$
{ }^{N} \operatorname{Ricci}\left(\varphi_{*} X_{2}, \varphi_{*} X_{2}\right)-{ }^{N} \operatorname{Ricci}\left(\varphi_{*} X_{3}, \varphi_{*} X_{3}\right)=(n-1)(n-2) X_{2}(\sigma)^{2} ;
$$

hence, $X_{2}(\sigma)$ is basic. Thus, if $X_{2}(\sigma) \neq 0$, since $X_{1}(\sigma) X_{2}(\sigma)$ is basic, we deduce that $X_{1}(\sigma)$ is also basic and the proof follows as before. There remains the case when $X_{2}(\sigma)=0$ which we now consider. Summing-up the previous discussion, we have that $n=2 k+1, k \geq 1$, and we are now on an open subset on which we have the following:

$$
\begin{aligned}
& \mu_{1}=0, \mu_{2}=\ldots=\mu_{n}=\mu, \\
& X_{2}(\sigma)=\ldots=X_{n}(\sigma)=0,
\end{aligned}
$$

$$
\mu \text { and } X_{1}(\sigma) \text { are not identically zero. }
$$

Moreover, we can assume that $\mu$ and $X_{1}(\sigma)$ are nowhere zero. Furthermore, because $\Omega^{2}\left(X_{1}\right)=-\mu_{1}^{2}$ we have that $\left|i_{X_{1}} \Omega\right|_{h}^{2}=\mu_{1}^{2}=0$. Hence $i_{X_{1}} \Omega=0$, equivalently $i_{\operatorname{grad}_{h} \sigma} \Omega=0$.

From this and A.2.24 it follows that we have for $i=1, \ldots, n$,

$$
\begin{equation*}
0=\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma}\left({ }^{h} \mathrm{~d}^{*} \Omega\right)\left(X_{i}\right)+(n-1) X_{i}(V(\sigma))-(n-1)(n-2) X_{i}(\sigma) V(\sigma) . \tag{3.4.4}
\end{equation*}
$$

Next, we compute $\left({ }^{h} \mathrm{~d}^{*} \Omega\right)\left(X_{1}\right)$.

$$
\begin{aligned}
\left({ }^{h} \mathrm{~d}^{*} \Omega\right)\left(X_{1}\right) & =-\sum_{j=1}^{2 k+1}\left(\nabla_{X_{j}} \Omega\right)\left(X_{j}, X_{1}\right) \\
& =-\sum_{j=1}^{2 k+1}\left\{X_{j}\left(\Omega\left(X_{j}, X_{1}\right)-\Omega\left(\nabla_{X_{j}} X_{j}, X_{1}\right)-\Omega\left(X_{j}, \nabla_{X_{j}} X_{1}\right)\right\}\right. \\
& =\sum_{j=1}^{2 k+1} \Omega\left(X_{j}, \nabla_{X_{j}} X_{1}\right)=\sum_{j=2}^{2 k+1} \Omega\left(X_{j}, \nabla_{X_{j}} X_{1}\right) \\
& =\sum_{j=1}^{k}\left\{\Omega\left(X_{2 j}, \nabla_{X_{2 j}} X_{1}\right)+\Omega\left(X_{2 j+1}, \nabla_{X_{2 j+1}}, X_{1}\right)\right\} .
\end{aligned}
$$

We can choose a basic orthonormal local frame $\left\{X_{1}, X_{2}, \ldots, X_{2 k+1}\right\}$ such that

$$
\left(\Omega_{i j}\right)=\left(\begin{array}{cccccc}
0 & 0 & 0 & \ldots & 0 & 0  \tag{3.4.5}\\
0 & 0 & -\mu & \ldots & 0 & 0 \\
0 & \mu & 0 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots \ldots . \ldots \ldots . \\
0 & 0 & 0 & \ldots & 0 & -\mu \\
0 & 0 & 0 & \ldots & \mu & 0
\end{array}\right)
$$

Then, from the above calculation we have

$$
\left({ }^{h} \mathrm{~d}^{*} \Omega\right)\left(X_{1}\right)=\sum_{j=1}^{k}\left\{h\left(\nabla_{X_{2 j}} X_{1}, X_{2 j+1}\right) \Omega_{2 j, 2 j+1}+h\left(\nabla_{X_{2 j+1}} X_{1}, X_{2 j}\right) \Omega_{2 j+1,2 j}\right\}
$$

$$
\begin{aligned}
& =\sum_{j=1}^{k}\left\{-\mu h\left(\nabla_{X_{2 j}} X_{1}, X_{2 j+1}\right)+\mu h\left(\nabla_{X_{2 j+1}} X_{1}, X_{2 j}\right)\right\} \\
& =\mu \sum_{j=1}^{k}\left\{h\left(X_{1}, \nabla_{X_{2 j}} X_{2 j+1}\right)-h\left(X_{1}, \nabla_{X_{2 j+1}} X_{2 j}\right)\right\} \\
& =\mu \sum_{j=1}^{k} h\left(X_{1},\left[X_{2 j}, X_{2 j+1}\right]\right) .
\end{aligned}
$$

Recall that $X_{j}(\sigma)=0$ for all $j \geq 2$; hence

$$
\begin{aligned}
{\left[X_{2 j}, X_{2 j+1}\right](\sigma)=0 } & \Longleftrightarrow-\mathscr{V}\left[X_{2 j}, X_{2 j+1}\right](\sigma)=\mathscr{H}\left[X_{2 j}, X_{2 j+1}\right](\sigma) \\
& \Longleftrightarrow \Omega\left(X_{2 j}, X_{2 j+1}\right) V(\sigma)=h\left(\left[X_{2 j}, X_{2 j+1}\right], \mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right) \\
& \Longleftrightarrow-\mu V(\sigma)=h\left(\left[X_{2 j}, X_{2 j+1}\right], X_{1}\right) X_{1}(\sigma) \\
& \Longleftrightarrow h\left(\left[X_{2 j}, X_{2 j+1}\right], X_{1}\right)=-\mu \frac{V(\sigma)}{X_{1}(\sigma)} .
\end{aligned}
$$

It follows from the last equation that

$$
\begin{equation*}
\left({ }^{h} \mathrm{~d}^{*} \Omega\right)\left(X_{1}\right)=-k \mu^{2} \frac{V(\sigma)}{X_{1}(\sigma)} \tag{3.4.6}
\end{equation*}
$$

From (3.4.4 and 3.4.6 we get

$$
0=-\frac{1}{2} k \mu^{2} \mathrm{e}^{4 k \sigma} \frac{V(\sigma)}{X_{1}(\sigma)}+2 k X_{1}(V(\sigma))-2 k(2 k-1) X_{1}(\sigma) V(\sigma)
$$

which is equivalent to

$$
\begin{equation*}
\mu^{2} \mathrm{e}^{4 k \sigma} V(\sigma)=4 X_{1}(\sigma) X_{1}(V(\sigma))-4(2 k-1) X_{1}(\sigma)^{2} V(\sigma) \tag{3.4.7}
\end{equation*}
$$

From (3.4.3) with $i=1, j=2$, we get that $\frac{1}{2} \mathrm{e}^{4 k \sigma} \mu^{2}-2 k(2 k-1) X_{1}(\sigma)^{2}$ is basic and hence on differentiating this with respect to $V$ we obtain

$$
2 k \mathrm{e}^{4 k \sigma} V(\sigma) \mu^{2}-4 k(2 k-1) X_{1}(\sigma) X_{1}(V(\sigma))=0
$$

which is equivalent to

$$
\begin{equation*}
\mu^{2} \mathrm{e}^{4 k \sigma} V(\sigma)=2(2 k-1) X_{1}(\sigma) X_{1}(V(\sigma)) \tag{3.4.8}
\end{equation*}
$$

From (3.4.7) and (3.4.8) we get that

$$
4 X_{1}(\sigma) X_{1}(V(\sigma))-4(2 k-1) X_{1}(\sigma)^{2} V(\sigma)=2(2 k-1) X_{1}(\sigma) X_{1}(V(\sigma))
$$

which, because $X_{1}(\sigma)$ is nowhere zero, is equivalent to

$$
\begin{equation*}
X_{1}(V(\sigma))=-\frac{2(2 k-1)}{2 k-3} X_{1}(\sigma) V(\sigma) . \tag{3.4.9}
\end{equation*}
$$

From (3.4.8) and (3.4.9) it follows that

$$
\mu^{2} \mathrm{e}^{4 k \sigma} V(\sigma)=-\frac{4(2 k-1)^{2}}{2 k-3} X_{1}(\sigma)^{2} V(\sigma)
$$

which, if $\mathscr{V}$ is not Riemannian (equivalently, $V(\sigma) \neq 0$ ), implies that

$$
\begin{equation*}
\mu^{2} \mathrm{e}^{4 k \sigma}=-\frac{4(2 k-1)^{2}}{2 k-3} X_{1}(\sigma)^{2} . \tag{3.4.10}
\end{equation*}
$$

This is impossible if $k \geq 2$, since $X_{1}(\sigma) \neq 0, \mu \neq 0$. The proof of the theorem is complete.

Remark 3.4.2. The same proof as above applies for the case $\operatorname{dim} M=4$ up to (3.4.10 . However, from (3.4.10) and $i_{X_{1}} \Omega=0$, we now have

$$
\begin{equation*}
\mathrm{d}_{\mathscr{\mathscr { C }}}\left(\lambda^{-2}\right)=*_{\mathscr{H}} \Omega, \tag{3.4.11}
\end{equation*}
$$

where $\mathrm{d}^{\mathscr{H}}$ is the differential composed with the horizontal projection and $* \mathscr{H}$ is the Hodge star-operator on $\left(\mathscr{H},\left.h\right|_{\mathscr{H}}\right)$ with respect to some orientation of $\mathscr{H}$.

From the proof of Theorem 3.4.1 it follows that if $\left(M^{4}, g\right)$ is a 4-dimensional Einstein manifold and $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ is a submersive harmonic morphism to a Riemannian 3-manifold which is not of type 1 or of type 2 (i.e. $\mathscr{V}=\operatorname{ker} \varphi_{*}$ is neither Riemannian, nor geodesic with integrable horizontal distribution), then the 'monopole equation' (3.4.11) must hold.

Since by [4 any harmonic morphism with one-dimensional fibres from a Riemannian manifold of dimension at least 5 is submersive, from Theorem 3.4.1 we obtain the following.

Corollary 3.4.3. Let $\left(M^{n+1}, g\right)$ be an Einstein manifold of dimension $n+1 \geq$ 5, and let $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right)$ be a non-constant harmonic morphism to a Riemannian manifold of dimension $n$.

Then either,
(i) the components of the fibres of $\varphi$ form a Riemannian foliation locally tangent to nowhere zero Killing vector fields, or
(ii) $\varphi$ is a horizontally homothetic submersion with geodesic fibres orthogonal to an umbilical foliation by hypersurfaces.

### 3.5. The four-dimensional case

To state the main result of this section we need a definition which is a trivial generalization to foliations of the well-known notion of principal connection on a principal bundle. For simplicity, we give this definition just for one-dimensional foliations.

Definition 3.5.1. Let $\mathscr{V}$ be a one-dimensional foliation and let $V \in \Gamma(\mathscr{V})$ be a nowhere zero vector field tangent to $\mathscr{V}$.

A principal connection for $\mathscr{V}$ (with respect to $V$ ) is a complementary distribution $\mathscr{H} \subseteq T M, \mathscr{H} \oplus \mathscr{V}=T M$ such that $V$ is an infinitesimal automorphism of $\mathscr{H}$ (i.e. $\mathscr{H}$ is invariant under the local flow of $V$ ).

The connection form $\theta$ of $\mathscr{H}$ is the 'vertical' dual of $V$ (i.e. $\theta(V)=1$ and $\left.\theta\right|_{\mathscr{H}}=0$ ) and the curvature form of $\mathscr{H}$ is $\Omega=\mathrm{d} \theta$. Note that $\Omega$ is basic and it can be interpreted as the integrability tensor of $\mathscr{H}$ (indeed $\Omega(X, Y) V=-\mathscr{V}([X, Y])$ for any horizontal vector fields $X$ and $Y)$.

It is obvious that a one form $\theta$ defines a principal connection for $\mathscr{V}$ with respect to $V$ if and only if $\theta(V)=1$ and $\mathcal{L}_{V} \theta=0$.

Example 3.5.2. Let $\mathscr{V}$ be an orientable one-dimensional geodesic foliation on $(M, g)$. Then $\mathscr{H}\left(=\mathscr{V}^{\perp}\right)$ is a principal connection for $\mathscr{V}$ with respect to $U \in \Gamma(\mathscr{V})$ where $g(U, U)=1$. The connection form is $U^{\mathrm{b}}$.

An orientable one-dimensional foliation $\mathscr{V}$ on $M$ admits a principal connection if and only if it is geodesible (i.e. there exists a Riemannian metric $h$ on $M$ such that the leaves of $\mathscr{V}$ are geodesics on $(M, h))$. Indeed, given the principal connection $\mathscr{H}$ (with respect to some $V \in \Gamma(\mathscr{V})$ ), if we choose any metric $h$ such that $h(V, V)=1$ and $h(V, X)=0$ for $X \in \mathscr{H}$ then the leaves of $\mathscr{V}$ are geodesics of $(M, h)$. Also the set of principal connections of $\mathscr{V}$ (if nonempty) with respect to a given nowhere zero vector field $V \in \Gamma(\mathscr{V})$ is an affine space over the linear space of basic one-forms: if $\theta_{j}, j=1,2$ are
connection forms then $\theta_{1}-\theta_{2}$ is locally the pull back by $\varphi: U \rightarrow N$ of a one-form $A \in \Gamma\left(T^{*} N\right)$ where $U$ is an open subset of $M$ and the fibres of $\varphi$ are open subsets of leaves of $\mathscr{V}$. Fix $V \in \Gamma(\mathscr{V})$. Then, in a neighbourhood of each point of $N$, a local section $s$ of $\varphi$ can be found which, in a neighbourhood of its image, defines a principal connection $\theta_{s}$ which is flat (i.e. $\mathrm{d} \theta_{s}=0$ ). If $\theta$ defines a principal connection then the one-form $A$ such that $\theta=\theta_{s}+\varphi^{*}(A)$ is the local connection form of $\theta$ with respect to $s$. Because $\mathscr{V}$ is one-dimensional we can define the local connection form of a principal connection with respect to a (local) flat principal connection by using any parallel section of the flat connection. Also note that the existence of a global flat principal connection imposes, severe restrictions on the topology of the foliation and of the manifold. For example, as is well known, if the leaves of $\mathscr{V}$ are the fibres of a principal bundle $\xi=\left(M, N, S^{1}\right)$ over a simply-conected $N$ and $\xi$ admits a flat principal connection then $\xi$ is trivial and, in particular, $M$ and $N \times S^{1}$ are diffeomorphic.

The orthogonal complement of a one-dimensional foliation which produces harmonic morphisms is a principal connection of it.

Proposition 3.5.3. Let $\mathscr{V}$ be a one-dimensional foliation which produces harmonic morphisms on $\left(M^{n+1}, g\right)$ where $\operatorname{dim} M=n+1$. Let $\rho=\mathrm{e}^{(2-n) \sigma}$ be a local density of $\mathscr{V}$. Suppose that $\mathscr{V}$ restricted to the domain of $\sigma$ is orientable and let $V \in \Gamma(\mathscr{V})$ be such that $g(V, V)=\mathrm{e}^{(2 n-4) \sigma}\left(=\rho^{-2}\right)$.

Then the horizontal distribution $\mathscr{H}\left(=\mathscr{V}^{\perp}\right)$ is a principal connection for $\sqrt{v}$ with respect to $V$.

Proof. This follows from Lemma 3.1.1.
We now state the main result of this section.
Theorem 3.5.4. Let $\left(M^{4}, g\right)$ be an Einstein manifold of dimension four and $\mathscr{V}$ a one-dimensional foliation which produces harmonic morphisms on $\left(M^{4}, g\right)$.

Then, one of the following assertions holds:
(i) $\mathscr{V}$ is Riemannian and locally generated by Killing fields;
(ii) $\mathscr{V}$ is a homothetic foliation by geodesics with integrable orthogonal complement;
(iii) $\left(M^{4}, g\right)$ is Ricci-flat and, up to homotheties, any harmonic morphism $\varphi:\left(U,\left.g\right|_{U}\right) \rightarrow\left(N^{3}, h\right)$, with dilation $\lambda$, produced by $\mathscr{V}$ such that $\left.\mathscr{V}\right|_{U}$ and $N^{3}$ are orientable is (locally) described as follows:
(a) $\left(N^{3}, h\right)$ has constant sectional curvature $k^{N}=1$,
(b) $\frac{1}{2} \mathrm{~d}\left(\lambda^{-2}\right)$ is a (flat) principal connection for $\mathscr{V}$ with respect to suitably chosen $V \in \Gamma(\mathscr{V})$ such that $g(V, V)=\lambda^{2}$,
(c) the local connection form $A$ of $\mathscr{H}$ with respect to $\frac{1}{2} \mathrm{~d}\left(\lambda^{-2}\right)$ satisfies the equation $\mathrm{d} A+2 * A=0$ on $\left(N^{3}, h\right)$ where $*$ is the Hodge star-operator of $\left(N^{3}, h\right)$ with respect to some orientation of $N^{3}$.

Moreover, only (i) and (ii) or (ii) and (iii) can occur simultaneously, in which case $\left(M^{4}, g\right)$ must be Ricci-flat.

From Theorem 3.5.4 we obtain the following.
Corollary 3.5.5. Let $\left(M^{4}, g\right)$ be an orientable Einstein manifold of dimension four, and $\left(N^{3}, h\right)$ an orientable Riemannian manifold of dimension three.

Let $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ be a submersive harmonic morphism; denote its dilation by $\lambda$ and let $V \in \Gamma(\mathscr{V})$ be such that $g(V, V)=\lambda^{2}$.

Then, one of the following assertions (i), (ii), (iii) holds:
(i) $V$ is a Killing field;
(ii) $\varphi$ is horizontally homothetic and has geodesic fibres orthogonal to an umbilical foliation by hypersurfaces;
(iii) (a) $\left(M^{4}, g\right)$ is Ricci-flat and $\left(N^{3}, h\right)$ has constant sectional curvature $k^{N}=\frac{c^{2}}{4}(c \neq 0)$,
(b) $\frac{1}{c} \mathrm{~d}\left(\lambda^{-2}\right)$ is a (flat) principal connection for $\operatorname{ker} \varphi_{*}$ with respect to $V$,
(c) the local connection form $A$ of $\left(\operatorname{ker} \varphi_{*}\right)^{\perp}$ with respect to $\frac{1}{c} \mathrm{~d}\left(\lambda^{-2}\right)$ satisfies $\mathrm{d} A+c * A=0$ on $\left(N^{3}, h\right)$ where $*$ is the Hodge star-operator of $\left(N^{3}, h\right)$ defined by some orientation of $N^{3}$.

Remark 3.5.6. 1) If $M^{4}$ is not orientable then we can replace $\left(M^{4}, g\right)$ by a Riemannian double covering $\left(\widetilde{M^{4}}, \widetilde{g}\right)$ such that $\widetilde{M}^{4}$ is orientable. Then we
replace $\varphi$ by $\widetilde{\varphi}=\varphi \circ \xi$ where $\xi:\left(\widetilde{M}^{4}, \widetilde{g}\right) \rightarrow\left(M^{4}, g\right)$ is the projection of the covering.
2) If $N^{3}$ is not orientable we can pull back $\varphi$ to a Riemannian doublecovering $\left(\widetilde{N}^{3}, \widetilde{h}\right)$ of $\left(N^{3}, h\right)$ such that $\widetilde{N}^{3}$ is orientable.

Proof of Theorem 3.5.4. Suppose that neither assertion (i) nor assertion (ii) holds. Then, by Remark 3.4 .2 if $\varphi:\left(O,\left.g\right|_{O}\right) \rightarrow(N, \bar{h}),(\operatorname{dim} N=3)$, is a harmonic morphism produced by $\mathscr{V}$ the relation (3.4.11) must hold.

If $\Omega=0$ on $O$ then, from (3.4.11), it follows that $\lambda$ is constant along horizontal curves which would imply that assertion (ii) holds.

If $\Omega \neq 0$, then, by analyticity, it is non-zero on a dense open subset of $O$. Now, note that the right hand side of (3.4.11) is basic. Hence $V\left(X\left(\lambda^{-2}\right)\right)=0$ for any basic vector field $X \in \Gamma(\mathscr{H})$. But $V$ commutes with basic vector fields and hence $V\left(\lambda^{-2}\right)$ is constant along horizontal curves. It follows that, if $V\left(\lambda^{-2}\right)$ is non-constant then $\mathscr{H}$ is integrable, equivalently, $\Omega=0$. Thus, $V\left(\lambda^{-2}\right)=c$ for some constant $c \in \mathbb{R}$. Furthermore, as $\mathscr{V}$ is not Riemannian, $c \neq 0$.

Then $(1 / c) \mathrm{d}\left(\lambda^{-2}\right)$ is a (flat) principal connection for $\mathscr{V}$. Let $A \in \Gamma\left(T^{*} N\right)$ be a local connection form of $\mathscr{H}$ with respect to $(1 / c) \mathrm{d}\left(\lambda^{-2}\right)$, that is, $A$ is the one-form on $N$ which satisfies

$$
\begin{equation*}
\theta=\frac{1}{c} \mathrm{~d}\left(\lambda^{-2}\right)+\varphi^{*}(A) . \tag{3.5.1}
\end{equation*}
$$

From (3.4.11) and (3.5.1), it follows that

$$
\begin{equation*}
-c \varphi^{*}(A)=\mathrm{d}^{\mathscr{H}}\left(\lambda^{-2}\right)=* \mathscr{H} \Omega=* \mathscr{H} \varphi^{*}(\mathrm{~d} A)=\varphi^{*}(* \mathrm{~d} A) . \tag{3.5.2}
\end{equation*}
$$

Hence $\mathrm{d} A+c * A=0$ which implies assertion (iii), except for the fact that ( $N, \bar{h}$ ) has constant sectional curvature equal to $c^{2} / 4$ which we shall now prove.

From Lemma A.2.5 we obtain that the Ricci tensors of $(M, g)$ and $(N, \bar{h})$ satisfy the following relations, on $O$ :

$$
\begin{align*}
& \left.{ }^{M} \operatorname{Ricci}\right|_{\mathscr{V} \otimes \mathscr{V}}=0,\left.\quad{ }^{M} \operatorname{Ricci}\right|_{\mathscr{V} \otimes \mathscr{H}}=0, \\
& \left.{ }^{M} \operatorname{Ricci}\right|_{\mathscr{H} \otimes \mathscr{H}}=\varphi^{*}\left({ }^{N} \operatorname{Ricci}\right)-\frac{c^{2}}{2} \varphi^{*}(\bar{h}) . \tag{3.5.3}
\end{align*}
$$

From (3.5.3 it follows easily that, if $\left(M^{4}, g\right)$ is Einstein and 3.4.11 holds, then it is Ricci-flat and $(N, \bar{h})$ has constant sectional curvature equal to $c^{2} / 4$.

Remark 3.5.7. 1) The equation $\mathrm{d} A+c * A=0$ for a one-form $A$ on an oriented three-dimensional Riemannian manifold is the Beltrami fields equation (see [34]). Obviously, if $A$ satisfies the Beltrami fields equation, with $c$ constant, then $\Delta A=c^{2} A$, where $\Delta$ is the Hodge Laplacian; also, $F=\mathrm{d} A$ satifies $\mathrm{d}^{*} F=c * F$.
2) The codomain of a harmonic morphism of type (iii) of Theorem 3.5.4 always has constant positive sectional curvature. In the limit, when this tends to zero, we obtain a harmonic morphism of type (i).
3) Harmonic morphisms of type (iii) are also of type (ii) if and only if $A=0$.
4) The results of Theorems 3.4.1 and 3.5 .4 shows that, on an Einstein manifold of dimension at least four, the nonlinear system of partial differential equations whose solutions are harmonic morphisms with fibres of dimension one can be reduced to one of three types of systems of linear partial differential equations of the first order. For type (i) this is Killing's equation, and for type (iii) it is the above mentioned Beltrami fields equation. Finally, the one-dimensional foliation $\mathscr{V}$ on $(M, g)$ is of type (ii) if and only if it is locally generated by vector fields $W \in \Gamma(\mathscr{V})$ which satisfies $4 \nabla W=\operatorname{div} W \operatorname{Id}_{T M}$ where $\nabla$ is the Levi-Civita connection of $(M, g)$ (see [51, Lemma 6.5]).

See [69], 42] for other situations in which the nonlinear system of partial differential equations whose solutions are harmonic morphisms can be reduced to a linear system of partial differential equations of the first order.

### 3.6. The third type

We shall say that a harmonic morphism $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ is of type (iii) ( (i), (ii) ) if its regular fibres form a foliation of type (iii) ( (i), (ii) ) of Theorem 3.5.4. In this section the harmonic morphisms of type (iii) will be the main object of study.

The first thing to note about the harmonic morphisms of type (iii) is that they are always submersive.

Proposition 3.6.1. Let $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ be a harmonic morphism of type (iii).

Then $\varphi$ is submersive.
Proof. By passing, if necessary, to a two-fold covering, we can suppose that the vertical distribution $\mathscr{V}$ (which is well-defined outside the set of critical points) is orientable. Then, as before, let $V \in \Gamma(\mathscr{V})$ be such that $g(V, V)=\lambda^{2}$ where $\lambda$ is the dilation of $\varphi$. Since, up to a multiplicative constant, $\mathrm{d}\left(\lambda^{-2}\right)$ is a (flat) principal connection with respect to $V$, we have that $V\left(\lambda^{-2}\right)$ is a nonzero constant. This implies that the connected components of any regular fibre of $\varphi$ are noncompact.

Suppose that $\varphi$ is not submersive and let $x_{0} \in M$ be a critical point of it. Recall that, by a result of P. Baird [4, Proposition 5.1], the set of critical points of $\varphi$ must be discrete. Then from the main result of [16] it follows that $\varphi$ is topologically locally equivalent at $x_{0}$ to the cone on the Hopf fibration $S^{3} \rightarrow S^{2}$. Hence, in a neighbourhood of $x_{0}$, the components of the regular fibres of $\varphi$ are diffeomorphic to $S^{1}$. But we have seen that all the regular fibres of $\varphi$ have noncompact components and hence $\varphi$ cannot have critical points.

Also, we have the following result.
Corollary 3.6.2. Let $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ be a harmonic morphism of type (iii).

Then the vertical distribution of $\varphi$ is orientable.
Proof. If $V$ is any local vertical vector field such that $g(V, V)=\lambda^{2}$ then $V\left(\lambda^{-2}\right)$ is a nonzero constant. Hence, there exists a unique vertical vector field $V$ on $M$ such that $g(V, V)=\lambda^{2}$ and $V\left(\lambda^{-2}\right)>0$.

Remark 3.6.3. Let $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ be a harmonic morphism of type (iii) with connected fibres; denote its dilation by $\lambda$. Then $\mathscr{H}^{*}\left(\mathrm{~d}\left(\lambda^{-2}\right)\right)$ is
a basic one-form; let $A \in \Gamma\left(T^{*} N\right)$ be such that $-\frac{1}{2} \mathscr{H}^{*}\left(\mathrm{~d}\left(\lambda^{-2}\right)\right)=\varphi^{*}(A)$. Because $\left(N^{3}, h\right)$ is of constant curvature it is an analytic manifold. But $A$ satisfies $\Delta A=4 A$ and so is analytic. Using this fact it is easy to see that, if $N^{3}$ is orientable, then there exists an orientation of it such that $\mathrm{d} A+2 * A=0$ on $N^{3}$.

From the proof of Theorem 3.5 .4 it follows that any harmonic morphism of type (iii) is locally determined by the local connection form $A$. This is also illustrated by the following example.

Example 3.6.4. Let $h$ be the canonical metric on the three-dimensional sphere $S^{3}$. Let $A=i^{*}\left(-x^{2} \mathrm{~d} x^{1}+x^{1} \mathrm{~d} x^{2}-x^{4} \mathrm{~d} x^{3}+x^{3} \mathrm{~d} x^{4}\right)$ where $i: S^{3} \hookrightarrow \mathbb{R}^{4}$ is the canonical inclusion.

Let $*$ be the Hodge star-operator on $\left(S^{3}, h\right)$ considered with the usual orientation of $S^{3}$. Then

$$
\mathrm{d} A-2 * A=0 .
$$

To show this, firstly note that $A$ is the canonical connection form on the Hopf bundle ( $S^{3}, S^{2}, S^{1}$ ). Also $|A|=1$ and thus it is sufficient to verify that $A \wedge \mathrm{~d} A=2 v_{S^{3}}$ where $v_{S^{3}}$ is the usual volume form on $S^{3}$.

For $a \in \mathbb{R}$ let $g_{a}$ be the Riemannian metric on $\mathbb{R}^{4} \backslash\{0\}=(0, \infty) \times S^{3}$ defined by

$$
g_{a}=\rho^{2} h+\rho^{-2}(\rho \mathrm{~d} \rho+a A)^{2} .
$$

Then, for any $a \neq 0$, the canonical projection $\varphi_{a}:\left(\mathbb{R}^{4} \backslash\{0\}, g_{a}\right) \rightarrow\left(S^{3}, h\right)$ is a harmonic morphism of type (iii) whilst $g_{0}$ is the restriction to $\mathbb{R}^{4} \backslash\{0\}$ of the canonical metric on $\mathbb{R}^{4}$ and thus $\varphi_{0}: \mathbb{R}^{4} \backslash\{0\} \rightarrow S^{3}$ is the usual radial projection; note that this is also of type (ii).

Note that $\left(\mathbb{R}^{4} \backslash\{0\}, g_{a}\right)$ is the Eguchi-Hanson II metric [19] and thus is Ricci-flat and anti-self-dual.

Let $\psi_{a}=\pi \circ \varphi_{a}$ where $\pi: S^{3} \rightarrow S^{2}$ is the Hopf fibration. Then $\psi_{a}$ is a harmonic morphism with totally geodesic fibres. Any fibre of it is isometric with $\left(\mathbb{R}^{2} \backslash\{0\}, \gamma_{a}\right)$ where $\gamma_{a}$ is given in polar coordinates $(\rho, \theta)$ by

$$
\gamma_{a}=\rho^{2} \mathrm{~d} \theta^{2}+\rho^{-2}(\rho \mathrm{~d} \rho+a \mathrm{~d} \theta)^{2} .
$$

It is easy to see that any point of $\mathbb{R}^{2} \backslash\{0\}$ is at finite distance from 0 with respect to $\gamma_{a}$. Hence $\left(\mathbb{R}^{2} \backslash\{0\}, \gamma_{a}\right)$ is not complete. Because the fibres of $\psi_{a}$ are closed and totally geodesic we obtain that $g_{a}$ is not complete for any $a \in \mathbb{R}$.

We shall prove that the $\varphi_{a}$ of Example 3.6 .4 are, essentially, the only surjective harmonic morphisms of type (iii) with connected fibres and complete simply-connected codomain. For this we need the following:

Proposition 3.6.5. Let $S^{3}(=\operatorname{Sp}(1))$ be the three-dimensional sphere endowed with its canonical metric and orientation and let $*$ be the Hodge star-operator on it.
(i) The space of solutions of the equation

$$
\begin{equation*}
\mathrm{d} A+2 * A=0, \quad A \in \Gamma\left(T^{*} S^{3}\right) \tag{3.6.1}
\end{equation*}
$$

is the space of left-invariant one-forms on $S^{3}$.
(ii) The space of solutions of the equation

$$
\begin{equation*}
\mathrm{d} A-2 * A=0, \quad A \in \Gamma\left(T^{*} S^{3}\right) \tag{3.6.2}
\end{equation*}
$$

is the space of right-invariant one-forms on $S^{3}$.
Proof. (i) Let $S^{3} \times \mathrm{Sp}(1) \rightarrow S^{3}$ be the unique spin structure on $S^{3}$ and let $S^{3} \times \mathbb{H} \rightarrow S^{3}$ be the spinor bundle induced by the action of the Clifford algebra $C l_{3}=\mathbb{H} \oplus \mathbb{H}$ on $\mathbb{H}$ given by $(x, y) \cdot q=x \cdot q$.

Consider the trivialization $T S^{3}=S^{3} \times \operatorname{Im} \mathbb{H}$ induced by the canonical left action of $S^{3}(=\operatorname{Sp}(1))$ on itself. Thus any one-form $A$ on $S^{3}$ can be viewed as a spinor field $A: S^{3} \rightarrow \operatorname{Im} \mathbb{H} \subseteq \mathbb{H}$ which is constant if and only if the corresponding one-form is left-invariant.

Consider the Dirac operator $D$ obtained by using the trivial flat connection on $S^{3} \times \mathbb{H} \rightarrow S^{3}$. Then it is easy to see that $A \in \Gamma\left(T^{*} S^{3}\right)$ satisfies (3.6.1) if and only if $D A=0$. Also a straightforward calculation gives $D^{2}=\Delta+2 D$ where $\Delta$ is the usual Laplacian acting on $\mathbb{H}$-valued functions on $S^{3}$. Thus, any solution $A$ of (3.6.1) induces a harmonic $\mathbb{H}$-valued function on $S^{3}$ which must be constant if $A$ is globally defined on $S^{3}$.
(ii) Since the isometry $x \mapsto x^{-1}$ of $S^{3}$ reverses the orientation, it pulls
back solutions of (3.6.1) to solutions of (3.6.2) . Thus the proof of (ii) follows from (i).

Remark 3.6.6. There are other ways to describe the solutions of the equations (3.6.1) and (3.6.2) . For example, since any orthogonal complex structure on $\mathbb{R}^{4}(=\mathbb{H})$ compatible with the canonical orientation can be described as left multiplication by imaginary quaternions of length one (see [15]) any solution of (3.6.2) is, up to a multiplicative constant, of the form

$$
A=i^{*}\left(\sum_{a, b} J_{b}^{a} x^{b} \mathrm{~d} x^{a}\right),
$$

where $J$ is any orthogonal complex structure which induces the canonical orientation on $\mathbb{R}^{4}$ (that is, if $\left\{u_{1}, u_{2}\right\}$ is a complex basis of $\left(\mathbb{R}^{4}, J\right)$ then ( $u_{1}, J u_{1}, u_{2}, J u_{2}$ ) is positively oriented) and $i: S^{3} \hookrightarrow \mathbb{R}^{4}$ is the canonical inclusion. This can also be checked directly.

Also, any solution $A$ of (3.6.2) can be written $A=* i^{*}(F)$ where $F \in$ $\Lambda_{+}^{2}\left(\mathbb{R}^{4}\right)$ is a self-dual two-form.

In fact, by using these characterisations an alternative proof for Proposition 3.6 .5 can be obtained. First, note that, for each one of the equations (3.6.1) and (3.6.2) we have a three dimensional space of solutions. Then, it is easy to see that if $A$ satisfies (3.6.1) or (3.6.2) then $A$ is coclosed and $\Delta A=4 A$ where $\Delta$ is the Hodge Laplacian on $S^{3}$. Thus $A$ is in the eigenspace corresponding to the first eigenvalue of $\Delta$ acting on coclosed one-forms of $S^{3}$ and it is well-known that this space is of dimension six (see [23, 7.2] or apply one of the results from [41, page 148] and [35, Chapter II, Theorem 2.3] ).

Proposition 3.6.7. Let $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ be a surjective harmonic morphism of type (iii) such that $\left(N^{3}, h\right)$ is complete, simply-connected and $\varphi$ has connected fibres.

Then there exists $a \in \mathbb{R}$ such that, up to homotheties, $\varphi$ is a restriction of $\varphi_{a}:\left(\mathbb{R}^{4} \backslash\{0\}, g_{a}\right) \rightarrow\left(S^{3}, h\right)$ from Example 3.6.4.

Proof. Up to a homothety, we can identify ( $N^{3}, h$ ) with $S^{3}$ considered with its canonical metric and orientation. Let $\lambda$ be the dilation of $\varphi$. Then, by Proposition 3.6.5, there exists $a \in \mathbb{R}$ such that, up to an isometry of $S^{3}$,
$-\frac{1}{2} \mathscr{H}^{*}\left(\mathrm{~d}\left(\lambda^{-2}\right)\right)=a \varphi^{*}(A)$ where $A \in \Gamma\left(T^{*} S^{3}\right)$ is as in Example 3.6.4. By Proposition 3.6.1, $\varphi$ is submersive and let $\mathscr{V}=\operatorname{ker} \varphi_{*}$. Because $\mathscr{V}$ is orientable we can find $V \in \Gamma(\mathscr{V})$ such that $g(V, V)=\lambda^{2}$.

Because $\varphi$ is of type (iii) we have that $V\left(\lambda^{-2}\right)$ is a nonzero constant. This implies that the restriction of $\lambda$ to any fibre of $\varphi$ is a diffeomorphism onto some open subinterval of $(0, \infty)$. Hence the map $\Phi: M^{4} \rightarrow S^{3} \times(0, \infty)$ defined by $\Phi(x)=\left(\varphi(x), \lambda(x)^{-1}\right), x \in M^{4}$, is a diffeomorphic embedding.

Then from the proof of Theorem 3.5.4 it follows that the map $\Phi$ from $\left(M^{4}, g\right)$ to $\left(S^{3} \times(0, \infty), g_{a}\right)$ is a local isometry and hence an isometric embedding. Also, it is obvious that $\varphi_{a} \circ \Phi=\varphi$.

Corollary 3.6.8. Let $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ be a surjective harmonic morphism of type (iii) with connected fibres and such that $\left(N^{3}, h\right)$ is complete.

Then $\left(M^{4}, g\right)$ is not complete.
Proof. Up to homotheties, the universal covering of $\left(N^{3}, h\right)$ is $S^{3}$ with its canonical metric and orientation. Then, $\varphi$ can be pulled back via $S^{3} \rightarrow N^{3}$ to a harmonic morphism whose total space is complete if and only if $\left(M^{4}, g\right)$ is complete. Define $\Phi$ as in the proof of Proposition 3.6.7. Then $\Phi$ is a local isometry and because $\left(\mathbb{R}^{4} \backslash\{0\}, g_{a}\right)$ is not complete $\left(M^{4}, g\right)$ is not complete.

### 3.7. Harmonic morphisms $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ between Einstein manifolds

In this section $\left(M^{4}, g\right)$ and $\left(N^{3}, h\right)$ will be Einstein manifolds of dimension four and three, respectively, (since $N^{3}$ is three-dimensional this means that $\left(N^{3}, h\right)$ is of constant curvature) and $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ will be a harmonic morphism. Recall that, by a result of P. Baird [4, Proposition 5.1], the set of critical points of $\varphi$ is discrete and hence, by the second axiom of countability, at most countable.

We now state one of the main results of this section enumerating all surjective submersive harmonic morphisms between complete simply-connected Einstein manifolds of dimension four and three, respectively.

Theorem 3.7.1. Let $\left(M^{4}, g\right)$ be a complete simply-connected Einstein manifold and let $\left(N^{3}, h\right)$ be complete, simply-connected and with constant curvature.

Let $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ be a surjective submersive harmonic morphism with connected fibres.

Then, up to homotheties, $\varphi$ is one of the following projections: $\mathbb{R}^{4} \rightarrow \mathbb{R}^{3}$, $H^{4} \rightarrow \mathbb{R}^{3}$, $H^{4} \rightarrow H^{3}$ induced by the following canonical warped-product decompositions $\mathbb{R}^{4}=\mathbb{R}^{1} \times \mathbb{R}^{3}$, $H^{4}=H^{1} \times_{r} \mathbb{R}^{3}$, $H^{4}=H^{1} \times_{s} H^{3}$ where $H^{k}$ denotes the hyperbolic space of dimension $k$.

Proof. First we prove that $\left(M^{4}, g\right)$ has constant curvature and that $\varphi$ has geodesic fibres and integrable horizontal distribution.

By Corollaries 3.5 .5 and 3.6 .8 either (i) the vertical distribution of $\varphi$ is Riemannian and locally generated by Killing fields or (ii) $\varphi$ has geodesic fibres and integrable horizontal distribution.

Suppose that case (i) holds. Then there exists a function $\check{\lambda}$ on $N^{3}$ such that $e^{\sigma}=\varphi^{*}(\check{\lambda})$. Also, there exists a two-form $F$ on $N^{3}$ such that $\Omega=\varphi^{*}(F)$. Furthermore, by Remark 3.4.2, we have

$$
\begin{equation*}
F=* \mathrm{~d} \check{\lambda}^{-2}, \tag{3.7.1}
\end{equation*}
$$

where $*$ is the Hodge star-operator on $\left(N^{3}, h\right)$.
But $\mathrm{d} F=0$ and thus (3.7.1) implies that $\check{\lambda}^{-2}$ is a positive harmonic function on $\left(N^{3}, h\right)$.

From (A.2.23) and A.2.25) we obtain

$$
\begin{equation*}
k^{N}=\check{\lambda}^{-2} c^{M} \tag{3.7.2}
\end{equation*}
$$

where $k^{N}$ is the constant sectional curvature of $\left(N^{3}, h\right)$ and $c^{M}$ is the Einstein constant of $\left(M^{4}, g\right)$. Thus either $\check{\lambda}$ is constant or $k^{N}=c^{M}=0$. But in the latter case, by Liouville's theorem, $\check{\lambda}^{-2}$ must be constant. Hence, $\varphi$ has geodesic fibres. Moreover, by (3.7.1), $F=0$; equivalently, $\mathscr{H}$ is integrable.

Thus, we always have case (ii). The fact that $\left(M^{4}, g\right)$ has constant curvature now follows from Corollary 3.3.2.

Note that $\mathscr{H}$ is an Ehresmann connection for $\varphi$ (see [9]); moreover, $\mathscr{H}$
is flat (i.e. integrable). Hence, any maximal integral submanifold of it is a covering space of $N^{3}$. But $N^{3}$ is simply-connected and hence $\varphi$ admits a (global) horizontal section. The proof of the theorem follows.

From the proof of Theorem 3.7.1 we, also, obtain.
Proposition 3.7.2. Let $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ be a harmonic morphism between Einstein manifolds and let $\lambda$ be its dilation. Suppose that the regular fibres of $\varphi$ form a Riemannian foliation.

Then, up to homotheties, $\varphi$ can be (locally) characterised as follows:

- $\left(M^{4}, g\right)$ is Ricci-flat and $\left(N^{3}, h\right)$ is flat;
- $\lambda^{-2}$ is the pull back of a local positive harmonic function $u$ on $\left(N^{3}, h\right)$ (in particular, $\lambda^{-2}$ is a harmonic function on ( $M^{4}, g$ ) ;
- Any local connection form $A\left(=s^{*} \theta\right)$ of the horizontal distribution satisfies

$$
\mathrm{d} A=* \mathrm{~d} u,
$$

where $*$ is the Hodge star-operator of $\left(N^{3}, h\right)$ with respect to some (local) orientation (equivalently, the curvature form $F=\mathrm{d} A$ satisfies the monopole equation $F=* \mathrm{~d} u)$;

- In a neighbourhood of the local section s of $\varphi$ where $\varphi$ is equivalent to a projection we have

$$
g=u h+u^{-1}(\mathrm{~d} t+A)^{2} .
$$

Remark 3.7.3. 1) Note that the metric $g$ of Proposition 3.7 .2 is constructed by applying the Gibbons-Hawking construction [31, [24] (cf. [40]).
2) Let $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ be a harmonic morphism between Einstein manifolds. If $\left(M^{4}, g\right)$ does not have constant curvature or the horizontal distribution is nonintegrable then $\left(M^{4}, g\right)$ is Ricci-flat and $\varphi$ is of type (i) (and hence locally given as in Proposition 3.7.2) or type (iii) of Theorem 3.5.4. This follows from Theorem 3.5 .4 and Corollary 3.3.2.

Let $a \geq 0$. If we apply the Gibbons-Hawking construction (with the convention $\mathrm{d} A=-* \mathrm{~d} u)$ to the harmonic function $u_{a}: \mathbb{R}^{3} \backslash\{0\} \rightarrow(0, \infty)$
defined by $u_{a}(y)=\frac{1}{4}\left(\frac{1}{|y|}+a\right), y \in \mathbb{R}^{3} \backslash\{0\}$, then the following metric is obtained [31], [24] (see [40]).

Definition 3.7.4. Let $a>0$. The Gibbons-Hawking Taub-NUT metric is the Riemannian metric on $\mathbb{R}^{4}$ defined by

$$
g_{a}=\left(a|x|^{2}+1\right) g_{0}-\frac{a\left(a|x|^{2}+2\right)}{a|x|^{2}+1}\left(-x^{2} \mathrm{~d} x^{1}+x^{1} \mathrm{~d} x^{2}-x^{4} \mathrm{~d} x^{3}+x^{3} \mathrm{~d} x^{4}\right)^{2} .
$$

For $a=0$ this gives the canonical metric $g_{0}$ on $\mathbb{R}^{4}$.
Note that $g_{1}$ is discussed in 40].
Remark 3.7.5. 1) For any $a \geq 0$ the Hopf polynomial map $\varphi:\left(\mathbb{R}^{4}, g_{a}\right) \rightarrow$ $\left(\mathbb{R}^{3}, h_{0}\right),\left(z^{1}, z^{2}\right) \mapsto\left(\left|z^{1}\right|^{2}-\left|z^{2}\right|^{2}, 2 z^{1} \overline{z^{2}}\right)$, can be thought of as the harmonic morphism induced by the isometric action (see Example 2.3.22(1)) of $S^{1}$ on $\left(\mathbb{R}^{4}, g_{a}\right)$ where $h_{0}$ is the canonical metric on $\mathbb{R}^{3}$. In particular, $\left(\mathbb{R}^{4}, g_{a}\right)$ is Ricci-flat for any $a \geq 0$; moreover, ( $\mathbb{R}^{4}, g_{a}$ ) is, also, self-dual (see 56]) and therefore hyper-Kähler.
2) Moreover, we can consider $a=\varphi^{*}(\check{a})$ to be the pull back of a nonnegative harmonic function $\check{a}$ defined in the neighbourhood of $0 \in \mathbb{R}^{3}$. Then, the resulting metric $g_{a}$ is still Ricci-flat self-dual and with respect to it the Hopf polynomial map, suitably restricted, is a harmonic morphism.

For the next construction we follow C.R. LeBrun's discussion [40] of the Gibbons-Hawking construction [31, [24].

Example 3.7.6. Let $u: \mathbb{R}^{3} \backslash C_{u} \rightarrow(0, \infty)$ be a positive harmonic function whose set of singularities $C_{u}=\left\{y_{j}\right\}_{j \in I}$ is discrete. Hence $I$ is finite or countable. Thus by applying Bôcher's theorem, the 'minimum' and Harnack's principles (see [2]) we obtain

$$
\begin{equation*}
u(y)=a+\sum_{j \in I} \frac{b_{j}}{\left|y-y_{j}\right|} \quad\left(y \in \mathbb{R}^{3}\right), \tag{3.7.3}
\end{equation*}
$$

where $a \geq 0$ and $b_{j} \geq 0$ are nonnegative constants. Suppose that $u$ has the same residue $b(>0)$ at each singular point, i.e. $b_{j}=b$ for each $j \in I$.

Let $F_{u} \in \Gamma\left(\Lambda^{2}\left(T^{*}\left(\mathbb{R}^{3} \backslash C_{u}\right)\right)\right)$ be defined by $F_{u}=-* \mathrm{~d} u$ where $*$ is the

Hodge star-operator on $\mathbb{R}^{3}$. Because $u$ is harmonic we have $\mathrm{d} F_{u}=0$. Then, taking $S^{1}=\mathbb{R} / 4 \pi b \mathbb{Z}$, the cohomology class

$$
\frac{1}{4 \pi b}\left[F_{u}\right] \in H^{2}\left(\mathbb{R}^{3} \backslash C_{u}, \mathbb{Z}\right)=H^{1}\left(\mathbb{R}^{3} \backslash C_{u}, S^{1}\right)
$$

is the first Chern class of a principal bundle $\xi_{u}=\left(P_{u}, \mathbb{R}^{3} \backslash C_{u}, S^{1}\right)$ with projection $\psi_{u}: P_{u} \rightarrow \mathbb{R}^{3} \backslash C_{u}$. It is not difficult to see, by using the homotopy sequence of $\xi_{u}$, that $P_{u}$ is simply-connected.

As is well-known, $F_{u}$ is the curvature form of a principal connection given by $\theta_{u} \in \Gamma\left(T^{*} P_{u}\right)$. Note that if $A$ is a local connection form of $\theta_{u}$ with respect to some local section of $\xi_{u}$, then $\mathrm{d} A=-* \mathrm{~d} u$.

Let $h_{0}$ be the canonical metric on $\mathbb{R}^{3}$ and define $\gamma_{u}=\psi_{u}^{*}\left(u h_{0}\right)+\psi_{u}^{*}\left(u^{-1}\right) \theta_{u}^{2}$. Then $\psi_{u}:\left(P_{u}, \gamma_{u}\right) \rightarrow\left(\mathbb{R}^{3} \backslash C_{u},\left.h_{0}\right|_{\mathbb{R}^{3}} \backslash C_{u}\right)$ is a harmonic morphism.

The key point of the construction is the fact that $\psi_{u}$ can be extended to a harmonic morphism whose codomain is $\mathbb{R}^{3}$.

To prove this, first note that if $C_{u}=\{0\}$ then $\xi_{u}$ is the cylinder on the Hopf bundle ( $S^{3}, S^{2}, S^{1}$ ) and hence $\psi_{u}$ is the restriction of the Hopf polynomial map to $\mathbb{R}^{4} \backslash\{0\}$. Moreover, one can easily verify that $\gamma_{u}$ is homothetic to the restriction of the Gibbons-Hawking Taub-NUT metric $g_{4 a}$ to $\mathbb{R}^{4} \backslash\{0\}$ where, from now on, we consider, for simplicity, that $b=\frac{1}{4}$.

Let $v(y)=\frac{b}{\left|y-y_{1}\right|}$ and $w=u-v$. Then

$$
\xi_{u}=\xi_{v+w}=\left.\left.\xi_{v}\right|_{\mathbb{R}^{3} \backslash C_{u}} \cdot \xi_{w}\right|_{\mathbb{R}^{3} \backslash C_{u}}
$$

where '.' denotes the group operation in $H^{1}\left(\mathbb{R}^{3} \backslash C_{u}, S^{1}\right)$. There exists a neighbourhood $U$ of $y_{1}$ such that $U \cap C_{u}=\left\{y_{1}\right\}$ and hence $\left.w\right|_{U}$ is a welldefined positive harmonic function. By taking $U$ to be contractible we get that $\left.\xi_{w}\right|_{U}$ is trivial (equivalently, it is the neutral element of $H^{1}\left(U, S^{1}\right)$ ). Then $\left.\xi_{u}\right|_{U \backslash\left\{y_{1}\right\}}=\left.\xi_{v}\right|_{U \backslash\left\{y_{1}\right\}}$ and hence $\psi_{u}$ can be extended so that its image contains $y_{1}$. More precisely, we can add a point $x_{1}$ to $\psi_{u}^{-1}(U)$ such that the extended map is smoothly equivalent in a neighbourhood of $x_{1}$ to the cone on the Hopf fibration $S^{3} \rightarrow S^{2}$. Moreover, because $w$ has no singularities in $U$ the metric $\gamma_{u}$ extends over $x_{1}$ to a metric which is homothetic, in the neighbourhood of $x_{1}$, to the metric $g_{4 w}$ of Remark 3.7.5(2).

In this way $\left(P_{u}, \gamma_{u}\right)$ can be extended to a Riemannian manifold ( $M_{u}, g_{u}$ ) and $\psi_{u}$ can be extended to a surjective harmonic morphism $\varphi_{u}:\left(M_{u}, g_{u}\right) \rightarrow$ $\left(\mathbb{R}^{3}, h_{0}\right)$ where $h_{0}$ is the canonical metric on $\mathbb{R}^{3}$. Note that $\left(M_{u}, g_{u}\right)$ is Ricciflat, simply-connected and that $\varphi_{u}$ is induced by an isometric action.

Proposition 3.7.7. Let $\left(M^{4}, g\right)$ be a four-dimensional Einstein manifold and let $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ be a harmonic morphism with one-dimensional fibres to a three-dimensional Riemannian manifold.

If $\varphi$ has critical points, then $\left(M^{4}, g\right)$ is Ricci-flat and the fibres of $\varphi$ are locally generated by Killing vector fields. If, further, the vertical distribution of $\varphi$ is orientable then there exists a (real-analytic) Killing vector field tangent to the fibres of $\varphi$ whose zero set is equal to the set of critical points of $\varphi$.

Proof. As $\varphi$ has critical points, by Corollary 3.5.5 and Proposition 3.6.1, either the fibres of $\varphi$ are locally generated by Killing vector fields or $\varphi$ is a horizontally homothetic submersion. But in the latter case, by a result of B. Fuglede [22], $\varphi$ would be submersive.

Suppose that the vertical distribution $\mathscr{V}$ of $\varphi$ is orientable and let $\lambda$ be the dilation of $\varphi$. Also, as above, let $V \in \Gamma(\mathscr{V})$ be such that $g(V, V)=\lambda^{2}$, let $\theta$ be its vertical dual and denote $\Omega=\mathrm{d} \theta$.

Obviously, $V$ can be extended to a continuous vector field on $M$ whose zero set is equal to $C_{\varphi}$, the set of critical points of $\varphi$.

Then Proposition 3.1 .3 implies that $V$ is a Killing field on $\left(M \backslash C_{\varphi},\left.g\right|_{M \backslash C_{\varphi}}\right)$. Hence it satisfies the equation

$$
\begin{equation*}
\nabla^{*} \nabla V={ }^{M} \operatorname{Ricci}(V) \tag{3.7.4}
\end{equation*}
$$

(see, for example, [35, page 44]), where ${ }^{M} \operatorname{Ricci} \in \Gamma\left(T M \otimes T^{*} M\right)$ denotes the $(1,1)$ tensor field associated to the Ricci tensor of $\left(M^{4}, g\right)$. From the regularity of solutions of elliptic equations (see [11, page 467]) it follows that $V$ is a smooth (in fact, analytic) vector field on $M$.

To complete the proof we have to show that $\left(M^{4}, g\right)$ is Ricci-flat, without the assumption that $\mathscr{V}$ is orientable. Nevertheless, as $C_{\varphi}$ is discrete, for any critical point $x_{0}$ there exists a neighbourhood $U$ such that $\mathscr{V}$ restricted to
$U \backslash\left\{x_{0}\right\}$ is orientable. Moreover, by [16], we may suppose that $\left.\varphi\right|_{U}$ is topologically equivalent to the cone over the Hopf fibration $S^{3} \rightarrow S^{2}$; in particular, $\left.\varphi\right|_{U}$ has connected fibres.

Then, on $U \backslash\left\{x_{0}\right\}$, because $\lambda$ and $\Omega$ are basic they are the pull-backs of a function and a two-form, respectively, which are defined on $N^{3}$. For simplicity, we shall denote the corresponding objects on $N^{3}$ by the same letters $\lambda$ and $\Omega$. Recall that ${ }^{M}$ Ricci $=c^{M} g$ and $\left.g\right|_{\mathscr{H}}=\lambda^{-2} \varphi^{*}(h)$. Thus A.2.23) can be written as an equation on $N^{3}$. Furthermore, by applying A.2.25, the corresponding equation on $N^{3}$ can be written as follows:

$$
\begin{equation*}
{ }^{N} \text { Ricci }=2 c^{M} \lambda^{-2} h-\frac{1}{2} \lambda^{4}(* \Omega) \otimes(* \Omega)+2 \lambda^{-2} \mathrm{~d} \lambda \otimes \mathrm{~d} \lambda . \tag{3.7.5}
\end{equation*}
$$

From (3.7.5), it follows that any vector orthogonal to both $(* \Omega)^{\sharp}$ and $\operatorname{grad} \lambda$ is an eigenvector for ${ }^{N}$ Ricci, the corresponding eigenvalue being $c^{M} \lambda^{-2}$. Hence, if $c^{M} \neq 0$, this eigenvalue tends to $\infty$ as we approach a critical value of $\varphi$, which is obviously impossible (apply, for example, Lemma A.3.1) . Thus $c^{M}=0$, i.e., ${ }^{M}$ Ricci $=0$.

We can now prove the next main result of this section enumerating all the surjective harmonic morphisms with critical points between complete, simplyconnected Einstein manifolds of dimensions four and three.

Theorem 3.7.8. Let $\left(M^{4}, g\right)$ be a complete Einstein manifold and let $\left(N^{3}, h\right)$ be complete, simply-connected and with constant curvature.

Let $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ be a surjective harmonic morphism; denote its dilation by $\lambda$. Suppose that $\varphi$ has critical points.

Then, up to homotheties, $\left(N^{3}, h\right)=\left(\mathbb{R}^{3}, h_{0}\right)$ where $h_{0}$ is the canonical metric on $\mathbb{R}^{3}$. Moreover, $\lambda^{-2}=\varphi^{*}(u)$ for a positive harmonic function $u: \mathbb{R}^{3} \backslash C_{u} \rightarrow(0, \infty)$ having the same (positive) residue at each (fundamental) pole $y \in C_{u},\left(M^{4}, g\right)=\left(M_{u}^{4}, g_{u}\right)$ and $\varphi=\varphi_{u}$.

Proof. By passing to a two-fold covering we can suppose that the vertical distribution of $\varphi$ is orentable. Then, by Proposition 3.7.7, there exists a Killing field $V$ on $(M, g)$ tangent to the fibres of $\varphi$.

Although $\varphi$ has critical points, an argument due to R. Hermann (see [11,
9.45]) can be adapted to prove that the horizontal distribution $\mathscr{H}$ (which is well-defined outside the set of critical points) is an Ehresmann connection [12] for $\varphi$ restricted to the set of regular points. By applying [11, 9.40], it is easy to see that $\varphi$ can be factorised into a harmonic morphism with connected fibres followed by a Riemannian covering over $\left(N^{3}, h\right)$. But the latter must be trivial because $N^{3}$ is simply-connected and hence $\varphi$ has connected fibres.

Now, as in the proof of Theorem 3.7.1 we obtain 3.7 .2 and the monopole equation 3.7.1 and hence $\check{\lambda}^{-2}$ is a harmonic function where $\lambda=\varphi^{*}(\check{\lambda})$.

Because $\varphi$ has critical points its dilation cannot be constant. This, together with (3.7.2), implies that $\left(M^{4}, g\right)$ is Ricci-flat and $\left(N^{3}, h\right)$ is flat. Hence, up to homotheties, $\left(N^{3}, h\right)=\left(\mathbb{R}^{3}, h_{0}\right)$ where $h_{0}$ is the canonical metric on $\mathbb{R}^{3}$.

Using the completeness of $\left(M^{4}, g\right)$ and the fact that $V$ is Killing it is not difficult to prove (directly or by using Theorem 3.1.9) that the restriction of $\varphi$ to the set of regular points is the projection of a principal bundle $\xi$ with group $(\mathbb{R},+)$ or $\left(S^{1}, \cdot\right)$ and the horizontal distribution is a principal connection on it. But $\varphi$ extends the projection of $\xi$ over the critical points. Hence in the neighbourhood of each critical point $\xi$ is a restriction of the cylinder on the Hopf bundle $\left(S^{3}, S^{2}, S^{1}\right)$ or its dual. Hence the structural group of $\xi$ is $S^{1}=\mathbb{R} / L \mathbb{Z}$ where $L(>0)$ is the period of the orbits of $V$.

Let $\left\{y_{j}\right\}_{j \in I}$ be the set of critical values of $\varphi$. Using the Chern-Weil morphism (see [36]) and (3.7.1) it is easy to see that the first Chern number of $\xi$ suitably restricted to a sphere about any $y_{j}$ is given by $c_{1}=-4 \pi b_{j} / L$ where $b_{j}(>0)$ is the residue of $\check{\lambda}^{-2}$ at $y_{j}$. But we must have $c_{1}= \pm 1$ and hence $b_{j}=b_{k}$ for any $j, k \in I$ and the proof follows.

Remark 3.7.9. Note that the period $L$ of $V$ is the mass of the regular fibres, i.e. $L=\int_{\text {fibre }} \rho$ where $\rho\left(=\lambda^{-1}\right)$ is the density of $\varphi$. Because $\rho$ is constant along the fibres we have that $L$ is equal to $\left.\rho\right|_{\text {fibre }}$ multiplied by the length of the considered fibre (see Definition 4.2.1).

We end this section with the following classification result.
Theorem 3.7.10. Let $\left(M^{4}, g\right)$ be a complete Einstein manifold and let $\left(N^{3}, h\right)$ be complete, simply-connected and with constant curvature.

Let $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ be a surjective harmonic morphism. Suppose that $\varphi$ has exactly one critical point.

Then there exists $a \geq 0$ such that, up to homotheties, $\varphi:\left(\mathbb{R}^{4}, g_{a}\right) \rightarrow$ $\left(\mathbb{R}^{3}, h_{0}\right)$ is the Hopf polynomial map with $g_{a}$ the Gibbons-Hawking Taub-NUT metric $(a>0)$ and $g_{0}$, $h_{0}$ the canonical metrics on $\mathbb{R}^{4}, \mathbb{R}^{3}$, respectively.

Proof. This follows from Theorem 3.7.8.

### 3.8. The classification on conformally-flat Riemannian manifolds

This section is devoted to the following result and its consequences.
Theorem 3.8.1. Let $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right)$ be a harmonic morphism between Riemannian manifolds, $(n \geq 3)$; denote by $\lambda$ the dilation of $\varphi$.

If $\left(M^{n+1}, g\right)$ is real-analytic and conformally-flat then either
(i) $\varphi$ is of Killing type, or
(ii) the horizontal distribution of $\varphi$ is integrable and its leaves endowed with the metrics induced by $\lambda^{-2 n+4} g$ have constant curvature.

Proof. By Corollary 1.3.3, at least away of the critical points (which may occur only if $n=3$, see [9] , we have $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right)$ real-analytic.

As the dimension of the intersection of (the complexification of) $\mathscr{H}$ with any isotropic two-dimensional space, on $\left(M^{n+1}, g\right)$, is at least 1, Proposition A.4.2 implies that $\left(M^{n+1}, g\right)$ is conformally-flat if and only if, for any $U \in \Gamma(\mathscr{V})$ and $X, Y \in \Gamma(\mathscr{H})$ with $g(U, U)=g(X, X), g(X, Y)=0, g(Y, Y)=0$, we have $R^{M}(U \pm \mathrm{i} X, Y, U \pm \mathrm{i} X, Y)=0$; equivalently,

$$
\begin{align*}
R^{M}(U, Y, U, Y) & =R^{M}(X, Y, X, Y) \\
R^{M}(U, Y, X, Y) & =0 \tag{3.8.1}
\end{align*}
$$

From (A.2.2), it follows quickly that the second relation of $(3.8 .1)$ is equivalent to

$$
\begin{equation*}
\left({ }^{h} \nabla_{Y} \Omega\right)(X, Y)+3(n-1) Y(\sigma) \Omega(X, Y)=0 . \tag{3.8.2}
\end{equation*}
$$

Thus, by assuming $X$ and $Y$ basic and using Lemma 3.1.1, we obtain

$$
\begin{equation*}
Y(V(\sigma)) \Omega(X, Y)=0, \tag{3.8.3}
\end{equation*}
$$

where $V$ is the fundamental vector field of $\varphi$.
Next, we shall use the first relation of (3.8.1). For this, we assume $X$ and $Y$ basic with $g(X, X)=\mathrm{e}^{-2 \sigma}$ (equivalently, $h(X, X)=1$ ), and $U=\mathrm{e}^{-(n-1) \sigma} V$ (so that, $g(U, U)=g(X, X)$ ). Thus, the first relation of (3.8.1) becomes

$$
\mathrm{e}^{-(2 n-2) \sigma} R^{M}(V, Y, V, Y)=R^{M}(X, Y, X, Y)
$$

which, by applying A.2.1 and A.2.3 , is equivalent to

$$
\begin{align*}
R^{N}(X, Y, X, Y)=-(n-1) h\left({ }^{h} \nabla_{Y}\right. & \left.\left(\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right), Y\right)-(n-1)^{2} Y(\sigma)^{2} \\
& +\frac{1}{4} \mathrm{e}^{(2 n-2) \sigma}\left\{h\left(i_{Y} \Omega, i_{Y} \Omega\right)+3 \Omega(X, Y)^{2}\right\}, \tag{3.8.4}
\end{align*}
$$

where we have denoted by the samy symbol $R^{N}$ and its pull-back by $\varphi$ to $M^{n+1}$.

We may assume that $Y$ is the horizontal lift of an isotropic geodesic (local) vector field on (the complexification of) ( $\left.N^{n}, h\right)$; equivalently, ${ }^{h} \nabla_{Y} Y=0$. Then (3.8.4) becomes

$$
\begin{align*}
R^{N}(X, Y, X, Y)=- & (n-1) Y(Y(\sigma))-(n-1)^{2} Y(\sigma)^{2} \\
& +\frac{1}{4} \mathrm{e}^{(2 n-2) \sigma}\left\{h\left(i_{Y} \Omega, i_{Y} \Omega\right)+3 \Omega(X, Y)^{2}\right\} . \tag{3.8.5}
\end{align*}
$$

As $R^{N}(X, Y, X, Y)$ is basic, from (3.8.3) and (3.8.5) it easily follows that either $\Omega=0$ or

$$
\begin{equation*}
V(\sigma)\left\{h\left(i_{Y} \Omega, i_{Y} \Omega\right)+3 \Omega(X, Y)^{2}\right\}=0 . \tag{3.8.6}
\end{equation*}
$$

Now, from $\Omega \neq 0$ it follows that there exist $Y \in \mathscr{H}$ isotropic and $X \in$ $Y^{\perp} \cap \mathscr{H}$ such that the second factor of the left hand side of (3.8.6) is not zero. Thus, we have proved that either $\Omega=0$ (equivalently, $\mathscr{H}$ is integrable) or $V(\sigma)=0$ (equivalently, $\varphi$ is of Killing type).

Next, we study the case $\Omega=0$. Then (3.8.2) (and hence, also, the second relation of (3.8.1) is automatically satisfied, whilst (3.8.4) is equivalent to

$$
\begin{equation*}
R^{N}(X, Y, X, Y)={ }^{h} \nabla\left(\mathrm{~d}^{\mathscr{H}} u\right)(Y, Y)-\left(\mathrm{d}^{\mathscr{H}} u\right)(Y)^{2}, \tag{3.8.7}
\end{equation*}
$$

where $u=-(n-1) \sigma$ and, recall that, $X$ and $Y$ are basic with $h(X, X)=1$, $h(X, Y)=0$ and $h(Y, Y)=0$.

Let $h_{1}=\left.\mathrm{e}^{2 u} h\right|_{\mathscr{H}}=\left.\mathrm{e}^{(-2 n+4) \sigma} g\right|_{\mathscr{H}}$.
We have proved that, if $\mathscr{H}$ is integrable, (3.8.1) is equivalent to the fact that the curvature tensor $R^{P}$ of any leaf $P$ of $\mathscr{H}$, endowed with the metric induced by $h_{1}$, satisfies $R^{P}(X, Y, X, Y)=0$.

It follows that if $\mathscr{H}$ is integrable then $h_{1}$ induces a conformally-flat Einstein metric on each leaf of $\mathscr{H}$; equivalently, $h_{1}$ induces a metric of constant curvature on each leaf of $\mathscr{H}$. The proof is complete.

Example 3.8.2. Let $\left(N^{n}, h\right)$ be $\mathbb{R}^{n}$, endowed with the canonical metric, and let

$$
M^{n+1}=\left\{(t, x) \in \mathbb{R} \times \mathbb{R}^{n}| | t x \mid<1\right\},
$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^{n}$.
Define $\lambda: M^{n+1} \rightarrow(0, \infty)$ by $\lambda(t, x)=\left(1-|t x|^{2}\right)^{\frac{1}{n-1}},(t, x) \in M^{n+1}$, and let $g=\lambda^{-2} h+\lambda^{2 n-4} \mathrm{~d} t^{2}$.

Then $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right),(t, x) \mapsto x$, is a harmonic morphism which satisfies assertion (ii) of Theorem 3.8.1; in particular, $\left(M^{n+1}, g\right)$ is conformally-flat, $(n \geq 3)$. Furthermore, $\varphi$ is neither of Killing type nor its fibres are geodesics.

Remark 3.8.3. If $n=3$ then Theorem 3.8.1 holds, also, in the complexanalytic category. Indeed, the only point in the proof of Theorem 3.8.1 where it is essential for $\varphi$ to be 'real' is when we deduce from $\Omega \neq 0$ that there exist $Y \in \mathscr{H}$ isotropic and $X \in Y^{\perp} \cap \mathscr{H}$ such that the second factor of the left hand side of (3.8.6) is not zero. But, if $n=3$ and $h(X, X)=1$ then

$$
h\left(i_{Y} \Omega, i_{Y} \Omega\right)+3 \Omega(X, Y)^{2}=4 \Omega(X, Y)^{2},
$$

which, also, in the complex-analytic category, is not zero, for suitable choices of $X$ and $Y$, if $\Omega \neq 0$.

For the proof of the following result the interested reader should consult 53 .

Corollary 3.8.4. The Hopf polynomial map $\varphi: \mathbb{R}^{4} \rightarrow \mathbb{R}^{3}$ is, up to local conformal diffeomorphisms with basic conformality factors, the only harmonic
morphism with one-dimensional fibres and nonintegrable horizontal distribution between conformally-flat Riemannian manifolds, of dimensions at least three.

We end this chapter with the following result of R.L. Bryant.
Corollary 3.8.5 ([14]). For $n \geq 3$ let $\left(M^{n+1}, g\right)$ be a Riemannian manifold with constant sectional curvature and let $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right)$ be a submersive harmonic morphism with orientable vertical distribution.

Then, either
(i) the fibres of $\varphi$ form a Riemannian foliation generated by a Killing field or
(ii) $\varphi$ is horizontally homothetic and has geodesic fibres orthogonal to an umbilical foliation by hypersurfaces.

Proof. This is a trivial consequence of Theorems 3.4.1, 3.5 .4 and 3.8.1.

## CHAPTER 4

## Harmonic morphisms on compact Riemannian manifolds

### 4.1. Mixed curvature and harmonic morphisms

In this section we give some applications of A.1.1) of Appendix A to harmonic morphisms.

The following Proposition is a generalization to conformal one-dimensional foliations of the corresponding results for Riemannian one-dimensional foliations from 58].

Proposition 4.1.1. Let $(M, g)$ be compact.
(i) If $(M, g)$ has nonpositive Ricci curvature, then any conformal onedimensional foliation is Riemannian and its orthogonal complement is a totally geodesic foliation. Further, $\operatorname{Ricci}(U, U)=0$ for any $U$ tangent to the foliation.
(ii) If $(M, g)$ has negative Ricci curvature then there exists no one-dimensional conformal foliation on it.

Proof. By passing to a finite covering, if necessary, we can suppose that both the foliation $\mathscr{V}$ and the manifold $M$ are oriented.

Since $\mathscr{V}$ is conformal we have ${ }^{\mathscr{H}} B_{0}=0$ where ${ }^{\mathscr{H}} B_{0}$ is the trace-free part of ${ }^{\mathscr{H}} B$. But, as for any codimension one foliation, $\mathscr{H}$ is also conformal. Hence ${ }^{\mathscr{V}} B_{0}=0$.

Next, note that, because $\mathscr{V}$ is one-dimensional, the mixed curvature is equal to the Ricci curvature restricted to $\mathscr{V}$.

Thus integrating (A.1.1) gives

$$
\int_{M} \operatorname{Ricci}(U, U) v_{g}=\int_{M}\left\{\frac{p-1}{p}\left|\operatorname{trace}\left({ }^{\mathcal{H}} B\right)\right|^{2}+\left.\left.\frac{1}{4}\right|^{\mathcal{H}} I\right|^{2}\right\} v_{g},
$$

where $U$ is a unit vector field tangent to $\mathscr{V}$ and $v_{g}$ is the volume element of $(M, g)$.

The proposition follows.
By a well-known result of S. Bochner (see [35, Ch. II, Theorem 4.3] ) any Killing field on a compact Riemannian manifold with nonpositive Ricci tensor is parallel. The following theorem can be viewed as an extension of that result.

Theorem 4.1.2. On a compact Riemannian manifold with nonpositive Ricci curvature any one-dimensional foliation which produces harmonic morphisms and admits a global density is locally generated by parallel vector fields. In particular, it is Riemannian, has geodesic leaves and its orthogonal complement is a totally geodesic Riemannian foliation. Hence the foliation corresponds to a local Riemannian product structure of the manifold. In particular, the universal cover of $(M, g)$ is a Riemannian product. If $M$ is simply-connected, or the foliation is simple and the base space is simply-connected, then the foliation corresponds to a Riemannian product structure on $(M, g)$.

Proof. If the dimension of the manifold is three then the leaves are geodesics. This together with Proposition 4.1.1, gives the result.

Assume that the manifold has dimension greater than three. As before, by passing to a finite covering if necessary, we can suppose that both the foliation $\mathcal{V}$ and the manifold $M$ are oriented.

By Proposition 4.1.1 the foliation is Riemannian; hence, by the proof of Proposition 3.1.3, it is globally generated by a Killing field, namely $\rho^{-1} U$ where $\rho$ is a global density for $\mathscr{V}$ and $U$ is a unit vertical vector field. Now, Bochner's result mentioned above implies that the foliation is generated by parallel vector fields. Hence $\mathscr{V}$ is a Riemannian foliation by geodesics and its orthogonal complement is a totally geodesic Riemannian foliation.

The fact that $\mathscr{V}$ induces on the universal cover of $(M, g)$ a Riemannian product structure follows from the de Rham decomposition theorem. If the foliation is simple then the leaves are compact and, hence, any curve in the base space admits (global) horizontal lifts, these induce an isometry between the fibres over the endpoints of the curve. Since the horizontal distribution is
integrable, this isometry depends only on the homotopy class of the curve. It follows that, when the base space is simply-connected, this isometry depends only on the two fibres and the theorem is proved.

Remark 4.1.3. 1) In Proposition 4.1.1(i) and Theorem 4.1 .2 we can replace the condition on the Ricci curvature by the condition: $\int_{M} \operatorname{Ricci}(U, U) v_{g} \leq 0$ for any vector field $U$ tangent to the foliation.
2) The nonexistence, due to S. Bochner, of Killing vector fields on compact Riemannian manifolds with negative Ricci tensor can be proved by using A.1.1) . In fact, if $V$ is a Killing field on $(M, g)$ which generates the (possibly singular) foliation $\mathscr{V}$, and $\sigma=|V|$ then $\sigma^{-1}$ is a dilation for the homothetic distribution $\mathscr{H}$ (see the proof of Proposition 3.1.3). By 1.4.1) we have $\operatorname{trace}\left({ }^{\mathscr{V}} B\right)=\operatorname{grad}\left(\log \left(\sigma^{-1}\right)\right)$. It is easy to see that, in this case, A.1.1) gives (sign convention for the Laplacian as in [11])

$$
\begin{equation*}
\sigma \Delta \sigma+|\operatorname{grad} \sigma|^{2}+\left.\left.\frac{1}{4} \sigma^{2}\right|^{\mathscr{H}} I\right|^{2}=\operatorname{Ricci}(V, V) . \tag{4.1.1}
\end{equation*}
$$

If $\sigma$ attains a maximum at a point where $V$ is not zero then the left hand side of 4.1.1) is nonnegative from which the result follows.
3) Recall that, by another well-known result of S. Bochner, on a compact Riemannian manifold with positive Ricci curvature there exists no harmonic one-forms (in particular, the first Betti number of such a manifold is zero). As is well-known (see [11]) this can be proved by using the Weitzenböck formula for the Hodge Laplacian acting on exterior forms. Also formula A.1.1) can be obtained from the Weitzenböck formula applied to a local volume form of one of the two distributions.

By Corollary 1.1.14, any foliation which produces harmonic morphisms on a simply-connected manifold admits a global density and hence, in this case, the hypotheses of the above theorem can be weakened. Also, we have the following:

Corollary 4.1.4. Any nonconstant submersive harmonic morphism with fibres of dimension one which is defined on a compact Riemannian manifold such that the Ricci curvature $\operatorname{Ricci}(U, U)$ is nonpositive when $U$ is tangent
to the fibres is totally geodesic (or up to a conformal transformation of the codomain if this is two-dimensional). Hence, if the total space or the base space is simply-connected, up to a homothety of the codomain (up to a conformal transformation of the codomain if this is two-dimensional), it is a projection of a Riemannian product.

In order to apply it to nonnegative curvature, note that formula A.1.1) can also be written

$$
\begin{align*}
\operatorname{div}\left(\operatorname{trace}\left({ }^{\mathcal{H}} B\right)\right) & +\operatorname{div}\left(\operatorname{trace}\left({ }^{\mathcal{V}} B\right)\right)+\left|\operatorname{trace}\left({ }^{\mathcal{H}} B\right)\right|^{2}+\left|\operatorname{trace}\left({ }^{\mathcal{V}} B\right)\right|^{2} \\
& +\left.\left.\frac{1}{4}\right|^{\mathcal{H}} I\right|^{2}+\left.\left.\frac{1}{4}\right|^{\mathcal{V}} I\right|^{2}=\left|{ }^{\mathcal{H}} B\right|^{2}+\left|{ }^{\mathcal{V}} B\right|^{2}+s_{\text {mix }} \tag{4.1.2}
\end{align*}
$$

The next result applies to arbitrary foliations, not necessary conformal.
Proposition 4.1.5. Let $(M, g)$ be a compact Riemannian manifold.
(i) Let $\mathscr{V}$ and $\mathscr{H}$ be two complementary orthogonal foliations whose mean curvatures are (globally) gradient vector fields. If the mixed curvature is nonnegative then $\mathscr{V}$ and $\mathscr{H}$ are totally geodesic and hence they induce on $(M, g)$ a local Riemannian product structure. Thus, the universal cover of $(M, g)$ is globally a Riemannian product.
(ii) If the mixed curvature is positive then there exists no pair of complementary orthogonal foliations on $(M, g)$ for which the mean curvatures are gradient vector fields.

Proof. If trace $\left({ }^{\mathscr{H}} B\right)=\operatorname{grad}(\log u)$ and $\operatorname{trace}\left({ }^{\mathscr{V}} B\right)=\operatorname{grad}(\log v)$ for some smooth positive functions $u$ and $v$ on $M$, then, (4.1.2) gives the following:
$-\Delta(\log u)-\Delta(\log v)+|\operatorname{grad}(\log u)|^{2}+|\operatorname{grad}(\log u)|^{2}=\left.\left.\right|^{\mathscr{H}} B\right|^{2}+\left.\left.\right|^{\mathscr{V}} B\right|^{2}+s_{\text {mix }}$.

Equation (4.1.3) can be written as follows:

$$
\begin{equation*}
-u^{-1} \Delta u-v^{-1} \Delta v=|\mathscr{H} B|^{2}+\left.\left.\right|^{\mathscr{V}} B\right|^{2}+s_{\text {mix }} . \tag{4.1.4}
\end{equation*}
$$

Since grad $u$ and $\operatorname{grad} v$ are orthogonal (the former being vertical whilst the latter is horizontal) relation (4.1.4) can be written as follows:

$$
-u^{-1} v^{-1} \Delta(u v)=\left|{ }^{\mathscr{H}} B\right|^{2}+\left.\left.\right|^{\mathscr{V}} B\right|^{2}+s_{\text {mix }} .
$$

The proof follows by multiplying by $u v$ and integrating over $M$.
Corollary 4.1.6. Let $(M, g)$ be compact and with zero first Betti number. Let $\mathscr{V}$ be a homothetic foliation with codim $\mathscr{V} \neq 2$ which produces harmonic morphisms on $(M, g)$ and has integrable orthogonal complement. Then
(i) the total mixed curvature $\int_{M} s_{\text {mix }} v_{g}$ is nonpositive.
(ii) if the mixed curvature is nonnegative then it is identically zero and $\mathscr{V}$ and $\mathscr{H}$ are totally geodesic. Hence, the universal cover of $(M, g)$ is globally a Riemannian product.

On a compact Riemannian manifold with positive sectional curvature there exists no homothetic foliation which produces harmonic morphisms and has integrable orthogonal complement.

In Theorem 4.2.11(i) we shall prove that for one-dimensional foliations the last assertion from above proposition is true without the integrability assumption on $\mathscr{H}$, when $\operatorname{dim} M$ is even and greater than two.

Proof. Since $\mathscr{V}$ is homothetic the mean curvature form of $\mathscr{H}$ is closed. But $\mathscr{V}$ produces harmonic morphisms and hence, by Corollary 1.4.1, the mean curvature form of $\mathscr{V}$ is also closed. Since the first Betti number of $M$ is zero both mean curvatures are globally gradient vector fields. The proof follows from Proposition 4.1.5.

Remark 4.1.7. 1) Recall (Remark 1.4.9) that Riemannian foliations are homothetic, as are the foliations with minimal leaves of codimension not equal to two and which produce harmonic morphisms.
2) If $\operatorname{codim} \mathscr{V}=1$ then the integrability assumption on the orthogonal complement of $\mathscr{V}$, made above, can be removed. Further, the mixed curvature is equal to the restriction of the Ricci curvature to its orthogonal complement.
3) Corollary 4.1.6 admits further consequences in a similar way to Corollary 4.1.4

Theorem 4.1.8. Let $(M, g)$ be a compact Riemannian manifold of dimension at least four, with zero first Betti number and with Ricci curvature of constant sign.

Then, there exists no orientable one-dimensional homothetic foliation which produces harmonic morphisms on $(M, g)$ and which has integrable orthogonal complement.

Note that, by Lemma 3.1.7, the integrability of the orthogonal complement above is automatic except on the set where $\mathscr{V}$ is Riemannian.

Proof. Suppose that there exists a foliation $\mathscr{V}$ with the stated properties. By Corollary 4.1.6 the Ricci curvature of $(M, g)$ is nonpositive. Since $M$ has zero Betti number, $\mathscr{V}$ admits a global density. From Theorem 4.1.2 it follows that $\mathscr{V}$ is locally generated by parallel vector fields. Moreover, being orientable and admiting a global density, as in the proof of Theorem 4.1.2, $\mathscr{V}$ must be globally generated by a parallel vector field. Hence, the first Betti number of $M$ is nonzero. This is a contradiction!

Corollary 4.1.9. On a compact Riemannian manifold with positive Ricci curvature, there exists no nonvanishing Killing field with integrable orthogonal complement.

The following immediate consequence of Proposition A.1.1 slightly improves Proposition 5.9 and Proposition 5.10 from [65] .

Corollary 4.1.10. If $\mathscr{V}$ is a Riemannian foliation on $(M, g)$, and $s_{m i x}<0$ at least at one point of $M$, then $\mathscr{V}$ cannot be totally geodesic.

### 4.2. Two integral formulae for harmonic morphisms with one-dimensional fibres

Throughout this section $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right), n \geq 1$, will denote a non-constant harmonic morphism defined on a compact Riemannian manifold. Recall that, by a result of P. Baird $4, \varphi$ is automatically submersive if $n \geq 4$. Since closed, all the fibres of $\varphi$ are compact. As is well known [21] if $\varphi$ is nonconstant then it is open, hence it is surjective and $N$ is also compact. Let $\lambda$ denote the dilation of $\varphi$; we shall denote by the same letter $h$ the metric on $N$ and the metric ${ }^{\lambda} g$ on $M$ of Definition 1.2.3. This metric should be seen just as an auxiliary tool and thus, whenever we denote a geometric object on
the total space of $\varphi$ without mentioning a metric then it should be understood that the metric considered is $g$.

Definition 4.2.1. We define the mass of a (regular) fibre of $\varphi$ to be the positive number

$$
\mathfrak{m}=\int_{\text {fibre }} \lambda^{2-n} v_{\text {fibre }}
$$

where $v_{\text {fibre }}$ is the volume measure of the fibre induced by the metric (see [39]).
By Proposition 1.1.11 the mass is independent of the (regular) fibre and it can be defined without any restriction on the dimensions.

Theorem 4.2.2. Let $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right),(n \geq 1)$ be a submersive harmonic morphism and let $\mathcal{S}^{M}=\int_{M} s^{M} v_{g}, \mathcal{S}^{N}=\int_{N} s^{N} v_{h}$ be the total scalar curvature of $(M, g)$ and $(N, h)$, respectively. Then

$$
\begin{align*}
\mathcal{S}^{M}-\mathfrak{m} \mathcal{S}^{N}= & n(n-1)\left\|\mathscr{V}\left(\operatorname{grad}_{g}(\log \lambda)\right)\right\|^{2} \\
& -(n-1)(n-2)\left\|\mathscr{H}\left(\operatorname{grad}_{g}(\log \lambda)\right)\right\|^{2}-\frac{1}{4}\|I\|^{2} . \tag{4.2.1}
\end{align*}
$$

Let $n \neq 2$. Then

$$
\begin{align*}
\int_{M} \lambda^{2}\left(s^{M}-\lambda^{2} s^{N}\right) v_{g}=n(n-5) \int_{M} \lambda^{2}\left|\mathscr{V}\left(\operatorname{grad}_{g}(\log \lambda)\right)\right|^{2} v_{g} \\
\quad-\left(n^{2}-3 n+6\right) \int_{M} \lambda^{2}\left|\mathscr{H}\left(\operatorname{grad}_{g}(\log \lambda)\right)\right|^{2} v_{g}-\frac{1}{4} \int_{M} \lambda^{2}|I|^{2} v_{g} . \tag{4.2.2}
\end{align*}
$$

Here, $I={ }^{\mathscr{H}}$ I is the integrability tensor of the horizontal distribution $\mathscr{H}$.
In what follows we shall use the horizontal Laplacian of the associated Riemannian submersion with geodesic fibres. This was introduced in 10 and it can be defined as follows:

Definition 4.2.3. If $\varphi: M \rightarrow(N, h)$ is a submersion endowed with a distribution $\mathscr{H}$ which is complementary to $\operatorname{ker} \varphi_{*}$, then the horizontal Laplacian $\varphi^{*} \Delta^{N}$ is the second-order differential operator which acts on a local function $f$ defined in the neighbourhood of the point $x \in M$ as follows:

$$
\left(\varphi^{*} \Delta^{N}\right)(f)=-\sum_{j}\left\{X_{j}\left(X_{j}(f)\right)-\left(\left(\varphi^{*} \nabla^{N}\right)_{X_{j}} X_{j}\right)(f)\right\} .
$$

Here, $\left\{X_{j}\right\}$ is a local orthonormal frame of $\mathscr{H}$ (endowed with the metric induced by h) formed of basic vector fields (i.e. sections of $\mathscr{H}$ which are projectable by $\varphi$ to vector fields on $N$ ), and $\nabla^{N}$ is the Levi-Civita connection of ( $N, h$ ).

Remark 4.2.4. Note that $\left(\varphi^{*} \Delta^{N}\right)(f \circ \varphi)=\left(\Delta^{N} f\right) \circ \varphi$ for any local smooth function $f$ on $N$.

Lemma 4.2.5. Let $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right)$ be a submersive harmonic morphism and let $\Delta^{M}$ and $\Delta^{N}$ be the Laplace operators on ( $M, g$ ) and ( $N, h$ ), respectively. Then

$$
\Delta^{M} f=\mathrm{e}^{2 \sigma}\left(\varphi^{*} \Delta^{N}\right)(f)-\mathrm{e}^{(-2 n+4) \sigma}\{V(V(f))-2(n-1) V(f) V(\sigma)\} .
$$

Remark 4.2.6. Note that $V(V(f))$ is just minus the 'vertical' Laplacian [10] applied to $f$ of the Riemannian submersion with geodesic fibres associated to $\varphi:(M, g) \rightarrow(N, h)$. More generally, the 'vertical' Laplacian of $\varphi:(M, g) \rightarrow(N, h)$ is defined by $\left(\Delta^{\text {fibre }} f\right)(x)=\left(\Delta^{\varphi^{-1}(\varphi(x))}\left(\left.f\right|_{\varphi^{-1}(\varphi(x))}\right)\right)(x)$ where $\Delta^{\varphi^{-1}(\varphi(x))}$ is the Laplacian of the fibre through $x$ endowed with the metric induced by $g$. If $\varphi$ is a Riemannian submersion with totally geodesic fibres, then the sum of the horizontal and the vertical Laplacians is equal to the Laplacian of the total space.

Proof of Theorem 4.2.2. Recall from the previous section that $s_{\text {mix }}$ is the sum of the sectional curvatures of all planes on $(M, g)$ spanned by a horizontal and a vertical vector from an orthonormal frame adapted to the decomposition $T M=\mathcal{H} \oplus \mathcal{V}$. Let $s^{\mathscr{H}}$ denote twice the sum of the sectional curvatures of all planes on $(M, g)$ spanned by the horizontal vectors of a frame as above.

Using the previous two lemmas and the fact that $I=V \otimes \Omega$, after a straightforward computation the following relation can be obtained. (Another way to obtain it is to use the previous two lemmas together with Corollary 2.2.4, from [27].)

$$
\begin{align*}
s^{\mathscr{H}} & -\mathrm{e}^{2 \sigma} s^{N}=-2(n-1) \Delta^{M} \sigma-(n-1)(n-2) \mathrm{e}^{2 \sigma}\left|\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right|^{2} \\
& -2(n-1) \mathrm{e}^{(-2 n+4) \sigma} V(V(\sigma))+(3 n-4)(n-1) \mathrm{e}^{(-2 n+4) \sigma} V(\sigma)^{2}-\frac{3}{4}|I|^{2} . \tag{4.2.3}
\end{align*}
$$

Using (1.1.3), (1.1.4) together with Lemma 4.2.5 and (A.1.1), we obtain:

$$
\begin{align*}
s_{m i x}=(n-2) \Delta^{M} \sigma+2( & n-1) \mathrm{e}^{(-2 n+4) \sigma} V(V(\sigma)) \\
& -(3 n-4)(n-1) \mathrm{e}^{(-2 n+4) \sigma} V(\sigma)^{2}+\frac{1}{4}|I|^{2} . \tag{4.2.4}
\end{align*}
$$

But it is obvious that $s^{M}=s^{\mathscr{H}}+2 s_{m i x}$ and hence from (4.2.3) and (4.2.4) we obtain

$$
\begin{align*}
& s^{M}-\mathrm{e}^{2 \sigma} s^{N}=-2 \Delta^{M} \sigma+\frac{2(n-1)}{n} \mathrm{e}^{n \sigma} \Delta^{\mathrm{fibre}}\left(\mathrm{e}^{-n \sigma}\right) \\
& \quad+n(n-1)\left|\mathscr{V}\left(\operatorname{grad}_{g} \sigma\right)\right|^{2}-(n-1)(n-2)\left|\mathscr{H}\left(\operatorname{grad}_{g} \sigma\right)\right|^{2}-\frac{1}{4}|I|^{2} . \tag{4.2.5}
\end{align*}
$$

Integrating 4.2.5 gives (4.2.1) Relation 4.2.5 can also be written as follows:

$$
\begin{align*}
& \mathrm{e}^{2 \sigma} s^{M}-\mathrm{e}^{4 \sigma} s^{N}=-\Delta^{M}\left(\mathrm{e}^{2 \sigma}\right)+\frac{2(n-1)}{n-2} \mathrm{e}^{n \sigma} \Delta^{\mathrm{fibre}}\left(\mathrm{e}^{(-n+2) \sigma}\right) \\
& \quad+n(n-5) \mathrm{e}^{2 \sigma}\left|\mathscr{V}\left(\operatorname{grad}_{g} \sigma\right)\right|^{2}-\left(n^{2}-3 n+6\right) \mathrm{e}^{2 \sigma}\left|\mathscr{H}\left(\operatorname{grad}_{g} \sigma\right)\right|^{2}-\frac{1}{4} \mathrm{e}^{2 \sigma}|I|^{2} . \tag{4.2.6}
\end{align*}
$$

Integrating 4.2.6) gives 4.2.2, since

$$
\int_{M} \mathrm{e}^{n \sigma} \Delta^{\mathrm{fibre}}\left(\mathrm{e}^{(-n+2) \sigma}\right) v_{g}=\int_{N} v_{h} \int_{\text {fibre }} \Delta^{\mathrm{fibre}}\left(\mathrm{e}^{(-n+2) \sigma}\right) v_{\text {fibre }}=0 .
$$

Remark 4.2.7. 1) Suppose that $n=1$, i.e. $\varphi:\left(M^{2}, g\right) \rightarrow \mathbb{R}$ is a harmonic function defined on the surface $\left(M^{2}, g\right)$. Then, equation 4.2.4) above reads:

$$
K=-\Delta(\log |\mathrm{d} \varphi|),
$$

where $K$ is the Gauss curvature of $(M, g)$. As is well-known this can also be proved by using the local isothermal coordinates induced by $\varphi$.
2) Computing $\lambda^{2}\left(s^{\mathscr{H}}+s_{m i x}\right)$, by adding (4.2.3) and 4.2.4) from the above proof, we can obtain formula (2.2) from [46] applied to harmonic morphisms with fibres of dimension one.

Proposition 4.2.8. Let $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right), n \geq 2$ be a submersive harmonic morphism. If $U$ is a unit vector field tangent to the fibres of $\varphi$ then

$$
\mathcal{S}^{M} \leq \mathfrak{m} \mathcal{S}^{N}+\int_{M} \operatorname{Ricci}(U, U) v_{g}
$$

and equality holds if and only if $\varphi$ has geodesic fibres and $\mathscr{H}$ is integrable.
Note that since, $\operatorname{Ricci}(U, U)$ is quadratic in $U$, we do not need $\mathscr{V}$ to be orientable.

Proof. First recall that $\operatorname{Ricci}(U, U)=s_{m i x}$, then take the sum of 4.2.3) and (4.2.4) and use the definition of $\mathfrak{m}$.

Corollary 4.2.9. Let $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right), n \geq 2$, be a submersive harmonic morphism.
(i) If $\varphi$ induces a Riemannian foliation on $(M, g)$ then $\mathcal{S}^{M} \leq \mathfrak{m} \mathcal{S}^{N}$ and equality holds if and only if $\varphi$ is totally geodesic (up to a conformal transformation of the codomain if $n=2$ ).
(ii) If $\varphi$ has geodesic fibres and $\mathscr{H}$ is integrable then $\mathcal{S}^{M} \geq \mathfrak{m} \mathcal{S}^{N}$ and equality holds if and only if $\varphi$ is totally geodesic (up to a conformal transformation of the codomain if $n=2$ ).

From Lemma 3.1.7 it follows that when the set of the points where $\mathscr{V}$ is Riemannian has measure zero then the integrability assumption on $\mathscr{H}$ in (ii) is superfluous.

Proof. (i) This is a trivial consequence of formula (4.2.1) from Theorem 4.2.2.
(ii) If $n=2$, then (4.2.1) from Theorem 4.2 .2 gives the result. If $n \neq 2$, by Proposition 1.1 .10 (b) we have $\mathscr{H}\left(\operatorname{grad}_{g} \lambda\right)=0$. Now apply formula 4.2.1).

The next corollary improves [46, Theorem 2.5] for the dimensions considered (see also [27, Corollary 2.2.6]).

Corollary 4.2.10. If $n \in\{3,4,5\}$ then $\int_{M} \lambda^{2}\left(s^{M}-\lambda^{2} s^{N}\right) v_{g} \leq 0$. For $n \in\{3,4\}$, equality holds if and only if $\varphi$ is totally geodesic and, for $n=5$,
equality holds if and only if $\varphi$ has geodesic fibres and $\mathscr{H}$ is integrable.
Therefore, for $n \in\{3,4,5\}$, if $\left(M^{n+1}, g\right),\left(N^{n}, h\right)$ are compact with $s^{M} \geq 0, s^{N} \leq 0$ and at least one of these inequalities is strict then there exists no nonconstant submersive harmonic morphism $\varphi:\left(M^{n+1}, g\right) \rightarrow\left(N^{n}, h\right)$.

Proof. This is an immediate consequence of formula $\sqrt{4.2 .2}$ ) from Theorem 4.2.2.

To end this section we prove two results on homothetic one-dimensional foliations which produce harmonic morphisms on compact manifolds, the first of them being a generalization (refered to in Section 3) of a well-known result of M. Berger (see [35, Ch. II, Corollary 5.7]) concerning Killing fields. To prove the first of these two results we shall use Lemma A.2.3.

Theorem 4.2.11. Let $M$ be compact with dimension at least four.
(i) If $\operatorname{dim} M$ is even and $(M, g)$ has positive sectional curvature then there exists no homothetic one-dimensional foliation which produces harmonic morphisms on $(M, g)$.
(ii) If $(M, g, J)$ is Kähler and has zero first Betti number then any homothetic one-dimensional foliation which produces harmonic morphisms on ( $M, g$ ) is Riemannian and locally generated by Killing fields.

Proof. (i) Since ( $M, g$ ) has positive sectional curvature it has positive Ricci curvature. Thus from Bochner's result (see Remark 4.1.3(3)) it follows that the first Betti number of $M$ is zero .

Suppose that there exists a homothetic foliation $\mathscr{V}$ which produces harmonic morphisms on $(M, g)$. Because the first Betti number of $M$ is zero, it follows from Corollary 1.1 .14 that $\mathscr{V}$ admits a global density $\lambda^{2-n}$ where $n+1=\operatorname{dim} M$. We shall denote by $h={ }^{\lambda} g$ the associated Riemannian metric on $M$ which makes $\mathscr{V}$ a Riemannian foliation with geodesic leaves.

Since $\mathscr{V}$ is homothetic, by Proposition 1.4 .2 we see that $\lambda$ is of the form $\mathrm{e}^{a+b}$ with $(\mathrm{d} a)^{\mathscr{y}}=(\mathrm{d} b)^{\mathscr{H}}=0$. At a point $x \in M$ where $a-b$ attains a minimum we have

$$
\begin{align*}
& 0 \leq\left({ }^{h} \nabla \mathrm{~d} a\right)(V, V)-\left({ }^{h} \nabla \mathrm{~d} b\right)(V, V)=-\left({ }^{h} \nabla \mathrm{~d} b\right)(V, V), \\
& 0 \leq\left({ }^{h} \nabla \mathrm{~d} a\right)(X, X)-\left({ }^{h} \nabla \mathrm{~d} b\right)(X, X)=\left({ }^{h} \nabla \mathrm{~d} a\right)(X, X), \tag{4.2.7}
\end{align*}
$$

where, $V$ is as in Lemma 3.1.1 and $X$ is any horizontal vector at $x$.
Now, evaluated at $x$, the first formula from Lemma A.2.3 gives

$$
\begin{align*}
R(X, V, X, V)=- & (n-2) \mathrm{e}^{(2 n-4)(a+b)}\left({ }^{h} \nabla \mathrm{~d} a\right)(X, X) \\
+ & \mathrm{e}^{-2(a+b)}\left({ }^{h} \nabla \mathrm{~d} b\right)(V, V) h(X, X)  \tag{4.2.8}\\
& +\frac{1}{4} \mathrm{e}^{(4 n-6)(a+b)} h\left(i_{X} \Omega, i_{X} \Omega\right),
\end{align*}
$$

for any horizontal vector $X$.
Since $\Omega$ is skew-symmetric it must have even rank at each point. But $\operatorname{dim} M$ is even and $i_{V} \Omega=0$ and hence there must be a horizontal vector $X$ at $x$ such that $i_{X} \Omega=0$. By (4.2.7) and (4.2.8), the sectional curvature of the plane spanned by $X$ and $V$ would be nonpositive, contradicting the hypothesis.
(ii) Since the first Betti number of $M$ is zero, the foliation $\mathscr{V}$ must admit a global density. By passing to a two-fold covering if necessary, we can suppose that $\mathscr{V}$ is oriented. Hence, by Proposition 3.1.5, it must be generated by a conformal vector field. But by a well-known result of A. Lichnerowicz (see [11, $2.125($ ii) $])$ any conformal vector field is Killing on a compact Kähler manifold of complex dimension greater than two.

The theorem is proved.
Remark 4.2.12. 1) Note that statement (i) from Theorem 4.2 .11 fails if $\operatorname{dim} M$ is odd, for example the Hopf maps $S^{2 n+1} \rightarrow \mathbb{C} P^{n}$ are harmonic Riemannian submersions.
2) In Theorem 4.2.11(i), if the sectional curvature $K$ of $\left(M^{n+1}, g\right)$ satisfies $K \geq a^{2}>0$ on $M$ for some positive constant $a$, then the compactness assumption on $M$ can be replaced by the weaker condition that $(M, g)$ be complete. This follows from a well-known result of S. B. Myers (see [11, 6.51]) noting that the Ricci curvature is $\geq n a^{2} g$. (A similar remark can be made for Corollary 4.1.9.)

Corollary 4.2.13. Let $(M, g)$ be a compact Riemannian manifold of even dimension at least four with zero first Betti number and with sectional curvature of constant sign.

Then, there exists no orientable one-dimensional homothetic foliation which produces harmonic morphisms on $(M, g)$.

Proof. This follows from Theorem 4.1.2 and Theorem 4.2.11(i).

### 4.3. Harmonic morphisms with one-dimensional fibres on compact Einstein manifolds

Firstly, we prove the following result.
Proposition 4.3.1. Let $(M, g)$ be a compact Einstein manifold of dimension at least five and let $\varphi:(M, g) \rightarrow(N, h)$ be a harmonic morphisms with onedimensional fibres.

Then the fibres of $\varphi$ form a Riemannian foliation locally generated by Killing vector fields.

Proof. By [4] (see [9) we have that $\varphi$ is submersive. Furthermore, by passing to a two-fold covering we may suppose that the vertical distribution $\mathscr{V}$ of $\varphi$ is orientable.

Suppose that assertion (ii) of Corollary 3.4 .3 holds and let $W$ be the (nowhere zero) vertical vector field such that $g(W, W)=\lambda^{-2}$, where $\lambda$ is the dilation of $\varphi$. It follows that $\nabla W=\mu \operatorname{Id}_{T M}$ where $\mu$ is a smooth function on $M$. Hence, $W$ is conformal and $\mathrm{d} W^{b}=0$. By a result of K. Yano and T. Nagano (see [35]), either $W$ is Killing or $(M, g)$ is $S^{m}$ with its canonical metric, where $m=\operatorname{dim} M$. But in the latter case $W=\operatorname{grad} u$ for some function $u$ on $S^{m}$; in particular, $W$ would have zeroes.

The proof is complete.
We end with the following theorem.
Theorem 4.3.2. Let $\left(M^{4}, g\right)$ be a compact four-dimensional Einstein manifold and let $\varphi:\left(M^{4}, g\right) \rightarrow\left(N^{3}, h\right)$ be a non-constant harmonic morphism to a Riemannian three-dimensional manifold.

Then, up to homotheties and Riemannian coverings, $\varphi$ is the canonical projection $T^{4} \rightarrow T^{3}$ between flat tori.

Proof. As above, up to a covering, the vertical distribution $\mathscr{V}$ of $\varphi$ is orientable.

If $\varphi$ has critical points then, by Proposition 3.7.7, we have that $\left(M^{4}, g\right)$ is Ricci-flat and there exists a Killing vector field $V$ tangent to the fibres of $\varphi$ whose zero set is equal to the set of critical points of $\varphi$. Then, by a well-known result of S. Bochner (see [11, 1.84] or apply (3.7.4) ) $V$ is parallel and hence nowhere zero; in particular, $\varphi$ is submersive.

As $\mathscr{V}$ is an orientable one-dimensional foliation on $M^{4}$, the Euler number of $M^{4}$ is zero. Hence, by a result of M. Berger, $\left(M^{4}, g\right)$ is flat (see [11, 6.32]).

Now, an argument similar to the proof of Proposition 4.3.1 shows that there exists a (nowhere zero) Killing vector field $V$ tangent to the fibres of $\varphi$.

Moreover, as $\left(M^{4}, g\right)$ is flat we have that $V$ is parallel. Hence, the horizontal distribution $\mathscr{H}$ is integrable and, by A.2.23), we have that, also, $(N, h)$ is flat.

From a well-known result of C. Ehresmann [20] it follows that the leaf space of the foliation formed by the connected components of the regular fibres of $\varphi$ is smooth. Thus by factorising $\varphi$, if necessary, into a harmonic morphism followed by a Riemannian covering we can suppose that $\varphi$ has connected fibres.

Then $\varphi$ is, up to homotheties, the quotient induced by $V$. Hence, $\varphi$ is the projection of an $S^{1}$-principal bundle and the horizontal distribution $\mathscr{H}$ is a flat principal connection on it. Then, each holonomy bundle $P$ of it is a regular covering over $N^{3}$ with group the holonomy group $H\left(\subseteq S^{1}\right)$ of $\mathscr{H}$. Moreover, $P$ considered with the metric induced by $g$ is flat (actually, up to homotheties, this is the unique metric with respect to which $P \rightarrow N$ becomes a Riemannian covering; in particular, $P$ with the considered metric is complete.) Hence $M=P \times{ }_{H} S^{1}$ and the pull back of $\varphi$ by $P \rightarrow N$ is the projection $P \times S^{1} \rightarrow P$.

Furthermore, from [36, Chapter V, Theorem 4.2] we deduce that $P$ is covered by an Euclidean cylinder or by a torus and the proof is complete.

## APPENDIX A

## Useful supplementary facts

## A.1. On pairs of complementary orthogonal distributions

In this appendix we recall a formula of P. Walczak [67] which relates the curvature of a Riemannian manifold to the geometric properties of a pair of complementary orthogonal distributions on it..

Let $(M, g)$ be a Riemannian manifold and $\mathscr{H}, \mathscr{V}$ a pair of complementary orthogonal distributions on it, with $\operatorname{dim} \mathscr{H}=p$ and $\operatorname{dim} \mathscr{V}=q$. As before, $\mathscr{H}$ and $\mathscr{V}$ will be called the horizontal and the vertical distribution, respectively, and, the corresponding projections will be denoted with the same letters $\mathscr{H}$ and $\mathscr{V}$. We shall denote by $X, Y$ horizontal vector fields and by $U, V$ vertical vector fields. Let ${ }^{\mathscr{V}} B$ and ${ }^{\mathscr{V}} I$ denote the second fundamental form and integrability tensor of $\mathscr{V}$, respectively. Recall that they are the unique $\mathscr{H}$-valued vertical tensor fields which satisfy

$$
\begin{aligned}
&{ } \\
& B(U, V)=\frac{1}{2} \mathscr{H}\left(\nabla_{U} V+\nabla_{V} U\right), \\
&{ }^{\prime} I(U, V)=-\mathscr{H}[U, V] .
\end{aligned}
$$

Note the minus sign in the last formula.
Recall that $\operatorname{trace}\left({ }^{\mathscr{V}} B\right)$ is the mean curvature of $\mathscr{V}$, i.e. if $\left\{U_{\alpha}\right\}$ is a local orthonormal frame for $\mathscr{V}$ then $\operatorname{trace}\left({ }^{\mathscr{V}} B\right)=\sum_{\alpha}{ }^{\mathscr{V}} B\left(U_{\alpha}, U_{\alpha}\right)$. We shall also consider the trace free part ${ }^{\mathscr{V}} B_{0}$ defined by

$$
{ }^{\mathscr{V}} B_{0}={ }^{\mathscr{V}} B-\frac{1}{q} \operatorname{trace}\left({ }^{\mathscr{V}} B\right) \otimes g^{\mathscr{V}},
$$

where $g^{\mathscr{V}}$ is the vertical component of $g$. Also, $\mathscr{H}_{B}, \mathscr{H}_{I},{ }^{\mathscr{H}} B_{0}$ are defined similarly by reversing the roles of $\mathscr{V}$ and $\mathscr{H}$.

Recall that $s_{m i x}$ denotes the mixed scalar curvature which is the sum of the sectional curvatures of all planes spanned by a horizontal and a vertical
vector from an orthonormal frame adapted to the orthogonal decomposition $T M=\mathscr{H} \oplus \mathscr{V}$. The following result is due to P. Walczak.

Proposition A.1.1 ([67]). With the notations above we have

$$
\begin{align*}
& \operatorname{div}\left(\operatorname{trace}\left({ }^{\mathscr{H}} B\right)\right)+\operatorname{div}\left(\operatorname{trace}\left({ }^{\mathscr{V}} B\right)\right)+\frac{p-1}{p}\left|\operatorname{trace}\left({ }^{\mathscr{H}} B\right)\right|^{2}+\frac{q-1}{q}\left|\operatorname{trace}\left({ }^{\mathscr{V}} B\right)\right|^{2} \\
&+\frac{1}{4}\left|\mathscr{H}_{I}\right|^{2}+\frac{1}{4}\left|{ }^{\mathscr{V}} I\right|^{2}=\left|\mathscr{\mathscr { H }}_{B_{0}}\right|^{2}+\left|{ }^{\mathscr{V}} B_{0}\right|^{2}+s_{\text {mix }} . \tag{A.1.1}
\end{align*}
$$

## A.2. Curvature formulae for harmonic morphisms with one-dimensional fibres

In this appendix we rewrite in a convenient way S. Gudmundsson's fundamental equations for horizontally conformal submersions. We do this only for submersive harmonic morphisms with one-dimensional fibres.

In this appendix $\mathscr{V}$ will always denote a one-dimensional foliation which produces harmonic morphisms on $\left(M^{n+1}, g\right)(n \geq 1)$ and $\rho=\mathrm{e}^{(2-n) \sigma}$ will denote a local density of it. As before, $h={ }^{\mathrm{e}^{\sigma}} g$ (see Definition 1.2.3) will denote the associated (local) metric on $M$ with respect to which $\mathscr{V}$ is Riemannian and has geodesic leaves and $\mathscr{H}$ will denote the orthogonal complement of $\mathscr{V}$.

The following lemma can be obtained by a straightforward computation.
Lemma A.2.1. Let $\sigma=\log \lambda$, and let $V$ be a local vertical field such that $g(V, V)=\mathrm{e}^{(2 n-4) \sigma}$, and $\theta$ its dual vertical 1 -form. Then, for any basic $X$ and $Y$ we have:

$$
\begin{aligned}
& \mathscr{H}\left({ }^{g} \nabla_{X} Y\right)=\mathscr{H}\left({ }^{h} \nabla_{X} Y\right)-X(\sigma) Y-Y(\sigma) X+\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right) h(X, Y), \\
& \mathscr{V}\left({ }^{g} \nabla_{X} Y\right)=\left\{\mathrm{e}^{(-2 n+2) \sigma} V(\sigma) h(X, Y)-\frac{1}{2} \Omega(X, Y)\right\} V, \\
& \mathscr{H}\left({ }^{g} \nabla_{V} X\right)=-V(\sigma) X+\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma}\left(i_{X} \Omega\right)^{\# h}, \\
& \mathscr{V}\left({ }^{g} \nabla_{V} X\right)=(n-2) X(\sigma) V, \\
& \mathscr{H}\left({ }^{g} \nabla_{V} V\right)=-(n-2) \mathrm{e}{ }^{(2 n-2) \sigma} \mathscr{H}\left(\operatorname{grad}_{h} \sigma\right), \\
& \mathscr{V}\left({ }^{g} \nabla_{V} V\right)=(n-2) V(\sigma) V .
\end{aligned}
$$

Here, ${ }^{g} \nabla$ and ${ }^{h} \nabla$ are, respectively, the Levi-Civita connections of $(M, g)$ and $(M, h), \Omega=\mathrm{d} \theta$ and ${ }^{\#_{h}}$ denotes the musical isomorphism defined by the metric $h$.

Remark A.2.2. Note that, if $\operatorname{dim} \mathscr{V}=0$, Lemma A.2.1 becomes a well-known formula (see [11, 1.159]).

Lemma A.2.3. Let $X, Y, Z, H$ be horizontal and $V$ vertical vectors on $M$, such that $g(V, V)=\mathrm{e}^{(2 n-4) \sigma}$; then the curvature tensor ${ }^{M} R$ of $(M, g)$ has the
following components

$$
\begin{align*}
{ }^{M} R(X, V, Y, V)= & -\frac{1}{2}(n-2) \mathrm{e}^{(2 n-4) \sigma}\left(\mathcal{L}_{\mathscr{H}}\left(\operatorname{grad}_{h} \sigma\right)\right. \\
& -(n-2) \mathrm{e}^{(2 n-4) \sigma}\left\{n X(\sigma) Y(\sigma)-\left|\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right|_{h}^{2} h(X, Y)\right\} \\
& +\mathrm{e}^{-2 \sigma}\left\{V(V(\sigma))-(n-1) V(\sigma)^{2}\right\} h(X, Y) \\
& +\frac{1}{4} \mathrm{e}^{(4 n-6) \sigma} h\left(i_{X} \Omega, i_{Y} \Omega\right),  \tag{A.2.1}\\
{ }^{M} R(X, Y, Z, V)= & -\frac{1}{2} \mathrm{e}^{(2 n-4) \sigma}\left({ }^{h} \nabla \Omega\right)(X, Y, Z) \\
+ & \frac{1}{2}(n-1) \mathrm{e}^{(2 n-4) \sigma}\{X(\sigma) \Omega(Y, Z)+Y(\sigma) \Omega(Z, X)-2 Z(\sigma) \Omega(X, Y)\} \\
- & \mathrm{e}^{-2 \sigma}\{X(V(\sigma))-(n-2) X(\sigma) V(\sigma)\} h(Y, Z) \\
+ & \mathrm{e}^{-2 \sigma}\{Y(V(\sigma))-(n-2) Y(\sigma) V(\sigma)\} h(X, Z) \\
+ & \frac{1}{2} \mathrm{e}^{(2 n-4) \sigma}\left\{\Omega\left(X, \operatorname{grad}_{h} \sigma\right) h(Y, Z)-\Omega\left(Y, \operatorname{grad}_{h} \sigma\right) h(X, Z)\right\}, \tag{A.2.2}
\end{align*}
$$

$$
\begin{align*}
& { }^{M} R(X, Y, Z, H)=\mathrm{e}^{-2 \sigma N^{N}} R\left(\varphi_{*} X, \varphi_{*} Y, \varphi_{*} Z, \varphi_{*} H\right) \\
& -\frac{1}{4} \mathrm{e}^{(2 n-4) \sigma}\{\Omega(H, X) \Omega(Y, Z)+\Omega(H, Y) \Omega(Z, X)-2 \Omega(H, Z) \Omega(X, Y)\} \\
& -\frac{1}{2} \mathrm{e}^{-2 \sigma} V(\sigma)\{-\Omega(Y, H) h(X, Z)+\Omega(X, H) h(Y, Z)-\Omega(X, Z) h(Y, H) \\
& +\Omega(Y, Z) h(X, H)\} \\
& -\mathrm{e}^{-2 \sigma}\{X(\sigma) H(\sigma) h(Y, Z)-X(\sigma) Z(\sigma) h(Y, H)-Y(\sigma) H(\sigma) h(X, Z) \\
& \quad+Y(\sigma) Z(\sigma) h(X, H)\} \\
& +\mathrm{e}^{-2 \sigma}\left\{h(X, Z) h\left({ }^{h} \nabla_{Y}\left(\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right), H\right)-h(Y, Z) h{ }^{h} \nabla_{X}\left(\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right), H\right) \\
& \left.\quad+h(Y, H) h\left({ }^{h} \nabla_{X}\left(\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right), Z\right)-h(X, H) h\left({ }^{h} \nabla_{Y}\left(\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right), Z\right)\right\} \\
& -\mathrm{e}^{-2 \sigma}\{h(X, Z) h(Y, H)-h(X, H) h(Y, Z)\}\left\{\mathrm{e}^{(-2 n+2) \sigma} V(\sigma)^{2}+\left|\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right|_{h}^{2}\right\} . \tag{A.2.3}
\end{align*}
$$

Here ${ }^{N} R$ is the Riemannian curvature tensor of the codomain of the harmonic morphism $\varphi:\left(O,\left.g\right|_{O}\right) \rightarrow(N, \bar{h})$ produced by $\mathscr{V}$ and ${ }^{h} \nabla$ denotes the Levi-Civita connection of $(M, h)$.

Proof. Let $\left\{X_{1}, \ldots, X_{n}\right\}$ be a local frame of $\mathscr{H}$ formed of basic vector fields and let $g_{j k}=g\left(X_{j}, X_{k}\right), h_{j k}=h\left(X_{j}, X_{k}\right), \Omega_{j k}=\Omega\left(X_{j}, X_{k}\right)$. Then $\left\{X_{1}, \ldots, X_{n}, V\right\}$ is a local frame on $M$. We shall always denote by $j, k, l, r, s$ 'horizontal' indices, by $\alpha, \beta$ 'vertical' indices whilst $a, b, c, d, a^{\prime}$ will be any kind of indices.

For $\left\{E_{a}\right\}$ a local frame on $M$ we denote, as usual, by $\left\{\Gamma_{a b}^{c}\right\}$ the corresponding Christoffel symbols of the Levi-Civita connection of $\left(M^{n+1}, g\right)$, i.e.

$$
{ }^{g} \nabla_{E_{a}} E_{b}=\Gamma_{b a}^{c} E_{c} .
$$

Then, the local connection forms $\left\{\Gamma_{a}^{b}\right\}$ are characterised by $\Gamma_{a}^{b}\left(E_{c}\right)=\Gamma_{a c}^{b}$. Recall that the components ${ }^{M} R_{a b c}^{d}$ of the curvature form of the Levi-Civita connection of $\left(M^{n+1}, g\right)$ are given by

$$
{ }^{M} R\left(E_{a}, E_{b}\right) E_{c}={ }^{M} R_{c a b}^{d} E_{d}
$$

where

$$
\begin{equation*}
{ }^{M} R_{a b c}^{d}=\left(\mathrm{d} \Gamma_{a}^{d}\right)_{b c}+\left(\Gamma_{a^{\prime}}^{d} \wedge \Gamma_{a}^{a^{\prime}}\right)_{b c} . \tag{A.2.4}
\end{equation*}
$$

Also, let $\left\{{ }^{h} \Gamma_{a b}^{c}\right\}$ be the Christoffel symbols of the Levi-Civita connection of ( $M^{n+1}, h$ ).

Then, from Lemma A.2.1, it follows that the Christoffel symbols of ( $M^{n+1}, g$ ), with respect to the local frame $\left\{X_{1}, \ldots, X_{n}, V\right\}$, are given by

$$
\begin{align*}
& \Gamma_{k j}^{l}={ }^{h} \Gamma_{k j}^{l}-X_{j}(\sigma) \delta_{k}^{l}-X_{k}(\sigma) \delta_{j}^{l}+X^{l}(\sigma) h_{j k} \\
& \Gamma_{k j}^{\alpha}=\mathrm{e}^{(-2 n+2) \sigma} V(\sigma) h_{j k}-\frac{1}{2} \Omega_{j k} \\
& \Gamma_{j \alpha}^{k}=-V(\sigma) \delta_{j}^{k}+\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} \Omega_{j}^{k}  \tag{A.2.5}\\
& \Gamma_{j \alpha}^{\alpha}=(n-2) X_{j}(\sigma) \\
& \Gamma_{\alpha \alpha}^{j}=-(n-2) \mathrm{e}^{(2 n-2) \sigma} X^{j}(\sigma) \\
& \Gamma_{\alpha \alpha}^{\alpha}=(n-2) V(\sigma)
\end{align*}
$$

where $X^{j}(\sigma)=X_{l}(\sigma) h^{l j}$ and $\Omega_{j}^{k}=\Omega_{j l} h^{l k}$.
Because $\left[V, X_{j}\right]=0, j=1, \ldots, n$, from A.2.4 we obtain

$$
{ }^{M} R_{\alpha l \alpha}^{k}=X_{l}\left(\Gamma_{\alpha \alpha}^{k}\right)-V\left(\Gamma_{\alpha l}^{k}\right)+\Gamma_{j l}^{k} \Gamma_{\alpha \alpha}^{j}+\Gamma_{\alpha l}^{k} \Gamma_{\alpha \alpha}^{\alpha}-\Gamma_{j \alpha}^{k} \Gamma_{\alpha l}^{j}-\Gamma_{\alpha \alpha}^{k} \Gamma_{\alpha l}^{\alpha}
$$

which, by applying A.2.5, becomes

$$
\begin{align*}
{ }^{M} R_{\alpha l \alpha}^{k}= & -(n-2) \mathrm{e}^{(2 n-2) \sigma} X_{l}\left(X^{k}(\sigma)\right)-2(n-1)(n-2) \mathrm{e}^{(2 n-2) \sigma} X_{l}(\sigma) X^{k}(\sigma) \\
& +V(V(\sigma)) \delta_{l}^{k}-(n-1) \mathrm{e}^{(2 n-2) \sigma} V(\sigma) \Omega_{l}^{k}-\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} V\left(\Omega_{l}^{k}\right) \\
& +\left\{{ }^{h} \Gamma_{j l}^{k}-X_{l}(\sigma) \delta_{j}^{k}-X_{j}(\sigma) \delta_{l}^{k}+X^{k}(\sigma) h_{j l}\right\}\left\{-(n-2) \mathrm{e}^{(2 n-2) \sigma} X^{j}(\sigma)\right\} \\
& +\left\{-V(\sigma) \delta_{l}^{k}+\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} \Omega_{l}^{k}\right\}(n-2) V(\sigma) \\
& -\left\{-V(\sigma) \delta_{j}^{k}+\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} \Omega_{j}^{k}\right\}\left\{-V(\sigma) \delta_{l}^{j}+\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} \Omega_{l}^{j}\right\} \\
& -\left\{-(n-2) \mathrm{e}^{(2 n-2) \sigma} X^{k}(\sigma)\right\}(n-2) X_{l}(\sigma) . \tag{A.2.6}
\end{align*}
$$

Because $\Omega$ and $X_{j}$ are basic we have that $\Omega_{l}^{k}$ is basic and thus $V\left(\Omega_{l}^{k}\right)=0$.
Lines two, four and five of A.2.6) contain the terms linear in $\Omega$. These are

$$
\begin{align*}
& -(n-1) \mathrm{e}^{(2 n-2) \sigma} V(\sigma) \Omega_{l}^{k}+\frac{1}{2}(n-2) \mathrm{e}^{(2 n-2) \sigma} V(\sigma) \Omega_{l}^{k}+\mathrm{e}^{(2 n-2) \sigma} V(\sigma) \Omega_{l}^{k} \\
= & -\frac{1}{2}(n-2) \mathrm{e}^{(2 n-2) \sigma} V(\sigma) \Omega_{l}^{k} . \tag{A.2.7}
\end{align*}
$$

The first terms of lines one and three of A.2.6 give

$$
\begin{align*}
& -(n-2) \mathrm{e}^{(2 n-2) \sigma} X_{l}\left(X^{k}(\sigma)\right)-(n-2) \mathrm{e}^{(2 n-2) \sigma h} \Gamma_{j l}^{k} X^{j}(\sigma) \\
= & -(n-2) \mathrm{e}^{(2 n-2) \sigma}\left\{X_{l}\left(X^{k}(\sigma)\right)+{ }^{h} \Gamma_{j l}^{k} X^{j}(\sigma)\right\} . \tag{A.2.8}
\end{align*}
$$

Now, it is easy to see that

$$
{ }^{h} \Gamma_{\alpha l}^{k}=\frac{1}{2} \Omega_{l}^{k}
$$

and hence, we can write

$$
\begin{align*}
\mathscr{H}\left({ }^{h} \nabla_{X_{l}}\left(\operatorname{grad}_{h} \sigma\right)\right) & =\mathscr{H}\left({ }^{h} \nabla_{X_{l}}\left(X^{j}(\sigma) X_{j}+V(\sigma) V\right)\right) \\
& =\left\{X_{l}\left(X^{k}(\sigma)\right)+X^{j}(\sigma)^{h} \Gamma_{j l}^{k}+V(\sigma)^{h} \Gamma_{\alpha l}^{k}\right\} X_{k}  \tag{A.2.9}\\
& =\left\{X_{l}\left(X^{k}(\sigma)\right)+X^{j}(\sigma)^{h} \Gamma_{j l}^{k}+\frac{1}{2} V(\sigma) \Omega_{l}^{k}\right\} X_{k} .
\end{align*}
$$

We use A.2.7, A.2.8 and A.2.9 to simplify A.2.6 thus obtaining

$$
\begin{align*}
{ }^{M} R_{\alpha l \alpha}^{k}= & -(n-2) \mathrm{e}^{(2 n-2) \sigma} h^{k r} h\left({ }^{h} \nabla_{X_{l}}\left(\operatorname{grad}_{h} \sigma\right), X_{r}\right) \\
& -2(n-1)(n-2) \mathrm{e}^{(2 n-2) \sigma} X_{l}(\sigma) X^{k}(\sigma)+(n-2)^{2} \mathrm{e}^{(2 n-2) \sigma} X_{l}(\sigma) X^{k}(\sigma) \\
& +(n-2) \mathrm{e}^{(2 n-2) \sigma} X_{l}(\sigma) X^{k}(\sigma)-(n-2) \mathrm{e}^{(2 n-2) \sigma} X_{l}(\sigma) X^{k}(\sigma) \\
& +(n-2) \mathrm{e}^{(2 n-2) \sigma}\left|\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right|_{h}^{2} \delta_{l}^{k} \\
& +V(V(\sigma)) \delta_{l}^{k}-V(\sigma)^{2} \delta_{l}^{k}-(n-2) V(\sigma)^{2} \delta_{l}^{k} \\
& -\frac{1}{4} \mathrm{e}^{(4 n-4) \sigma} \Omega_{j}^{k} \Omega_{l}^{j} . \tag{A.2.10}
\end{align*}
$$

Hence

$$
\begin{align*}
{ }^{M} R_{s a l \alpha}= & g_{s k}{ }^{M} R_{\alpha l \alpha}^{k}=\mathrm{e}^{-2 \sigma} h_{s k}{ }^{M} R_{\text {ald }}^{k} \\
= & -(n-2) \mathrm{e}^{(2 n-4) \sigma} h\left({ }^{h} \nabla_{X_{l}}\left(\operatorname{grad}_{h} \sigma\right), X_{s}\right) \\
& -(n-2) \mathrm{e}^{(2 n-4) \sigma}\left\{n X_{s}(\sigma) X_{k}(\sigma)-\left|\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right|_{h}^{2} h_{s l}\right\}  \tag{A.2.11}\\
& +\mathrm{e}^{-2 \sigma}\left\{V(V(\sigma))-(n-1) V(\sigma)^{2}\right\} h_{s l} \\
& +\frac{1}{4} \mathrm{e}^{(4 n-6) \sigma} \Omega_{s j} \Omega_{l}^{j} .
\end{align*}
$$

It is easy to see that A.2.11) is equivalent to A.2.1.
Next, we prove A.2.2. Because $\left[V, X_{j}\right]=0$, the equation A.2.4 gives

$$
\begin{equation*}
{ }^{M} R_{k l \alpha}^{j}=X_{l}\left(\Gamma_{k \alpha}^{j}\right)-V\left(\Gamma_{k l}^{j}\right)+\Gamma_{r l}^{j} \Gamma_{k \alpha}^{r}+\Gamma_{\alpha l}^{j} \Gamma_{k \alpha}^{\alpha}-\Gamma_{r \alpha}^{j} \Gamma_{k l}^{r}-\Gamma_{\alpha \alpha}^{j} \Gamma_{k l}^{\alpha} \tag{A.2.12}
\end{equation*}
$$

which by applying (A.2.5) becomes

$$
\begin{align*}
{ }^{M} R_{k l \alpha}^{j}= & -X_{l}(V(\sigma)) \delta_{k}^{j}+(n-1) \mathrm{e}^{(2 n-2) \sigma} X_{l}(\sigma) \Omega_{k}^{j}+\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} X_{l}\left(\Omega_{k}^{j}\right) \\
& -V\left({ }^{h} \Gamma_{k l}^{j}\right)+V\left(X_{k}(\sigma)\right) \delta_{l}^{j}+V\left(X_{l}(\sigma)\right) \delta_{k}^{j}-V\left(X^{j}(\sigma)\right) h_{k l} \\
& +\left\{{ }^{h} \Gamma_{r l}^{j}-X_{r}(\sigma) \delta_{l}^{j}-X_{l}(\sigma) \delta_{r}^{j}+X^{j}(\sigma) h_{r l}\right\}\left\{-V(\sigma) \delta_{k}^{r}+\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} \Omega_{k}^{r}\right\} \\
& +\left\{-V(\sigma) \delta_{l}^{j}+\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} \Omega_{l}^{j}\right\}(n-2) X_{k}(\sigma) \\
& -\left\{-V(\sigma) \delta_{r}^{j}+\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} \Omega_{r}^{j}\right\}\left\{{ }^{h} \Gamma_{k l}^{r}-X_{k}(\sigma) \delta_{l}^{r}-X_{l}(\sigma) \delta_{k}^{r}+X^{r}(\sigma) h_{k l}\right\} \\
& +(n-2) \mathrm{e}^{(2 n-2) \sigma} X^{j}(\sigma)\left\{\mathrm{e}^{(-2 n+2) \sigma} V(\sigma) h_{k l}-\frac{1}{2} \Omega_{l k}\right\} . \tag{A.2.13}
\end{align*}
$$

Because $X_{j}$ are basic and $\mathscr{V}$ is a Riemannian foliation with respect to $h$, we have $V\left({ }^{h} \Gamma_{k l}^{j}\right)=0$. Also, because $\left[V, X_{l}\right]=0$, the first term of the right hand side of A.2.13 and the second term of the second line of A.2.13 cancel. Hence, A.2.13 can be written as follows

$$
\begin{align*}
& { }^{M} R_{k l \alpha}^{j}=(n-1) \mathrm{e}^{(2 n-2) \sigma} X_{l}(\sigma) \Omega_{k}^{j}+\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} X_{l}\left(\Omega_{k}^{j}\right) \\
& +V\left(X_{k}(\sigma)\right) \delta_{l}^{j}-V\left(X^{j}(\sigma)\right) h_{k l} \\
& -V(\sigma)^{h} \Gamma_{k l}^{j}+V(\sigma) X_{k}(\sigma) \delta_{l}^{j}+V(\sigma) X_{l}(\sigma) \delta_{k}^{j}-V(\sigma) X^{j}(\sigma) h_{k l} \\
& +\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma}{ }^{h} \Gamma_{r l}^{j} \Omega_{k}^{r}-\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} X_{r}(\sigma) \Omega_{k}^{r} \delta_{l}^{j} \\
& -\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} X_{l}(\sigma) \Omega_{k}^{j}+\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} X^{j}(\sigma) \Omega_{k}^{r} h_{r l} \\
& -(n-2) V(\sigma) X_{k}(\sigma) \delta_{l}^{j}+\frac{1}{2}(n-2) \mathrm{e}^{(2 n-2) \sigma} X_{k}(s) \Omega_{l}^{j} \\
& +V(\sigma)^{h} \Gamma_{k l}^{j}-\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} \Omega_{r}^{j} \Gamma_{k l}^{r} \\
& -V(\sigma) X_{k}(\sigma) \delta_{l}^{j}-V(\sigma) X_{l}(\sigma) \delta_{k}^{j}+V(\sigma) X^{j}(\sigma) h_{k l} \\
& +\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} \Omega_{l}^{j} X_{k}(\sigma)+\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} \Omega_{k}^{j} X_{l}(\sigma)-\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} \Omega_{r}^{j} X^{r}(\sigma) h_{k l} \\
& +(n-2) X^{j}(\sigma) V(\sigma) h_{k l}+\frac{1}{2}(n-2) \mathrm{e}^{(2 n-2) \sigma} X^{j}(\sigma) \Omega_{k l} \tag{A.2.14}
\end{align*}
$$

where we have used that $\Omega_{l k}=-\Omega_{k l}$.
After cancelling the corresponding terms, A.2.14 becomes

$$
\begin{align*}
& { }^{M} R_{k l \alpha}^{j}=\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma}\left\{X_{l}\left(\Omega_{k}^{j}\right)+{ }^{h} \Gamma_{r l}^{j} \Omega_{k}^{r}-{ }^{h} \Gamma_{k l}^{r} \Omega_{r}^{j}\right\} \\
& +\frac{1}{2}(n-1) \mathrm{e}^{(2 n-2) \sigma} X_{k}(\sigma) \Omega_{l}^{j}+\frac{1}{2}(n-1) \mathrm{e}^{(2 n-2) \sigma} X^{j}(\sigma) \Omega_{k l} \\
& +(n-1) \mathrm{e}^{(2 n-2) \sigma} X_{l}(\sigma) \Omega_{k}^{j}-\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma}\left\{X_{r}(\sigma) \Omega_{k}^{r} \delta_{l}^{j}+\Omega_{r}^{j} X^{r}(\sigma) h_{k l}\right\} \\
& +V\left(X_{k}(\sigma)\right) \delta_{l}^{j}-V\left(X^{j}(\sigma)\right) h_{k l}-(n-2) V(\sigma) X_{k}(\sigma) \delta_{l}^{j} \\
& +(n-2) X^{j}(\sigma) V(\sigma) h_{k l} \tag{A.2.15}
\end{align*}
$$

Now, recall that

$$
\begin{align*}
-\left({ }^{h} \nabla \Omega\right)\left(X_{j}, X_{k}, X_{l}\right) & =-\left({ }^{h} \nabla_{X_{l}} \Omega\right)\left(X_{j}, X_{k}\right)=\left({ }^{h} \nabla_{X_{l}} \Omega\right)\left(X_{k}, X_{j}\right) \\
& =\left({ }^{h} \nabla_{X_{l}} \Omega\right)_{k j}=\left({ }^{h} \nabla_{X_{l}} \Omega\right){ }_{k}^{r} h_{r j}  \tag{A.2.16}\\
& =h_{j r}\left\{X_{l}\left(\Omega_{k}^{r}\right)+{ }^{h} \Gamma_{s l}^{r} \Omega_{k}^{s}-{ }^{h} \Gamma_{k l}^{s} \Omega_{s}^{r}\right\} .
\end{align*}
$$

From A.2.16, it easily follows that A.2.2) is equivalent to A.2.15.
To prove A.2.3) we use the corresponding fundamental equation for horizontally conformal submersions [27].

Firstly, from $\lambda=\mathrm{e}^{\sigma}$, it follows that

$$
\begin{equation*}
\operatorname{grad}_{g}\left(\frac{1}{\lambda^{2}}\right)=-2 \mathrm{e}^{(-2 n+2) \sigma} V(\sigma) V-2 \mathscr{H}\left(\operatorname{grad}_{h} \sigma\right) \tag{A.2.17}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\left|\operatorname{grad}_{g}\left(\frac{1}{\lambda^{2}}\right)\right|_{g}^{2}=4 \mathrm{e}^{-2 n \sigma} V(\sigma)^{2}+4 \mathrm{e}^{-2 \sigma}\left|\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right|_{h}^{2} . \tag{A.2.18}
\end{equation*}
$$

Next, a straightforward calculation using Lemma A.2.1 and A.2.17) gives

$$
\begin{align*}
\mathscr{H}\left({ }^{g} \nabla_{X}\left(\operatorname{grad}_{g}\left(\frac{1}{\lambda^{2}}\right)\right)\right)= & 2 \mathrm{e}^{(-2 n+2) \sigma} V(\sigma)^{2} X-V(\sigma)\left(i_{X} \Omega\right)^{\#_{h}} \\
& -2 \mathscr{H}\left({ }^{h} \nabla_{X}\left(\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right)\right)+2\left|\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right|_{h}^{2} X, \tag{A.2.19}
\end{align*}
$$

where $X$ is any basic vector field. Hence, if $Z$ is a basic vector field then

$$
\begin{align*}
g\left({ }^{g} \nabla_{X}\left(\operatorname{grad}_{g}\left(\frac{1}{\lambda^{2}}\right)\right), Z\right) & =2 \mathrm{e}^{-2 n \sigma} V(\sigma)^{2} h(X, Z)+2 \mathrm{e}^{-2 \sigma}\left|\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right|_{h}^{2} h(X, Z) \\
& -\mathrm{e}^{-2 \sigma} V(\sigma) \Omega(X, Z)-2 \mathrm{e}^{-2 \sigma} h\left({ }^{h} \nabla_{X}\left(\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right), Z\right) . \tag{A.2.20}
\end{align*}
$$

Recall that $\Omega=\mathrm{d} \theta$ and thus $\Omega(X, Y)=-\theta([X, Y])$ which, because $\theta$ is the vertical dual of $V$, is equivalent to

$$
\begin{equation*}
\mathscr{V}([X, Y])=-\Omega(X, Y) V . \tag{A.2.21}
\end{equation*}
$$

Now, let $\varphi:\left(O,\left.g\right|_{O}\right) \rightarrow(N, \bar{h})$ be a harmonic morphism produced by $\mathscr{V}$, with dilation $\lambda=\mathrm{e}^{\sigma}$. If we put A.2.20 and A.2.21 into the corresponding
fundamental equation for horizontally conformal submersions [27] we obtain

$$
\begin{align*}
&-{ }^{M} R(X, Y, Z, H)=-\mathrm{e}^{-2 \sigma} \varphi^{*}\left({ }^{N} R\right)(X, Y, Z, H) \\
&+\frac{1}{4} \mathrm{e}^{(2 n-4) \sigma}\{\Omega(X, Z) \Omega(Y, H)-\Omega(Y, Z) \Omega(X, H)+2 \Omega(X, Y) \Omega(Z, H)\} \\
&+ \frac{1}{2} h(X, Z)\left\{h(Y, H) 2 \mathrm{e}^{-2 n \sigma} V(\sigma)^{2}-\mathrm{e}^{-2 \sigma} \Omega(Y, H) V(\sigma)\right. \\
&\left.+2 \mathrm{e}^{-2 \sigma}\left|\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right|_{h}^{2} h(Y, H)-2 \mathrm{e}^{-2 \sigma} h\left({ }^{h} \nabla_{Y}\left(\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right), H\right)\right\} \\
&- \frac{1}{2} h(Y, Z)\left\{h(X, H) 2 \mathrm{e}^{-2 n \sigma} V(\sigma)^{2}-\mathrm{e}^{-2 \sigma} \Omega(X, H) V(\sigma)\right. \\
&+2 \mathrm{e}^{-2 \sigma}\left|\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right|_{h}^{2} h(X, H)-2 \mathrm{e}^{-2 \sigma} h\left({ }^{h} \nabla_{X}\left(\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right), H\right)\right\} \\
&+ \frac{1}{2} h(Y, H)\left\{h(X, Z) 2 \mathrm{e}^{-2 n \sigma} V(\sigma)^{2}-\mathrm{e}^{-2 \sigma} \Omega(X, Z) V(\sigma)\right. \\
&\left.+2 \mathrm{e}^{-2 \sigma}\left|\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right|_{h}^{2} h(X, Z)-2 \mathrm{e}^{-2 \sigma} h\left({ }^{h} \nabla_{X}\left(\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right), Z\right)\right\} \\
&-\frac{1}{2} h(X, H)\left\{h(Y, Z) 2 \mathrm{e}^{-2 n \sigma} V(\sigma)^{2}-\mathrm{e}^{-2 \sigma} \Omega(Y, Z) V(\sigma)\right. \\
&\left.+2 \mathrm{e}^{-2 \sigma}\left|\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right|_{h}^{2} h(Y, Z)-2 \mathrm{e}^{-2 \sigma} h\left({ }^{h} \nabla_{Y}\left(\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right), Z\right)\right\} \\
&+\{h(X, H) h(Y, Z)-h(Y, H) h(X, Z)\}\left\{\mathrm{e}^{-2 n \sigma} V(\sigma)^{2}+\mathrm{e}^{-2 \sigma}\left|\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right|_{h}^{2}\right\} \\
&+ \mathrm{e}^{-2 \sigma} h(X(\sigma) Y-Y(\sigma) X, H(\sigma) Z-Z(\sigma) H) \\
&+\{h(X, Z) h(Y, H)-h(X, H) h(Y, Z)\}\left\{\mathrm{e}^{-2 n \sigma} V(\sigma)^{2}+\mathrm{e}^{-2 \sigma}\left|\mathscr{H}\left(\operatorname{grad}_{h} \sigma\right)\right|_{h}^{2}\right\}, \tag{A.2.22}
\end{align*}
$$

where $X, Y, Z, H$ are basic vector fields.
Now, A.2.3 follows from A.2.22 after cancelling the corresponding terms.

Remark A.2.4. The formula A.2.3) can be also proved directly in a similar way to A.2.1) and A.2.2.

The first formula of the following lemma follows after a straightforward computation using A.2.1 and A.2.3 whilst the second formula follows from A.2.2) . The third formula of the following lemma follows from (1.1.3), 1.1.4) together with Lemma 4.2.5 and A.1.1,

Lemma A.2.5. Suppose that $\mathscr{V}$ restricted to the domain of the local density $\rho=e^{(2-n) \sigma}$ is a simple foliation (i.e. the leaves are the fibres of a submersion) and let $\varphi:\left(O,\left.g\right|_{O}\right) \rightarrow(N, \bar{h})$ be the induced harmonic morphism. If ${ }^{M}$ Ricci denotes the Ricci tensor of $(M, g)$ and ${ }^{N}$ Ricci denotes the Ricci tensor of $(N, \bar{h})$, then,

$$
\begin{gather*}
{ }^{M} \operatorname{Ricci}(X, Y)=\left({ }^{N} \operatorname{Ricci}\right)\left(\varphi_{*} X, \varphi_{*} Y\right)-\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma} h\left(i_{X} \Omega, i_{Y} \Omega\right)  \tag{A.2.23}\\
-\mathrm{e}^{-2 \sigma}\left(\Delta^{M} \sigma\right) h(X, Y)-(n-1)(n-2) X(\sigma) Y(\sigma), \\
\begin{aligned}
{ }^{M} \operatorname{Ricci}(X, V) & =\frac{1}{2} \mathrm{e}^{(2 n-2) \sigma}\left({ }^{h} \mathrm{~d}^{*} \Omega\right)(X)+(n-1) \mathrm{e}^{(2 n-2) \sigma} \Omega\left(X, \operatorname{grad}_{h} \sigma\right) \\
& \quad+(n-1) X(V(\sigma))-(n-1)(n-2) X(\sigma) V(\sigma),
\end{aligned}
\end{gather*}
$$

${ }^{M} \operatorname{Ricci}(V, V)=(n-2) \mathrm{e}^{(2 n-4) \sigma} \Delta^{M} \sigma+2(n-1) V(V(\sigma))$

$$
\begin{equation*}
-(3 n-4)(n-1) V(\sigma)^{2}+\frac{1}{4} \mathrm{e}^{(4 n-4) \sigma}|\Omega|_{h}^{2} \tag{A.2.25}
\end{equation*}
$$

where ${ }^{h} \mathrm{~d}^{*}$ is the codifferential on $(M, h)$.
Remark A.2.6. Putting $n=2$ in the above formula we obtain particular cases of formulae of P. Baird and J.C. Wood [8, Proposition 4.2].

## A.3. A diagonalization result for self-adjoint bundle endomorphisms

In this appendix we prove two results used in Chapter 3 .
Lemma A.3.1. For $n \geq 1$, let $P: \mathbb{C}^{n} \times \mathbb{C} \rightarrow \mathbb{C}$ be defined by

$$
P(\mathbf{a}, \lambda)=\lambda^{n}+a_{1} \lambda^{n-1}+\cdots+a_{n} \quad\left(\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right) \in \mathbb{C}^{n}, \lambda \in \mathbb{C}\right)
$$

Let $\left(\mathbf{a}^{(k)}\right) \subseteq \mathbb{C}^{n}$ be a convergent sequence.
Then the set $S=\left\{\lambda \in \mathbb{C} \mid \exists k: P\left(\mathbf{a}^{(k)}, \lambda\right)=0\right\}$ is bounded.
Proof. Let $\widetilde{P}: \mathbb{C}^{n} \times \mathbb{C} P^{1} \rightarrow \mathbb{C} P^{1}$ be defined by
$\widetilde{P}\left(\mathbf{a},\left[\lambda_{0}, \lambda_{1}\right]\right)=\left[\lambda_{0}^{n}, \lambda_{1}^{n}+a_{1} \lambda_{0} \lambda_{1}^{n-1}+\cdots+a_{n} \lambda_{0}^{n}\right] \quad\left(\mathbf{a} \in \mathbb{C}^{n},\left[\lambda_{0}, \lambda_{1}\right] \in \mathbb{C} P^{1}\right)$.
Obviously $S=\bigcup_{k} S_{k}$ where $S_{k}=\left\{\lambda \in \mathbb{C} \mid \widetilde{P}\left(\mathbf{a}^{(k)},[1, \lambda]\right)=[1,0]\right\} ;$ suppose that $S$ is unbounded. For each $k$, let $\lambda^{(k)} \in S_{k}$ be such that $|\lambda| \leq$ $\left|\lambda^{(k)}\right|$ for every $\lambda \in S_{k}$. Because $S$ is unbounded we must have that $\left(\lambda^{(k)}\right)$ is unbounded. By passing to a subsequence, if necessary, we can suppose that $\lambda^{(k)} \rightarrow \infty$.

Because $\lambda^{(k)} \in S_{k}$ we have $\widetilde{P}\left(\mathbf{a}^{(k)},\left[1, \lambda^{(k)}\right]\right)=[1,0]$. But

$$
\lim _{k \rightarrow \infty} \widetilde{P}\left(\mathbf{a}^{(k)},\left[1, \lambda^{(k)}\right]\right)=\widetilde{P}(\mathbf{b},[0,1])=[0,1],
$$

where $\mathbf{b}=\lim _{k \rightarrow \infty} \mathbf{a}^{(k)}$, a contradiction. Hence $S$ must be bounded.
For the second lemma we need the following definition.
Definition A.3.2. Let $E \rightarrow M$ be a vector bundle endowed with a Riemannian metric $h$, and let $T \in \Gamma\left(\odot^{2} E^{*}\right)$. Say that $T$ can be consistently diagonalized at $x_{0} \in M$ if there exists an open neighbourhood $U$ of $x_{0}$ and a local orthonormal frame $\left\{e^{j}\right\}$ on $U$ for $E^{*}$ such that

$$
T=\sum_{j, k} \mu_{j} \delta_{j k} e^{j} \otimes e^{k}
$$

for some smooth functions $\mu_{j}: U \rightarrow \mathbb{R}$.
A similar definition can be made for a field of self-adjoint endomorphisms $\widetilde{T} \in \Gamma(\operatorname{End} E)$.

Lemma A.3.3. Let $E \rightarrow M$ be a vector bundle endowed with a Riemannian metric $h$, and let $T \in \Gamma\left(\odot^{2} E^{*}\right)$.

Then $T$ can be consistently diagonalized at each point of a dense open subset of $M$.

Proof. The proof is by induction on the rank (fibre dimension) of $E$.
For $\operatorname{rank} E=1$ the lemma is trivial.
Suppose that the assertion of the lemma is true for $\operatorname{rank} E<n$; we shall prove that the assertion is true for $\operatorname{rank} E=n$.

Let $P(x, \lambda)=P_{x}(\lambda)$ be the characteristic polynomial of $T_{x}$, with respect to $h_{x}(x \in M)$. For $p=1, \ldots, n$, set

$$
G_{p}=\left\{x \in M \mid P_{x} \text { has a root of order at most } p\right\},
$$

and set $G_{0}=\emptyset$. Because

$$
M=\bigcup_{p=1}^{n} \bar{G}_{p} \backslash \bar{G}_{p-1} \subseteq \bigcup_{p=1}^{n} \overline{G_{p} \backslash \bar{G}_{p-1}}=\overline{\bigcup_{p=1}^{n} G_{p} \backslash \bar{G}_{p-1}} \subseteq M
$$

we have that

$$
M=\overline{\bigcup_{p=1}^{n} G_{p} \backslash \bar{G}_{p-1}}
$$

where $\bar{A}$ denotes the closure of the set $A$. To complete the proof, it suffices to prove that each $x \in \bigcup_{p=1}^{n} G_{p} \backslash \bar{G}_{p-1}$ has an open neighbourhood $U$ such that $\left.E\right|_{U}=E_{1} \oplus E_{2}$ where $E_{1}$ and $E_{2}$ are complementary orthogonal vector subbundles of $E$ of positive rank such that $\left.T\right|_{E_{1} \otimes E_{2}}=0$.

Let $p \in\{1, \ldots, n\}$ be such that $G_{p} \backslash \bar{G}_{p-1} \neq \emptyset$ and let $x_{0} \in G_{p} \backslash \bar{G}_{p-1}$. Let $\lambda_{0}$ be a root of $P_{x_{0}}$ of order at most $p$. Because $x_{0}$ is not in $\bar{G}_{p-1}$ we have that $\lambda_{0}$ has order $p$. Then, by the Malgrange Preparation Theorem (see [25, Chapter IV]), in an open neighbourhood $U \subseteq M \backslash \bar{G}_{p-1}$ of $x_{0}$ we have $P(x, \lambda)=Q(x, \lambda) R(x, \lambda)$ where $Q$ is a polynomial of degree $p$ in $\lambda$ such that $Q\left(x_{0}, \lambda\right)=\left(\lambda-\lambda_{0}\right)^{p}$ and $R\left(x_{0}, \lambda_{0}\right) \neq 0$. From the fact that $P$ and $Q$ are both polynomials in $\lambda$ (with coefficients smooth functions of $x$ ), it follows that $R$ is also polynomial in $\lambda$.

We shall show that there exists an open neighbourhood $V \subseteq U$ of $x_{0}$ such
that, for each $x \in V, Q_{x}$ has a root of order $p$. Suppose not. Let $\left(x^{(k)}\right) \subseteq U$ be such that $\lim _{k \rightarrow \infty} x^{(k)}=x_{0}$ and, for each $k$, there exists $\mu^{(k)} \in \mathbb{R}$ such that $Q\left(x^{(k)}, \mu^{(k)}\right)=0$ with $\mu^{(k)}$ a root of $Q_{x^{(k)}}$ of order less than $p$. Obviously, $\mu^{(k)}$ is also a root of $P_{x^{(k)}}$ and, because $x^{(k)} \in U \subseteq M \backslash \bar{G}_{p-1}, \mu^{(k)}$ is a root of order at least $p$ of $P_{x^{(k)}}$. Hence $R\left(x^{(k)}, \mu^{(k)}\right)=0$. Now, by Lemma A.3.1, the sequence $\left(\mu^{(k)}\right)$ is bounded and hence, by passing to a subsequence if necessary, we can suppose that $\lim _{k \rightarrow \infty} \mu^{(k)}=\mu_{0}$ with $\mu_{0} \in \mathbb{R}$. Then $R\left(x_{0}, \mu_{0}\right)=$ $\lim _{k \rightarrow \infty} R\left(x^{(k)}, \mu^{(k)}\right)=0$ and also $Q\left(x_{0}, \mu_{0}\right)=\lim _{k \rightarrow \infty} Q\left(x^{(k)}, \mu^{(k)}\right)=0$. Because $R\left(x_{0}, \lambda_{0}\right) \neq 0$ we have that $\mu_{0} \neq \lambda_{0}$. But this implies that $\lambda_{0}$ is not a root of order $p$ of $Q_{x_{0}}$. It follows that, in an open neighbourhood $V$ of $x_{0}$, we have that $Q_{x}$ has only roots or order $p$ for any $x \in V$. Thus we can write $Q(x, \lambda)=(\lambda-\mu(x))^{p},((x, \lambda) \in V \times \mathbb{R})$, where $\mu(x)$ is the root of $\partial^{p-1} Q / \partial \lambda^{p-1}(x, \cdot)$ so that $\mu$ is smooth. Hence

$$
P(x, \lambda)=(\lambda-\mu(x))^{p} R(x, \lambda) \quad((x, \lambda) \in V \times \mathbb{R}) .
$$

Moreover, because $\partial^{p} P / \partial \lambda^{p}\left(x_{0}, \lambda_{0}\right) \neq 0$ we can suppose that $\partial^{p} P / \partial \lambda^{p}(x, \mu(x))$ is non-zero for any $x \in V$. It follows that $\mu(x)$ is an eigenvalue of order $p$ for $T_{x}$ for any $x \in V$.

Let $\left(E_{1}\right)_{x}$ be the eigenspace of $\mu(x)$ and let $\left(E_{2}\right)_{x}$ be its orthogonal complement. It is easy to see that $E_{j}=\bigcup_{x \in V}\left(E_{j}\right)_{x},(j=1,2)$, are smooth subbundles of $E$ which have the required properties. The lemma follows.

## A.4. Conformally-flat Riemannian manifolds

Firstly, we recall (see [38]) the definition of the Weyl tensor of a Riemannian manifold.

Let $\left(M^{m}, g\right)$ be a Riemannian manifold. For $h$ and $k$ sections of $\odot^{2}\left(T^{*} M\right)$ (that is, $h$ and $k$ are symmetric covariant tensor fields of degree two on $M^{m}$ ), we shall denote by $h \otimes k$ the section of $\odot^{2}\left(\Lambda^{2}\left(T^{*} M\right)\right)$ defined by

$$
\begin{aligned}
(h \otimes k)(T, X, Y, Z)= & h(T, Y) k(X, Z)+h(X, Z) k(T, Y) \\
& -h(T, Z) k(X, Y)-h(X, Y) k(T, Z),
\end{aligned}
$$

for any $T, X, Y, Z \in T M$.
If $S$ is a $(1,3)$-tensor field on $(M, g)$ then we shall denote by the same symbol $S$ the (0,4)-tensor field defined by $S(T, X, Y, Z)=-g(S(T, X, Y), Z)$, for any $T, X, Y, Z \in T M$.

The Weyl (curvature) tensor of $\left(M^{m}, g\right)$ is the $(1,3)$-tensor field $W$ characterised by the following two conditions:

1) $\operatorname{trace}(X \mapsto W(X, Y) Z)=0$, for any $Y, Z \in T M$,
2) $R=g \otimes r+W$ for some (necessarily unique) section $r$ of $\odot^{2}\left(T^{*} M\right)$, where $R$ is the curvature tensor of $\left(M^{m}, g\right)$.

The Weyl tensor is conformally invariant; that is, if we denote by $W^{g}$ the Weyl tensor of $\left(M^{m}, g\right)$ then $W^{\lambda^{2} g}=W^{g}$, for any positive function $\lambda$ on $M^{m}$.

The Riemannian manifold $\left(M^{m}, g\right)$ is called (locally) conformally-flat if for each point of $M^{m}$ there exists an open neighbourhood $U$ and a conformal diffeomorphism $\varphi$ from $U$ onto some open set of $\mathbb{R}^{m}$ (endowed with its canonical Riemannian metric); the local coordinates on $U$ induced by $\varphi$ are called flat.

From Liouville's theorem on local conformal diffeomorphisms between Euclidean spaces (see [9] ), it follows easily that if $\left(M^{m}, g\right)$ is conformally-flat then $M^{m}$ is real-analytic in flat local coordinates $(m \geq 2)$.

The following theorem is due to H . Weyl (see [38] ).
Theorem A.4.1. A Riemannian manifold, of dimension at least four, is conformally-flat if and only if its Weyl tensor is zero.
(See [38] for the case when the dimension is less than four.)

The following result is used in Section 3.8.
Proposition A.4.2. Let $\left(M^{m}, g\right)$ be a Riemannian manifold, $(m \geq 4)$. The following assertions are equivalent.
(i) $\left(M^{m}, g\right)$ is conformally-flat.
(ii) $R(X, Y, X, Y)=0$ for any $X, Y \in T M$ spanning an isotropic space on $(M, g)$, where $R$ is the curvature tensor of $(M, g)$, and $T M$ now denotes the complexified tangent bundle.

Proof. Clearly, assertion (ii) is equivalent to $W(X, Y, X, Y)=0$ for any $X, Y \in T M$ spanning an isotropic space on $(M, g)$, where $W$ is the Weyl tensor of $(M, g)$. Therefore, by Theorem A.4.1, we have $(\mathrm{i}) \Longrightarrow$ (ii).

Suppose that (ii) holds and let ( $X_{1}, \ldots, X_{m}$ ) be an orthonormal frame on $\left(M^{m}, g\right)$. Then for any distinct $i, j, k, l \in\{1, \ldots, m\}$ we have

$$
W\left(X_{i} \pm \mathrm{i} X_{j}, X_{k}+\mathrm{i} X_{l}, X_{i} \pm \mathrm{i} X_{j}, X_{k}+\mathrm{i} X_{l}\right)=0 .
$$

This is equivalent to the following two relations

$$
\begin{gather*}
W\left(X_{i}, X_{k}+\mathrm{i} X_{l}, X_{i}, X_{k}+\mathrm{i} X_{l}\right)=W\left(X_{j}, X_{k}+\mathrm{i} X_{l}, X_{j}, X_{k}+\mathrm{i} X_{l}\right),  \tag{А.4.1}\\
W\left(X_{i}, X_{k}+\mathrm{i} X_{l}, X_{j}, X_{k}+\mathrm{i} X_{l}\right)=0 . \tag{A.4.2}
\end{gather*}
$$

Also, by applying condition (2) of the definition of the Weyl tensor, we obtain

$$
\begin{equation*}
\sum_{r=1}^{m} W\left(X_{r}, X_{k}+\mathrm{i} X_{l}, X_{r}, X_{k}+\mathrm{i} X_{l}\right)=0 . \tag{А.4.3}
\end{equation*}
$$

From A.4.1) and A.4.3) , it follows that $W\left(X_{j}, X_{k}+\mathrm{i} X_{l}, X_{j}, X_{k}+\mathrm{i} X_{l}\right)=$ 0 and, hence, $W_{j k j k}=W_{j l j l}$, for any distinct $j, k, l \in\{1, \ldots, m\}$. Therefore, for any distinct $i, j \in\{1, \ldots, m\}$, we have

$$
(m-1) W_{i j i j}=\sum_{r=1}^{m} W_{i r i r}=0 .
$$

From A.4.2 we obtain that, for any distinct $i, j, k, l \in\{1, \ldots, m\}$, we have

$$
\begin{align*}
W_{i k j k} & =W_{i l j l}  \tag{A.4.4}\\
W_{i k j l} & =-W_{i l j k}
\end{align*}
$$

The first relation of A.4.4 implies $W_{i k j k}=0$, whilst from the second relation of A.4.4 and the algebraic Bianchi identity it follows quickly that $W_{i j k l}=0$, for any distinct $i, j, k, l \in\{1, \ldots, m\}$.

Thus, if (ii) holds then $W=0$ which, by Theorem A.4.1, is equivalent to (i).

If $\mathscr{H}$ is a distribution on a Riemannian manifold $\left(M^{m}, g\right)$ we shall denote by $I^{\mathscr{H}}$ the integrability tensor of $\mathscr{H}$, which is the $\mathscr{V}$-valued horizontal twoform on $M^{m}$ defined by $I^{\mathscr{H}}(X, Y)=-\mathscr{V}[X, Y]$, for any horizontal vector fields $X$ and $Y$, where $\mathscr{V}=\mathscr{H}^{\perp}$.

Next, we prove the following:
Proposition A.4.3. Let $\varphi:\left(M^{m}, g\right) \rightarrow\left(N^{n}, h\right)$ be a horizontally conformal submersion between conformally-flat Riemannian manifolds, $(m \geq n \geq 4)$.

Then $g\left(I^{\mathscr{H}}(X, Y), I^{\mathscr{H}}(X, Y)\right)=0$, for any horizontal vectors $X$ and $Y$ spanning an isotropic space on $\left(M^{m}, g\right)$.

Proof. As both the hypothesis and the conclusion are conformally-invariant, we may suppose that $\varphi:\left(M^{m}, g\right) \rightarrow\left(N^{n}, h\right)$ is a Riemannian submersion. Then the proof follows easily from Proposition A.4.2 and the following wellknown relation of B. O'Neill (see 9]):

$$
R^{M}(X, Y, X, Y)=\varphi^{*}\left(R^{N}\right)(X, Y, X, Y)-\frac{3}{4} g(\mathscr{V}[X, Y], \mathscr{V}[X, Y]),
$$

for any horizontal vector fields $X$ and $Y$.
Corollary A.4.4. Any horizontally conformal submersion, with fibres of dimension at most two, between conformally-flat Riemannian manifolds has integrable horizontal distribution, if the codomain has dimension at least four.

Proof. Let $\varphi:\left(M^{m}, g\right) \rightarrow\left(N^{n}, h\right)$ be a horizontally conformal submersion between conformally-flat Riemannian manifolds, $m \geq n \geq 4$.

Let $x \in M$ and let $E \subseteq T_{x} M$ be an oriented four-dimensional subspace. From Proposition A.4.3, it follows that $I_{x}^{\mathscr{H}}: \Lambda_{+}^{2} E \rightarrow \mathscr{V}_{x}$ is conformal, where $\Lambda_{+}^{2} E$ is the space of self-dual bivectors on $\left(E,\left.g\right|_{E}\right)$. As $\Lambda_{+}^{2} E$ is threedimensional, we obtain that either $I_{x}^{\mathscr{H}}=0$ or $\operatorname{dim}\left(\mathscr{V}_{x}\right) \geq 3$.

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