

Submersive Harmonic Maps and Morphisms

Radu Pantilie

Submitted in accordance with the requirements for the degree of
Ph.D. in Pure Mathematics.

University of Leeds,
School of Mathematics,
Department of Pure Mathematics,
September 2000.

*The candidate confirms that the work submitted is his own and
that appropriate credit has been given where reference has been made
to the work of others.*

Acknowledgements

Firstly, I'd like to thank to L. Crosilla who helped me whilst I was learning L^AT_EX.

I am also grateful to P. Baird, S. Gudmundsson and S. Montaldo for generous help and very useful discussions.

I gratefully acknowledge the support of the O.R.S. Scheme Awards, the School of Mathematics of the University of Leeds, the Tetley and Lupton Scholarships and the Edward Boyle Bursary.

I would like to express my deep gratitude to J.C. Wood for thorough guidance. I am deeply indebted to him for numerous suggestions which led to significant improvements of both the style and the content of the thesis.

All this work would have not been possible without the love and support of my wife Iuliana and my daughter Ana.

SUBMERSIVE HARMONIC MAPS AND MORPHISMS

R. Pantilie

Submitted in accordance with the requirements

for the degree of PhD

September 2000

ABSTRACT

A *harmonic morphism* between Riemannian manifolds is a map which preserves Laplace's equation. Harmonic morphisms have been characterised, by B. Fuglede and T. Ishihara, as a special class of *harmonic maps*.

Following an idea of J.C. Wood, we study submersive harmonic morphisms (mainly, to Riemannian manifolds of dimension greater than or equal to three) by placing them into the context of *conformal foliations*. In this way we obtain:

(i) *New examples and constructions of harmonic morphisms* based on:

- Riemannian foliations locally generated by Killing vector fields. These examples are related to a basic type of harmonic morphism due to R.L. Bryant.

- Homothetic foliations. These constructions are related to a basic type of harmonic morphism due to P. Baird and J. Eells.

- The Hawking Taub-Nut metric and a related new metric construction, in dimension four, which contains as a particular case the Eguchi-Hanson II metric.

(ii) *Classification results* for:

- Harmonic morphisms with one-dimensional fibres on Einstein manifolds.

Along the way an entirely new proof of a fundamental classification theorem of R.L. Bryant is obtained.

- Submersive harmonic morphisms induced by (infinitesimal) conformal actions on Einstein manifolds.

- Submersive harmonic morphisms between compact Riemannian manifolds whose curvatures satisfy certain (non)positivity conditions.

Contents

1	Foliations which produce harmonic morphisms	11
1.1	Foliations and harmonic morphisms	11
1.2	Metric deformations	16
1.3	Characterisation of the conformal foliations which produce harmonic morphisms	20
1.4	Homothetic foliations	21
2	Group actions and harmonic morphisms	28
2.1	Mean curvature forms and adapted Bott connections	28
2.2	Characterisation of the conformal actions which produce harmonic morphisms	36
2.3	Isometric actions and harmonic morphisms	39
2.4	Homothetic foliations locally generated by conformal vector fields	50
2.5	Homothetic actions and harmonic morphisms	53
2.6	Conformal actions and harmonic morphisms on Einstein manifolds	59
3	Harmonic morphisms with one-dimensional fibres	65
3.1	Some basic facts on harmonic morphisms with one-dimensional fibres	65
3.2	Some local and global results for one-dimensional foliations which produce harmonic morphisms	72
3.3	Some constructions of one-dimensional foliations which produce harmonic morphisms on Einstein manifolds	84
3.3.1	Homothetic foliations with geodesic leaves	85
3.3.2	Riemannian foliations locally generated by Killing fields . .	86

3.4	Foliations of dimension one which produce harmonic morphisms on four-dimensional Einstein manifolds	87
3.5	The third type	98
3.6	A few applications	102
3.7	Harmonic morphisms $\varphi : (M^4, g) \rightarrow (N^3, h)$ between Einstein manifolds	107
4	Harmonic morphisms on compact Riemannian manifolds	115
4.1	Mixed curvature and harmonic morphisms	115
4.2	Two integral formulae for harmonic morphisms with one-dimensional fibres	120
A	A formula related to a pair of complementary orthogonal distri- butions	128
B	Curvature formulae for the total space of a harmonic morphism with 1-dim fibres	130

Introduction

Harmonic morphisms between Riemannian manifolds are smooth maps which pull back harmonic functions to harmonic functions.

By the fundamental theorem of B. Fuglede [23] and T. Ishihara [35], harmonic morphisms were characterised as harmonic maps which are horizontally weakly conformal.

The next important step towards a better understanding of harmonic morphisms was done by P. Baird and J. Eells [6] who found the necessary and sufficient condition for a harmonic morphism to have minimal (regular) fibres (see Proposition 1.1.10, below). In the particular case of a map $\varphi : (M^m, g) \rightarrow (N^2, h)$ to a Riemannian manifold (N^2, h) of dimension two the result of P. Baird and J. Eells states that φ is a harmonic morphism if and only if φ is horizontally weakly conformal and its regular fibres are minimal submanifolds of (M^m, g) . This result was then elegantly formulated in the language of conformal foliations by J.C. Wood [65].

The first classification results for harmonic morphisms were due to P. Baird and J.C. Wood (see [9]) who completely classified harmonic morphisms with one-dimensional fibres from three-dimensional Riemannian manifolds with constant curvature.

More recently, R.L. Bryant [13] proved that from a Riemannian manifold (M^{n+1}, g) , $n \geq 3$, with constant sectional curvature there are just *two* types of submersive harmonic morphisms $\varphi : (M^{n+1}, g) \rightarrow (N^n, h)$ with one-dimensional fibres, namely either

- (i) there exists a nowhere zero Killing vector field tangent to the fibres of φ

or

(ii) φ has geodesic fibres orthogonal to an umbilical foliation by hypersurfaces.

Moreover, the type (i) was new (whilst type (ii) was known to P. Baird and J. Eells [6]). However Bryant's proof is based on long calculations involving the theory of exterior differential systems.

We study harmonic morphisms by placing them into the framework of conformal foliations, an idea due to J.C. Wood [65]. Unless otherwise stated, we consider only foliations of codimension greater than or equal to three.

In Chapter 1 we present some basic facts on *foliations which produce harmonic morphisms* (i.e. foliations which can be locally defined by submersive harmonic morphisms). We recall Bryant's result [13] which states that a conformal foliation produces harmonic morphisms if and only if a certain one-form is closed (see Proposition 1.3.1). In particular, if the mean curvature forms of a conformal foliation $\mathcal{V}$ and of its orthogonal complement $\mathcal{H}$ are closed then $\mathcal{V}$ produces harmonic morphisms. Following a suggestion of J.C. Wood we call a foliation *homothetic* if it can be locally defined by horizontally homothetic submersions. This is equivalent to the condition that the foliation is conformal and its orthogonal complement has closed mean curvature form.

Note also that, although a Riemannian one-dimensional foliation produces harmonic morphisms if and only if it is locally generated by Killing vector fields [13], a one-dimensional foliation locally generated by conformal vector fields produces harmonic morphisms if and only if it is a homothetic foliation (a consequence of Proposition 3.1.5). This starts to show the importance of homothetic foliations. Another equivalent condition for a conformal foliation $\mathcal{V}$ to be homothetic is that any local dilation (Definition 1.1.8) of it can be locally written as the product of a function constant along leaves and a function constant along horizontal curves (in particular, a conformal foliation $\mathcal{V}$ is homothetic if and only if, in a neighbourhood of each point, a local dilation for $\mathcal{V}$ can be found which is constant along horizontal curves). This latter property is also satisfied by any positive smooth function f defined on the Riemannian manifold (M, g) and which has the property that $\mathcal{V}$ produces harmonic morphisms on both (M, g) and

$(M, f^2 g)$. This is a particular case of a result which answers a question formulated by X. Mo (cf. [32]).

For a foliation $\mathcal{V}$ on M there exists a metric g on M with respect to which $\mathcal{V}$ is a homothetic foliation if and only if $\mathcal{V}$ is a foliation of type (A) in the sense of I. Vaisman [62]. However, all the above geometrical properties of the homothetic foliations appear to be new.

In terms of foliations which produce harmonic morphisms the types (i) and (ii) above correspond to:

- (i) Riemannian one-dimensional foliations locally generated by Killing vector fields,
- (ii) homothetic foliations with geodesic leaves orthogonal to an umbilical foliation by hypersurfaces.

The starting point of the results of Chapter 2 was to try to generalise the type (i) above to foliations of higher dimensions. In Theorem 2.2.6 we give necessary and sufficient conditions for a conformal foliation locally generated by conformal vector fields to produce harmonic morphisms. In the particular case when the foliation is Riemannian and locally generated by Killing vector fields this condition depends only on the induced infinitesimal action and on the integrability tensor of the horizontal distribution. Many natural constructions of Riemannian foliations which produce harmonic morphisms can be thus obtained. Moreover, most of the constructions can be generalised to homothetic foliations locally generated by conformal vector fields.

In Chapter 2 we also study foliations locally generated by homothetic vector fields and their relations to homothetic foliations and harmonic morphisms. This is motivated by the fact that, on a Ricci-flat manifold, given any foliation $\mathcal{V}$ locally generated by conformal vector fields and which produces harmonic morphisms, *either* $\mathcal{V}$ is locally generated by homothetic vector fields *or* any harmonic morphism produced by $\mathcal{V}$ can be locally decomposed into a totally geodesic harmonic morphism followed by another harmonic morphism. This is shown in Section 2.6 where other similar factorisation results are obtained.

One of the main achievements of Chapter 3 is a new (and more geometrical) proof of the above mentioned fundamental result of R.L. Bryant. In fact we slightly generalise this result by replacing, in the hypothesis, the simply-connectedness of the domain with the orientability of the vertical distribution. Also, we have looked for extensions of R.L. Bryant's result to one-dimensional foliations $\mathcal{V}$ which produce harmonic morphisms on Einstein manifolds (M, g) , $\dim M \geq 4$. In fact we prove the following:

- $\mathcal{V}$ must be of type (i) or type (ii) if its orthogonal complement is integrable (Theorem 3.2.4).

- $\mathcal{V}$ must be of type (i) or type (ii) if it is homothetic (Corollary 3.2.6).

- If $\dim M = 4$ then $\mathcal{V}$ is of one of the types (i), (ii) or (iii) (Theorem 3.4.4) where types (i) and (ii) are as before and type (iii) is the following: (M^4, g) is Ricci-flat and, up to homotheties, any harmonic morphism $\varphi : (U, g|_U) \rightarrow (N^3, h)$, with dilation λ , produced by $\mathcal{V}$ such that $\mathcal{V}|_U$ and N^3 are orientable is (locally) described by:

- (a) (N^3, h) has constant sectional curvature equal to one;

- (b) $\frac{1}{2}d(\lambda^{-2})$ is a (flat) principal connection for $\mathcal{V}$ with respect to suitably chosen $V \in \Gamma(\mathcal{V})$ such that $g(V, V) = \lambda^2$ (Definition 3.4.1);

- (c) the local connection form A of $\mathcal{H}$ with respect to $\frac{1}{2}d(\lambda^{-2})$ satisfies the equation $dA + 2 * A = 0$ on (N^3, h) where $*$ is the Hodge star-operator of (N^3, h) with respect to some orientation of N^3 .

The construction of the metric g in type (iii) above appears to be in the spirit of work of P.E. Jones and K.P. Tod (see [16]).

Examples of harmonic morphisms of type (iii) are given in Section 3.5. There we prove that these are always submersive (Proposition 3.5.1). Also, we show that any surjective harmonic morphism of type (iii) with connected fibres and complete codomain is, up to homotheties and Riemannian coverings, the restriction of the radial projection $(\mathbb{R}^4 \setminus \{0\}, g_a) \rightarrow S^3$ where g_a is the Eguchi-Hanson II metric [21] (when $a = 0$, g_a is the restriction of the canonical metric on $\mathbb{R}^4$ and φ_0 is a well-known harmonic morphism simultaneously of type (ii) and (iii)). In particular, there exists no surjective harmonic morphism of type (iii) whose domain and codomain are both complete.

In Section 3.7 we study surjective harmonic morphisms $\varphi : (M^4, g) \rightarrow (N^3, h)$ (the necessity of surjectivity was pointed out by J.C. Wood) between *complete* Einstein manifolds of dimension four and three, respectively. If M^4 and N^3 are simply-connected we prove the following:

- If φ is submersive then, up to homotheties, it is one of the following projections $\mathbb{R}^4 \rightarrow \mathbb{R}^3$, $H^4 \rightarrow \mathbb{R}^3$, $H^4 \rightarrow H^3$ induced by the following canonical warped-product decompositions $\mathbb{R}^4 = \mathbb{R}^1 \times \mathbb{R}^3$, $H^4 = H^1 \times_r \mathbb{R}^3$, $H^4 = H^1 \times_s H^3$ where H^k is the hyperbolic space of dimension k (Theorem 3.7.1).

- If φ has exactly one critical point then there exists $a \geq 0$ such that, up to homotheties, $\varphi : (\mathbb{R}^4, g_a) \rightarrow (\mathbb{R}^3, h_0)$ is the Hopf polynomial with g_a the Hawking Taub-NUT metric ($a > 0$) and g_0 , h_0 the canonical metrics on $\mathbb{R}^4$, $\mathbb{R}^3$, respectively (Theorem 3.7.10).

- If M^4 is compact and φ non-constant then (M^4, g) and (N^3, h) are flat and, up to homotheties and Riemannian coverings, φ is the canonical projection between flat tori $T^4 \rightarrow T^3$ (Theorem 3.7.11).

The necessary curvature formulae for Chapter 3 are given in Appendix B.

In Chapter 4 we present some results on harmonic morphisms defined on compact Riemannian manifolds. In Section 1 these are based on a known formula of P. Walczak [64] which we recall in Appendix A. The results of Section 2 are based on two integral formulae which are proved in Theorem 4.2.2. From Chapter 4 we mention here the following results:

- On a compact Riemannian manifold with nonpositive Ricci curvature any one-dimensional foliation which produces harmonic morphisms and admits a global density is locally generated by parallel vector fields (Theorem 4.1.2).

- On a compact Riemannian manifold with positive sectional curvature there exists no homothetic foliation which produces harmonic morphisms and has integrable orthogonal complement (Corollary 4.1.6).

- For $n \in \{3, 4, 5\}$ if (M^{n+1}, g) , (N^n, h) are compact with scalar curvatures $s^M \geq 0$, $s^N \leq 0$ and, if at least one of these inequalities is strict, then there exists no nonconstant submersive harmonic morphism $\varphi : (M^{n+1}, g) \rightarrow (N^n, h)$ (Corollary 4.2.10).

Chapter 1

Foliations which produce harmonic morphisms

1.1 Foliations and harmonic morphisms

We recall some basic definitions and results on harmonic morphisms and conformal foliations.

Definition 1.1.1 ([23],[35]). A *harmonic morphism* between Riemannian manifolds is a smooth map $\varphi : (M^m, g) \rightarrow (N^n, h)$ such that for any harmonic function $f : (U, h|_U) \rightarrow \mathbb{R}$, defined on some open Riemannian submanifold $(U, h|_U)$ of (N, h) with $\varphi^{-1}(U) \neq \emptyset$, the pull back $f \circ \varphi : (\varphi^{-1}(U), g|_{\varphi^{-1}(U)}) \rightarrow \mathbb{R}$ is a harmonic function.

By the fundamental result of B. Fuglede [23] and T. Ishihara [35] the harmonic morphisms form a special class of harmonic maps which we now describe.

Definition 1.1.2 ([23],[35]). A smooth map $\varphi : (M^m, g) \rightarrow (N^n, h)$ between Riemannian manifolds is called *horizontally weakly conformal* if at each point $x \in M$ either φ is submersive (i.e. $\varphi_{*,x} : T_x M \rightarrow T_{\varphi(x)} N$ is surjective) and $\varphi_*|_{\mathcal{H}_x}$ is conformal, where $\mathcal{H}_x = (\ker \varphi_{*,x})^\perp$, or $\varphi_{*,x} = 0$.

Let $\lambda : M \rightarrow \mathbb{R}$ be the nonnegative function such that $\lambda(x)$ is the conformal factor of $\varphi_*|_{\mathcal{H}_x}$ if $x \in M$ is a regular point of φ and $\lambda(x) = 0$ if x is a critical point of φ . Then λ is called the *dilation* of φ .

Remark 1.1.3 ([23],[35]). Let λ be the dilation of the horizontally weakly conformal map $\varphi : (M^m, g) \rightarrow (N^n, h)$. Then, because $\lambda^2 = \frac{1}{n}|\varphi_*|^2$, λ is continuous on M and smooth outside the set of critical points; λ^2 is smooth on M .

Theorem 1.1.4 ([23],[35]). *A map $\varphi : (M, g) \rightarrow (N, h)$ is a harmonic morphism if and only if φ is a harmonic map which is horizontally weakly conformal.*

Let (M, g) be a Riemannian manifold (it will always be assumed that M is paracompact and connected) and $\mathcal{V}$ (the tangent bundle of) a foliation or, more generally, a distribution on it. The orthogonal complement of $\mathcal{V}$ will be denoted by $\mathcal{H}$. Then, $\mathcal{H}$ and $\mathcal{V}$ will be called the *horizontal* and *vertical* distributions, respectively. Following [11], we shall denote the corresponding projections by the same letters $\mathcal{H}$ and $\mathcal{V}$; we shall denote by X, Y horizontal vector fields, i.e. sections of $\mathcal{H}$ and by U, V vertical vector fields, i.e. sections of $\mathcal{V}$.

Definition 1.1.5 (see [65]). A foliation $\mathcal{V}$ on (M, g) is called *conformal* if for any vertical vector field $U \in \Gamma(\mathcal{V})$ and horizontal vector fields $X, Y \in \Gamma(\mathcal{H})$ we have

$$(\mathcal{L}_U g)(X, Y) = \mu(U) g(X, Y)$$

for some vertical one-form $\mu \in \Gamma(T^*M)$ and where $\mathcal{L}$ denotes the Lie differentiation.

By Theorem 1.1.4 the connected components of a submersive harmonic morphism $\varphi : (M, g) \rightarrow (N, h)$ form a conformal foliation. Conversely, we make the following definition:

Definition 1.1.6 (cf. [65]). Let (M, g) be a Riemannian manifold and let $\mathcal{V}$ be a foliation on it.

We shall say that $\mathcal{V}$ *produces harmonic morphisms on (M, g)* if $\mathcal{V}$ is locally defined by submersive harmonic morphisms (i.e. each point of M has an open neighbourhood U which is the domain of a submersive harmonic morphism $\varphi : (U, g|_U) \rightarrow (N, h)$ whose fibres are open subsets of the leaves of $\mathcal{V}$).

Remark 1.1.7. When $\text{codim}\mathcal{V} = 2$, $\mathcal{V}$ produces harmonic morphisms if and only if it is conformal and its leaves are minimal [65]; in this case any local submersion φ on M whose fibres are open subsets of the leaves can be made into a harmonic morphism. Indeed, it suffices to choose a metric on the codomain such that φ is horizontally conformal.

We shall see (as an immediate consequence of Corollary 1.1.14) that, when $\text{codim}\mathcal{V} \neq 2$ and $\mathcal{V}$ produces harmonic morphisms, then each submersion $\varphi : U \rightarrow N$ defined on an open subset $U \subseteq M$ with zero first Betti number and which locally defines $\mathcal{V}$ can be made into a harmonic morphism (i.e. there exists a Riemannian metric h on N such that $\varphi : (U, g|_U) \rightarrow (N, h)$ is a harmonic morphism).

Definition 1.1.8 (cf. [62, (1.4)]). Let $\mathcal{V}$ be a conformal foliation on the Riemannian manifold (M, g) . A smooth positive function $\lambda : O \rightarrow \mathbb{R}$ on an open subset O of M will be called a *local dilation* of $\mathcal{V}$ if $\mathcal{V}|_O$ is a Riemannian foliation on $(O, \lambda^2 g|_O)$. If $O = M$ then we shall call λ a (*global*) *dilation* of $\mathcal{V}$.

Remark 1.1.9. 1) It is obvious that local dilations for a conformal foliation $\mathcal{V}$ can be found in the neighbourhood of each point; in fact, this is equivalent to the definition of its conformality. If $\mathcal{V}$ is *simple*, i.e. its leaves are the fibres of a (horizontally conformal) submersion φ , then it admits a (global) dilation, for example, the dilation of φ .

2) A smooth positive function λ is a local dilation for $\mathcal{V}$ if and only if

$$(\mathcal{L}_U(\lambda^2 g))(X, Y) = 0$$

for any vertical vector field U and any horizontal vector fields X, Y . Hence, if we multiply a local dilation of a conformal foliation by a smooth positive function which is constant along the leaves then we obtain another local dilation of the foliation. Conversely, if two local dilations λ_j , $j = 1, 2$, of a conformal foliation $\mathcal{V}$ have the same domain then $\lambda_2 = \lambda_1 \rho$ where the factor ρ is a smooth positive function, constant along the leaves of $\mathcal{V}$.

3) Let $\mathcal{V}$ be a foliation, not necessarily conformal. Let $\mathcal{H}$ denote its orthogonal

complement. Recall that its second fundamental form ${}^{\mathcal{H}}B$ is the horizontal $\mathcal{V}$ -valued tensor field defined by

$${}^{\mathcal{H}}B(X, Y) = \frac{1}{2}\mathcal{V}(\nabla_X Y + \nabla_Y X) , \quad (1.1.1)$$

where X, Y are horizontal vector fields (see [56, Ch.IV, 3.16]). A simple calculation (see [9]) gives the following formula

$$(\mathcal{L}_U g)(X, Y) = -2g({}^{\mathcal{H}}B(X, Y), U) , \quad (1.1.2)$$

where U, X, Y are as above.

It follows quickly (see [9]) that any local dilation λ of a conformal foliation $\mathcal{V}$ is characterised by the relation

$$\text{trace}({}^{\mathcal{H}}B) = n\mathcal{V}(\text{grad}(\log \lambda)) , \quad (1.1.3)$$

where $n = \text{codim} \mathcal{V}$.

Note that formula (1.1.2) shows that $\mathcal{V}$ is a conformal foliation if and only if $\mathcal{H}$ is an umbilical distribution, i.e. ${}^{\mathcal{H}}B(X, X)$ is independent of X for $g(X, X) = 1$, if $\mathcal{H}$ is integrable this condition says that its integral submanifolds are umbilical (see [56]).

Let $\varphi : (M^m, g) \rightarrow (N^n, h)$ be a horizontally conformal submersion with dilation λ . Let τ denote the tension field of φ and ${}^{\mathcal{V}}B$ the second fundamental of the foliation $\mathcal{V}$ induced by the fibres, then we have the *fundamental equation* [6] (see [9] for a different proof):

$$\tau + \text{trace}({}^{\mathcal{V}}B) + (n - 2)\mathcal{H}(\text{grad}(\log \lambda)) = 0 . \quad (1.1.4)$$

From this, P. Baird and J. Eells concluded:

Proposition 1.1.10 ([6]). (a) *When $n = 2$, φ is a harmonic morphism if and only if its fibres are minimal.*

(b) *When $n \neq 2$ any two of the following assertions imply the remaining assertion:*

- (i) *φ is a harmonic morphism,*
- (ii) *φ has minimal fibres,*
- (iii) *φ is horizontally homothetic (i.e. λ is constant along horizontal curves).*

Note that in above proposition it is unnecessary for φ to be submersive (see [9]).

Let ω denote a local volume form of $\mathcal{V}$. It can easily be seen that, the fundamental equation (1.1.4) is equivalent to

$$\mathcal{V}^*(\mathcal{L}_X(\lambda^{2-n}\omega)) = \lambda^{2-n}g(X, \tau)\omega, \quad (1.1.5)$$

for any horizontal vector field X . Thus, we have the following:

Proposition 1.1.11 ([13],[46]). *A horizontally conformal submersion with dilation λ is a harmonic morphism if and only if the parallel displacement defined by the horizontal distribution preserves the mass of the fibres, where the fibres are given the mass density λ^{2-n} .*

Definition 1.1.12. Let $\mathcal{V}$ be a foliation of codimension n , which produces harmonic morphisms on (M, g) . Let λ be a local dilation of $\mathcal{V}$ which restricts to give dilations of harmonic morphisms which locally define $\mathcal{V}$. Then $\rho = \lambda^{2-n}$ is called a *local density* of $\mathcal{V}$. If λ is globally defined on M then ρ is called a (*global density*).

Proposition 1.1.13. *Let $\mathcal{V}$ be a foliation which produces harmonic morphisms on (M, g) . Then there exists a Riemannian regular covering $\xi : (\widetilde{M}, \widetilde{g}) \rightarrow (M, g)$ with the following properties:*

- (i) $\xi^*(\mathcal{V})$ admits a global density.
- (ii) If $\eta : (P, k) \rightarrow (M, g)$ is any Riemannian regular covering such that $\eta^*(\mathcal{V})$ admits a global density then there exists a unique Riemannian regular covering $\sigma : (P, k) \rightarrow (\widetilde{M}, \widetilde{g})$ such that $\eta = \xi \circ \sigma$.

Moreover, ξ is the unique Riemannian regular covering satisfying (i) and (ii).

Proof. Let $[a] \in H^1(M, \mathbb{R})$ be the cohomology class defined by the differentials of the logarithms of the local densities of $\mathcal{V}$ and let $\xi : \widetilde{M} \rightarrow M$ be the regular covering corresponding to it ($\widetilde{M}$ is connected).

Let $\widetilde{g} = \xi^*(g)$. It is obvious that $\xi^*(\mathcal{V})$ produces harmonic morphisms on $(\widetilde{M}, \widetilde{g})$.

Also $\xi^*[a] = 0 \in H^1(\widetilde{M}, \mathbb{R})$. Hence, there exists a positive smooth function

$\rho : \widetilde{M} \rightarrow (0, \infty)$ such that $\xi^*(a) = d \log \rho$.

Then ρ is a global density of $\xi^*(\mathcal{V})$.

Let $\eta : (P, k) \rightarrow (M, g)$ be any other Riemannian regular covering such that $\eta^*(\mathcal{V})$ admits a global density. If H and K are the groups of ξ and η , respectively, then ξ and η correspond to surjective group morphisms $\pi_1(M) \rightarrow H$ and $\pi_1(M) \rightarrow K$, respectively, where $\pi_1(M)$ is the fundamental group of M (see [57, Part I, §14.6]).

Now, $\eta^*(\mathcal{V})$ admits a global density if and only if $\eta^*(\xi)$ is a trivial covering and this happens if and only if the image of the injective group morphism $\pi_1(P) \rightarrow \pi_1(M)$ is contained in the kernel of the group morphism $\pi_1(M) \rightarrow H$. But the image of $\pi_1(P) \rightarrow \pi_1(M)$ is equal to the kernel of $\pi_1(M) \rightarrow K$ and hence the surjective group morphism $\pi_1(M) \rightarrow H$ can be factorised $\pi_1(M) \rightarrow K \rightarrow H$. The surjective group morphism $K \rightarrow H$ induces a Riemannian regular covering $\sigma : (P, k) \rightarrow (\widetilde{M}, \widetilde{g})$ having the required properties.

The uniqueness of ξ is obvious. □

Corollary 1.1.14. *Let (M, g) be a Riemannian manifold with zero first Betti number. Let $\mathcal{V}$ be a foliation of codimension not equal to two which produces harmonic morphisms on (M, g) .*

Then, $\mathcal{V}$ admits a global density λ^{2-n} .

1.2 Metric deformations

We next discuss how much the metric of M can be changed preserving the property of producing harmonic morphisms (cf. [4], [43]).

Proposition 1.2.1 (cf. [43, Theorem 5.1]). *Let $\mathcal{V}$ be a foliation on (M, g) , with $\dim \mathcal{V} = p$ and $\text{codim} \mathcal{V} = n$. Let r and s be smooth positive functions on M . Let $g^{\mathcal{H}}$ and $g^{\mathcal{V}}$ denote the horizontal and the vertical components of g , and set $\tilde{g} = s^2 g^{\mathcal{H}} + r^2 g^{\mathcal{V}}$.*

(a) *If $n \neq 2$, then, any two of the following assertions imply the remaining assertion:*

- (i) $\mathcal{V}$ produces harmonic morphisms on (M, g) ,

- (ii) $\mathcal{V}$ produces harmonic morphisms on $(M, \tilde{g})$,
- (iii) $r^p s^{n-2}$ is locally the product of a function constant on horizontal curves and a function constant on vertical curves.
- (b) If $n = 2$, then the same implications are true after replacing (iii) with:
 - (iii') r is constant along horizontal curves.

Proof. Suppose that $\mathcal{V}$ is conformal and let λ and $\tilde{\lambda}$ be local dilations of $\mathcal{V}$ with respect to g and $\tilde{g}$, respectively. Then $\tilde{\lambda} = a s^{-1} \lambda$, where a is a smooth positive function which is constant along vertical curves.

Let ω and $\tilde{\omega}$ be local volume forms of $\mathcal{V}$ with respect to g and $\tilde{g}$, respectively. Then $\tilde{\omega} = r^p \omega$.

It follows that

$$\tilde{\lambda}^{2-n} \tilde{\omega} = a^{2-n} (s^{n-2} r^p) (\lambda^{2-n} \omega). \quad (1.2.1)$$

To prove that (i),(ii) $\Rightarrow$ (iii) note that (1.2.1) and Proposition 1.1.11 implies that if λ^{2-n} and $\tilde{\lambda}^{2-n}$ are local densities of $\mathcal{V}$ with respect to g and $\tilde{g}$, respectively, then

$$s^{n-2} r^p = a^{n-2} b, \quad (1.2.2)$$

where b is a smooth positive function constant along horizontal curves.

To prove that (i),(iii) $\Rightarrow$ (ii), suppose that λ^{2-n} is a local density of $\mathcal{V}$ with respect to g and choose smooth positive functions a and b which satisfy (1.2.2) and such that a is constant along vertical curves and b is constant along horizontal curves. Now, (1.2.1) implies that $\tilde{\lambda} = a s^{-1} \lambda$ corresponds to a local density of $\mathcal{V}$ with respect to $\tilde{g}$.

The proof of (ii),(iii) $\Rightarrow$ (i) is similar. $\square$

Corollary 1.2.2 (cf. [43]). *Let $\mathcal{V}$ be a foliation with $\text{codim} \mathcal{V} \neq 2$ on (M, g) . Let a and b be smooth positive functions on M such that a is constant along vertical curves and b is constant along horizontal curves. Then the following assertions are equivalent:*

- (i) $\mathcal{V}$ produces harmonic morphisms on (M, g) ,
- (ii) $\mathcal{V}$ produces harmonic morphisms on $(M, a^2 b^2 g)$.

If $\text{codim} \mathcal{V} = 2$ then (i) $\iff$ (ii) if and only if the function a is constant on M .

Proof. This is an immediate consequence of Proposition 1.2.1. $\square$

Propositions 1.1.11 and 1.2.1 suggests the following:

Definition 1.2.3 ([43], cf. [46]). Let $\mathcal{V}$ be a distribution of dimension p and codimension n on the Riemannian manifold (M, g) . For a positive smooth function σ on M we define the metric ${}^\sigma g$ by

$${}^\sigma g = \sigma^2 g^{\mathcal{H}} + \sigma^{\frac{4-2n}{p}} g^{\mathcal{V}},$$

where $g^{\mathcal{H}}$ and $g^{\mathcal{V}}$ are the horizontal and the vertical components of g , respectively.

Proposition 1.2.4 (cf. [43]). *Let $\mathcal{V}$ be a conformal foliation on (M, g) and let σ be a positive smooth function on M . Then,*

(i) *$\mathcal{V}$ is also a conformal foliation on $(M, {}^\sigma g)$. Furthermore, λ is a local dilation of $\mathcal{V}$ with respect to g if and only if $\lambda \sigma^{-1}$ is a local dilation with respect to ${}^\sigma g$.*

(ii) *$\mathcal{V}$ produces harmonic morphisms on (M, g) if and only if it produces harmonic morphisms on $(M, {}^\sigma g)$.*

(iii) *If $\mathcal{V}$ produces harmonic morphisms and admits a global dilation λ such that λ^{2-n} is a density for $\mathcal{V}$ with respect to g then, $\mathcal{V}$ is a Riemannian foliation with minimal leaves on $(M, {}^\lambda g)$.*

Proof. Statement (i) follows from Remark 1.1.9(2) whilst (ii) follows from Proposition 1.2.1.

If $\text{codim} \mathcal{V} = 2$ assertion (iii) is obvious. If $\text{codim} \mathcal{V} \neq 2$ first note that if λ is a local dilation of $\mathcal{V}$ with respect to g then $\mathcal{V}$ is a Riemannian foliation on $(M, {}^\lambda g)$. Now the proof of Proposition 1.2.1 shows that if λ^{2-n} is a density for $\mathcal{V}$ then the constant function $\tilde{\lambda} = 1$ is a dilation which corresponds to a density for $\mathcal{V}$ with respect to ${}^\lambda g$. Thus, by (1.1.4) the leaves of $\mathcal{V}$ are minimal submanifolds of $(M, {}^\lambda g)$. $\square$

The next result shows that the metric on the codomain is much more rigid.

Proposition 1.2.5 ([13]). *Let $\varphi_j : (M, g) \rightarrow (N, h_j)$, $j = 1, 2$, be nonconstant harmonic morphisms having the same fibres. Suppose that N is connected and*

$\dim N \neq 2$.

Then, h_1 and h_2 are homothetic.

Proof. Let λ_j be the dilation of φ_j ($j = 1, 2$). Then $\lambda_2 = \lambda_1 \sigma$ where $\sigma : M \rightarrow \mathbb{R}$ is a smooth positive function, constant along the fibres.

Recall from Proposition 1.1.11 that the property that φ_j is a harmonic morphism is equivalent to the property that the parallel displacement defined by the horizontal distribution preserves $\lambda_j^{2-n} \omega$, where $n = \dim N$ and ω is a local volume form for the vertical distribution. Hence, σ is also constant along horizontal curves.

It follows that σ is constant on M , and the proposition is proved. $\square$

An immediate consequence of Proposition 1.2.5 is the following:

Corollary 1.2.6. *A foliation of codimension $q \neq 2$ which produces harmonic morphisms is given by a Haefliger structure [33] with values in the groupoid of germs of homothetic diffeomorphisms of the sheaf of germs of Riemannian metrics on $\mathbb{R}^q$.*

In the following theorem we attach assertion (iv) to a well-known list of equivalent assertions (see [44, Appendix B]).

Theorem 1.2.7. *Let M be a compact manifold with zero first Betti number.*

For a foliation $\mathcal{V}$ on M with compact leaves the following assertions are equivalent:

- (i) *the holonomy group of each leaf of $\mathcal{V}$ is finite,*
- (ii) *there exists a metric g on M such that $\mathcal{V}$ is a Riemannian foliation on (M, g) ,*
- (iii) *there exists a metric g on M such that the leaves of $\mathcal{V}$ are minimal submanifolds of (M, g) ,*
- (iv) *there exists a metric g on M such that $\mathcal{V}$ produces harmonic morphisms on (M, g) .*

Moreover, if $\text{codim } \mathcal{V} = 2$, is not necessary to assume that the first Betti number of M is zero.

Proof. It is well-known that the assertions (i), (ii) and (iii) are equivalent. Moreover, if any of these properties holds then there exists a metric g on M such that $\mathcal{V}$ is a Riemannian foliation with minimal leaves on (M, g) . (To see this let h be a metric on M with respect to which $\mathcal{V}$ has minimal leaves and let k be a metric on M with respect to which $\mathcal{V}$ is Riemannian. If $\mathcal{H}$ is the orthogonal complement of $\mathcal{V}$ with respect to h and $\mathcal{K}$ is the orthogonal complement of $\mathcal{V}$ with respect to k let $g = h^\mathcal{V} + k^\mathcal{H}$ where, $h^\mathcal{V}$ is the vertical component of h and $k^\mathcal{H}$ is the metric on $\mathcal{H}$ induced by the restriction of k to $\mathcal{K}$ via the canonical isomorphisms of vector bundles $\mathcal{H} \rightarrow TM/\mathcal{V} \rightarrow \mathcal{K}$.) But any Riemannian foliation with minimal leaves produces harmonic morphisms.

Conversely, suppose that $\mathcal{V}$ produces harmonic morphisms on (M, g) . If $\text{codim}\mathcal{V} = 2$ then by Proposition 1.1.10, $\mathcal{V}$ has minimal leaves (see [65]). If $\text{codim}\mathcal{V} \neq 2$ then by Corollary 1.1.14, there exists a global density λ^{2-n} of $\mathcal{V}$, and $\mathcal{V}$ is a Riemannian foliation with minimal leaves on $(M, {}^\lambda g)$. $\square$

1.3 Characterisation of the conformal foliations which produce harmonic morphisms

We now characterise conformal foliations which produce harmonic morphisms. Recall [65] that a conformal foliation $\mathcal{V}$ of $\text{codim}\mathcal{V} = 2$ produces harmonic morphisms if and only if its leaves are minimal. For $\text{codim}\mathcal{V} \neq 2$, the situation is more complicated and we have the following reformulation of a result of R.L. Bryant [13] (see [9] for another treatment).

Proposition 1.3.1 ([13]). *Let $\mathcal{V}$ be a conformal foliation on (M, g) of $\text{codim}\mathcal{V} \neq 2$ and $\mathcal{H}$ its orthogonal complement. Let ${}^\mathcal{V}B$ and ${}^\mathcal{H}B$ be the second fundamental forms of $\mathcal{V}$ and $\mathcal{H}$, respectively.*

Then, $\mathcal{V}$ produces harmonic morphisms on (M, g) if and only if the vector field

$$(n - 2) \text{trace}({}^\mathcal{H}B) - n \text{trace}({}^\mathcal{V}B)$$

is locally a gradient vector field. (Here $\text{trace}({}^\mathcal{H}B) = \sum_j {}^\mathcal{H}B(X_j, X_j)$, $\text{trace}({}^\mathcal{V}B) =$

$\sum_{\alpha} {}^{\mathcal{V}}B(U_{\alpha}, U_{\alpha})$ for local orthonormal frames $\{X_j\}$ and $\{U_{\alpha}\}$ of $\mathcal{H}$ and $\mathcal{V}$, respectively.)

Proof. Note that the following relation holds

$$(n-2) \operatorname{trace}({}^{\mathcal{H}}B) - n \operatorname{trace}({}^{\mathcal{V}}B) = n(n-2) \operatorname{grad}(\log \lambda), \quad (1.3.1)$$

if and only if:

$$\operatorname{trace}({}^{\mathcal{H}}B) = n \mathcal{V}(\operatorname{grad}(\log \lambda)), \quad (1.3.1a)$$

and

$$\operatorname{trace}({}^{\mathcal{V}}B) = -(n-2) \mathcal{H}(\operatorname{grad}(\log \lambda)). \quad (1.3.1b)$$

By Remark 1.1.9(3), (1.3.1a) holds if and only if λ is a local dilation of $\mathcal{V}$. This together with the fundamental equation (1.1.4), imply that (1.3.1a) and (1.3.1b) hold if and only if $\mathcal{V}$, restricted to the domain of λ , produces harmonic morphisms and λ^{2-n} is a density of it. $\square$

Note that Corollary 1.1.14 can be proved using (1.3.1).

1.4 Homothetic foliations

By the *mean curvature form* of $\mathcal{V}$ we mean the one-form $(\operatorname{trace}({}^{\mathcal{V}}B))^{\flat}$ obtained by applying the musical isomorphism (see [11]) ${}^{\flat} : TM \rightarrow T^*M$ with similar terminology for $\mathcal{H}$ (see [60]). Then we have.

Corollary 1.4.1. *Let $\mathcal{V}$ be a conformal foliation with $\operatorname{codim} \mathcal{V} \neq 2$ and let $\mathcal{H}$ be its orthogonal complement. Then any two of the following assertions imply the other one:*

- (i) $\mathcal{V}$ produces harmonic morphisms,
- (ii) $\mathcal{V}$ has closed mean curvature form,
- (iii) $\mathcal{H}$ has closed mean curvature form.

Proof. This is an immediate consequence of Proposition 1.3.1 and the fact that $(\text{trace}({}^{\mathcal{V}}B))^{\flat}$ is closed if and only if $\text{trace}({}^{\mathcal{V}}B)$ is locally a gradient and similarly for $\mathcal{H}$. $\square$

We now introduce a new sort of foliation midway between conformal and Riemannian foliations.

Proposition 1.4.2. *For a conformal foliation $\mathcal{V}$ on the Riemannian manifold (M, g) the following assertions are equivalent:*

- (i) *the leaves of $\mathcal{V}$ can be, locally, given as fibres of horizontally homothetic submersion;*
- (ii) *each point of M has an open neighbourhood on which there can be defined a local dilation of $\mathcal{V}$ which is constant along horizontal curves;*
- (iii) *any local dilation λ of $\mathcal{V}$, defined on an open subset O with zero first Betti number, is a product $\lambda = ab$, where a and b are positive smooth functions such that a is constant along vertical curves and b is constant along horizontal curves;*
- (iv) *the mean curvature of the orthogonal complement of $\mathcal{V}$ is locally a gradient vector field.*

Proof. The equivalence (i) $\iff$ (ii) is obvious. Also, by Remark 1.1.9(2) it follows that (ii) $\iff$ (iii).

By the same remark, if (ii) holds then, any local dilation $\lambda : O \rightarrow \mathbb{R}$ of $\mathcal{V}$ is, locally, a product $\lambda = ab$ as in (iv). If on the same open subset of O , we also have $\lambda = a_1 b_1$ then, $a^{-1} a_1 = b b_1^{-1} = \text{const.}$. Hence the differentials of the logarithms of the factors a, b from the local decompositions of λ define closed one-forms on O . If the first Betti number of O is zero then these one-forms are exact and the implication (ii) $\implies$ (iii) is proved.

The equivalence (ii) $\iff$ (iv) follows from Remark 1.1.9(3). $\square$

Remark 1.4.3. 1) If a conformal foliation satisfies one of the properties from the above proposition then the holonomy groupoid of each leaf is formed of germs of homothetic diffeomorphisms.

2) In (iii) above instead of $H^1(O; \mathbb{R}) = 0$ we could ask that the first basic cohomology group (see [60]) of $\mathcal{V}|_O$ be zero. This follows from the fact that the

set of differentials $\{da\}$ define a closed *basic* one-form on O .

3) Alternatively, it is sufficient to ask that the orthogonal complement $\mathcal{H}|_O$ of $\mathcal{V}|_O$ is an Ehresmann connection [12] with trivial holonomy (in particular, $\mathcal{H}|_O$ is integrable). To see this, note that the set of differentials $\{d(\log b)\}$ define a closed one-form which, when restricted to a leaf L , is exact (because it coincide with $d((\log \lambda)|_L)$). When $\mathcal{H}$ is an Ehresmann connection with trivial holonomy these exact forms can be matched together to define an exact form on O .

The notion of *local dilations* for conformal foliations can be generalized to conformal *distributions*, although, in this case, these might not exist. Nevertheless, if a conformal distribution admits local dilations then these share the same properties (Remark 1.1.9(2) and (3)) as the local dilations of a conformal foliation. Moreover, assertions (ii), (iii) and (iv) from Proposition 1.4.2 remain equivalent for conformal distributions which admit local dilations in a neighbourhood of each point.

Proposition 1.4.2 suggests the following definition.

Definition 1.4.4 (cf. [62, §2]). Let $\mathcal{V}$ be a distribution on the Riemannian manifold (M, g) . We shall say that $\mathcal{V}$ is *homothetic* if it is conformal and the mean curvature of its orthogonal complement is locally a gradient vector field.

Remark 1.4.5. 1) Let $\mathcal{V}$ be a foliation on M . Then, there exists a metric g on M such that $\mathcal{V}$ is a homothetic foliation on (M, g) if and only if $\mathcal{V}$ is a foliation of type (A) in the sense of I. Vaisman [62].

2) Note that, unlike conformal distributions, *homothetic* distributions always admit local dilations, even if nonintegrable. Indeed, if $\mathcal{V}$ is a homothetic distribution with $\text{codim} \mathcal{V} = n$ and $\mathcal{H}$ is its orthogonal complement, then any local smooth positive function λ on M which has the property

$$n \text{grad}(\log \lambda) = \text{trace}({}^{\mathcal{H}}B), \quad (1.4.1)$$

is a local dilation of $\mathcal{V}$.

Moreover, if $\mathcal{V}$ is a homothetic distribution and λ is a local dilation of it defined on an open set O such that $\lambda = ab$ as in (iii) from Proposition 1.4.2, then any other local dilation defined on O is of this form.

Proposition 1.4.6. *Let $\mathcal{V}$ be a foliation on (M, g) with orthogonal complement $\mathcal{H}$.*

If $\mathcal{H}$ is a homothetic distribution then the parallel displacement defined by it consists of (local) homothetic diffeomorphisms between leaves of $\mathcal{V}$.

Conversely, if $\mathcal{H}$ is conformal, integrable and the parallel displacement defined by it is formed of (local) homothetic diffeomorphisms between leaves of $\mathcal{V}$, then it is a homothetic distribution.

Proof. If $\mathcal{H}$ is conformal and integrable then it admits local dilations. Let λ be a local dilation of $\mathcal{H}$. Then for any horizontal vector field X invariant under the holonomy of $\mathcal{V}$ (i.e. for any basic vector field X) we have:

$$\mathcal{L}_X(\lambda^2(g|_{\mathcal{V}})) = 0. \quad (1.4.2)$$

If (ψ_t) is the local flow of X , then (1.4.2) is equivalent to the fact that for any t we have

$$(\psi_t)^*(\lambda^2(g|_{\mathcal{V}})) = \lambda^2(g|_{\mathcal{V}}). \quad (1.4.3)$$

The proof follows from (1.4.3) by using the fact that, if $\mathcal{H}$ is conformal, then it is homothetic if and only if in the neighbourhood of each point a local dilation can be found, which is constant along the leaves of $\mathcal{V}$. $\square$

Proposition 1.4.7. *Let $\mathcal{V}$ be a homothetic foliation of codimension not equal to two on (M, g) . Then, the following assertions are equivalent:*

- (i) $\mathcal{V}$ produces harmonic morphisms;
- (ii) the mean curvature of $\mathcal{V}$ is locally a gradient vector field.

In particular, a homothetic foliation whose orthogonal complement is a homothetic distribution produces harmonic morphisms with umbilical fibres.

Proof. Since $\mathcal{V}$ is homothetic we have that $\text{trace}({}^{\mathcal{H}}B)$ is, locally, a gradient vector field. From Proposition 1.3.1 it follows that $\mathcal{V}$ produces harmonic morphisms if and only if $\text{trace}({}^{\mathcal{V}}B)$ is, locally, a gradient vector field.

The last assertion follows from the fact that (see Remark 1.1.9(3)) $\mathcal{V}$ is umbilical if and only if $\mathcal{H}$ is a conformal distribution. $\square$

Corollary 1.4.8. *Let $\mathcal{V}$ be a foliation with minimal leaves and $\text{codim}\mathcal{V} \neq 2$. Then, $\mathcal{V}$ produces harmonic morphisms if and only if it is homothetic.*

See [2] for other relations between harmonic maps and minimal submanifolds.

Remark 1.4.9. 1) In Proposition 1.4.7 condition (ii) is a bit more general than saying $\mathcal{H}$ is homothetic since it is not assumed to be conformal.

2) Any Riemannian foliation is homothetic.

3) Given a horizontally homothetic submersion the connected components of its fibres form a homothetic foliation. Conversely, note that if $\varphi : (M, g) \rightarrow (N, h)$ is a horizontally conformal submersion such that its fibres form a homothetic foliation then it is not true, in general, that φ is horizontally homothetic. If the first Betti number of M is zero then φ can be factorised into a horizontally homothetic submersion followed by a conformal diffeomorphism. (By Remark 1.4.3(2)(3) this factorization can also be done when the first Betti number of N is zero or when the horizontal distribution $\mathcal{H}$ is an Ehresmann connection with trivial holonomy, in which case $\mathcal{H}$ is integrable and M is diffeomorphic to the product of N and the fibre.)

Example 1.4.10. 1) *Doubly-warped-products* (see [53], and the references therein). If (M^p, g) and (N^q, h) are Riemannian manifolds and $r : M \rightarrow \mathbb{R}$ and $s : N \rightarrow \mathbb{R}$ are positive smooth functions then the doubly-warped-product of (M^p, g) and (N^q, h) is defined to be:

$$M_s \times_r N = (M \times N, s^2 \pi_M^*(g) + r^2 \pi_N^*(h)) ,$$

where π_M and π_N are the projections onto M and N , respectively.

The projections $\pi_M : M_s \times_r N \rightarrow (M, g)$ and $\pi_N : M_s \times_r N \rightarrow (N, h)$ are horizontally homothetic so their fibres define a pair of complementary orthogonal homothetic foliations. Conversely, any Riemannian manifold endowed with a pair of complementary orthogonal homothetic foliations is canonically locally isometric to a double-warped-product. Hence, by Proposition 1.4.7, when $p, q \neq 2$ both of the foliations induced by the factors of a double-warped-product produce harmonic morphisms. More precisely, if $p \neq 2$ the following projection is a harmonic

morphism with umbilical fibres:

$$\pi_M : M_s \times_r N \longrightarrow (M, r^{\frac{2q}{p-2}} g) . \quad (1.4.4)$$

The fact that the above projection is a harmonic morphism also follows from Proposition 1.2.4(iii) .

A concrete example of a warped product is provided by the open subsets in spheres $(S^{p+q} \setminus S_0^{p-1}, g_{p+q})$, where $S_0^{p-1} = \mathbb{R}^p \cap S^{p+q}$ and $\mathbb{R}^p \equiv \{x \in \mathbb{R}^{p+q+1} \mid x^{p+1} = \dots = x^{p+q+1} = 0\}$ and g_{p+q} is the restriction of the canonical metric on S^{p+q} . The warped-product is the one induced by the following diffeomorphism:

$$\begin{aligned} \Phi : S^{p+q} \setminus S_0^{p-1} &\longrightarrow S_+^p \times S^q \\ \Phi(x, y) &= ((x, |y|), \frac{1}{|y|} y) . \end{aligned}$$

where $S_+^p = \{x \in S^{p+1} \mid x^{p+1} > 0\}$. To make Φ an isometry we must give $S_+^p \times S^q$ the warped-product structure $S_+^p \times_r S^q$ where $r(x^1, \dots, x^{p+1}) = x^{p+1}$. Thus, if $p \neq 2$, (1.4.4) particularises to give the following harmonic morphism with umbilical fibres:

$$\begin{aligned} \varphi : (S^{p+q} \setminus S_0^{p-1}, g_{p+q}) &\longrightarrow (S_+^p, r^{\frac{2q}{p-2}} g_p) \\ \varphi(x^1, \dots, x^{p+q+1}) &= (x^1, \dots, x^p, \sqrt{(x^{p+1})^2 + \dots + (x^{p+q+1})^2}) . \end{aligned}$$

By Proposition 1.2.5 any other metric on S_+^p with respect to which φ above is a harmonic morphism is homothetic to the one considered. Also, note that, although φ can be extended to a continuous map $\tilde{\varphi}$ on S^{p+q} , the considered metric on the codomain cannot be extended to the range of $\tilde{\varphi}$.

2) *Doubly-twisted-products*(see [53], and the references therein). These are defined in the same way as doubly-warped-products, but now $r, s : M \times N \rightarrow \mathbb{R}$. It is easy to see that a Riemannian manifold endowed with a pair of complementary orthogonal foliations which are both umbilical (equivalently, both conformal) is, canonically, locally isometric to a double-twisted-product.

If $p \neq 2$, it follows from Proposition 1.2.1 that the foliation $\mathcal{V}$ induced by the second factor of the double-twisted-product $M_s \times_r N$ produces harmonic morphisms if and only if the function $r^q s^{p-2}$ is, locally, the product of a function

constant on M and a function constant on N .

If $p = 2$, then $\mathcal{V}$ produces harmonic morphisms if and only if r is a function defined on N . In this case, $\mathcal{V}$ has totally geodesic leaves and $M_s \times_r N$ is isometric to the twisted product $M_s \times \tilde{N}$ where $\tilde{N} = (N, r^2 h)$.

It follows that a pair of complementary orthogonal umbilical foliations both of codimension not equal to two are both homothetic if and only if each of them produces harmonic morphisms.

Chapter 2

Group actions and harmonic morphisms

2.1 Mean curvature forms and adapted Bott connections

In this section we shall establish the fact that the exterior derivative of the mean curvature form of a distribution is the curvature form of the connection induced on the determinant bundle of the distribution by its adapted Bott connection (see [55]). By using this or by direct calculation we obtain formulae for the exterior derivative of the mean curvature form of a distribution. (We do not imagine that these formulae are new, but we could not find them in the literature.) Some of these formulae apply to prove that if a conformal foliation $\mathcal{V}$ has integrable orthogonal complement $\mathcal{H}$ and if $\mathcal{V}$ and $\mathcal{H}$ both have basic mean curvature forms then $\mathcal{V}$ produces harmonic morphisms.

The following simple lemma will be used later on.

Lemma 2.1.1 (cf. [56, Chapter IV, Example 4.10]). *Let $\mathcal{V}$ be a foliation on (M, g) and let $V \in \Gamma(\mathcal{V})$ be a conformal vector field.*

Then $[V, X] = 0$ for any basic vector field X .

Proof. Let $\mathcal{H} = \mathcal{V}^\perp$ and $X \in \Gamma(\mathcal{H})$ a basic vector field. Then $[V, X] \in \Gamma(\mathcal{V})$.

But V is conformal and hence we can write

$$0 = (\mathcal{L}_V g)(W, X) = -g(W, [V, X])$$

for any $W \in \Gamma(\mathcal{V})$. Hence $[V, X] = 0$. $\square$

The result of Lemma 2.1.1 is equivalent to the fact that any conformal vector field tangent to a foliation $\mathcal{V}$ is an infinitesimal automorphism of the orthogonal complement of $\mathcal{V}$ (see [56] or [36] for the definition of the infinitesimal automorphism of a distribution).

Let $\mathcal{V}$ and $\mathcal{H}$ be two complementary orthogonal distributions (not necessarily integrable) on (M, g) .

Definition 2.1.2 (see [60]). The adapted Bott connection $\overset{\mathcal{H}}{\nabla}$ on $\mathcal{H}$ is defined by

$$\overset{\mathcal{H}}{\nabla}_E X = \mathcal{H}[\mathcal{V}E, X] + \mathcal{H}(\nabla_{\mathcal{H}E} X)$$

for $E \in \Gamma(TM)$, $X \in \Gamma(\mathcal{H})$ where ∇ is the Levi-Civita connection of (M, g) .

The adapted Bott connection $\overset{\mathcal{V}}{\nabla}$ on $\mathcal{V}$ is defined similarly by reversing the roles of $\mathcal{V}$ and $\mathcal{H}$.

Remark 2.1.3 (see [60]). It is easy to see that $\overset{\mathcal{H}}{\nabla}$ is compatible with the metric induced by g on $\mathcal{H}$ if and only if $\mathcal{H}$ is totally geodesic. Nevertheless, since $\overset{\mathcal{H}}{\nabla}_X = \mathcal{H}\nabla_X$ for any $X \in \Gamma(\mathcal{H})$ we have that $\overset{\mathcal{H}}{\nabla}_X(g|_{\mathcal{H}}) = 0$.

Let ${}^{\mathcal{H}}I$ be the integrability tensor of $\mathcal{H}$ which is the $\mathcal{V}$ -valued horizontal two-form defined by ${}^{\mathcal{H}}I(X, Y) = -\mathcal{V}[X, Y]$ for $X, Y \in \Gamma(\mathcal{H})$.

Proposition 2.1.4. *Let $\mathcal{H}$ be a distribution on (M, g) . Then*

$$d(\text{trace}({}^{\mathcal{H}}B)^b)(X, Y) = g(\text{trace}({}^{\mathcal{H}}B), {}^{\mathcal{H}}I(X, Y)) , \quad (2.1.1)$$

$$d(\text{trace}({}^{\mathcal{H}}B)^b)(X, V) = \overset{\mathcal{V}}{\nabla}_X(\text{trace}({}^{\mathcal{H}}B)^b)(V) \quad (2.1.2)$$

for any horizontal vectors X, Y and vertical vector V .

Proof. This is a straightforward calculation using the fact that $\text{trace}({}^{\mathcal{H}}B)$ is a vertical vector field. $\square$

Let $n = \dim \mathcal{H}$ and let $\bigwedge^n \mathcal{H}$ be the determinant line bundle of $\mathcal{H}$.

Let ${}^{\mathcal{H}}R \in \Gamma(\text{End}(\mathcal{H}) \otimes \bigwedge^2(T^*M))$ be the curvature form of $\overset{\mathcal{H}}{\nabla}$. Then the curvature form of the connection induced by $\overset{\mathcal{H}}{\nabla}$ on $\bigwedge^n \mathcal{H}$ is $\text{trace}({}^{\mathcal{H}}R) \in \Gamma(\bigwedge^2(T^*M))$.

Proposition 2.1.5 (see [55]). *Let $\mathcal{H}$ be a distribution on (M, g) . Then*

$$\text{trace}({}^{\mathcal{H}}R) = d(\text{trace}({}^{\mathcal{H}}B)^{\flat}).$$

Proof. Let $\overset{\mathcal{H}}{\omega}$ be a local volume form of $\mathcal{H}$ considered with respect to the metric induced by g .

Recall that $\text{trace}({}^{\mathcal{H}}R) = dA$ where A is any local connection form of the connection induced by $\overset{\mathcal{H}}{\nabla}$ on $\bigwedge^n \mathcal{H}$. Thus it suffices to show that

$$\overset{\mathcal{H}}{\nabla}_E \overset{\mathcal{H}}{\omega} = -g(E, \text{trace}({}^{\mathcal{H}}B)) \overset{\mathcal{H}}{\omega} \quad (2.1.3)$$

for any $E \in TM$.

If $E \in \mathcal{H}$ then the right hand side of (2.1.3) is zero. Also, the left hand side is zero because if $E \in \mathcal{H}$ then $\overset{\mathcal{H}}{\nabla}_E(g|_{\mathcal{H}}) = 0$ (see Remark 2.1.3).

If $E \in \mathcal{V}$ then $\overset{\mathcal{H}}{\nabla}_E \overset{\mathcal{H}}{\omega} = \mathcal{H}^*(\mathcal{L}_E \overset{\mathcal{H}}{\omega})$. Thus, if $E \in \mathcal{V}$ then (2.1.3) reduces to a well-known formula (see [65]). $\square$

Proposition 2.1.6 (see [56],[60]). *Let $X \in \Gamma(\mathcal{H})$ be a horizontal vector field. Then the following assertions are equivalent.*

- (i) $\overset{\mathcal{H}}{\nabla}_V X = 0$ for any $V \in \Gamma(\mathcal{V})$;
- (ii) $\mathcal{H}[X, V] = 0$ for any $V \in \Gamma(\mathcal{V})$;
- (iii) $\mathcal{L}_X(\Gamma(\mathcal{V})) \subseteq \Gamma(\mathcal{V})$;
- (iv) X is an infinitesimal automorphism of $\mathcal{V}$.

Proof. The equivalences (i) $\iff$ (ii) and (ii) $\iff$ (iii) are trivial. Also, (ii) $\iff$ (iv) follows easily from [37, Chapter 1, Corollary 1.10]. $\square$

Remark 2.1.7. If in Proposition 2.1.6 we further assume that $\mathcal{H}$ is integrable then the following assertions can be added.

- (v) $\mathcal{L}_X \circ \mathcal{V} = \mathcal{V} \circ \mathcal{L}_X$;
- (vi) $\mathcal{L}_X \circ \mathcal{H} = \mathcal{H} \circ \mathcal{L}_X$.

Example 2.1.8 (see [60]). Suppose that $\mathcal{V}$ is integrable. Then any basic vector field $X \in \Gamma(\mathcal{H})$ for $\mathcal{V}$ with respect to $\mathcal{H}$ is an infinitesimal automorphism of $\mathcal{V}$; in fact, if $\mathcal{V}$ is integrable a horizontal vector field $X \in \Gamma(\mathcal{H})$ is basic if and only if any of the assertions (i), (ii), (iii) or (iv) holds.

The following definition does not require any assumption on the distribution $\mathcal{H}$.

Definition 2.1.9 (see [60, (4.34)]). Let $E \in \Gamma(TM)$. The *horizontal divergence* $\operatorname{div}_{\mathcal{H}} E$ of E is defined by

$$\mathcal{H}^*(\mathcal{L}_E \overset{\mathcal{H}}{\omega}) = (\operatorname{div}_{\mathcal{H}} E) \overset{\mathcal{H}}{\omega}$$

where $\overset{\mathcal{H}}{\omega}$ is any local volume form of $\mathcal{H}$ (endowed with the metric induced by g).

The vertical divergence $\operatorname{div}_{\mathcal{V}}$ is defined similarly (note that

$$\operatorname{div} E = \operatorname{div}_{\mathcal{H}} E + \operatorname{div}_{\mathcal{V}} E$$

for any $E \in \Gamma(TM)$).

A standard calculation gives the following proposition.

Proposition 2.1.10. Let $E \in \Gamma(TM)$. Then $\operatorname{div}_{\mathcal{H}} E$ is the (pointwise) trace of the linear endomorphism $\mathcal{H} \longrightarrow \mathcal{H}$ defined by $Y \longmapsto \mathcal{H}(\nabla_Y E)$.

Hence, $\operatorname{div}_{\mathcal{H}} E$ is globally well-defined (i.e. it does not depend on $\overset{\mathcal{H}}{\omega}$).

Remark 2.1.11. 1) Obviously, if $\mathcal{H}$ is integrable and $X \in \Gamma(\mathcal{H})$ then the restriction of $\operatorname{div}_{\mathcal{H}} X$ to each leaf L of $\mathcal{H}$ is equal to the divergence of the restriction of X to $(L, g|_L)$.

2) If $V \in \Gamma(\mathcal{V})$ then $\operatorname{div}_{\mathcal{H}} V = -g(\operatorname{trace}({}^{\mathcal{H}}B), V)$.

Lemma 2.1.12. (a) Suppose that $X \in \Gamma(\mathcal{H})$ is an infinitesimal automorphism of $\mathcal{V}$. Then

$$\mathcal{H}R(V, W)X = \mathcal{H}(\nabla_{\nu_{I(V, W)}}X)$$

for any vertical V and W .

(b) Suppose that $\mathcal{H}$ is integrable and let $X, Y \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$ be such that $[V, X] = 0 = [V, Y]$. Then

$$\mathcal{H}R(V, X)Y = \overset{\mathcal{H}}{\nabla}_V(\mathcal{H}\nabla_Y X).$$

Proof. (a) Let $V, W \in \Gamma(\mathcal{V})$. Then

$$\begin{aligned} \mathcal{H}R(V, W)X &= [\overset{\mathcal{H}}{\nabla}_V, \overset{\mathcal{H}}{\nabla}_W]X - \overset{\mathcal{H}}{\nabla}_{[V, W]}X \\ &= -\overset{\mathcal{H}}{\nabla}_{\nu_{[V, W]}}X - \overset{\mathcal{H}}{\nabla}_{\mathcal{H}[V, W]}X = \mathcal{H}(\nabla_{\nu_{I(V, W)}}X). \end{aligned}$$

(b) We have

$$\begin{aligned} \mathcal{H}R(V, X)Y &= [\overset{\mathcal{H}}{\nabla}_V, \overset{\mathcal{H}}{\nabla}_X]Y - \overset{\mathcal{H}}{\nabla}_{[V, X]}Y \\ &= \overset{\mathcal{H}}{\nabla}_V(\overset{\mathcal{H}}{\nabla}_X Y) = \overset{\mathcal{H}}{\nabla}_V(\mathcal{H}\nabla_X Y) \\ &= \overset{\mathcal{H}}{\nabla}_V(\mathcal{H}[X, Y] + \mathcal{H}\nabla_Y X). \end{aligned}$$

Because $\mathcal{H}$ is integrable we have that $\mathcal{H}[X, Y] = [X, Y]$ and from $[V, X] = 0 = [V, Y]$, by using the Jacobi identity, we obtain that $[V, [X, Y]] = 0$. Hence $\overset{\mathcal{H}}{\nabla}_V(\mathcal{H}[X, Y]) = 0$ and the lemma follows. $\square$

Let $\{X_a\}$ be a local frame for $\mathcal{H}$ over the open subset $U \subseteq M$ and let $\{V_r\}$ be a local frame for $\mathcal{V}$ over U . We shall denote ‘horizontal’ indices by a, b, c and ‘vertical’ indices by r, s, t .

Lemma 2.1.13. (a) Suppose that the X_a are infinitesimal automorphisms of $\mathcal{V}$. Then

$$d(\text{trace}(\mathcal{H}B)^b)_{rs} = (c_{ba}^a + \text{div}_{\mathcal{H}}X_b) \nu_{I_{rs}}^b \quad (2.1.4)$$

where $\{c_{ab}^c\}$ are defined by $\mathcal{H}[X_a, X_b] = c_{ab}^c X_c$.

(b) If both $\mathcal{H}$ and $\mathcal{V}$ are locally generated by infinitesimal automorphisms of $\mathcal{V}$ and $\mathcal{H}$, respectively, and $\mathcal{H}$ is integrable then

$$\mathcal{V}^*(d(\text{trace}(\mathcal{H}B)^b)) = \text{div}_{\mathcal{H}}(\nu I).$$

Proof. (a) This follows from Proposition 2.1.5, Proposition 2.1.10 and Lemma 2.1.12(a).

(b) This follows from (a) and the fact that $\mathcal{V}^*(\mathcal{L}_X(\mathcal{V}I)) = 0$ for any infinitesimal automorphism $X \in \Gamma(\mathcal{H})$ of $\mathcal{V}$. $\square$

(Note that it seems to be impossible to formulate invariantly assertion (a) of Lemma 2.1.13.)

Proposition 2.1.14 (see [55]). *If $\mathcal{V}$ is integrable then $d(\text{trace}(\mathcal{H}B)^\flat)(V, W) = 0$ for any vertical V and W .*

Proof. This follows from Lemma 2.1.13 because if $\mathcal{V}$ is integrable then any basic vector field for $\mathcal{V}$ is an infinitesimal automorphism of $\mathcal{V}$. $\square$

Proposition 2.1.15. *Suppose that both $\mathcal{V}$ and $\mathcal{H}$ are integrable. Then the following assertions are equivalent.*

- (i) *The mean curvature form of $\mathcal{H}$ is closed;*
- (ii) *The mean curvature form of $\mathcal{H}$ is basic (for $\mathcal{V}$).*

Proof. This follows from Proposition 2.1.4 and Proposition 2.1.14. $\square$

Proposition 2.1.16. *Suppose that $\mathcal{H}$ is integrable and locally generated by infinitesimal automorphisms of $\mathcal{V}$. Let $V \in \Gamma(\mathcal{V})$ and let $X \in \Gamma(\mathcal{H})$ be an infinitesimal automorphism of $\mathcal{V}$. Then*

$$d(\text{trace}(\mathcal{H}B)^\flat)(V, X) = V(\text{div}_{\mathcal{H}}X) = -\overset{\mathcal{V}}{\nabla}_X(\text{trace}(\mathcal{H}B)^\flat)(V). \quad (2.1.5)$$

Proof. Note that (2.1.5) is tensorial in V and thus we can suppose that V is basic for $\mathcal{H}$. Then the proof follows from Proposition 2.1.4, Proposition 2.1.5, Proposition 2.1.10 and Lemma 2.1.12(b). $\square$

By reversing the roles of $\mathcal{V}$ and $\mathcal{H}$ in Proposition 2.1.4 and Lemma 2.1.13 and Proposition 2.1.16 we obtain the corresponding formulae for $d(\text{trace}(\mathcal{V}B)^\flat)$.

The following simple lemma holds for any complementary orthogonal distributions $\mathcal{H}$ and $\mathcal{V}$.

Lemma 2.1.17. *Let f be any smooth function on M . Then*

$$\overset{\mathcal{H}}{\nabla}_V(\mathcal{H}^*(df))(X) = \overset{\mathcal{V}}{\nabla}_X(\mathcal{V}^*(df))(V) \quad (2.1.6)$$

for any vertical V and horizontal X .

Proof. Let X and V be vector fields which are horizontal and vertical, respectively. The following relation is trivial

$$V(X(f)) - X(V(f)) - [V, X](f) = 0. \quad (2.1.7)$$

But (2.1.7) is equivalent to the following

$$V(X(f)) - \mathcal{H}[V, X](f) = X(V(f)) - \mathcal{V}[X, V](f)$$

which is obviously equivalent to (2.1.6). $\square$

Proposition 2.1.18. *Let $\mathcal{V}$ be a foliation of $\text{codim } \mathcal{V} = n \neq 2$ which produces harmonic morphisms on (M, g) . Then the following assertions are equivalent.*

- (i) *The mean curvature form of $\mathcal{V}$ is basic;*
- (ii) *The mean curvature form of $\mathcal{H}$ is invariant under the parallel displacement determined by $\mathcal{H}$ (i.e. $\overset{\mathcal{V}}{\nabla}_X(\text{trace}(\mathcal{H}B)^\flat)(V) = 0$).*

Proof. Recall that $\text{trace}(\mathcal{H}B)^\flat = n\mathcal{V}^*(d \log \lambda)$ (see [9]) for any local dilation λ of $\mathcal{V}$.

Also, from the fundamental equation (1.1.4) of P. Baird and J. Eells it follows that $\text{trace}(\mathcal{V}B)^\flat = -(n-2)\mathcal{H}^*(d \log \lambda)$ for any local density λ^{2-n} of $\mathcal{V}$.

Now the equivalence (i) $\iff$ (ii) follows from Lemma 2.1.17. $\square$

Theorem 2.1.19. *Let $\mathcal{V}$ be a conformal foliation on (M, g) of $\text{codim } \mathcal{V} \neq 2$. Suppose that the orthogonal complement $\mathcal{H}$ of $\mathcal{V}$ is integrable.*

Then any two of the following assertions imply the remaining assertion.

- (i) *$\mathcal{V}$ produces harmonic morphisms;*
- (ii) *The mean curvature form of $\mathcal{V}$ is basic (for $\mathcal{H}$);*
- (iii) *The mean curvature form of $\mathcal{H}$ is basic (for $\mathcal{V}$).*

Moreover, if any two of (i), (ii) or (iii) hold then both $\mathcal{V}$ and $\mathcal{H}$ have closed mean curvature forms.

Proof. If (i) holds then the equivalence (ii) $\iff$ (iii) follows from Proposition 2.1.18.

Suppose that both the assertions (ii) and (iii) hold. Then (i) follows from Proposition 2.1.15 and Proposition 1.3.1. $\square$

Proposition 2.1.20. (a) *If $\mathcal{V}$ is integrable then*

$$d(\text{trace}(\mathcal{H}B)^\flat)(V, W) = 0 = d(\text{trace}(\mathcal{V}B)^\flat)(V, W) \quad (2.1.8)$$

for any vertical V and W .

(b) *Let $\mathcal{V}$ be a conformal foliation on (M, g) . Then the following assertions are equivalent.*

(i) *For any local dilation λ of $\mathcal{V}$ the one-form $\text{trace}(\mathcal{V}B)^\flat + (n-2)\mathcal{H}^*(d \log \lambda)$ is basic ($n = \dim \mathcal{H}$);*

(ii) *For any horizontal X and vertical V we have*

$$d((n-2)\text{trace}(\mathcal{H}B)^\flat - n\text{trace}(\mathcal{V}B)^\flat)(X, V) = 0. \quad (2.1.9)$$

Proof. (a) The first equality of (2.1.8) follows from Proposition 2.1.14.

The second equality of (2.1.8) follows from (2.1.1) of Proposition 2.1.4 by reversing the roles of $\mathcal{H}$ and $\mathcal{V}$.

(b) Let λ be a local dilation of $\mathcal{V}$ and recall that $\text{trace}(\mathcal{H}B) = n\mathcal{V}(\text{grad}(\log \lambda))$. Hence, by applying (2.1.2) of Proposition 2.1.4 and Lemma 2.1.17 we obtain that

$$\begin{aligned} & ((n-2)d(\text{trace}(\mathcal{H}B)^\flat) - n d(\text{trace}(\mathcal{V}B)^\flat))(X, V) \\ &= (n-2)\overset{\mathcal{V}}{\nabla}_X(\text{trace}(\mathcal{H}B)^\flat)(V) + n\overset{\mathcal{H}}{\nabla}_V(\text{trace}(\mathcal{V}B)^\flat)(X) \\ &= n(n-2)\overset{\mathcal{V}}{\nabla}_X(\mathcal{V}^*(d \log \lambda))(V) + n\overset{\mathcal{H}}{\nabla}_V(\text{trace}(\mathcal{V}B)^\flat)(X) \\ &= n(n-2)\overset{\mathcal{H}}{\nabla}_V(\mathcal{H}^*(d \log \lambda))(X) + n\overset{\mathcal{H}}{\nabla}_V(\text{trace}(\mathcal{V}B)^\flat)(X) \end{aligned}$$

for any horizontal X and vertical V and the proof of (i) $\iff$ (ii) follows from the fact that the basic vector fields (for $\mathcal{V}$) are precisely those horizontal vector fields which are infinitesimal automorphisms of $\mathcal{V}$ (see Example 2.1.8(1)). $\square$

2.2 Characterisation of the conformal actions which produce harmonic morphisms

On a two-dimensional Riemannian manifold a foliation (of dimension one) produces harmonic morphisms if and only if it is locally generated by conformal vector fields. This follows from the well-known fact that a harmonic morphism to a one-dimensional Riemannian manifold is essentially a harmonic function and, if the domain is two-dimensional, this is locally the real part of a conformal map. If the manifold has dimension greater than two then it is not true that any foliation locally generated by conformal vector fields produces harmonic morphisms. In this section we shall give necessary and sufficient conditions for a foliation locally generated by conformal vector fields to produce harmonic morphisms (Theorem 2.2.6). To state this result we need some preparations.

Firstly, we recall the following definition.

Definition 2.2.1 (cf. [59]). Let $\mathcal{V}$ be a foliation on the Riemannian manifold (M, g) .

Then $\mathcal{V}$ is *locally generated by conformal (resp. Killing) vector fields* if in the neighbourhood of each point a local frame for $\mathcal{V}$ can be found which is formed of conformal (resp. Killing) vector fields.

We also need the following:

Definition 2.2.2. Let $\mathcal{V}$ be an orientable foliation of dimension p on a smooth manifold M . Let $\mathcal{H}$ be a complementary distribution (i.e. $\mathcal{V} \oplus \mathcal{H} = TM$). Let ${}^{\mathcal{H}}I$ be its integrability tensor. Let $\overset{\mathcal{V}}{\omega}$ be a volume form on $\mathcal{V}$ (i.e. a vertical nonvanishing p -form).

Suppose that $\mathcal{V}$ is locally generated by local frames $\{V_r\}$ such that

- (1) $\mathcal{V}^*(\mathcal{L}_{V_r} \overset{\mathcal{V}}{\omega}) = 0$,
- (2) V_r is an infinitesimal automorphism of $\mathcal{H}$, for any r .

We define the two-form $\text{trace}(\text{ad}({}^{\mathcal{H}}I))$ on M by

$$\text{trace}(\text{ad}({}^{\mathcal{H}}I)) = c_{rs}^s {}^{\mathcal{H}}I^r$$

where ${}^{\mathcal{H}}I = V_r \otimes {}^{\mathcal{H}}I^r$ and $[V_r, V_s] = c_{rs}^t V_t$.

Example 2.2.3. Let $\mathcal{V}$ be an orientable foliation on (M, g) , $\mathcal{H} = \mathcal{V}^\perp$ and $\overset{\mathcal{V}}{\omega}$ a volume form for $\mathcal{V}$ with respect to g .

Then, (i) if $\{V_r\}$ is a local frame for $\mathcal{V}$ made up of Killing fields then they satisfy (1) and (2) of Definition 2.2.2; (ii) more generally, the $\{V_r\}$ satisfy (1) of Definition 2.2.2 if and only if their restrictions to each leaf are divergence-free.

It can be shown, directly, that the above definition is independent of the local frame $\{V_r\}$ of $\mathcal{V}$ such that (1) and (2) above, hold. This also follows from the following proposition.

Proposition 2.2.4. *Let $\mathcal{V}$, $\mathcal{H}$, $\mathcal{H}I$, $\overset{\mathcal{V}}{\omega}$, $\{V_r\}$ be as in Definition 2.2.2 and let $\mathcal{V}R$ be the curvature form of $\overset{\mathcal{V}}{\nabla}$.*

Then,

- (a) $\text{trace}(\text{ad}(\mathcal{H}I)) = \text{trace}(\mathcal{V}R)$;
- (b) $\mathcal{V}^*(\mathcal{L}_{\mathcal{H}I(X,Y)}\overset{\mathcal{V}}{\omega}) = \text{trace}(\text{ad}(\mathcal{H}I))(X, Y) \overset{\mathcal{V}}{\omega}$ for any basic vector fields X and Y .

Further, the following assertions are equivalent:

- (i) $\text{trace}(\text{ad}(\mathcal{H}I)) = 0$,
- (ii) *At least locally there can be defined smooth positive basic functions ρ such that $\mathcal{V}^*(\mathcal{L}_X(\rho \overset{\mathcal{V}}{\omega})) = 0$ for any horizontal field X ,*
- (iii) $\mathcal{V}^*(\mathcal{L}_{\mathcal{H}I(X,Y)}\overset{\mathcal{V}}{\omega}) = 0$ for any basic vector fields X and Y .

In particular, if the first Betti number of M is zero and (i) holds then $\mathcal{V}$ is taut (i.e. there exists a Riemannian metric on M with respect to which the leaves of $\mathcal{V}$ are minimal).

Proof. Let g be a Riemannian metric on M such that $\mathcal{H} = \mathcal{V}^\perp$ and $\overset{\mathcal{V}}{\omega}$ is equal to the induced volume form on $\mathcal{V}$, and let $\overset{\mathcal{V}}{\nabla}$ be the adapted Bott connection on $\mathcal{V}$ corresponding to g .

Note that (1) of Definition 2.2.2 is equivalent to the fact that $\text{div}_{\mathcal{V}} V_\alpha = 0$.

The proof of (a) and (b) follows from Proposition 2.1.5, Lemma 2.1.13, Proposition 2.1.14 and Proposition 2.1.16 reversing the roles of $\mathcal{H}$ and $\mathcal{V}$.

Assertion (a) is equivalent to the fact that $\text{trace}(\text{ad}(\mathcal{H}I))$ is the curvature form of the connection induced by $\overset{\mathcal{V}}{\nabla}$ on $\Lambda^r(\mathcal{V})$. It is easy to see that this connection

is flat if and only if assertion (ii) holds. The equivalence (i) $\iff$ (ii) now follows from (a).

The equivalence (i) $\iff$ (iii) follows from (b). $\square$

Remark 2.2.5. Note that $\text{trace}(\text{ad}({}^{\mathcal{H}}I))$ is well-defined also for nonorientable foliations (in which case $\overset{\mathcal{V}}{\omega}$ is defined just up to the sign). Also, note that $\text{trace}(\text{ad}({}^{\mathcal{H}}I))$ is well-defined for foliations locally generated by conformal vector fields.

We now state the main result of this section.

Theorem 2.2.6. *Let $\mathcal{V}$ be a conformal foliation of $\text{codim } \mathcal{V} = n \neq 2$ on (M^m, g) , $m \geq 3$. Suppose that $\mathcal{V}$ is locally generated by conformal vector fields and let $\mathcal{H}$ denote its orthogonal complement.*

Then the following assertions are equivalent:

- (i) $\mathcal{V}$ produces harmonic morphisms.
- (ii) The mean curvature form of $\mathcal{V}$ is basic and the following relation holds:

$$\text{trace}(\text{ad}({}^{\mathcal{H}}I)) = \frac{m-2}{n} g(\text{trace}({}^{\mathcal{H}}B), {}^{\mathcal{H}}I) . \quad (2.2.1)$$

Proof. By Proposition 1.3.1, $\mathcal{V}$ produces harmonic morphisms if and only if

$$(n-2) d(\text{trace}({}^{\mathcal{H}}B)^{\flat}) - n d(\text{trace}({}^{\mathcal{V}}B)^{\flat}) = 0 . \quad (2.2.2)$$

By Proposition 2.1.20(a) the left hand side of (2.2.2) is automatically zero when evaluated on a pair of vertical vectors.

Let λ be a local dilation of $\mathcal{V}$.

Let $V \in \Gamma(\mathcal{V})$ be a conformal vector field on (M, g) . It is obvious that V restricted to any leaf L of $\mathcal{V}$ is a conformal vector field on $(L, g|_L)$. Using this it is easy to see that

$$\text{div}_{\mathcal{V}} V = -(m-n) V(\log \lambda) . \quad (2.2.3)$$

Let $X \in \Gamma(\mathcal{H})$ be a basic vector field. Then

$$\begin{aligned} \overset{\mathcal{H}}{\nabla}_V(\mathcal{H}^*(d \log \lambda))(X) &= V(X(\log \lambda)) = X(V(\log \lambda)) \\ &= -\frac{1}{m-n} X(\text{div}_{\mathcal{V}} V) = \frac{1}{m-n} \overset{\mathcal{H}}{\nabla}_V(\text{trace}({}^{\mathcal{V}}B)^{\flat})(X) \end{aligned} \quad (2.2.4)$$

where we have also applied Lemma 2.1.1 and Proposition 2.1.16 (reversing the roles of $\mathcal{H}$ and $\mathcal{V}$ in the latter). From Proposition 2.1.20(b) and (2.2.4) it follows that the left hand side of (2.2.2) is zero when evaluated on a pair made up of a vertical vector and a horizontal vector if and only if $\mathcal{V}$ has basic mean curvature form.

Now, using Lemma 2.1.13(a) (with the roles of $\mathcal{H}$ and $\mathcal{V}$ reversed) and (2.2.3) it is easy to see that

$$d(\text{trace}({}^{\mathcal{V}}B)^b)(X, Y) = \text{trace}(\text{ad}({}^{\mathcal{H}}I))(X, Y) - \frac{m-n}{n}g(\text{trace}({}^{\mathcal{H}}B), {}^{\mathcal{H}}I(X, Y)) \quad (2.2.5)$$

for any horizontal X and Y . By combining (2.2.5) and Proposition 2.1.4 we obtain that the left hand side of (2.2.2) is zero when evaluated on a pair of horizontal vectors if and only if (2.2.1) holds. The theorem is proved. $\square$

Remark 2.2.7. The first condition of Theorem 2.2.6(ii) (i.e. $\mathcal{V}$ has basic mean curvature form) can be replaced with the fact that the mean curvature form of $\mathcal{H}$ is invariant under the parallel displacement determined by $\mathcal{H}$.

Corollary 2.2.8. *Let $\mathcal{V}$ be a foliation on (M^m, g) , $m \geq 3$, which is locally generated by conformal vector fields and has integrable orthogonal complement. Then the following assertions are equivalent.*

- (i) $\mathcal{V}$ produces harmonic morphisms;
- (ii) $\mathcal{V}$ has basic mean curvature form.

Moreover, if either assertion (i) or (ii) holds then both $\mathcal{V}$ and its orthogonal complement have closed mean curvature forms.

Proof. The equivalence (i) $\iff$ (ii) is an immediate consequence of Theorem 2.2.6.

The last assertion follows from Proposition 2.1.4, Lemma 2.1.13, Proposition 2.1.16 and Theorem 2.2.6. $\square$

2.3 Isometric actions and harmonic morphisms

From Theorem 2.2.6 we obtain the following.

Corollary 2.3.1. *Let $\mathcal{V}$ be a Riemannian foliation of $\text{codim } \mathcal{V} \neq 2$ on (M^m, g) , $m \geq 3$, and let ${}^{\mathcal{H}}I$ be the integrability tensor field of its orthogonal complement. Suppose that $\mathcal{V}$ is locally generated by Killing fields.*

Then the following assertions are equivalent:

- (i) $\mathcal{V}$ produces harmonic morphisms,
- (ii) $\text{trace}(\text{ad}({}^{\mathcal{H}}I)) = 0$.

Proof. It is obvious that $\mathcal{V}$ has basic mean curvature. Also, $\text{trace}({}^{\mathcal{H}}B) = 0$ and the proof follows from Theorem 2.2.6. $\square$

Remark 2.3.2. It is easy to see that Corollary 2.3.1 holds more generally for Riemannian foliations locally generated by infinitesimal automorphisms of the horizontal distribution which when restricted to any leaf are divergence-free.

When the foliation is simple the result of Corollary 2.3.1 takes a more concrete form.

Corollary 2.3.3. *Let $\varphi : (M, g) \rightarrow (N, h)$, $\dim N \neq 2$, be a Riemannian submersion whose fibres are connected and locally generated by Killing fields and let ${}^{\mathcal{H}}I$ be the integrability tensor of the horizontal distribution. Then the following assertions are equivalent.*

- (i) φ lifts to a harmonic morphism $\tilde{\varphi} : (\tilde{M}, \tilde{g}) \rightarrow (\tilde{N}, \tilde{h})$ where $(\tilde{M}, \tilde{g}) \rightarrow (M, g)$ is a Riemannian regular covering and $\tilde{N} \rightarrow N$ is a regular covering such that $\tilde{h}$ and the pull-back of h to $\tilde{N}$ are conformally equivalent.
- (ii) $\text{trace}(\text{ad}({}^{\mathcal{H}}I)) = 0$.

Proof. From Corollary 2.3.1 it follows that it is sufficient to prove that if $\mathcal{V} (= \ker \varphi_*)$ produces harmonic morphisms then (i) holds.

Let $\tilde{M} \rightarrow M$ be the regular covering which corresponds to the cohomology class $[a] \in H^1(M; \mathbb{R})$ induced by the differentials of the logarithms of the dilations of the (local) harmonic morphisms produced by $\mathcal{V} (= \ker \varphi_*)$. (From the fundamental equation it follows that a can be also defined as the one-form obtained by applying the musical isomorphism $\flat$ to $\frac{-1}{n-2} \text{trace}({}^{\mathcal{V}}B)$.) It is obvious that the pull-back of $[a]$ to $\tilde{M}$ is zero; let λ be a smooth positive function on $\tilde{M}$ such that

$d(\log \lambda)$ is equal to the pull-back of a to $\widetilde{M}$.

Since a is basic (see Remark 2.3.2(1)) there exists a regular covering $\widetilde{N} \rightarrow N$ whose pull-back by φ is $\widetilde{M} \rightarrow M$. It is obvious that φ lifts to a smooth map $\widetilde{\varphi} : \widetilde{M} \rightarrow \widetilde{N}$. Note that we can suppose that λ is constant along the fibres of $\widetilde{\varphi}$ and hence there exists a positive smooth function $\bar{\lambda}$ on $\widetilde{N}$ such that $\widetilde{\varphi}^*(\bar{\lambda}) = \lambda$.

Let $\widetilde{g}$ be the pull-back of g to $\widetilde{M}$ and $\bar{h}$ be the pull-back of h to $\widetilde{N}$. Then $\widetilde{\varphi} : (\widetilde{M}, \widetilde{g}) \rightarrow (\widetilde{N}, \bar{\lambda}^2 \bar{h})$ is a harmonic morphism and the corollary is proved. $\square$

Remark 2.3.4. If in Corollary 2.3.3 we have that $H^1(M; \mathbb{R}) = 0$ or $H^1(N; \mathbb{R}) = 0$ then assertion (i) can be replaced by the following stronger assertion:

(i') There exists a Riemannian metric h_1 on N which is conformally equivalent to h such that $\varphi : (M, g) \rightarrow (N, h_1)$ is a harmonic morphism.

The same improvement can be made if the foliation formed by the fibres is generated by a *commuting* family of Killing fields $\{V_1, \dots, V_r\}$ (in particular, if the foliation is generated by an Abelian Lie group of isometries). To see this define λ by $g(V_1 \wedge \dots \wedge V_r, V_1 \wedge \dots \wedge V_r) = \lambda^{2n-4}$, $n = \dim N$. Then λ is the dilation of the induced harmonic morphism.

In some cases $\text{trace}(\text{ad}(\mathcal{H}I))$ can be defined in a different way:

Lemma 2.3.5. *Let $\mathcal{V}$ be a Riemannian foliation on (M, g) generated by the action of a closed subgroup G of the isometry group of (M, g) . Let $\{V_r\}$ be a local frame of $\mathcal{V}$ made up of Killing vector fields induced by the action of G and let $\{c_{rs}^t\}$ be defined by $[V_r, V_s] = c_{rs}^t V_t$.*

Then there exists a well-defined vertical one-form $\text{trace} \circ \text{ad} \in \Gamma(\mathcal{V}^)$ such that*

$$(\text{trace} \circ \text{ad})(V_r) = c_{rs}^s.$$

Moreover, we have $\text{trace}(\text{ad}(\mathcal{H}I)) = (\text{trace} \circ \text{ad})(\mathcal{H}I)$.

Proof. Let $x \in M$ and let $\mathfrak{h}_x$ be the Lie algebra of the isotropy group H_x at x of the action of G on M . Because G is a closed subgroup of the isometry group, the isotropy groups are compact and hence we can find a subspace $\mathfrak{m}_x$ of the Lie algebra $\mathfrak{g}$ of G such that $\mathfrak{g} = \mathfrak{h}_x \oplus \mathfrak{m}_x$ and $[\mathfrak{h}_x, \mathfrak{m}_x] \subseteq \mathfrak{m}_x$ (for example, take $\mathfrak{m}_x$ to be the orthogonal complement of $\mathfrak{h}_x$ in $\mathfrak{g}$ with respect to an $\text{Ad}H_x$ -invariant

metric on $\mathfrak{g}$).

Then, by identifying, as usual, $\mathfrak{m}_x = \mathcal{V}_x$, $(\text{trace} \circ \text{ad})_x$ is the restriction to $\mathfrak{m}_x$ of the trace of the adjoint representation of $\mathfrak{g}$.

If $y = xa$, $a \in G$, is another point on the same leaf, then we can take $\mathfrak{m}_y = (\text{Ad } a^{-1})(\mathfrak{m}_x)$.

If $y \in M$ can be joined to x by a horizontal curve then from Lemma 2.1.1 it follows that $H_y = H_x$.

Hence $\text{trace} \circ \text{ad}$ is well-defined.

The last assertion of the proposition is obvious. $\square$

Let $\mathcal{V}$ be a foliation on (M, g) locally generated by Killing fields. Although $\text{trace} \circ \text{ad}$ is not always well-defined, to simplify the exposition, we shall write $\text{trace} \circ \text{ad} = 0$ to mean that in the neighbourhood of each point a local frame $\{V_r\}$ for $\mathcal{V}$ can be found which is made up of Killing fields and is such that $c_{rs}^s = 0$ where c_{rs}^t are defined by $[V_r, V_s] = c_{rs}^t V_t$.

The following two corollaries follow immediately from Corollary 2.3.1.

Corollary 2.3.6. *A foliation of codimension not equal to two which is locally generated by Killing fields and which has integrable orthogonal complement produces harmonic morphisms.*

Corollary 2.3.7. *A foliation of codimension not equal to two which is locally generated by Killing fields and for which $\text{trace} \circ \text{ad} = 0$ produces harmonic morphisms.*

The above result admits the following partial converse.

Proposition 2.3.8. *Let $\mathcal{V}$ be a foliation of codimension not equal to two which produces harmonic morphisms on (M, g) and is locally generated by Killing fields. Let ${}^{\mathcal{H}}I$ be the integrability tensor of the orthogonal complement $\mathcal{H}$ of $\mathcal{V}$.*

Suppose that on each leaf L of $\mathcal{V}$ a point $x \in L$ can be found such that $\mathcal{V}_x$ is spanned by $\{{}^{\mathcal{H}}I(X, Y) \mid X, Y \in \mathcal{H}_x\}$.

Then $\text{trace} \circ \text{ad} = 0$.

Proof. From Corollary 2.3.1 it follows that it is sufficient to prove that $\mathcal{V}_x$ is spanned by $\{{}^{\mathcal{H}}I(X, Y) \mid X, Y \in \mathcal{H}_x\}$ at each point $x \in M$. From Lemma 2.1.1 it

follows that $[V, {}^{\mathcal{H}}I(X, Y)] = 0$ for any Killing field $V \in \Gamma(\mathcal{V})$ and any basic vector fields $X, Y \in \Gamma(\mathcal{H})$.

The proof follows from the fact that any two points of a leaf can be joined by a curve which is piecewisely an integral curve of a Killing vector field $V \in \Gamma(\mathcal{V})$ (see [56, Chapter I, Theorem 4.4]). $\square$

Next we give an example of a Riemannian foliation locally generated by Killing fields for which $\text{trace}(\text{ad}({}^{\mathcal{H}}I)) = 0$ but ${}^{\mathcal{H}}I \neq 0$ and $\text{trace} \circ \text{ad} \neq 0$.

Example 2.3.9. Let $\mathcal{F}$ be a Riemannian foliation locally generated by Killing fields and which produces harmonic morphisms on (M, g) . Suppose that the orthogonal complement of $\mathcal{F}$ is not integrable (see examples, below).

Let G be the Lie group defined by

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a > 0, b \in \mathbb{R} \right\}.$$

Endow G with a right invariant metric γ and consider the Riemannian product manifold $(M \times G, \pi_M^*(g) + \pi_G^*(\gamma))$ where π_M and π_G are the projections onto M and G , respectively. Let $\mathcal{V} = \mathcal{F} \times TG$. It is obvious that $\mathcal{V}$ is a foliation locally generated by Killing fields and which produces harmonic morphisms on $(M \times G, \pi_M^*(g) + \pi_G^*(\gamma))$.

Notice, however, that the orthogonal complement $\mathcal{H}$ of $\mathcal{V}$ is nonintegrable so ${}^{\mathcal{H}}I \neq 0$, and also $\text{trace} \circ \text{ad} \neq 0$.

For the next application we recall the following definition (cf. [11, 7.84]).

Definition 2.3.10. Let (L^p, h) be a locally homogeneous Riemannian manifold (i.e. a Riemannian manifold whose tangent bundle admits, in a neighbourhood of each point, local frames made up of Killing fields).

Then (L^p, h) is called *naturally reductive* if each point $x \in L$ has an open neighbourhood on which a local frame $\{V_r\}_{r=1, \dots, p}$ made up of Killing fields can be found such that $h([V_r, V_s], V_t) + h(V_s, [V_r, V_t]) = 0$ at x .

The following well-known lemma follows from the fact that any skew-symmetric endomorphism is trace-free.

Lemma 2.3.11. *Let (L, h) be a naturally reductive locally-homogeneous Riemannian manifold. Then $\text{trace} \circ \text{ad} = 0$.*

The following result follows from Corollary 2.3.7 and Lemma 2.3.11.

Proposition 2.3.12. *A foliation of codimension not equal to two which is locally generated by Killing fields and whose leaves are naturally reductive produces harmonic morphisms.*

Theorem 2.3.13. *Let G be a Lie group which acts as an isometry group on the Riemannian manifold (M, g) .*

Suppose that the following conditions are satisfied:

- (i) *The orbits of the action of G on M have the same codimension not equal to two.*
- (ii) *There exists on G a bi-invariant Riemannian metric.*
- (iii) *The canonical representation of an isotropy group is irreducible.*

Then, the connected components of the orbits form a Riemannian foliation with umbilical leaves which produces harmonic morphisms.

Proof. The fact that (i) implies that the connected components of the orbits form a Riemannian foliation is well-known (see [56, Chapter IV, Example 4.10]). Let $\mathcal{V}$ be this foliation.

By choosing an $\text{Ad } G$ invariant metric on the Lie algebra of G and restricting it to the orthogonal complement of the Lie algebra of the isotropy group at $x \in M$ we can induce a metric $\bar{h}_x$ on $\mathcal{V}_x$ which by (iii) must be homothetic to $g_x|_{\mathcal{V}_x}$ (see [37, vol.I, Appendix 5]). Then $\bar{h}$ is a metric on $\mathcal{V}$ which can be extended to a metric h on M such that $h|_{\mathcal{H}} = g|_{\mathcal{H}}$ where $\mathcal{H}$ is the orthogonal complement of $\mathcal{V}$.

Since $h|_{\mathcal{V}}$ is induced by an $\text{Ad } G$ invariant metric, $\mathcal{V}$ has naturally reductive leaves with respect to h . But g and h are homothetic when restricted to a leaf and hence the leaves of $\mathcal{V}$ are also naturally reductive with respect to g . Moreover, from the fact that $\mathcal{V}$ has totally geodesic leaves with respect to h , and g and h are conformal when restricted to $\mathcal{V}$ and equal when restricted to $\mathcal{H}$ it follows that the leaves of $\mathcal{V}$ are umbilical with respect to g . $\square$

Remark 2.3.14. Note that the same argument as above can be applied to show that the Ricci tensor of each leaf is proportional to the induced metric (see [11, 7.44]); hence, in the above theorem, each leaf is an Einstein manifold.

Theorem 2.3.15. *Let G be a closed subgroup of the isometry group of (M, g) which generates a foliation $\mathcal{V}$ of codimension not equal to two.*

(i) *Suppose that the Lie algebra $\mathfrak{g}$ of G satisfies $\text{trace}(\text{ad } \mathfrak{g}) = 0$. Then, $\mathcal{V}$ produces harmonic morphisms.*

(ii) *Conversely, if $\mathcal{V}$ produces harmonic morphisms and on each orbit Q a point $x \in Q$ can be found such that $\mathcal{V}_x$ is spanned by $\{\mathcal{H}I(X, Y) \mid X, Y \in \mathcal{H}_x\}$ where $\mathcal{H}I$ is the integrability tensor of the orthogonal complement of $\mathcal{V}$, then $\text{trace}(\text{ad } \mathfrak{g}) = 0$.*

Proof. (i) It is sufficient to prove that $\text{trace} \circ \text{ad} = 0$.

Let H be the isotropy group of G at $x \in M$. Since G is a closed subgroup of the isometry group we have that H is compact and hence we can find $\mathfrak{m} \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$.

Let $\{A_1, \dots, A_r\}$ be a basis of $\mathfrak{h}$ and $\{A_{r+1}, \dots, A_s\}$ a basis of $\mathfrak{m}$. Let $\{c_{\alpha\beta}^\gamma\}$ be the corresponding structural constants of $\mathfrak{g}$.

Because $\text{trace}(\text{ad } \mathfrak{g}) = 0$ we have that $\sum_{\alpha=1}^s c_{\alpha\beta}^\alpha = 0$ for $\beta = 1, \dots, s$.

Also, $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ implies that $c_{\alpha\beta}^\gamma = 0$ for $\alpha, \gamma = 1, \dots, r$ and $\beta = r+1, \dots, s$.

Let $\beta \in \{r+1, \dots, s\}$. Then $\sum_{\alpha=r+1}^s c_{\alpha\beta}^\alpha = -\sum_{\alpha=1}^r c_{\alpha\beta}^\alpha = 0$ and it follows that $(\text{trace} \circ \text{ad})_x = 0$.

(ii) Note that if a point of an orbit has the assumed property then on each component of that orbit a point can be found with the same property. The proof now follows from Proposition 2.3.8. $\square$

Corollary 2.3.16. *Let G be a compact Lie group of isometries of (M, g) . If the principal orbits of G have codimension not equal to two then their connected components form a Riemannian foliation which produces harmonic morphisms.*

In particular, if (M, g) is compact and the principal orbits of the isometry group are of codimension not equal to two then their connected components form a Riemannian foliation which produces harmonic morphisms.

Remark 2.3.17. Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Recall that if G is connected then $\text{trace}(\text{ad } \mathfrak{g}) = 0$ if and only if G is unimodular, i.e. its left and right invariant Haar measures (which are unique up to multiplicative constants) are equal.

Theorem 2.3.18. *Let $\xi = (P, N, G)$ be a principal bundle, $\dim N \neq 2$, whose total space P is endowed with a Riemannian metric g which is invariant under the action of G .*

Let $\mathcal{H}$ be the induced principal connection on ξ and h the (unique) Riemannian metric on N such that the projection $\pi : (P, g) \rightarrow (N, h)$ is a Riemannian submersion.

Then the following assertions are equivalent:

(i) *The connection induced by $\mathcal{H}$ on the determinant bundle of the adjoint bundle $\text{Ad } \xi$ is flat.*

(ii) *The projection π lifts to a harmonic morphism $\tilde{\pi} : (\tilde{P}, \tilde{g}) \rightarrow (\tilde{N}, \tilde{h})$ where $(\tilde{P}, \tilde{g}) \rightarrow (P, g)$ is a Riemannian regular covering and $\tilde{N} \rightarrow N$ is a regular covering such that $\tilde{h}$ and the pull-back of h to $\tilde{N}$ are conformally equivalent. Moreover, $(\tilde{P}, \tilde{N}, G)$ is, in a natural way, a principal bundle.*

Proof. Let $\Omega \in \Gamma(\mathfrak{g} \otimes \Lambda^2(T^*P))$ be the curvature form of $\mathcal{H}$. It is obvious that $\text{trace}(\text{ad } \mathcal{H}) = (\text{trace} \circ \text{ad})(\Omega)$. But $(\text{trace} \circ \text{ad})(\Omega)$ is the pull back, by π , of the curvature form of the determinant bundle of $\text{Ad } \xi$. Hence, (i) is equivalent to the fact that the fibres of π form a (Riemannian) foliation which produces harmonic morphisms.

Now, from the Corollary 2.3.1 it follows that it is sufficient to prove that $(\tilde{P}, \tilde{N}, G)$ is, in a natural way, a principal bundle. Using the same notations as in Corollary 2.3.3 (with $P = M$) this follows from the fact that $\tilde{P}$ is the total space of $\xi + \eta \in H^1(N, G \times K)$, where $\eta \in H^1(N, K)$ is the regular covering corresponding to $[b] \in H^1(N, \mathbb{R})$ and $[b]$ is such that $\pi^*[b] = [a]$. $\square$

Remark 2.3.19. 1) From the holonomy theorem (see [37]) it follows that assertion (i) of Theorem 2.3.18 is equivalent to the fact that the identity component H of the holonomy group of $\mathcal{H}$ satisfies $\det(\text{Ad}_G H) = 1$.

2) Let $\mathcal{V}$ be a foliation on (M, g) generated by the action of a closed subgroup G of the isometry group of (M, g) . Then it can be proved, directly by using the mass invariance characteristic property of harmonic morphisms (see Proposition 1.1.11), that $\mathcal{V}$ produces harmonic morphisms if and only if the identity component H of the holonomy group at x of the orthogonal complement of $\mathcal{V}$ satisfies $\det(\text{Ad}_G H) = 1$. (The holonomy group of $\mathcal{H}(= \mathcal{V}^\perp)$ at $x \in M$ consists of those $a \in G$ such that x and xa can be joined by a horizontal path, cf. [12].) In this way another proof can be obtained for the result of Corollary 2.3.1 applied to foliations *globally* generated by closed subgroups of the isometry group.

Corollary 2.3.20. *Let (P, N, G) be a principal bundle, $\dim N \geq 3$, whose total space P is endowed with a Riemannian metric g which is invariant under the action of G .*

Let $H \subseteq G$ be a closed subgroup and suppose that $\text{trace}(\text{ad } \mathfrak{h}) = 0$ and $\text{trace}(\text{ad } \mathfrak{g}) = 0$. Let $E = P \times_G G/H$ be the total space of the associated bundle.

Then in a neighbourhood of any point, there exist Riemannian metrics on E and N with respect to which the restriction of the natural projection $E \rightarrow N$ to that neighbourhood is a harmonic morphism.

Proof. It is well-known that (P, E, H) is in a natural way a principal bundle (see [37]). From Theorem 2.3.18 it follows that both of the foliations induced on P by the actions of G and of H produces harmonic morphisms. Thus, at least locally, we can find metrics on E and on N with respect to which the projections $P \rightarrow E$ and $P \rightarrow N$ are harmonic morphisms. Since the first of these is surjective this implies that the projection $E \rightarrow N$ can also be made, at least locally, a harmonic morphism (see [19, 2.31] and [30, Proposition 1.1]). $\square$

It is obvious that Corollary 2.3.20 still holds if $\dim N = 1 \neq \dim G - \dim H$.

Corollary 2.3.21. *Let (G, g) be a Lie group endowed with a right (left) invariant Riemannian metric. Let $K \subseteq H \subseteq G$ be closed subgroups such that $\text{trace}(\text{ad } \mathfrak{k}) = 0$, $\text{trace}(\text{ad } \mathfrak{h}) = 0$ and $\dim G - \dim H \geq 3$.*

Then in a neighbourhood of any point, there exist Riemannian metrics on G/H and G/K with respect to which the restriction of the natural projection $G/K \rightarrow G/H$ to that neighbourhood is a harmonic morphism.

It is obvious that Corollary 2.3.21 still holds if $\dim G - \dim H = 1 \neq \dim H - \dim K$.

Example 2.3.22. 1) A one-dimensional Riemannian foliation produces harmonic morphisms if and only if it is locally generated by Killing vector fields. This result is due to R.L. Bryant [13] and the ‘if’ part follows also from Corollary 2.3.7 (see also Proposition 3.1.3, below).

Let $\varphi : (M^{n+1}, g) \rightarrow N^n$, $n \geq 3$, be a submersion with connected one-dimensional fibres and let $\mathcal{V}$ be the foliation formed by the fibres of φ . Suppose that there exists a nowhere zero Killing vector field $V \in \Gamma(\mathcal{V})$. Let $\bar{g}$ be the unique metric on N such that $\varphi : (M^{n+1}, g) \rightarrow (N^n, \bar{g})$ is a Riemannian submersion and let λ be the positive smooth function such that $g(V, V) = \lambda^{2n-4}$. Then, because V is Killing, λ is basic, i.e. there exists a positive smooth function $\bar{\lambda}$ on N^n such that $\lambda = \bar{\lambda} \circ \varphi$. Then, $\varphi : (M, g) \rightarrow (N, \bar{\lambda}^2 \bar{g})$ is a harmonic morphism [13]. We shall say that φ is induced by an (infinitesimal) isometric action.

2) The foliation formed on (an open subset of) a hypersphere S^n by the intersections with it of a parallel family of planes in $\mathbb{R}^{n+1}$ of codimension not equal to three produces harmonic morphisms. This follows from Corollary 2.3.6 or from Theorem 2.3.13. This can also be proved by noting that the foliation is induced by one of the projections of a warped-product (see Example 1.4.10). Similar examples can be obtained on Euclidean spaces and on hyperbolic spaces.

3) Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and consider on $Gl_n(\mathbb{K})$, $n \geq 2$, the following well-known right invariant Riemannian metric

$$g = \sum_{i,j=1}^n \left| dx_k^i \cdot (x^{-1})_j^k \right|^2.$$

Let $K \subseteq H \subseteq G \subseteq Gl_n(\mathbb{K})$ be closed subgroups such that $\text{trace}(\text{ad } \mathfrak{k}) = 0$, $\text{trace}(\text{ad } \mathfrak{h}) = 0$ and $\dim G - \dim H \neq 2 \neq \dim G - \dim K$. Then, at least locally, a metric can be found on G/H (which is unique up to homotheties) such that the projection $G \rightarrow G/H$ becomes, suitably restricted, a harmonic morphism. (If G or G/H has zero first Betti number then this metric can be defined globally on G/H .) Also, at least locally, a metric can be found on the total space of the projection $G/K \rightarrow G/H$ such that the induced foliation produces harmonic

morphisms. (If G/K and G/H both have zero first Betti number then there can be defined (global) metrics on them such that the projection $G/K \rightarrow G/H$ becomes a harmonic morphism.)

For example, the foliations formed by the fibres of the following natural maps produce harmonic morphisms:

$$\begin{aligned} Gl_{p+q}(\mathbb{K}) &\rightarrow G_{p+q,p}(\mathbb{K}) \times G_{p+q,q}(\mathbb{K}), \\ Gl_{p+q}(\mathbb{K}) &\rightarrow V_{p+q,p}(\mathbb{K}) \times G_{p+q,q}(\mathbb{K}), \\ Gl_{p+q}(\mathbb{K}) &\rightarrow PGL_{p+q}(\mathbb{K}) \end{aligned}$$

where, for $p, q \geq 1$, $G_{p+q,p}(\mathbb{K})$ is the Grassmanian manifold of p -dimensional subspaces of $\mathbb{K}^{p+q}$, $V_{p+q,p}(\mathbb{K})$ is the Stiefel manifold of p -frames on $\mathbb{K}^{p+q}$ and for the first projection $p + q \geq 3$ if $\mathbb{K} = \mathbb{R}$. If $\mathbb{K} = \mathbb{H}$, or $\mathbb{K} = \mathbb{R}$ and $p + q \geq 3$, then the first Betti number of $Gl_{p+q}(\mathbb{K})$ is zero and hence in these cases on the image of each of the above maps a metric can be found such that the induced map becomes a harmonic morphism.

In particular, consider on $Gl_2^+(\mathbb{R})$ the coordinates given by $x = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$. With respect to these coordinates we have

$$\begin{aligned} g = \frac{1}{(x_1x_4 - x_2x_3)^2} \{ &(x_3^2 + x_4^2)(dx_1^2 + dx_2^2) + (x_1^2 + x_2^2)(dx_3^2 + dx_4^2) \\ &- 2(x_1x_3 + x_2x_4)(dx_1 dx_3 + dx_2 dx_4) \}. \end{aligned}$$

Then on the images of the following maps there exist Riemannian metrics, unique up to homotheties, with respect to which the induced maps are harmonic morphisms.

$\varphi_1 : Gl_2^+(\mathbb{R}) \rightarrow PGL_2^+(\mathbb{R})$, given by the natural projection,

$\varphi_2 : Gl_2^+(\mathbb{R}) \rightarrow \mathbb{R}^2 \times \mathbb{R}P^1$, given by $\varphi_2(x_1, x_2, x_3, x_4) = ((x_1, x_2), [x_3 : x_4])$.

If $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$ then we can also consider the foliation induced by the map $Gl_2(\mathbb{K}) \rightarrow \mathbb{K}P^1 \times \mathbb{K}P^1$.

Other examples can be obtained by considering other linear Lie groups.

2.4 Homothetic foliations locally generated by conformal vector fields

From results of Section 2 and 3 we obtain necessary and sufficient conditions for a foliation to be homothetic.

Corollary 2.4.1. *Let $\mathcal{V}$ be a conformal foliation on (M, g) with integrable orthogonal complement $\mathcal{H}$. If both $\mathcal{V}$ and $\mathcal{H}$ have basic mean curvature forms then $\mathcal{V}$ is a homothetic foliation. If, further, $\text{codim } \mathcal{V} \neq 2$ and $\dim M \geq 3$ then $\mathcal{V}$ produces harmonic morphisms.*

Proof. This is an immediate consequence of Theorem 2.1.19. $\square$

Proposition 2.4.2. *Let $\mathcal{V}$ be a foliation of $\text{codim } \mathcal{V} = n$ on (M^m, g) which is locally generated by conformal vector fields and let $\mathcal{H}$ denote its orthogonal complement.*

If $\mathcal{V}$ is homothetic then $g(\text{trace}({}^{\mathcal{H}}B), {}^{\mathcal{H}}I) = 0$.

Conversely, if $g(\text{trace}({}^{\mathcal{H}}B), {}^{\mathcal{H}}I) = 0$ then the following assertions are equivalent:

- (i) $\mathcal{V}$ is a homothetic foliation;
- (ii) The mean curvature form of $\mathcal{V}$ is basic;
- (iii) The mean curvature form of $\mathcal{H}$ is invariant under the parallel displacement determined by $\mathcal{H}$;
- (iv) In the neighbourhood of each point of M there exists a local dilation λ of $\mathcal{V}$ such that for any horizontal vector X and conformal vector field V tangent to $\mathcal{V}$ we have $X(V(\log \lambda)) = 0$.

Proof. The first assertion is an immediate consequence of formula (2.1.1) from Proposition 2.1.4.

Suppose now that $g(\text{trace}({}^{\mathcal{H}}B), {}^{\mathcal{H}}I) = 0$.

The equivalence (i) $\iff$ (iii) follows from Proposition 2.1.4.

From the proof of Theorem 2.2.6 it follows that

$$\frac{1}{m-n} \nabla_V^{\mathcal{V}}(\text{trace}({}^{\mathcal{V}}B)^{\flat})(X) = X(V(\log \lambda)) = \nabla_V^{\mathcal{H}}(\mathcal{H}^*(d \log \lambda))(X) \quad (2.4.1)$$

for any $X \in \Gamma(\mathcal{H})$, any conformal vector field $V \in \Gamma(\mathcal{V})$ and any local dilation λ of $\mathcal{V}$. The first equality of (2.4.1) implies that (ii) $\iff$ (iv).

From the second equality of (2.4.1) and (2.1.2) of Proposition 2.1.4 we obtain the equivalence (iii) $\iff$ (iv) and the proposition is proved. $\square$

Theorem 2.4.3. *Let $\mathcal{V}$ be a foliation of $\text{codim} \mathcal{V} \neq 2$ on (M^m, g) , $m \geq 3$, which is locally generated by conformal vector fields. Then any two of the following assertions imply the other one.*

- (i) $\mathcal{V}$ produces harmonic morphisms;
- (ii) $\mathcal{V}$ is homothetic;
- (iii) $\text{trace}(\text{ad}(\mathcal{H}I)) = 0$.

Proof. This is a consequence of Proposition 2.4.2 and of Theorem 2.2.6. $\square$

Remark 2.4.4. 1) There is another way to prove that in Theorem 2.4.3 if (ii) holds then (i) $\iff$ (iii). To see this let λ be a local dilation of $\mathcal{V}$ which is constant along horizontal curves. Then, with respect to $\lambda^2 g$, $\mathcal{V}$ is a Riemannian foliation locally generated by Killing fields. Thus, condition (ii) of Theorem 2.2.6 says the same thing when written for a local frame made up of fields which are conformal with respect to g and when written for the same frame with the metric $\lambda^2 g$ so that the fields are now Killing fields. Moreover, since λ is constant along horizontal curves, $\mathcal{V}$ produces harmonic morphisms with respect to g if and only if it produces harmonic morphisms with respect to $\lambda^2 g$ (Corollary 1.2.2). The proof of the theorem now follows from Corollary 2.3.1 (which can be proved directly).

2) It is not difficult to see using Theorem 2.4.3 that the following classes of foliations of codimension not equal to two produce harmonic morphisms:

- homothetic foliations locally generated by conformal fields and with integrable orthogonal complement;
- homothetic foliations generated by the local action of an Abelian Lie group of conformal transformations;
- homothetic foliations generated by the action of a unimodular closed subgroup of the group of conformal transformations;

- homothetic foliations formed by the components of the fibres of principal bundles for which the total space is endowed with a metric such that the structural group acts by conformal transformations and the connection induced on the determinant bundle of the adjoint bundle is flat.

From Theorem 2.4.3 we obtain the following.

Corollary 2.4.5. *Let $\mathcal{V}$ be a foliation of $\text{codim } \mathcal{V} \neq 2$ on (M^m, g) , $m \geq 3$ which is locally generated by conformal vector fields. Suppose that the orthogonal complement $\mathcal{H}$ of $\mathcal{V}$ is integrable. Then the following assertions are equivalent.*

- (i) $\mathcal{V}$ produces harmonic morphisms;
- (ii) $\mathcal{V}$ is homothetic.

Remark 2.4.6. We shall see (Proposition 3.1.5) that if $\dim \mathcal{V} = 1$ then (i) and (ii) of Corollary 2.4.5 imply that $\mathcal{V}$ is locally generated by conformal vector fields.

Next we give a construction of a foliation which produces harmonic morphisms which has basic mean curvature form but is nowhere homothetic.

Example 2.4.7. Let $\varphi : (M^{n+1}, h) \rightarrow (N^n, \bar{h})$, $n \geq 1$, be a Riemannian submersion with geodesic fibres and let $\mathcal{V}$ be the foliation formed by the fibres of φ .

Suppose that V is a local vertical field such that $h(V, V) = 1$. Because φ has geodesic leaves we have that $[V, X] = 0$ for any basic X .

Let $\theta = V^\flat$ and $\Omega = d\theta$. It is easy to see that $\Omega = 0$ if and only if the horizontal distribution $\mathcal{H}$ is integrable. Also, Ω is basic and since $d\Omega = dd\theta = 0$, at least locally, we can find a basic one-form A such that $\Omega = -dA$.

Thus $d\theta = \Omega = -dA$ and hence $d(A + \theta) = 0$. It follows that, at least locally, we can write $A + \theta = d\sigma$ for some smooth local function σ on M . Note that the horizontal component of $d\sigma$ is basic, being equal to A .

Supposing that σ is defined on the whole M , let g^σ be the Riemannian metric on M defined by

$$g^\sigma = e^{-2\sigma} \varphi^*(\bar{h}) + e^{(2n-4)\sigma} \theta^2.$$

Then $\varphi : (M, g^\sigma) \rightarrow (N, \bar{h})$ is a harmonic morphism [13]. Moreover, the mean curvature form of $\mathcal{V}$ with respect to g^σ is $(2 - n)A$ and therefore is basic. However, from Proposition 2.1.4 and Proposition 2.1.14 it follows that the connected

components of the fibres of φ form a homothetic foliation with respect to g^σ only over the set of points where the horizontal distribution is integrable. Thus if $\mathcal{H}$ is nowhere integrable then $\mathcal{V}$ is nowhere homothetic with respect to g^σ .

Let ρ be any other function which has the same properties as σ (i.e. $\varphi : (M, g^\rho) \rightarrow (N, \bar{h})$ is a harmonic morphism, the mean curvature form of $\mathcal{V}$ with respect to g^ρ is basic and $\mathcal{V}$ is nowhere homothetic on (M, g^ρ)). Then, there exists a unique constant $c \in \mathbb{R}$ such that $\rho - c\sigma$ is, at least locally, a basic function.

To see this note that because the induced foliation is nowhere homothetic then we must have:

(i) $M = \{x \in M \mid V(V(\rho)) = 0\}$ (otherwise on some open subset of M the level hypersurfaces of $V(\rho)$ would be integral submanifolds of the horizontal distribution);

(ii) the interior of the set $\{x \in M \mid V(\rho) = 0\}$ is empty (otherwise the restriction of $\mathcal{V}$ to some open subset of M would be Riemannian).

Thus we have $V(\rho) = c$, for some constant $c \neq 0$. Hence $d\rho = c\theta + B$.

Then B must be basic (because $X(V(\rho)) = 0$ for any horizontal X) and hence $0 = c d\theta + dB$ which is equivalent to $dB = -c\Omega$.

It follows that $d(\rho - c\sigma) = d\rho - c d\sigma = c\theta + B - c\theta - cA = B - cA$.

Because $B - cA$ is a closed basic one-form, at least locally, we can find a basic function whose differential is equal to $d(\rho - c\sigma)$ and hence $\rho - c\sigma$ is, at least locally, a basic function.

2.5 Homothetic actions and harmonic morphisms

Recall that a vector field V on a Riemannian manifold (M, g) is homothetic if $\mathcal{L}_V g = a g$ for some constant $a \in \mathbb{R}$ (see [67]).

The first thing to note about a foliation locally generated by homothetic vector fields is the following.

Proposition 2.5.1. *Let $\mathcal{V}$ be a foliation on (M, g) locally generated by homothetic vector fields. Then either $\mathcal{V}$ is Riemannian and locally generated by Killing vector fields or $\mathcal{V}$ is nowhere Riemannian.*

Proof. Let $P = \{x \in M \mid \mathcal{V} \text{ is Riemannian at } x\}$.

It is obvious that P is closed. Also let $\{V_r\}$ be a local frame of $\mathcal{V}$, defined on a connected open subset U and made up of homothetic vector fields. It is obvious that if $U \cap P \neq \emptyset$ then V_r are Killing fields and thus $U \subseteq P$. Hence P is also open and, since M is connected, either $P = M$ or $P = \emptyset$. $\square$

From Proposition 2.4.2 we obtain the following.

Corollary 2.5.2. *Let $\mathcal{V}$ be a foliation locally generated by homothetic vector fields on the Riemannian manifold (M, g) .*

Then $\mathcal{V}$ is a homothetic foliation if and only if $g(\text{trace}(\mathcal{H}B), \mathcal{H}I) = 0$.

In particular, a foliation locally generated by homothetic vector fields and with integrable orthogonal complement is a homothetic foliation.

Proof. Let V be a homothetic vector field on (M, g) which is tangent to the foliation. Then it is easy to see that $\mathcal{L}_V g = -2V(\log \lambda)g$ where λ is any local dilation of the foliation. But V is homothetic and hence $V(\log \lambda)$ is a constant function. The proof now follows from (i) $\iff$ (iv) of Proposition 2.4.2. $\square$

Remark 2.5.3. Let $\mathcal{V}$ be a conformal foliation on (M, g) and define the vertical one-form μ by the relation $(\mathcal{L}_V g)(X, Y) = \mu(V)g(X, Y)$ where V is vertical and X, Y are horizontal [62] (see [9]). Then $\mu = -2\mathcal{V}^*(d(\log \lambda))$ where λ is a local dilation of $\mathcal{V}$, and, because $\text{trace}(\mathcal{H}B) = n\mathcal{V}(\text{grad}(\log \lambda))$, we have $\mu = -\frac{2}{n} \text{trace}(\mathcal{H}B)^\flat$ (see [9]).

By Corollary 2.5.2, if $\mathcal{V}$ is locally generated by homothetic vector fields then $\mathcal{V}$ is a homothetic foliation if and only if $\mu(\mathcal{H}I) = 0$.

Proposition 2.5.4. *Let $\mathcal{V}$ be a one-dimensional foliation of $\text{codim } \mathcal{V} \neq 2$ on (M^m, g) , $m \geq 3$, which is not a Riemannian foliation and which is locally generated by homothetic vector fields. Then the following assertions are equivalent:*

- (i) $\mathcal{V}$ produces harmonic morphisms,
- (ii) $\mathcal{V}$ is a homothetic foliation,
- (iii) $\mathcal{H}$ is integrable.

Proof. It is easy to see that because $\mathcal{V}$ is locally generated by homothetic vector fields its mean curvature form is basic. The proof now follows from Theorem 2.2.6 and Corollary 2.5.2. $\square$

Note that in Proposition 2.5.4 the equivalence (ii) $\iff$ (iii) holds also when $\text{codim } \mathcal{V} = 2$.

Proposition 2.5.5 (cf. [62, Proposition 2.8]). *Let $\mathcal{V}$ be a foliation on (M^m, g) locally generated by homothetic vector fields. Then there exists a Riemannian foliation $\mathcal{W} \subseteq \mathcal{V}$ locally generated by Killing fields. Moreover, if $\mathcal{V}$ is not Riemannian then $\dim \mathcal{V} = \dim \mathcal{W} + 1$.*

Proof. Suppose that $\mathcal{V}$ is not Riemannian. Then by Proposition 2.5.1 the foliation $\mathcal{V}$ is nowhere Riemannian. Since $\mathcal{V}$ is conformal we can find local dilations of it in the neighbourhood of each point. Let λ be a local dilation of $\mathcal{V}$ defined on the open subset $U \subseteq M$. For $x \in U$ let

$$\mathcal{W}_x = \{ V \in \mathcal{V}_x \mid V(\log \lambda) = 0 \} = \mathcal{V}_x \cap (\text{grad}(\log \lambda)_x)^\perp.$$

Since any two local dilations of $\mathcal{V}$ differ locally by a factor which is constant along the leaves it follows that $\mathcal{W}_x$ does not depend on λ . Because $\mathcal{V}$ is nowhere Riemannian, $\mathcal{W}_x \neq \mathcal{V}_x$. Also $\text{grad}(\log \lambda)$ is nonvanishing and hence $\dim((\text{grad}(\log \lambda)_x)^\perp) = m - 1$ where $m = \dim M$. We have

$$\begin{aligned} \dim \mathcal{W}_x &= \dim(\mathcal{V}_x \cap (\text{grad}(\log \lambda)_x)^\perp) \\ &= \dim \mathcal{V}_x + \dim((\text{grad}(\log \lambda)_x)^\perp) - \dim(\mathcal{V}_x + (\text{grad}(\log \lambda)_x)^\perp). \end{aligned}$$

It follows that the minimum value of $\dim \mathcal{W}_x$ occurs precisely when $\mathcal{V}_x + (\text{grad}(\log \lambda)_x)^\perp = T_x M$. If this is the case, then $\dim \mathcal{W}_x = \dim \mathcal{V}_x + (m - 1) - m = \dim \mathcal{V}_x - 1$. Since $\mathcal{W}_x \subset \mathcal{V}_x$, $\mathcal{W}_x \neq \mathcal{V}_x$ it follows that $\dim \mathcal{W}_x = \dim \mathcal{V}_x - 1$. Thus $\mathcal{W} = (\mathcal{W}_x)_{x \in M}$ defines a distribution on M . Then $\mathcal{W}$ is integrable because it is the intersection of two transversal foliations.

Let $V \in \Gamma(\mathcal{V})$ be a homothetic vector field. It is easy to see that if $V_x \in \mathcal{W}_x$ then $V \in \Gamma(\mathcal{W})$. Since $\mathcal{V}$ is locally generated by homothetic vector fields it follows that $\mathcal{W}$ is locally generated by Killing fields. (This also implies that $\mathcal{W}$ is

integrable since any Killing field which is tangent to $\mathcal{V}$ must be tangent to $\mathcal{W}$ and the bracket of any two Killing fields is a Killing field.) $\square$

We can now characterise geometrically the homothetic (infinitesimal) actions which induce homothetic foliations and their relations with harmonic morphisms.

Theorem 2.5.6. *Let $\mathcal{V}$ be a foliation locally generated by homothetic vector fields and let $\mathcal{H}$ be its orthogonal complement.*

Then the following assertions are equivalent:

- (a) $\mathcal{V}$ is a homothetic foliation;
- (b) either $\mathcal{V}$ is Riemannian and locally generated by Killing fields or there exists a Riemannian foliation $\mathcal{W} \subseteq \mathcal{V}$ locally generated by Killing fields such that $\dim \mathcal{V} = \dim \mathcal{W} + 1$ and the distribution $\mathcal{F} = \mathcal{W} \oplus \mathcal{H}$ is integrable.

Moreover, if (a) or (b) hold and $\dim \mathcal{V} \geq 2$, $\text{codim } \mathcal{V} \geq 3$ then the following assertions are equivalent:

- (i) $\mathcal{V}$ produces harmonic morphisms,
- (ii) The restriction of $\mathcal{W}$ to any leaf of $\mathcal{F}$ produces harmonic morphisms.

Proof. The equivalence (a) $\iff$ (b) follows from Corollary 2.5.2 and Proposition 2.5.5.

Suppose that (a) or (b) hold and $\mathcal{V}$ is not Riemannian. Let $V \in \Gamma(\mathcal{V})$ be a homothetic vector field which is not Killing. (Such a vector field can be found in the neighbourhood of each point of M because $\mathcal{V}$ is locally generated by homothetic vector fields and $\mathcal{W} \neq \mathcal{V}$.) Note that, for any Killing field $W \in \Gamma(\mathcal{W})$ we have that $[V, W]$ is also Killing and hence $[V, W] \in \Gamma(\mathcal{W})$. Using this fact together with Theorem 2.2.6 and Corollary 2.5.2 it is not difficult to see that the assertions (i) and (ii) are equivalent. $\square$

Remark 2.5.7. 1) If $\mathcal{V}$ is homothetic then the leaves of $\mathcal{F}$ are level hypersurfaces of the local dilations of $\mathcal{V}$ which are constant along horizontal curves.

2) In Theorem 2.5.6 if (a) or (b) hold and $\text{codim } \mathcal{V} = 1$ then (i) $\iff$ (ii).

Let G be a Lie group which acts to the right by homotheties on (M, g) and for $a \in G$ let $\rho(a) \in (0, \infty)$ be the conformal factor of the homothetic transformation

induced by $a \in G$ on (M, g) . Then it is easy to see that $\rho : G \rightarrow (0, \infty)$ is a morphism of Lie groups (hence, if ρ is nonconstant, G is isomorphic to a semi-direct product of $\ker \rho$ and $((0, \infty), \cdot)$). In particular, if G is compact then ρ is constant. Nevertheless, if G is compact, then there might exist *local* morphisms of Lie groups $G \rightarrow (0, \infty)$ (see Example 2.5.8(3), below) which can be used to construct homothetic *local* actions.

Here are a few well-known examples of morphisms of Lie groups $\rho : G \rightarrow (0, \infty)$.

Example 2.5.8. 1) For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ define $\rho : GL_n(\mathbb{K}) \rightarrow (0, \infty)$ by $\rho(a) = |\det a|$.

2) For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ define $\rho : CO_n(\mathbb{K}) \rightarrow (0, \infty)$ by $|Au| = \rho(A)|u|$ for $u \in \mathbb{K}^n$ and $A \in CO_n(\mathbb{K})$.

3) The canonical morphisms $\det : U_n \rightarrow S^1$ and $Spin_n^c = Spin_n \times_{\mathbb{Z}_2} S^1 \rightarrow S^1$, $[a, z] \mapsto z^2$, when composed with the exponential of $\arg : S^1 \setminus \{-1\} \rightarrow (-\pi, \pi)$ induce *local* morphisms of Lie groups $U_n \rightarrow (0, \infty)$ and $Spin_n^c \rightarrow (0, \infty)$, respectively.

From now on we shall suppose that G acts *freely* on M . In this case there exists a natural isomorphism of vector bundles $\mathcal{V} = M \times \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of G .

Hence $\mathcal{H}I$ can be viewed as a $\mathfrak{g}$ -valued two form on M which has properties similar to the properties of the curvature form of a principal connection (in particular, $R_a^*(\mathcal{H}I) = \text{Ad } a^{-1} \cdot \mathcal{H}I$ where R_a is the transformation induced by $a \in G$ on M).

Also ρ_* can be viewed as a vertical one form on M . Moreover, we have that $\rho_* = \mu$ (see Remark 2.5.3 for the definition of μ).

It follows from Corollary 2.5.2 that the foliation induced by the free action of G on (M, g) is homothetic if and only if $\rho_*(\mathcal{H}I) = 0$.

By identifying G with an orbit we can induce on it a metric which we shall denote by γ . Then it is easy to see that $\rho^{-2}\gamma$ is right invariant.

Suppose that ρ is nonconstant and let $\mathcal{V}$ be the foliation on G formed by the components of the fibres of ρ . This is generated by the action of the normal

subgroup $H = \ker \rho$. Then it is obvious that H acts by isometries on (G, γ) and hence $\mathcal{V}$ is a Riemannian foliation on it.

Also, $\mathcal{H} (= \mathcal{V}^\perp)$ is a (one-dimensional) homothetic foliation with geodesic leaves for which ρ^{-1} is a global dilation.

Thus both $\mathcal{V}$ and $\mathcal{H}$ produce harmonic morphisms and, in particular, ρ induces a harmonic function on (G, γ) (which gives another argument for the fact that if G is compact then ρ cannot be globally defined unless it is constant).

Example 2.5.9. Let G and ρ be as in Example 2.5.8(1) or (2) and define $h = |dx \cdot x^{-1}|^2$. Then $g = \rho^2 h$ has all the above properties.

Next we show that the results of Theorem 2.5.6 takes a more concrete form in the case of homothetic free actions.

Proposition 2.5.10. *Let G be a connected Lie group which acts freely to the right by homotheties on (M, g) and let $\mathcal{V}^G$ be the induced foliation. Let $\rho : G \rightarrow (0, \infty)$ be the corresponding morphism of Lie groups and let $H = \ker \rho$.*

Then the following assertions are equivalent:

- (i) $\mathcal{V}^G$ is a homothetic foliation;
- (ii) *there exists a hypersurface N of M such that H acts by isometries on $(N, g|_N)$ to generate a Riemannian foliation $\mathcal{V}^H$ and such that $M = N \times_H G$.*

Further, if (i) or (ii) hold and $2 \leq \dim G \leq \dim M - 3$, then $\mathcal{V}^G$ produces harmonic morphisms if and only if $\mathcal{V}^H$ produces harmonic morphisms.

Proof. Let $\mathcal{H}$ be the orthogonal complement of $\mathcal{V}^G$ and let $\mathcal{W}$ be the foliation induced by the *isometric* (free) action of H on M . Then, assertion (i) is equivalent to the fact that $\rho_*(\mathcal{H}) = 0$ which, since $\mathcal{H}$ is the integrability tensor of $\mathcal{H}$, is equivalent to the fact that the distribution $\mathcal{F} = \mathcal{W} \oplus \mathcal{H}$ is integrable.

Suppose that (i) holds and let N be a leaf of $\mathcal{F}$. Then $\mathcal{V}^H = \mathcal{W}|_N$ and the implication (i) $\Rightarrow$ (ii) follows.

The implication (i) $\Leftarrow$ (ii) is now obvious.

The last assertion follows from the fact that if (i) holds then $\mathcal{H}$ is $\mathfrak{h}$ -valued and $R_a^*(\mathcal{H}) = \text{Ad } a^{-1} \cdot \mathcal{H}$. $\square$

Remark 2.5.11. In Proposition 2.5.10 we also have that if (i), or (ii), holds then g is determined by ρ and the induced metric h on N .

To see this recall that we have considered the metric γ induced on G by identifying it with an orbit. Suppose that this identification and N were chosen such that the identity element of G is contained in N . Then H acts by isometries on $(N \times G, \pi_N^*(h) + \pi_G^*(\gamma))$ (where $\pi_N : N \times G \rightarrow N$ and $\pi_G : N \times G \rightarrow G$ are the canonical projections) and (M, g) is the induced isometric quotient.

Example 2.5.12. Let $\rho : G \rightarrow (0, \infty)$ be as in Example 2.5.8 and let $H = \ker \rho$. Then $\text{trace}(\text{ad } \mathfrak{h}) = 0$ (here, as above, $\mathfrak{h}$ is the Lie algebra of H).

Let (Q, M, H) be a principal bundle whose total space is endowed with a Riemannian metric h such that H acts by isometries on (Q, h) . (Obviously, any such h corresponds to a triple $(\gamma, \mathcal{H}, k)$ where γ is a Riemannian metric on the vector bundle $\text{Ad } Q \rightarrow M$, $\mathcal{H}$ is a principal connection on (Q, M, H) and k is a Riemannian metric on M .)

From Remark 2.5.11 it follows that a metric (and just a local metric, for ρ from Example 2.5.8(3)) can be found on $P = Q \times_H G$ with respect to which the foliation induced by G is homothetic (but not Riemannian) and produces harmonic morphisms.

2.6 Conformal actions and harmonic morphisms on Einstein manifolds

In this section we study foliations which are locally generated by conformal vector fields and produce harmonic morphisms on Einstein manifolds. Note that, as before, no compactness or completeness assumptions are made. The main results of this section are the following:

Theorem 2.6.1. *Let (M^m, g) , $m \geq 3$ be an Einstein manifold (${}^M\text{Ricci} = c g$, $c \in \mathbb{R}$). Let $\mathcal{V}$ be a foliation with $\text{codim } \mathcal{V} \neq 2$ which is locally generated by conformal vector fields.*

Suppose that $\mathcal{V}$ produces harmonic morphisms on (M^m, g) . Then either $\mathcal{V}$ is

Riemannian and locally generated by Killing fields or the set of points where $\mathcal{V}$ is Riemannian has empty interior. Moreover, we have the following:

(i) If $c > 0$ then either $\mathcal{V}$ is Riemannian and locally generated by Killing vector fields or any harmonic morphism produced by $\mathcal{V}$ can be locally decomposed into a harmonic morphism with geodesic fibres and integrable horizontal distribution followed by another harmonic morphism.

(ii) If $c < 0$ then, at least outside the points where $\mathcal{V}$ is Riemannian, any harmonic morphism produced by $\mathcal{V}$ can be locally decomposed into a harmonic morphism with geodesic fibres and integrable horizontal distribution followed by another harmonic morphism.

(iii) If $c = 0$ then either $\mathcal{V}$ is locally generated by homothetic vector fields or any harmonic morphism produced by $\mathcal{V}$ can be locally decomposed into a harmonic morphism with geodesic fibres, constant dilation and integrable horizontal distribution followed by another harmonic morphism.

Corollary 2.6.2. *Let (M^m, g) , $m \geq 3$ be an Einstein manifold (${}^M\text{Ricci} = cg$) and let $\mathcal{V}$ be a foliation on it with $\text{codim } \mathcal{V} \neq 2$ which is locally generated by conformal vector fields.*

Suppose that $\mathcal{V}$ produces harmonic morphisms on (M^m, g) .

(i) *If $c > 0$ then any harmonic morphism produced by $\mathcal{V}$ can be locally decomposed into two harmonic morphisms in which the first one either has geodesic fibres and integrable horizontal distribution or is induced by an isometric quotient.*

(ii) *If $c < 0$ then at least outside a set with empty interior any harmonic morphism produced by $\mathcal{V}$ can be locally decomposed into two harmonic morphisms in which the first one either has geodesic fibres and integrable horizontal distribution or is induced by an isometric quotient.*

Remark 2.6.3. We shall see (Corollary 3.2.6) that when $\dim \mathcal{V} = 1$ the assertion (i) above holds for $c \in \mathbb{R}$.

Theorem 2.6.4. *Let (M^m, g) be a Ricci-flat Riemannian manifold and let $\mathcal{V}$ be a homothetic foliation on it with $\dim \mathcal{V} \geq 2$, $\text{codim } \mathcal{V} \geq 3$ which is locally generated by conformal vector fields.*

Suppose that $\mathcal{V}$ produces harmonic morphisms on (M^m, g) . Then one of the

following assertions holds.

- (a) $\mathcal{V}$ is Riemannian and locally generated by Killing fields;
- (b) there exists a Riemannian foliation $\mathcal{W} \subseteq \mathcal{V}$, $\dim \mathcal{W} = \dim \mathcal{V} - 1$, locally generated by Killing fields such that $\mathcal{F} = \mathcal{W} \oplus \mathcal{H}$ is integrable and the restriction of $\mathcal{W}$ to any leaf of $\mathcal{F}$ produces harmonic morphisms;
- (c) any harmonic morphism produced by $\mathcal{V}$ can be locally decomposed into two harmonic morphisms in which the first one has geodesic fibres, constant dilation and integrable horizontal distribution.

Corollary 2.6.5. *Let (M^m, g) be a Ricci-flat Riemannian manifold and let $\mathcal{V}$ be a foliation on it with $\dim \mathcal{V} \geq 2$, $\text{codim } \mathcal{V} \geq 3$ which is locally generated by conformal vector fields and with integrable orthogonal complement, $\dim \mathcal{V} \geq 2$, $\text{codim } \mathcal{V} \geq 3$.*

Suppose that $\mathcal{V}$ produces harmonic morphisms on (M^m, g) . Then one of the following assertions holds.

- (a) $\mathcal{V}$ is Riemannian and locally generated by Killing fields;
- (b) there exists a Riemannian foliation $\mathcal{W} \subseteq \mathcal{V}$, $\dim \mathcal{W} = \dim \mathcal{V} - 1$, locally generated by Killing fields such that $\mathcal{F} = \mathcal{W} \oplus \mathcal{H}$ is integrable and the restriction of $\mathcal{W}$ to any leaf of $\mathcal{F}$ produces harmonic morphisms;
- (c) any harmonic morphism produced by $\mathcal{V}$ can be locally decomposed into two harmonic morphisms in which the first one has geodesic fibres, constant dilation and integrable horizontal distribution.

The proofs of the above results are based on results obtained in the previous sections. We also need a few lemmas some of which are well-known.

In the proof of the next simple lemma we use a result (Proposition 3.1.5) which we shall prove in Chapter 3. Also, note that Lemma 2.6.6 reformulates, in terms of homothetic foliations, a well-known fact.

Lemma 2.6.6 (cf. [38]). *Let $\mathcal{V}$ be a one-dimensional foliation on a Riemannian manifold (M, g) . Then the following assertions are equivalent.*

- (i) $\mathcal{V}$ is a homothetic foliation with geodesic leaves and integrable orthogonal complement;

(ii) $\mathcal{V}$ is locally generated by (nowhere zero) conformal vector fields $V \in \Gamma(\mathcal{V})$ such that $dV^\flat = 0$.

Proof. A vector field $V \in \Gamma(TM)$ is conformal if and only if $\mathcal{L}_V g = 2\mu g$ for some function μ ; then $dV^\flat = 0$ if and only if $\nabla V = \mu Id_{TM}$. Now it is obvious that any such V which is nowhere zero generates a conformal foliation with geodesic leaves and integrable orthogonal complement. Moreover, $|V|^{-1}$ is a local dilation for it whose gradient is tangent to the leaves and thus V generates a homothetic foliation.

Conversely, if $\mathcal{V}$ satisfies (i) then it produces harmonic morphisms. But $\mathcal{V}$ is homothetic and hence $\mathcal{V}$ is locally generated by conformal vector fields (see Proposition 3.1.5, below). Now, if $V \in \Gamma(\mathcal{V})$ is conformal and nowhere zero and $X \in \Gamma(\mathcal{H})$ is basic then by applying Lemma 2.1.1 we obtain

$$(dV^\flat)(V, X) = 2g(X, \nabla_V V) = 2g(X, \nabla_U U)g(V, V) = 0 \quad (2.6.1)$$

where $U = \frac{1}{|V|} V$. Also, from Proposition 2.1.4 we obtain

$$(dV^\flat)(X, Y) = g(\text{trace}({}^{\mathcal{H}}B), {}^{\mathcal{H}}I(X, Y)) = 0. \quad (2.6.2)$$

From (2.6.1) and (2.6.2) it follows that $dV^\flat = 0$ and the lemma is proved. $\square$

The following lemma is well-known.

Lemma 2.6.7 (see [67]). *Let V be a conformal vector field on an Einstein manifold (M^m, g) ; write $\mathcal{L}_V g = 2\sigma g$, ${}^M\text{Ricci} = c g$. Then*

$$\nabla d\sigma = -\frac{c}{m-1} \sigma g. \quad (2.6.3)$$

Hence

$$\Delta \sigma = \frac{cm}{m-1} \sigma. \quad (2.6.4)$$

Proof. Formula (2.6.3) follows after a straightforward but tedious computation (see [67]). $\square$

The following simple lemma is well-known.

Lemma 2.6.8. *Let f be a smooth function on a Riemannian manifold (M, g) such that $\nabla df = -kf g$ for some constant $k \in \mathbb{R}$. Then*

$$kf^2 + |df|^2 = \text{constant}.$$

Proof. Simply compute the differential $d(kf^2 + |df|^2)$. □

The following simple lemma seems to be less well-known.

Lemma 2.6.9. *Let $\mathcal{V}$ be a foliation ($\text{codim } \mathcal{V} > 0$) on a Riemannian manifold (M, g) and let $V \in \Gamma(\mathcal{V})$ be such that $\nabla V^\flat = \mu g$ for some smooth function μ on M .*

If for some $x \in M$ we have that $V_x = 0$ then $\mu(x) = 0$.

Proof. Let $X \in \Gamma(\mathcal{V}^\perp)$ be a basic vector field. Then $[V, X] \in \Gamma(\mathcal{V})$. But

$$[V, X] = \nabla_V X - \nabla_X V = \nabla_V X - \mu X.$$

Thus if $V_x = 0$ then $[V, X]_x = -\mu(x)X_x \in \mathcal{V}_x$ and hence $\mu(x) = 0$. □

The following simple lemma is an immediate consequence of Lemma 2.6.8 and Lemma 2.6.9.

Lemma 2.6.10. *Let $\mathcal{V}$ be a foliation on a Riemannian manifold (M, g) and let $\text{grad } f \in \Gamma(\mathcal{V})$ be such that $\nabla df = -kf g$ for some nonnegative constant $k \geq 0$.*

If for some $x \in M$ we have that $(\text{grad } f)_x = 0$ then $\text{grad } f = 0$. Moreover, if $k > 0$ then $f = 0$.

Proof. If $k = 0$ then $|\text{grad } f| = \text{constant}$.

If $k > 0$ and for some $x \in M$ we have that $(\text{grad } f)_x = 0$ then by Lemma 2.6.9 we have $f(x) = 0$. The proof now follows from Lemma 2.6.8. □

Proof of Theorem 2.6.1. Let $\mathcal{H}$ be the orthogonal complement of $\mathcal{V}$. Let $V \in \Gamma(\mathcal{V})$ be a conformal vector field. Then at least locally we can write $\mathcal{L}_V g = -2V(\log \lambda)g$ for some local dilation λ of $\mathcal{V}$. By Lemma 2.1.1 we have that V is an infinitesimal automorphism of $\mathcal{H}$.

Because $\mathcal{V}$ produces harmonic morphisms from Theorem 2.2.6 it follows that

the mean curvature form of $\mathcal{V}$ is basic. Applying Proposition 2.1.16 with the roles of $\mathcal{V}$ and $\mathcal{H}$ reversed we obtain that, for any basic vector field $X \in \Gamma(\mathcal{H})$, we have that $X(V(\log \lambda)) = 0$. It follows that $\text{grad}(V(\log \lambda)) \in \Gamma(\mathcal{V})$.

Now recall that (M, g) is an Einstein manifold and thus it is an analytic manifold (see [11]). From the regularity of solutions for elliptic operators and (2.6.4) it follows that $V(\log \lambda)$ is an analytic function. Hence either $\mathcal{V}$ is Riemannian or the interior of the set where $\mathcal{V}$ is Riemannian is empty.

From Lemma 2.6.6 and Lemma 2.6.7 it follows that if $\text{grad}(V(\log \lambda))$ is nowhere zero then it generates a one-dimensional homothetic foliation $\mathcal{F}$ with geodesic leaves and integrable orthogonal complement. Moreover $\mathcal{F} \subseteq \mathcal{V}$. Also, note that, if $c = 0$, then $\text{grad}(V(\log \lambda))$ is a parallel vector field.

Let $x \in M$ and suppose that for any conformal vector field $V \in \Gamma(\mathcal{V})$ we have $\text{grad}(V(\log \lambda))_x = 0$.

If $c \neq 0$, then from Lemma 2.6.9 it follows that $\mathcal{V}$ is Riemannian at x . This establishes assertion (ii).

If $c > 0$ then from Lemma 2.6.10 it follows that $\mathcal{V}$ is Riemannian in a neighbourhood of x and this establishes assertion (i).

If $c = 0$, let $\mathcal{U}$ be a locally finite open covering of M such that each $U \in \mathcal{U}$ is connected and there exists a local frame $\{V_r^U\}_{r=1, \dots, \dim \mathcal{V}}$ for $\mathcal{V}$, over U , made up of conformal vector fields: $\mathcal{L}_{V_r^U} g = \sigma_r^U g$, $r = 1, \dots, \dim \mathcal{V}$. Let F be the set of points $x \in M$ at which $(d\sigma_r^U)_x = 0$, $r = 1, \dots, \dim \mathcal{V}$, for all $U \in \mathcal{U}$ with $x \in U$. From Lemma 2.6.10 it follows that if $U \in \mathcal{U}$ is such that $F \cap U \neq \emptyset$ then V_r^U are homothetic vector fields. Since $\mathcal{U}$ is locally finite, this implies that F is open and closed which, because M is connected, establishes assertion (iii). $\square$

Proof of Corollary 2.6.2. This follows from assertions (i) and (ii) of Theorem 2.6.1. $\square$

Proof of Theorem 2.6.4. This follows from Theorem 2.5.6 and Theorem 2.6.1. $\square$

Proof of Corollary 2.6.5. This follows from Proposition 2.4.2 and Theorem 2.6.4. $\square$

Chapter 3

Harmonic morphisms with one-dimensional fibres

3.1 Some basic facts on harmonic morphisms with one-dimensional fibres

In this section we present, for later use, a few facts about one-dimensional foliations which produce harmonic morphisms. Here, $\mathcal{V}$ will always denote a one-dimensional foliation.

The following lemma will be used several times in this chapter. The case $n = 2$ was used by P. Baird and J.C. Wood in [8, §3].

Lemma 3.1.1. *Let $\mathcal{V}$ be a conformal one-dimensional foliation on (M^{n+1}, g) . Then, the following assertions are equivalent.*

- (i) $\mathcal{V}$ produces harmonic morphisms;
- (ii) *each point has a neighbourhood on which a local dilation λ of $\mathcal{V}$ can be found such that, if V is a vertical field with $g(V, V) = \lambda^{2n-4}$, then $[V, X] = 0$ for any basic field X .*

Proof. From (1.1.5) it follows that assertion (i) is equivalent to the possibility of finding in the neighbourhood of each point a local dilation λ of $\mathcal{V}$ such that

$$\mathcal{V}^*(\mathcal{L}_X(\lambda^{2-n}\omega)) = 0 \tag{3.1.1}$$

for any basic vector field X and where ω is a local volume of $\mathcal{V}$.

If V is as in (ii) and θ is its dual vertical one-form (i.e. θ is the unique vertical one-form such that $\theta(V) = 1$) then $\lambda^{n-2}\theta$ is a local volume form of $\mathcal{V}$. Hence (3.1.1) is equivalent to $(\mathcal{L}_X \theta)(V) = 0$ which is equivalent to $[V, X] = 0$. $\square$

Remark 3.1.2. 1) From the proof above we see that (ii) is a characterisation of those local dilations which restrict to give dilations of harmonic morphisms which locally define the foliation.

2) If V is as above, let θ be its dual vertical one-form. Using the fact that $[V, X] = 0$ for any basic vector field X , it follows that the two-form $\Omega = d\theta$ is basic. (In fact, θ and $\Omega = d\theta$ are, respectively, the connection form and the curvature form of a principal (local) connection, see Theorem 3.1.9).

The equivalence (iii) $\iff$ (i) from the following proposition is due to R.L. Bryant [13].

Proposition 3.1.3. *For $n \neq 2$, let $\mathcal{V}$ be a one-dimensional Riemannian foliation on (M^{n+1}, g) and let $\mathcal{H}$ be its orthogonal complement. Then, the following assertions are equivalent:*

- (i) $\mathcal{V}$ produces harmonic morphisms,
- (ii) $\mathcal{H}$ is a homothetic distribution,
- (iii) $\mathcal{V}$ is locally generated by Killing fields.

Furthermore, if $\mathcal{V}$ is orientable and the first Betti number of M is zero then (iii) above can be replaced by

- (iii') $\mathcal{V}$ is globally generated by a Killing field.

Proof. (i) $\iff$ (ii) This follows from Proposition 1.4.7, since, being Riemannian, $\mathcal{V}$ is homothetic and, being of codimension one, $\mathcal{H}$ is conformal.

(ii) $\implies$ (iii) Let ρ be a local dilation of $\mathcal{H}$ which is constant along the leaves of $\mathcal{V}$ and let V be a local vertical field such that $g(V, V) = \rho^{-2}$.

Because ρ is constant along the leaves of $\mathcal{V}$ we have

$$(\mathcal{L}_V g)(V, V) = 0. \quad (3.1.2)$$

Because ρ is a local dilation of $\mathcal{H}$ we have

$$(\mathcal{L}_X(\rho^2 g))(V, V) = 0 \quad (3.1.3)$$

for any horizontal vector field X . It is easy to see that (3.1.3) is equivalent to $g([X, V], V) = 0$. This implies that for any horizontal vector field we have

$$(\mathcal{L}_V g)(V, X) = 0. \quad (3.1.4)$$

Since $\mathcal{V}$ is Riemannian we have

$$(\mathcal{L}_V g)(X, Y) = 0, \quad (3.1.5)$$

for any horizontal vector fields X and Y .

Equations (3.1.2), (3.1.4) and (3.1.5) show that V is a Killing field.

(iii) $\Rightarrow$ (ii) Since $\dim \mathcal{V} = 1$, the orthogonal complement $\mathcal{H}$ of $\mathcal{V}$ is a conformal distribution.

If V is a (local) nonvanishing Killing field, which (locally) generates $\mathcal{V}$, and $|V|$ its norm then $|V|^{-1}$ is a local dilation for the horizontal distribution $\mathcal{H}$. Moreover, $\mathcal{H}$ is homothetic, since $|V|$ is constant along the leaves of $\mathcal{V}$.

The last assertion follows from the fact that when the first Betti number of M is zero and $\mathcal{V}$ is orientable we can find a global density λ^{2-n} of $\mathcal{V}$ (which is also a local dilation for $\mathcal{H}$) and a vertical vector field V defined on M , such that $g(V, V) = \lambda^{2n-4}$. $\square$

Remark 3.1.4. 1) Note that if $n = 2$ then (i) $\Rightarrow$ (ii) $\iff$ (iii). In fact, in this case, a one-dimensional foliation $\mathcal{V}$ produces harmonic morphisms on (M^3, g) if and only if its leaves are geodesics (see [8]). Thus, being of codimension one, $\mathcal{H}$ is a Riemannian distribution. However, if $n = 2$ then (ii) $\Rightarrow$ (i) fails, as simple examples show.

2) If in the above proposition we further assume that $\mathcal{H}$ is integrable then $\mathcal{V}$ induces, locally, a warped product structure on (M, g) .

One might guess that a similar proposition to the one above holds in general for any conformal one-dimensional foliation, just by replacing ‘Killing fields’, with ‘conformal fields’. It is not difficult to see that this is not true, the actual situation being described by the following:

Proposition 3.1.5. *For $n \geq 3$, let $\mathcal{V}$ be a one-dimensional foliation on (M^{n+1}, g) . Then any two of the following assertions imply the remaining assertion.*

- (i) $\mathcal{V}$ produces harmonic morphisms,
- (ii) $\mathcal{V}$ (or $\mathcal{H}$) is homothetic,
- (iii) $\mathcal{V}$ is locally generated by conformal vector fields.

Furthermore, if $\mathcal{V}$ is orientable and the first Betti number of M is zero then

- (iii) above can be replaced by
- (iii') $\mathcal{V}$ is (globally) generated by a conformal field.

Proof. (i),(ii) $\Rightarrow$ (iii) Let λ^{2-n} be a local density for $\mathcal{V}$. By Proposition 1.4.2, we can suppose that $\lambda = a b$, where a is constant along leaves and b is constant along horizontal curves.

Let W be a local vertical vector field such that $g(W, W) = a^{2n-4} b^{-2}$. It is a straightforward calculation to check that W is a local conformal vector field on (M, g) .

(ii),(iii) $\Rightarrow$ (i) Since $\mathcal{V}$ is homothetic, by Proposition 1.4.2, we can find a local dilation b of $\mathcal{V}$ which is constant along horizontal curves.

Let W be a local conformal vector field which (locally) generates $\mathcal{V}$. We can suppose that b and W are defined on the same open subset of M . It is easy to see that, since W is conformal, we have that $b^2 g(W, W)$ is constant along leaves.

We can choose a smooth positive local function a on M such that $g(W, W) = a^{2n-4} b^{-2}$. Hence a is constant along the leaves and thus $\lambda = a b$ is a local dilation of $\mathcal{V}$.

If V is a local field, tangent to the leaves and such that $g(V, V) = \lambda^{2n-4}$ then, from the fact that W is conformal it follows that $[V, X] = 0$ for any basic X . Hence, by Lemma 3.1.1, $\mathcal{V}$ is a foliation which produces harmonic morphisms.

(iii),(i) $\Rightarrow$ (ii) Let λ^{2-n} be a local density for $\mathcal{V}$. Let V be a local vector field, tangent to the leaves and such that $g(V, V) = \lambda^{2n-4}$, and let W be a local conformal vector field tangent to the leaves. We can suppose that V and W are defined on the same open set.

Since W is conformal, for any basic X we have $(\mathcal{L}_W g)(W, X) = 0$, and hence, $[W, X] = 0$. But, by Lemma 3.1.1 we also have $[V, X] = 0$ for any basic X . Hence if b is such that $W = bV$, then b is constant on horizontal curves.

Since λ is a local dilation of the conformal foliation $\mathcal{V}$, from Remark 1.1.9,

we see that

$$(\mathcal{L}_W g)(V, V) = W(\log(\lambda^{-2}))g(V, V) . \quad (3.1.6)$$

Relation (3.1.6) together with $W = bV$ implies after a straightforward calculation that $\lambda^{n-1} b$ is constant along leaves. Thus, we can write $\lambda = r s$ where r, s are positive smooth functions on M such that r is constant along the leaves and s is constant along horizontal curves. From Proposition 1.4.2, we get that $\mathcal{V}$ is a homothetic foliation. $\square$

Remark 3.1.6. Note that, if in Proposition 3.1.5 we have $n = 2$, the implication (ii),(iii) $\Rightarrow$ (i) fails, the other implications still holding.

If $n = 1$, then (i) $\Longleftrightarrow$ (iii) but they do not imply (ii).

Lemma 3.1.7 (cf. [8, Remark 5.3]). *Let $\mathcal{V}$ be a one-dimensional homothetic foliation on (M, g) . Then, at least away of the points where $\mathcal{V}$ is Riemannian, its orthogonal complement is integrable.*

Proof. By Proposition 1.4.2, $\mathcal{V}$ admits a local dilation λ whose gradient is vertical. The points $x \in M$, where $\mathcal{V}$ is not Riemannian are characterised by $(\text{grad } \lambda)_x \neq 0$. Hence, in a neighbourhood of such a point, the level hypersurfaces of λ are integral submanifolds of the horizontal distribution. $\square$

Lemma 3.1.8. *Let $\varphi : (M^{n+1}, g) \rightarrow (N^n, h)$ be a harmonic morphism with one-dimensional fibres. Let λ denote the dilation of φ and let V be a (local) vertical vector field on M such that $g(V, V) = \lambda^{2n-4}$.*

Then, the following assertions are equivalent:

- (i) *the fibres of φ form a homothetic foliation at least on the complement of the interior of the set $\{x \in M \mid d(V(\log \lambda))(x) = 0 \neq (V(\log \lambda))(x)\}$;*
- (ii) *for any basic field X , we have $V(X(\log \lambda)) = 0$.*

Proof. Let μ be the vertical one-form on M such that for any horizontal fields X, Y we have $(\mathcal{L}_V g)(X, Y) = \mu(V)g(X, Y)$. Hence, by the definition of λ we have $\mu(V) = -2V(\log \lambda)$.

Let $\mathcal{H}$ be the horizontal distribution and ${}^{\mathcal{H}}B$ its second fundamental form. Using (1.1.3) we obtain the following relation:

$$\mu = -\frac{2}{n}(\text{trace}({}^{\mathcal{H}}B))^{\flat} .$$

Hence, $\mathcal{V}$ is homothetic if and only if μ is closed.

By Lemma 3.1.1, for any basic X we have $[V, X] = 0$, and hence:

$$\begin{aligned} (d\mu)(V, X) &= -V(\mu(X)) - X(\mu(V)) - \mu[V, X] = -X(\mu(V)) \\ &= 2X(V(\log \lambda)) = 2V(X(\log \lambda)) . \end{aligned}$$

The lemma follows. □

In [8, Proposition 3.5], P. Baird and J.C. Wood gave a *global* description of the metric of a Riemannian manifold of dimension three, on which a harmonic morphism can be defined. In [13, Theorem 1], R.L. Bryant gave a *local* description of the metric of the total space of a submersive harmonic morphism with one-dimensional fibres (with no restriction on the dimension of the total space). The following theorem explains how the latter result can be globalized, giving also a simpler proof of Bryant's local result.

Theorem 3.1.9 (cf. [13, Theorem 1]). *Let $\varphi : (M^{n+1}, g) \rightarrow (N^n, h)$, $n \geq 1$, be a submersive harmonic morphism with connected one-dimensional fibres of the same homotopy type. Let λ be the dilation of φ and suppose that $\mathcal{V} (= \ker \varphi_*)$ is orientable.*

Then, there exists:

- (i) *a principal bundle $\pi : P \rightarrow N$ with group $G = (\mathbb{R}, +)$ or $G = (S^1, \cdot)$,*
- (ii) *a principal connection $\theta \in \Gamma(T^*P)$ on π ,*
- (iii) *a diffeomorphic embedding $\iota : M \rightarrow P$*

such that:

- 1) $\pi \circ \iota = \varphi$,
- 2) $g = \lambda^{-2}(\varphi^*h) + \lambda^{2n-4}(\iota^*\theta)^2$.

Furthermore, if the fibres are all diffeomorphic to circles, or are all complete with respect to the metric induced by ${}^\lambda g$, then ι is onto, and hence, φ itself is a principal bundle and the horizontal distribution is a principal connection on it.

Note that, by the result of P. Baird [13], we know that φ is automatically submersive except when $n \leq 3$.

Proof. Let V be a vertical field such that $g(V, V) = \lambda^{2n-4}$. By Lemma 3.1.1, the horizontal distribution $\mathcal{H}$ is invariant under the local flow of V . Thus, the integral curves of V are the fibres of a local principal bundle, and $\mathcal{H}$ is a principal connection on it. If θ is the vertical one-form dual to V then, it is obvious that $\lambda g = \varphi^* h + \theta^2$. (This establishes [13, Theorem 1].)

To end the proof we shall prove the following assertions:

- (a) if the fibres are diffeomorphic to circles then φ is a principal bundle with group $(S^1, \cdot)$;
- (b) if the fibres are diffeomorphic to $\mathbb{R}$ then, there exists a diffeomorphic embedding $\iota : M \rightarrow N \times \mathbb{R}$, such that $\pi_1 \circ \iota = \varphi$, and a principal connection on the trivial principal bundle $\pi_1 : N \times \mathbb{R} \rightarrow N$, with group $(\mathbb{R}, +)$, such that $\mathcal{H}$ is the restriction to M of it.

From now on, all the considerations which will be made in this proof will be done with respect to the metric λg on M .

For $x \in M$, let $I_x \subseteq \mathbb{R}$ be the open interval which is the domain of the (maximal) geodesic with velocity V_x . Let $Q = \{(x, r) \in M \times \mathbb{R} \mid r \in I_x\}$, and define $\Psi : Q \rightarrow M$, by $\Psi(x, r) = \exp r V_x$.

If the fibres are all circles then $Q = M \times \mathbb{R}$. Since $d\theta(V, X) = 0$ for any horizontal vector field X , by applying the Stokes theorem we obtain that the fibres have the same length. Hence Ψ descends to a map $M \times S^1 \rightarrow M$ which is a free action of $(S^1, \cdot)$ on M . Thus assertion (a) is proved.

Suppose now that the fibres are diffeomorphic to $\mathbb{R}$. If they are all complete with respect to the metric induced from λg then, $Q = M \times \mathbb{R}$ and Ψ represents a free action of $(\mathbb{R}, +)$ on M , and thus the proof of the theorem is finished. Otherwise, since φ is a submersion, we can find local sections of it in the neighbourhood of each point of N . Let $\mathcal{S}$ be a family of such sections whose domains form an open covering $\{O_s\}_{s \in \mathcal{S}}$ of N .

Let $s, t \in \mathcal{S}$. For $x \in O_s \cap O_t$, let $a_{st}(x)$ be the (unique) real number such that $t(x) = \Psi(s(x), a_{st}(x))$.

It is obvious that $\{a_{st}\}_{s, t \in \mathcal{S}}$ is a cocycle with values in $(\mathbb{R}, +)$, which induces a principal bundle. This bundle is trivial because $\mathbb{R}$ is contractible.

Moreover, set $A_s = s^* \theta$; then the family of one-forms $\{A_s\}_{s \in \mathcal{S}}$, defines a prin-

cial connection on this bundle.

The total space $N \times \mathbb{R}$ of this bundle, can be retrieved, as usual from the cocycle $\{a_{st}\}_{s,t \in \mathcal{S}}$ as the space of equivalence classes $[x, r]$, under the identifications $[x, r] \equiv [x, a_{st}(x)r]$, $x \in O_s \cap O_t$.

For $x \in M$, let $s \in \mathcal{S}$ be such that $\varphi(x) \in O_s$ and, $r_x \in I_{s(\varphi(x))}$ be the real number which satisfies $x = \Psi(s(\varphi(x)), r_x)$. (Note that r_x depends just on x and s .) We can define $\iota : M \rightarrow N \times \mathbb{R}$ by $\iota(x) = [\varphi(x), r_x]$, $x \in M$ and the theorem follows. $\square$

Remark 3.1.10. The proof of above theorem, can be simplified considerably when $\mathcal{H}$ is an Ehresmann connection (see [12] for the definition of Ehresmann connection). It is not difficult to prove that a sufficient condition for $\mathcal{H}$ to be an Ehresmann connection is that V be a complete vector field.

3.2 Some local and global results for one-dimensional foliations which produce harmonic morphisms

All of the main results of this section hold for Riemannian manifolds of dimension at least four. None of the results of this section requires the compactness or the completeness of the manifold.

If the manifold has dimension at least four then by Proposition 3.1.3 a one-dimensional Riemannian foliation produces harmonic morphisms on it if and only if it is locally generated by Killing fields, and from Corollary 1.4.8 it follows that a foliation by geodesics produces harmonic morphisms if and only if it is homothetic.

The first main result of this section concerns a one-dimensional foliation $\mathcal{V}$ which has integrable orthogonal complement and which produces harmonic morphisms on an Einstein manifold. We prove that $\mathcal{V}$ is one of these two types. Also, we prove that this still holds if we replace the integrability assumption on the orthogonal complement by the condition that the foliation be homothetic.

In [13, Theorem 3] R.L. Bryant considered a submersive harmonic morphism φ with connected one-dimensional fibres defined on a simply-connected Riemannian

nian manifold of dimension at least four and with constant sectional curvature. Bryant's result is that φ is one of the following two types: either there exists a nowhere zero Killing field tangent to the fibres of φ or the fibres are geodesics orthogonal to an umbilical foliation by hypersurfaces. We improve this result by showing that, on a Riemannian manifold with constant sectional curvature any orientable one-dimensional foliation which produces harmonic morphisms and admits a global density is either Riemannian and has leaves generated by a nonvanishing Killing field or is homothetic and has leaves which are geodesics orthogonal to an umbilical foliation by hypersurfaces. In this way an entirely new proof for Bryant's result is obtained.

In this section $\mathcal{V}$ will always denote a one-dimensional foliation which produces harmonic morphisms on (M^{n+1}, g) ($n \geq 1$) and $\rho = e^{(2-n)\sigma}$ will denote a local density of it. As before, $h = e^\sigma g$ will denote the associated (local) metric on M with respect to which $\mathcal{V}$ is Riemannian and has geodesic leaves and $\mathcal{H}$ will denote the orthogonal complement of $\mathcal{V}$.

Proposition 3.2.1 (cf. [11, Chapter 9, §J]). *For $n \geq 3$, let $\varphi : (M^{n+1}, g) \rightarrow (N^n, \bar{h})$ be a harmonic morphism with one-dimensional geodesic fibres.*

(a) *If $\mathcal{H}$ is integrable then the following assertions are equivalent:*

- (i) *(M, g) is Einstein,*
- (ii) *$(N, \bar{h})$ is Einstein and the following relation holds*

$$\frac{c^M}{n} \lambda^2 - \frac{c^N}{n-1} \lambda^4 + (U(\lambda))^2 + \frac{3}{4} \lambda^{2n+2} |\Omega|_h^2 = 0 \quad (3.2.1)$$

where c^N is the Einstein constant of $(N^n, \bar{h})$, U is a vertical local vector field such that $g(U, U) = 1$ and $c^M = {}^M\text{Ricci}(U, U)$.

Moreover, if (i) or (ii) holds then

$$K_{X \wedge Y}^M - \frac{c^M}{n} = \lambda^2 \left(K_{\varphi_* X \wedge \varphi_* Y}^N - \frac{c^N}{n-1} \right) \quad (3.2.2)$$

where K^M and K^N are the sectional curvature of (M, g) and (N, h) , respectively, and X, Y are horizontal.

(b) When $n = 4$ and M and N are oriented consider also the following assertion:

(iii) Ω is the pull back of a (anti-)self-dual form on $(N, \bar{h})$.

Then, any two of the assertions (i), (ii) and (iii) imply the remaining assertion.

Proof. (a) By Proposition 1.1.10, we have that $X(\sigma) = 0$ for any horizontal X . Lemma 3.1.1 implies that $[V, X] = 0$ and hence $X(V(\sigma)) = V(X(\sigma)) = 0$. By hypothesis, $\Omega = 0$ so, from (B.1.24) for any horizontal X we have $\text{Ricci}(X, V) = 0$. Similarly, from (B.1.23) we get:

$${}^M\text{Ricci}(X, Y) = ({}^N\text{Ricci})(\varphi_*X, \varphi_*Y) - e^{-2\sigma} (\Delta^M \sigma) h(X, Y). \quad (3.2.3)$$

It follows that (M, g) is Einstein if and only if $(N, \bar{h})$ is Einstein and (3.2.1) holds.

If (i) or (ii) holds then (3.2.2) follows from (3.2.1) and the following formula

$$\lambda^2 K_{X \wedge Y}^M - \lambda^4 K_{\varphi_* X \wedge \varphi_* Y}^N + (U(\lambda))^2 = 0$$

which can be obtained directly or as a consequence of a formula of S. Gudmundsson [29].

(b) Let $\bar{\Omega}$ be the two-form on N such that $\varphi^*(\bar{\Omega}) = \Omega$. Note that $({}^h d^* \Omega)|_{\mathcal{H}} = 0$ if and only if $\bar{\Omega}$ is coclosed on $(N, \bar{h})$.

If (iii) holds the equivalence (i) $\iff$ (ii) can be proved in a similar way to (a), using the fact that any closed (anti-)self-dual form is coclosed and that, for any two-form ω on a four-dimensional oriented Euclidean space $(E^4, <, >)$ and $u, v \in E$, we have (see [14]):

$$< i_u \omega, i_v \omega > = \frac{1}{2} |\omega|^2 < u, v > + 2 < i_u \omega_+, i_v \omega_- > \quad (3.2.4)$$

where, ω_+ and ω_- are, respectively, the self-dual and the anti-self-dual components of ω .

To prove (i), (ii) $\implies$ (iii) we adapt a method of [14].

First note that, by (B.1.24), $\bar{\Omega}$ is coclosed.

Now, recall from [14], that (3.2.4) gives the decomposition of the symmetric bilinear map $(u, v) \mapsto < i_u \omega, i_v \omega >$ into its ‘spherical’ part and its ‘trace-free’

part. Also, the bilinear map $(u, v) \mapsto \langle i_u \alpha, i_v \beta \rangle$ induces a natural isomorphism between the space of ‘trace-free’ symmetric bilinear maps and $\Lambda_+^2(E) \otimes \Lambda_-^2(E)$ (see [14]). Using these facts it is easy to see that at each point $\bar{\Omega}$ is either self-dual or anti-self-dual.

If $N_{\pm} = \{y \in N \mid (\bar{\Omega}_{\pm})_y = 0\}$ then by the Baire category theorem at least one of the two sets N_+ and N_- has nonempty interior. If N_+ has nonempty interior then, following [14], we apply Aronszajn’s unique continuation theorem (see [18] noting that $\bar{\Omega}$, and hence also $\bar{\Omega}_+$, is closed and coclosed) to obtain $\bar{\Omega}_+ = 0$. Hence $\bar{\Omega}$ is anti-self-dual. $\square$

Remark 3.2.2. 1) From Lemma 3.1.7 we see that if the foliation given by the fibres of φ is nowhere Riemannian then $\mathcal{H}$ is automatically integrable.

2) Since the decomposition of two-forms into self-dual and anti-self-dual parts is conformally invariant, the condition that Ω be the pull back of a (anti-)self-dual form is equivalent to the condition that Ω restricted to the horizontal distribution be (anti-)self-dual.

The following elementary algebraic lemma will be used several times in this section.

Lemma 3.2.3. *Let E be an Euclidean linear space of dimension at least two and α a linear function on it such that, for any pair of orthogonal vectors $\{u, v\}$ we have $\alpha(u)\alpha(v) = 0$.*

Then $\alpha = 0$.

Proof. Let $u, v \in E$ be orthogonal and such that $|u| = |v|$. Since $u + v, u - v$ are also orthogonal we get that $0 = \alpha(u + v)\alpha(u - v) = \alpha(u)^2 - \alpha(v)^2$.

Thus $\alpha(u) = \pm\alpha(v)$ and since by hypothesis at least one of must be zero they are both zero. The lemma is proven. $\square$

Recall that on a Riemannian manifold of dimension at least four a Riemannian foliation with one-dimensional leaves produces harmonic morphisms if and only if it is locally generated by Killing fields (Proposition 3.1.3) and a foliation by

geodesics produces harmonic morphisms if and only if it is homothetic (Corollary 1.4.8). The next few results give conditions under which these are the only possible types of one-dimensional foliations which produces harmonic morphisms. They require a technical lemma which will be proved at the end of the section (Lemma 3.2.14).

Theorem 3.2.4. *Let (M^{n+1}, g) be an Einstein manifold of dimension $n+1 \geq 4$, and let $\mathcal{V}$ be a one-dimensional foliation with integrable orthogonal complement.*

Then $\mathcal{V}$ produces harmonic morphisms if and only if either

- (i) *$\mathcal{V}$ is Riemannian and locally generated by Killing fields or*
- (ii) *$\mathcal{V}$ is homothetic and its leaves are geodesics.*

Moreover, if both (i) and (ii) occur then (M^{n+1}, g) is Ricci-flat.

Proof. By passing to a Riemannian covering if necessary, we can suppose that $\mathcal{V}$ admits a global density.

By the remarks above we need to prove just the ‘only if’ part.

If $\mathcal{H}$ is integrable, then from (B.1.23) of Appendix B, for any orthogonal pair $\{X, Y\}$ formed of basic vector fields we have:

$${}^N \text{Ricci}(\varphi_* X, \varphi_* Y) = (n-1)(n-2)X(\sigma)Y(\sigma). \quad (3.2.5)$$

Since $n \geq 3$ and the left-hand side of (3.2.5) is a basic function we get that $X(\sigma)Y(\sigma)$ is a basic function.

Also, from (B.1.24) we obtain that $V(X(\sigma)) = X(V(\sigma)) = (n-2)V(\sigma)X(\sigma)$.

Hence:

$$\begin{aligned} 0 &= V(X(\sigma)Y(\sigma)) = V(X(\sigma))Y(\sigma) + X(\sigma)V(Y(\sigma)) \\ &= 2(n-2)V(\sigma)X(\sigma)Y(\sigma). \end{aligned}$$

If, at a point x we have that $V(\sigma)(x) \neq 0$, then this holds in an open neighbourhood O of x . It follows that $X(\sigma)Y(\sigma) = 0$ on O .

Using Lemma 3.2.3 we see that $\text{grad } \sigma$ restricted to O is vertical and hence, from Proposition 1.1.10, it follows that $\mathcal{V}$ restricted to O has geodesic leaves. Thus, we have proved that at each point $\mathcal{V}$ is either Riemannian or geodesic. Using Lemma 3.2.14 we get that locally either (i) or (ii) holds.

To prove that this alternative holds globally first recall that, being Einstein, (M, g) is analytic (see [11, 5.26]).

On an open subset where $\mathcal{V}$ has geodesic leaves, by (A.1.1), σ satisfies the equation

$$-n\Delta\sigma + n(n-1)|\operatorname{grad}_g\sigma|^2 = c \quad (3.2.6)$$

where c is the Einstein constant of (M, g) and Δ is the Laplacian of (M, g) . Hence, by the regularity of solutions of elliptic equations (see [11, page 467]) we get that σ is analytic on any open set where $\mathcal{V}$ has geodesic leaves.

Also, σ is analytic on an open set where $\mathcal{V}$ is Riemannian since, by (A.1.1), on such open sets σ satisfies the equation

$$\Delta\sigma = \frac{1}{n-2}c. \quad (3.2.7)$$

Because the alternative (i) or (ii) holds locally, σ is analytic on M .

Now suppose that locally both (i) and (ii) occur. In this case, being analytic, σ satisfies both (3.2.6) and (3.2.7) on M . It is easy to see that this implies that $|\operatorname{grad}_g\sigma|^2 = 2c/n(n-2)$ is constant on M . Also, if locally both (i) and (ii) occur, there must be at least one point $x \in M$ where $(\operatorname{grad}\sigma)_x = 0$. Thus, if locally both situations (i) and (ii) occur then σ is constant on M and (M, g) is Ricci flat. $\square$

Remark 3.2.5. In the above proof the fact that σ is analytic on an open subset of M where $\mathcal{V}$ has geodesic leaves also follows from the fact that if (M, g) is Einstein and φ a local harmonic morphism with geodesic fibres produced by $\mathcal{V}$ then by Proposition 3.2.1 the codomain of φ is also Einstein and hence is analytic in harmonic coordinates (see [11, 5.26]). Thus, φ is an analytic horizontally conformal (actually, homothetic) submersion between analytic Riemannian manifolds. Hence, the dilation of φ is analytic.

Next we prove that Theorem 3.2.4 still holds if we replace the integrability assumption on $\mathcal{H}$ with the condition that $\mathcal{V}$ be a homothetic foliation.

Corollary 3.2.6. *Let (M, g) be an Einstein manifold of dimension at least four endowed with a one-dimensional homothetic foliation $\mathcal{V}$.*

Then $\mathcal{V}$ produces harmonic morphisms if and only if either

- (i) $\mathcal{V}$ is Riemannian and locally generated by Killing fields or
- (ii) $\mathcal{V}$ has geodesic leaves and integrable orthogonal complement.

Moreover, if both (i) and (ii) occur then (M, g) is Ricci flat.

Proof. Again we need to prove just the ‘only if’ part.

Let

$$F_1 = \{x \in M \mid \mathcal{V} \text{ is Riemannian at } x\},$$

$$F_2 = \{x \in M \mid \mathcal{V} \text{ is geodesic at } x\}.$$

By Lemma 3.1.7 we have that $\mathcal{H}$, the orthogonal complement of $\mathcal{V}$, is integrable at least on $M \setminus F_1$. From Theorem 3.2.4 it follows that on $M \setminus F_1$ either $\mathcal{V}$ is Riemannian or geodesic. Using Lemma 3.2.14 we get that locally either $\mathcal{V}$ is Riemannian or geodesic, i.e. $M = \overset{\circ}{F}_1 \cup \overset{\circ}{F}_2$, where $\overset{\circ}{F}$ means the interior of F . If $\overset{\circ}{F}_1 \cap \overset{\circ}{F}_2 \neq \emptyset$ then, by the proof of Theorem 3.2.4, $\overset{\circ}{F}_2 \subseteq \overset{\circ}{F}_1$. Hence, $\mathcal{V}$ is either Riemannian or geodesic, on M . If $\mathcal{V}$ is not Riemannian then $\mathcal{H}$ is integrable, by Lemma 3.1.7.

The last assertion follows from Theorem 3.2.4. □

Remark 3.2.7. For the alternative (i) or (ii) from the above corollary to hold locally it is enough to suppose that the Ricci tensor takes the value zero when evaluated on any pair formed by a horizontal and a vertical vector.

In the next two propositions we shall assume curvature conditions which are automatically satisfied when (M, g) has constant sectional curvature.

Proposition 3.2.8. *For $n \geq 3$, let $\mathcal{V}$ be a one-dimensional foliation on (M^{n+1}, g) . Suppose that the orthogonal complement of $\mathcal{V}$ is integrable, that for any horizontal vector X we have ${}^M\text{Ricci}(X, V) = 0$ and that for any pair $\{X, Y\}$ of orthogonal basic vector fields the function $e^{(-2n+4)\sigma} {}^MR(X, V, Y, V)$ is basic.*

Then $\mathcal{V}$ produces harmonic morphisms if and only if locally either

- (i) $\mathcal{V}$ is Riemannian and generated by a Killing field or
- (ii) $\mathcal{V}$ is homothetic and its leaves are geodesic.

Proof. Again we need to prove just the ‘only if’ part.

By Lemma 3.2.14 and Proposition 3.1.3, it is sufficient to prove that at each point $\mathcal{V}$ is either Riemannian or geodesic.

Let $\{X, Y\}$ be a pair of basic vector fields orthonormal with respect to h . Then the hypothesis together with (B.1.1) implies that

$$e^{(-2n+4)\sigma} {}^M R(X, V, Y, V) = -\frac{1}{2}(n-2)(\mathcal{L}_{\mathcal{H}(\text{grad}_h \sigma)} h)(X, Y) - n(n-2)X(\sigma)Y(\sigma). \quad (3.2.8)$$

Also, with the given hypotheses, from (B.1.24) we see that

$$V(X(\sigma)) = X(V(\sigma)) = (n-2)X(\sigma)V(\sigma). \quad (3.2.9)$$

Since the left-hand side of (3.2.8) is basic by hypothesis, we can write

$$0 = \frac{1}{2}V((\mathcal{L}_{\mathcal{H}(\text{grad}_h \sigma)} h)(X, Y)) + nV(X(\sigma)Y(\sigma)).$$

Hence, using the properties of the Lie derivative together with (3.2.9) we get

$$\begin{aligned} 0 &= \frac{1}{2}(\mathcal{L}_V(\mathcal{L}_{\mathcal{H}(\text{grad}_h \sigma)} h))(X, Y) + \frac{1}{2}(\mathcal{L}_{\mathcal{H}(\text{grad}_h \sigma)} h)([V, X], Y) \\ &+ \frac{1}{2}(\mathcal{L}_{\mathcal{H}(\text{grad}_h \sigma)} h)(X, [V, Y]) + 2n(n-2)V(\sigma)X(\sigma)Y(\sigma). \end{aligned}$$

Since $[\mathcal{L}_A, \mathcal{L}_B] = \mathcal{L}_{[A, B]}$, for any vector fields A, B , and $[V, X] = 0$ for any basic field X , we get

$$\begin{aligned} 0 &= \frac{1}{2}(\mathcal{L}_{[V, \mathcal{H}(\text{grad}_h \sigma)]} h)(X, Y) + \frac{1}{2}(\mathcal{L}_{\mathcal{H}(\text{grad}_h \sigma)}(\mathcal{L}_V h))(X, Y) \\ &+ 2n(n-2)V(\sigma)X(\sigma)Y(\sigma). \end{aligned}$$

Since $\mathcal{V}$ is a Riemannian foliation with respect to h , the last relation becomes

$$0 = \frac{1}{2}(\mathcal{L}_{[V, \mathcal{H}(\text{grad}_h \sigma)]} h)(X, Y) + 2n(n-2)V(\sigma)X(\sigma)Y(\sigma).$$

Now, from (3.2.9) one can easily obtain

$$[V, \mathcal{H}(\text{grad}_h \sigma)] = (n-2)V(\sigma)\mathcal{H}(\text{grad}_h \sigma).$$

From the last two relations we obtain

$$\begin{aligned} 0 &= \frac{1}{2}(n-2)V(\sigma)(\mathcal{L}_{\mathcal{H}(\text{grad}_h \sigma)} h)(X, Y) + \frac{1}{2}(n-2)X(V(\sigma))Y(\sigma) \\ &+ \frac{1}{2}(n-2)Y(V(\sigma))X(\sigma) + 2n(n-2)V(\sigma)X(\sigma)Y(\sigma). \end{aligned}$$

The last relation together with (3.2.8) and (3.2.9) gives

$$e^{(-2n+4)\sigma} V(\sigma) {}^M R(X, V, Y, V) = 2(n-1)(n-2)V(\sigma)X(\sigma)Y(\sigma). \quad (3.2.10)$$

If at a point x we have that $\mathcal{V}$ is not Riemannian (i.e. $V(\sigma)(x) \neq 0$) then (3.2.10) together with the fact that $e^{(-2n+4)\sigma} {}^M R(X, V, Y, V)$ is basic gives us that in a neighbourhood O of x we have $V(X(\sigma)Y(\sigma)) = 0$. Using (3.2.9), this implies that $X(\sigma)Y(\sigma) = 0$. Using Lemma 3.2.3, we get that in O , $\text{grad}_h \sigma$ is vertical. Now apply Lemma 3.2.14. $\square$

Proposition 3.2.9. *Let $\mathcal{V}$ be a one-dimensional foliation on a Riemannian manifold (M^{n+1}, g) and let R denote the Riemannian curvature tensor of (M^{n+1}, g) . Suppose that $\mathcal{V}$ produces harmonic morphisms and that, for any triple $\{X, Y, Z\}$ of orthogonal basic vector fields, the function $e^{(-2n+4)\sigma} {}^M R(X, Y, Z, V)$ is basic.*

(i) *If $n \geq 4$, then, except possibly at the points where the orthogonal complement of $\mathcal{V}$ is integrable, $\mathcal{V}$ has basic mean curvature form.*

(ii) *If $n = 3$, and ${}^M \text{Ricci}(X, Y)$ is basic for any pair of orthogonal basic vector fields $\{X, Y\}$, then $\mathcal{V}$ has basic mean curvature form.*

Proof. Let $\{X, Y, Z\}$ be orthogonal basic fields. Put them into (B.1.2) and multiply by $e^{(-2n+4)\sigma}$. Then, the first term on the right hand side is basic by a simple calculation. The terms in the last three lines are zero and we conclude that the second term on the right hand side of (B.1.2) is basic.

Hence, for any triple $\{X, Y, Z\}$ of orthogonal basic fields the following relation holds:

$$V(X(\sigma))\Omega(Y, Z) + V(Y(\sigma))\Omega(Z, X) = 2V(Z(\sigma))\Omega(X, Y).$$

Rewriting this after a circular permutation of the vectors in the frame and then subtracting the second relation from the first gives the following:

$$V(X(\sigma))\Omega(Y, Z) = V(Z(\sigma))\Omega(X, Y). \quad (3.2.11)$$

Hence, if $\{T, X, Y, Z\}$ are orthogonal basic fields we have the following:

$$\begin{aligned} V(T(\sigma))V(X(\sigma))\Omega(Y, Z) &= V(T(\sigma))V(Z(\sigma))\Omega(X, Y) \\ &= -V(Z(\sigma))V(T(\sigma))\Omega(Y, X) = -V(Z(\sigma))V(X(\sigma))\Omega(T, Y) \\ &= -V(X(\sigma))V(Z(\sigma))\Omega(T, Y) = -V(X(\sigma))V(T(\sigma))\Omega(Y, Z), \end{aligned}$$

so that

$$V(T(\sigma))V(X(\sigma))\Omega(Y, Z) = 0 .$$

It follows that on $S = \{x \in M \mid \Omega_x \neq 0\}$, for any pair $\{X, Y\}$ of orthogonal basic fields we have $V(X(\sigma))V(Y(\sigma)) = 0$. By Lemma 3.2.3 this implies that on S we have $V(X(\sigma)) = 0$ for any basic vector field X .

Suppose now that $\dim M = 4$ and let S be defined as above. Then, it suffices to prove that $\mathcal{V}$ has basic mean curvature form on S and on the interior of $M \setminus S$.

Let $\{X, Y\}$ be a pair of orthogonal basic vector fields defined on the interior of $M \setminus S$. Then from (B.1.23) we obtain that $X(\sigma)Y(\sigma)$ is a basic function. If X and Y have the same length with respect to h then this applies also to the pair $\{X + Y, X - Y\}$ and hence $X(\sigma)^2 - Y(\sigma)^2$ is a basic function. Hence $X(\sigma)^2$ and $Y(\sigma)^2$ are basic and thus $X(\sigma)$ and $Y(\sigma)$ are basic.

Since $\dim M = 4$ and $i_V \Omega = 0$ (where, as usual, i denotes the interior product) each point of S has an open neighbourhood on which a unique basic vector field Z can be defined such that $i_Z \Omega = 0$ and $h(Z, Z) = 1$.

We can choose basic vector fields X and Y such that $\{X, Y, Z\}$ is an orthonormal local frame for $\mathcal{H}$ with respect to h . Since, on S , $\Omega \neq 0$ we have that $\Omega(X, Y) \neq 0$. By (3.2.11) and the way Z was chosen we have:

$$V(Z(\sigma))\Omega(X, Y) = V(X(\sigma))\Omega(Y, Z) = 0 .$$

Hence $V(Z(\sigma)) = 0$.

Now, a simple calculation gives the following relations:

$$\begin{aligned} h(i_X \Omega, i_Y \Omega) &= 0 , \\ h(i_X \Omega, i_X \Omega) &= h(i_Y \Omega, i_Y \Omega) . \end{aligned}$$

The first of the above relations together with (B.1.23) applied to $\{X, Y\}$ gives that the function $X(\sigma)Y(\sigma)$ is basic. The second of the above relations together with (B.1.23) applied to $\{X + Y, X - Y\}$ gives that the function $X(\sigma)^2 - Y(\sigma)^2$ is basic. Hence $X(\sigma)$ and $Y(\sigma)$ are basic. $\square$

Proposition 3.2.10. *Let $\mathcal{V}$ be a one-dimensional foliation which produces harmonic morphisms on (M^{n+1}, g) . Suppose that the following conditions are satisfied, for any horizontal X :*

- (i) $X(V(\sigma)) = 0$,
- (ii) ${}^M\text{Ricci}(X, V) = 0$.

Then $\mathcal{V}$ is homothetic.

Proof. By Lemma 3.1.8 it is sufficient to prove that $\mathcal{V}$ is homothetic on the interior S of the set $\{x \in M \mid d(V(\sigma))(x) = 0 \neq (V(\sigma))(x)\}$.

By (B.1.24) on S we have

$$e^{(2n-2)\sigma} \left\{ \frac{1}{2}({}^h d^* \Omega)(X) + (n-1)\Omega(X, \text{grad}_h \sigma) \right\} = (n-1)(n-2)X(\sigma)V(\sigma), \quad (3.2.12)$$

for any basic vector field X .

By hypothesis the right hand side above is a basic function on S . Also, the second factor from the left hand side of (3.2.12) is basic and thus, if this second factor is nonzero, then $e^{(2n-2)\sigma}$ is a basic function. This implies that $\mathcal{V}$ is Riemannian and, in particular, homothetic.

If the second factor from the left hand side of (3.2.12) is zero on an open subset S_0 of S then the right hand side is also zero and hence $\mathcal{V}$ has geodesic fibres on S_0 . From Corollary 1.4.8 it follows that $\mathcal{V}$ is homothetic on S_0 . $\square$

It is obvious that a space with constant sectional curvature satisfies the curvature assumptions of all of the previous theorems of this section. In fact, in this case, we have the following:

Theorem 3.2.11. *For $n \geq 3$ let (M^{n+1}, g) be a Riemannian manifold with constant sectional curvature and $\mathcal{V}$ a one-dimensional foliation on (M^{n+1}, g) .*

Then, $\mathcal{V}$ produces harmonic morphisms if and only if either

- (i) $\mathcal{V}$ is Riemannian and locally generated by Killing fields or
- (ii) $\mathcal{V}$ is a homothetic foliation by geodesics with integrable orthogonal complement.

Moreover, both (i) and (ii) can occur only if (M^{n+1}, g) is flat.

Furthermore, if $\mathcal{V}$ is orientable and admits a global density then (i) can be replaced by

- (i') $\mathcal{V}$ is (globally) generated by a nonvanishing Killing field.

Proof. Again we need to prove just the ‘only if’ part.

From Theorem 3.2.4, Proposition 3.2.9 and Proposition 3.2.10 we obtain that at each point either $\mathcal{V}$ is Riemannian or geodesic.

Using Lemma 3.2.14 we get that, locally, either $\mathcal{V}$ is Riemannian or geodesic. In particular, $\mathcal{V}$ is homothetic. Now the proof of the alternative (i) or (ii) follows from Corollary 3.2.6.

If both (i) and (ii) hold then, from formula (B.1.1), it follows that (M, g) is flat.

The fact that (i) can be replaced by (i') when $\mathcal{V}$ is orientable and admits a global density follows from the proof of Proposition 3.1.3. $\square$

Corollary 3.2.12. *For $n \geq 3$ let (M^{n+1}, g) be a Riemannian manifold with constant sectional curvature and let $\varphi : (M^{n+1}, g) \rightarrow (N^n, h)$ be a submersive harmonic morphism with orientable vertical distribution.*

Then, either

- (i) *the fibres of φ form a Riemannian foliation generated by a Killing field or*
- (ii) *φ is horizontally homothetic and has geodesic fibres orthogonal to an umbilical foliation by hypersurfaces.*

Lemma 3.2.13. *Let M^m be a manifold and $F_1, F_2 \subseteq M$, closed subsets such that $M = F_1 \cup F_2$.*

Then, $M = \text{cl}(\overset{\circ}{F}_1) \cup \text{cl}(\overset{\circ}{F}_2)$.

Proof. Suppose that there exists $x \notin \text{cl}(\overset{\circ}{F}_1) \cup \text{cl}(\overset{\circ}{F}_2)$. Then there exists an open neighbourhood O of x , which is homeomorphic with $\mathbb{R}^m$ and such that $O \cap \text{cl}(\overset{\circ}{F}_j) = \emptyset$, $j = 1, 2$.

For $j = 1, 2$, let $G_j = O \cap F_j$. Since F_j is closed, G_j is closed in O . Also, since O is open we have that the interior of G_j in O is the same as its interior in M , which is empty by the way O was chosen.

Hence, G_j is nowhere dense in O . From $M = F_1 \cup F_2$ we get that $\mathbb{R}^m \approx O = G_1 \cup G_2$. But this is impossible, by the Baire category theorem. $\square$

Lemma 3.2.14. *Let $\mathcal{V}$ be a one-dimensional foliation with geodesic leaves on (M, g) and let $\mathcal{H}$ be its orthogonal complement. Let $f : M \rightarrow \mathbb{R}$ be a smooth*

function such that at each point $x \in M$ either $\mathcal{V}(\text{grad } f)_x = 0$ or $\mathcal{H}(\text{grad } f)_x = 0$.

Then each point of M has an open neighbourhood O such that either $\mathcal{V}(\text{grad } f) = 0$ on O or $\mathcal{H}(\text{grad } f) = 0$ on O .

Proof. Let

$$F_1 = \{ x \in M \mid \mathcal{V}(\text{grad } f)_x = 0 \} ,$$

$$F_2 = \{ x \in M \mid \mathcal{H}(\text{grad } f)_x = 0 \} .$$

By hypothesis, $M = F_1 \cup F_2$ and hence, by Lemma 3.2.13, we have that

$$M = \text{cl}(\overset{\circ}{F}_1) \cup \text{cl}(\overset{\circ}{F}_2) . \quad (3.2.13)$$

Let V be a local unit vertical vector field, then, by Lemma 3.1.1, $[V, X] = 0$ for any basic vector field X . Also, using (3.2.13) we see that for any basic vector field X , we have:

$$V(X(f)) = X(V(f)) = 0 . \quad (3.2.14)$$

Note that $\mathcal{H}$ is integrable on $M \setminus F_1$, because on this set the level hypersurfaces of f are integral manifolds for $\mathcal{H}$. Using this fact together with (3.2.14) and the hypothesis it is easy to see that $\mathcal{H}(\text{grad } f)$ and $\mathcal{V}(\text{grad } f)$ are, locally, gradient vector fields. Thus, we can find local smooth functions a and b such that:

$$\begin{aligned} f &= a + b \quad \text{and,} \\ \mathcal{V}(\text{grad } a) &= 0 = \mathcal{H}(\text{grad } b) \end{aligned}$$

It follows that F_1 is the set of critical points of b , and F_2 is the set of critical points of a . Since locally, M is diffeomorphic to the product between an open subset of a leaf and a local base, the lemma quickly follows. $\square$

3.3 Some constructions of one-dimensional foliations which produce harmonic morphisms on Einstein manifolds

In this section we use well-known results on warped-products and conformal vector fields (see [38], [11, Chapter 9, §J]) to obtain one-dimensional foliations with

integrable orthogonal complement and which produce harmonic morphisms on Einstein manifolds which are not of constant curvature (thus answering to a question formulated by S. Gudmundsson).

3.3.1 Homothetic foliations with geodesic leaves

Proposition 3.3.1 (cf. [38, Lemma 13(iv)]). *Let $\varphi : (M^{n+1}, g) \rightarrow (N^n, h)$, $n \geq 3$, be a nonconstant harmonic morphism with geodesic leaves and integrable horizontal distribution.*

- (i) *If (M^{n+1}, g) has constant curvature then (N^n, h) has constant curvature.*
- (ii) *If (M^{n+1}, g) is Einstein and (N^n, h) has constant curvature then (M^{n+1}, g) has constant curvature.*

Proof. Assertion (i) is an immediate consequence of (3.2.2).

If (M^{n+1}, g) is Einstein then from (3.2.1) and from (B.1.1) we obtain

$$K_{X \wedge U}^M = \frac{c^M}{n} \quad (3.3.1)$$

where X is any horizontal vector.

The proof of (ii) follows from (3.2.2) and (3.3.1). $\square$

Corollary 3.3.2 (cf. [38, Corollary 15]). *Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a harmonic morphism with one-dimensional geodesic leaves and integrable horizontal distribution.*

If (M^4, g) is Einstein then both (M^4, g) and (N^3, h) have constant curvature.

Proof. If (M^4, g) is Einstein then by Proposition 3.2.1, (N^3, h) is Einstein. But N^3 is three-dimensional and thus (N^3, h) has constant curvature. The proof follows from Proposition 3.3.1(ii). $\square$

Corollary 3.3.3 (cf. [38],[11, Chapter 9, §J]). *Given any Einstein manifold (N^n, h) of dimension n there exists an Einstein manifold (M^{n+1}, g) of dimension $n + 1$ and a harmonic morphism $\varphi : (M^{n+1}, g) \rightarrow (N^n, h)$ with geodesic fibres and integrable horizontal distribution.*

If $n \geq 4$ and (N^n, h) does not have constant curvature then (M^{n+1}, g) does not have constant curvature.

Proof. Let (N^n, h) be Einstein and let λ be a (local) solution of (3.2.1) (see [11, 9.109]).

Let $M^{n+1} = \mathbb{R} \times N^n$ and $g = dt^2 + \lambda^{-2} h$. It is obvious that the canonical projection $(M^{n+1}, g) \rightarrow (N^n, h)$ is a harmonic morphism with geodesic leaves and integrable horizontal distribution. Also (M^{n+1}, g) is an Einstein manifold by Proposition 3.2.1. Moreover, if (N^n, h) does not have constant curvature then, by Proposition 3.3.1, (M^{n+1}, g) does not have constant curvature. $\square$

3.3.2 Riemannian foliations locally generated by Killing fields

The following results are consequences of [11, Corollary 9.107, 9.108, 9.109].

Proposition 3.3.4 ([11]). *Let (N^n, h) be a Riemannian manifold and $\rho : N^n \rightarrow (0, \infty)$ a smooth positive function.*

Let $M^{n+1} = \mathbb{R} \times N^n$ and $g = \rho^2 dt^2 + h$. Then, the following assertions are equivalent.

- (i) *(M^{n+1}, g) is Einstein (${}^M\text{Ricci} = c^M g$).*
- (ii) *(N^n, h) has constant scalar curvature $s^N = (n-1)c^M$ and the following relation holds*

$${}^N\nabla d\rho = -\frac{c^M}{n} \rho h + \rho Z^N \quad (3.3.2)$$

where ${}^N\nabla$ is the Levi-Civita connection on (N^n, h) and $Z^N = {}^N\text{Ricci} - (s^N/n)h$ is the trace-free part of ${}^N\text{Ricci}$.

Proof. From [11, 9.106a, 9.106c] or, by a straightforward calculation the following equations can be obtained:

$$\begin{aligned} ({}^M\text{Ricci})|_{TN} &= {}^N\text{Ricci} - \rho^{-1} {}^N\nabla d\rho \\ ({}^M\text{Ricci})|_{T\mathbb{R}} &= \rho^{-1} (\Delta^N \rho) g|_{T\mathbb{R}}. \end{aligned} \quad (3.3.3)$$

Also, [11, 9.106b] gives that ${}^M\text{Ricci}(X, \partial/\partial t) = 0$ for any $X \in \Gamma(TN)$. From this and (3.3.3) the proof easily follows. $\square$

Corollary 3.3.5. *For each $n \geq 5$ there exists Einstein manifolds (M^{n+1}, g) not of constant curvature, endowed with a nowhere zero Killing field which has integrable orthogonal complement. Moreover, the construction can be done in such a way that the (locally) induced isometric quotients are also Einstein.*

Proof. If the equation $\overset{N}{\nabla} d\rho = a\rho h$, where $a \in \mathbb{R}$, has solutions then there exists a homothetic one-dimensional foliation with geodesic leaves and integrable orthogonal complement (see Lemma 2.6.6). Recall that $Z^N = 0$ if and only if (N^n, h) is Einstein (see [11, 1.118]). Hence, [11, 9.109] and Corollary 3.3.3 implies that there exists an Einstein manifold (N^n, h) , $n \geq 5$, not of constant curvature, on which (3.3.2) has a (local) solution ρ which is positive. Then, by Proposition 3.3.4, (M^{n+1}, g) (where $M^{n+1} = \mathbb{R} \times N^n$ and $g = \rho^2 dt^2 + h$) is Einstein. Clearly $V = \partial/\partial t$ is a nowhere zero Killing field on (M^{n+1}, g) . $\square$

3.4 Foliations of dimension one which produce harmonic morphisms on four-dimensional Einstein manifolds

To state the main result of this section we need a definition which is a trivial generalization to foliations of the well-known notion of principal connection on a principal bundle. For simplicity, we give this definition just for one-dimensional foliations.

Definition 3.4.1. Let $\mathcal{V}$ be a one-dimensional foliation and let $V \in \Gamma(\mathcal{V})$ be a nowhere zero vector field tangent to $\mathcal{V}$.

A *principal connection* for $\mathcal{V}$ (with respect to V) is a complementary distribution $\mathcal{H} \subseteq TM$, $\mathcal{H} \oplus \mathcal{V} = TM$ such that V is an infinitesimal automorphism of $\mathcal{H}$ (i.e. $\mathcal{H}$ is invariant under the local flow of V).

The *connection form* θ of $\mathcal{H}$ is the ‘vertical’ dual of V (i.e. $\theta(V) = 1$ and $\theta|_{\mathcal{H}} = 0$) and the *curvature form* of $\mathcal{H}$ is $\Omega = d\theta$. Note that Ω is basic and it can be interpreted as the integrability tensor of $\mathcal{H}$ (indeed $\Omega(X, Y)V = -\mathcal{V}([X, Y])$ for any horizontal vector fields X and Y).

It is obvious that a one form θ defines a principal connection for $\mathcal{V}$ with respect to V if and only if $\theta(V) = 1$ and $\mathcal{L}_V \theta = 0$.

Example 3.4.2. Let $\mathcal{V}$ be an orientable one-dimensional geodesic foliation on (M, g) . Then $\mathcal{H}(= \mathcal{V}^\perp)$ is a principal connection for $\mathcal{V}$ with respect to $U \in \Gamma(\mathcal{V})$ where $g(U, U) = 1$. The connection form is $U^\flat$.

An orientable one-dimensional foliation $\mathcal{V}$ on M admits a principal connection if and only if it is geodesible (i.e. there exists a Riemannian metric h on M such that the leaves of $\mathcal{V}$ are geodesics on (M, h)). Indeed, given the principal connection $\mathcal{H}$ (with respect to some $V \in \Gamma(\mathcal{V})$), if we choose any metric h such that $h(V, V) = 1$ and $h(V, X) = 0$ for $X \in \mathcal{H}$ then the leaves of $\mathcal{V}$ are geodesics of (M, h) . Also the set of principal connections of $\mathcal{V}$ (if nonempty) with respect to a given nowhere zero vector field $V \in \Gamma(\mathcal{V})$ is an affine space over the linear space of basic one-forms: if θ_j , $j = 1, 2$ are connection forms then $\theta_1 - \theta_2$ is locally the pull back by $\varphi : U \rightarrow N$ of a one-form $A \in \Gamma(T^*N)$ where U is an open subset of M and the fibres of φ are open subsets of leaves of $\mathcal{V}$. Fix $V \in \Gamma(\mathcal{V})$. Then, in a neighbourhood of each point of N , a local section s of φ can be found which, in a neighbourhood of its image, defines a principal connection θ_s which is flat (i.e. $d\theta_s = 0$). If θ defines a principal connection then the one-form A such that $\theta = \theta_s + \varphi^*(A)$ is the *local* connection form of θ with respect to s . Because $\mathcal{V}$ is one-dimensional we can define the *local* connection form of a principal connection with respect to a (local) flat principal connection by using any parallel section of the flat connection. Also note that the existence of a *global* flat principal connection imposes, severe restrictions on the topology of the foliation and of the manifold. For example, as is well known, if the leaves of $\mathcal{V}$ are the fibres of a principal bundle $\xi = (M, N, S^1)$ over a simply-connected N and ξ admits a flat principal connection then ξ is trivial and, in particular, M and $N \times S^1$ are diffeomorphic.

The orthogonal complement of a one-dimensional foliation which produces harmonic morphisms is a principal connection of it.

Proposition 3.4.3. *Let $\mathcal{V}$ be a one-dimensional foliation which produces harmonic morphisms on (M^{n+1}, g) where $\dim M = n + 1$. Let $\rho = e^{(2-n)\sigma}$ be a local*

density of $\mathcal{V}$. Suppose that $\mathcal{V}$ restricted to the domain of σ is orientable and let $V \in \Gamma(\mathcal{V})$ be such that $g(V, V) = e^{(2n-4)\sigma} (= \rho^{-2})$.

Then the horizontal distribution $\mathcal{H}(= \mathcal{V}^\perp)$ is a principal connection for $\mathcal{V}$ with respect to V .

Proof. This follows from Lemma 3.1.1. □

We now state the main result of this section.

Theorem 3.4.4. *Let (M^4, g) be an Einstein manifold of dimension four and $\mathcal{V}$ a one-dimensional foliation which produces harmonic morphisms on (M^4, g) .*

Then, one of the following assertions holds:

- (i) $\mathcal{V}$ is Riemannian and locally generated by Killing fields;
- (ii) $\mathcal{V}$ is a homothetic foliation by geodesics with integrable orthogonal complement;
- (iii) (M^4, g) is Ricci-flat and, up to homotheties, any harmonic morphism $\varphi : (U, g|_U) \rightarrow (N^3, h)$, with dilation λ , produced by $\mathcal{V}$ such that $\mathcal{V}|_U$ and N^3 are orientable is (locally) described as follows:

- (a) (N^3, h) has constant sectional curvature $k^N = 1$,
- (b) $\frac{1}{2} d(\lambda^{-2})$ is a (flat) principal connection for $\mathcal{V}$ with respect to suitably chosen $V \in \Gamma(\mathcal{V})$ such that $g(V, V) = \lambda^2$,
- (c) the local connection form A of $\mathcal{H}$ with respect to $\frac{1}{2} d(\lambda^{-2})$ satisfies the equation $dA + 2 * A = 0$ on (N^3, h) where $*$ is the Hodge star-operator of (N^3, h) with respect to some orientation of N^3 .

Moreover, only (i) and (ii) or (ii) and (iii) can occur simultaneously, in which case (M^4, g) must be Ricci-flat.

From Theorem 3.4.4 we obtain the following.

Corollary 3.4.5. *Let (M^4, g) be an orientable Einstein manifold of dimension four, and (N^3, h) an orientable Riemannian manifold of dimension three.*

Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a submersive harmonic morphism; denote its dilation by λ and let $V \in \Gamma(\mathcal{V})$ be such that $g(V, V) = \lambda^2$.

Then, one of the following assertions (i), (ii), (iii) holds:

- (i) V is a Killing field;
- (ii) φ is horizontally homothetic and has geodesic fibres orthogonal to an umbilical foliation by hypersurfaces;
- (iii) (a) (M^4, g) is Ricci-flat and (N^3, h) has constant sectional curvature $k^N = \frac{c^2}{4}$ ($c \neq 0$),
 (b) $\frac{1}{c} d(\lambda^{-2})$ is a (flat) principal connection for $\ker \varphi_*$ with respect to V ,
 (c) the local connection form A of $(\ker \varphi_*)^\perp$ with respect to $\frac{1}{c} d(\lambda^{-2})$ satisfies $dA + c * A = 0$ on (N^3, h) where $*$ is the Hodge star-operator of (N^3, h) defined by some orientation of N^3 .

Remark 3.4.6. 1) If M^4 is not orientable then we can replace (M^4, g) by a Riemannian double covering $(\widetilde{M}^4, \widetilde{g})$ such that $\widetilde{M}^4$ is orientable. Then we replace φ by $\widetilde{\varphi} = \varphi \circ \xi$ where $\xi : (\widetilde{M}^4, \widetilde{g}) \rightarrow (M^4, g)$ is the projection of the covering.

2) If N^3 is not orientable we can pull back φ to a Riemannian double-covering $(\widetilde{N}^3, \widetilde{h})$ of (N^3, h) such that $\widetilde{N}^3$ is orientable.

Before proving Theorem 3.4.4 we need some further preparations.

Let $\mathcal{V}$ be a one-dimensional foliation which produces harmonic morphisms on a Riemannian manifold (M^4, g) , $\dim M = 4$ and let $\rho = e^{-\sigma}$ be a positive smooth function. We define the Riemannian metric h on M by

$$h = e^{2\sigma} g^{\mathcal{H}} + e^{-2\sigma} g^{\mathcal{V}}$$

where $g^{\mathcal{H}}$ and $g^{\mathcal{V}}$ are the horizontal and the vertical part of g , respectively. Then from Proposition 1.2.4 it follows that $\mathcal{V}$ produces harmonic morphisms on (M, h) as well. Furthermore if $\rho = e^{-\sigma}$ is a (local) density of $\mathcal{V}$ then $\mathcal{V}$ is Riemannian and has geodesic leaves with respect to h .

Assuming that $\mathcal{V}$ restricted to the domain O of the local density ρ is orientable let $V \in \Gamma(\mathcal{V}|_O)$ be such that $g(V, V) = \rho^{-2}$. Then, by Proposition 3.4.3, $\mathcal{H}$ is a principal connection for $\mathcal{V}$, with respect to V . We shall always denote by Ω the curvature form of $\mathcal{H}$.

Next we relate the Ricci tensors ${}^M\text{Ricci}$ and ${}^N\text{Ricci}$ of (M^4, g) and $(N^3, \bar{h})$, respectively, where $(N^3, \bar{h})$ is the codomain of a harmonic morphism produced by $\mathcal{V}$.

Lemma 3.4.7. *Let O be an open subset of M^4 and let $\varphi : (O, g|_O) \rightarrow (N, \bar{h})$ be a harmonic morphism produced by $\mathcal{V}$. Then,*

$$\begin{aligned} {}^M\text{Ricci}(X, Y) = & {}^N\text{Ricci}(\varphi_*X, \varphi_*Y) - \frac{1}{2} e^{4\sigma} h(i_X\Omega, i_Y\Omega) \\ & - e^{-2\sigma} \Delta^M \sigma h(X, Y) - 2X(\sigma)Y(\sigma), \end{aligned} \quad (3.4.1)$$

$$\begin{aligned} {}^M\text{Ricci}(X, V) = & \frac{1}{2} e^{4\sigma} ({}^h\text{d}^*\Omega)(X) + 2e^{4\sigma} \Omega(X, \text{grad}_h \sigma) \\ & + 2X(V(\sigma)) - 2X(\sigma)V(\sigma), \end{aligned} \quad (3.4.2)$$

$${}^M\text{Ricci}(V, V) = e^{2\sigma} \Delta^M \sigma + \frac{1}{4} e^{8\sigma} |\Omega|_h^2 + 4V(V(\sigma)) - 10V(\sigma)^2 \quad (3.4.3)$$

for any horizontal vectors X, Y , where e^σ is the dilation of φ , Δ^M is the Laplacian on (M, g) and ${}^h\text{d}^*$ denotes the codifferential on (M, h) .

Proof. These equations follow from Lemma B.1.5. □

The following proposition, which will be used later on, holds for manifolds of any dimension.

Proposition 3.4.8. *Let (M, g) be an Einstein manifold and $\mathcal{V}$ a one-dimensional foliation of codimension not equal to two which produces harmonic morphisms on (M, g) .*

Then the following assertions are equivalent:

- (i) $\mathcal{V}$ has basic mean curvature form;
- (ii) $\mathcal{V}$ is a homothetic foliation.

Proof. This follows from Proposition 3.2.10. □

As mentioned before, Theorem 3.4.4 extends Theorem 3.2.4 and Corollary 3.2.6 in the four-dimensional case. From Theorem 3.2.4 and Corollary 3.2.6 it follows the following:

Theorem 3.4.9. *Let (M, g) be an Einstein manifold of dimension at least four endowed with a one-dimensional foliation $\mathcal{V}$. Suppose that either $\mathcal{V}$ has integrable orthogonal complement or $\mathcal{V}$ is homothetic.*

Then $\mathcal{V}$ produces harmonic morphisms if and only if either

- (i) $\mathcal{V}$ is Riemannian and locally generated by Killing fields or
 - (ii) $\mathcal{V}$ is homothetic, has geodesic leaves and integrable orthogonal complement.
- Moreover, if both (i) and (ii) hold then (M, g) is Ricci-flat.

Proof of Theorem 3.4.4. By passing to a two-fold covering, if necessary, we can suppose that $\mathcal{V}$ is oriented. Also, by passing to a regular covering, if necessary, we can suppose that $\mathcal{V}$ admits a global density $\rho = e^{-\sigma}$. Thus there exists $V \in \Gamma(\mathcal{V})$ such that $g(V, V) = e^{2\sigma}$.

If $\Omega = 0$ then we are done by Theorem 3.4.9. So suppose that $\Omega \neq 0$. Then, since Ω is basic, we can choose a local orthonormal frame $\{X, Y, Z\}$ of $\mathcal{H}$ with respect to h , made up of basic fields and such that

$$i_Z \Omega = 0 \quad \text{and} \quad \Omega(X, Y) \neq 0.$$

Then it is easy to see that

$$h(i_X \Omega, i_Y \Omega) = h(i_Y \Omega, i_Z \Omega) = h(i_Z \Omega, i_X \Omega) = 0, \quad (3.4.4)$$

$$h(i_X \Omega, i_X \Omega) = h(i_Y \Omega, i_Y \Omega) = \Omega(X, Y)^2. \quad (3.4.5)$$

Now recall that (M^4, g) is Einstein and thus ${}^M\text{Ricci} = c^M g$ for some real number $c^M \in \mathbb{R}$. Then from (3.4.1) and (3.4.4) it follows that

$$X(\sigma)Y(\sigma), Y(\sigma)Z(\sigma), Z(\sigma)X(\sigma)$$

are basic functions. Also, because ${}^M\text{Ricci}(X, X) = {}^M\text{Ricci}(Y, Y)$, it follows from (3.4.1) and (3.4.5) that $X(\sigma)^2 - Y(\sigma)^2$ is a basic function. Hence $X(\sigma), Y(\sigma)$ are basic functions and, moreover, outside the set

$$S = \{x \in M \mid X_x(\sigma) = Y_x(\sigma) = 0\},$$

we have that $Z(\sigma)$ is also basic. It follows that, at least, outside the interior of S , the foliation $\mathcal{V}$ is homothetic by Proposition 3.4.8. From Theorem 3.4.9 we obtain that the alternative (i) or (ii) of Theorem 3.4.4 holds locally at least on $M \setminus S$ (just *locally* because $M \setminus S$ might be disconnected). If also $Z(\sigma) = 0$ on S then we are done because then $\mathcal{V}$ is homothetic on M and we can again apply

Theorem 3.4.9.

From now on in this proof we shall work on $\overset{\circ}{S} \cap \{x \in M \mid Z_x(\sigma) \neq 0\}$. There we can write $[X, Y](\sigma) = X(Y(\sigma)) - Y(X(\sigma)) = 0$ and hence

$$\begin{aligned} -\mathcal{V}[X, Y](\sigma) &= \mathcal{H}[X, Y](\sigma) \\ \iff \Omega(X, Y) V(\sigma) &= h(\mathcal{H}[X, Y], \text{grad}_h \sigma) . \end{aligned}$$

Since we are in the interior of S the last equality is equivalent to

$$\Omega(X, Y) V(\sigma) = Z(\sigma) h([X, Y], Z) . \quad (3.4.6)$$

Next we compute $({}^h\text{d}^*\Omega)(Z)$ (note that by (3.4.2) the other components of $\mathcal{H}^*({}^h\text{d}^*\Omega)$ are zero):

$$({}^h\text{d}^*\Omega)(Z) = -({}^h\nabla_X\Omega)(X, Z) - ({}^h\nabla_Y\Omega)(Y, Z) - ({}^h\nabla_Z\Omega)(Z, Z) - ({}^h\nabla_V\Omega)(V, Z) .$$

Because $i_Z\Omega = 0$ we have $({}^h\nabla_Z\Omega)(Z, Z) = 0$ and because Ω is basic and $\mathcal{V}$ is geodesic with respect to h we have $({}^h\nabla_V\Omega)(V, Z) = 0$. Also

$$\begin{aligned} ({}^h\nabla_X\Omega)(X, Z) &= X(\Omega(X, Z)) - \Omega({}^h\nabla_X X, Z) - \Omega(X, {}^h\nabla_X Z) \\ &= -\Omega(X, {}^h\nabla_X Z) . \end{aligned}$$

Since $h({}^h\nabla_X Z, Y) = -h(Z, {}^h\nabla_X Y)$ we have

$$({}^h\nabla_X\Omega)(X, Z) = \Omega(X, Y) h(Z, {}^h\nabla_X Y) .$$

Similarly $({}^h\nabla_Y\Omega)(Y, Z) = \Omega(Y, X) h(Z, {}^h\nabla_Y X)$ and we obtain that

$$\begin{aligned} ({}^h\text{d}^*\Omega)(Z) &= -\Omega(X, Y) h(Z, {}^h\nabla_X Y) - \Omega(Y, X) h(Z, {}^h\nabla_Y X) \\ &= -\Omega(X, Y) \{h(Z, {}^h\nabla_X Y) - h(Z, {}^h\nabla_Y X)\} . \end{aligned}$$

We have proved that

$$({}^h\text{d}^*\Omega)(Z) = -\Omega(X, Y) h(Z, [X, Y]) . \quad (3.4.7)$$

Now by (3.4.2) (with X replaced by Z) we have

$$\frac{1}{2} e^{4\sigma} ({}^h\text{d}^*\Omega)(Z) + 2 Z(V(\sigma)) - 2 Z(\sigma) V(\sigma) = 0 . \quad (3.4.8)$$

From (3.4.7) and (3.4.8) it follows that on $\overset{\circ}{S} \cap \{x \in M \mid Z_x(\sigma) \neq 0\}$ we have

$$-\frac{1}{2} e^{4\sigma} \Omega(X, Y) h(Z, [X, Y]) + 2 Z(V(\sigma)) - 2 Z(\sigma) V(\sigma) = 0$$

which together with (3.4.6) gives

$$-\frac{1}{2} e^{4\sigma} \frac{V(\sigma)}{Z(\sigma)} \Omega(X, Y)^2 + 2 Z(V(\sigma)) - 2 Z(\sigma) V(\sigma) = 0 . \quad (3.4.9)$$

Because ${}^M\text{Ricci}(X, X) = {}^M\text{Ricci}(Z, Z)$ from relation (3.4.1) we obtain that the function $-\frac{1}{2} e^{4\sigma} h(i_X \Omega, i_X \Omega) + 2 Z(\sigma)^2$ is basic, equivalently, $-\frac{1}{2} e^{4\sigma} \Omega(X, Y)^2 + 2 Z(\sigma)^2$ is basic. This implies that

$$e^{4\sigma} V(\sigma) \Omega(X, Y)^2 = 2 V(Z(\sigma)) Z(\sigma) . \quad (3.4.10)$$

From (3.4.9) and (3.4.10) it follows that

$$V(Z(\sigma)) = 2 V(\sigma) Z(\sigma) \quad (3.4.11)$$

which is equivalent to the fact that $e^{-2\sigma} Z(\sigma)$ is basic. Hence $Z(V(e^{-2\sigma})) = V(Z(e^{-2\sigma})) = 0$. This implies that, if $V(e^{-2\sigma})$ is nonconstant, then its level hypersurfaces are horizontal and hence $\mathcal{H}$ is integrable; then the proof follows from Theorem 3.4.9. There remains to be considered the case when $V(e^{-2\sigma})$ is a constant, say $c \in \mathbb{R}$. By replacing, if necessary, V with $-V$ we can assume that $c \geq 0$. Since $V(e^{-2\sigma})$ is basic, $V(V(e^{-2\sigma})) = 0$, equivalently,

$$V(V(\sigma)) = 2 V(\sigma)^2 . \quad (3.4.12)$$

From (3.4.8) and (3.4.11) it follows that

$$({}^h\text{d}^* \Omega)(Z) = -4 e^{-4\sigma} Z(\sigma) V(\sigma) . \quad (3.4.13)$$

From (3.4.10) and (3.4.11) it follows that either

$$(a) \ V(\sigma) = 0 \quad \text{or} \quad (b) \ \Omega(X, Y)^2 = 4 e^{-4\sigma} Z(\sigma)^2$$

which, after replacing, if necessary, one of the vector fields X, Y or Z with its negative, is equivalent to

$$\Omega(X, Y) = Z(e^{-2\sigma}) . \quad (3.4.14)$$

In case (a), $\mathcal{V}$ is Riemannian and the proof follows from Theorem 3.4.9. So suppose from now on that (3.4.14) holds.

By (3.4.11) the horizontal component of $d(e^{-2\sigma})$ is basic and thus we can locally write

$$\mathcal{H}^*(d(e^{-2\sigma})) = \varphi^*(B) \quad (3.4.15)$$

where φ is as in Lemma 3.4.7 and B is a one-form on N . Hence $\varphi^*(dB) + c d\theta = 0$ where θ is, as before, the vertical dual of V . But Ω is also basic and hence $\Omega = \varphi^*(F)$ for some two-form F on N . It follows that

$$dB = -c F. \quad (3.4.16)$$

Because $\mathcal{V}$ is Riemannian and has geodesic leaves with respect to h we have that $\varphi^*(\bar{h}d^*F) = \mathcal{H}^*({}^h d^*\Omega)$ where $\bar{h}d^*$ is the codifferential on $(N, \bar{h})$.

Now (3.4.13) can be written:

$$\bar{h}d^*F = -c B. \quad (3.4.17)$$

Also (3.4.14) can be written:

$$F = *B \quad (\text{equivalently, } *F = B) \quad (3.4.18)$$

where $*$ is the Hodge star-operator (locally) induced on $(N, \bar{h})$ by the (local) orientation corresponding to $\{\varphi_*X, \varphi_*Y, \varphi_*Z\}$.

From (3.4.1) it easily follows that

$$\begin{aligned} {}^N\text{Ricci}(\varphi_*X, \varphi_*Y) &= {}^N\text{Ricci}(\varphi_*Y, \varphi_*Z) = {}^N\text{Ricci}(\varphi_*Z, \varphi_*X) = 0, \\ {}^N\text{Ricci}(\varphi_*X, \varphi_*X) &= {}^N\text{Ricci}(\varphi_*Y, \varphi_*Y). \end{aligned}$$

Also by (3.4.1), ${}^N\text{Ricci}(\varphi_*X, \varphi_*X) = {}^N\text{Ricci}(\varphi_*Z, \varphi_*Z)$ if and only if (3.4.14) holds. Thus $(N^3, \bar{h})$ is Einstein and, because it is three-dimensional, it is of constant sectional curvature k^N . Then ${}^N\text{Ricci} = 2k^N \bar{h}$.

Now from (3.4.3) we obtain

$$e^{2\sigma} c^M = e^{2\sigma} \Delta^M \sigma + \frac{1}{2} e^{8\sigma} \Omega(X, Y)^2 + 4V(V(\sigma)) - 10V(\sigma)^2$$

which together with (3.4.12) gives

$$e^{-2\sigma} c^M = e^{-2\sigma} \Delta^M \sigma + \frac{1}{2} e^{4\sigma} \Omega(X, Y)^2 - 2 e^{-4\sigma} V(\sigma)^2 . \quad (3.4.19)$$

It is easy to see by using (3.4.14) that (3.4.19) implies

$$\frac{1}{2} \Delta^M (e^{-2\sigma}) + e^{-2\sigma} c^M + c^2 = 0 . \quad (3.4.20)$$

Also by (3.4.1) we have

$$e^{-2\sigma} c^M = {}^N\text{Ricci}(\varphi_* X, \varphi_* X) - \frac{1}{2} e^{4\sigma} \Omega(X, Y)^2 - e^{-2\sigma} \Delta^M \sigma . \quad (3.4.21)$$

From formulae (3.4.19) and (3.4.21) it follows that

$$\begin{aligned} {}^N\text{Ricci}(\varphi_* X, \varphi_* X) &= e^{-2\sigma} c^M + \frac{1}{2} e^{4\sigma} \Omega(X, Y)^2 + e^{-2\sigma} \Delta^M \sigma \\ &= e^{-2\sigma} c^M + (e^{-2\sigma} c^M + 2 e^{-4\sigma} V(\sigma)^2) \end{aligned}$$

and thus

$${}^N\text{Ricci}(\varphi_* X, \varphi_* X) = 2 e^{-2\sigma} c^M + 2 e^{-4\sigma} V(\sigma)^2 . \quad (3.4.22)$$

Recall also that $(N, \bar{h})$ is of constant sectional curvature k^N and thus (3.4.22) is equivalent to

$$k^N = e^{-2\sigma} c^M + e^{-4\sigma} V(\sigma)^2 . \quad (3.4.23)$$

Now recall that $V(e^{-2\sigma}) = c = \text{constant}$ and thus (3.4.23) reads

$$k^N = e^{-2\sigma} c^M + \frac{c^2}{4} . \quad (3.4.24)$$

Now, if $\mathcal{V}$ is not Riemannian (i.e. $c > 0$), then $e^{-2\sigma}$ cannot be constant, and hence $c^M = 0$ and $k^N = \frac{c^2}{4} > 0$. After a homothetic transformation if necessary, we can suppose that $k^N = 1$ and hence $c = 2$. Then $\frac{1}{2} d(e^{-2\sigma})$ is a flat principal connection for $\mathcal{V}$ with respect to V . Put $A = -\frac{1}{2} B$. From equation (3.4.15) and the fact that $V(e^{-2\sigma}) = 2$ it follows that

$$\theta = \frac{1}{2} d(e^{-2\sigma}) + \varphi^*(A)$$

and hence A is a local connection form of θ with respect to $\frac{1}{2} d(e^{-2\sigma})$. Also $dA = F$ and this together with (3.4.18) implies that

$$dA + 2 * A = 0 . \quad (3.4.25)$$

We have proved that, at each point of M , one of the situations (i), (ii) or (iii) occurs.

Now by (3.4.20) it is obvious that there cannot exist any point where both (i) and (iii) occur.

Suppose that in an open connected subset $O \subseteq M$ which is the domain of the harmonic morphism $\varphi : (O, g|_O) \rightarrow (N, \bar{h})$ both situations (ii) and (iii) occurs. Then on the image (by φ) of the set where (iii) occurs from (3.4.25) it follows that

$$\Delta A = 4A . \quad (3.4.26)$$

But, on the image of the interior of the set where (ii) holds, the equation (3.4.26) is trivially satisfied. Thus (3.4.26) is satisfied on N ; this is an analytic manifold because $(N, \bar{h})$ has constant curvature. By the regularity of solutions of elliptic equations (see [11, p. 467]) we have that A is analytic and hence, if O contains interior points of the set where (ii) occurs, we have that $A = 0$ on N . Hence, if the set where (iii) occurs is nonempty and the interior of the set where (ii) occurs is nonempty, then the set where (iii) occurs is contained by the set of points where (ii) occurs.

Since by Theorem 3.4.9 on each connected component of the complement of the set where (iii) occurs the alternative (i) or (ii) holds globally the theorem is proved. $\square$

Remark 3.4.10. 1) Note that if A satisfies (3.4.25) then $F = dA$ satisfies

$$\bar{h} \, d^* F + 2 * F = 0 .$$

2) The codomain of a harmonic morphism of type (iii) of Theorem 3.4.4 always has constant positive sectional curvature. In the limit, when this tends to zero, we obtain a harmonic morphism of type (i). This follows from (3.4.24).

3) Harmonic morphisms of type (iii) are also of type (ii) if and only if $A = 0$.

4) The result of Theorem 3.4.4 shows that, on an Einstein manifold of dimension four, the nonlinear system of partial differential equations whose solutions are harmonic morphisms with fibres of dimension one can be reduced to one of three types of systems of linear partial differential equations of the first order.

For type (i) this is Killing's equation, and for type (iii) it is (3.4.25). Finally, the one-dimensional foliation $\mathcal{V}$ on (M, g) is of type (ii) if and only if it is locally generated by vector fields $W \in \Gamma(\mathcal{V})$ which satisfies $4\nabla W = \operatorname{div} W \operatorname{Id}_{TM}$ where ∇ is the Levi-Civita connection of (M, g) (see [49, Lemma 6.5]).

See [66] for other situations in which the nonlinear system of partial differential equations whose solutions are harmonic morphisms can be reduced to a linear system of partial differential equations of the first order.

3.5 The third type

We shall say that a harmonic morphism $\varphi : (M^4, g) \rightarrow (N^3, h)$ is of type (iii) ((i), (ii)) if its regular fibres form a foliation of type (iii) ((i), (ii)) of Theorem 3.4.4. In this section the harmonic morphisms of type (iii) will be the main object of study.

The first thing to note about the harmonic morphisms of type (iii) is that they are always submersive.

Proposition 3.5.1. *Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a harmonic morphism of type (iii).*

Then φ is submersive.

Proof. By passing, if necessary, to a two-fold covering, we can suppose that the vertical distribution $\mathcal{V}$ (which is well-defined outside the set of critical points) is orientable. Then, as before, let $V \in \Gamma(\mathcal{V})$ be such that $g(V, V) = \lambda^2$ where λ is the dilation of φ . Since, up to a multiplicative constant, $d(\lambda^{-2})$ is a (flat) principal connection with respect to V , we have that $V(\lambda^{-2})$ is a nonzero constant. This implies that the connected components of any regular fibre of φ are noncompact.

Suppose that φ is not submersive and let $x_0 \in M$ be a critical point of it. Recall that, by a result of P. Baird [5, Proposition 5.1], the set of critical points of φ must be discrete. Then from the main result of [17] it follows that φ is topologically locally equivalent at x_0 to the cone on the Hopf fibration $S^3 \rightarrow S^2$. Hence, in a neighbourhood of x_0 , the components of the regular fibres of φ are

diffeomorphic to S^1 . But we have seen that all the regular fibres of φ have noncompact components and hence φ cannot have critical points. $\square$

Remark 3.5.2. Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a harmonic morphism of type (iii); denote its dilation by λ . Then $\mathcal{H}^*(d(\lambda^{-2}))$ is a basic one-form; let $A \in \Gamma(T^*N)$ be such that $-\frac{1}{2} \mathcal{H}^*(d(\lambda^{-2})) = \varphi^*(A)$. Because (N^3, h) is of constant curvature it is an analytic manifold. But A satisfies (3.4.26) and so is analytic. Using this fact it is easy to see that, if N^3 is orientable, then there exists an orientation of it such that $dA + 2 * A = 0$ on N^3 .

From the proof of Theorem 3.4.4 it follows that any harmonic morphism of type (iii) is locally determined by the local connection form A . This is also illustrated by the following example.

Example 3.5.3. Let h be the canonical metric on the three-dimensional sphere S^3 . Let $A = i^*(-x^2 dx^1 + x^1 dx^2 - x^4 dx^3 + x^3 dx^4)$ where $i : S^3 \hookrightarrow \mathbb{R}^4$ is the canonical inclusion.

Let $*$ be the Hodge star-operator on (S^3, h) considered with the usual orientation of S^3 . Then

$$dA - 2 * A = 0.$$

To show this, first note that A is the canonical connection form on the Hopf bundle (S^3, S^2, S^1) . Also $|A| = 1$ and thus it suffices to verify that $A \wedge dA = 2 v_{S^3}$ where v_{S^3} is the usual volume form on S^3 .

For $a \in \mathbb{R}$ let g_a be the Riemannian metric on $\mathbb{R}^4 \setminus \{0\} = (0, \infty) \times S^3$ defined by

$$g_a = \rho^2 h + \rho^{-2} (\rho d\rho + aA)^2.$$

Then, for any $a \neq 0$, the canonical projection $\varphi_a : (\mathbb{R}^4 \setminus \{0\}, g_a) \rightarrow (S^3, h)$ is a harmonic morphism of type (iii) whilst g_0 is the restriction to $\mathbb{R}^4 \setminus \{0\}$ of the canonical metric on $\mathbb{R}^4$ and thus $\varphi_0 : \mathbb{R}^4 \setminus \{0\} \rightarrow S^3$ is the usual radial projection; note that this is also of type (ii).

Note that $(\mathbb{R}^4 \setminus \{0\}, g_a)$ is the Eguchi-Hanson II metric [21] and thus is Ricci-flat and anti-self-dual.

Let $\psi_a = \pi \circ \varphi_a$ where $\pi : S^3 \rightarrow S^2$ is the Hopf fibration. Then ψ_a is a

harmonic morphism with totally geodesic fibres. Any fibre of it is isometric with $(\mathbb{R}^2 \setminus \{0\}, \gamma_a)$ where γ_a is given in polar coordinates (ρ, θ) by

$$\gamma_a = \rho^2 d\theta^2 + \rho^{-2} (\rho d\rho + a d\theta)^2.$$

It is easy to see that any point of $\mathbb{R}^2 \setminus \{0\}$ is at finite distance from 0 with respect to γ_a . Hence $(\mathbb{R}^2 \setminus \{0\}, \gamma_a)$ is not complete. Because the fibres of ψ_a are closed and totally geodesic we obtain that g_a is not complete for any $a \in \mathbb{R}$.

We shall prove that the φ_a of Example 3.5.3 are, essentially, the only surjective harmonic morphisms of type (iii) with connected fibres and complete simply-connected codomain. For this we need the following:

Proposition 3.5.4. *Let $S^3 (= \text{Sp}(1))$ be the three-dimensional sphere endowed with its canonical metric and orientation and let $*$ be the Hodge star-operator on it.*

(i) *The space of solutions of the equation*

$$dA + 2 * A = 0, \quad A \in \Gamma(T^*S^3) \tag{3.5.1}$$

is the space of left-invariant one-forms on S^3 .

(ii) *The space of solutions of the equation*

$$dA - 2 * A = 0, \quad A \in \Gamma(T^*S^3) \tag{3.5.2}$$

is the space of right-invariant one-forms on S^3 .

Proof. (i) Let $S^3 \times \text{Sp}(1) \rightarrow S^3$ be the unique spin structure on S^3 and let $S^3 \times \mathbb{H} \rightarrow S^3$ be the spinor bundle induced by the action of the Clifford algebra $Cl_3 = \mathbb{H} \oplus \mathbb{H}$ on $\mathbb{H}$ given by $(x, y) \cdot q = x \cdot q$.

Consider the trivialization $TS^3 = S^3 \times \text{Im } \mathbb{H}$ induced by the canonical left action of $S^3 (= \text{Sp}(1))$ on itself. Thus any one-form A on S^3 can be viewed as a spinor field $A : S^3 \rightarrow \text{Im } \mathbb{H} \subseteq \mathbb{H}$ which is constant if and only if the corresponding one-form is left-invariant.

Consider the Dirac operator D obtained by using the trivial flat connection on $S^3 \times \mathbb{H} \rightarrow S^3$. Then it is easy to see that $A \in \Gamma(T^*S^3)$ satisfies (3.5.1) if and

only if $DA = 0$. Also a straightforward calculation gives $D^2 = \Delta + 2D$ where Δ is the usual Laplacian acting on $\mathbb{H}$ -valued functions on S^3 . Thus, any solution A of (3.5.1) induces a harmonic $\mathbb{H}$ -valued function on S^3 which must be constant if A is globally defined on S^3 .

(ii) Since the isometry $x \mapsto x^{-1}$ of S^3 reverses the orientation, it pulls back solutions of (3.5.1) to solutions of (3.5.2). Thus the proof of (ii) follows from (i). $\square$

Remark 3.5.5. There are other ways to describe the solutions of the equations (3.5.1) and (3.5.2). For example, since any orthogonal complex structure on $\mathbb{R}^4 (= \mathbb{H})$ compatible with the canonical orientation can be described as left multiplication by imaginary quaternions of length one (see [15]) any solution of (3.5.2) is, up to a multiplicative constant, of the form

$$A = i^* \left(\sum_{a,b} J_b^a x^b dx^a \right)$$

where J is any orthogonal complex structure which induces the canonical orientation on $\mathbb{R}^4$ (i.e. if $\{u_1, u_2\}$ is a complex basis of $(\mathbb{R}^4, J)$ then $\{u_1, Ju_1, u_2, Ju_2\}$ is positively oriented) and $i : S^3 \hookrightarrow \mathbb{R}^4$ is the canonical inclusion. This can also be checked directly.

Also, any solution A of (3.5.2) can be written $A = *i^*(F)$ where $F \in \Lambda_+^2(\mathbb{R}^4)$ is a self-dual two-form.

In fact, by using these characterisations an alternative proof for Proposition 3.5.4 can be obtained. First, note that, for each one of the equations (3.5.1) and (3.5.2) we have a three dimensional space of solutions. Then, it is easy to see that if A satisfies (3.5.1) or (3.5.2) then A is coclosed and $\Delta A = 4A$ where Δ is the Hodge Laplacian on S^3 . Thus A is in the eigenspace corresponding to the first eigenvalue of Δ acting on coclosed one-forms of S^3 and it is well-known that this space is of dimension six (see [25, 7.2] or apply one of the results from [42, page 148] and [36, Chapter II, Theorem 2.3]).

Proposition 3.5.6. *Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a surjective harmonic morphism of type (iii) such that (N^3, h) is complete, simply-connected and φ has*

connected fibres and orientable vertical distribution.

Then there exists $a \in \mathbb{R}$ such that, up to homotheties, φ is a restriction of $\varphi_a : (\mathbb{R}^4 \setminus \{0\}, g_a) \rightarrow (S^3, h)$ from Example 3.5.3.

Proof. Up to a homothety, we can identify (N^3, h) with S^3 considered with its canonical metric and orientation. Let λ be the dilation of φ . Then, by Proposition 3.5.4, there exists $a \in \mathbb{R}$ such that, up to an isometry of S^3 , $-\frac{1}{2}\mathcal{H}^*(d(\lambda^{-2})) = a\varphi^*(A)$ where $A \in \Gamma(T^*S^3)$ is as in Example 3.5.3. By Proposition 3.5.1, φ is submersive and let $\mathcal{V} = \ker \varphi_*$. Because $\mathcal{V}$ is orientable we can find $V \in \Gamma(\mathcal{V})$ such that $g(V, V) = \lambda^2$.

Because φ is of type (iii) we have that $V(\lambda^{-2})$ is a nonzero constant. This implies that the restriction of λ to any fibre of φ is a diffeomorphism onto some open subinterval of $(0, \infty)$. Hence the map $\Phi : M^4 \rightarrow S^3 \times (0, \infty)$ defined by $\Phi(x) = (\varphi(x), \lambda(x)^{-1})$, $x \in M^4$, is a diffeomorphic embedding.

Then from the proof of Theorem 3.4.4 it follows that $\Phi : (M^4, g) \rightarrow (S^3 \times (0, \infty), g_a)$ is a local isometry and hence an isometric embedding. Also, it is obvious that $\varphi_a \circ \Phi = \varphi$. $\square$

Corollary 3.5.7. *Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a surjective harmonic morphism of type (iii) such that (N^3, h) is complete.*

Then (M^4, g) is not complete.

Proof. By passing, if necessary, to a two-fold covering of M^4 we can suppose that $\mathcal{V}$ is orientable.

Up to homotheties, the universal covering of (N^3, h) is S^3 with its canonical metric and orientation. Then, φ can be pulled back via $S^3 \rightarrow N^3$ to a harmonic morphism whose total space is complete if and only if (M^4, g) is complete. Define Φ as in the proof of Proposition 3.5.6. Then Φ is a local isometry and because $(\mathbb{R}^4 \setminus \{0\}, g_a)$ is not complete (M^4, g) is not complete. $\square$

3.6 A few applications

From Theorem 3.4.4 and Corollary 3.5.7 we can exclude the third type in the complete case as follows:

Theorem 3.6.1. *Let (M^4, g) be a complete Einstein manifold of dimension four and let (N^3, h) be a complete Riemannian manifold of dimension three.*

Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a surjective harmonic morphism.

Then, either:

- (i) the regular fibres of φ form a Riemannian foliation locally generated by Killing fields or*
- (ii) φ is horizontally homothetic and has geodesic fibres orthogonal to an umbilical foliation by hypersurfaces.*

Recall that for a horizontally weakly conformal map the vertical distribution is well-defined outside the set of critical points. We analyse the behaviour of φ at a critical point. The model is the cone on the Hopf fibration $S^3 \rightarrow S^2$ which, as is well-known, can be written as a quadratic polynomial:

Definition 3.6.2 (see [9]). The *Hopf polynomial* is the harmonic morphism $\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ defined by

$$\varphi(z^1, z^2) = (|z^1|^2 - |z^2|^2, 2z^1 \overline{z^2})$$

via the standard identifications $\mathbb{R}^4 = \mathbb{C}^2$, $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$.

Note that the Hopf polynomial has an isolated critical point at the origin of $\mathbb{R}^4$.

Corollary 3.6.3. *Let (M^4, g) be an Einstein manifold of dimension four.*

Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a nonconstant harmonic morphism which has critical points and orientable vertical distribution.

Then there exists a Killing field $V \in \Gamma(TM)$ tangent to the fibres of φ which vanishes precisely at the critical points of φ . Moreover, in a neighbourhood of each critical point, φ is smoothly equivalent to the Hopf polynomial.

Proof. Because φ has critical points and (M^4, g) is Einstein, φ must be of type (i). (By Proposition 3.5.1 it cannot be of type (iii) and from the main result of [24] it follows that φ cannot be of type (ii).)

Let λ be the dilation of φ and let $\mathcal{V}$ be the vertical distribution of φ . Let $V \in \Gamma(\mathcal{V})$ be such that $g(V, V) = \lambda^2$.

Obviously, V can be extended to a continuous vector field on M whose zero set is equal to C_φ , the set of critical points of φ .

Then V is a Killing field on $(M \setminus C_\varphi, g|_{M \setminus C_\varphi})$ [13] (cf. [9], [47]). Hence it satisfies the equation

$$\nabla^* \nabla V = {}^M \text{Ricci}(V) \quad (3.6.1)$$

(see, for example, [36, page 44]), where ${}^M \text{Ricci} \in \Gamma(TM \otimes T^*M)$ denotes the (1,1) tensor field associated to the Ricci tensor of (M^4, g) . From the regularity of solutions of elliptic equations (see [11, page 467]) it follows that V is a smooth (in fact, analytic) vector field on M .

By Baird's result ([5, Proposition 5.1]) φ has isolated critical points. Then in a neighbourhood of the critical point $x \in M$ the local flow of V is equivalent, via the exponential map at x , with the flow (ξ_t) on $T_x M$ given by the vector field with value $\nabla_w V$ at $w \in T_x M$. Recall that $(\nabla V)_x : T_x M \rightarrow T_x M$ induces an orthogonal complex structure on $(T_x M, g_x)$ (see [36]). Hence (ξ_t) induces an S^1 action on $T_x M$ which must be free outside zero since its quotient is equivalent to φ in a neighbourhood of x . From this fact the last assertion of the proposition easily follows. $\square$

Remark 3.6.4. Since (3.6.1) holds in general, the conclusion of the above corollary holds for any harmonic morphism $\varphi : (M^4, g) \rightarrow (N^3, h)$ of type (i) and with orientable vertical distribution.

Corollary 3.6.5. *Let (M^4, g) be a complete Einstein manifold of dimension four and let (N^3, h) be a complete Riemannian manifold of dimension three.*

Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a surjective submersive harmonic morphism with orientable vertical distribution; denote its dilation by λ . Let W be a vertical vector field such that $g(W, W) = \lambda^{-2}$ and suppose that W is complete.

Then φ is of type (i) and there exists a globally defined nowhere zero Killing field tangent to the fibres of φ .

Proof. Suppose that (ii) of Theorem 3.6.1 occurs. Then, either directly or by using [47, Lemma 4.3], it can be shown that $\nabla W = \mu \text{Id}_{TM}$ where μ is a smooth

function on M^4 . In particular, W is conformal. By a result of K. Yano and T. Nagano (see [36]), either W is Killing or (M^4, g) is S^4 with its canonical metric. But in the latter case W would have two zeroes. The proof follows from Theorem 3.6.1. $\square$

Proposition 3.6.6. *Let (M^4, g) be a compact Einstein manifold (${}^M\text{Ricci} = c^M g$) of dimension four and let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a nonconstant harmonic morphism with orientable vertical distribution.*

Then there exists a Killing field tangent to the fibres of φ and nowhere zero on the set of regular points. In particular, the Euler number of M^4 is equal to the number of critical points of φ .

If φ has critical points, then $c^M > 0$ and ${}^Nb_1 = {}^Nb_2 = 0$, where Nb_1 and Nb_2 denote the Betti numbers of N^3 .

Proof. Because φ has compact fibres it cannot be of the third type.

From Corollary 3.6.3 and Corollary 3.6.5 it follows that there exists a Killing vector field tangent to the fibres of φ . Then the fact that $c^M > 0$ follows from Corollary 3.6.3 and equation (3.6.1). Also by a well-known result of S. Bochner we have ${}^Mb_1 = 0$. From [18, 7.14] it follows that ${}^Nb_1 = 0$ and by Poincaré duality ${}^Nb_2 = {}^Nb_1 = 0$. $\square$

Remark 3.6.7. More generally, if M^4 is an arbitrary compact manifold then the number of critical points of $\varphi : M^4 \rightarrow N^3$ is equal to the Euler number of M^4 provided that the smooth map φ has isolated critical points [58, Proposition 4.2(iii)].

Recall that by the well-known result of M. Berger the Euler number of a compact Einstein manifold (M^4, g) is nonnegative and is zero if and only if (M^4, g) is flat (see [11, 6.32]).

We can now completely describe submersive harmonic morphisms with one-dimensional fibres defined on a compact Einstein four-manifold as follows.

Theorem 3.6.8. *Let (M^4, g) be a compact Einstein manifold of dimension four. Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a submersive harmonic morphism.*

Then (M^4, g) and (N^3, h) are flat and the fibres of φ are locally generated by parallel vector fields. In particular, up to homotheties and Riemannian coverings, φ is the canonical projection between flat tori $T^4 \rightarrow T^3$.

Proof. By passing, to a two-fold covering if necessary, we can suppose that the vertical distribution of φ is orientable. Then, from Proposition 3.6.6 it follows that there exists a Killing vector field V on (M^4, g) tangent to the fibres of φ and the Euler number of M^4 is zero. Hence (M^4, g) is flat by the above mentioned result of Berger. From a result of Bochner (see [11, 1.84]) V is parallel.

The fact that (N^3, h) is flat follows from Lemma B.1.5.

Because φ has compact regular fibres, from a well-known result of C. Ehresmann [22], it follows that the leaf space of the foliation whose leaves are the components of the regular fibres of φ is smooth. Thus by factorising φ , if necessary, into a harmonic morphism followed by a Riemannian covering we can suppose that φ has connected fibres. Thus, φ is, up to homotheties, the quotient induced by V . Hence, φ is the projection of a S^1 -principal bundle and the horizontal distribution $\mathcal{H}$ is a flat principal connection on it. Then, each holonomy bundle P of it is a regular covering over N^3 with group the holonomy group $H(\subseteq S^1)$ of $\mathcal{H}$. Moreover, because (M^4, g) is flat, P considered with the metric induced by g is flat (actually, up to homotheties, this is the unique metric with respect to which $P \rightarrow N$ becomes a Riemannian covering; in particular, P with the considered metric is complete.) Hence $M = P \times_H S^1$ and the pull back of φ by $P \rightarrow N$ is the projection $P \times S^1 \rightarrow P$. To end the proof, recall ([37, Chapter V, Theorem 4.2]) that P is covered by an Euclidean cylinder or by a torus. $\square$

Remark 3.6.9. 1) The condition that φ be submersive, in Theorem 3.6.8, can be removed if we assume that (N^3, h) is of constant curvature (see Theorem 3.7.11).

2) Recall that K3 surfaces and the tori cannot carry any metric of positive scalar curvature (see [11, 4.34]). Hence, from Proposition 3.6.6 and Theorem 3.6.8 we obtain the following:

- There exists no K3 surface endowed with an Einstein metric which is the domain of a harmonic morphism whose regular fibres are of dimension one.
- Let g be an Einstein metric on the torus T^4 and let $\varphi : (T^4, g) \rightarrow (N^3, h)$

be a nonconstant harmonic morphism. Then g and h are flat, φ is submersive and, up to homotheties and Riemannian coverings, φ is the canonical projection $T^4 \rightarrow T^3$.

Corollary 3.6.10. *Let (M^4, g) be a compact Einstein manifold of dimension four.*

Then there exists no submersive harmonic morphism $\varphi : (M^4, g) \rightarrow (N^3, h)$ if either ${}^Nb_1 = 0$ or ${}^Mb_1 = 0$.

3.7 Harmonic morphisms $\varphi : (M^4, g) \rightarrow (N^3, h)$ between Einstein manifolds

In this section (M^4, g) and (N^3, h) will be Einstein manifolds of dimension four and three, respectively, (since N^3 is three-dimensional this means that (N^3, h) is of constant curvature) and $\varphi : (M^4, g) \rightarrow (N^3, h)$ will be a harmonic morphism. Recall that, by a result of P. Baird [5, Proposition 5.1], the set of critical points of φ is discrete and hence, by the second axiom of countability, at most countable.

We now state one of the main results of this section enumerating all surjective submersive harmonic morphisms between complete simply-connected Einstein manifolds of dimension four and three, respectively.

Theorem 3.7.1. *Let (M^4, g) be a complete simply-connected Einstein manifold and let (N^3, h) be complete, simply-connected and with constant curvature.*

Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a surjective submersive harmonic morphism.

Then, up to homotheties, φ is one of the following projections: $\mathbb{R}^4 \rightarrow \mathbb{R}^3$, $H^4 \rightarrow \mathbb{R}^3$, $H^4 \rightarrow H^3$ induced by the following canonical warped-product decompositions $\mathbb{R}^4 = \mathbb{R}^1 \times \mathbb{R}^3$, $H^4 = H^1 \times_r \mathbb{R}^3$, $H^4 = H^1 \times_s H^3$ where H^k denotes the hyperbolic space of dimension k .

Proof. First we prove that (M^4, g) has constant curvature and that φ has geodesic fibres and integrable horizontal distribution.

By Theorem 3.6.1, either (i) the vertical distribution of φ is Riemannian and locally generated by Killing fields or (ii) φ has geodesic fibres and integrable

horizontal distribution. Suppose that case (i) holds and let $\mathcal{V} = \ker \varphi_*$ be the vertical distribution. We can choose a local orthonormal frame $\{X, Y, Z\}$ for $\mathcal{H} (= \mathcal{V}^\perp)$ with respect to h made up of basic fields and such that $i_Z \Omega = 0$.

Because (M^4, g) and (N^3, h) are both Einstein, from (3.4.1) we obtain

$$X(\sigma)Y(\sigma) = 0 \quad \text{and} \quad X(\sigma)^2 = Y(\sigma)^2. \quad (3.7.1)$$

Thus $X(\sigma) = Y(\sigma) = 0$.

From (3.4.1) it follows that (3.4.14) holds after, replacing, if necessary, one of the vector fields X, Y, Z or V by its negative. Choose the (local) orientation on N^3 such that $\{\varphi_*X, \varphi_*Y, \varphi_*Z\}$ is positively oriented.

Let $\check{\lambda}$ be the function on N^3 such that $e^\sigma = \varphi^*(\check{\lambda})$ and let F be the two-form on N^3 such that $\Omega = \varphi^*(F)$. Then (3.4.14) implies that

$$F = * d\check{\lambda}^{-2}. \quad (3.7.2)$$

where $*$ is the Hodge star-operator on (N^3, h) .

But $dF = 0$ and thus (3.7.2) implies that $\check{\lambda}^{-2}$ is a positive harmonic function on (N^3, h) .

From (3.4.1) and (3.4.3) we obtain

$$k^N = \check{\lambda}^{-2} c^M \quad (3.7.3)$$

where k^N is the constant sectional curvature of (N^3, h) and c^M is the Einstein constant of (M^4, g) . Thus either $\check{\lambda}$ is constant or $k^N = c^M = 0$. But in the latter case, by Liouville's theorem, $\check{\lambda}^{-2}$ must be constant. Hence $\check{\lambda}$ is constant and, by [6, Theorem 5.2], φ has geodesic fibres. Moreover, by (3.7.2), $F = 0$ and thus $\mathcal{H}$ is integrable.

Thus, we always have case (ii). The fact that (M^4, g) has constant curvature now follows from Corollary 3.3.2.

Next we prove that the horizontal distribution $\mathcal{H}$ is an Ehresmann connection for φ (see [11, 9.39] for the definition). Let $\check{\gamma} : [0, 1] \rightarrow N^3$ be a curve in N^3 and let $x \in \varphi^{-1}(\gamma(0))$. Recall that locally horizontal lifts exist and are unique. Thus there is a maximal interval $[0, b) \subseteq [0, 1]$ on which there can be defined a

horizontal lift $\gamma : [0, b) \rightarrow M^4$ of $\tilde{\gamma}|_{[0, b)}$ such that $\gamma(0) = x$. Obviously, $\tilde{\gamma}([0, 1])$ is a bounded subset of (N^3, h) . Because φ is horizontally homothetic this implies that $\gamma([0, b))$ is bounded in (M^4, g) . But (M^4, g) is complete and hence the closure of $\gamma([0, b))$ is compact. Hence the closure of $\gamma([0, b))$ (which is obviously connected) is contained in an integral manifold of $\mathcal{H}$ with the property that its intersection with any fibre of φ is discrete. This implies that γ can be extended to a continuous curve defined on $[0, b]$ which we shall denote again by γ . Now, φ restricted to a neighbourhood of $\gamma(b)$ is the projection of a warped-product. Using this, it is easy to see that $\gamma : [0, b] \rightarrow M^4$ is a (smooth) horizontal lift of $\tilde{\gamma}|_{[0, b]}$. If we had $b < 1$ then γ could be extended to a horizontal lift $[0, b + \epsilon) \rightarrow M^4$ of $\tilde{\gamma}|_{[0, b + \epsilon)}$ which would contradict the maximality of $[0, b) \subseteq [0, 1]$. Thus $b = 1$ and hence $\mathcal{H}$ is an Ehresmann connection for φ .

Because $\mathcal{H}$ is a flat (i.e. integrable) Ehresmann connection, any maximal integral submanifold of it is a covering space of N^3 . But N^3 is simply-connected and hence φ admits a (global) horizontal section. The proof of the theorem follows. $\square$

Corollary 3.7.2. *Let (M^4, g) be a complete simply-connected Einstein manifold and let (N^3, h) be complete, simply-connected and of constant curvature.*

Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a surjective submersive harmonic morphism; denote its dilation by λ . Let W be the vertical vector field such that $g(W, W) = \lambda^{-2}$. Suppose that W is complete.

Then, up to homotheties, φ is the orthogonal projection $\mathbb{R}^4 \rightarrow \mathbb{R}^3$.

Proof. This follows from Corollary 3.6.5 and Theorem 3.7.1. $\square$

Proposition 3.7.3. *Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a harmonic morphism between Einstein manifolds and let λ be its dilation. Suppose that the regular fibres of φ form a Riemannian foliation.*

Then, up to homotheties, φ can be (locally) characterised as follows:

- *(M^4, g) is Ricci-flat and (N^3, h) is flat;*
- *λ^{-2} is the pull back of a local positive harmonic function u on (N^3, h) (in particular, λ^{-2} is a harmonic function on (M^4, g));*

- Any local connection form $A (= s^*\theta)$ of the horizontal distribution satisfies

$$dA = * du$$

where $*$ is the Hodge star-operator of (N^3, h) with respect to some (local) orientation (equivalently, the curvature form $F = dA$ satisfies the monopole equation $F = * du$);

- In a neighbourhood of the local section s of φ where φ is equivalent to a projection we have

$$g = u h + u^{-1}(dt + A)^2 .$$

Proof. This follows from the proof of Theorem 3.7.1. □

Remark 3.7.4. 1) Note that the metric g of Proposition 3.7.3 is constructed by applying S.W. Hawking's ansatz [34] (cf. [41]).

2) Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a harmonic morphism between Einstein manifolds. If (M^4, g) does not have constant curvature or the horizontal distribution is nonintegrable then (M^4, g) is Ricci-flat and φ is of type (i) (and hence locally given as in Proposition 3.7.3) or type (iii) of Theorem 3.4.4. This follows from Theorem 3.4.4 and Corollary 3.3.2.

Let $a \geq 0$. If we apply the Hawking's ansatz (with the convention $dA = - * du$) to the harmonic function $u_a : \mathbb{R}^3 \setminus \{0\} \rightarrow (0, \infty)$ defined by $u_a(y) = \frac{1}{4}(\frac{1}{|y|} + a)$, $y \in \mathbb{R}^3 \setminus \{0\}$, then the following metric is obtained [34] (see [41]).

Definition 3.7.5. Let $a > 0$. The *Hawking Taub-NUT metric* is the Riemannian metric on $\mathbb{R}^4$ defined by

$$g_a = (a|x|^2 + 1) g_0 - \frac{a(a|x|^2 + 2)}{a|x|^2 + 1} (-x^2 dx^1 + x^1 dx^2 - x^4 dx^3 + x^3 dx^4)^2 .$$

For $a = 0$ this gives the canonical metric g_0 on $\mathbb{R}^4$.

Note that g_1 is discussed in [41].

Remark 3.7.6. 1) For any $a \geq 0$ the Hopf polynomial $\varphi : (\mathbb{R}^4, g_a) \rightarrow (\mathbb{R}^3, h_0)$ (see Definition 3.6.2) can be thought of as the harmonic morphism induced by

the isometric action (see Example 2.3.22(1)) of S^1 on $(\mathbb{R}^4, g_a)$ where h_0 is the canonical metric on $\mathbb{R}^3$. In particular, $(\mathbb{R}^4, g_a)$ is Ricci-flat for any $a \geq 0$.

2) Moreover, we can consider $a = \varphi^*(\check{a})$ to be the pull back of a nonnegative harmonic function $\check{a}$ defined in the neighbourhood of $0 \in \mathbb{R}^3$. Then, the resulting metric g_a is still Ricci-flat and with respect to it the Hopf polynomial, suitably restricted, is a harmonic morphism.

For the next construction we follow C. LeBrun's discussion [41] of Hawking's ansatz [34].

Example 3.7.7. Let $u : \mathbb{R}^3 \setminus C_u \rightarrow (0, \infty)$ be a positive harmonic function whose set of singularities $C_u = \{y_j\}_{j \in I}$ is discrete. Hence I is finite or countable. Thus by applying Bôcher's theorem, the 'minimum' and Harnack's principles (see [3]) we obtain

$$u(y) = a + \sum_{j \in I} \frac{b_j}{|y - y_j|} \quad (y \in \mathbb{R}^3) \quad (3.7.4)$$

where $a \geq 0$ and $b_j \geq 0$ are nonnegative constants. Suppose that u has the same residue $b (> 0)$ at each singular point, i.e. $b_j = b$ for each $j \in I$.

Let $F_u \in \Gamma(\Lambda^2(T^*(\mathbb{R}^3 \setminus C_u)))$ be defined by $F_u = - * du$ where $*$ is the Hodge star-operator on $\mathbb{R}^3$. Because u is harmonic we have $dF_u = 0$. Then, taking $S^1 = \mathbb{R}/4\pi b\mathbb{Z}$, the cohomology class

$$\frac{1}{4\pi b} [F_u] \in H^2(\mathbb{R}^3 \setminus C_u, \mathbb{Z}) = H^1(\mathbb{R}^3 \setminus C_u, S^1)$$

is the first Chern class of a principal bundle $\xi_u = (P_u, \mathbb{R}^3 \setminus C_u, S^1)$ with projection $\psi_u : P_u \rightarrow \mathbb{R}^3 \setminus C_u$. It is not difficult to see, by using the homotopy sequence of ξ_u , that P_u is simply-connected.

As is well-known, F_u is the curvature form of a principal connection given by $\theta_u \in \Gamma(T^*P_u)$. Note that if A is a local connection form of θ_u with respect to some local section of ξ_u , then $dA = - * du$.

Let h_0 be the canonical metric on $\mathbb{R}^3$ and define $\gamma_u = \psi_u^*(u h_0) + \psi_u^*(u^{-1})\theta_u^2$. Then $\psi_u : (P_u, \gamma_u) \rightarrow (\mathbb{R}^3 \setminus C_u, h_0|_{\mathbb{R}^3 \setminus C_u})$ is a harmonic morphism.

The key point of the construction is the fact that ψ_u can be extended to a harmonic morphism whose codomain is $\mathbb{R}^3$.

To prove this, first note that if $C_u = \{0\}$ then ξ_u is the cylinder on the Hopf bundle (S^3, S^2, S^1) and hence ψ_u is the restriction of the Hopf polynomial to $\mathbb{R}^4 \setminus \{0\}$. Moreover, one can easily verify that γ_u is homothetic to the restriction of the Hawking Taub-NUT metric g_{4a} to $\mathbb{R}^4 \setminus \{0\}$ where, from now on, we consider, for simplicity, that $b = \frac{1}{4}$.

Let $v(y) = \frac{b}{|y-y_1|}$ and $w = u - v$. Then

$$\xi_u = \xi_{v+w} = \xi_v|_{\mathbb{R}^3 \setminus C_u} \cdot \xi_w|_{\mathbb{R}^3 \setminus C_u}$$

where ‘ $\cdot$ ’ denotes the group operation in $H^1(\mathbb{R}^3 \setminus C_u, S^1)$. There exists a neighbourhood U of y_1 such that $U \cap C_u = \{y_1\}$ and hence $w|_U$ is a well-defined positive harmonic function. By taking U to be contractible we get that $\xi_w|_U$ is trivial (equivalently, it is the neutral element of $H^1(U, S^1)$). Then $\xi_u|_{U \setminus \{y_1\}} = \xi_v|_{U \setminus \{y_1\}}$ and hence ψ_u can be extended so that its image contains y_1 . More precisely, we can add a point x_1 to $\psi_u^{-1}(U)$ such that the extended map is smoothly equivalent in a neighbourhood of x_1 to the cone on the Hopf fibration $S^3 \rightarrow S^2$. Moreover, because w has no singularities in U the metric γ_u extends over x_1 to a metric which is homothetic, in the neighbourhood of x_1 , to the metric g_{4w} of Remark 3.7.6(2).

In this way (P_u, γ_u) can be extended to a Riemannian manifold (M_u, g_u) and ψ_u can be extended to a surjective harmonic morphism $\varphi_u : (M_u, g_u) \rightarrow (\mathbb{R}^3, h_0)$ where h_0 is the canonical metric on $\mathbb{R}^3$. Note that (M_u, g_u) is Ricci-flat, simply-connected and that φ_u is induced by an isometric action.

We can now state the next main result of this section enumerating all the surjective harmonic morphisms with critical points between complete, simply-connected Einstein manifolds of dimension four and three.

Theorem 3.7.8. *Let (M^4, g) be a complete Einstein manifold and let (N^3, h) be complete, simply-connected and with constant curvature.*

Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a surjective harmonic morphism; denote its dilation by λ . Suppose that φ has critical points.

Then, up to homotheties, $(N^3, h) = (\mathbb{R}^3, h_0)$ where h_0 is the canonical metric on $\mathbb{R}^3$. Moreover, $\lambda^{-2} = \varphi^(u)$ for a positive harmonic function $u : \mathbb{R}^3 \setminus C_u \rightarrow (0, \infty)$*

having the same (positive) residue at each (fundamental) pole $y \in C_u$, $(M^4, g) = (M_u^4, g_u)$ and $\varphi = \varphi_u$.

Proof. By Corollary 3.6.3 there exists a Killing field V on (M, g) tangent to the fibres of φ .

Although φ has critical points, an argument due to R. Hermann (see [11, 9.45]) can be adapted to prove that the horizontal distribution $\mathcal{H}$ (which is well-defined outside the set of critical points) is an Ehresmann connection [12] for φ restricted to the set of regular points. By applying [11, 9.40], it is easy to see that φ can be factorised into a harmonic morphism with connected fibres followed by a Riemannian covering over (N^3, h) . But the latter must be trivial because N^3 is simply-connected and hence φ has connected fibres.

Now, as in the proof of Theorem 3.7.1 we obtain (3.7.3) and the monopole equation (3.7.2) and hence $\check{\lambda}^{-2}$ is a harmonic function where $\lambda = \varphi^*(\check{\lambda})$.

Because φ has critical points its dilation cannot be constant. This, together with (3.7.3), implies that (M^4, g) is Ricci-flat and (N^3, h) is flat. Hence, up to homotheties, $(N^3, h) = (\mathbb{R}^3, h_0)$ where h_0 is the canonical metric on $\mathbb{R}^3$.

Using the completeness of (M^4, g) and the fact that V is Killing it is not difficult to prove (directly or by using Theorem 3.1.9) that the restriction of φ to the set of regular points is the projection of a principal bundle ξ with group $(\mathbb{R}, +)$ or $(S^1, \cdot)$ and the horizontal distribution is a principal connection on it. But φ extends the projection of ξ over the critical points. Hence in the neighbourhood of each critical point ξ is a restriction of the cylinder on the Hopf bundle (S^3, S^2, S^1) or its dual. Hence the structural group of ξ is $S^1 = \mathbb{R}/L\mathbb{Z}$ where $L (> 0)$ is the period of the orbits of V .

Let $\{y_j\}_{j \in I}$ be the set of critical values of φ . Using the Chern-Weil morphism (see [37]) and (3.7.2) it is easy to see that the first Chern number of ξ suitably restricted to a sphere about any y_j is given by $c_1 = -4\pi b_j/L$ where $b_j (> 0)$ is the residue of $\check{\lambda}^{-2}$ at y_j . But we must have $c_1 = \pm 1$ and hence $b_j = b_k$ for any $j, k \in I$ and the proof follows. $\square$

Remark 3.7.9. Note that the period L of V is the *mass* of the regular fibres, i.e. $L = \int_{\text{fibre}} \rho$ where $\rho (= \lambda^{-1})$ is the density of φ . Because ρ is constant along the

fibres we have that L is equal to $\rho|_{\text{fibre}}$ multiplied by the length of the considered fibre (see Definition 4.2.1).

We end with two other classification results:

Theorem 3.7.10. *Let (M^4, g) be a complete Einstein manifold and let (N^3, h) be complete, simply-connected and with constant curvature.*

Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a surjective harmonic morphism. Suppose that φ has exactly one critical point.

Then there exists $a \geq 0$ such that, up to homotheties, $\varphi : (\mathbb{R}^4, g_a) \rightarrow (\mathbb{R}^3, h_0)$ is the Hopf polynomial with g_a the Hawking Taub-NUT metric ($a > 0$) and g_0 , h_0 the canonical metrics on $\mathbb{R}^4$, $\mathbb{R}^3$, respectively.

Proof. This follows from Theorem 3.7.8. □

Theorem 3.7.11. *Let (M^4, g) be a compact Einstein manifold of dimension four. Let (N^3, h) be a Riemannian manifold of dimension three with constant curvature. Let $\varphi : (M^4, g) \rightarrow (N^3, h)$ be a nonconstant harmonic morphism.*

Then (M^4, g) and (N^3, h) are flat, φ is submersive and its fibres are locally generated by parallel vector fields. In particular, up to homotheties and Riemannian coverings, φ is the canonical projection $T^4 \rightarrow T^3$ between flat tori.

In particular there exists no harmonic morphism with one-dimensional fibres from a compact Einstein manifold of dimension four to S^3 .

Proof. Suppose that φ has critical points. Then by Proposition 3.6.6 we have that $c^M > 0$. From (3.7.3) it follows that the dilation of φ is constant and hence φ cannot have critical points.

Hence φ is submersive and the proof follows from Theorem 3.6.8. □

Remark 3.7.12. We do not know any example of a harmonic morphism with critical points and one-dimensional regular fibres which is defined on a compact four-dimensional Einstein manifold.

Chapter 4

Harmonic morphisms on compact Riemannian manifolds

4.1 Mixed curvature and harmonic morphisms

In this section we give some applications of (A.1.1) of Appendix A to harmonic morphisms.

The following Proposition is a generalization to conformal one-dimensional foliations of the corresponding results for Riemannian one-dimensional foliations from [54].

Proposition 4.1.1. *Let (M, g) be compact.*

(i) *If (M, g) has nonpositive Ricci curvature, then any conformal one-dimensional foliation is Riemannian and its orthogonal complement is a totally geodesic foliation. Further, $\text{Ricci}(U, U) = 0$ for any U tangent to the foliation.*

(ii) *If (M, g) has negative Ricci curvature then there exists no one-dimensional conformal foliation on it.*

Proof. By passing to a finite covering, if necessary, we can suppose that both the foliation $\mathcal{V}$ and the manifold M are oriented.

Since $\mathcal{V}$ is conformal we have ${}^{\mathcal{H}}B_0 = 0$ where ${}^{\mathcal{H}}B_0$ is the trace-free part of ${}^{\mathcal{H}}B$. But, as for any codimension one foliation, $\mathcal{H}$ is also conformal. Hence ${}^{\mathcal{V}}B_0 = 0$.

Next, note that, because $\mathcal{V}$ is one-dimensional, the mixed curvature is equal

to the Ricci curvature restricted to $\mathcal{V}$.

Thus integrating (A.1.1) gives

$$\int_M \text{Ricci}(U, U) v_g = \int_M \left\{ \frac{p-1}{p} |\text{trace}({}^{\mathcal{H}}B)|^2 + \frac{1}{4} |{}^{\mathcal{H}}I|^2 \right\} v_g$$

where U is a unit vector field tangent to $\mathcal{V}$ and v_g is the volume element of (M, g) .

The proposition follows. $\square$

By a well-known result of S. Bochner (see [36, Ch.II, Theorem 4.3]) any Killing field on a compact Riemannian manifold with nonpositive Ricci tensor is parallel. The following theorem can be viewed as an extension of that result.

Theorem 4.1.2. *On a compact Riemannian manifold with nonpositive Ricci curvature any one-dimensional foliation which produces harmonic morphisms and admits a global density is locally generated by parallel vector fields. In particular, it is Riemannian, has geodesic leaves and its orthogonal complement is a totally geodesic Riemannian foliation. Hence the foliation corresponds to a local Riemannian product structure of the manifold. In particular, the universal cover of (M, g) is a Riemannian product. If M is simply-connected, or the foliation is simple and the base space is simply-connected, then the foliation corresponds to a Riemannian product structure on (M, g) .*

Proof. If the dimension of the manifold is three then the leaves are geodesics. This together with Proposition 4.1.1, gives the result.

Assume that the manifold has dimension greater than three. As before, by passing to a finite covering if necessary, we can suppose that both the foliation $\mathcal{V}$ and the manifold M are oriented.

By Proposition 4.1.1 the foliation is Riemannian; hence, by the proof of Proposition 3.1.3, it is globally generated by a Killing field, namely $\rho^{-1}U$ where ρ is a global density for $\mathcal{V}$ and U is a unit vertical vector field. Now, Bochner's result mentioned above implies that the foliation is generated by parallel vector fields. Hence $\mathcal{V}$ is a Riemannian foliation by geodesics and its orthogonal complement is a totally geodesic Riemannian foliation.

The fact that $\mathcal{V}$ induces on the universal cover of (M, g) a Riemannian product structure follows from the de Rham decomposition theorem. If the foliation

is simple then the leaves are compact and, hence, any curve in the base space admits (global) horizontal lifts, these induce an isometry between the fibres over the endpoints of the curve. Since the horizontal distribution is integrable, this isometry depends only on the homotopy class of the curve. It follows that, when the base space is simply-connected, this isometry depends only on the two fibres and the theorem is proved. $\square$

Remark 4.1.3. 1) In Proposition 4.1.1(i) and Theorem 4.1.2 we can replace the condition on the Ricci curvature by the condition: $\int_M \text{Ricci}(U, U) v_g \leq 0$ for any vector field U tangent to the foliation.

2) The above mentioned result of S. Bochner can be proved by using (A.1.1). In fact, if V is a Killing field on (M, g) which generates the (possibly singular) foliation $\mathcal{V}$, and $\sigma = |V|$ then σ^{-1} is a dilation for the homothetic distribution $\mathcal{H}$ (see the proof of Proposition 3.1.3). By (1.4.1) we have $\text{trace}({}^{\mathcal{V}}B) = \text{grad}(\log(\sigma^{-1}))$. It is easy to see that, in this case, (A.1.1) gives (sign convention for the Laplacian as in [11])

$$\sigma \Delta \sigma + |\text{grad } \sigma|^2 + \frac{1}{4} \sigma^2 |{}^{\mathcal{H}}I|^2 = \text{Ricci}(V, V) . \quad (4.1.1)$$

If σ attains a maximum at a point where V is not zero then the left hand side of (4.1.1) is nonnegative from which the result follows.

3) Recall that, by another well-known result of S. Bochner, on a compact Riemannian manifold with positive Ricci curvature there exists no harmonic one-forms (in particular, the first Betti number of such a manifold is zero). As is well-known (see [11]) this can be proved by using the Weitzenböck formula for the Hodge Laplacian acting on exterior forms. Also formula (A.1.1) can be obtained from the Weitzenböck formula applied to a local volume form of one of the two distributions.

By Corollary 1.1.14, any foliation which produces harmonic morphisms on a simply-connected manifold admits a global density and hence, in this case, the hypotheses of the above theorem can be weakened. Also, we have the following:

Corollary 4.1.4. *Any nonconstant submersive harmonic morphism with fibres of dimension one which is defined on a compact Riemannian manifold such that*

the Ricci curvature $\text{Ricci}(U, U)$ is nonpositive when U is tangent to the fibres is totally geodesic (or up to a conformal transformation of the codomain if this is two-dimensional). Hence, if the total space or the base space is simply-connected, up to a homothety of the codomain (up to a conformal transformation of the codomain if this is two-dimensional), it is a projection of a Riemannian product.

In order to apply it to *nonnegative* curvature, note that formula (A.1.1) can also be written

$$\begin{aligned} \text{div}(\text{trace}({}^{\mathcal{H}}B)) + \text{div}(\text{trace}({}^{\mathcal{V}}B)) + |\text{trace}({}^{\mathcal{H}}B)|^2 + |\text{trace}({}^{\mathcal{V}}B)|^2 \\ + \frac{1}{4}|{}^{\mathcal{H}}I|^2 + \frac{1}{4}|{}^{\mathcal{V}}I|^2 = |{}^{\mathcal{H}}B|^2 + |{}^{\mathcal{V}}B|^2 + s_{mix}. \end{aligned} \quad (4.1.2)$$

The next result applies to arbitrary foliations, not necessary conformal.

Proposition 4.1.5. *Let (M, g) be a compact Riemannian manifold.*

(i) *Let $\mathcal{V}$ and $\mathcal{H}$ be two complementary orthogonal foliations whose mean curvatures are (globally) gradient vector fields. If the mixed curvature is nonnegative then $\mathcal{V}$ and $\mathcal{H}$ are totally geodesic and hence they induce on (M, g) a local Riemannian product structure. Thus, the universal cover of (M, g) is globally a Riemannian product.*

(ii) *If the mixed curvature is positive then there exists no pair of complementary orthogonal foliations on (M, g) for which the mean curvatures are gradient vector fields.*

Proof. If $\text{trace}({}^{\mathcal{H}}B) = \text{grad}(\log u)$ and $\text{trace}({}^{\mathcal{V}}B) = \text{grad}(\log v)$ for some smooth positive functions u and v on M , then, (4.1.2) gives the following:

$$-\Delta(\log u) - \Delta(\log v) + |\text{grad}(\log u)|^2 + |\text{grad}(\log v)|^2 = |{}^{\mathcal{H}}B|^2 + |{}^{\mathcal{V}}B|^2 + s_{mix}. \quad (4.1.3)$$

Equation (4.1.3) can be written as follows:

$$-u^{-1}\Delta u - v^{-1}\Delta v = |{}^{\mathcal{H}}B|^2 + |{}^{\mathcal{V}}B|^2 + s_{mix}. \quad (4.1.4)$$

Since $\text{grad } u$ and $\text{grad } v$ are orthogonal (the former being vertical whilst the latter is horizontal) relation (4.1.4) can be written as follows:

$$-u^{-1}v^{-1}\Delta(uv) = |{}^{\mathcal{H}}B|^2 + |{}^{\mathcal{V}}B|^2 + s_{mix}.$$

The proof follows by multiplying by $u v$ and integrating over M . $\square$

Corollary 4.1.6. *Let (M, g) be compact and with zero first Betti number. Let $\mathcal{V}$ be a homothetic foliation with $\text{codim} \mathcal{V} \neq 2$ which produces harmonic morphisms on (M, g) and has integrable orthogonal complement. Then*

- (i) *the total mixed curvature $\int_M s_{\text{mix}} v_g$ is nonpositive.*
- (ii) *if the mixed curvature is nonnegative then it is identically zero and $\mathcal{V}$ and $\mathcal{H}$ are totally geodesic. Hence, the universal cover of (M, g) is globally a Riemannian product.*

On a compact Riemannian manifold with positive sectional curvature there exists no homothetic foliation which produces harmonic morphisms and has integrable orthogonal complement.

In Theorem 4.2.11(i) we shall prove that for one-dimensional foliations the last assertion from above proposition is true without the integrability assumption on $\mathcal{H}$, when $\dim M$ is even and greater than two.

Proof. Since $\mathcal{V}$ is homothetic the mean curvature form of $\mathcal{H}$ is closed. But $\mathcal{V}$ produces harmonic morphisms and hence, by Corollary 1.4.1, the mean curvature form of $\mathcal{V}$ is also closed. Since the first Betti number of M is zero both mean curvatures are globally gradient vector fields. The proof follows from Proposition 4.1.5. $\square$

Remark 4.1.7. 1) Recall (Remark 1.4.9) that Riemannian foliations are homothetic, as are the foliations with minimal leaves of codimension not equal to two and which produce harmonic morphisms.

2) If $\text{codim} \mathcal{V} = 1$ then the integrability assumption on the orthogonal complement of $\mathcal{V}$, made above, can be removed. Further, the mixed curvature is equal to the restriction of the Ricci curvature to its orthogonal complement.

3) Corollary 4.1.6 admits further consequences in a similar way to Corollary 4.1.4.

Theorem 4.1.8. *Let (M, g) be a compact Riemannian manifold of dimension at least four, with zero first Betti number and with Ricci curvature of constant sign.*

Then, there exists no orientable one-dimensional homothetic foliation which produces harmonic morphisms on (M, g) and which has integrable orthogonal complement.

Note that, by Lemma 3.1.7, the integrability of the orthogonal complement above is automatic except on the set where $\mathcal{V}$ is Riemannian.

Proof. Suppose that there exists a foliation $\mathcal{V}$ with the stated properties. By Corollary 4.1.6 the Ricci curvature of (M, g) is nonpositive. Since M has zero Betti number, $\mathcal{V}$ admits a global density. From Theorem 4.1.2 it follows that $\mathcal{V}$ is locally generated by parallel vector fields. Moreover, being orientable and admitting a global density, as in the proof of Theorem 4.1.2, $\mathcal{V}$ must be *globally* generated by a parallel vector field. Hence, the first Betti number of M is nonzero. This is a contradiction! $\square$

Corollary 4.1.9. *On a compact Riemannian manifold with positive Ricci curvature, there exists no nonvanishing Killing field with integrable orthogonal complement.*

The following immediate consequence of Proposition A.1.1 slightly improves Proposition 5.9 and Proposition 5.10 from [61].

Corollary 4.1.10. *If $\mathcal{V}$ is a Riemannian foliation on (M, g) , and $s_{mix} < 0$ at least at one point of M , then $\mathcal{V}$ cannot be totally geodesic.*

4.2 Two integral formulae for harmonic morphisms with one-dimensional fibres

Throughout this section $\varphi : (M^{n+1}, g) \rightarrow (N^n, h)$, $n \geq 1$, will denote a non-constant harmonic morphism defined on a compact Riemannian manifold. Recall that, by a result of P. Baird [5], φ is automatically submersive if $n \geq 4$. Since closed, all the fibres of φ are compact. As is well known [23] if φ is nonconstant then it is open, hence it is surjective and N is also compact. Let λ denote the dilation of φ ; we shall denote by the same letter h the metric on N and the metric

λg on M of Definition 1.2.3. This metric should be seen just as an auxiliary tool and thus, whenever we denote a geometric object on the total space of φ without mentioning a metric then it should be understood that the metric considered is g .

Definition 4.2.1. We define the *mass of a (regular) fibre* of φ to be the positive number

$$\mathbf{m} = \int_{\text{fibre}} \lambda^{2-n} v_{\text{fibre}} .$$

Where v_{fibre} is the volume measure of the fibre induced by the metric (see [40]).

By Proposition 1.1.11 the mass is independent of the (regular) fibre and it can be defined without any restriction on the dimensions.

Theorem 4.2.2. Let $\varphi : (M^{n+1}, g) \rightarrow (N^n, h)$, $(n \geq 1)$ be a submersive harmonic morphism and let $\mathcal{S}^M = \int_M s^M v_g$, $\mathcal{S}^N = \int_N s^N v_h$ be the total scalar curvature of (M, g) and (N, h) , respectively. Then

$$\begin{aligned} \mathcal{S}^M - \mathbf{m} \mathcal{S}^N &= n(n-1) \|\mathcal{V}(\text{grad}_g(\log \lambda))\|^2 \\ &\quad - (n-1)(n-2) \|\mathcal{H}(\text{grad}_g(\log \lambda))\|^2 - \frac{1}{4} \|I\|^2 . \end{aligned} \quad (4.2.1)$$

Let $n \neq 2$. Then

$$\begin{aligned} \int_M \lambda^2 (s^M - \lambda^2 s^N) v_g &= n(n-5) \int_M \lambda^2 |\mathcal{V}(\text{grad}_g(\log \lambda))|^2 v_g \\ &\quad - (n^2 - 3n + 6) \int_M \lambda^2 |\mathcal{H}(\text{grad}_g(\log \lambda))|^2 v_g - \frac{1}{4} \int_M \lambda^2 |I|^2 v_g . \end{aligned} \quad (4.2.2)$$

Here, $I = {}^{\mathcal{H}}I$ is the integrability tensor of the horizontal distribution $\mathcal{H}$.

In what follows we shall use the *horizontal Laplacian* of the associated Riemannian submersion with geodesic fibres. This was introduced in [10] and it can be defined as follows:

Definition 4.2.3. If $\varphi : M \rightarrow (N, h)$ is a submersion endowed with a distribution $\mathcal{H}$ which is complementary to $\ker \varphi_*$, then the *horizontal Laplacian* $\varphi^* \Delta^N$ is the

second-order differential operator which acts on a local function f defined in the neighbourhood of the point $x \in M$ as follows:

$$(\varphi^* \Delta^N)(f) = - \sum_j \{X_j(X_j(f)) - ((\varphi^* \nabla^N)_{X_j} X_j)(f)\}.$$

Here, $\{X_j\}$ is a local orthonormal frame of $\mathcal{H}$ (endowed with the metric induced by h) formed of basic vector fields (i.e. sections of $\mathcal{H}$ which are projectable by φ to vector fields on N), and ∇^N is the Levi-Civita connection of (N, h) .

Remark 4.2.4. Note that $(\varphi^* \Delta^N)(f \circ \varphi) = (\Delta^N f) \circ \varphi$ for any local smooth function f on N .

Lemma 4.2.5. Let $\varphi : (M^{n+1}, g) \rightarrow (N^n, h)$ be a submersive harmonic morphism and let Δ^M and Δ^N be the Laplace operators on (M, g) and (N, h) , respectively. Then

$$\Delta^M f = e^{2\sigma} (\varphi^* \Delta^N)(f) - e^{(-2n+4)\sigma} \{V(V(f)) - 2(n-1)V(f)V(\sigma)\}.$$

Remark 4.2.6. Note that $V(V(f))$ is just minus the ‘vertical’ Laplacian [10] applied to f of the Riemannian submersion with geodesic fibres associated to $\varphi : (M, g) \rightarrow (N, h)$. More generally, the ‘vertical’ Laplacian of $\varphi : (M, g) \rightarrow (N, h)$ is defined by $(\Delta^{\text{fibre}} f)(x) = (\Delta^{\varphi^{-1}(\varphi(x))}(f|_{\varphi^{-1}(\varphi(x))}))(x)$ where $\Delta^{\varphi^{-1}(\varphi(x))}$ is the Laplacian of the fibre through x endowed with the metric induced by g . If φ is a Riemannian submersion with totally geodesic fibres, then the sum of the horizontal and the vertical Laplacians is equal to the Laplacian of the total space.

Proof of Theorem 4.2.2. Recall from the previous section that s_{mix} is the sum of the sectional curvatures of all planes on (M, g) spanned by a horizontal and a vertical vector from an orthonormal frame adapted to the decomposition $TM = \mathcal{H} \oplus \mathcal{V}$. Let $s^{\mathcal{H}}$ denote twice the sum of the sectional curvatures of all planes on (M, g) spanned by the horizontal vectors of a frame as above.

Using the previous two lemmas and the fact that $I = V \otimes \Omega$, after a straightforward computation the following relation can be obtained. (Another way to obtain it is to use the previous two lemmas together with Corollary 2.2.4, from

[29] .)

$$\begin{aligned} s^{\mathcal{H}} - e^{2\sigma} s^N &= -2(n-1)\Delta^M \sigma - (n-1)(n-2)e^{2\sigma} |\mathcal{H}(\text{grad}_h \sigma)|^2 \\ &\quad - 2(n-1)e^{(-2n+4)\sigma} V(V(\sigma)) + (3n-4)(n-1)e^{(-2n+4)\sigma} V(\sigma)^2 - \frac{3}{4}|I|^2 . \end{aligned} \quad (4.2.3)$$

Using (1.1.3), (1.1.4) together with Lemma 4.2.5 and (A.1.1), we obtain:

$$\begin{aligned} s_{mix} &= (n-2)\Delta^M \sigma + 2(n-1)e^{(-2n+4)\sigma} V(V(\sigma)) \\ &\quad - (3n-4)(n-1)e^{(-2n+4)\sigma} V(\sigma)^2 + \frac{1}{4}|I|^2 . \end{aligned} \quad (4.2.4)$$

But it is obvious that $s^M = s^{\mathcal{H}} + 2s_{mix}$ and hence from (4.2.3) and (4.2.4) we obtain

$$\begin{aligned} s^M - e^{2\sigma} s^N &= -2\Delta^M \sigma + \frac{2(n-1)}{n} e^{n\sigma} \Delta^{\text{fibre}}(e^{-n\sigma}) \\ &\quad + n(n-1)|\mathcal{V}(\text{grad}_g \sigma)|^2 - (n-1)(n-2)|\mathcal{H}(\text{grad}_g \sigma)|^2 - \frac{1}{4}|I|^2 . \end{aligned} \quad (4.2.5)$$

Integrating (4.2.5) gives (4.2.1). Relation (4.2.5) can also be written as follows:

$$\begin{aligned} e^{2\sigma} s^M - e^{4\sigma} s^N &= -\Delta^M(e^{2\sigma}) + \frac{2(n-1)}{n-2} e^{n\sigma} \Delta^{\text{fibre}}(e^{(-n+2)\sigma}) \\ &\quad + n(n-5)e^{2\sigma} |\mathcal{V}(\text{grad}_g \sigma)|^2 - (n^2 - 3n + 6)e^{2\sigma} |\mathcal{H}(\text{grad}_g \sigma)|^2 - \frac{1}{4}e^{2\sigma} |I|^2 . \end{aligned} \quad (4.2.6)$$

Integrating (4.2.6) gives (4.2.2), since

$$\int_M e^{n\sigma} \Delta^{\text{fibre}}(e^{(-n+2)\sigma}) v_g = \int_N v_h \int_{\text{fibre}} \Delta^{\text{fibre}}(e^{(-n+2)\sigma}) v_{\text{fibre}} = 0$$

□

Remark 4.2.7. 1) Suppose that $n = 1$, i.e. $\varphi : (M^2, g) \rightarrow \mathbb{R}$ is a harmonic function defined on the surface (M^2, g) . Then, equation (4.2.4) above reads:

$$K = -\Delta(\log |d\varphi|) ,$$

where K is the Gauss curvature of (M, g) . As is well-known this can also be proved by using the local isothermal coordinates induced by φ .

2) Computing $\lambda^2(s^{\mathcal{H}} + s_{mix})$, by adding (4.2.3) and (4.2.4) from the above proof, we can obtain formula (2.2) from [45] applied to harmonic morphisms with fibres of dimension one.

Proposition 4.2.8. *Let $\varphi : (M^{n+1}, g) \rightarrow (N^n, h)$, $n \geq 2$ be a submersive harmonic morphism. If U is a unit vector field tangent to the fibres of φ then*

$$\mathcal{S}^M \leq \mathfrak{m} \mathcal{S}^N + \int_M \text{Ricci}(U, U) v_g$$

and equality holds if and only if φ has geodesic fibres and $\mathcal{H}$ is integrable.

Note that since, $\text{Ricci}(U, U)$ is quadratic in U , we do not need $\mathcal{V}$ to be orientable.

Proof. First recall that $\text{Ricci}(U, U) = s_{mix}$, then take the sum of (4.2.3) and (4.2.4) and use the definition of $\mathfrak{m}$. $\square$

Corollary 4.2.9. *Let $\varphi : (M^{n+1}, g) \rightarrow (N^n, h)$, $n \geq 2$, be a submersive harmonic morphism.*

(i) *If φ induces a Riemannian foliation on (M, g) then $\mathcal{S}^M \leq \mathfrak{m} \mathcal{S}^N$ and equality holds if and only if φ is totally geodesic (up to a conformal transformation of the codomain if $n = 2$).*

(ii) *If φ has geodesic fibres and $\mathcal{H}$ is integrable then $\mathcal{S}^M \geq \mathfrak{m} \mathcal{S}^N$ and equality holds if and only if φ is totally geodesic (up to a conformal transformation of the codomain if $n = 2$).*

From Lemma 3.1.7 it follows that when the set of the points where $\mathcal{V}$ is Riemannian has measure zero then the integrability assumption on $\mathcal{H}$ in (ii) is superfluous.

Proof. (i) This is a trivial consequence of formula (4.2.1) from Theorem 4.2.2.
(ii) If $n = 2$, then (4.2.1) from Theorem 4.2.2 gives the result. If $n \neq 2$, by Proposition 1.1.10(b) we have $\mathcal{H}(\text{grad}_g \lambda) = 0$. Now apply formula (4.2.1). $\square$

The next corollary improves [45, Theorem 2.5] for the dimensions considered (see also [29, Corollary 2.2.6]).

Corollary 4.2.10. *If $n \in \{3, 4, 5\}$ then $\int_M \lambda^2(s^M - \lambda^2 s^N) v_g \leq 0$. For $n \in \{3, 4\}$, equality holds if and only if φ is totally geodesic and, for $n = 5$, equality holds if and only if φ has geodesic fibres and $\mathcal{H}$ is integrable.*

Therefore, for $n \in \{3, 4, 5\}$, if (M^{n+1}, g) , (N^n, h) are compact with $s^M \geq 0$, $s^N \leq 0$ and at least one of these inequalities is strict then there exists no nonconstant submersive harmonic morphism $\varphi : (M^{n+1}, g) \rightarrow (N^n, h)$.

Proof. This is an immediate consequence of formula (4.2.2) from Theorem 4.2.2. □

To end this section we prove two results on homothetic one-dimensional foliations which produce harmonic morphisms on compact manifolds, the first of them being a generalization (referred to in Section 3) of a well-known result of M. Berger (see [36, Ch.II, Corollary 5.7]) concerning Killing fields. To prove the first of these two results we shall use Lemma B.1.3.

Theorem 4.2.11. *Let M be compact with dimension at least four.*

(i) *If $\dim M$ is even and (M, g) has positive sectional curvature then there exists no homothetic one-dimensional foliation which produces harmonic morphisms on (M, g) .*

(ii) *If (M, g, J) is Kähler and has zero first Betti number then any homothetic one-dimensional foliation which produces harmonic morphisms on (M, g) is Riemannian and locally generated by Killing fields.*

Proof. (i) Since (M, g) has positive sectional curvature it has positive Ricci curvature. Thus from Bochner's result (see Remark 4.1.3(3)) it follows that the first Betti number of M is zero.

Suppose that there exists a homothetic foliation $\mathcal{V}$ which produces harmonic morphisms on (M, g) . Because the first Betti number of M is zero, it follows from Corollary 1.1.14 that $\mathcal{V}$ admits a global density λ^{2-n} where $n+1 = \dim M$. We shall denote by $h = {}^\lambda g$ the associated Riemannian metric on M which makes $\mathcal{V}$ a Riemannian foliation with geodesic leaves.

Since $\mathcal{V}$ is homothetic, by Proposition 1.4.2 we see that λ is of the form e^{a+b}

with $(da)^\mathcal{V} = (db)^\mathcal{H} = 0$. At a point $x \in M$ where $a - b$ attains a minimum we have

$$\begin{aligned} 0 &\leq ({}^h\nabla da)(V, V) - ({}^h\nabla db)(V, V) = -({}^h\nabla db)(V, V), \\ 0 &\leq ({}^h\nabla da)(X, X) - ({}^h\nabla db)(X, X) = ({}^h\nabla da)(X, X), \end{aligned} \quad (4.2.7)$$

where, V is as in Lemma 3.1.1 and X is any horizontal vector at x .

Now, evaluated at x , the first formula from Lemma B.1.3 gives

$$\begin{aligned} R(X, V, X, V) &= -(n-2)e^{(2n-4)(a+b)} ({}^h\nabla da)(X, X) \\ &\quad + e^{-2(a+b)} ({}^h\nabla db)(V, V) h(X, X) + \frac{1}{4}e^{(4n-6)(a+b)} h(i_X\Omega, i_X\Omega), \end{aligned} \quad (4.2.8)$$

for any horizontal vector X .

Since Ω is skew-symmetric it must have even rank at each point. But $\dim M$ is even and $i_V\Omega = 0$ and hence there must be a horizontal vector X at x such that $i_X\Omega = 0$. By (4.2.7) and (4.2.8), the sectional curvature of the plane spanned by X and V would be nonpositive, contradicting the hypothesis.

(ii) Since the first Betti number of M is zero, the foliation $\mathcal{V}$ must admit a global density. By passing to a two-fold covering if necessary, we can suppose that $\mathcal{V}$ is oriented. Hence, by Proposition 3.1.5, it must be generated by a conformal vector field. But by a well-known result of A. Lichnerowicz (see [11, 2.125(ii)]) any conformal vector field is Killing on a compact Kähler manifold of complex dimension greater than two.

The theorem is proved. $\square$

Remark 4.2.12. 1) Note that statement (i) from Theorem 4.2.11 fails if $\dim M$ is *odd*, for example the Hopf maps $S^{2n+1} \rightarrow \mathbb{C}P^n$ are harmonic Riemannian submersions.

2) In Theorem 4.2.11(i), if the sectional curvature K of (M^{n+1}, g) satisfies $K \geq a^2 > 0$ on M for some positive constant a , then the compactness assumption on M can be replaced by the weaker condition that (M, g) be complete. This follows from a well-known result of S. B. Myers (see [11, 6.51]) noting that the Ricci curvature is $\geq na^2 g$. (A similar remark can be made for Corollary 4.1.9.)

Corollary 4.2.13. *Let (M, g) be a compact Riemannian manifold of even dimension at least four with zero first Betti number and with sectional curvature of constant sign.*

Then, there exists no orientable one-dimensional homothetic foliation which produces harmonic morphisms on (M, g) .

Proof. This follows from Theorem 4.1.2 and Theorem 4.2.11(i). □

Appendix A

A formula related to a pair of complementary orthogonal distributions

In this appendix we recall a formula of P. Walczak [64] which relates the curvature of a Riemannian manifold to the geometric properties of a pair of complementary orthogonal distributions on it. This formula is used in Chapter 4.

Let (M, g) be a Riemannian manifold and $\mathcal{H}, \mathcal{V}$ a pair of complementary orthogonal distributions on it, with $\dim \mathcal{H} = p$ and $\dim \mathcal{V} = q$. As before, $\mathcal{H}$ and $\mathcal{V}$ will be called the *horizontal* and the *vertical* distribution, respectively, and, the corresponding projections will be denoted with the same letters $\mathcal{H}$ and $\mathcal{V}$. We shall denote by X, Y horizontal vector fields and by U, V vertical vector fields. Let ${}^{\mathcal{V}}B$ and ${}^{\mathcal{V}}I$ denote the second fundamental form and integrability tensor of $\mathcal{V}$, respectively. Recall that they are the unique $\mathcal{H}$ -valued vertical tensor fields which satisfy

$$\begin{aligned} {}^{\mathcal{V}}B(U, V) &= \frac{1}{2}\mathcal{H}(\nabla_U V + \nabla_V U), \\ {}^{\mathcal{V}}I(U, V) &= -\mathcal{H}[U, V]. \end{aligned}$$

Note the minus sign in the last formula.

Recall that $\text{trace}({}^{\mathcal{V}}B)$ is q times the mean curvature of $\mathcal{V}$, i.e. if $\{U_\alpha\}$ is a local orthonormal frame for $\mathcal{V}$ then $\text{trace}({}^{\mathcal{V}}B) = \sum_\alpha {}^{\mathcal{V}}B(U_\alpha, U_\alpha)$. We shall also

consider the trace free part ${}^{\mathcal{V}}B_0$ defined by

$${}^{\mathcal{V}}B_0 = {}^{\mathcal{V}}B - \frac{1}{q} \text{trace}({}^{\mathcal{V}}B) \otimes g^{\mathcal{V}}$$

where $g^{\mathcal{V}}$ is the vertical component of g . Also, ${}^{\mathcal{H}}B, {}^{\mathcal{H}}I, {}^{\mathcal{H}}B_0$ are defined similarly by reversing the roles of $\mathcal{V}$ and $\mathcal{H}$.

Recall that s_{mix} denotes the mixed scalar curvature which is the sum of the sectional curvatures of all planes spanned by a horizontal and a vertical vector from an orthonormal frame adapted to the decomposition $TM = \mathcal{H} \oplus \mathcal{V}$. The following result is due to P. Walczak.

Proposition A.1.1 ([64]). *With the notations above we have*

$$\begin{aligned} \text{div}(\text{trace}({}^{\mathcal{H}}B)) + \text{div}(\text{trace}({}^{\mathcal{V}}B)) + \frac{p-1}{p} |\text{trace}({}^{\mathcal{H}}B)|^2 + \frac{q-1}{q} |\text{trace}({}^{\mathcal{V}}B)|^2 \\ + \frac{1}{4} |{}^{\mathcal{H}}I|^2 + \frac{1}{4} |{}^{\mathcal{V}}I|^2 = |{}^{\mathcal{H}}B_0|^2 + |{}^{\mathcal{V}}B_0|^2 + s_{mix}. \end{aligned} \quad (\text{A.1.1})$$

Appendix B

Curvature formulae for the total space of a harmonic morphism with 1-dim fibres

In this appendix we rewrite in a convenient way S. Gudmundsson's fundamental equations for horizontally conformal submersions. We do this only for submersive harmonic morphisms with one-dimensional fibres.

In this appendix $\mathcal{V}$ will always denote a one-dimensional foliation which produces harmonic morphisms on (M^{n+1}, g) ($n \geq 1$) and $\rho = e^{(2-n)\sigma}$ will denote a local density of it. As before, $h = e^\sigma g$ (see Definition 1.2.3) will denote the associated (local) metric on M with respect to which $\mathcal{V}$ is Riemannian and has geodesic leaves and $\mathcal{H}$ will denote the orthogonal complement of $\mathcal{V}$.

The following lemma can be obtained by a straightforward computation.

Lemma B.1.1. *Let $\sigma = \log \lambda$, and let V be a local vertical field such that $g(V, V) = e^{(2n-4)\sigma}$, and θ its dual vertical 1-form. Then, for any basic X and Y we have:*

$$\begin{aligned}\mathcal{H}(^g\nabla_X Y) &= \mathcal{H}(^h\nabla_X Y) - X(\sigma)Y - Y(\sigma)X + \mathcal{H}(\text{grad}_h \sigma)h(X, Y) , \\ \mathcal{V}(^g\nabla_X Y) &= \{e^{(-2n+2)\sigma} V(\sigma)h(X, Y) - \frac{1}{2}\Omega(X, Y)\}V ,\end{aligned}$$

$$\begin{aligned}
\mathcal{H}({}^g\nabla_V X) &= -V(\sigma)X + \frac{1}{2}e^{(2n-2)\sigma}(i_X\Omega)^{\#_h}, \\
\mathcal{V}({}^g\nabla_V X) &= (n-2)X(\sigma)V, \\
\mathcal{H}({}^g\nabla_V V) &= -(n-2)e^{(2n-2)\sigma}\mathcal{H}(\text{grad}_h\sigma), \\
\mathcal{V}({}^g\nabla_V V) &= (n-2)V(\sigma)V.
\end{aligned}$$

Here, ${}^g\nabla$ and ${}^h\nabla$ are, respectively, the Levi-Civita connections of (M, g) and (M, h) , $\Omega = d\theta$ and $\#_h$ denotes the musical isomorphism defined by the metric h .

Remark B.1.2. Note that, if $\dim \mathcal{V} = 0$, Lemma B.1.1 becomes a well-known formula (see [11, 1.159]).

Lemma B.1.3. Let X, Y, Z, H be horizontal and V vertical vectors on M , such that $g(V, V) = e^{(2n-4)\sigma}$; then the curvature tensor ${}^M R$ of (M, g) has the following components

$$\begin{aligned}
{}^M R(X, V, Y, V) &= -\frac{1}{2}(n-2)e^{(2n-4)\sigma}(\mathcal{L}_{\mathcal{H}(\text{grad}_h\sigma)}h)(X, Y) \\
&\quad - (n-2)e^{(2n-4)\sigma}\{nX(\sigma)Y(\sigma) - |\mathcal{H}(\text{grad}_h\sigma)|_h^2 h(X, Y)\} \\
&\quad + e^{-2\sigma}\{V(V(\sigma)) - (n-1)V(\sigma)^2\}h(X, Y) \\
&\quad + \frac{1}{4}e^{(4n-6)\sigma}h(i_X\Omega, i_Y\Omega),
\end{aligned} \tag{B.1.1}$$

$$\begin{aligned}
{}^M R(X, Y, Z, V) &= -\frac{1}{2}e^{(2n-4)\sigma}({}^h\nabla\Omega)(X, Y, Z) \\
&\quad + \frac{1}{2}(n-1)e^{(2n-4)\sigma}\{X(\sigma)\Omega(Y, Z) + Y(\sigma)\Omega(Z, X) - 2Z(\sigma)\Omega(X, Y)\} \\
&\quad - e^{-2\sigma}\{X(V(\sigma)) - (n-2)X(\sigma)V(\sigma)\}h(Y, Z) \\
&\quad + e^{-2\sigma}\{Y(V(\sigma)) - (n-2)Y(\sigma)V(\sigma)\}h(X, Z) \\
&\quad + \frac{1}{2}e^{(2n-4)\sigma}\{\Omega(X, \text{grad}_h\sigma)h(Y, Z) - \Omega(Y, \text{grad}_h\sigma)h(X, Z)\},
\end{aligned} \tag{B.1.2}$$

$$\begin{aligned}
{}^M R(X, Y, Z, H) &= e^{-2\sigma} {}^N R(\varphi_* X, \varphi_* Y, \varphi_* Z, \varphi_* H) \\
&- \frac{1}{4} e^{(2n-4)\sigma} \{ \Omega(H, X) \Omega(Y, Z) + \Omega(H, Y) \Omega(Z, X) - 2\Omega(H, Z) \Omega(X, Y) \} \\
&- \frac{1}{2} e^{-2\sigma} V(\sigma) \{ -\Omega(Y, H) h(X, Z) + \Omega(X, H) h(Y, Z) - \Omega(X, Z) h(Y, H) + \Omega(Y, Z) h(X, H) \} \\
&- e^{-2\sigma} \{ X(\sigma) H(\sigma) h(Y, Z) - X(\sigma) Z(\sigma) h(Y, H) - Y(\sigma) H(\sigma) h(X, Z) + Y(\sigma) Z(\sigma) h(X, H) \} \\
&+ e^{-2\sigma} \{ h(X, Z) h({}^h \nabla_Y (\mathcal{H}(\text{grad}_h \sigma)), H) - h(Y, Z) h({}^h \nabla_X (\mathcal{H}(\text{grad}_h \sigma)), H) \\
&+ h(Y, H) h({}^h \nabla_X (\mathcal{H}(\text{grad}_h \sigma)), Z) - h(X, H) h({}^h \nabla_Y (\mathcal{H}(\text{grad}_h \sigma)), Z) \} \\
&- e^{-2\sigma} \{ h(X, Z) h(Y, H) - h(X, H) h(Y, Z) \} \{ e^{(-2n+2)\sigma} V(\sigma)^2 + |\mathcal{H}(\text{grad}_h \sigma)|_h^2 \} .
\end{aligned} \tag{B.1.3}$$

Here ${}^N R$ is the Riemannian curvature tensor of the codomain of the harmonic morphism $\varphi : (O, g|_O) \rightarrow (N, \bar{h})$ produced by $\mathcal{V}$ and ${}^h \nabla$ denotes the Levi-Civita connection of (M, h) .

Proof. Let $\{X_1, \dots, X_n\}$ be a local frame of $\mathcal{H}$ formed of basic vector fields and let $g_{jk} = g(X_j, X_k)$, $h_{jk} = h(X_j, X_k)$, $\Omega_{jk} = \Omega(X_j, X_k)$. Then $\{X_1, \dots, X_n, V\}$ is a local frame on M . We shall always denote by j, k, l, r, s ‘horizontal’ indices, by α, β ‘vertical’ indices whilst a, b, c, d, a' will be any kind of indices.

For $\{E_a\}$ a local frame on M we denote, as usual, by $\{\Gamma_{ab}^c\}$ the corresponding Christoffel symbols of the Levi-Civita connection of (M^{n+1}, g) , i.e.

$${}^g \nabla_{E_a} E_b = \Gamma_{ba}^c E_c .$$

Then, the local connection forms $\{\Gamma_a^b\}$ are characterised by $\Gamma_a^b(E_c) = \Gamma_{ac}^b$. Recall that the components ${}^M R_{abc}^d$ of the curvature form of the Levi-Civita connection of (M^{n+1}, g) are given by

$${}^M R(E_a, E_b) E_c = {}^M R_{cab}^d E_d$$

where

$${}^M R_{abc}^d = (d\Gamma_a^d)_{bc} + (\Gamma_{a'}^d \wedge \Gamma_a^{a'})_{bc} . \tag{B.1.4}$$

Also, let $\{{}^h \Gamma_{ab}^c\}$ be the Christoffel symbols of the Levi-Civita connection of (M^{n+1}, h) . Then, from Lemma B.1.1, it follows that the Christoffel symbols of (M^{n+1}, g) ,

with respect to the local frame $\{X_1, \dots, X_n, V\}$, are given by

$$\begin{aligned}
\Gamma_{kj}^l &= T_{kj}^l - X_j(\sigma) \delta_k^l - X_k(\sigma) \delta_j^l + X^l(\sigma) h_{jk} \\
\Gamma_{kj}^\alpha &= e^{(-2n+2)\sigma} V(\sigma) h_{jk} - \frac{1}{2} \Omega_{jk} \\
\Gamma_{j\alpha}^k &= -V(\sigma) \delta_j^k + \frac{1}{2} e^{(2n-2)\sigma} \Omega_j^k \\
\Gamma_{j\alpha}^\alpha &= (n-2) X_j(\sigma) \\
\Gamma_{\alpha\alpha}^j &= -(n-2) e^{(2n-2)\sigma} X^j(\sigma) \\
\Gamma_{\alpha\alpha}^\alpha &= (n-2) V(\sigma)
\end{aligned} \tag{B.1.5}$$

where $X^j(\sigma) = X_l(\sigma) h^{lj}$ and $\Omega_j^k = \Omega_{jl} h^{lk}$.

Because $[V, X_j] = 0$, $j = 1, \dots, n$, from (B.1.4) we obtain

$${}^M R_{\alpha l \alpha}^k = X_l(\Gamma_{\alpha\alpha}^k) - V(\Gamma_{\alpha l}^k) + \Gamma_{jl}^k \Gamma_{\alpha\alpha}^j + \Gamma_{\alpha l}^k \Gamma_{\alpha\alpha}^\alpha - \Gamma_{j\alpha}^k \Gamma_{\alpha l}^j - \Gamma_{\alpha\alpha}^k \Gamma_{\alpha l}^\alpha$$

which, by applying (B.1.5), becomes

$$\begin{aligned}
{}^M R_{\alpha l \alpha}^k &= -(n-2) e^{(2n-2)\sigma} X_l(X^k(\sigma)) - 2(n-1)(n-2) e^{(2n-2)\sigma} X_l(\sigma) X^k(\sigma) \\
&\quad + V(V(\sigma)) \delta_l^k - (n-1) e^{(2n-2)\sigma} V(\sigma) \Omega_l^k - \frac{1}{2} e^{(2n-2)\sigma} V(\Omega_l^k) \\
&\quad + \{T_{jl}^k - X_l(\sigma) \delta_j^k - X_j(\sigma) \delta_l^k + X^k(\sigma) h_{jl}\} \{-(n-2) e^{(2n-2)\sigma} X^j(\sigma)\} \\
&\quad + \{-V(\sigma) \delta_l^k + \frac{1}{2} e^{(2n-2)\sigma} \Omega_l^k\} (n-2) V(\sigma) \\
&\quad - \{-V(\sigma) \delta_j^k + \frac{1}{2} e^{(2n-2)\sigma} \Omega_j^k\} \{-V(\sigma) \delta_l^j + \frac{1}{2} e^{(2n-2)\sigma} \Omega_l^j\} \\
&\quad - \{-(n-2) e^{(2n-2)\sigma} X^k(\sigma)\} (n-2) X_l(\sigma) .
\end{aligned} \tag{B.1.6}$$

Because Ω and X_j are basic we have that Ω_l^k is basic and thus $V(\Omega_l^k) = 0$.

Lines two, four and five of (B.1.6) contain the terms linear in Ω . These are

$$\begin{aligned}
&-(n-1) e^{(2n-2)\sigma} V(\sigma) \Omega_l^k + \frac{1}{2} (n-2) e^{(2n-2)\sigma} V(\sigma) \Omega_l^k + e^{(2n-2)\sigma} V(\sigma) \Omega_l^k \\
&= -\frac{1}{2} (n-2) e^{(2n-2)\sigma} V(\sigma) \Omega_l^k .
\end{aligned} \tag{B.1.7}$$

The first terms of lines one and three of (B.1.6) give

$$\begin{aligned} & - (n-2)e^{(2n-2)\sigma} X_l(X^k(\sigma)) - (n-2)e^{(2n-2)\sigma} {}^h\Gamma_{jl}^k X^j(\sigma) \\ & = - (n-2)e^{(2n-2)\sigma} \{X_l(X^k(\sigma)) + {}^h\Gamma_{jl}^k X^j(\sigma)\} . \end{aligned} \quad (\text{B.1.8})$$

Now, it is easy to see that

$${}^h\Gamma_{\alpha l}^k = \frac{1}{2} \Omega_l^k$$

and hence, we can write

$$\begin{aligned} \mathcal{H}({}^h\nabla_{X_l}(\text{grad}_h \sigma)) &= \mathcal{H}({}^h\nabla_{X_l}(X^j(\sigma)X_j + V(\sigma)V)) \\ &= \{X_l(X^k(\sigma)) + X^j(\sigma) {}^h\Gamma_{jl}^k + V(\sigma) {}^h\Gamma_{\alpha l}^k\} X_k \\ &= \{X_l(X^k(\sigma)) + X^j(\sigma) {}^h\Gamma_{jl}^k + \frac{1}{2} V(\sigma) \Omega_l^k\} X_k . \end{aligned} \quad (\text{B.1.9})$$

We use (B.1.7), (B.1.8) and (B.1.9) to simplify (B.1.6) thus obtaining

$$\begin{aligned} {}^M R_{\alpha l \alpha}^k &= - (n-2)e^{(2n-2)\sigma} h^{kr} h({}^h\nabla_{X_l}(\text{grad}_h \sigma), X_r) \\ &\quad - 2(n-1)(n-2)e^{(2n-2)\sigma} X_l(\sigma)X^k(\sigma) + (n-2)^2 e^{(2n-2)\sigma} X_l(\sigma)X^k(\sigma) \\ &\quad + (n-2)e^{(2n-2)\sigma} X_l(\sigma)X^k(\sigma) - (n-2)e^{(2n-2)\sigma} X_l(\sigma)X^k(\sigma) \\ &\quad + (n-2)e^{(2n-2)\sigma} |\mathcal{H}(\text{grad}_h \sigma)|_h^2 \delta_l^k \\ &\quad + V(V(\sigma)) \delta_l^k - V(\sigma)^2 \delta_l^k - (n-2) V(\sigma)^2 \delta_l^k \\ &\quad - \frac{1}{4} e^{(4n-4)\sigma} \Omega_j^k \Omega_l^j . \end{aligned} \quad (\text{B.1.10})$$

Hence

$$\begin{aligned} {}^M R_{s \alpha l \alpha} &= g_{sk} {}^M R_{\alpha l \alpha}^k = e^{-2\sigma} h_{sk} {}^M R_{\alpha l \alpha}^k \\ &= - (n-2)e^{(2n-4)\sigma} h({}^h\nabla_{X_l}(\text{grad}_h \sigma), X_s) \\ &\quad - (n-2)e^{(2n-4)\sigma} \{nX_s(\sigma)X_k(\sigma) - |\mathcal{H}(\text{grad}_h \sigma)|_h^2 h_{sl}\} \\ &\quad + e^{-2\sigma} \{V(V(\sigma)) - (n-1)V(\sigma)^2\} h_{sl} \\ &\quad + \frac{1}{4} e^{(4n-6)\sigma} \Omega_{sj} \Omega_l^j . \end{aligned} \quad (\text{B.1.11})$$

It is easy to see that (B.1.11) is equivalent to (B.1.1).

Next, we prove (B.1.2). Because $[V, X_j] = 0$, the equation (B.1.4) gives

$${}^M R_{kl \alpha}^j = X_l(\Gamma_{k \alpha}^j) - V(\Gamma_{kl}^j) + \Gamma_{rl}^j \Gamma_{k \alpha}^r + \Gamma_{\alpha l}^j \Gamma_{k \alpha}^r - \Gamma_{r \alpha}^j \Gamma_{kl}^r - \Gamma_{\alpha \alpha}^j \Gamma_{kl}^r \quad (\text{B.1.12})$$

which by applying (B.1.5) becomes

$$\begin{aligned}
{}^M R_{kl\alpha}^j = & -X_l(V(\sigma))\delta_k^j + (n-1)e^{(2n-2)\sigma}X_l(\sigma)\Omega_k^j + \frac{1}{2}e^{(2n-2)\sigma}X_l(\Omega_k^j) \\
& -V({}^h\Gamma_{kl}^j) + V(X_k(\sigma))\delta_l^j + V(X_l(\sigma))\delta_k^j - V(X^j(\sigma))h_{kl} \\
& + \{{}^h\Gamma_{rl}^j - X_r(\sigma)\delta_l^j - X_l(\sigma)\delta_r^j + X^j(\sigma)h_{rl}\}\{-V(\sigma)\delta_k^r + \frac{1}{2}e^{(2n-2)\sigma}\Omega_k^r\} \\
& + \{-V(\sigma)\delta_l^j + \frac{1}{2}e^{(2n-2)\sigma}\Omega_l^j\}(n-2)X_k(\sigma) \\
& - \{-V(\sigma)\delta_r^j + \frac{1}{2}e^{(2n-2)\sigma}\Omega_r^j\}\{{}^h\Gamma_{kl}^r - X_k(\sigma)\delta_l^r - X_l(\sigma)\delta_k^r + X^r(\sigma)h_{kl}\} \\
& + (n-2)e^{(2n-2)\sigma}X^j(\sigma)\{e^{(-2n+2)\sigma}V(\sigma)h_{kl} - \frac{1}{2}\Omega_{lk}\}.
\end{aligned} \tag{B.1.13}$$

Because X_j are basic and $\mathcal{V}$ is a Riemannian foliation with respect to h , we have $V({}^h\Gamma_{kl}^j) = 0$. Also, because $[V, X_l] = 0$, the first term of the right hand side of (B.1.13) and the second term of the second line of (B.1.13) cancel. Hence, (B.1.13) can be written as follows

$$\begin{aligned}
{}^M R_{kl\alpha}^j = & (n-1)e^{(2n-2)\sigma}X_l(\sigma)\Omega_k^j + \frac{1}{2}e^{(2n-2)\sigma}X_l(\Omega_k^j) \\
& + V(X_k(\sigma))\delta_l^j - V(X^j(\sigma))h_{kl} \\
& - V(\sigma){}^h\Gamma_{kl}^j + V(\sigma)X_k(\sigma)\delta_l^j + V(\sigma)X_l(\sigma)\delta_k^j - V(\sigma)X^j(\sigma)h_{kl} \\
& + \frac{1}{2}e^{(2n-2)\sigma}{}^h\Gamma_{rl}^j\Omega_k^r - \frac{1}{2}e^{(2n-2)\sigma}X_r(\sigma)\Omega_k^r\delta_l^j - \frac{1}{2}e^{(2n-2)\sigma}X_l(\sigma)\Omega_k^j + \frac{1}{2}e^{(2n-2)\sigma}X^j(\sigma)\Omega_k^r h_{rl} \\
& - (n-2)V(\sigma)X_k(\sigma)\delta_l^j + \frac{1}{2}(n-2)e^{(2n-2)\sigma}X_k(s)\Omega_l^j \\
& + V(\sigma){}^h\Gamma_{kl}^j - \frac{1}{2}e^{(2n-2)\sigma}\Omega_r^j{}^h\Gamma_{kl}^r \\
& - V(\sigma)X_k(\sigma)\delta_l^j - V(\sigma)X_l(\sigma)\delta_k^j + V(\sigma)X^j(\sigma)h_{kl} \\
& + \frac{1}{2}e^{(2n-2)\sigma}\Omega_l^jX_k(\sigma) + \frac{1}{2}e^{(2n-2)\sigma}\Omega_k^jX_l(\sigma) - \frac{1}{2}e^{(2n-2)\sigma}\Omega_r^jX^r(\sigma)h_{kl} \\
& + (n-2)X^j(\sigma)V(\sigma)h_{kl} + \frac{1}{2}(n-2)e^{(2n-2)\sigma}X^j(\sigma)\Omega_{kl}
\end{aligned} \tag{B.1.14}$$

where we have used that $\Omega_{lk} = -\Omega_{kl}$.

After cancelling the corresponding terms, (B.1.14) becomes

$$\begin{aligned}
{}^M R_{kl\alpha}^j &= \frac{1}{2} e^{(2n-2)\sigma} \{X_l(\Omega_k^j) + {}^h\Gamma_{rl}^j \Omega_k^r - {}^h\Gamma_{kl}^r \Omega_r^j\} \\
&+ \frac{1}{2} (n-1) e^{(2n-2)\sigma} X_k(\sigma) \Omega_l^j + \frac{1}{2} (n-1) e^{(2n-2)\sigma} X^j(\sigma) \Omega_{kl} + (n-1) e^{(2n-2)\sigma} X_l(\sigma) \Omega_k^j \\
&- \frac{1}{2} e^{(2n-2)\sigma} \{X_r(\sigma) \Omega_k^r \delta_l^j + \Omega_r^j X^r(\sigma) h_{kl}\} \\
&+ V(X_k(\sigma)) \delta_l^j - V(X^j(\sigma)) h_{kl} - (n-2) V(\sigma) X_k(\sigma) \delta_l^j + (n-2) X^j(\sigma) V(\sigma) h_{kl} .
\end{aligned} \tag{B.1.15}$$

Now, recall that

$$\begin{aligned}
-({}^h\nabla\Omega)(X_j, X_k, X_l) &= -({}^h\nabla_{X_l}\Omega)(X_j, X_k) = ({}^h\nabla_{X_l}\Omega)(X_k, X_j) \\
&= ({}^h\nabla_{X_l}\Omega)_{kj} = ({}^h\nabla_{X_l}\Omega)_k^r h_{rj} \\
&= h_{jr} \{X_l(\Omega_k^r) + {}^h\Gamma_{sl}^r \Omega_k^s - {}^h\Gamma_{kl}^s \Omega_s^r\} .
\end{aligned} \tag{B.1.16}$$

From (B.1.16), it easily follows that (B.1.2) is equivalent to (B.1.15).

To prove (B.1.3) we use the corresponding fundamental equation for horizontally conformal submersions [29].

Firstly, from $\lambda = e^\sigma$, it follows that

$$\text{grad}_g\left(\frac{1}{\lambda^2}\right) = -2e^{(-2n+2)\sigma} V(\sigma) V - 2\mathcal{H}(\text{grad}_h \sigma) \tag{B.1.17}$$

and hence

$$\left|\text{grad}_g\left(\frac{1}{\lambda^2}\right)\right|_g^2 = 4e^{-2n\sigma} V(\sigma)^2 + 4e^{-2\sigma} |\mathcal{H}(\text{grad}_h \sigma)|_h^2 . \tag{B.1.18}$$

Next, a straightforward calculation using Lemma B.1.1 and (B.1.17) gives

$$\begin{aligned}
\mathcal{H}(g\nabla_X(\text{grad}_g(\frac{1}{\lambda^2}))) &= 2e^{(-2n+2)\sigma} V(\sigma)^2 X - V(\sigma)(i_X\Omega)^{\#_h} \\
&- 2\mathcal{H}({}^h\nabla_X(\mathcal{H}(\text{grad}_h \sigma))) + 2|\mathcal{H}(\text{grad}_h \sigma)|_h^2 X
\end{aligned} \tag{B.1.19}$$

where X is any basic vector field. Hence, if Z is a basic vector field then

$$\begin{aligned}
g(g\nabla_X(\text{grad}_g(\frac{1}{\lambda^2})), Z) &= 2e^{-2n\sigma} V(\sigma)^2 h(X, Z) + 2e^{-2\sigma} |\mathcal{H}(\text{grad}_h \sigma)|_h^2 h(X, Z) \\
&- e^{-2\sigma} V(\sigma) \Omega(X, Z) - 2e^{-2\sigma} h({}^h\nabla_X(\mathcal{H}(\text{grad}_h \sigma)), Z) .
\end{aligned} \tag{B.1.20}$$

Recall that $\Omega = d\theta$ and thus $\Omega(X, Y) = -\theta([X, Y])$ which, because θ is the vertical dual of V , is equivalent to

$$\mathcal{V}([X, Y]) = -\Omega(X, Y) V. \quad (\text{B.1.21})$$

Now, let $\varphi : (O, g|_O) \rightarrow (N, \bar{h})$ be a harmonic morphism with dilation $\lambda = e^\sigma$ produced by $\mathcal{V}$. If we put (B.1.20) and (B.1.21) into the corresponding fundamental equation for horizontally conformal submersions [29] we obtain

$$\begin{aligned} -{}^M R(X, Y, Z, H) = & -e^{-2\sigma} \varphi^*({}^N R)(X, Y, Z, H) \\ & + \frac{1}{4} e^{(2n-4)\sigma} \{ \Omega(X, Z) \Omega(Y, H) - \Omega(Y, Z) \Omega(X, H) + 2\Omega(X, Y) \Omega(Z, H) \} \\ & + \frac{1}{2} h(X, Z) \{ h(Y, H) 2e^{-2n\sigma} V(\sigma)^2 - e^{-2\sigma} \Omega(Y, H) V(\sigma) \\ & \quad + 2e^{-2\sigma} |\mathcal{H}(\text{grad}_h \sigma)|_h^2 h(Y, H) - 2e^{-2\sigma} h({}^h \nabla_Y (\mathcal{H}(\text{grad}_h \sigma)), H) \} \\ & - \frac{1}{2} h(Y, Z) \{ h(X, H) 2e^{-2n\sigma} V(\sigma)^2 - e^{-2\sigma} \Omega(X, H) V(\sigma) \\ & \quad + 2e^{-2\sigma} |\mathcal{H}(\text{grad}_h \sigma)|_h^2 h(X, H) - 2e^{-2\sigma} h({}^h \nabla_X (\mathcal{H}(\text{grad}_h \sigma)), H) \} \\ & + \frac{1}{2} h(Y, H) \{ h(X, Z) 2e^{-2n\sigma} V(\sigma)^2 - e^{-2\sigma} \Omega(X, Z) V(\sigma) \\ & \quad + 2e^{-2\sigma} |\mathcal{H}(\text{grad}_h \sigma)|_h^2 h(X, Z) - 2e^{-2\sigma} h({}^h \nabla_X (\mathcal{H}(\text{grad}_h \sigma)), Z) \} \\ & - \frac{1}{2} h(X, H) \{ h(Y, Z) 2e^{-2n\sigma} V(\sigma)^2 - e^{-2\sigma} \Omega(Y, Z) V(\sigma) \\ & \quad + 2e^{-2\sigma} |\mathcal{H}(\text{grad}_h \sigma)|_h^2 h(Y, Z) - 2e^{-2\sigma} h({}^h \nabla_Y (\mathcal{H}(\text{grad}_h \sigma)), Z) \} \\ & + \{ h(X, H) h(Y, Z) - h(Y, H) h(X, Z) \} \{ e^{-2n\sigma} V(\sigma)^2 + e^{-2\sigma} |\mathcal{H}(\text{grad}_h \sigma)|_h^2 \} \\ & + e^{-2\sigma} h(X(\sigma)Y - Y(\sigma)X, H(\sigma)Z - Z(\sigma)H) \\ & + \{ h(X, Z) h(Y, H) - h(X, H) h(Y, Z) \} \{ e^{-2n\sigma} V(\sigma)^2 + e^{-2\sigma} |\mathcal{H}(\text{grad}_h \sigma)|_h^2 \} \end{aligned} \quad (\text{B.1.22})$$

where X, Y, Z, H are basic vector fields.

Now, (B.1.3) follows from (B.1.22) after cancelling the corresponding terms. $\square$

Remark B.1.4. The formula (B.1.3) can be also proved directly in a similar way to (B.1.1) and (B.1.2).

The first formula of the following lemma follows after a straightforward computation using (B.1.1) and (B.1.3) whilst the second formula follows from (B.1.2). The third formula of the following lemma follows from (1.1.3), (1.1.4) together with Lemma 4.2.5 and (A.1.1),

Lemma B.1.5. *Suppose that $\mathcal{V}$ restricted to the domain of the local density $\rho = e^{(2-n)\sigma}$ is a simple foliation (i.e. the leaves are the fibres of a submersion) and let $\varphi : (O, g|_O) \rightarrow (N, \bar{h})$ be the induced harmonic morphism. If ${}^M\text{Ricci}$ denotes the Ricci tensor of (M, g) and ${}^N\text{Ricci}$ denotes the Ricci tensor of $(N, \bar{h})$, then,*

$$\begin{aligned} {}^M\text{Ricci}(X, Y) &= ({}^N\text{Ricci})(\varphi_* X, \varphi_* Y) - \frac{1}{2} e^{(2n-2)\sigma} h(i_X \Omega, i_Y \Omega) \\ &\quad - e^{-2\sigma} (\Delta^M \sigma) h(X, Y) - (n-1)(n-2)X(\sigma)Y(\sigma), \end{aligned} \quad (\text{B.1.23})$$

$$\begin{aligned} {}^M\text{Ricci}(X, V) &= \frac{1}{2} e^{(2n-2)\sigma} ({}^h d^* \Omega)(X) + (n-1)e^{(2n-2)\sigma} \Omega(X, \text{grad}_h \sigma) \\ &\quad + (n-1)X(V(\sigma)) - (n-1)(n-2)X(\sigma)V(\sigma), \end{aligned} \quad (\text{B.1.24})$$

$$\begin{aligned} {}^M\text{Ricci}(V, V) &= (n-2)e^{(2n-4)\sigma} \Delta^M \sigma + 2(n-1)V(V(\sigma)) \\ &\quad - (3n-4)(n-1)V(\sigma)^2 + \frac{1}{4} e^{(4n-4)\sigma} |\Omega|_h^2. \end{aligned} \quad (\text{B.1.25})$$

where ${}^h d^*$ is the codifferential on (M, h) .

Remark B.1.6. Putting $n = 2$ in the above formula we obtain particular cases of formulae of P. Baird and J.C. Wood [8, Proposition 4.2].

Bibliography

- [1] C.K. Anand, P. Baird, E. Loubeau, J.C. Wood (editors), *Harmonic morphisms, harmonic maps and related topics*, Brest 1997, Pitman Research Notes in Mathematics, CRC Press (1999).
- [2] M.A. Aprodu, M. Aprodu, V. Brînzănescu, A class of harmonic submersions and minimal submanifolds, Preprint, I.H.E.S., 1998.
- [3] S. Axler, P. Bourdon, W. Ramey, *Harmonic function theory*, Graduate Texts in Mathematics, 137, Springer-Verlag, New-York, 1992.
- [4] P. Baird, *Harmonic maps with symmetry, harmonic morphisms and deformations of metrics*, Research Notes in Mathematics, **87**, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1983.
- [5] P. Baird, Harmonic morphisms and circle actions on 3- and 4-manifolds, *Ann. Inst. Fourier*, **40** (1990) 177–212.
- [6] P. Baird, J. Eells, A conservation law for harmonic maps, *Geometry Symposium. Utrecht 1980*, Lecture Notes in Math. **894**, Springer-Verlag, Berlin, Heidelberg, New York, 1981, 1–25.
- [7] P. Baird, J.C. Wood, Harmonic morphisms and conformal foliations by geodesics of three-dimensional space forms, *J. Austral. Math. Soc. (A)*, **51** (1991) 118–153.
- [8] P. Baird, J.C. Wood, Harmonic morphisms, Seifert fibre spaces and conformal foliations, *Proc. London Math. Soc.*, **64** (1992) 170–196.

- [9] P. Baird, J.C. Wood, *Harmonic morphisms between Riemannian manifolds*, Oxford University Press, London Mathematical Society Monographs, (to appear).
- [10] L.Bérard-Bergery, J.-P. Bourguignon, Laplacians and Riemannian submersions with totally geodesic fibres, *Illinois J. Math.*, **26** (1982) 181–200.
- [11] A.L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 10, Springer-Verlag, Berlin-New York, 1987.
- [12] R.A. Blumenthal, J.J. Hebda, Ehresmann connections for foliations, *Indiana Univ. Math. J.*, **33** (1984) 597–611.
- [13] R.L. Bryant, Harmonic morphisms with fibres of dimension one, *Comm. Anal. Geom.*, **8** (2000), (to appear).
- [14] J.-P. Bourguignon, H.B. Lawson Jr., Stability and isolation phenomena for Yang-Mills fields, *Comm. Math. Phys.*, **79** (1981) 189–230.
- [15] F. Burstall, D. Ferrus, K. Leshke, F. Pedit, U. Pinkall, Conformal Geometry of Surfaces in the 4-Sphere and Quaternions, Preprint, (<http://xxx.lanl.gov/abs/math.DG/0002075>).
- [16] D.M.J. Calderbank, Selfdual Einstein metrics and conformal submersions, Preprint, (<http://xxx.lanl.gov/abs/math.DG/0001041>)
- [17] P.T. Church, J.G. Timourian, Maps with 0-dimensional critical set, *Pacific J. Math.*, **57** (1975) 59–66.
- [18] J. Eells, L. Lemaire, *Selected topics in harmonic maps*, CBMS Regional Conference Series in Mathematics, 50, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1983.
- [19] J. Eells, L. Lemaire, Another report on harmonic maps, *Bull. London Math. Soc.*, **20** (1988) 385–524.

- [20] J. Eells, J.H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.*, **86** (1964) 109–160.
- [21] T. Eguchi, A.J. Hanson, Asimptotically flat self-dual solutions to Euclidean gravity, *Phys. Lett. B*, 74 (1978) 249–251.
- [22] C. Ehresmann, Les connexions infinitésimales dans un espace fibré différentiable, *Colloque de topologie (espaces fibrés), Bruxelles, 1950*, Georges Thone, Liège; Masson et Cie., Paris, 1951, 29–55.
- [23] B. Fuglede, Harmonic morphisms between Riemannian manifolds, *Ann. Inst. Fourier*, **28** (1978) 107–144.
- [24] B. Fuglede, A criterion of non-vanishing differential of a smooth map, *Bull. London Math. Soc.*, **14** (1982) 98–102.
- [25] S. Gallot, D. Meyer, Opérateur de courbure et laplacien des formes différentielles d’une variété riemannienne, *J. Math. Pures Appl. (9)*, **54** (1975), no. 3, 259–284.
- [26] D. Gromoll, K. Grove, One-dimensional metric foliations in constant curvature spaces, *Differential geometry and complex analysis*, Springer, Berlin-New York, 1985, 165–168.
- [27] S. Gudmundsson, On the geometry of harmonic morphisms, *Math. Proc. Cambridge Philos. Soc.*, **108** (1990) 461–466.
- [28] S. Gudmundsson, Harmonic morphisms between spaces of constant curvature, *Proc. Edinb. Math. Soc.*, **36** (1992) 133–143.
- [29] S. Gudmundsson, *The Geometry of Harmonic Morphisms*, Ph.D. Thesis, University of Leeds, 1992.
- [30] S. Gudmundsson, Harmonic morphisms from complex projective spaces, *Geom. Dedicata*, **53** (1994) 155–161.

- [31] S. Gudmundsson, *The Bibliography of Harmonic Morphisms*,
[http://www.maths.lth.se/matematiklu/personal/sigma/harmonic/
 bibliography.html](http://www.maths.lth.se/matematiklu/personal/sigma/harmonic/bibliography.html)
- [32] S. Gudmundsson, E. Loubeau, S. Montaldo, M.T. Mustafa, *The Atlas of Harmonic Morphisms*,
[http://www.maths.lth.se/matematiklu/personal/sigma/harmonic/
 atlas.html](http://www.maths.lth.se/matematiklu/personal/sigma/harmonic/atlas.html)
- [33] A. Haefliger, Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes, *Comment. Math. Helv.*, **32** (1958) 248–329.
- [34] S.W. Hawking, Gravitational Instantons, *Phys. Lett. A*, **60** (1977) 81–83.
- [35] T. Ishihara, A mapping of Riemannian manifolds which preserves harmonic functions, *J. Math. Kyoto Univ.* **19** (1979) 215–229.
- [36] S. Kobayashi, *Transformation groups in differential geometry*, reprint of the 1972 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.
- [37] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, I, II, Interscience Tracts in Pure and Applied Math. 15, Interscience Publ., New York, London, Sydney, 1963, 1969.
- [38] W. Kühnel, Conformal transformations between Einstein spaces, *Conformal geometry. Papers from the seminar held in Bonn, 1985–1986*, R.S. Kulkarni, U. Pinkall (editors), Aspects of Mathematics, E12, Friedr. Vieweg & Sohn, Braunschweig, 1988, 105–146.
- [39] H.B. Lawson, Jr., M.L. Michelsohn, *Spin geometry*, Princeton Mathematical Series, 38, Princeton University Press, Princeton, NJ, 1989.
- [40] S. Lang, *Differential and Riemannian manifolds*, Graduate Texts in Mathematics, 160, Springer-Verlag, New York, 3rd edition, 1995.
- [41] C. LeBrun, Complete Ricci-flat Kähler metrics on $\mathbb{C}^n$ need not be flat, *Several complex variables and complex geometry, Part 2 (Santa Cruz, CA,*

1989), 297–304, Proc. Sympos. Pure Math., 52, Part 2, Amer. Math. Soc., Providence, RI, 1991.

- [42] A. Lichnerowicz, *Geometry of groups of transformations*. Translated from French and edited by Michael Cole. Noordhoff International Publishing, Leyden, 1977.
- [43] X. Mo, Horizontally conformal maps and harmonic morphisms, *Chinese Journal of Contemporary Mathematics*, **17** (1996) 245–252.
- [44] P. Molino, *Riemannian foliations*, translated from French by G. Cairns, with appendices by G. Cairns, Y. Carrière, É. Ghys and V. Sergiescu, Progress in Mathematics, 73, Birkhäuser Boston, Inc., Boston, MA, 1988.
- [45] M.T. Mustafa, A Bochner technique for harmonic morphisms, *J. London Math. Soc.*, (to appear).
- [46] R. Pantilie, On submersive harmonic morphisms, *Harmonic morphisms, harmonic maps and related topics*, Brest 1997, C.K. Anand, P. Baird, E. Loubeau and J.C. Wood (editors), Pitman Research Notes in Mathematics, CRC Press (1999), 23–29.
- [47] R. Pantilie, Harmonic morphisms with one-dimensional fibres, *Internat. J. Math.*, **10** (1999) 457–501.
- [48] R. Pantilie, Isometric actions and harmonic morphisms, *Michigan Math. J.*, (to appear).
- [49] R. Pantilie, Conformal actions and harmonic morphisms, *Math. Proc. Cambridge Philos. Soc.*, (to appear).
- [50] R. Pantilie, Harmonic morphisms with 1-dim fibres on 4-dim Einstein manifolds, *Comm. Anal. Geom.*, (to appear).
- [51] R. Pantilie, J.C. Wood, Topological restrictions for circle actions and harmonic morphisms, Preprint, University of Leeds, 2000.

- [52] R. Pantilie, J.C. Wood, New results on harmonic morphisms with one-dimensional fibres, Preprint, University of Leeds, 2000.
- [53] R. Ponge, H. Reckziegel, Twisted products in pseudo-Riemannian geometry, *Geom. Dedicata*, **48** (1993) 15–25.
- [54] A. Ranjan, Structural equations and an integral formula for foliated manifolds, *Geom. Dedicata*, **20** (1986) 85–91.
- [55] B.L. Reinhart, The second fundamental form of a plane field, *J. Differential Geometry*, **12** (1977) 619–627.
- [56] B.L. Reinhart, *Differential geometry of foliations. The fundamental integrability problem*, Ergebnisse der Mathematik und Ihrer Grenzgebiete, 99, Springer-Verlag, Berlin-New York, 1983.
- [57] N.E. Steenrod, *The topology of fibre bundles*, Princeton Mathematical Series 14. Princeton: Princeton University Press 1951.
- [58] J.G. Timourian, Fibre bundles with discrete singular set, *J. Math. Mech.*, **18** (1968) 61–70.
- [59] Ph. Tondeur, A characterisation of Riemannian flows, *Proc. Amer. Math. Soc.*, **125** (1997) 3403–3405.
- [60] Ph. Tondeur, *Geometry of foliations*, Monographs in Mathematics, 90, Birkhäuser Verlag, Basel, 1997.
- [61] Ph. Tondeur, L. Vanhecke, Jacobi fields, Riccati equation and Riemannian foliations, *Illinois J. Math.*, **40** (1996) 211–225.
- [62] I. Vaisman, Conformal foliations, *Kodai Math. J.*, **2** (1979) 26–37.
- [63] J. Vilms, Totally geodesic maps, *J. Differential Geometry*, **4** (1970) 73–79.
- [64] P. Walczak, An integral formula for a Riemannian manifold with two orthogonal complementary distributions, *Colloq. Math.*, **LVIII** (1990) 243–252.

- [65] J.C. Wood, Harmonic morphisms, foliations and Gauss maps, *Complex differential geometry and non-linear differential equations*, Contemp. Math. 49, Amer Math. Soc., Providence, R.I., 1986, 145–183.
- [66] J.C. Wood, Harmonic morphisms and Hermitian structures on Einstein 4-manifolds, *Internat. J. Math.*, **3** (1992) 415–439.
- [67] K. Yano, *The theory of Lie derivatives and its applications*, North-Holland Publishing Co., Amsterdam, P. Noordhoff Ltd, Groningen, Interscience Publishers Inc., New York, 1957.