Institute of Mathematics "Simion Stoilow" of the Romanian Academy

HABILITATION THESIS

Spectral analysis of elliptic operators in hyperbolic geometry

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”[...] Forma fără fond nu numai că nu aduce nici un folos, dar este de-a dreptul stricăcioasă, fiindcă nimiceste un mijloc puternic de cultură. [...] Mai bine să nu facem de loc academii, cu sectiunile lor, cu sedintele solemne, cu discursurile de receptiune, cu analele pentru elaborate decât să le facem toate aceste fără maturitatea stiintifică ce singură le dă ratiunea de a fi.

Titu Maiorescu, 1868.

Contents

Part I

Abstract

Chapter 1

Abstract of the thesis

1.1 Motivation

I have defended my Habilitation thesis more than 8 years ago, in June 2004 at the Université Paul Sabatier in Toulouse, France. Due to the demanding standards of the bureaucracy in the ministry of research, that thesis cannot be recognized in our country. The Habilitation procedure was introduced only in 2011, together with guidelines on how to recognize automatically Habilitation degrees obtained in the European Union. Unfortunately these guidelines (art. 3 (2) from the ministry of education and research order nr. 5.690/13.10.2011) require the candidate to have already directed at least one PhD student abroad. In my case, since I work in Bucharest, such a requirement cannot possibly be fulfilled. Knowing the well-deserved place our country holds in Europe in terms of scientific prestige and of research output per capita, it is perhaps justified that Habilitation theses from perypheral countries like France are not recognized automatically. It appears therefore necessary for me to gather a new Habilitation thesis in order to finally earn the right to direct PhD theses in Romania.

1.2 Abstract

Hyperbolic 3-manifolds may have conical singularities along geodesic curves. We consider such hyperbolic cone-manifolds with infinite-length singular curves. We prove that when all the angles around the singularities are

less than π, the hyperbolic cone-manifold is inifnitesimally rigid when the cone angles are fixed and when the natural conformal structure with marked points, corresponding to the end of the singular lines, is also fixed. Furthermore, small deformations of the cone angles and of the marked conformal structure at infinity correspond to unique small deformations of the cone-manifold structure. Under our assumption on the angles, the singular locus consists of closed geodesics, complete geodesics, and of a graph with vertices of valence at most 3, with possibly half-infinite edges.

On non-compact smooth hyperbolic 3 manifolds of finite geometry without cusps, or more generally on convex co-compact odd-dimensional hyperbolic manifolds, we analyze the geometric Selberg zeta function $Z^{o}_{\Gamma,\Sigma}(\lambda)$ corresponding to the spinor bundle. We prove the meromorphic extension to the complex plane and describe the zeros and the poles of $Z^{o}_{\Gamma,\Sigma}(\lambda)$. In the proof, we construct the resolvent of the Dirac operator on convex co-compact hyperbolic manifolds, prove its meromorphic extension and derive the existence and properties of Eisenstein and scattering operators as corollary. We remark that there exists a naturally-defined spectral eta invariant of the Dirac operator on convex cocompact hyperbolic manifolds obtained as quotients $\Gamma\backslash\mathbb{H}^{2n+1}$. We prove that this invariant is related to the Selberg zeta function: $\exp(\pi i\eta(D)) = Z^{o}_{\Gamma,\Sigma}(0)$. Formally this extends a result of Millson from the case of closed manifolds. Assuming that the exponent of convergence of the fundamental group Γ is small enough, the same identities hold for the od signature operator. In the particular case of Schottky 3-manifolds, the eta invariant, which depends just on the point in the Schottky moduli space corresponding to the conformal structure on the underlying Riemann surface, is the argument of a holomorphic function appearing in the Zograf factorization formula as the ratio of two kähler potentials for the Weil-Petersson metric.

Chapter 2

Rezumat

2.1 Motivatie

Mi-am sustinut teza de Abilitare (Habilitation à Diriger des Recherches) cu 8 ani inaintea prezentei lucrari, la Universitatea Paul Sabatier din Toulouse, Franta in iunie 2004. Datorita standardelor ridicate ale birocratiei din ministerul educatiei, teza mea nu poate fi insa, din pacate, recunoscuta pe plan local. Procedura de Abilitare a fost introdusa in 2011, urmata de modalitati practice de recunoastere automata a diplomelor de Abilitare obtinute in alte tari UE. Din nefericire, articolul 3 (2) din ordinul 5.690/13.10.2011 cere candidatului sa fi condus deja cel putin un doctorant in strainatate. In cazul meu, intrucat lucrez in Romania, aceasta cerinta nu poate fi indeplinita. Tinand cont de locul bine-meritat pe care il ocupa tara noasta in Europa prin prestigiul stiintific si prin rezultatele concrete ale cercetarii, e fara indoiala pe deplin justificat ca teze de Abilitare din tari mai putin dezvoltate matematic precum Franta sa nu fie recunoscute automat. Sunt astfel nevoit sa redactez prezenta teza pentru a obtine in final dreptul de a conduce doctorate in Romania.

2.2 Rezumat stiintific

Varietatile hiperbolice de dimensiune 3 pot avea singularitati conice de-a lungul unor curbe geodezice. Consideram varietati de dimensiune 3 hyperbolice care sunt convex co-compacte cu singularitati conice de-a lungul unor geodezice posibil infinite sau de-a lungul unui graf geodezic. Astfel de singularitati sunt folosite de fizicieni ca model pentru particule masive fara spin. Demonstram un rezultat de rigiditate infinitezimala cand unghiurile in jurul curbelor singulare sunt mai mici decat π: orice deformare infinitezimala schimba sau aceste unghiuri, sau structura conforma la infinit cu puncte marcate corespunzand directiilor singulare. Mai mult, orice variatie suficient de mica a structurii conforme marcate la infinit sau a unghiurilor singulare poate fi realizata printr-o unica deformare a structurii de varietate conica hyperbolica.

Aratam ca functia zeta Selberg de tip impar $Z^{\mathrm{o}}_{\Gamma,\Sigma}(\lambda)$ pe o varietate de dimensiune impara convex co-compacta $X = \Gamma\backslash\mathbb{H}^{2n+1}$ asociata fibratului de spinori admite o extensie meromorfa la planul complex si descriem structura polilor si a zerourilor. Ca unealta de abordare analizam spectrul operatorului Dirac si dezvoltam teoria difuziei (scattering) pe varietati asimptotic hiperbolice. Aratam ca exista un invariant eta $\eta(D)$ asociat natural operatorului Dirac peste varietati hyperbolice convex co-compacte si demonstram identitatea $\exp(\pi i\eta(D)) = Z^{\mathrm{o}}_{\Gamma,\Sigma}(0)$, extinzand astfel formula Millson din cazul compact. Sub ipoteza ca exponentul grupului convex co-compact Γ este suficient de mic, definim un invariant eta pentru operatorul de signatura impar si aratam ca pe varietati Schottky de dimensiune 3 invariantul eta este argumentul unei functii olomorfe care apare in formula Zograf de factorizare care leaga doua potentiale Kähler naturale pentru metrica Weil-Petersson pe spatiul de moduli de varietati Schottky.

Part II

Works on elliptic operators in hyperbolic geometry

In my thesis *Residue functionals on the algebra of adiabatic pseudo-differential operators*, defended in 1999 at MIT (and recognized in Romania in 2004) with professor Richard Melrose, I studied the Hochschild homology of the algebra of adiabatic pseudo-differential operators. In the subsequent papers [55], [59] and [62] I extended this study to other classes of pseudo-differential operators, namely fibered-cusp and double-edge operators. The study of Hochschild homology of pseudo-differential algebra is essentially motivated by index theory. I studied Fredholm index problems in [54], [56] and [63]. Together with Robert Lauter we find an index formula for pseudo-differential operators on manifolds with corners and on manifolds with fibered boundary. A related subject of study was the K-theory of the algebra of suspended operators [58], which allowed me to prove a cobordism invariance statement for the index of elliptic operators on manifolds with corners in [64], and also to give an analytic proof of the cobordism invariance for elliptic operators on closed manifolds [67].

I gradually shifted from index theory towards more general studies of the spectrum of linear elliptic operators. My actual Habilitation thesis [53], defended in 2004, originated from my works on the spectrum of elliptic operators on noncompact manifolds, namely [72], [57], [61]. In [72] I find that the spin Dirac operator on a hyperbolic manifold of finite volume continues to obey a sort of Weyl asymptotic formula for eigenvalues classically known on compact manifolds. In [57] I prove that the eta and zeta functions of elliptic first-order differential operators on the total space of a fibration with compact base have an adiabatic limit, generalizing works by Bismut-Fried and Cheeger. Together with Christian Bär we described in [60] the small-time heat kernel asymptotics for roots of Laplace type operators on closed manifolds, displaying some non-local terms in the asymptotic expansion. With André Legrand [66] we found an L^p index formula for conical manifolds, retrieving and extending the Chou index formula in the L^2 case.

In this current Habilitation thesis I will only detail some of the directions in my recent research activity dealing with the interaction between hyperbolic geometry and the analysis of elliptic operators. I will therefore not explain in detail my other subjects of research. Below I will briefly list some projects I have carried out in different areas revolving around elliptic differential operators on Riemannian manifolds.

In [74] and [77], together with Sylvain Golénia, I have analyzed the spectrum of magnetic Laplacians and of the Laplacians on differential forms. We find very interesting geometric examples of noncompact Riemannain manifolds where the spectrum of the Hodge Laplacian on forms is pure-point and obeys the Weyl law. The examples include certain complete hyperbolic manifolds of finite volume. We thus correct a result from the literature concerning the spectrum of the Hodge laplacian on geometrically finite hyperbolic manifolds.

I have studied the generalized Taub-NUT metrics together with Ion Cotaescu, Andrei Moroianu and Mihai Visinescu [65], [68], [80]. We compute the eta invariant of the Dirac operator on different domains, thus determining the index of the Dirac operator with the Atyiah-Patodi-Singer boundary condition. On the whole space we prove that the index is finite although the Dirac operator itself is not Fredholm. Finally on a wider class of spaces we show that there do not exist L^2 harmonic spinors.

I have studied the Dirac spectrum on manifolds with gradient conformal vector fields in the joint paper [71] with Andrei Moroianu, generalizing a result of J. Lott about zero modes. We prove that the there do not exist L^2 eigenspinors, which is the opposite case when compared to [74]. Interestingly, the setting in [71] includes infinite volume convex co-compact hyperbolic manifolds, while [74] includes finite volume hyperbolic manifolds with cusps.

The index formulae from [70] and [73], the last one in collaboration with Victor Nistor, are by now quite old. I have more or less turned away from index theory in recent years.

The results on the generic presence of singularities of spectral eta and zeta functions on closed manifolds from [82] close an interesting question, that of invariants appearing as values of zeta-type spectral functions at integer points. Such a value is meaningful whenever the associated zeta function is regular, in particular it then becomes conformally invariant.

The new conformally covariant series of differential operators starting with the Dirac operator discovered in [81] are very new and surprising. Even the third-order operator we get is entirely new: the operator

$$L := D^3 - \frac{\mathrm{cl}(\mathrm{d}(\mathrm{scal}))}{2(n-1)} - \frac{2\,\mathrm{cl}\circ\mathrm{Ric}\circ\nabla}{n-2} + \frac{\mathrm{scal}}{(n-1)(n-2)}D,$$

defined in terms of the metric g, is conformally covariant, in the sense that

$$\hat{L}_1 = e^{-\frac{n+3}{2}\omega} L_1 e^{\frac{n-3}{2}}$$

if $\hat{L}_1$ is defined in terms of the conformal metric $\hat{h} = e^{2\omega}h$.

In the rest of this presentation I will focus on my current research revolving around hyperbolic geometry. Already the paper [74] was in the setting of finite-volume hyperbolic manifolds. Together with Paul Loya and Jinsung Park we examined in [75] the adiabatic limit of the eta invariant on circle fibrations over noncompact hyperbolic surfaces. The regularity of the eta function on finite-volume hyperbolic manifolds was investigated in [78] jointly with the same co-authors as above. All these works set the stage for the two papers I define as the core of this Habilitation thesis, namely [76] by J.-M. Schlenker, Sergiu Moroianu, *Quasi-fuchsian manifolds with particles,* published in the Journal of Differential Geometry **83** (2009), 75–129, and [79] by C. Guillarmou, Sergiu Moroianu, J. Park, *Eta invariant and Selberg Zeta function of odd type over convex co-compact hyperbolic manifolds,* published in Advances in Mathematics **225** (2010), no. 5, 2464–2516.

2.3 Hyperbolic surfaces

I should start this presentation with the most elementary and perhaps most fascinating objects of geometry, namely surfaces. A surface is a topological space Σ locally diffeomorphic to $\mathbb{R}^2$. Every surface admits a *smooth structure*, namely an atlas of such diffeomorphisms whose change of charts consists of smooth local automorphisms of $\mathbb{R}^2$. Every two such atlases differ by a homeomorphism of Σ. Every smooth atlas contains a *conformal* atlas, in the sense that the changes of charts are conformal maps with respect to the canonical flat metric on $\mathbb{R}^2$, however these conformal structures are no longer unique up to diffeomorphism. In fact, their equivalence classes modulo isotopy gives rise to one of the most intriguing objects in mathematics, the celebrated Teichmüller space.

For simplicity we consider here only the case of orientable connected surfaces, since the language of Riemann surfaces comes handy in that setting. In general one needs to include anti-holomorphic maps in atlases of generalized holomorphic structure. Every orientable surface Σ admits a holomorphic atlas. By the Riemann mapping theorem (proved in a final form in 1907 by Poincaré and Koebe) its universal cover is therefore bi-holomorphic to one of the three simply connected Riemann surfaces – the sphere, the plane $\mathbb{C}$ or the hyperbolic plane $\mathbb{H}^2$. Only the last deserves some explanation. By definition,

$(\mathbb{H}^2, g_{\mathbb{H}^2})$ is the upper half-plane endowed with the Poincaré metric:

$$\mathbb{H}^2 = \{z \in \mathbb{C}; y = \Im(z) > 0)\}, \qquad g_{\mathbb{H}^2} = y^{-2}|dz|^2.$$

It is a complete metric of Gaussian curvature -1. Of course, its conformal structure is the same as the one defined by the euclidean metric $|dz|^2$ on the upper half-plane. Clearly, a self-isometry of $(\mathbb{H}^2, g_{\mathbb{H}^2})$ is either holomorphic or anti-holomorphic. The group of orientation-preserving isometries of $(\mathbb{H}^2, g_{\mathbb{H}^2})$ can be identified with $\mathrm{PSL}_2(\mathbb{R})$ by the action

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

However, it is true conversely that *every bi-holomorphic automorphism of* $\mathbb{H}^2$ *is a hyperbolic isometry*, in other words it belongs to $\mathrm{PSL}_2(\mathbb{R})$.

The surface Σ can be identified with the quotient of its universal cover by the action of the fundamental group, which must act without fixed points. The only bi-holomorphism of $\mathbb{C}P^1$ without fixed points is the identity, hence in the first case Σ itself must be $\mathbb{C}P^1$. In the second case when $\tilde{\Sigma} = \mathbb{C}$, the bi-holomorphisms are given by affine transformations of the form

$$z \mapsto az + b, \qquad a, b \in \mathbb{C}, \qquad a \neq 0.$$

The only such maps without fixed points are when $a = 1$, hence translations, which have the pleasant property of being also isometries for the Euclidean metric $|dz|^2$. It follows that Σ is the quotient of $\mathbb{C}$ by a discrete group of translations, hence (according to the rank of the group) it must be $\mathbb{C}$, a cylinder or a torus, and it inherits a flat metric from $\mathbb{C}$. The most interesting case is the remaining case when $\tilde{\Sigma} = \mathbb{H}^2$. In this case Σ is the quotient of $\mathbb{H}^2$ by a discrete subgroup in $\mathrm{PSL}_2(\mathbb{R})$ acting properly discontinuously (without fixed points). Therefore Σ inherits a complete hyperbolic metric.

Theorem 2.1 (Riemann-Poincaré-Koebe Uniformization theorem)**.** *Every connected Riemann surface* Σ *other than* $\mathbb{C}P^1$*,* $\mathbb{C}$*, a cylinder or a torus, admits in its conformal class a unique complete metric of constant Gaussian curvature* -1*.*

Even for compact surfaces, there exists a non-trivial deformation space of hyperbolic metrics (or equivalently of Riemann surface structures). Assume therefore that Σ is a compact orientable surface of genus $g \geq 2$ with a fixed smooth structure. The Riemann moduli space $\mathcal{M}_g$ is the set of equivalence classes of holomorphic structures modulo the group Diff of diffeomorphisms of Σ. The Teichmüller space $\mathcal{T}_g$ is the quotient of the set of holomorphic structures by the smaller group Diff_0 of *isotopies*, namely diffeomorphisms homotopic (inside Diff) to the identity map of Σ. The quotient $\mathcal{G} := \mathrm{Diff}/\mathrm{Diff}_0$ is the mapping class group of Σ, and clearly $\mathcal{M} = \mathcal{T}/\mathcal{G}$. We can view $\mathcal{T}$ as the symplectic reduction of the space of metrics on Σ by the semi-direct product of Diff_0 and of $\mathcal{C}^\infty(\Sigma, \mathbb{R})$, the last one acting by conformal transformations on metrics, i.e.,

$$f \cdot g := e^f g.$$

Hence $\mathcal{T}$ inherits a manifold structure. Alternately, $\mathcal{T}$ is the space of equivalence classes of hyperbolic metrics on Σ modulo isotopy. Its tangent space at a given holomorphic structure on Σ is the space of holomorphic quadratic differentials, i.e., of holomorphic sections in the square of the line bundle $(T^*)^{1,0}$. By the Riemann-Roch formula this space has dimension equal to $3g-3$, hence $\mathcal{T}$ is a smooth complex manifold of dimension $3g-3$. By using Fenchel-Nielsen coordinates given by any pants decompositions, we can see that $\mathcal{T}$ is topologically a ball of dimension $6g-6$. Moreover $\mathcal{T}$ carries a natural Kähler metric, the Weil-Petersson metric defined by the L^2 inner product of two holomorphic quadratic differentials using the unique underlying hyperbolic metric on Σ.

In conclusion, the Riemann moduli space is the quotient of a contractible Kähler manifold by a discrete group of isometries, the mapping class group, acting properly discontinuously but with some fixed points at those complex structures admitting non-trivial automorphisms (for instance, hyperelliptic curves). The isotropy subgrup of any point in $\mathbb{T}_g$ is finite, hence $\mathcal{M}$ is a Kähler orbifold. Its properties are sometimes studied under the guise of $\mathcal{G}$-invariant objects on $\mathcal{T}$.

Let us point out that the mapping class group is a very interesting object in its own right. Representations of $\mathcal{G}$ can be obtained from equivariant objects on $\mathbb{T}_g$, like the holomorphic section in the Chern-Simons line bundle over $\mathbb{T}_g$ recently constructed in a preprint by C. Guillarmou and this author.

2.4 Hyperbolic 3-manifolds

Hyperbolic 3-manifolds are a central object of current research in mathematics. Although it is not the focus of my work in this thesis, I should pay at least lip service to the geometrization program, featuring compact and finite-volume hyperbolic manifolds. According to Thurston's hyperbolization conjecture, every closed aspherical and atoroidal 3-manifold with infinite fundamental group admits a hyperbolic metric. The hyperbolization conjecture was recently proved based on Perelman's ideas on the Ricci flow, completing a program started by Hamilton in the 1980's. Many leading mathematicians have contributed to this achievement, both before and after Perelman's work. The hyperbolization conjecture is a part of the geometrization conjecture, which states that every 3-manifold can be decomposed in essentially unique "geometric" pieces by cutting along embedded spheres and tori. Each of the geometric pieces admits a complete geometry (in the sense of Cartan) of finite volume from a list of eight maximal geometries. Such a geometry is an atlas modeled on (X, G) where X is a model simply-connected 3-manifold and G is a Lie group acting transitively on X, maximal with this property and with compact stabilizers.

One instance of such a geometry is the spherical geometry modeled on $(S^3, O(4))$. The geometrization conjecture (or rather theorem) implies the celebrated Poincaré conjecture, stating that every compact simply connected smooth 3-manifold is diffeomorphic to S^3.

However by far most examples of geometrizable 3-manifolds are hyperbolic. Oriented hyperbolic manifolds are modelled on $(\mathbb{H}^3, \mathrm{PSL}_2(\mathbb{C}))$. Here we consider the upper half-space model for hyperbolic space as embedded inside quaternions orthogonal on k:

$$\mathbb{H}^3 = \{(z + jt); z \in \mathbb{C}, t > 0\}$$

and the Möbius group $\mathrm{PSL}_2(\mathbb{C})$ acts by quaternion linear fractional transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : q = z + jt \mapsto (aq + b)(cq + d)^{-1}.$$

One can check that this action preserves the metric $t^{-2}|dq|^2$ of constant sectional curvature -1.

In dimension 3 a finite-volume hyperbolic metric on a given manifold, when it exists, is unique by Mostow rigidity. Therefore there cannot be any analog of the Teichmüller space and its rich Kähler structure. Nevertheless something stronger than an analogy - an *identification* - does occur between Teichmüller space and the space of deformations of certain infinite-volume hyperbolic 3-manifold of finite geometry.

In this thesis we are interested in two classes of manifolds which are locally modelled on $(\mathbb{H}^3, \mathrm{PSL}_2(\mathbb{C}))$ but which are neither necessarily complete nor of finite volume (in fact, they are always of infinite volume). The typical examples are convex co-compact hyperbolic manifolds, which we describe below.

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ be a discrete group acting properly discontinuously and freely on $\mathbb{H}^3$ which admits a hyperbolic polyhedron (with a finite number of vertices, edges and faces) as fundamental domain. Then the quotient $X := \Gamma \backslash \mathbb{H}^3$ is a complete hyperbolic manifold of finite geometry. We assume that X is non-compact and that every vertex at infinity (i.e., on the Riemann sphere viewed as compactification of $\mathbb{C} \times \{0\}$) lies in a 2-dimensional face at infinity. Such a manifold X (and the group Γ) are called *convex co-compact*, and then the volume of X is necessarily infinite. An alternate description is requiring X to admit a compact geodesically convex subset (a subset $C \subset X$ is called geodesically convex if every geodesic arc with end-points in C is entirely contained in C).

A convex co-compact manifold X is diffeomorphic to the interior of a compact manifold $\bar{X}$ with boundary Σ. To describe the metric near the boundary we choose a boundary-defining function $x : \bar{X} \to [0, \infty)$, i.e., a smooth function vanishing precisely on the boundary $\Sigma = \{x = 0\}$ and such that $dx \neq 0$ in Σ. Near $x = 0$ the hyperbolic metric on X takes the form

$$g = x^{-2}(dx^2 + h(x)), \qquad h(x) = h_0 + x^2 h_2 + x^4 h_4$$

where h_0 is a metric on Σ, and h_2, h_4 are symmetric two tensors on Σ.

There exist two fundamental examples of convex co-compact hyperbolic 3-manifolds: quasi-fuchsian manifolds, and Schottky manifolds.

2.4.1 Quasi-fuchsian manifolds

A discrete subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is called a *fuchsian group*. If we assume that Γ does not contain elliptic or parabolic elements (i.e., elements γ with $|\mathrm{tr}(\gamma)| \leq 2$) then the quotient $\Sigma := \Gamma \backslash \mathbb{H}^2$ is a smooth complete hyperbolic surface. The hyperbolic metric from $\mathbb{H}^2$ is Γ-invariant so it descends to a metric g_Σ on Σ of gaussian curvature -1. Since $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{PSL}_2(\mathbb{C})$, we can also consider the quotient $X := \Gamma\backslash\mathbb{H}^3$. It is isometric to a cylinder

$$(\mathbb{R} \times \Sigma, g), \qquad\qquad g = dr^2 + \cosh(r)^2 g_\Sigma.$$

If Σ is compact then X is called a *Fuchsian* 3*-manifold*. It is clearly complete and non-compact. Moreover, it has a natural compactification by adding two copies of Σ at $r = \pm\infty$, corresponding to the set of directions of geodesics escaping to infinity (two such geodesics are identified if they get closer to each other in time, which implies that they become exponentially closer to each other). The two faces $\Sigma^\pm$ can also be realized as the quotient of the upper and of the lower half-planes by the action of Γ. This action is in fact an action on $\mathbb{C}P^1$ preserving the real circle $\mathbb{R} \sqcup \{\infty\}$. The fuchsian group Γ has therefore the property that its action on $\mathbb{C}P^1$ is properly discontinuous on two simply connected domains separated by a Jordan curve which is precisely the limit set of Γ.

By a fundamental result of Alhfors and Bers, the group Γ can be deformed inside $\mathrm{PSL}_2(\mathbb{C})$ as a convex co-compact subgroup. More precisely, given two Riemann surface structures (or equivalently two conformal structures) $c^\pm$ on a fixed smooth surface Σ, there exists a faithful representation of Γ in $\mathrm{PSL}_2(\mathbb{C})$ without elliptic or parabolic elements, unique up to conjugacy, such that the resulting 3-manifold $X := \rho(\Gamma)\backslash\mathbb{H}^3$ induces the conformal structures $\mathrm{c}^\pm$ at its two ideal boundary components $\Sigma^\pm$. Let us explain this geometrically. Choose boundary-defining functions x near each boundary component so that the induced metrics $h_0^\pm := \lim_{x\to 0} x^2 g$ on the boundaries $\Sigma^\pm$ are hyperbolic (this is always possible by the uniformization theorem, and x is unique up to second order errors at the boundary). A quasi-fuchsian manifold turns out to be a hyperbolic manifold diffeomorphic to $\mathbb{R} \times \Sigma$, such that near $t = \pm\infty$ the metric takes the form

$$g = x^{-2}(dx^2 + h(x)), \qquad\qquad h(x) = h_0^\pm + x^2 h_2^\pm + x^4 h_4^\pm$$

where $h_0^\pm$ are hyperbolic metrics on $\{\pm\infty\} \times \Sigma$, and $h_2^\pm$ are divergence-free symmetric two tensors on Σ of constant trace 1.

2.4.2 Schottky manifolds

Every Riemann surface Σ of genus g can be realized as the ideal boundary of a convex cocompact handlebody. To see this, choose disjoint simple curves

α_1, α_g on Σ such that the cut along them leads to a topological sphere with $2g$ disks removed. The Schottky reverse cut theorem states that this surface is biholomorphic to the complement of g pairs of mutually disjoint disks D_j, D'_j, $j = 1, \ldots, g$ in $\mathbb{C}P^1$. The interior of D_j can be mapped onto the interior of D'_j by an element $\gamma_j \in \mathrm{PSL}_2(\mathbb{C})$, and the elements $\gamma_1, \ldots, \gamma_g$ generate a free group Γ of rank g. The quotient of $\mathbb{H}^3$ by this group is called a Schottky manifold, and its boundary at infinity can be identified with the initial surface Σ.

In both cases, Schottky and quasi-fuchsian, the conformal structure at infinity determines uniquely the hyperbolic convec co-compact metric in the interior. Therefore, the Schottky, respectively the quasi-fuchsian spaces are identified with $\mathcal{T}_g$, respectively $\mathcal{T}_g^2$. Actually this fact remains true for every geometrically finite hyperbolic 3-manifold without rank 3 cusps: the space of its deformations is in bijection with the Teichmüller space of its ideal boundary. This correspondence may be interpreted as a rigidity statement: convex co-compact manifolds are rigid unde deformations preserving the conformal structure at infinity. One of the results emphasized in this thesis deals with the extension of this phenomenon to convex co-compact manifolds with conical singularities.

2.5 Hyperbolic cone manifolds

A general notion of hyperbolic cone-manifolds was introduced by Thurston in [95]. They are stratified spaces whose open startum is a hyperbolic manifold. In dimension 2 the sigularities are easy to describe, as they are isolated points. In this case, a conical point has a neighborhood isometric to a model neighborhood V_α defined as follows: consider the universal cover U of the complement of the point i in $\mathbb{H}^2$. The group $\mathbb{R}$ acts by isometries on this cover, lifting the action of the isotropy group $S^1 \subset \mathrm{PSL}_2(\mathbb{R})$ of i. The model conical manifold V_α is defined as the quotient of U by translation with $\alpha \in [0, \infty)$. This set is incomplete but we can define a one-point completion by adding the tip of the cone. If $\alpha = 2\pi$ we obtain $V_{2\pi} = \mathbb{H}^2$, for other angles the metric is singular at the tip of the cone.

In the case of 3-dimensional manifolds, the singular set can be a disjoint union of geodesic curves, defined as in the previous case. Then the metric in a neighborhood of each singular point is determined by the angle, a real number which is locally constant along the singular locus. The metric takes the explicit form

$$V_\alpha = (S_\alpha \times \mathbb{R}_{>0}, dt^2 + \sinh^2(t)h) \ ,$$

where (S_α, h) is the spherical metric surface with two conical points of angle α. However the singularities can also occur along a geodesic *graph*. An important remark due to Weiss [99] is that when all conical angles are less than π, the singular graph has all vertices of valence 3.

As explained above, convex co-compact manifods without conical singularities are rigid relative to the conformal structure at the ideal boundary, so they are parametrized (on a given smooth manifold) by those conformal structures, i.e., by Teichmüller space. A natural question is to see to what extent conical hyperbolic manifolds are rigid in that sense.

Compact conical hyperbolic manifolds were considered by Hodgson and Kerckhoff [31]. In their setting the singular set is a disjoint union of simple closed geodesic curves and each of these singularities is determined by four real parameters: the angle and the translation component of holonomy around the singular curve (which form together the so-called complex angle), and the lenght and the twist along the curve (which determine the complex length). Their result states that for real angles less than 2π the cone manifold is infinitesimally rigid by deformations preserving the complex angle. Moreover, they showed that changes in the complex angles parametrize small deformations of the hyperbolic cone-manifold structure. Weiss [99] proved a similar result for cone manifolds with graph singularities, however he must assume that the angles are less than π.

The infinitesimal rigidity result of Hodgson and Kerkhoff was generalized by Bromberg [8] to the setting of convex co-compact cone manifolds, with singularities again along closed curves. His results states that any infinitesimal deformation of such a cone-manifold structure must imply a deformation of the angles (supposed to be less than π or of the conformal structure at infinity.

In my paper [76] with Jean-Marc Schlenker we examine hyperbolic 3-manifolds with singularities along graphs with possibly infinite-length edges, i.e., half-infinite geodesics stretching in the funnels. We must assume that the angles around the half-infinite geodescs are bounded above by π for several technical resons. Although possible in principle, we have prefered not to allow cusps of rank 2 in the manifold, mainly because their presence would make the presentation more cumbersome.

As mentioned above, a subset C in a hyperbolic cone-manifold M is called *geodesically convex* if it is non-empty and any geodesic segment in M with endpoints in C is contained in C. This notion is *global* and thus stronger than the usual condition of convexity on the boundary, for instance points are not geodesically convex unless M is contactible. Directly from the definition, any nonempty intersection of geodesically convex sets is again geodesically convex. We prove that when the conical angles are less than π, every closed geodesic is contained in every geodesically convex set. This allows us to define a smallest geodesically convex set, containing all closed geodesics, playing the role of the convex core in convex co-compact manifolds. We just need to assume that M contains a compact convex set. If M is a complete, hyperbolic cone-manifold of dimension 3 with cone singularities of angles at most π along a finite graph with possibly infinite-length edges, we will call it a *convex co-compact manifold with particles* if it contains a compact subset C which is convex.

Such a manifold is always homeomorphic to the interior of a compact manifold with boundary, which can be chosen to be the smalles geodesically convex subset of M. The vertices of valence 1 in the singular graph correspond to half-infinite edges going to infinity.

It follows that M is then homeomorphic to the interior of a compact manifold with boundary which we will call N (N is actually homeomorphic to the compact convex subset C in the definition). The singular set of M corresponds under the homeomorphism with a graph Γ embedded in N, such that vertices of Γ adjacent to only one edge are in the boundary of N.

Infinitesimal deformations of possibly incomplete hyperbolic manifolds are parametrized by the 1-cohomology group of a flat bundle of "infinitesimal Killing vector fields". This idea goes back to Weil [97] in this setting and is an instance of the so-called Kodaira-Spencer deformation theory. Basically, an infinitesimal deformation gives rise to a closed 1-form on M twisted by the bundle E of infinitesimal Killing vector fields, well defined up to addition of an exact form. The bundle E is the bundle associated to the holonomy representation of $\pi_1(M)$ into $\mathrm{PSL}_2(\mathbb{C})$, unique up to conjugacy, and to the adjoint representation of $\mathrm{PSL}_2(\mathbb{C})$ on its lie algebra. Geometrically E can also be identified with the complexification of the tangent bundle, endowed with an explicit flat connection. Any trivial deformation, i.e., the infinitesimal action of a vector field, leads to an exact 1-form, hence the twisted de Rham cohomology class of the deformation 1-form is well defined. This construction was recently used for cone-manifolds by Hodgson and Kerckhoff [31], with a special emphasis on the square-integrability of representatives for the deformation cohomology class.

We prove two related results about the rigidy of hyperbolic convex co-compact manifolds with particles.

Theorem 2.2. *The metric on a convex co-compact manifold with particles* (M, g) *is infinitesimally rigid under deformations preserving the cone angles and the marked conformal structure at infinity.*

To state the second main result, let $\mathcal{R}(M_r)$ be the representation variety of $\pi_1(M_r)$ into $PSL_2(\mathbb{C})$ and $\rho : \pi_1(M_r) \to \mathrm{PSL}_2(\mathbb{C})$ the holonomy representation of the regular part of M. We call $\mathcal{R}_{cone}(M_r)$ the subset of those representations for which the holonomy of meridians of the singular curves have no translation component, that is, these holonomies are pure rotations. Thus $\rho \in \mathcal{R}_{cone}(M_r)$, and, in the neighborhood of ρ, the points of $\mathcal{R}_{cone}(M_r)$ are precisely the holonomies of cone-manifolds in the sense of our definition.

Theorem 2.3. *Let* (M, g) *be a convex co-compact manifold with particles,* c *the induced marked conformal structure at infinity, and* $\theta_1, \cdots, \theta_N \in (0, \pi)$ *the conical angles. In the neighborhood of the holonomy representation* ρ*, the moduli space of convex co-compact manifolds with particles structures on* M*, i.e., the quotient of* $\mathcal{R}_{cone}(M_r)$ *by* $PSL(2, \mathbb{C})$*, is parameterized by the parameters* $c, \theta_1, \cdots, \theta_n$*.*

The proof of the rigidity theorem is based, as we mentioned above, on the analysis of the deformation class in $H^1(M, E)$. We first show that the deformation can be normalized, modulo trivial deformations, into a form where the 1-form ω is square integrable. The main body of work is essentially showing that the L^2 cohomology group $H^1_{L^2}(M, E)$ vanishes. For this we use again in a crucial way the hypothesis that the cone angles are less than π to deduce the essential self-adjointness of a twisted Dirac operator. Thus the crux of the articles lies in the fine analysis of certain elliptic operators appearing naturally in the problem. In general, rigidity would follow from the vanishing of $H^1(M, E)$. In the case of a closed manifold, this vanishing reduces, by Hodge theory, to the vanishing of the space of harmonic 1-forms twisted by E. This Hodge cohomology space turns out to be zero since the Hodge laplacian twisted by E is strictly positive on 1-forms, using a Weitzenböck formula due to Matsushima and Murakami. This formal positivity is typical for negative curvature. When the manifold M is no longer compact, it is no longer true that $H^1(M, E)$ is zero since as we have seen there do exist deformations changing the conformal structure at infinity. One works instead inside the Hilbert space of L^2 forms twisted by E. First we prove that modulo trivial deformations, any deformation preserving the angles and the conformal structure at infinity marked by the endpoints of outgoing conical geodesics can be represented by a L^2 form. Then we show that this form must be exact, with an L^2 primitive.

The second theorem is a consequence of the first. Its proof consists in a computation of dimensions for the deformation space (without restrictions on the angles) and the fact that the global infinitesimal deformations (i.e., deformations of the whole manifold) form a Lagrangian subspace in the space of germs of deformations near the singular locus. This idea of a Lagrangian space inside the moduli space appears also in the second work featured in this thesis.

2.6 Odd Selberg zeta function of the Dirac operator on hyperbolic manifolds

We place ourselves in the same setting as in the previous section but without “particles”, namely we study convex co-compact hyperbolic manifolds and their spectral zeta functions.

First let us recall a few basic things about the spectrum of elliptic operators on a closed manifold. Let D be a symmetric first-order elliptic differential operator on a closed manifold M, acting on sections of a vector bundle E. We can view D as an unbounded linear operator acting in the Hilbert space of square-integrable section in E, with domain $\mathcal{C}^\infty(M, E)$. Then D is essentially self-adjoint, and its spectrum is formed of real eigenvalues accumulating in absolute value to infinity. Moreover, if $\dim(M) = n$ and Rank(E) = k, then the counting function N for the eigenvalues of D,

$$N(x) := \#\{\lambda \in \mathrm{Spec(D)}; |\lambda| < \mathrm{x}\},$$

satisfies the Weyl asymptotic formula

$$\lim_{x\to\infty} \frac{N(x)}{x^n} = \frac{1}{(2\pi)^n} \int_{S^*M} |\sigma(D)|$$

where $\sigma(D)$ is the principal simbol of D and S^*M the sphere inside the tangent bundle to M. This asymptotic law implies that the series defining the spectral zeta function

$$\zeta(D,S) := \sum_{0\neq\lambda\in\mathrm{Spec(D)}} |\lambda|^{-s}$$

is absolutely convergent for $\Re(s) > n$. Similarly, the eta function

$$\zeta(D,S) := \sum_{\lambda\in\mathrm{Spec(D)}} \lambda|\lambda|^{-s-1}$$

is absolutely convergent for $\Re(s) > n$. Both these functions extend meromorphically to the complex plane with a regular point in the origin. The eta invariant is the value of the eta function at $s = 0$. It is the boundary term in the index formula for compact manifolds with boundary [4] in the case where D is the tangential component of an elliptic operator on a compact manifold X with boundary M. By the definition of the Gamma function (or more sophisticatedly using the Mellin transform) one can write

$$\eta(D,s) = \mathrm{Tr}\left(D(D^2)^{-\frac{s+1}{2}}\right) = \frac{1}{\Gamma((s+1)/2)}\int_0^\infty t^{s-\frac{1}{2}}\,\mathrm{Tr}\left(De^{-tD^2}\right)dt,$$

One can think of $\eta(D) := \eta(D,0)$ as computing the asymmetry "$\mathrm{Tr}(D|D|^{-1})$" of the spectrum, or the diference between the number of positive and of negative eigenvalues.

On non-compact manifolds the situation is drastically different in general. What makes things difficult is that symmetric elliptic operators may admit several self-adjoint extensions, or none. The spectrum may be pure-point and obey the Weyl law, as in [72], or it can be purely continuous without any eigenvalue as in [81]. The eta function may exist, but it can have a pole at $s = 0$. Even in the case of hyperbolic surfaces of finite volume, it is not known, with the notable exception of arithmetic surfaces, whether there exist eigenvalues at all other than 0.

2.6.1 The Selberg trace formula

The original Selberg trace formula is a fundamental and surprising relationship between the spectrum of the Laplacian on a hyperbolic surface and the so called length spectrum, i.e., the set of lengths of closed geodesics. Selberg proved this formula in particular as a way to show that on arithmetic surfaces there exist enough eigenvalues to satisfy the growth rate predicted by the Weyl law. However the formula applies to locally symmetric spaces in quite wide generality.

Before applying it to Dirac operators on manifolds of infinite volume, let us review at least the simplest instance of the Selberg trace formula, that of compact hyperbolic surfaces. Let S be a closed hyperbolic surface, $\Delta = d^*d$ the associated scalar Laplacian, and f an even holomorphic function, which in practice will be $f(z) = \exp(-t(s^2+\frac{1}{4}))$ and $R_{s_0}(s) = (s_0^2 - s^2)^{-1}$ for $t > 0$ and $\Im(s_0) > 0$. Let F be the holomorphic function defined by $F(s^2 + \frac{1}{4}) = f(s)$ and define the trace of $F(\Delta)$ as

$$\mathrm{tr}(\Delta) = \sum_{\lambda \in \mathrm{Spec}(\Delta)} F(\lambda).$$

The operator $F(\Delta)$ makes sense for every F, is bounded when F is bounded on the positive real line, and is trace-class when $F(\lambda)$ decreases faster than $|\lambda|^{-1-\epsilon}$, again on the real line towards $+\infty$. Thus $\mathrm{tr}(\Delta)$ is under these hypotheses the actual operator trace of $F(\Delta)$. Let now $\Delta^{\mathbb{H}^2}$ be the Laplacian on the universal cover $\mathbb{H}^2$ of S, and $F(\Delta^{\mathbb{H}^2})$ the coresponding operator on $L^2(\mathbb{H}^2)$. This operator has a Schwartz kernel $A(z, z')$, possibly singular on the diagonal $z = z'$ and decreasing away from it, which is $\mathrm{PSL}_2(\mathbb{R})$-bivariant and therefore depends only on the distance between z and z'. Thus there exists a convolution kernel G such that $A(z, z') = G(2\cosh d(z, z') - 2)$, with G decreasing at infinity. This function G can be computed explicitly in terms of f and *vice-versa*, using the so-called Abel transform.

The operator $F(\Delta)$ also has a Schwartz kernel $A^S(m, m')$ defined on $S \times S$ with singularities on the diagonal. In fact, one can write the lift of A^S to $\mathbb{H}^2 \times \mathbb{H}^2$ as a sum over the fundamental group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$:

$$A^s(z, z') = \sum_{\gamma \in \Gamma} A(z, \gamma z).$$

This can be re-written in terms of the convolution kernel G as $\sum_{\gamma\in\Gamma} G(2\cosh d(z, \gamma z') - 2)$. Now the trace of a pseudodifferential operator of order strictly less than $2 = \dim(S)$ with kernel A^S is computed by

$$\mathrm{tr}(F(\Delta)) = \int_S A^S(m, m) dm.$$

Thus by choosing a fundamental domain $\Omega \subset \mathbb{H}^2$, we get

$$\operatorname{tr}(F(\Delta)) = \int_\Omega \sum_{\gamma\in\Gamma} G(2\cosh d(z,\gamma z) - 2).$$

Under suitable hypothesis on f the above series is absolutely convergent in L^1 so we reverse the order of integration and summation. Moreover, we group the terms according to conjugacy classes inside Γ:

$$\operatorname{tr}(F(\Delta)) = \sum_{[\gamma]\in\Gamma'} I_{[\gamma]}, \qquad I_{[\gamma]} := \sum_{\gamma\in[\gamma]} \int_\Omega G(2\cosh d(z,\gamma z) - 2)dz.$$

Each conjugacy class is either the class of the identity, or a hyperbolic class since we assume S has no cusps. The conjugacy class of the identity contains just itself. The centralizer of an elliptic element $\gamma \neq 1$ in a Fuchsian group is a cyclic group containing γ, so $\gamma = \gamma_0^n$ for some $n \geq 1$ and γ_0 a primitive element of Γ. Thus the conjugacy class of γ is in bijection with $\Gamma/\langle\gamma_0\rangle$, hence by the bi-invariance of the distance function, we can rewrite

$$I_{[\gamma]} = \int_{\langle\gamma_0\rangle\backslash\mathbb{H}^2} G(2\cosh d(z,\gamma z) - 2)dz.$$

In conclusion, we have written

$$\operatorname{tr}(F(\Delta)) = I_1 + \sum_{1\neq[\gamma_0]\in\Gamma'} \sum_{n\geq 1} \int_{\langle\gamma_0\rangle\backslash\mathbb{H}^2} G(2\cosh d(z,\gamma_0^n z) - 2)dz.$$

Recall now that conjugacy classes of hyperbolic elements in Γ are in bijection with oriented closed geodesics in $S = \Gamma\backslash\mathbb{H}^2$, and that the trace of a group element is $\cosh(\frac{d}{2})$ where d is le length of the corresponding oriented closed geodesic. We let $\mathfrak{L}$ denote the set of length (with multiplicities) of the closed geodesics, also called the geometric spectrum. After some manipulations with Abel transforms, from the above we get the Trace formula:

$$\operatorname{tr}(F(\Delta)) = (g-1)\int_{\mathbb{R}} f(\xi)\xi\tanh(\pi\xi)d\xi + \sum_{d\in\mathfrak{L}} \frac{d\hat{f}(nd)}{2\sinh(\frac{nd}{2})}$$

where $\hat{f}$ is the Fourier transform of f.

One remarkable offshot of the trace formula is the analytic extension of the Selberg zeta function. This is a geometrically defined holomorphic function in terms of the length spectrum of S:

$$\zeta(z) := \prod_{d\in\mathcal{L}} \zeta(d,z), \qquad \zeta(d,z) = \prod_{n=0}^{\infty}(1-e^{-(n+z)d}).$$

The product is absolutely convergent for $\mathrm{Re}(z) > 1$ and hence the zeta function is holomorphic there. The Selberg zeta function for the resolvent function proves that ζ extends holomorphically to $\mathbb{C}$ with zeros at points determined by the spectrum of Δ. Explicitly, except for some "trivial" zeros at the non-negative integers, ζ has zeros precisely at the points $\frac{1}{2}+iz \in \mathbb{C}$ such that $\frac{1}{4}+z^2 \in \mathrm{Spec}(\Delta)$, with multiplicity equal to that of the corresponding eignevalue. Moreover the Selberg zeta function satisfies a functional equation making it symmetric around $\frac{1}{2}$. This provides a hyperbolic counterpart of the Riemann hypothesis, satisfied by the Selberg zeta function, with the exception of a finite number of real zeros corresponding to the eigenvalues of Δ which are less than $\frac{1}{4}$ (the so-called small eigenvalues). Note also the identity

$$\zeta(z) = \exp\left(-\sum_{m=1}^{\infty}\sum_{d\in\mathcal{L}} \frac{e^{-zmd}}{m(1-e^{-md})}\right).$$

There is no conceptual difficulty in extending the above reasoning to closed hyperbolic surfaces of higher dimensions, or more generally to locally symmetric spaces [89]. Also, the trace formula can be applied to operators acting in homogeneous bundles, like the Dirac operator on spinors. This has been carried out by Millson [49], who showed that the fractional part of the eta invariant of the odd signature operator A on odd forms $\Lambda^{\mathrm{odd}} = \oplus_{p=0}^{2m}\Lambda^{2p-1}$ on a hyperbolic $4k-1$-dimensional manifold $X_\Gamma := \Gamma\backslash\mathbb{H}^{4m-1}$ is determined by the central value of the corresponding zeta function of odd type. To define the zeta function, clearly more information than the length of the closed geodesics is needed. In dimension 2 the trace is the unique invariant of a hyperbolic group element, in higher dimensions and when we deal with vector bundles various holonomies appear naturally. Millson defines

$$Z^{o}_{\Gamma,\Lambda}(\lambda) := \exp\left(-\sum_{\gamma\in\mathcal{P}}\sum_{k=1}^{\infty} \frac{\chi_+(R(\gamma)^k)-\chi_-(R(\gamma)^k)}{|\det(\mathrm{Id}-P(\gamma)^k)|^{\frac{1}{2}}}\frac{e^{-\lambda k d(\gamma)}}{k}\right)$$

where $\mathcal{P}$ is the set of primitive closed geodesics in X_Γ, or equivalently the set of conjugacy classes in γ; $R(\gamma) \in \mathrm{SO}(4m-2)$ is the holonomy in the form bundle along a geodesic γ, $\chi_\pm$ denotes the character associated to the two irreducible representations of $\mathrm{SO}(4m-2)$ corresponding to the $\pm i$ eigenspace of the Hodge operator $\star$ acting on Λ^{2m-1}, $P(\gamma)$ is the linear Poincaré map along γ, and $d(\gamma)$ is the length of the closed geodesic γ. The function $Z^o_{\Gamma,\Lambda}(\lambda)$ extends meromorphically to $\lambda \in \mathbb{C}$, its zeros and poles occur on the line $\Re(\lambda) = 0$ with order given in terms of the multiplicity of the eigenvalues of A, and the following remarkable identity holds:

$$e^{\pi i \eta(A)} = Z^o_{\Gamma,\Lambda}(0).$$

Moscovici and Stanton [85] extended this result to closed locally symmetric spaces of higher rank.

The geometric Selberg zeta function makes perfect sense on certain non-compact hyperbolic manifolds, at least in the domain of absolute convergence, since closed geodesics continue to be related to the traces of hyperbolic elements on the fundamental group. One example of this is the case of finite-volume surfaces, treated by Selberg himslef. Another example is the analysis of the odd-type zeta function for the spin Dirac operator carried out by J. Park in [86]. There the functional equation of the zeta function involves the determinant of the scattering matrix. Together with C. Guillarmou and J. Park I have studied the Selberg zeta function on convex co-compact hyperbolic manifolds associated to the spin Dirac operator. We then specialize to the case of dimension 3 and we extend our results, basically without change, to the odd signature operator. For Schottky manifolds we get an interesting relation to Teichmüller theory, which is developed further in a joint paper with C. Guillarmou.

We start our investigation by studying the meromorphic extension of the resolvent of the Dirac operator on a general asymptotically hyperbolic manifold. These are complete Riemannian manifolds (X, g) such that X is the interior of a smooth manifold with boundary $\bar{X}$, and the metric g is of the form $g = x^{-2}\bar{g}$ where $\bar{g}$ is a smooth Riemannian metric on $\bar{X}$, and x is a boundary-defining function, $x \in \mathcal{C}^\infty(\bar{X}, [0, \infty))$ such that $x^{-1}(0) = \partial\bar{X}$ and dx does not vanish on $\partial\bar{X}$, and finally dx has length 1 with respect to $\bar{g}$. Then the sectional curvatures of g tend to -1 near $\partial\bar{X}$, ie., $\bar{X}$ is indeed asymtotically hyperbolic. Clearly convex co-compact manifolds are particular cases of asymptotically hyperbolic manifolds.

The Dirac operator D acting on the spinor bundle Σ on a spin asymptotically hyperbolic manifold X_Γ has real spectrum (it is essentially self-adjoint since the manifold is complete), and one can define its resolvent for $\Re(\lambda) > 0$ in two ways

$$R_+(\lambda) := (D + i\lambda)^{-1}, \qquad R_-(\lambda) := (D - i\lambda)^{-1}$$

as analytic families of bounded operators acting on $L^2(X_\Gamma; \Sigma)$. Our first task is to prove meromorphic extension of these analytic families: the resolvents $R_\pm(\lambda)$ extend meromorphically to $\lambda \in \mathbb{C}$ as operators from $C_0^\infty(X_\Gamma, \Sigma)$ to the dual space $C^{-\infty}(X_\Gamma, \Sigma^*)$, and moreover the polar part of the meromorphic extensions is of finite rank. This is proved by actually constructing the resolvent inside the calculus of "zero" pseudodifferential operators of Mazzeo-Melrose [46].

The first step in the Melrose construction of resolvents in a wide class of pseudo-differential calculi is, besides, the obvious principal symbol, the resolvent of the so-called normal operator, which is a model operator at the boundary obtained by freezing coefficients. In our case this normal operator turns out to be precisely the Dirac operator on the hyperbolic space $\mathbb{H}^{d+1}$. The explicit computation of Camporesi [13] shows that the Schwartz kernel $R^{\mathbb{H}^{d+1}}(\lambda; m, m')$ of the resolvent $(D^2 + \lambda^2)^{-1}$ on $\mathbb{H}^{d+1}$, defined for $\Re(\lambda) > 0$, is given essentially by a hypergeometric function times the parallel transport from m' to m. We remark that the parallel transport lifts smoothly on the so-called zero double space, which is the (real) blow-up of the boundary diagonal in $\bar{X} \times \bar{X}$. Thus $R^{\mathbb{H}^{d+1}}(\lambda; m, m')$ is a conormal distribution on the zero double space. Using this, we can show that $R^\pm(\lambda; m, m')$ extends analytically to $\lambda \in \mathbb{C}$ as a conormal distribution in $\Psi_0^{-1,\lambda+\frac{d}{2},\lambda+\frac{d}{2}}(\bar{X}, \Sigma_0)$, where Σ_0 is the spinor bundle on $\bar{X}$ with respect to $\bar{g}$. More precisely, the asymptotic structure is as follows: let ρ, ρ' be defining functions for the left and right hyperfaces of $\bar{X}_0^2$. Then $(\rho\rho')^{-\lambda-\frac{d}{2}} R^\pm(\lambda; m, m')$ has a conormal singularity at the lifted diagonal corresponding to the pseudodifferential order -1 and is smooth to the three boundary faces, in the sense that is has Taylor series there. We now remark that the formula is analytic in $\lambda \in \mathbb{C}$, in other words $R^{\mathbb{H}^{d+1}}(\lambda; m, m')$ extends analytically to $\mathbb{C}$. Clearly when $\Re(\lambda) \leq 0$ the associated operator is no longer bounded on L^2, but its Schwartz kernel is analytic as a family of conormal distributions.

Once the first step of the machinery is established, the second step consists in constructing the resolvent modulo residual (smoothing) terms.

This relies on the composition rules of $\Psi_0(X)$. We obtain a parametrix in $\Psi_0^{-1,\lambda+\frac{d}{2},\lambda+\frac{d}{2}}(\bar{X},\Sigma_0)$ for the resolvent $R^{\pm}(\lambda;m,m')$, with an error in $\rho_{\mathrm{ff}}\Psi_0^{-\infty;\infty;\lambda+\frac{d}{2}}$. This error term is already compact, we use then a Neumann series argument to solve away the Taylor series at the front face. This may in principle introduce logarithmic terms in $\rho,\rho',\rho_{\mathrm{ff}}$ but they are shown not to occur by the indicial equation of $D\pm i\lambda$.

This structure theorem for the resolvent implies that the spectrum of D on asymptotically hyperbolic manifolds is absolutely continuous and given by $\mathbb{R}$. The Eisenstein series are defined directly from the resolvent, by constructing generalized eigenspinors with prescribed asymptotic behaviour at infinity. The scattering operator $S(\lambda):C^\infty(\partial\bar{X},\Sigma)\to C^\infty(\partial\bar{X},\Sigma)$ is just the leading "outgoing" asymptotic term in the Eisenstein series. It is a meromorphic family of pseudodifferential operators on $\partial\bar{X}$ of order 2λ, with principal symbol essentially the same as that of the Dirac operator. In [81] we show that the intrinsic conformal covariance of the scattering operator leads to the construction of conformally covariant differential operators on any spin manifold M, of any odd order $2k+1\leq\dim(M)$, coniciding with the $2k+1$-th power of the Dirac operator up to lower order terms. The fact that the scattering operator is invertible outside some isolated points where it has zeros or poles gives a new proof of the cobordism invariance of the index. Also $\frac{1}{2}(\mathrm{Id}-S(0))$ turns out to be the Calderon projector for the Dirac operator, i.e., the orthogonal L^2 projector on the Cauchy data space, or the boundary values of harmonic spinors on $\bar{X}$. The scattering operator plays an important role in the study of the odd-type Selberg zeta function.

The exponent of a convex co-compact group Γ is defined using an arbitrary $m\in\mathbb{H}^{d+1}$ by

$$\inf\left\{\lambda\in\mathbb{R}|\sum_{\gamma\in\Gamma}e^{-\lambda d(m,\gamma m)}<\infty\right\}.$$

For $\lambda>\delta_\Gamma-n$, we define the Selberg zeta function of odd type $Z^o_{\Gamma,\Sigma}(\lambda)$ associated to the spinor bundle Σ exactly like Millson did for the odd signature operator, except that $R(\gamma)$ denotes now the holonomy in the spinor bundle Σ along γ, and $\chi_\pm$ denotes the character of the two irreducible representations of $\mathrm{Spin}(2n)$ corresponding to the $\pm i$ eigenspaces of the Clifford multiplication $\mathrm{cl}(T_\gamma)$ with the tangent vector field T_γ to γ.

One of our two main results in [81] is the pole structure of the zeta function:

Theorem 2.4. *The odd-type Selberg zeta function $Z^o_{\Gamma,\Sigma}(\lambda)$ on an odd dimensional spin convex co-compact hyperbolic manifold $X_\Gamma = \Gamma\backslash\mathbb{H}^{2n+1}$ has a meromorphic extension to $\mathbb{C}$, and it is analytic near $\{\Re(\lambda) \geq 0\}$. A point λ_0 with $\Re(\lambda_0) < 0\}$ is a zero or a pole of $Z^o_{\Gamma,\Sigma}(\lambda)$ if and only if the meromorphic extension of $R_+(\lambda)$ or of $R_-(\lambda)$ has a pole at λ_0, and in that case the multiplicity of λ_0 is*

$$\operatorname{rank}\operatorname{Res}_{\lambda_0} R_-(\lambda) - \operatorname{rank}\operatorname{Res}_{\lambda_0} R_+(\lambda).$$

The proof of this theorem uses of course the Selberg trace formula, for the particular case of the heat kernel. The main problem is perhaps that the odd heat operator $D\exp(-tD^2)$ on a asymptotically hyperbolic manifold is not of trace class. We can try to regularize the pointwise trace using the Hadamard regularization procedure, but in fact we remark that for convex co-compact hyperbolic manifolds this pointwise trace turns out to be absolutely integrable, as a consequence of the explicit formula of Camporesi-Pedon [14] of the odd heat kernel on hyperbolic space (which in particular has vanishing pointwise trace), and a second construction of a parametric for the resolvent when the sectional curvatures are -1 near infinity showing that the error from the model case is rapidly decreasing at infinity.

Let us comment about the novelty element here. Freed [20] proved the meromorphic extension of a large class of dynamical zeta functions on compact hyperbolic manifolds. His approach applies also to the convex co-compact setting, as noted by Patterson-Perry [87], but without giving any information about the structure of the poles and zeros. Since the original work of Selberg one suspects that these are related to spectral and topological invariants of the manifold. In the scalar case a description of the poles exists by the recent work of Patterson-Perry [87] and Bunke-Olbrich [10], but for general homogeneous bundles the zeta function is not well understood.

Our second main result proves that Millson's formula linking the eta invariant and the Selberg zeta function holds on convex co-compact manifolds for the Dirac operator, and also for the odd signature operator under the assumption that the exponent δ_Γ is small enough. The eta invariant can no longer be defined in terms of eigenvalues (as there are none in our setting, the spectrum being purely absolutely continuous!) but the integral

$$\eta(D) := \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \Big(\int_{X_\Gamma} \operatorname{tr}(De^{-tD^2})(m)\, \mathrm{dv}(m) \Big)\, dt,$$

is still absolutely continuous, again from the cancellations in the formula for the odd heat kernel on hyperbolic space due to Camporesi and Pedon [14].

Theorem 2.5. *Let $X_\Gamma = \Gamma\backslash\mathbb{H}^{2n+1}$ be an odd dimensional spin convex co-compact hyperbolic manifold. Then*

$$e^{\pi i \eta(D)} = Z^o_{\Gamma,\Sigma}(0). \tag{2.1}$$

If X is of dimension $4k$ the eta invariant on the boundary can also be defined for the odd signature operator, under the technical condition on the exponent $\delta_\Gamma < 2k-1$. Then the identity $e^{\pi i \eta(A)} = Z^o_{\Gamma,\Lambda}(0)$ continues to hold. We expect this identity to be valid irrespective of δ_Γ. The technical problems appearing in our proof for this last case appear since the continuous spectrum of the signature operator has two layers, coresponding to closed and co-closed forms. Such a phenomenon occurs also in the analysis of the Laplacian on forms on hyperbolic manifolds of finite volume [77].

Part III

Research perspectives

After two years at the University of Bucharest, where I had graduated the first three years of study, I have obtained my DEA in the Ecole Polytechnique in Paris, France and my PhD in MIT, USA. I had the chance to study in the best Romanian university and in two leading research institutions in the world, with some of the best Romanian and foreign mathematicians of our times: Kostake Teleman, Paul Gauduchon, Richard Melrose. I am indebted to them for teaching me some great pieces of mathematics. After my thesis I have spent two years at IMAR between 1999–2001, then I have taken post-doc positions in Hamburg in the research group of Christian Bär, the current president of the German Mathematical Society, and in Toulouse, where I have worked with André Legrand and Jean-Marc Schlenker. The Graduate School in Toulouse accepted my Habilitation project, I have defended my Habilitation á Diriger des Recherches thesis in June 2004. My jury consisted of A. Legrand, E. Leichtnam, V. Nistor, M. Hilsum and J.-M. Schlenker.

My activity in the last eight years has been mainly carried out while a researcher (CS II) at the Institute of Mathematics "Simion Stoilow" of the Romanian Academy. Between 2006–2012 I have been a Invited Professor in several top research institutions in France: University of Toulouse, Ecole Normale Superieure, Ecole Polytechnique, and a visiting scientist in the Institut des Hautes Etudes Scientifiques and Institut Henri Poincaré. My teaching activity has consisted in two courses at the Scoala Normala Superioara Bucharest on "Hyperbolic Geometry" and on "Uniformization of Surfaces", and a mini-course at a summer school in KIAS - Korea Institute for Advanced Studies. I have co-directed (together with Colin Guillarmou) in 2011 and 2012 two groups of two students each from the Ecole Normale Superieure (Ulm) for their Licence-3 memoirs. I had a post-doc on my Marie Curie ERG grant (Sylvain Golénia, presently Maître de Conferences in Bordeaux) with whom I wrote two papers published in Ann H Poincaré and Trans Am Math Soc. I directed a BS thesis of a student in Bucharest University. We had to invoke a different nominal advisor for bureaucratic resons, a fact which I disliked.

Since 2011 I am a member of the Mathematics committee in charge of validating university degrees (CNATDCU), including Habilitation theses. I am currently on two Habilitation thesis committees nominated by CNATDCU.

I have obtained two national grants in the last five years as Principal Investigator, and three international grants since 2001 as PI for the Romanian side.

1. Grant PNII-TE-0053/2011 ”Quantum invariants in hyperbolic geometry”, 2012-2014.

2. Grant PNII-ID-1188/2009 ”Geometric and quantum invariants of 3-manifolds and applications”, 2009-2011.

3. CNRS – Romanian Academy grant, 2006–2007.

4. Marie Curie European reintegration grant MERG-006375, 2004-2005.

5. DFG grant 436 RUM 17/7/01, 2001.

The works on hyperbolic geometry included in this thesis were partially developed in the framework of the grant PNII-ID-1188/2009. They open new perspectives on this important objects by furthering the knowledge on the deformation theory of hyperbolic cone-manifolds, and on the spectral theory of the Dirac operator on convex co-compact hyperblic manifolds. Some of the resulting directions of research are covered by my grant PNII-TE-0053/2011, and they deal notably with Chern-Simons theory on Teichmüller space, which is the subject of a current joint project with Colin Guillarmou. We essentially believe that complex Chern-Simons invariants will include simultaneously the information encoded in the eta invariant and in the renormalized volume. Our research program could result in a representation of the mapping class group in a Hilbert space of holomorphic section in the Hermitian Chern-Simons line bundle over the Teichmüller space. Such a representation would be of substantial interest for TQFT, since up-to-date it is not known whether the mapping class group of genus g surfaces admits any faithful finite-dimensional representations.

Outside hyperbolic geometry, I am currently interested in the more general notion of Einstein metrics. Together with Bernd Ammann and Andrei Moroianu we have proved that the Cauchy problem for the Einstein equation in Riemannian geometry, i.e., the problem of constructing an Einstein metric starting from a hypersurface and a symmetric tensor playing the role of a second fundamental form, always has solutions if the initial data are real analytic. We also give examples when the solution exists *if and only if* the initial metric is real analytic. This problem, or at least its Lorentzian counterpart, has been at the center of mathematical relativity since the work of Choquet-Bruhat some 50 years ago. In the semi-Riemannian framework the Cauchy problem always has solutions, since it behaves essentially like a wave equation. In the Riemannian case we discovered that this is not the case. Further work is needed to find weaker conditions than real-analyticity to ensure existence of the solutions.

In another project with Andrei Moroianu we examine Ricci surfaces, that is, surfaces whose Gaussian curvature satisfies the nonlinear elliptic partial differential equation

$$K\Delta K + |dK|^2 + 4K^3 = 0.$$

We show that every Ricci surface can be locally embedded either in the flat Euclidean space $\mathbb{R}^3$ as a minimal surface, or in the flat Lorentz space $\mathbb{R}^{2,1}$ as a maximal surface. This completes the answer to a question raised and partially answered by Ricci-Curbastro in 1895, who solved the easy case, that of non-vanishing curvature. An intersting question which we do not solve is what genus g surfaces admit Ricci metrics. We show that when g is odd, every hyperelliptic surface admits a Ricci metric. This is one of my projects for future research, that I could share with a prospective student.

In a fourth ongoing project, this time with Colin Guillarmou and Jean-Marc Schlenker, we look at the renormalized volume for asymptotically hyperbolic Einstein manifolds. We discovered that in a given conformal class at infinity the renormalized volume is maximized when a certain quantity v_n, a higher-dimensional analog of the Gauss curvature and of the Q-curvature, is constant. This renormalized volume plays therefore an analog role to the renormalized volume from convex co-compact manifolds, which is a Kähler potential on the Teichmüller space of the ideal boundary. The study of conformal manifolds admitting metrics with v_n constant could be a very fruitful thesis subject.

All the above projects could lead to reasonably difficult and meaningful PhD thesis subjects. The virtue of these subjects, as compared to the sort of thesis subjects I encounter in my activity on the CNATDCU mathematics committee, is being actually connected to the main stream of mathematical research, following a definite goal to prove an actual mathematical result of interest to the scientific community, as opposed to the prevailing attitude in our country in the last 10-20 years, which means focusing on some oral exams, some conspects from the literature, culminating in a lengthy manuscript which too often does not go beyond the depth level of a MS thesis.

Closing remarks

I chose to return to Romania after my PhD at MIT (1999) in the hope of contributing towards a world-level mathematical atmosphere in Romania. My PhD thesis took 5 years to be recognized by the bureaucracy in the ministry. I was available to direct PhD theses in 2004, after finishing my post-doctoral fellowships in Germany and France and returning to Bucharest with an Habilitation degree. Most of my Romanian colleagues who held similar positions to mine (associate professor) in the US or France could, and did, direct PhD students. Unfortunately, myself I could not get the "right to direct PhD's" since I was not a full professor. My application for a full professor position was turned down because I had not had written 2 books, as the twisted standards of that time required. I preferred instead to concentrate on my research, hoping for better times. Then the notion of Habilitation was finally introduced in Romania in 2011. In my grant application PNII-TE-0053/2011 I had anticipated hiring a PhD student... only to realize that my Habiliation from France (2004) was after all still not good enough to be allowed to direct theses in Romania! I am therefore presenting this new thesis, although I find it irresponsible to ask someone with a PhD from MIT and HDR from Toulouse to write yet another thesis for bureaucratic reasons, even more in light of the abysmal level of research in post-1989 Romania as compared to the US or France, the leading mathematical countries of our times.

Although my French Habilitation is not recognized by our authorities, I am currently on two Habilitation committees in Romania. Indeed, foreign members on these committees cannot be reasonably required to possess a Romanian Habilitation, but just any foreign Habilitation or equivalent diploma. The criteria are stated uniformly for the whole committee, not just for its foreign members. I thus fulfill the criteria thanks to my French Habilitation!

The Habilitation process in the hands of Romanian bureaucracy has good chances of becoming yet another spectacular illustration of Titu Maiorescu's celebrated "forme fără fond" theory, an unfortunate mainstay of Romanian culture in the past two centuries.

Despite these difficulties I am convinced that doing mathematical research in Romania is still possible today. My intention is to develop my career here and I aim to create a strong group in geometric analysis. Being allowed to direct PhD theses will be a step in the right direction.

Part IV

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