Simion Stoilow Institute of Mathematics of the Romanian Academy
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Ph. D. Thesis
Abstract

Probabilistic and deterministic models for fracture type phenomena

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The branching phenomena arise in applicative fields such as mechanics (avalanches, disintegration of snow blocs) or astrophysics (formation and fragmentation of planets). This thesis is related to the present scientific effort in developing the connection between branching processes and the fragmentation phenomenon with applications to material ruptures models, as the avalanche phenomena. We construct and study measure-valued branching processes and their nonlinear partial differential equations, we associate Markov processes to the subordination in the sense of Bochner of the L^p-semigroups, and we investigate the nonlinear partial differential equations modeling the flow onset of dense avalanches (visco-plastic Saint-Venant model with topography), we emphasize a numerical approach.

The branching processes describe the time evolution of a system of particles, located in an Euclidean domain. A branching process is interpreted in a suggestive way as describing the evolution of a random cloud. It turns out that, in order to describe such Markovian time evolution phenomena, the natural state space to be considered is a set of positive finite measures on that domain. Therefore, the methods of constructing measure-valued Markov processes are of particular interest. We establish natural connexions with nonlinear partial differential equations of the type $\Delta u - u = -\sum_{k=1}^{\infty} q_k u^k$, where the coefficients q_k are positive and $\sum_{k=1}^{\infty} q_k = 1$, associated with the transition functions of the branching processes.

The subordination in the sense of Bochner is a convenient way of transforming the semigroups of operators and their infinitesimal generators. Recall that in particular, this is a method of studying the fractional powers of the Laplace operator. The second objective from the stochastic processes part of the thesis is to show that any subordination in the sense of Bochner of a sub-Markovian L^p-semigroups is actually produced by the subordination of a Markov process. It turns out that an enlargement of the base space is necessary. Our approach to the branching processes and Bochner subordination uses analytic and probabilistic tools from potential theory.

Modeling avalanche formation of soils, snow, granular materials or other geomaterials, is a complex task. The main method in modeling the shallow avalanche onset is the study of a global non smooth optimization problem, called the safety factor (of limit load) problem. This optimization problem is reconsidered in the space of bounded deformations functions, a suitable space to capture the discontinuities of the onset velocity field and the velocity boundary conditions are relaxed. We prove that the initial optimization problem is not changed and the reformulated safety factor problem has at least one solution, modeling the onset flow field of the avalanche. We also develop a Discontinuous Velocity Domain Splitting method (DVDS method), a numerical technique to solve the safety factor problem through a shape optimization problem.

Our numerical approach is illustrated by numerical simulations of some limit load problems.

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The thesis has two parts. The organization of the first part of the thesis is the following. In Chapter 1 we collect some preliminaries on the resolvents of kernels and basic notions of potential theory. We present in Section 1.3 a suitable result on the existence of a càdlàg, quasi-left continuous strong Markov processes, given a resolvent of kernels, imposing conditions on the resolvent.

The description of a discrete branching process is as follows (cf. e.g., [52], page 235, and [19]). An initial particle starts at a point of a set E and moves according to a standard Markov process with state space E (called base process) until a random terminal time when it is destroyed and replaced by a finite number of new particles, its direct descendants. Each direct descendant moves according to the same right standard process until its own terminal time when it too is destroyed and replaced by second generation particles and the process continues in this manner. N. Ikeda, M. Nagasawa, S. Watanabe, and M. L. Silverstein (cf. [28], [29], and [52]) indicated the natural connection between discrete branching processes and nonlinear partial differential operators Λ of the type

$$\Lambda u := \mathcal{L}u + \sum_{k=1}^{\infty} q_k u^k,$$

where $\mathcal{L}$ is the infinitesimal generator of the given base process and the coefficients q_k are positive, Borelian functions with $\sum_{k=1}^{\infty} q_k = 1$. In the description of a branching process this particular case means that each direct descendant starts at the terminal position of the parent particle and $q_k(x)$ is the probability that a particle destroyed at x has precisely k descendants. It is possible to consider a more general nonlinear part for the above operator, generated by a branching kernel B; the descendants start to move from their birthplaces which have been distributed according to B. Thus, these processes are also called "non-local branching" (cf. [19]); from the literature about branching processes we indicate the classical monographs [25], [4], [3], the recent one [40], and the lecture notes [39] and [20].

In Chapter 2 we construct discrete branching Markov processes associated to operators of the type Λ, using several analytic and probabilistic potential theoretical methods. The base space of the process is the set S of all finite configurations of E. The branching kernels on the space of finite configurations are introduced in Section 2.1.

Section 2.5 is devoted to the construction of the measure-valued discrete branching processes. The first step is to solve the nonlinear evolution equation induced by Λ. Then, using a technique of absolutely monotonic operators developed in [52], it is possible to construct a Markovian transition function on S, formed by branching kernels. We follow the classical approach from [28] and [52], but we consider a more general frame, the given topological space E being a Lusin one and not more compact (see Ch. 5 in [3] for the locally compact space situation).

The second step is to show that the transition function we constructed on S is indeed associated to a standard process with state space S, the proof involves the entrance space S_1, an extension of S constructed by using a Ray type compactification method. We apply the mentioned results from Section 1.3 and Section 1.2, showing that the required imposed conditions (from Section 1.3) are satisfied by the resolvent of kernels on S associated with the branching semigroup constructed in the previous step. We emphasize relations between a class of excessive functions with respect to the base process X and two classes of excessive functions (defined on S) with respect to the forthcoming branching process: the linear and the exponential type excessive functions. A particular linear excessive function for the branching process becomes a function having compact level sets (called compact Lyapunov function) and will lead to the tightness property of the capacity on S. It turns out that it is necessary to make a perturbation of $\mathcal{L}$ with a kernel induced by the given branching kernel.

The above mentioned tools were useful in the case of the continuous branching processes too (cf. [9] and [14]), e.g., for the superprocesses in the sense of E. B. Dynkin (cf. [22]; see Section 3.1 for the basic definitions), like the super-Brownian motion, processes on the space of all finite measures on E induced by operators of the form $\mathcal{L}u - u^{\alpha}$ with $1 < \alpha \leqslant 2$. We establish several links with the nonlinear partial differential equations associated with the branching semigroups and we point out connections between the continuous and discrete branching processes. Note that a cumulant semigroup (similar to the continuous branching case) is introduced. In particular, when the base process X is the Brownian motion, then the cumulant semigroup of the induced discrete branching process formally satisfies a nonlinear evolution equation involving the square of the gradient. Finally, recall that the method of finding a convenient compact Lyapunov function was originally applied in order to obtain martingale solutions for singular stochastic differential equations on Hilbert spaces (cf. [10] and [15]) and for proving the standardness property of infinite dimensional Lévy processes on Hilbert spaces (see [12]).

We complete the main result of Chapter 2 with an application as suggested in [25], page 50, where T. E. Harris emphasizes the interest for branching processes for which "each object is a very complicated entity; e.g., an object may itself be a population".

More precisely, because we may consider base processes with general state space, it might be a continuous branching process playing this role. In Section 2.6, we obtain in this way a branching Markov process, having the space of finite configurations of positive finite measures on E as base space; an additional suggestive interpretation of this branching process is exposed. Note that in [8] new branching processes are generated starting with a superprocess and using an appropriate subordination theory.

In Chapter 3 we study the subordination in the sense of Bochner for L^p-semigroups and the associated Markov processes. Let $(P_t)_{t\geq 0}$ be a C_0-semigroup on a Banach space B and $\mu = (\mu_t)_{t\geq 0}$ be a convolution semigroup on $\mathbb{R}_+$. Recall that the subordinate of $(P_t)_{t\geq 0}$ in the sense of Bochner is the C_0-semigroup $(P_t^\mu)_{t\geq 0}$ on B defined as

$$P_t^\mu u := \int_0^\infty P_s u \, \mu_t(\,\mathrm{d}s), \; t \geq 0, \, u \in B.$$

The probabilistic counterpart of the Bochner subordination is a procedure of introducing jumps in the evolution of a given Markov process, by means of a positive real-valued stationary stochastic process $(\xi_t)_{t\geqslant 0}$, with independent nonnegative increments (called subordinator), induced by $\mu = (\mu_t)_{t\geqslant 0}$. More precisely, if $X = (X_t)_{t\geqslant 0}$, is a (Borel) right Markov process with state space E, then define the subordinate process $X^\xi = (X_t^\xi)_{t\geqslant 0}$ as

$$X_t^\xi := X_{\xi_t}, \;\; t \geqslant 0.$$

A specific problem is to show that regularity properties are transferred from the given process X to the subordinate one X^ξ; for example the Feller or strongly Feller properties of the corresponding transition functions (see, e.g., [35] and Section 3 from Ch. V in [16]). Applications to the Dirichlet forms are developed in [37] and [1], to the pseudo differential operators in [35] and [36], while to establish Harnack inequalities in [24]. Recall that the classical example of such an operator obtained by Bochner subordination is the square root of the Laplace operator. For other various developments and applications see [50], [26], [18], [21], [51], and [17].

Recall that if $(P_t)_{t\geqslant 0}$ is the C_0-semigroup on $L^p(E,m)$ induced by the transition function of $X = (X_t)_{t\geqslant 0}$ (where m is a σ-finite P_t-*subinvariant measure*, i.e., $\int_E P_t f \,\mathrm{d}m \leqslant \int_E f \,\mathrm{d}m$ for all $f \in L^p(E,m)$, $f \geqslant 0$, and $t \geqslant 0$), then the transition function of the subordinate process $X^\xi = (X_t^\xi)_{t\geqslant 0}$ is $(P_t^\mu)_{t\geqslant 0}$. A converse of this statement is the main result of Chapter 3 and it is given in Section 3.4.

Consequently, one can apply results on the subordination (in the sense of Bochner) of the Markov processes for the subordination of the L^p-semigroups on arbitrary Lusin measurable state spaces. As an example, the subordinate process X^ξ may be regarded as the solution of the martingale problem associate with the infinitesimal generator $\mathcal{L}^\mu$ of the subordinate semigroup $(P_t^\mu)_{t\geqslant 0}$.

A second consequence of the main result is the validity of the quasi continuity property for the subordinate semigroup $(P_t^\mu)_{t\geqslant 0}$, with respect to the Choquet capacity associated with the given C_0-semigroup $(P_t)_{t\geqslant 0}$. Recall that this property is analogous to the quasi-regularity condition from the Dirichlet forms theory (cf. [42]); the role of the capacity induced by the energy is played in this L^p frame by the capacity associated to the process. We give more details in Section 3.4.

The crucial argument in proving the main result is the association of a right process to a C_0-resolvent of sub-Markovian contractions on an L^p-space, proved in [11], where the necessity of the enlargement of the space E is also discussed; for the reader's convenience we present in Section 3.4, some details about the construction of the larger space E_1. A second main argument is to show that if X is a transient (Borel) right process with state space a Lusin topological space then X^ξ is a (Borel) right process with the same state space and topology, in particular, it is strong Markov. Results of this type were obtained by Bouleau in [18]. We present here a different approach, based on a characterization of the property of a resolvent of kernels to be associated to a right process, in terms of excessive measures (due to Steffens, see [53]), combined with a result of Sharpe on the preservation of the properties of a process under change of realization (cf. Theorem (19.3) from [51]). Note that according with [11] (Theorem 1.3 and the comment before it) there are some difficulties in applying Steffens' result in the non-transient case, therefore we present it in Section 1.1. Note also that the paper [27] contains a related result, namely, it is shown that the resolvent of a semigroup of kernels obtained by subordination from the transition function of a transient right process is the resolvent of a right process, but possible in some different topology and assuming in addition that it is proper; however, no information about the subordinate process is given and no process is associated to a given transition function (semigroup of kernels), as we are doing in Section 3.3. The above mentioned result of Steffens is an essential argument in [27] too.

A preliminary result is to show that the min-stability property of the excessive functions is preserved under the subordination by a convolution semigroup. We complete in this way results from [15], where this property was supposed to be satisfied by the subordinate resolvent, in order to associate to it a càdlàg Markov process; see the Example following Corollary 5.4 from [15]. Recall that from the probabilistic point of view the stability to the point-wise infimum of the convex cone of all excessive functions is precisely the property that all the points of the state space of the process are nonbranch points.

This first part of the thesis is essentially based on results from the papers [13] and [41].

The second part of the thesis is devoted to the modeling of the shallow avalanche onset.

An important issue in geophysics is the understanding of the phenomena related to shallow avalanche of soils, snow or other geomaterials ([2, 45]). The real problem is three dimensional and the mathematical and numerical modeling is very complex. For that, reduced 2-D models (called also Saint-Venant models) are introduced (see [5, 6, 7, 48, 49, 57, 43, 44]) to capture the principal features of the flow.

Figure 1: A snow avalanche flow.

Natural avalanches and debris flows are often associated with complicated mountain topologies, which makes the prediction very difficult. For that, a lot of studies include the bottom curvature effect into the classical Saint-Venant equations to describe channelized flows along talwegs [23, 58, 46, 57] (see also the review [47]) or flows on more general basal geometries [38]. Very recently, the model obtained in [30] for plane slopes, was extended in [31] to the case of a general basal topography by using a local base given by the bottom geometry and the associated differential operators.

The first goal of the second part of the thesis is to show that the model obtained in [31] can describe also the avalanche onset of a flow. We introduce here a simple criterion, which relates the yield limit (material resistance) to the external forces distribution, able to distinguish if an avalanche occurs or not. This criterion is related with the maximum of the loading parameter such that the fluid/solid can withstand without collapsing. The safety factor is the solution of an global optimization problem, called limit load analysis. The velocity field, solution of this optimization problem, is called the collapse flow or onset velocity field.

In many applications, the strains are localized on some surfaces where the velocity

Figure 2: The onset of a snow avalanche as a fracture process.

of the collapse flow exhibits discontinuities. From mathematical and numerical points of view, the avalanche onset modeling was and remains a difficult problem. The second objective is to prove the existence of an onset velocity field in an appropriate functional space. Since the functional involved in the global optimization is non-smooth, and non coercive in classical Sobolev spaces, we have to consider it in the space of bounded tangential deformation functions (i.e., the space of velocities which have their tangential rate of deformation in the space of bounded measures), similar to the space introduced in [54, 56].

The third objective is to propose a numerical strategy to solve the limit load problem and to get the onset flow field of the avalanche. The numerical solutions methods in limit analysis are based on the discretization of the kinematic or static variational principles using the finite element method technics and the convex and linear programming. Despite great progress in the last decades (X-FEM, re-meshing techniques), the finite element method remains associated to continous fields and it is not so well adapted for modeling strain localization and velocities discontinuities on unknown surfaces. For that, we will adapt here the discontinuous velocity domain splitting (DVDS) method, introduced in [34]. DVDS is a mesh free method which does not use a finite element discretization of the solid. It focuses on the strain localization and completely neglect the bulk deformations. The limit load problem is thus reduced to the minimization of a shape-dependent functional (plastic dissipation power). The avalanche collapse flow velocity field, which is discontinuous, is associated to an optimum sub-domain and a rigid flow. It has localized deformations only, at the boundary of the sub-domain.

The main novelty of this part consists in finding the appropriate functional space of the limit load problem, in obtaining an existence result for the onset velocity field. The specific Stokes formula are proved, the variational formulation of the velocity field by using the tangential plane Stokes formula associated to these operators, and the set of tangential rigid velocities are deduced. As far as we know, the use of a mesh free technique (DVDS) for a numerical approach of the shallow avalanche onset problem is also new. All the new results of this part can be found in [32, 33].

The second part of the thesis is organized as follows: In Chapter 4 we present the 3-D dimensional mechanical problem and we discuss the choice of the visco-plastic model adopted here.

In Chapter 5 we introduce the shallow flow problem. Firstly, we give a geometrical description of the bottom surface and the expressions of the differential operators acting in the tangential plane. We prove here specific Stokes formula and we deduce the set of tangential rigid velocities. We recall from [31] the boundary-value problem for the visco-plastic Saint-Venant model with topography formulated on the local base associated to the bottom surface. Finaly, we give the variational formulation of the velocity field by using the tangential plane Stokes formula associated to these operators.

Chapter 6 is devoted to the mathematical approach of the limit load problem obtained from the variational formulation, described before. We introduce a global optimization problem (called the limit load analysis or safety factor problem) on classical Sobolev spaces to study the link between the yield limit, the external forces and the thickness distributions for which the shallow flow of a visco-plastic fluid does, or does not occur. Then, we consider the same optimization problem in the space of bounded tangential deformation functions. The boundary conditions, expressed for smooth functions have to be relaxed for non-smooth velocity fields considered in this new functional framework. For that, we have to add some additional boundary integrals on the plastic dissipation functionals and external forces power. In these integrals, which are modeling a discontinuity of a non-smooth velocity field located at boundary, we have to define the tangential normal on a bottom boundary. We prove that the above relaxation of the boundary conditions does not change the initial optimization problem. At the end of this chapter we prove that the reformulated safety factor problem has at least a solution, modeling the avalanche onset. For that, we have to study coercivity properties of the plastic dissipation functional and to describe the kernel of the tangential deformation operator by using the space of tangential rigid velocities introduced before.

In Chapter 7 we adapted the DVDS numerical technique ([34]) to solve the safety factor problem. This last problem is reduced to a shape optimization problem. The description of the subset shape is given by a level set of a Fourier function and we use genetic algorithms to solve the resulted non convex and non-smooth global optimization problem. We illustrate the proposed numerical approach in solving some safety factor problems. First, we consider the case of a plane slope with a non-uniform thickness distribution. For a circular dome geometry of a Bingham (Von-Mises plasticity) fluid we give a comparison between our results and a dynamic finite element/finite volume method. Then, we analyze the avalanche of a square dome of a Drucker-Prager fluid, and the last example concerns the avalanche of a thick Bingham fluid over an obstacle. Finally, we illustrate our technique in the case of a complex basal topography. For a half-sphere et quarter of a sphere covered with a Bingham fluid/solid with constant thickness distribution we compute the safety factor and the avalanche onset. In the last example we analyze the case of a quarter of a ellipsoid filled with a Drucker-Prager fluid/solid with a non-uniform thickness distribution.

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