Simion Stoilow Institute of Mathematics of the Romanian Academy

HABILITATION THESIS

PROOF MINING IN NONLINEAR ANALYSIS AND ERGODIC THEORY

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Abstract

In this thesis we present applications of proof mining to nonlinear analysis and ergodic theory. The results presented in the thesis are based on the papers [142, 143, 144, 145, 146, 147] (written jointly with Ulrich Kohlenbach), [2] (written jointly with David Ariza-Ruiz and Genaro López-Acedo) and [163, 164, 165, 166]. Before presenting these applications, let us give a very short presentation of proof mining. We refer to 3 for more details.

Proof mining is a research paradigm concerned with the extraction of hidden finitary and combinatorial content from proofs that make use of highly infinitary principles. This new information is obtained after a logical analysis using proof-theoretic tools and can be both of quantitative nature, such as algorithms and effective bounds, as well as of qualitative nature, such as uniformities in the bounds or weakening the premises. This line of research, developed by Ulrich Kohlenbach in the 90's, has its roots in Kreisel's program on *unwinding of proofs*.

The main proof-theoretic technique used in proof mining is Kohlenbach's monotone functional interpretation [124], which systematically transforms any statement in a given proof into a new version for which explicit bounds are provided.

Recently, Terence Tao [234] arrived at a proposal of so-called *hard analysis* (as opposed to *soft analysis*), inspired by the finitary arguments used by him and Green [90] in their proof that there are arithmetic progressions of arbitrary length in the prime numbers. As Kohlenbach points out in [132], Tao's hard analysis could be viewed as carrying out, using monotone functional interpretation, analysis on the level of uniform bounds. In many cases, this allows one to *finitize* the proofs and to arrive at qualitatively stronger results.

In Chapter 2 we present a class of geodesic spaces, called by us W-hyperbolic spaces. We introduce the class UCW-hyperbolic spaces as a natural generalization both of uniformly convex Banach spaces and CAT(0) spaces. These spaces are an appropriate setting for the study of the metric fixed point theory of nonexpansive mappings and provide the context of many of our applications.

A survey of the logical tools and a presentation of general logical metatheorems for different classes of spaces are given in Chapter 3.

I. Proof mining in nonlinear analysis

The first part of the thesis presents applications of proof mining to the asymptotic behaviour of nonlinear iterations.

The main result of Section 4.3 is an effective uniform quantitative version of a

well-known theorem due to Borwein, Reich and Shafrir [21] on the asymptotic behaviour of Krasnoselski-Mann iterations of nonexpansive mappings. Inspired by this result, we introduce in Section 4.5 the notions of *uniform approximate fixed point property* and *uniform asymptotic regularity property.* Our quantitative Borwein-Reich-Shafrir theorem is the key ingredient in Chapter 5, where we generalize to (families of) unbounded convex subsets C of W-hyperbolic spaces results obtained by Kirk and Espínola [62, 115] on approximate fixed points of nonexpansive mappings in product spaces $(C \times M)_\infty$, where M is a metric space and C is a convex closed bounded subset of a normed or a CAT(0)-space. Furthermore, using our notion of uniform approximate fixed point property, we give some partial answers to an open problem of Kirk [115, Problem 27].

Another important application of our quantitative Borwein-Reich-Shafrir Theorem is a uniform rate of asymptotic regularity for the Krasnoselski-Mann iterations of nonexpansive mappings in general W-hyperbolic spaces. As a consequence, we get an exponential rate of asymptotic regularity in the case of CAT(0) spaces. In Section 4.4, we show that we can get a quadratic rate of asymptotic regularity for CAT(0) spaces, but following a completely different approach, inspired by the results on asymptotic regularity obtained by Groetsch [91] in the setting of uniformly convex Banach spaces. The method we use is to find explicit uniform bounds on the rate of asymptotic regularity in the general setting of UCW-hyperbolic spaces and then to specialize them to CAT(0) spaces.

In Chapter 6 we get effective rates of asymptotic regularity for Ishikawa iterations of nonexpansive self-mappings of closed convex subsets in UCW-hyperbolic spaces. These effective results are new even for uniformly convex Banach spaces.

Asymptotically nonexpansive mappings were introduced by Goebel and Kirk [80] as a generalization of the nonexpansive ones. We present in Chapter 7 a fixed point theorem for asymptotically nonexpansive mappings in UCW-hyperbolic spaces, generalizing results of of Goebel-Kirk [80] and Kirk [115]. The main part of Chapter 7 is devoted to getting effective results on the asymptotic regularity of Krasnoselski-Mann iterations for this class of mappings.

Chapter 8 is dedicated to the study of firmly nonexpansive mappings in different classes of geodesic spaces, such as UCW-hyperbolic spaces, Busemann spaces and CAT(0) spaces. Firmly nonexpansive mappings play an important role in nonlinear functional analyis and optimization due to their correspondence with maximal monotone operators. We do a thorough study of fixed point theory and the asymptotic behaviour of Picard iterates of these mappings. We apply methods of proof mining to obtain an effective rate of asymptotic regularity for the Picard iterations, which turns to be quadratic in the case of CAT(0) spaces.

II. Proof mining in (nonlinear) ergodic theory

In the second part of the thesis we obtain effective results on the asymptotic behaviour of (nonlinear) ergodic averages.

We provide in Chapter 9 a finitary version of the the generalization to uniformly convex Banach spaces of the von Neumann mean ergodic theorem. Thus, we obtain an explicit rate of metastability (as defined by Tao [234, 236]) of ergodic averages in uniformly convex Banach spaces, generalizing similar results obtained by Avigad, Gerhardy and Towsner [5] for Hilbert spaces and by Tao [236] for a particular finitary dynamical system, as part of his proof of the generalization of the mean ergodic theorem to commuting families of invertible measure-preserving transformations. Despite of our result being significantly more general then the Hilbert space case treated in [5], the extraction of our bound is considerably easier compared to [5] and even numerically better.

In Chapter 10 we obtain effective rates of metastability for Halpern iterations, nonlinear generalizations of ergodic averages. The most important result on the strong convergence of Halpern iterations in Hilbert spaces was proved by Wittmann [245]. In this chapter we get finitary versions of the generalizations of Wittmann's result obtained by Saejung [210] for CAT(0) spaces and Shioji and Takahashi [218] for spaces with a uniformly Gaâteaux differentiable norm.

These results constitute a significant extension of the hitherto context of proof mining, as the proofs in [210, 218] use Banach limits. The existence of Banach limits is either proved by applying the Hahn-Banach theorem to l^∞, which due to the nonseparability of that space needs the axiom of choice, or via ultralimits which, again, needs choice.

Nevertheless, we develop a method to convert such proofs into more elementary proofs that no longer rely on Banach limits and can be analysed by the existing logical machinery. As the way Banach limits are used in these proofs seems to be rather typical for other proofs in nonlinear ergodic theory, our method may also be seen as providing a blueprint for doing similar unwindings in those cases as well.

The last chapter of the thesis presents future research directions and some further plans regarding the evolution of the professional and scientific career

Rezumat

În această teză prezentăm aplicaţii ale proof mining în analiza neliniară şi teoria ergodică. Rezultatele prezentate în teză sunt bazate pe articolele [142, 143, 144, 145, 146, 147] (scrise în colaborare cu Ulrich Kohlenbach), [2] (scris în colaborare cu David Ariza-Ruiz şi Genaro López-Acedo) şi [163, 164, 165, 166]. Înaintea prezentării acestor aplicaţii, vom da o scurtă prezentare a proof mining. Ne referim la Capitolul 3 pentru mai multe detalii.

Proof mining este o paradigmă de cercetare având ca scop extragerea conţinutului finitar şi combinatorial din demonstraţii care folosesc principii infinitare puternice. Această nouă informaţie este obţinută după o analiză logică folosind unelte din teoria demonstraţiei şi poate fi atât de natură cantitativă cum ar fi algoritmi şi margini efective, dar şi de natură calitativă, cum ar fi uniformităţi ale marginilor şi premize mai slabe. Această linie de cercetare, dezvoltată de Ulrich Kohlenbach în anii 90, îşi are rădăcinile în programul lui Kreisel de *desfăşurare a demonstraţiilor (unwinding of proofs)*.

Principala tehnică de teoria demonstraţiei folosită în proof mining este interpretarea funcţională monotonă introdusă de Kohlenbach [124], care transformă sistematic orice pas intermediar dintr-o demonstraţie dată într-o versiune nouă, cu margini explicite

Recent, Terence Tao [234] a propus ca direcţie de cercetare *analiza "hard"* (în opoziţie cu *analiza "soft"*), având ca inspiraţie argumentele finitare folosite de el şi Green [90] în demonstraţia faptului că există în mulţimea numerelor prime progresii aritmetice de lungime arbitrară. După cum observă Kohlenbach în [132], analiza "hard" propusă de Tao poate fi văzută şi ca o efectuare, folosind interpretarea funcţională monotonă, a analizei la nivelul marginilor uniforme. În multe cazuri, aceasta ne permite să *finitizăm* demonstraţiile şi să ajungem la rezultate mai tari din punct de vedere calitativ.

În Capitolul 2 prezentăm o clasă de spaţii geodezice, numite de noi spaţii W-hiperbolice. Introducem clasa spaţiilor UCW-hiperbolice ca o generalizare naturală a spaţiilor Banach uniform convexe şi a spaţiilor CAT(0). Aceste spaţii constituie un cadru corespunzător pentru studiul teoriei metrice de punct fix a funcţiilor nonexpansive şi apar în multe din aplicaţiile noastre.

Instrumentele logice şi metateoremele logice pentru diferite clase de structuri sunt prezentate în Capitolul 3.

I. Proof mining în analiza neliniară

Prima parte a tezei prezintă aplicaţii ale proof mining la comportarea asimptotică a iteraţiilor neliniare.

Principalul rezultat al Secţiunii 4.3 este o versiune cantitativă uniformă efectivă a unei binecunoscute teoreme a lui Borwein, Reich şi Shafrir [21] despre comportarea asimptotică a iteraţiilor Krasnoselski-Mann ale funcţiilor nonexpansive. Inspiraţi de acest rezultat introducem în Secţiunea 4.5 noţiunile de *proprietate de punct fix aproximativ uniform* şi *proprietate de asimptotic regularitate uniformă*. Versiunea noastră cantitativă a teoremei Borwein-Reich-Shafrir este instrumentul cheie în Capitolul 5, unde generalizăm la (familii de) submulţimi convexe nemărginite C ale spaţiilor W-hiperbolice rezultate obţinute de Kirk şi Espínola [62, 115] despre puncte fixe aproximative ale funcţiilor nonexpansive în spaţii produs $(C \times M)_\infty$, unde M este un spaţiu metric şi C este o submulţime convexă închisă mărginită a unui spaţiu normat sau CAT(0). Mai departe, folosind proprietatea de punct fix aproximativ uniform, dăm un răspuns parţial unei probleme deschise a lui Kirk [115, Problem 27].

O altă aplicaţie importantă a versiunii noastre cantitative a teoremei Borwein-Reich-Shafrir este o rată uniformă de asimptotic regularitate pentru iteraţiile Krasnoselski-Mann în spaţii W-hiperbolice generale. Ca o consecinţă, obţinem o rată exponenţială de asimptotic regularitate în cazul spaţiilor CAT(0). În Secţiunea 4.4, arătăm că putem obţine o rată pătratică de asimptotic regularitate pentru spaţii CAT(0), dar folosind o abordare complet diferită, inspirată de rezultate de asimptotic regularitate obţinute de Groetsch [91] pentru spaţii Banach uniform convexe. Metoda folosită de noi este de a calcula margini explicite uniforme pentru rata de asimptotic regularitate în contextul general al spaţiilor UCW-hiperbolice şi apoi să le specializăm la spaţii CAT(0).

În Capitolul 6 obţinem rate efective de asimptotic regularitate pentru iteraţiile Ishikawa ale funcţiilor nonexpansive pe submulţimi convexe închise în spaţii UCW-hiperbolice. Aceste rezultate efective sunt noi chiar şi pentru spaţii Banach uniform convexe.

Funcţiile asimptotic nonexpansive au fost introduse de Goebel şi Kirk [80] ca generalizări ale celor nonexpansive. Prezentăm în Capitolul 7 o teoremă de punct fix pentru funcţiile asimptotic nonexpansive în spaţii UCW-hiperbolice, generalizând rezultate ale lui Goebel-Kirk [80] şi Kirk [115]. Cea mai importantă parte a Capitolului 7 este dedicată obţinerii de rezultate efective pentru asimptotic regularitatea iteraţiilor Krasnoselski-Mann pentru această clasă de funcţii.

Capitolul 8 este dedicat studiului funcţiilor ferm nonexpansive în diferite clase de spaţii geodezice, cum ar fi spaţiile UCW-hiperbolice, spaţiile Busemann şi spaţiile CAT(0). Funcţiile ferm nonexpansive joacă un rol foarte important în analiza funcţională neliniară şi optimizare datorită corespondenţei cu operatorii maximal monotoni. Studiem teoria de punct fix şi comportarea asimptotică a iteraţiilor Picard ale acestor funcţii. Aplicăm metode de proof mining pentru a obţine o rată efectivă de asimptotic regularitate pentru iteraţiile Picard, care este pătratică în cazul spaţiilor CAT(0).

II. Proof mining în teoria ergodică neliniară

În a doua parte a tezei obţinem rezultate efective relativ la comportarea asimptotică a mediilor ergodice (neliniare).

Obţinem în Capitolul 9 o versiune finitară a generalizării la spaţii Banach uniform convexe a teoremei ergodice medii a lui von Neumann. Astfel, obţinem o rată explicită de metastabilitate (definită de Tao [234, 236]) a mediilor ergodice în spaţii Banach uniform convexe, generalizând rezultate similare obţinute de Avigad, Gerhardy şi Towsner [5] pentru spaţii Hilbert şi de Tao [236] pentru un sistem dinamic finitar particular, ca parte a demonstraţiei sale a generalizării teoremei ergodice medii la familii de transformări invertibile care păstrează măsura. Cu toate că rezultatul nostru este semnificativ mai general decât cel din spaţii Hilbert obţinut în [5], extragerea marginilor este considerabil mai uşoară în comparaţie cu [5] şi chiar mai bună din punct de vedere numeric.

În Capitolul 10 obţinem rate efective de metastabilitate pentru iteraţiile Halpern, generalizări neliniare ale mediilor ergodice. Cel mai important rezultat cu privire la convergenţa tare a iteraţiilor Halpern în spaţii Hilbert a fost demonstrat de Wittmann [245]. În acest capitol obţinem versiuni finitare ale generalizărilor rezultatului lui Wittmann obţinute de Saejung [210] pentru spaţii CAT(0) şi Shioji şi Takahashi [218] pentru spaţii cu normă uniform Gâteaux diferenţiabilă.

Aceste rezultate constituie o extensie semnificativă a contextului actual din proof mining, deoarece demonstraţiile din [210, 218] folosesc limite Banach. Existenţa limitelor Banach este demonstrată fie aplicând teorema Hahn-Banach spaţiului l^∞, care datorită neseparabilităţii spaţiului necesită axioma alegerii, fie via ultralimite, care iarăşi necesită axioma alegerii.

Cu toate acestea dezvoltăm o metodă de a converti astfel de demonstraţii în unele elementare care nu se mai bazează pe limite Banach şi pot fi analizate cu maşinăria logică existentă. Modul în care limitele Banach sunt folosite în aceste demonstraţii pare a fi tipic pentru alte rezultate din teoria ergodică neliniară. Prin urmare, metoda noastră poate fi folosită pentru a obţine rezultate similare şi în acele cazuri.

Ultimul capitol al tezei prezintă direcţii de cercetare viitoare şi planuri privind evoluţia carierei profesionale şi ştiinţifice.

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Chapter 1

Introduction

Proof mining is a new paradigm of research, concerned with the extraction of hidden finitary and combinatorial content from proofs that make use of highly infinitary principles. This new information is obtained after a logical analysis, using proof-theoretic tools and can be both of quantitative nature, such as algorithms and effective bounds, as well as of qualitative nature, such as uniformities in the bounds or weakening the premises. Thus, even if one is not particularly interested in the numerical details of the bounds themselves, in many cases such explicit bounds immediately show the independence of the quantity in question from certain input data.

The main proof-theoretic techniques in proof mining are the so-called *proof interpretations*. A proof interpretation I maps proofs p in theories $\mathcal{T}$ of theorems A into new proofs p^I in theories $\mathcal{T}^I$ of the interpretation A^I of A. In this way, the original mathematical proof is transformed into a new enriched proof of a stronger result, from which the desired additional information can be read off. While the soundness of these methods rests on results in mathematical logic, the new proof can again be written in ordinary mathematics.

This line of research, developed by Ulrich Kohlenbach in the 90's, has its roots in Kreisel's program on *unwinding of proofs*. Already in the 50's, Kreisel had asked

"What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?"

Kreisel proposed to apply proof-theoretic techniques - originally developed for foundational purposes - to analyze concrete proofs in mathematics and unwind the extra information hidden in them; see for example [156, 176] and, more recently, [178]. Unwinding of proofs has had applications in number theory [155, 177], algebra [52, 51, 49, 50, 173] and combinatorics [14, 78, 109, 242].

However, the most systematic development of proof mining took place in connection with applications to approximation theory [121, 122, 123, 185, 149], asymptotic behaviour of nonlinear iterations [127, 128, 142, 141, 130, 73, 25, 26, 143, 164, 165, 27, 145, 166, 136, 28, 29, 138, 2, 184], as well as (nonlinear) ergodic theory [5, 144, 135, 211, 137, 146, 147], topological dynamics and Ramsey theory [74, 139, 140]. We refer to Kohlenbach's book [134] for a comprehensive reference for proof mining.

In the context of these applications, general logical metatheorems were developed starting in [129] and continued in [76, 163, 134]. These logical metatheorems have the following form: If certain $\forall\exists$-sentences are proved in some formal systems associated to abstract structures, then from a given proof one can extract an effective bound which holds in arbitrary such spaces and is uniform for all parameters meeting very weak local boundedness conditions. The proofs of the logical metatheorems are based on extensions to the new theories of two proof interpretations developed by Gödel: *functional* (or *Dialectica*) *interpretation* [87] and *double-negation* interpretation [86]. Structures treated so far are:

(i) bounded metric, hyperbolic and CAT(0)-spaces, (real) normed, uniformly convex and inner product spaces also with abstract bounded convex subsets $C \subseteq X$ in the normed case [129];

(ii) unbounded metric, hyperbolic and CAT(0)-spaces and (real) normed spaces also with unbounded convex subsets [76];

(iii) Gromov δ-hyperbolic spaces, $\mathbb{R}$-trees and uniformly convex hyperbolic spaces [163];

(iv) complete metric and normed spaces [134].

The importance of the metatheorems is that they can be used to infer new uniform existence results without having to carry out any actual proof analysis. The logical metatheorems apply to formal systems and they guarantee that additional information can be extracted based on transformations of formalized proofs. However, in applications of proof mining, one does not formalize completely the mathematical proofs in order to analyze them. In these applications, we put the statement of the theorem and the main concepts into a suitable logical form and then identify the steps in the proof that require a computational interpretation. As a result, we get direct proofs for the explicit quantitative versions of the original results, that is proofs that no longer rely on any logical tools.

In applications of proof mining, Kohlenbach's *monotone* functional interpretation (see [125] or [134, Chapter 9] for details) is crucially used, since it systematically transforms any statement in a given proof into a new version for which explicit bounds are provided. As it is argued in [148], monotone functional interpretation provides in many cases the *right* notion of *numerical implication* in analysis.

Recently, Terence Tao [234] arrived at a proposal of so-called *hard analysis* (as opposed to *soft analysis*), inspired by the finitary arguments used recently by him and Green [90] in their proof that there are arithmetic progressions of arbitrary length in the prime numbers, as well as by him alone in a series of papers [232, 235, 236, 238]. In the essay [234], Tao illustrates his ideas using two examples: a *finite convergence* principle and a *finitary* infinite pigeonhole principle. It turns out that both the former and a variant of the latter directly result from monotone functional interpretation [132]. Thus, as Kohlenbach points out in [132], Tao's hard analysis could be viewed as carrying out, using monotone functional interpretation, analysis on the level of uniform bounds. In many cases allows one to *finitize* the proofs and to arrive at qualitatively stronger results.

Chapter 2

W-hyperbolic spaces and nonexpansive mappings

We work in the setting of hyperbolic spaces as introduced by Kohlenbach [129]. In order to distinguish them from Gromov hyperbolic spaces [24] or from other notions of hyperbolic space that can be found in the literature (see for example [112, 81, 205]), we shall call them W-hyperbolic spaces.

A *W-hyperbolic space* (X, d, W) is a metric space (X, d) together with a *convexity mapping* $W : X \times X \times [0, 1] \to X$ satisfying

$$
\begin{array}{ll}
(W1) & d(z, W(x, y, \lambda)) \le (1-\lambda)d(z, x) + \lambda d(z, y),\\
(W2) & d(W(x, y, \lambda), W(x, y, \tilde{\lambda})) = |\lambda - \tilde{\lambda}| \cdot d(x, y),\\
(W3) & W(x, y, \lambda) = W(y, x, 1-\lambda),\\
(W4) & d(W(x, z, \lambda), W(y, w, \lambda)) \le (1-\lambda)d(x, y) + \lambda d(z, w).
\end{array}
$$

The convexity mapping W was first considered by Takahashi in [228], where a triple (X, d, W) satisfying $(W1)$ is called a *convex metric space*. If (X, d, W) satisfies $(W1) - (W3)$, then we get the notion of *space of hyperbolic type* in the sense of Goebel and Kirk [81]. $(W4)$ was already considered by Itoh [106] under the name "condition III" and it is used by Reich and Shafrir [205] and Kirk [112] to define their notions of hyperbolic space. We refer to [134, p. 384–387] for a detailed discussion.

Obviously, any normed space is a W-hyperbolic space: just define $W(x, y, \lambda) = (1-\lambda)x + \lambda y$. Furthermore, any convex subset of a normed space is a W-hyperbolic space. We shall see in Section 2.2 other examples of W-hyperbolic spaces.

We shall denote a W-hyperbolic space simply by X, when the metric d and the mapping W are clear from the context. One can easily see that

$$d(x, W(x, y, \lambda)) = \lambda d(x, y) \quad \text{and} \quad d(y, W(x, y, \lambda)) = (1-\lambda)d(x, y). \tag{2.1}$$

Furthermore, $W(x, y, 0) = x$, $W(x, y, 1) = y$ and $W(x, x, \lambda) = x$.

Let us recall now some notions concerning geodesics. Let (X, d) be a metric space. A *geodesic path in X* (*geodesic in* X for short) is a map $\gamma : [a, b] \to X$ satisfying

$$d(\gamma(s), \gamma(t)) = |s-t| \quad \text{for all} \quad s, t \in [a, b]. \tag{2.2}$$

A *geodesic ray* in X is a distance-preserving map $\gamma : [0, \infty) \to X$ and a *geodesic line* in X is a distance-preserving map $\gamma : \mathbb{R} \to X$. A *geodesic segment* in X is the image of a geodesic in X, while a *straight line* in X is the image of a geodesic line in X. If $\gamma : [a, b] \to \mathbb{R}$ is a geodesic in X, $\gamma(a) = x$ and $\gamma(b) = y$, we say that the geodesic γ *joins* x *and* y or that the geodesic segment $\gamma([a, b])$ *joins* x *and* y; x and y are also called the *endpoints* of γ.

A metric space (X, d) is said to be a *(uniquely) geodesic space* if every two points are joined by a (unique) geodesic segment.

If $\gamma([a, b])$ is a geodesic segment joining x and y and $\lambda \in [0, 1]$, $z := \gamma((1-\lambda)a+\lambda b)$ is the unique point in $\gamma([a, b])$ satisfying

$$d(z, x) = \lambda d(x, y) \quad \text{and} \quad d(z, y) = (1 - \lambda)d(x, y). \tag{2.3}$$

In the sequel, we shall use the notation $[x, y]$ for the geodesic segment $\gamma([a, b])$ and we shall denote this z by $(1-\lambda)x \oplus \lambda y$, provided that there is no possible ambiguity.

Given three points x, y, z in a metric space (X, d), we say that y *lies between* x and z if these points are pairwise distinct and if we have $d(x, z) = d(x, y) + d(y, z)$. Obviously, if y lies between x and z, then y also lies between z and x.

The next lemmas collect some well-known properties of geodesic spaces. We refer to [190] for details.

Lemma 2.0.1. *Let (X, d) be a geodesic space.*

(i) For every pairwise distinct points x, y, z in X, y lies between x and z if and only if there exists a geodesic segment $[x, z]$ containing y.

(ii) For every points x, y, z, w and any geodesic segment $[x, y]$, if $z, w \in [x, y]$, then either $d(x, z) + d(z, w) = d(x, w)$ or $d(w, z) + d(z, y) = d(w, y)$.

(iii) For every geodesic segment $[x, y]$ in X and $\lambda, \tilde{\lambda} \in [0, 1]$,

$$d\left((1 - \lambda)x \oplus \lambda y, (1 - \tilde{\lambda})x \oplus \tilde{\lambda} y\right) = |\lambda - \tilde{\lambda}| d(x, y).$$

(iv) Let $\gamma : [a, b] \to X$ be a geodesic that joins x and y. Define

$$\gamma^- : [a, b] \to X, \quad \gamma^-(s) = \gamma(a + b - s).$$

Then γ^- is a geodesic that joins y and x such that $\gamma^-([a, b]) = \gamma([a, b])$.

(v) Let $\gamma, \eta : [a, b] \to X$ be geodesics. If $\gamma([a, b]) = \eta([a, b])$ and $\gamma(a) = \eta(a)$ (or $\gamma(b) = \eta(b)$), then $\gamma = \eta$.

(vi) The following two statements are equivalent:

(a) X is uniquely geodesic.

(b) For any $x \neq y \in X$ and any $\lambda \in [0, 1]$ there exists a unique element $z \in X$ such that

$$d(x, z) = \lambda d(x, y) \quad \textit{and} \quad d(y, z) = (1 - \lambda)d(x, y).$$

Lemma 2.0.2. *Let X be a uniquely geodesic space.*

(i) For all $x, y \in X$, $[x, y] = \{(1-\lambda)x \oplus \lambda y \mid \lambda \in [0,1]\}$.

(ii) For every pairwise distinct points x, y, z in X, y lies between x and z if and only if $y \in [x, z]$.

Following [228], we call a W-hyperbolic space *strictly convex* if for any $x \neq y \in X$ and any $\lambda \in (0,1)$ there exists a unique element $z \in X$ (namely $z = W(x, y, \lambda)$) such that

$$d(x, z) = \lambda d(x, y) \quad \text{and} \quad d(y, z) = (1-\lambda)d(x, y). \tag{2.4}$$

Proposition 2.0.3. *Let (X, d, W) be a W-hyperbolic space. Then*

(i) X is a geodesic space and for all $x \neq y \in X$, $[x, y]_W$ is a geodesic segment joining x and y.

(ii) X is a uniquely geodesic space if and only if it is strictly convex.

(iii) If X is uniquely geodesic, then

(a) W is the unique convexity mapping that makes (X, d, W) a W-hyperbolic space.

(b) For all $x, y \in X$ and $\lambda \in [0,1]$, $W(x, y, \lambda) = (1-\lambda)x \oplus \lambda y$.

Proof. (i) For $x \neq y \in X$, the map $W_{xy} : [0, d(x, y)] \to,$

$$W_{xy}(\alpha) = W\left(x, y, \frac{\alpha}{d(x, y)}\right). \tag{2.5}$$

is a geodesic satisfying $W_{xy}([0, d(x, y)]) = [x, w]_W$.

(ii) By Lemma 2.0.1.(vi).

(iii) (b) is obvious. We prove in the sequel (a). Let $W' : X \times X \times [0,1] \to X$ be another convexity mapping such that (X, d, W') is a W-hyperbolic space. For $\lambda \in [0,1]$ and $x \in X$ one has $W(x, x, \lambda) = W'(x, x, \lambda) = x$. Let $x, y \in X, x \neq y$. Then $[x, y]_W$ and $[x, y]_{W'}$ are geodesic segments that join x and y, hence we must have that $[x, y]_W = [x, y]_{W'}$, that is $W_{xy}([0, d(x, y)]) = W'_{xy}([0, d(x, y)])$. Since $W_{xy}(0) = W'_{xy}(0) = x$, we can apply Lemma 2.0.1.(v) to get that $W_{xy} = W'_{xy}$, so that $W(x, y, \lambda) = W'(x, y, \lambda)$.

□

Let (X, d, W) be a W-hyperbolic space. For all $x, y \in X$, let us define

$$[x, y]_W := \{W(x, y, \lambda) \mid \lambda \in [0,1]\}. \tag{2.6}$$

Then $[x, x]_W = \{x\}$ for all $x \in X$. A subset $C \subseteq X$ is *convex* if $[x, y]_W \subseteq C$ for all $x, y \in C$. A nice feature of our setting is that any convex subset is itself a W-hyperbolic space. It is easy to see that open and closed balls are convex and that the intersection of any family of convex sets is again convex. Moreover, using (W4), we get that the closure of a convex subset of a W-hyperbolic space is again convex.

If C is a convex subset of X, then a function $f : C \to \mathbb{R}$ is said to be *convex* if

$$f\left(W(x, y, \lambda)\right) \leq (1-\lambda) f(x) + \lambda f(y) \tag{2.7}$$

for all $x, y \in C, \lambda \in [0,1]$. f is said to be *strictly convex* if strict inequality holds in (2.7) for $x \neq y$ and $\lambda \in (0,1)$.

Convention: Given a W-hyperbolic space (X, d, W) and $x, y \in X$, $\lambda \in [0,1]$, we shall use from now on the notation $(1-\lambda)x \oplus \lambda y$ for $W(x, y, \lambda)$.

2.1 UCW-hyperbolic spaces

One of the most important classes of Banach spaces are the uniformly convex ones, introduced by Clarkson in the 30's [46]. Following Goebel and Reich [84, p. 105], we can define uniform convexity for W-hyperbolic spaces too.

A W-hyperbolic space (X, d, W) is *uniformly convex* [164] if for any $r > 0$ and any $\varepsilon \in (0,2]$ there exists $\delta \in (0,1]$ such that for all $a, x, y \in X$,

$$\left.\begin{array}{l} d(x,a) \leq r \\ d(y,a) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \quad \Rightarrow \quad d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq (1-\delta) r. \tag{2.8}$$

A mapping $\eta : (0, \infty) \times (0,2] \to (0,1]$ providing such a $\delta := \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$ is called a *modulus of uniform convexity*. We call η *monotone* if it decreases with r (for a fixed ε).

Proposition 2.1.1. *[164] Any uniformly convex W-hyperbolic space is strictly convex.*

Lemma 2.1.2. *[164, 145] Let (X, d, W) be a uniformly convex W-hyperbolic space and η be a modulus of uniform convexity. Assume that $r > 0, \varepsilon \in (0,2], a, x, y \in X$ are such that*

$$d(x,a) \leq r,\ d(y,a) \leq r \text{ and } d(x,y) \geq \varepsilon r.$$

Then for any $\lambda \in [0,1]$,

(i) $d((1-\lambda)x \oplus \lambda y, a) \leq \big(1 - 2\lambda(1-\lambda)\eta(r, \varepsilon)\big) r$;

(ii) for any $\psi \in (0,2]$ such that $\psi \leq \varepsilon$,

$$d((1-\lambda)x \oplus \lambda y, a) \leq \big(1 - 2\lambda(1-\lambda)\eta(r, \psi)\big) r\,;$$

(iii) for any $s \geq r$,

$$d((1-\lambda)x \oplus \lambda y, a) \leq \left(1 - 2\lambda(1-\lambda)\eta\left(s, \frac{\varepsilon r}{s}\right)\right) s\,;$$

(iv) if η is monotone, then for any $s \geq r$,

$$d((1-\lambda)x \oplus \lambda y, a) \leq \left(1 - 2\lambda(1-\lambda)\eta\left(s, \varepsilon\right)\right) r\,.$$

Following [166], we shall refer to uniformly convex W-hyperbolic spaces with a monotone modulus of uniform convexity as *UCW-hyperbolic spaces*. Furthermore, we shall also use the notation (X, d, W, η) for a UCW-hyperbolic space having η as a monotone modulus of uniform convexity.

We shall see in Subsection 2.2 that $CAT(0)$ spaces are UCW-hyperbolic spaces with modulus of uniform convexity $\eta(r, \varepsilon) = \varepsilon^2/8$ quadratic in ε. Thus, UCW-hyperbolic spaces are a natural generalization of both uniformly convex normed spaces and $CAT(0)$ spaces.

Moreover, as we shall see in the sequel, complete UCW-hyperbolic spaces have very nice properties. For the rest of this section, (X, d, W) is a complete UCW-hyperbolic space.

Proposition 2.1.3. *[145] The intersection of any decreasing sequence of nonempty bounded closed convex subsets of X is nonempty.*

The next result is inspired by [84, Proposition 2.2].

Proposition 2.1.4. *[166] Let C be a closed convex subset of X, $f : C \to [0, \infty)$ be convex and lower semicontinuous. Assume moreover that for all sequences (x_n) in C,*

$$\lim_{n\to\infty} d(x_n, a) = \infty \text{ for some } a \in X \text{ implies } \lim_{n\to\infty} f(x_n) = \infty.$$

Then f attains its minimum on C. If, in addition, for all $x \neq y$,

$$f\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) < \max\{f(x), f(y)\}$$

then f attains its minimum at exactly one point.

Let us recall that a subset C of a metric space (X, d) is called a *Chebyshev set* if to each point $x \in X$ there corresponds a unique point $z \in C$ such that $d(x, z) = d(x, C)(= \inf\{d(x, y) \mid y \in C\})$. If C is a Chebyshev set, *metric projection* $P_C : X \to C$ can be defined by assigning z to x.

Proposition 2.1.5. *[166] Every closed convex subset C of X is a Chebyshev set.*

2.2 Some related structures

Spaces of hyperbolic type

Spaces of hyperbolic type were introduced by Goebel and Kirk [81] (see also [112]). Let (X, d) be a metric space and S be a family of geodesic segments in X. We say that the structure (X, d, S) is a *space of hyperbolic type* if the following conditions are satisfied:

(i) for each two points $x, y \in X$ there exists a unique geodesic segment from S that joins them, denoted $[x, y]$;

(ii) if $p, x, y \in M$ and if $m \in [x, y]$ satisfies $d(x, m) = \lambda d(x, y)$ for some $\lambda \in [0, 1]$, then

$$d(p, m) \leq (1 - \lambda)d(p, x) + \lambda d(p, y).$$

The following result shows that spaces of hyperbolic type are exactly the metric spaces with a convexity mapping W satisfying $(W1), (W2), (W3)$.

Proposition 2.2.1. *Let (X, d) be a metric space. The following are equivalent.*

(i) There exists a family S of metric segments such that (X, d, S) is a space of hyperbolic type.

(ii) There exists a a convexity mapping W such that (X, d, W) satisfies $(W1)$,$(W2)$ and $(W3)$.

Proof. $(i) \Rightarrow (ii)$ Define $W : X \times X \times [0, 1] \to X$ by $W(x, y, \lambda) = (1 - \lambda)x \oplus \lambda y$. Then (X, d, W) satisfies $(W1), (W2), (W3)$.
$(ii) \Rightarrow (i)$ For all $x, y \in X, x \neq y$, consider the geodesic W_{xy} joining x and y, defined by (2.5). For $x = y$, let $W_{xx} : \{0\} \to X, W_{xx}(0) = x$. Taking $S := \{W_{xy} \mid x, y \in X\}$, we obtain that (X, d, S) is a space of hyperbolic type. □

As a consequence, any W-hyperbolic space is a space of hyperbolic type. In fact, W-hyperbolic spaces are exactly the spaces of hyperbolic type satisfying $(W4)$.

Hyperbolic spaces in the sense of Reich and Shafrir

The class of hyperbolic spaces presented in this section was defined by Reich and Shafrir [205] as an appropriate context for the study of operator theory in general, and of iterative processes for nonexpansive mappings in particular.

Let (X, d) be a metric space and M be a nonempty family of straight lines in X with the following property: for each two distinct points $x, y \in X$ there is a unique straight line from M which passes through x, y.

We shall denote by (X, d, M) a metric space (X, d) together with a family M as above. Since $M \neq \emptyset$, there is at least one geodesic line $\gamma : \mathbb{R} \to X$ with $\gamma(\mathbb{R}) \in M$, so $card(X) \geq card(\mathbb{R}) = \aleph_1$, as γ is injective. Furthermore, the metric space (X, d) must be unbounded.

The following lemma collects some useful properties; we refer to [142] for the proofs.

Lemma 2.2.2.

(i) For any $x \in X$ there is at least one straight line from M that passes through x.

(ii) For any distinct points x and y in X, the unique straight line that passes through x and y determines in a unique way a geodesic segment joining x and y, denoted by $[x, y]$.

(iii) For all $x, y \in X$ and all $\lambda \in [0, 1]$ there is a unique point $z \in [x, y]$ satisfying

$$d(x, z) = \lambda d(x, y) \quad \text{and} \quad d(y, z) = (1 - \lambda)d(x, y). \tag{2.9}$$

The unique point z satisfying (2.9) will be denoted by $(1 - \lambda)x \oplus \lambda y$.

We say that the structure (X, d, M) is a *hyperbolic space* if the following inequality is satisfied

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\right) \leq \frac{1}{2}d(y, z). \tag{2.10}$$

Proposition 2.2.3. *Let (X, d, M) be a hyperbolic space. Then*

$$d((1-\lambda)x \oplus \lambda z, (1-\lambda)y \oplus \lambda w) \leq (1-\lambda)d(x, y) + \lambda d(z, w) \tag{2.11}$$

for all $x, y, z, w \in X$ and any $\lambda \in [0, 1]$.

If we define

$$W : X \times X \times [0, 1] \to X, \quad W(x, y, \lambda) = (1-\lambda)x \oplus \lambda y,$$

it is easy to see that (X, d, W) is a W-hyperbolic space. Thus, any hyperbolic space in the sense of Reich and Shafrir is a W-hyperbolic space.

Busemann spaces

An important class of W-hyperbolic spaces are the so-called Busemann spaces, used by Busemann [40, 41] to define a notion of 'nonpositively curved space'. We refer to [190] for an extensive study. Let us recall that a map $\gamma : [a, b] \to X$ is an *affinely reparametrized geodesic* if γ is a constant path or there exist an interval $[c, d]$ and a geodesic $\gamma' : [c, d] \to X$ such that $\gamma = \gamma' \circ \psi$, where $\psi : [a, b] \to [c, d]$ is the unique affine homeomorphism between the intervals $[a, b]$ and $[c, d]$.

A geodesic space (X, d) is a *Busemann space* if for any two affinely reparametrized geodesics $\gamma : [a, b] \to X$ and $\gamma' : [c, d] \to X$, the map

$$D_{\gamma,\gamma'} : [a, b] \times [c, d] \to \mathbb{R}, \quad D_{\gamma,\gamma'}(s, t) = d(\gamma(s), \gamma'(t)) \tag{2.12}$$

is convex. Examples of Busemann spaces are strictly convex normed spaces. In fact, a normed space is a Busemann space if and only if it is strictly convex.

Proposition 2.2.4. *Let (X, d) be a metric space. The following two statements are equivalent:*

(i) X is a Busemann space.

(ii) There exists a (unique) convexity mapping W such that (X, d, W) is a uniquely geodesic W-hyperbolic space.

Proof. $(i) \Rightarrow (ii)$ Assume that X is Busemann. By [190, Proposition 8.1.4], any Busemann space is uniquely geodesic. For any $x, y \in X$, let $[x, y]$ be the unique geodesic segment that joins x and y and define

$$W : X \times X \times [0, 1] \to X, \quad W(x, y, \lambda) = (1-\lambda)x \oplus \lambda y. \tag{2.13}$$

Let us verify (W1)-(W4): (W4) follows from [190, Proposition 8.1.2.(ii)]; (W2) follows from Lemma 2.0.1.(iii); (W1) follows from (W4) applied with $z = x$ and the fact that $W(x, x, \lambda) = x$; (W3) follows by Lemma 2.0.1.(iv).

$(ii) \Rightarrow (i)$ Apply [190, Proposition 8.1.2.(ii)] and (W4). □

$CAT(0)$ spaces

In this section we give a very brief exposition of $CAT(0)$ spaces. We refer to the monograph by Bridson and Haefliger [24] for an extensive study of this important class of spaces.

Let (X, d) be a geodesic space. A *geodesic triangle* in X consists of three points $p, q, r \in X$, its *vertices*, and a choice of three geodesic segments $[p, q], [q, r], [r, s]$ joining them, its *sides*. Such a geodesic triangle will be denoted $\Delta([p, q], [q, r], [r, s])$. If a point lies in the union of $[p, q], [q, r], [r, s]$, then we write $x \in \Delta$.

A triangle $\overline{\Delta} = \Delta(\overline{p}, \overline{q}, \overline{r})$ in $\mathbb{R}^2$ is called a *comparison triangle* for the geodesic triangle $\Delta([p, q], [q, r], [r, s])$ if $d_{\mathbb{R}^2}(\overline{p}, \overline{q}) = d(p, q), d_{\mathbb{R}^2}(\overline{q}, \overline{r}) = d(q, r)$ and $d_{\mathbb{R}^2}(\overline{r}, \overline{p}) = d(p, r)$. Such a triangle $\overline{\Delta}$ always exists and it is unique up to isometry [24, Lemma I.2.14]. We write $\overline{\Delta} = \overline{\Delta}(p, q, r)$ or $\Delta(\overline{p}, \overline{q}, \overline{r})$ according to whether a specific choice of $\overline{p}, \overline{q}, \overline{r}$ is required. A point $\overline{x} \in [\overline{p}, \overline{q}]$ is called a *comparison point* for $x \in [p, q]$ if $d(p, x) = d_{\mathbb{R}^2}(\overline{p}, \overline{x})$. Comparison points on $[\overline{q}, \overline{r}]$ and $[\overline{r}, \overline{p}]$ are defined similarly.

Let Δ be a geodesic triangle in X and $\overline{\Delta}$ be a comparison triangle for Δ in $\mathbb{R}^2$. Then Δ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \Delta$ and for all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{2.14}$$

A geodesic space X is said to be a $CAT(0)$ *space* if all geodesic triangles satisfy the $CAT(0)$ *inequality*. Complete $CAT(0)$ spaces are often called *Hadamard spaces*. It can be shown that $CAT(0)$ spaces are uniquely geodesic and that a normed space is a $CAT(0)$-space if and only if it is a pre-Hilbert space.

In the sequel, we give an equivalent characterization of $CAT(0)$ spaces, using the so-called: *CN inequality of Bruhat-Tits* [39]: for all $x, y, z \in X$ and all $m \in X$ with $d(x, m) = d(y, m) = \dfrac{1}{2}d(x, y)$,

$$d(z, m)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2. \tag{2.15}$$

In the setting of W-hyperbolic spaces, we consider the following reformulation of the CN inequality, which is nicer from the point of view of the logical metatheorems to be presented in Chapter 3: for all $x, y, z \in X$,

$$CN^- : \qquad d\left(z, \frac{1}{2}x \oplus \frac{1}{2}y\right)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2. \tag{2.16}$$

We refer to [24, p. 163] and to [134, p. 386-388] for the proof of the following result.

Proposition 2.2.5. *Let (X, d) be a metric space. The following are equivalent.*

(i) X is a CAT(0)-space.

(ii) X is a geodesic space that satisfies the CN inequality (2.15);

(iii) There exists a a convexity mapping W such that (X, d, W) is a W-hyperbolic space satisfying the CN inequality (2.15).

(iv) There exists a a convexity mapping W such that (X,d,W) is a W-hyperbolic space satisfying the CN^- inequality (2.16).

Thus, $CAT(0)$ spaces are exactly the W-hyperbolic spaces satisfying the CN inequality. Furthermore

Proposition 2.2.6. *[164] $CAT(0)$ spaces are UCW-hyperbolic spaces with a monotone modulus of uniform convexity*

$$\eta(\varepsilon, r) = \frac{\varepsilon^2}{8},$$

that does not depend on r.

The Hilbert ball

Let H be a complex Hilbert space, and let $\mathbb{B}$ be the open unit ball in H. We consider the Poincaré metric on $\mathbb{B}$, defined by

$$\rho(x,y) := \operatorname{argtanh}(1-\sigma(x,y))^{1/2}, \quad \text{where} \quad \sigma(x,y) = \frac{(1-\|x\|^2)(1-\|y\|^2)}{|1-\langle x,y\rangle|^2}. \tag{2.17}$$

The metric space $(\mathbb{B}, \rho)$ is called the *Hilbert ball*.

The Hilbert ball is a uniquely geodesic space (see [161, Theorem 4.1] or [85]). Moreover, by the inequality (4.2) in [205], the CN inequality is satisfied. Applying Proposition 2.2.5.(ii), it follows that the Hilbert ball is a $CAT(0)$ space.

We refer to Goebel and Reich's book [84] for an extensive study of the Hilbert ball.

Gromov hyperbolic spaces

Gromov's theory of hyperbolic spaces is set out in [94]. The study of Gromov hyperbolic spaces has been largely motivated and dominated by questions about (Gromov) hyperbolic groups, one of the main object of study in geometric group theory. In the sequel, we review some definitions and elementary facts concerning Gromov hyperbolic spaces. For a more detailed account of this material, the reader is referred to [94, 77, 24].

Let (X,d) be a metric space. Given three points x,y,w, the *Gromov product* of x and y with respect to the *base point* w is defined to be:

$$(x \cdot y)_w = \frac{1}{2}(d(x,w) + d(y,w) - d(x,y)). \tag{2.18}$$

It measures the failure of the triangle inequality to be an equality and it is always nonnegative.

Definition 2.2.7. *Let $\delta \geq 0$. X is called $\delta -$ hyperbolic if for all $x,y,z,w \in X$,*

$$(x \cdot y)_w \geq \min\{(x \cdot z)_w, (y \cdot z)_w\} - \delta. \tag{2.19}$$

We say that X is hyperbolic *if it is (δ)-hyperbolic for some $\delta \geq 0$.*

It turns out that the definition is independent of the choice of the base point w in the sense that if there exists *some* $w \in X$ such that the above inequality holds for all $x, y, z \in X$, then X is 2δ-hyperbolic.

By the definition of Gromov product, (2.19) can be rewritten as a 4-point condition: for all $x, y, z, w \in X$,

$$d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\} + 2\delta. \tag{2.20}$$

$\mathbb{R}$-trees

The notion of $\mathbb{R}$-tree was introduced by Tits [239], as a generalization of the notion of local Bruhat-Tits building for rank-one groups, which itself generalizes the notion of simplicial tree. A more general concept, that of a Λ-tree, where Λ is a totally ordered abelian group, made its appearance as an essential tool in the study of groups acting on hyperbolic manifolds in the work of Morgan and Shalen [182]. For detailed informations about $\mathbb{R}(\Lambda)$-trees, we refer to [17, 44].

Definition 2.2.8. *[239] An* $\mathbb{R}$-tree *is a geodesic space containing no homeomorphic image of a circle.*

We remark that in the initial definition, Tits only considered $\mathbb{R}$-trees that are complete as metric spaces, but the assumption of completeness is usually irrelevant. The following proposition gives some equivalent characterizations of $\mathbb{R}$-trees, which can be found in the literature.

Proposition 2.2.9. *(see, for example, [1, 17, 77])*
Let (X, d) be a metric space. The following are equivalent:

(i) X is an $\mathbb{R}$-tree,

(ii) X is uniquely geodesic and for all $x, y, z \in X$,

$$[y, x] \cap [x, z] = \{x\} \Rightarrow [y, x] \cup [x, z] = [y, z].$$

(i.e., if two geodesic segments intersect in a single point, then their union is a geodesic segment.)

(iii) X is a geodesic space that is (Gromov) 0-hyperbolic, i.e. satisfies the inequality (2.20) with $\delta = 0$.

The fact that $\mathbb{R}$-trees are exactly the geodesic 0-hyperbolic spaces follows from a very important result of Alperin and Bass [1, Theorem 3.17] (see also [77, Chapter 2, Exercise 8] and is the basic ingredient for proving the following characterization of $\mathbb{R}$-trees using our notion of W-hyperbolic space.

Proposition 2.2.10. *Let (X, d) be a metric space. The following are equivalent:*

(i) X is an $\mathbb{R}$-tree;

(ii) there exists a convexity mapping W such that (X, d, W) is a W-hyperbolic space satisfying for all $x, y, z, w \in X$,

$$d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\}.$$

2.3 Fixed point theory of nonexpansive mappings

We refer to [119, Chapter 3] or to [82, 84] for a comprehensive treatment of the fixed point theory of nonexpansive mappings.

The notion of nonexpansive mapping can be introduced in the very general setting of metric spaces. Thus, if (X, d) is a metric space, and $C \subseteq X$ a nonempty subset, then a mapping $T : C \to C$ is said to be *nonexpansive* if for all $x, y \in C$,

$$d(Tx, Ty) \leq d(x, y).$$

We shall denote with $Fix(T)$ the set of fixed points of T. The subset C is said to have the *fixed point property for nonexpansive mappings*, FPP for short, if $Fix(T) \neq \emptyset$ for any nonexpansive mapping $T : C \to C$.

While an abstract metric space is all that is needed to define the concept of nonexpansive mapping, the most interesting results were obtained in the setting of Banach spaces.

Fixed point theory of contractions is, even from a computational point of view, essentially trivial, due to Banach's Contraction Mapping Principle. Anyway, the picture known for contractions breaks down for nonexpansive mappings, as we indicate below:

(i) Nonexpansive mappings need not to have fixed points: just take $T : \mathbb{R} \to \mathbb{R},\ T(x) = x + 1$.

(ii) Even when C is compact (and therefore fixed points exist by the fixed point theorems of Brouwer and Schauder), they are not unique: take $T : \mathbb{R} \to \mathbb{R},\ T(x) = x$.

(iii) Even when the fixed point is unique, it will in general not be approximated by the Picard iteration $x_{n+1} = Tx_n$: if we l et $T : [0, 1] \to [0, 1]$, $T(x) = 1 - x$ and $x_0 = 0$, then T has a unique fixed point $\frac{1}{2}$, while x_n alternates between 0 and 1.

Fixed point theory for nonexpansive mappings has been a very active research area in nonlinear analysis beginning with the 60's, when the most widely known result in the theory, the so-called Browder-Göhde-Kirk Theorem, was published.

Theorem 2.3.1. *If C is a bounded closed and convex subset of a uniformly convex Banach space X and $T : C \to C$ is nonexpansive, then T has a fixed point.*

The above theorem was proved independently by Browder [30] and Göhde [88] in the form stated above, and by Kirk [111] in a more general form. Browder and Kirk used the same line of argument, which in fact yields a more general result - while the proof of Göhde relies on properties essentially unique to uniformly convex Banach spaces.

2.3.1 Asymptotic centers and a fixed point theorem

The asymptotic center technique, introduced by Edelstein [58, 59], is one of the most useful tools in metric fixed point theory of nonexpansive mappings in uniformly convex Banach spaces, due to the fact that bounded sequences have unique asymptotic centers with respect to closed convex subsets.

Let us recall basic facts about asymptotic centers. We refer to [58, 59, 84, 82] for details.

Let (X,d) be a metric space, (x_n) be a bounded sequence in X and $C \subseteq X$ be a nonempty subset of X. We define the following functionals:

$$\begin{aligned} r_m(\cdot,(x_n)) : X \to [0,\infty), \quad r_m(y,(x_n)) &= \sup\{d(y,x_n) \mid n \geq m\} \\ &\quad \text{for } m \in \mathbb{N}, \\ r(\cdot,(x_n)) : X \to [0,\infty), \quad r(y,(x_n)) &= \limsup_n d(y,x_n) = \inf_m r_m(y,(x_n)) \\ &= \lim_{m\to\infty} r_m(y,(x_n)). \end{aligned}$$

The following lemma collects some basic properties of the above functionals.

Lemma 2.3.2. *Let $y \in X$.*

(i) $r_m(\cdot,(x_n))$ *is nonexpansive for all* $m \in \mathbb{N}$*;*

(ii) $r(\cdot,(x_n))$ *is continuous and* $r(y,(x_n)) \to \infty$ *whenever* $d(y,a) \to \infty$ *for some* $a \in X$*;*

(iii) $r(y,(x_n)) = 0$ *if and only if* $\lim_{n\to\infty} x_n = y$*;*

(iv) if (X,d,W) *is a convex metric space and* C *is convex, then* $r(\cdot,(x_n))$ *is a convex function.*

The *asymptotic radius of* (x_n) *with respect to* C is defined by

$$r(C,(x_n)) = \inf\{r(y,(x_n)) \mid y \in C\}.$$

The *asymptotic radius* of (x_n), denoted by $r((x_n))$, is the asymptotic radius of (x_n) with respect to X, that is $r((x_n)) = r(X,(x_n))$.

A point $c \in C$ is said to be an *asymptotic center* of (x_n) *with respect to* C if

$$r(c,(x_n)) = r(C,(x_n)) = \min\{r(y,(x_n)) \mid y \in C\}.$$

We denote with $A(C,(x_n))$ the set of asymptotic centers of (x_n) with respect to C. When $C = X$, we call c an *asymptotic center* of (x_n) and we use the notation $A((x_n))$ for $A(X,(x_n))$.

The following lemma, inspired by [59, Theorem 1], turns out to be very useful in proving the uniqueness of asymptotic centers.

Lemma 2.3.3. *[166] Let* (x_n) *be a bounded sequence in* X *with* $A(C,(x_n)) = \{c\}$ *and* $(\alpha_n), (\beta_n)$ *be real sequences such that* $\alpha_n \geq 0$ *for all* $n \in \mathbb{N}$, $\limsup_n \alpha_n \leq 1$ *and* $\limsup_n \beta_n \leq 0$.
Assume that $y \in C$ *is such that there exist* $p, N \in \mathbb{N}$ *satisfying*

$$\forall n \geq N \bigg(d(y,x_{n+p}) \leq \alpha_n d(c,x_n) + \beta_n \bigg).$$

Then $y = c$.

In general, the set $A(C,(x_n))$ of asymptotic centers of a bounded sequence (x_n) with respect to $C \subseteq X$ may be empty or, on the contrary, contain infinitely many points.

A classical result is the fact that in uniformly convex Banach spaces, bounded sequences have unique asymptotic centers with respect to closed convex subsets. For the Hilbert ball, this was proved in [84, Proposition 21.1]. The following result shows that the same is true for complete UCW-hyperbolic spaces.

Proposition 2.3.4. *[166] Let (X,d,W) be a complete UCW-hyperbolic space. Every bounded sequence (x_n) in X has a unique asymptotic center with respect to any closed convex subset C of X.*

As an application of Proposition 2.3.4 and Lemma 2.3.3, we can prove the following characterization of the fact that a nonexpansive mapping $T : C \to C$ has fixed points.

Theorem 2.3.5. *[166] Let C be a convex closed subset of a complete UCW-hyperbolic space (X,d,W) and $T : C \to C$ be nonexpansive. The following are equivalent.*

(i) T has fixed points;

(ii) T has a bounded approximate fixed point sequence;

(iii) for all $x \in C$ there exists $b > 0$ such that T has approximate fixed points in a b-neighborhood of x;

(iv) there exist $x \in C$ and $b > 0$ such that T has approximate fixed points in a b-neighborhood of x;

(v) the sequence $(T^n x)$ of Picard iterates is bounded for some $x \in C$;

(vi) the sequence $(T^n x)$ of Picard iterates is bounded for all $x \in C$.

As an immediate consequence we obtain the generalization to complete UCW-hyperbolic spaces of the Browder-Göhde-Kirk Theorem.

Corollary 2.3.6. *Let C be a bounded convex closed subset of a complete UCW-hyperbolic space (X,d,W) and $T : C \to C$ be nonexpansive. Then T has fixed points.*

2.4 The approximate fixed point property

Let (X,d) be a metric space, $C \subseteq X$ and $T : C \to C$. The *minimal displacement of T* is defined as

$$r_C(T) := \inf\{d(x,Tx) \mid x \in C\}. \tag{2.21}$$

A sequence (x_n) in C is an *approximate fixed point sequence* of T if $\lim\limits_{n\to\infty} d(x_n,Tx_n) = 0$. We say that T is *approximately fixed* [21], or that T *has approximate fixed points*, if T has an approximate fixed point sequence.

Given $\varepsilon > 0$, a point $x \in C$ is said to be an *ε-fixed point* of T if $d(x, Tx) < \varepsilon$. We shall denote with $Fix_\varepsilon(T)$ the set of ε-fixed points of T.

It is easy to see that T is approximately fixed if and only if $r_C(T) = 0$ if and only if $Fix_\varepsilon(T) \neq \emptyset$ for any $\varepsilon > 0$.

A related notion is the following. For $x \in C$ and $b, \varepsilon > 0$, let us denote

$$Fix_\varepsilon(T, x, b) := \{y \in C \mid d(y, x) \leq b \text{ and } d(y, Ty) < \varepsilon\}.$$

If $Fix_\varepsilon(T, x, b) \neq \emptyset$ for all $\varepsilon > 0$, we say that T *has approximate fixed points* in a b-neighborhood of x.

Lemma 2.4.1. *The following are equivalent.*

(i) T has a bounded approximate fixed point sequence;

(ii) for all $x \in C$ there exists $b > 0$ such that T has approximate fixed points in a b-neighborhood of x;

(iii) there exist $x \in C$ and $b > 0$ such that T has approximate fixed points in a b-neighborhood of x.

A subset C of a metric space (X, d) is said to have the *approximate fixed point property for nonexpansive mappings*, AFPP for short, if each nonexpansive mapping $T : C \to C$ is approximately fixed. It is well-known that *bounded* closed convex subsets of Banach spaces have the AFPP for nonexpansive mappings (see, for example, [119, Chapter 3, Lemma 2.4].

Goebel and Kuczumow [83] were the first to remark that there exist *unbounded* sets in Hilbert spaces that have this property. Namely, they proved that any closed convex set C contained in a block has the AFPP; a set $K \subseteq \ell_2$ is a *block* if K is of the form $K = \{x \in \ell_2 \mid | < x, e_n > | \leq M_n, n = 1, 2, \dots, \}$, where $\{e_n\}$ is some orthogonal basis and (M_n) is a sequence of positive reals. More recently, Kuczumow gave in [160] an example of an unbounded closed convex subset of ℓ_2 that has the AFPP, but it is not contained in a block for any orthogonal basis of ℓ_2.

Goebel and Kuczumow' result was extended by Ray [191] to include all linearly bounded subsets of ℓ_p, $1 < p < \infty$. A subset C of a normed space X is said to be *linearly bounded* if it has bounded intersection with all lines in X. Subsequently, Ray obtained the following characterization of the FPP in Hilbert spaces, answering an open problem of Kirk.

Theorem 2.4.2. *[192] A closed convex subset of a real Hilbert space has the FPP for nonexpansive mappings if and only if it is bounded.*

In [199], Reich proved the following remarkable theorem.

Theorem 2.4.3. *[199] A closed convex subset of a reflexive Banach space has the AFPP for nonexpansive mappings if and only if it is linearly bounded.*

If the Banach space X is finite-dimensional, then any linearly bounded subset C of X is, in fact, bounded. Thus, in this case, either C is bounded and has the FPP, or C is unbounded and does not even have the AFPP for nonexpansive mappings.

As it was already noted in [199], the above theorem can not be extended to all Banach spaces: just take $X = \ell_1$, $C = \{x \in \ell_1 \mid |x_n| \leq 1 \text{ for all } n\}$ and define $T : C \to C$ by $T(x_1, x_2, \ldots) = (1, x_2.x_3, \ldots)$. Then C is linearly bounded and T is an isometry, but $r_C(T) = 1$, hence T is not approximately fixed.

In [216], Shafrir gave a more general geometric characterization of the AFPP which is true in an arbitrary Banach space or even for the more general class of complete hyperbolic spaces in the sense of [205]. In order to do this, he introduced the concept of a *directionally bounded set*.

A *directional curve* in a metric space (X, d) is a curve $\gamma : [0, \infty) \to X$ for which there exists $b > 0$ such that for each $t \geq s \geq 0$,

$$t - s - b \leq d(\gamma(s), \gamma(t)) \leq t - s.$$

A convex subset of a Banach space is called *directionally bounded* if it contains no directional curve. Since a line is a directional curve with $b = 0$, directionally bounded sets are always linearly bounded. Shafrir proved two important results.

Theorem 2.4.4. *[216]*

(i) A convex subset of a Banach space has the AFPP if and only if it is directionally bounded.

(ii) A Banach space X is reflexive if and only if every closed convex linearly bounded subset of X is directionally bounded.

Therefore, the characterization for the AFPP from Reich Theorem 2.4.3 is true for a Banach space X if and only if X is reflexive.

Answering an open question of Shafrir [216], in [180] Matoušková and Reich showed that any infinite-dimensional Banach space contains an unbounded convex subset which has the AFPP for nonexpansive mappings; Shafrir [216] had proved this only for infinite-dimensional Banach spaces which do not contain an isomorphic copy of ℓ_1.

Chapter 3

Logical metatheorems

In this section we give an informal presentation of the general logical metatheorems proved by Kohlenbach [129] and Gerhardy-Kohlenbach [76]. We refer to Kohlenbach's book [134] for a comprehensive treatment.

The system $\mathcal{A}^\omega$ of so-called *weakly extensional* classical analysis goes back to Spector [223]. It is formulated in the language of functionals of finite types and consists of a finite type extension $\mathbf{PA}^\omega$ of first order Peano arithmetic $\mathbf{PA}$ and the axiom schemas of quantifier-free choice and of dependent choice in all types, which implies countable choice and hence comprehension over natural numbers. Full second order arithmetic in the sense of reverse mathematics [219] is contained in $\mathcal{A}^\omega$ if we identify subsets of $\mathbb{N}$ with their characteristic functions.

Let us recall the so-called *Axiom of Countable Choice*: For each set B and each binary relation $P \subseteq \mathbb{N} \times B$ between natural numbers and members of B,

$$\forall n \in \mathbb{N}\, \exists y \in B\, P(n, y) \quad \Rightarrow \quad \exists f : \mathbb{N} \to B\, \forall n \in \mathbb{N}\, P(n, f(n)).$$

In contrast to the full Axiom of Choice which demands the existence of choice functions $f : A \to B$ for arbitrary sets A, B, the Axiom of Countable Choice justifies only a sequence of independent choices from an arbitrary set B which successively satisfy the conditions

$$P(0, f(0)),\ P(1, f(1)),\ P(2, f(2)), \ldots$$

A stronger axiom is the *Axiom of Dependent Choice (DC)*: For each set A, each relation $P \subseteq A \times A$ and all $a \in A$

$$\forall x \in A\, \exists y \in A\, P(x, y) \ \Rightarrow\ \exists f : \mathbb{N} \to A\big[f(0) = a \text{ and } \forall n \in \mathbb{N}\, P(f(n), f(n+1))\big].$$

The Axiom of Dependent Choice also justifies only a sequence of choices, where, however, each of them may depend on the previous one, since they must now satisfy the conditions

$$P(f(0), f(1)),\ P(f(1), f(2)),\ P(f(2), f(3)), \ldots$$

It is easy to see that the Axiom of Choice implies the Axiom of Dependent Choice, which implies further the Axiom of Countable Choice.

The axiom scheme of *Comprehension over natural numbers* says that

$$\exists f : \mathbb{N} \to \mathbb{N}\, \forall n \in \mathbb{N}\big(f(n) = 0 \Leftrightarrow A(n)\big),$$

where $A(n)$ is an arbitrary formula in our language, not containing f free but otherwise with arbitrary parameters. We refer to the very nice monograph [183] for details on set theory.

The set $\mathbf{T}$ of all *finite types* is defined inductively by the clauses:

(i) $0 \in \mathbf{T}$;

(ii) if $\rho, \tau \in \mathbf{T}$ then $(\rho \to \tau) \in \mathbf{T}$.

We usually omit the outermost parentheses for types. The intended interpretation of the base type 0 is the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Objects of type $\rho \to \tau$ are functions which map objects of type ρ to objects of type τ. For example, $0 \to 0$ is the type of functions $f : \mathbb{N} \to \mathbb{N}$ and $(0 \to 0) \to 0$ is the type of operations F mapping such functions f to natural numbers.

Any type $\rho \neq 0$ can be uniquely written in the normal form $\rho = \rho_1 \to (\rho_2 \to \ldots \to (\rho_n \to 0)\ldots)$ (for suitable $n \geq 1$ and types $\rho_1, \ldots, \rho_n$), which is usually abbreviated by $\rho = \rho_1 \to \rho_2 \to \ldots \to \rho_n \to 0$ if it is clear to which types $\rho_1, \ldots, \rho_n$ we refer and there is no danger of confusion.

We use the notation $\underline{x}$ for tuples of variables $\underline{x} = x_1, \ldots, x_n$ and $\underline{\rho}$ for tuples of types $\underline{\rho} = \rho_1, \ldots, \rho_n$. When we write $\underline{x}^{\underline{\rho}}$ we mean that each x_i has type ρ_i. The notations $\underline{x}^{\rho}$ or $\underline{x} \in \rho$ mean that each x_i is of type ρ.

The set $\mathbf{P} \subset \mathbf{T}$ of *pure types* is defined inductively by: (i) $0 \in \mathbf{P}$ and (ii) if $\rho \in \mathbf{P}$, then $\rho \to 0 \in \mathbf{P}$. Pure types are often denoted by natural numbers: $0 \to 0 = 1$, $(0 \to 0) \to 0 = 2$, in general $n \to 0 = n + 1$.

The *degree* (or *type level*) $deg(\rho)$ of a type ρ is defined as

$$deg(0) := 0, \quad deg(\rho \to \tau) := \max(deg(\tau), deg(\rho) + 1).$$

Note that for pure types ρ, $deg(\rho)$ is just the number which denotes ρ. Objects of type ρ with $deg(\rho) > 1$ are usually called *functionals*.

We shall denote formulas with $A, B, C, \ldots$ and quantifier-free formulas with $A_0, B_0, C_0, \ldots$. A formula A is said to be *universal* if it has the form $A \equiv \forall \underline{x}\, A_0(\underline{x}, \underline{a})$, where $\underline{x}, \underline{a}$ are tuples of variables. Similarly, A is an *existential* formula if $A \equiv \exists\, \underline{x}\, A_0(\underline{x}, \underline{a})$.

Furthermore, A is called a Π^0_n-formula if it has n-alternating blocks of equal quantifiers starting with a block of universal quantifiers, that is

$$\forall \underline{x_1}\, \exists \underline{x_2} \ldots \forall/\exists \underline{x_n} A_0(\underline{x_1}, \ldots, \underline{x_n}, \underline{a}).$$

If the formula starts with a block of existential quantifiers, that is

$$\exists \underline{x_1}\, \forall \underline{x_2} \ldots \forall/\exists \underline{x_n} A_0(\underline{x_1}, \ldots, \underline{x_n} \underline{a}),$$

it is called a Σ^0_n-formula.

We only include equality $=_0$ between objects of type 0 as a primitive predicate. Equality between objects of higher types is defined *extensionally*: if $\rho = \rho_1 \to \ldots \to \rho_n \to 0$ and s, t are terms of type ρ, then

$$s =_\rho t \;:=\; \forall y_1^{\rho_1}, \ldots, y_n^{\rho_n} \big(s y_1 \ldots y_n =_0 t y_1 \ldots y_n \big),$$

where $y_1, \dots, y_n$ are variables not occurring in s, t.

Instead of the full axiom of extensionality in all types, the system $\mathcal{A}^\omega$ only has a *quantifier-free rule of extensionality*:

$$\frac{A_0 \to s =_\rho t}{A_0 \to r[s/x] =_\tau r[t/x]},$$

where A_0 is a quantifier-free formula, s, t are terms of type ρ, r is a term of type τ, and $r[s/x]$ (resp. $r[t/x]$) is the result of replacing every occurrence of x in r by s (resp. t). We refer to [129] for an extensive discussion of extensionality issues.

In the sequel, we briefly recall the representation of real numbers in $\mathcal{A}^\omega$. We refer to [134, Chapter 4] for details.

We will most times use $\mathbb{N}$ instead of 0 and $\mathbb{N}^\mathbb{N}$ instead of 1, say "natural numbers" instead of "objects of type 0", and write $n \in \mathbb{N}$ or $n^\mathbb{N}$ instead of n^0, respectively $f : \mathbb{N} \to \mathbb{N}$ or $f^{\mathbb{N}\to\mathbb{N}}$ instead of f^1.

Rational numbers are represented as codes $j(n, m)$ of pairs of natural numbers: $j(n, m)$ represents the rational number $\dfrac{\frac{n}{2}}{m+1}$ if n is even, and the negative rational number $-\dfrac{\frac{n+1}{2}}{m+1}$ otherwise. Here we use the surjective Cantor pairing j, defined by $j(n, m) = \dfrac{1}{2}(n + m)(n + m + 1) + m$.

As a consequence, each natural number codes a uniquely determined rational number. An equality $=_\mathbb{Q}$ on the representatives of the rational numbers (i.e. on $\mathbb{N}$) together with operations $+_\mathbb{Q}, -_\mathbb{Q}, \cdot_\mathbb{Q}$ and predicates $<_\mathbb{Q}, \leq_Q$ are defined primitive recursively in a natural way.

In order to express the statement that n represents the rational r, we write $n =_\mathbb{Q} \langle r\rangle$ or simply $n = \langle r\rangle$. Since a rational number r possesses infinitely many representatives, $\langle\cdot\rangle$ is not a function. In fact, rational numbers are equivalence classes on $\mathbb{N}$ with respect to $=_\mathbb{Q}$, but one can avoid formally introducing the set $\mathbb{Q}$ of all these equivalence classes. An alternative is to select a *canonical representative* by defining

$$c : \mathbb{N} \to \mathbb{N}, \quad c(n) :=_0 \min m \leq_0 n[n =_\mathbb{Q} m]. \tag{3.1}$$

Then $c(n)$ is the code of the irreducible fraction representing the rational number encoded by n. It is clear that $c(n) =_\mathbb{Q} n$ and $n =_\mathbb{Q} m \to c(n) =_\mathbb{Q} c(m)$.

$\mathbb{N}$ can be naturally embedded into our representation of $\mathbb{Q}$ via $n \mapsto \langle n\rangle := j(2n, 0)$, $0_\mathbb{Q} := \langle 0\rangle$, $1_\mathbb{Q} := \langle 1\rangle$. Then $(\mathbb{N}, +_\mathbb{Q}, \cdot_\mathbb{Q}, 0_\mathbb{Q}, 1_\mathbb{Q}, <_\mathbb{Q})$ is an ordered field, which represents $(\mathbb{Q}, +, \cdot, 0, 1, <)$ in $\mathcal{A}^\omega$.

Each function $f : \mathbb{N} \to \mathbb{N}$ can be conceived of as an infinite sequence of codes of rationals and therefore as a representative of a sequence of rationals. Real numbers are represented by functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$\forall n \in \mathbb{N}\big(|f(n+1) -_\mathbb{Q} f(n)|_\mathbb{Q} <_\mathbb{Q} 2^{-n}\big) \tag{3.2}$$

For better readability, we usually write 2^{-n} instead of its (canonical) code $\langle 2^{-n}\rangle := j\left(2, 2^n - 1\right)$.

(3.2) implies that for all $m, n, p \in \mathbb{N}$ with $m \geq n$,

$$|f(m+p) -_{\mathbb{Q}} f(m)|_{\mathbb{Q}} \leq_{\mathbb{Q}} \sum_{i=m}^{m+p-1} |f(i+1) -_{\mathbb{Q}} f(i)|_{\mathbb{Q}} \leq_{\mathbb{Q}} \sum_{i=n}^{\infty} |f(i+1) -_{\mathbb{Q}} f(i)|_{\mathbb{Q}} < 2^{-n},$$

hence each f satisfying (3.2) in fact represents a Cauchy sequence of rationals with Cauchy modulus 2^{-n}. In order to guarantee that each function $f : \mathbb{N} \to \mathbb{N}$ codes a real number, we use the following construction:

$$\widehat{f}(n) := \begin{cases} f(n) \text{ if } \forall k < n\big(|f(k+1) -_{\mathbb{Q}} f(k)|_{\mathbb{Q}} <_{\mathbb{Q}} 2^{-k-1}\big), \\ f(k) \text{ for the least } k < n \text{ with } |f(k+1) -_{\mathbb{Q}} f(k)|_{\mathbb{Q}} \geq_{\mathbb{Q}} 2^{-k-1} \quad \text{otherwise.} \end{cases}$$

Then $\widehat{f}$ always satisfies (3.2) and, moreover, if (3.2) is already valid for f, then $\forall n(fn =_0 \widehat{f}n)$. Thus each function $f : \mathbb{N} \to \mathbb{N}$ codes a uniquely determined real number, namely the real number which is given by the Cauchy sequence coded by $\widehat{f}$. The construction $f \mapsto \widehat{f}$ allows us to reduce quantification over $\mathbb{R}$ to $\forall f : \mathbb{N} \to \mathbb{N}$ resp. $\exists f : \mathbb{N} \to \mathbb{N}$ without adding further quantifiers. This also holds for the operations on $\mathbb{R}$ defined below.

On the representatives of real numbers, i.e. on the functions $f_1, f_2 : \mathbb{N} \to \mathbb{N}$, one defines the relations $=_{\mathbb{R}}, <_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$:

$$\begin{aligned} f_1 =_{\mathbb{R}} f_2 &:\equiv \forall n\big(|\widehat{f_1}(n+1) -_{\mathbb{Q}} \widehat{f_2}(n+1)|_{\mathbb{Q}} <_{\mathbb{Q}} 2^{-n}\big), \\ f_1 <_{\mathbb{R}} f_2 &:\equiv \exists n\big(\widehat{f_2}(n+1) -_{\mathbb{Q}} \widehat{f_1}(n+1) \geq_{\mathbb{Q}} 2^{-n}\big), \\ f_1 \leq_{\mathbb{R}} f_2 &:\equiv \neg(f_2 <_{\mathbb{R}} f_1). \end{aligned}$$

Hence, the relations $=_{\mathbb{R}}, \leq_{\mathbb{R}}$ are given by Π^0_1 predicates, while $<_{\mathbb{R}}$ is given by a Σ^0_1 predicate.

The operations $+_{\mathbb{R}}, -_{\mathbb{R}}, \cdot_{\mathbb{R}}$, etc. on representatives of real numbers can be defined by primitive recursive functionals. If $n = \langle r \rangle$ codes the rational number r, then $\lambda k.n$ represents r as a real number. Thus, $0_{\mathbb{R}} := \lambda k.0_{\mathbb{Q}}$, $1_{\mathbb{R}} := \lambda k.1_{\mathbb{Q}}$ and $(2^{-n})_{\mathbb{R}} := \lambda k.j(2.2^n - 1)$; we shall write simply 2^{-n} for $(2^{-n})_{\mathbb{R}}$. $\mathbb{R}$ denotes the set of all equivalence classes on $\mathbb{N}^{\mathbb{N}}$ with respect to $=_{\mathbb{R}}$. As in the case of $\mathbb{Q}$, we use $\mathbb{R}$ only informally and deal exclusively with the representatives and the operations defined on them. One can verify that $\big(\mathbb{N}^{\mathbb{N}}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, 0_{\mathbb{R}}, 1_{\mathbb{R}}, <_{\mathbb{R}}\big)$ is an Archimedean ordered field which represents $(\mathbb{R}, +, \cdot, 0, 1, <)$ in $\mathcal{A}^\omega$.

In the sequel, we need a semantic operator which for any real number $x \in [0, \infty)$ selects out of all the representatives $f : \mathbb{N} \to \mathbb{N}$ of x a unique representative $(x)_\circ$ satisfying some "nice" properties. For any $x \in [0, \infty)$, $(x)_\circ : \mathbb{N} \to \mathbb{N}$ is defined by

$$(x)_0(n) := j(2k_0, 2^{n+1} - 1), \quad \text{where } \; k_0 := \max k \left[\frac{k}{2^{n+1}} \leq x\right]. \tag{3.3}$$

Lemma 3.0.5. *[134, Lemma 17.8] Let $x \in [0, \infty)$. Then*

(i) $(x)_\circ$ is a representative of x, so $\widehat{(x)_\circ} =_{\mathbb{N}\to\mathbb{N}} (x)_\circ$;

(ii) if $x, y \in [0, \infty)$ and $x \leq y$, then $(x)_\circ \leq_{\mathbb{R}} (y)_\circ$ and $(x)_\circ \leq_{\mathbb{N}\to\mathbb{N}} (y)_\circ$, i.e. $\forall n \in N\big((x)_\circ(n) \leq_{\mathbb{N}} (y)_\circ(n)\big)$;

(iii) $(x)_\circ$ *is monotone, that is* $\forall n \in N\big((x)_\circ(n) \leq_{\mathbb{N}} (x)_\circ(n+1)\big)$.

Since the interval $[0,1]$ will play a very important role in the theory of W-hyperbolic spaces, we use for it a special representation by number theoretic functions $\mathbb{N} \to \mathbb{N}$. For every $\lambda : \mathbb{N} \to \mathbb{N}$, let us define

$$\tilde{\lambda} := \lambda n.j(2k_0, 2^{n+2}-1), \quad \text{where} \;\; k_0 = \max k \leq 2^{n+2}\left[\frac{k}{2^{n+2}} \leq_{\mathbb{Q}} \widehat{\lambda}(n+2)\right] \tag{3.4}$$

($k_0 := 0$ if no such k exists; recall that $j(2k_0, 2^{n+2}-1)$ encodes the rational number $k_0/2^{n+2}$).

It is easy to verify the following properties.

Lemma 3.0.6. *[134, Lemma 4.25] Provably in* $\mathcal{A}^\omega$*, for all* $\lambda, \theta : \mathbb{N} \to \mathbb{N}$*:*

(i) $0_{\mathbb{R}} \leq_{\mathbb{R}} \lambda \leq_{\mathbb{R}} 1_{\mathbb{R}} \to \tilde{\lambda} =_{\mathbb{R}} \lambda$, $\;\lambda >_{\mathbb{R}} 1_{\mathbb{R}} \to \tilde{\lambda} =_{\mathbb{R}} 1_{\mathbb{R}}$ *and* $\lambda <_{\mathbb{R}} 0_{\mathbb{R}} \to \tilde{\lambda} =_{\mathbb{R}} 0_{\mathbb{R}}$*,*

(ii) $0_{\mathbb{R}} \leq_{\mathbb{R}} \tilde{\lambda} \leq_{\mathbb{R}} 1_{\mathbb{R}}$*,*

(iii) $\lambda =_{\mathbb{R}} \theta \to \widehat{\lambda} =_{\mathbb{R}} \widehat{\theta}$*,*

(iv) $\tilde{\lambda} \leq_1 M := \lambda n.j(2^{n+3}, 2^{n+2}-1)$*.*

3.1 Metatheorems for metric and W-hyperbolic spaces

In order to be able to talk about arbitrary metric spaces, we axiomatically add general metric spaces (X,d) to our system $\mathcal{A}^\omega$, resulting in a theory $\mathcal{A}^\omega[X,d]_{-b}$ which is based on two ground types $\mathbb{N}, X$ rather than only $\mathbb{N}$. Hence, the theory $\mathcal{A}^\omega[X,d]_{-b}$ for abstract metric spaces is an extension of $\mathcal{A}^\omega$ defined as follows:

(i) extend $\mathbf{T}$ to the set $\mathbf{T}^X$ of all finite types over the ground types $\mathbb{N}$ and X, that is:
$$\mathbb{N}, X \in \mathbf{T}^X \quad \text{and} \quad \rho, \tau \in \mathbf{T}^X \;\Rightarrow\; \rho \to \tau \in \mathbf{T}^X;$$

(ii) extend all the axioms and rules of $\mathcal{A}^\omega$ to the new set of types $\mathbf{T}^X$;

(iii) add a constant 0_X of type X;

(iv) add a new constant d_X of type $X \to X \to \mathbb{N}^{\mathbb{N}}$ together with the axioms

(M1) $\forall x^X \left(d_X(x,x) =_{\mathbb{R}} 0_{\mathbb{R}}\right)$,
(M2) $\forall x^X, y^X \left(d_X(x,y) =_{\mathbb{R}} d_X(y,x)\right)$,
(M3) $\forall x^X, y^X, z^X \left(d_X(x,z) \leq_{\mathbb{R}} d_X(x,y) +_{\mathbb{R}} d_X(y,z)\right)$.

We use the subscript $_{-b}$ here and for the theories defined in the sequel in order to be consistent with the notations from [134].

Equality $=_X$ between objects of type X is defined by $x =_X y :\equiv d_X(x,y) =_{\mathbb{R}} 0_{\mathbb{R}}$ and equality for complex types is defined as before as extensional equality using $=_{\mathbb{N}}$ and $=_X$ for the base cases. The new axioms (M1)-(M3) of $\mathcal{A}^\omega[X,d]_{-b}$ express that d_X represents a pseudo-metric d on the domain the variables of type X are ranging over.

Thus, d_X represents a metric on the set of equivalence classes generated by $=_X$. We do not form these classes explicitly, but talk instead about representatives x^X, y^X. As a consequence, we have to keep in mind that a functional $f^{X\to X}$ represents a function $X \to X$ only if it respects this equivalence relation, i.e. $\forall x^X, y^X \big(x =_X y \to f(x) =_X f(y)\big)$. However, the mathematical properties of the functions considered in applications of proof mining usually imply their full extensionality.

The theory $\mathcal{A}^\omega[X,d,W]_{-b}$ for W-hyperbolic spaces results from $\mathcal{A}^\omega[X,d]_{-b}$ by adding a new constant W_X of type $X \to X \to \mathbb{N}^\mathbb{N} \to X$ together with the axioms:

$$\forall x^X, y^X, z^X\, \forall \lambda^{\mathbb{N}\to\mathbb{N}} \Big(d_X(z, W_X(x,y,\lambda)) \le_\mathbb{R} (1_\mathbb{R} -_\mathbb{R} \tilde{\lambda}) \cdot_\mathbb{R} d_X(z,x) +_\mathbb{R} \tilde{\lambda} \cdot_\mathbb{R} d_X(z,y)\Big),$$

$$\forall x^X, y^X\, \forall \lambda_1^{\mathbb{N}\to\mathbb{N}}, \lambda_2^{\mathbb{N}\to\mathbb{N}} \Big(d_X(W_X(x,y,\lambda_1), W_X(x,y,\lambda_2)) =_\mathbb{R} |\tilde{\lambda_1} -_\mathbb{R} \tilde{\lambda_2}|_\mathbb{R} \cdot_\mathbb{R} d_X(x,y)\Big),$$

$$\forall x^X, y^X\, \forall \lambda^{\mathbb{N}\to\mathbb{N}} \Big(W_X(x,y,\lambda) =_X W_X(y,x,1_\mathbb{R} -_\mathbb{R} \lambda)\Big),$$

$$\forall x^X, y^X, z^X, w^W\, \forall \lambda^{\mathbb{N}\to\mathbb{N}} \Big(d_X(W_X(x,z,\lambda), W_X(y,w,\lambda)) \le_\mathbb{R} (1_\mathbb{R} -_\mathbb{R} \tilde{\lambda}) \cdot_\mathbb{R} d_X(x,y) +_\mathbb{R} \tilde{\lambda} \cdot_\mathbb{R} d_X(z,w)\Big).$$

In the above axioms, $\tilde{\lambda}$ is defined by (3.4).

Definition 3.1.1. *Let X be a nonempty set. The* full-theoretic type structure *$S^{\omega,X} := \langle S_\rho \rangle_{\rho \in \mathbf{T}^X}$ over $\mathbb{N}$ and X is defined as follows:*

$$S_\mathbb{N} := \mathbb{N}, \quad S_X := X \quad \textit{and} \quad S_{\rho\to\tau} := S_\tau^{S_\rho},$$

where $S_\tau^{S_\rho}$ is the set of all set-theoretic functions $S_\rho \to S_\tau$.

Let (X,d) be a metric space. $S^{\omega,X}$ becomes a model of $\mathcal{A}^\omega[X,d]_{-b}$ by letting the variables of type ρ range over S_ρ, giving the natural interpretations to the constants of $\mathcal{A}^\omega$, interpreting 0_X by an arbitrary element in X and $d_X(x,y)$ (for $x,y \in X$) by $(d(x,y))_\circ$, where $(\cdot)_\circ$ refers to (3.3).

If, moreover, (X,d,W) is a W-hyperbolic space, then $S^{\omega,X}$ becomes a model of $\mathcal{A}^\omega[X,d,W]_{-b}$ if we interpret $W_X(x,y,\lambda)$ (for $x,y \in X, \lambda : \mathbb{N} \to \mathbb{N}$) as $W(x,y,r_{\tilde{\lambda}})$ where $r_{\tilde{\lambda}}$ is the uniquely determined real number in $[0,1]$ which is represented by $\tilde{\lambda}$.

Definition 3.1.2. *We say that a sentence in the language $\mathcal{L}(\mathcal{A}^\omega[X,d]_{-b})$ holds in a nonempty metric space (X,d) if it holds in the models of $\mathcal{A}^\omega[X,d]_{-b}$ obtained from $S^{\omega,X}$ as specified above.*
The notion that a sentence in $\mathcal{L}(\mathcal{A}^\omega[X,d,W]_{-b})$ holds in a nonempty W-hyperbolic space is defined similarly.

From now on, in order to improve readability, we shall usually omit the subscripts $_{\mathbb{N}}, _{\mathbb{R}}, _{\mathbb{Q}}, _X$ excepting the cases where such an omission could create confusions. We shall write, for example, $x \in X, T : X \to X$ instead of $x^X, T^{X\to X}$ and sometimes $x \in \rho$ instead of x^ρ.

The notion of *majorizability* was originally introduced by Howard [102], and subsequently modified by Bezem [18]. For any type $\rho \in \mathbf{T}^X$, we define the type

$\widehat{\rho} \in \mathbf{T}$, which is the result of replacing all occurrences of the type X in ρ by $\mathbb{N}$. Based on Bezem's notion of *strong majorizability s-maj* [18], Gerhardy and Kohlenbach [76] defined a parametrized *a-majorization* relation $\gtrsim^a_\rho$ between objects of type $\rho \in \mathbf{T}^X$ and their majorants of type $\widehat{\rho} \in \mathbf{T}$, where the parameter a of type X serves as a reference point for comparing and majorizing elements of X:

(i) $x^* \gtrsim^a_{\mathbb{N}} x :\equiv x^* \geq x$ for $x, x^* \in \mathbb{N}$

(ii) $x^* \gtrsim^a_X x :\equiv (x^*)_{\mathbb{R}} \geq_{\mathbb{R}} d(x, a)$ for $x^* \in \mathbb{N}, x \in X$,

(iii) $x^* \gtrsim^a_{\rho \to \tau} x :\equiv \forall y^*, y(y^* \gtrsim^a_\rho y \to x^*y^* \gtrsim^a_\tau xy) \wedge \forall z^*, z(z^* \gtrsim^a_{\hat{\rho}} z \to x^*z^* \gtrsim^a_{\hat{\tau}} x^*z)$.

Restricted to the types $\mathbf{T}$ the relation $\gtrsim^a$ is identical with Bezems's strong majorizability *s-maj* and, hence, for $\rho \in \mathbf{T}$ we write *s-maj*$_\rho$ instead of $\gtrsim^a_\rho$, since in this case the parameter a is irrelevant.

If $t^* \gtrsim^a t$ for terms t^*, t, we say that t^* *a-majorizes* t or that t^* is an *a-majorant* of t. A term t is said to be *a-majorizable* if it has an a-majorant and t is said to be *majorizable* if it is a-majorizable for some $a \in X$. Since it can be shown that if a term t is *a-majorizable* for some $a \in X$, then this is true for all $a \in X$ [134, Lemma 17.78], it follows that t is majorizable if and only if it is a-majorizable for each $a \in X$. Although the question whether or not a certain term is a-majorizable is independent from the particular choice of $a \in X$, the complexity and possible uniformities of the majorants may depend crucially on that choice. If t^* a-majorizes t and does not depend on a, then we say that t^* *uniformly* a-majorizes t. We will in general look for uniform majorants so as to produce uniform bounds.

Lemma 3.1.3. *Let $T : X \to X$. The following are equivalent.*

(i) T is majorizable;

(ii) for all $x \in \mathbb{N}$ there exists $\Omega : \mathbb{N} \to \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, y \in X\Big(d(x, y) < n \to d(x, Ty) \leq \Omega(n)\Big) \tag{3.5}$$

(iii) for all $x \in \mathbb{N}$ there exists $\Omega : \mathbb{N} \to \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, y \in X\Big(d(x, y) \leq n \to d(x, Ty) \leq \Omega(n)\Big) \tag{3.6}$$

Proof. T is majorizable if and only if T is x-majorizable for each $x \in X$ if and only if for each $x \in X$ there exists a function $T^* : \mathbb{N} \to \mathbb{N}$ such that T^* is monotone and satisfies

$$\forall n \in \mathbb{N}\, \forall y \in X\Big(d(x, y) \leq n \to d(x, Ty) \leq T^*n\Big).$$

$(i) \Rightarrow (iii)$ is obvious: take $\Omega := T^*$. For the implication $(iii) \Rightarrow (i)$, given, for $x \in X$, Ω satisfying (3.6), define $T^*n := \max\limits_{k \leq n} \Omega(k)$.

$(iii) \Rightarrow (ii)$ is again obvious. For the converse implication, given Ω satisfying (3.5) define $\tilde{\Omega}(n) := \Omega(n + 1)$. Then $\tilde{\Omega}$ satisfies (3.6) □

In the sequel, given a majorizable function $T : X \to X$, an Ω satisfying (3.6) will be called a *modulus of majorizability at x* of T; we say also that T is *x-majorizable with modulus* Ω. We gave in the lemma above the equivalent condition (3.5) for logical reasons: since $<_{\mathbb{R}}$ is a Σ^0_1 predicate and $\leq_{\mathbb{R}}$ is a Π^0_1 predicate, the formula in (3.5) is (equivalent to) a universal sentence.

The following lemma shows that natural classes of mappings in metric or W-hyperbolic spaces are majorizable with a very "nice" modulus; its proof is implicit in the proof of [134, Corollary 17.55].

Lemma 3.1.4. *Let (X, d) be a metric space.*

(i) If (X, d) is bounded with diameter d_X, then any function $T : \mathbb{N} \to \mathbb{N}$ is majorizable with modulus of majorizabiliy $\Omega(n) := \lceil d_X \rceil$ for each $x \in X$.

(ii) If $T : X \to X$ is L-Lipschitz, then T is majorizable with modulus at x given by $\Omega(n) := n + L^*b$, where $b, L^* \in \mathbb{N}$ are such that $d(x, Tx) \leq b$ and $L \leq L^*$. In particular, any nonexpansive mapping is majorizable with modulus $\Omega(n) := n + b$.

(iii) If (X, d, W) is a W-hyperbolic space, then and uniformly continuous mapping $T : X \to X$ is majorizable with modulus $\Omega(n) := n \cdot 2^{\alpha_T(0)} + 1 + b$ at x, where $d(x, Tx) \leq b \in \mathbb{N}$ and α_T is a modulus of uniform continuity of T, i.e. $\alpha_T : \mathbb{N} \to \mathbb{N}$ satisfies

$$\forall x, y \in X\, \forall k \in \mathbb{N} \left(d(x, y) \leq 2^{-\alpha_T(k)} \to d(Tx, Ty) \leq 2^{-k} \right).$$

Before stating the main logical metatheorem, let us give a couple of definitions. Let $\rho \in \mathbf{T}^X$ be a type. We say that

(i) ρ has degree $(0, X)$ if $\rho = X$ or $\rho = \mathbb{N} \to \ldots \to \mathbb{N} \to X$;

(ii) ρ is of degree $(1, X)$ if $\rho = X$ or has the form $\rho = \rho_1 \to \ldots \to \rho_n \to X$, where $n \geq 1$ and each ρ_i has degree ≤ 1 or $(0, X)$.

(iii) ρ has degree 1^* if $deg(\widehat{\rho}) \leq 1$.

A formula A is called a *$\forall$-formula* (resp. *$\exists$-formula*) if it has the form

$$A \equiv \forall \underline{x}^{\underline{\sigma}} A_0(\underline{x}, \underline{a}) \quad (\text{resp. } A \equiv \exists \underline{x}^{\underline{\sigma}} A_0(\underline{x}, \underline{a})),$$

where A_0 is a quantifier free formula and the types in $\underline{\sigma}$ are of degree 1^* or $(1, X)$.

We assume in the following that the constant 0_X does not occur in the formulas we consider. This is no restriction, since 0_X is just an arbitrary constant which could have been replaced by any new variable of type X that, by taking universal closure, would just add another input that had to be a-majorized. Whenever we write $A(\underline{x})$, we mean that A is a formula in our language which has only the variables $\underline{x}$ free.

Very general metatheorems were proved first by Kohlenbach [129] for bounded metric (W-hyperbolic) spaces, and then generalized to the unbounded case by Gerhardy and Kohlenbach [76]. In the following we give a simplified version of these metatheorems, specially designed for concrete applications in mathematics.

Theorem 3.1.5. *(see [134, Corollary 17.54])*
Let P be $\mathbb{N}$ or $\mathbb{N}^{\mathbb{N}}$, K be an $\mathcal{A}^\omega$-definable compact metric space, ρ be of degree 1^*, $B_\forall(u,y,z,n)$ be a $\forall$-formula and $C_\exists(u,y,z,N)$ be a $\exists$-formula.

Assume that $\mathcal{A}^\omega[X,d]_{-b}$ proves that

$$\forall u \in P \forall y \in K \forall z^\rho \Big(\forall n \in \mathbb{N}\, B_\forall \to \exists N \in \mathbb{N}\, C_\exists \Big). \tag{3.7}$$

Then one can extract a computable functional $\Phi : P \times \mathbb{N}^{(\mathbb{N}\times \ldots \times \mathbb{N})} \to \mathbb{N}$ such that the following holds in all nonempty metric spaces (X,d):

for all $z \in S_\rho$, $z^* \in \mathbb{N}^{(\mathbb{N}\times\ldots\times\mathbb{N})}$, if there exists $a \in X$ such that $z^* \gtrsim^a_\rho z$, then

$$\forall u \in P \forall y \in K \Big(\forall n \le \Phi(u,z^*)\, B_\forall \to \exists N \le \Phi(u,z^*)\, C_\exists \Big).$$

Remark 3.1.6. *(i) The above theorem holds for $\mathcal{A}^\omega[X,d,W]_{-b}$ and nonempty W-hyperbolic spaces (X,d,W) too.*

(ii) Instead of single variables u,y,n and single premises $\forall n B_\forall(u,y,z,n)$ we may have tuples $\underline{u} \in P, \underline{y} \in K, \underline{n} \in \mathbb{N}$ of variables and finite conjunctions of premises. Moreover, we can have also $\underline{z}^{\underline{\rho}} = z_1^{\rho_1}, \ldots z_k^{\rho_k}$ as long as all the types $\rho_1, \ldots, \rho_k$ are of degree 1^* and in the conclusion is assumed that $z_i^* \gtrsim^a_{\rho_i} z_i$ for a common *$a \in X$ for all $i = 1, \ldots, k$. Furthermore, the bound Φ depends now on all the a-majorants $z_1^*, \ldots, z_k^*$.*

Remark 3.1.7. *The theory $\mathcal{A}^\omega[X, \|\cdot\|]$ of normed spaces and $\mathcal{A}^\omega[X, \|\cdot\|, \eta]$ corresponding to uniformly convex normed spaces were defined by Kohlenbach [129] and similar logical metatheorems were obtained for these theories too. We refer to [129] or to [134, Section 17.3] for details.*

The proof of the logical metatheorem is based on an extension to $\mathcal{A}^\omega[X,d]_{-b}$, resp. $\mathcal{A}^\omega[X,d,W]_{-b}$, of Spector's [223] interpretation of classical analysis $\mathcal{A}^\omega$ by *bar-recursive* functionals followed by an interpretation of these functionals in an extension of Bezem's [18] type structure of hereditarily strongly majorizable functionals to all types $\mathbf{T}^X$, based on the a-majorization relation $\gtrsim^a$, parametrized by $a \in X$. Spector's work generalizes Gödel's well-known *functional interpretation* [87] for intuitionistic and - via Gödel's *double-negation* interpretation [86] as intermediate step - classical arithmetic to classical analysis. We refer to [133] for a recent survey on applied aspects of functional interpretation and to [175] for a book treatment of Spector's bar recursion.

Moreover, the proof of the metatheorem actually provides an extraction algorithm for the functional Φ, which can always be defined in the calculus of bar-recursive functionals. However, as we shall see in thesis, for concrete applications usually small fragments of $\mathcal{A}^\omega[X,d,W]_{-b}$ or $\mathcal{A}^\omega[X,d]_{-b}$ (corresponding to fragments of $\mathcal{A}^\omega$) are needed to formalize the proof. In particular, it follows from results of Kohlenbach [124, 125] that a single use of sequential compactness (over a sufficiently weak base system) only gives rise to at most primitive recursive complexity in the sense of Kleene, often only simple exponential complexity. This corresponds to the complexity of the bounds obtained in our applications from this thesis.

In these applications, one actually is interested in the extraction of bounds which, in order to be useful, should be *uniform*, i.e. independent from various parameters. This can be achieved by using Kohlenbach's *monotone* functional interpretation, introduced in [125] (see [134, Chapter 9] for details), that systematically transforms any statement in a given proof into a new version for which explicit bounds are provided. In recent years, other "bounds-oriented" variants of functional interpretation were defined, as *bounded* functional interpretation introduced by Ferreira and Oliva [65, 66] or the very recent Shoenfield-like bounded functional interpretation of Ferreira [64], that gives a direct interpretation of classical theories and so could be suitable for proof mining.

We give now a very useful corollary of Theorem 3.1.5.

Corollary 3.1.8. *(see [134, Corollary 17.54])*
Let P be $\mathbb{N}$ or $\mathbb{N}^{\mathbb{N}}$, K be a $\mathcal{A}^\omega$-definable compact metric space, $B_\forall(\underline{u}, \underline{y}, x, x^*, T, n)$ be a $\forall$-formula and $C_\exists(\underline{u}, \underline{y}, x, x^*, T, N)$ a $\exists$-formula. Assume that $\mathcal{A}^\omega[X, d, W]_{-b}$ proves that

$$\forall \underline{u} \in P\, \forall \Omega : \mathbb{N} \to \mathbb{N}\, \forall \underline{y} \in K\, \forall x, x^* \in X\, \forall T : X \to X \\ \Big(T \text{ is } x\text{-majorizable with modulus } \Omega \wedge \forall n \in \mathbb{N}\, B_\forall \to \exists N \in \mathbb{N}\, C_\exists\Big).$$

Then one can extract a computable functional Φ such that for all $b \in \mathbb{N}$,

$$\forall \underline{u} \in P\, \forall \Omega : \mathbb{N} \to \mathbb{N}\, \forall \underline{y} \in K\, \forall x, x^* \in X\, \forall T : X \to X \\ \Big(T \text{ is } x\text{-majorizable with modulus } \Omega \wedge d(x, x^*) \leq b \wedge \forall n \leq \Phi(\underline{u}, b, \Omega)\, B_\forall \\ \to \exists N \leq \Phi(\underline{u}, b, \Omega)\, C_\exists\Big).$$

holds in all nonempty W-hyperbolic spaces (X, d, W).

Proof. The premise "T x-majorizable with modulus Ω" is a $\forall$-formula, by (3.5). Furthermore, 0 x-majorizes x, b is a x-majorant for x^*, since $d(x, x^*) \leq b$, and $T^* := \lambda n. \max\limits_{k \leq n} \Omega(k)$ x-majorizes T, by the proof of Lemma 3.1.3. Apply now Theorem 3.1.5 $\square$

Remark 3.1.9. *As in the case of Theorem 3.1.5, instead of single $n \in \mathbb{N}$ and a single premise $\forall n B_\forall$ we could have tuples $\underline{n} = n_1, \ldots, n_k$ and a conjunction of premises $\forall n_1 B^1_\forall \wedge \ldots \wedge \forall n_k B^k_\forall$. In this case, in the conclusion we shall have in the premise $\forall n_1 \leq \Phi\, B^1_\forall \wedge \ldots \wedge \forall n_k \leq \Phi\, B^k_\forall$.*

Corollary 3.1.8 will be used for our first application in metric fixed point theory, a quantitative version of Borwein-Reich-Shafrir Theorem (see Section 4.3). In fact, a simplified version of it suffices for this application, namely for T nonexpansive. In this case, as we have seen in Lemma 3.1.4, a modulus of majorizability at x is given by $\Omega(n) = n + b$, where $b \geq d(x, Tx)$, so the bound Φ will depend only on the parameters $\underline{u} \in P$ and $b \in \mathbb{N}$ such that $d(x, Tx), d(x, x^*) \le b$.

A remarkable feature of the (proof of the) above logical metatheorem is the fact the same results hold true for extensions of the theories $\mathcal{A}^\omega[X,d]_{-b}, \mathcal{A}^\omega[X,d,W]_{-b}$ obtained as follows:

(i) the theory may be extended by new axioms that have the form of $\forall$-sentences;

(ii) the language may be extended by new majorizable constants, in particular constants of type $\mathbb{N}$ or $\mathbb{N}^{\mathbb{N}}$ which are uniformly majorizable. In this case, the extracted bounds then additionally depend on a-majorants for the new constants.

Then the conclusion holds in all metric spaces (X,d), resp. W-hyperbolic spaces (X,d,W), satisfying these axioms (under a suitable interpretation of the new constants if any).

We shall exemplify this with three classes of spaces discussed in Section 2.2: Gromov hyperbolic spaces, $CAT(0)$ spaces and $\mathbb{R}$-trees.

The theory of δ-hyperbolic spaces, $\mathcal{A}^\omega[X,d,\delta\text{-hyperbolic}]_{-b}$ is an extension of $\mathcal{A}^\omega[X,d]_{-b}$ defined as follows:

(i) add a constant $\delta_{\mathbb{R}}$ of type $\mathbb{N} \to \mathbb{N}$ (representing the nonnegative real δ);

(ii) add the axioms: $\delta_{\mathbb{R}} \ge_{\mathbb{R}} 0_{\mathbb{R}}$ and

$$\forall x,y,z,w \in X \left(d_X(x,y) +_{\mathbb{R}} d_X(z,w) \le_{\mathbb{R}} \max{}_{\mathbb{R}}\{d_X(x,z) +_{\mathbb{R}} d_X(y,w), d_X(x,w) +_{\mathbb{R}} d_X(y,z)\} \right.$$
$$\left. +_{\mathbb{R}} 2 \cdot_{\mathbb{R}} \delta_R \right).$$

The notion that a sentence of $\mathcal{L}(\mathcal{A}^\omega[X,d,\delta\text{-hyperbolic}]_{-b})$ holds in a nonempty δ-hyperbolic space (X,d) is defined as in Definition 3.1.2, by interpreting the new constant $\delta_{\mathbb{R}}$ as $(\delta)_0$.

Since $\le_{\mathbb{R}}$ is Π^0_1, the two axioms are $\forall$-sentences. Thus, in order to adapt Theorem 3.1.5 to the theory of Gromov δ-hyperbolic spaces, we need to show that the new constant $\delta_{\mathbb{R}}$ is strongly majorizable. It is easy to see that if (X,d) is a δ-hyperbolic space, and $k \in \mathbb{N}$ is such that $k \ge \delta$, then

$$\delta^*_{\mathbb{R}} := \lambda n.j(k \cdot 2^{n+2}, 2^{n+1} - 1) \text{ s-maj}_1 (\delta)_\circ.$$

Theorem 3.1.10. *Theorem 3.1.5 holds also for the theory $\mathcal{A}^\omega[X,d,\delta\text{-hyperbolic}]_{-b}$ and nonempty Gromov δ-hyperbolic spaces (X,d), with the bound Φ depending additionally on $k \in \mathbb{N}$ such that $k \ge \delta$.*

Let us consider the case of $CAT(0)$ spaces. As we have seen in Subsection 2.2, we can define the theory $\mathcal{A}^\omega[X,d,W,CAT(0)]_{-b}$ for $CAT(0)$ spaces by adding to $\mathcal{A}^\omega[X,d,W]_{-b}$ the formalized form of the CN^- inequality, which is a $\forall$-sentence.

$$\forall x,y,z \in X \left(d_X\left(z, W_X\left(x,y,\frac{1}{2}\right)\right)^2 \le_{\mathbb{R}} \frac{1}{2} d_X(z,x)^2 +_{\mathbb{R}} \frac{1}{2} d_X(z,y)^2 -_{\mathbb{R}} \frac{1}{4} d_X(x,y)^2 \right).$$

Theorem 3.1.11. *Theorem 3.1.5 holds also for the theory $\mathcal{A}^\omega[X, d, W, CAT(0)]_{-b}$ and nonempty $CAT(0)$ spaces.*

Following Proposition 2.2.10, the theory $\mathcal{A}^\omega[X, d, W, \mathbb{R}\text{-tree}]_{-b}$ of $\mathbb{R}$-trees results from the theory $\mathcal{A}^\omega[X, d, W]_{-b}$ by adding a $\forall$-axiom:

$$\forall x, y, z, w \in X \Bigg(d_X(x, y) +_{\mathbb{R}} d_X(z, w) \leq_{\mathbb{R}} \\ \max_{\mathbb{R}}\{d_X(x, z) +_{\mathbb{R}} d_X(y, w), d_X(x, w) +_{\mathbb{R}} d_X(y, z)\} \Bigg).$$

As a consequence

Theorem 3.1.12. *Theorem 3.1.5 holds also for $\mathcal{A}^\omega[X, d, W, R\text{-tree}]_{-b}$ and nonempty $\mathbb{R}$-trees.*

3.2 Logical metatheorems for UCW-hyperbolic spaces

In the sequel, we shall see that the logical metatheorem from the previous section can be easily adapted to UCW-hyperbolic spaces [163].

The theory $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$, corresponding to the class of UCW-hyperbolic spaces is obtained from $\mathcal{A}^\omega[X, d, W]_{-b}$ by adding a constant $\eta_X : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ together with axioms

$$\forall r, k \in \mathbb{N} \forall x, y, a \in X \Bigg(d_X(x, a) <_{\mathbb{R}} r \wedge d_X(y, a) <_{\mathbb{R}} r \wedge \\ d_X(W_X(x, y, 1/2), a) >_{\mathbb{R}} \left(1 -_{\mathbb{R}} 2^{-\eta_X(r,k)}\right) \cdot_{\mathbb{R}} r \to d_X(x, y) \leq_{\mathbb{R}} 2^{-k} \cdot_{\mathbb{R}} r \Bigg), \\ \forall r_1, r_2, k \in \mathbb{N}\big(r_1 \leq_{\mathbb{Q}} r_2 \to \eta(r_1, k) \geq_0 \eta(r_2, k)\big), \\ \forall r, k \in \mathbb{N}\big(\eta_X(r, k) =_0 \eta_X(c(r), k)\big).$$

The first two axioms express the fact that $\eta_X : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ represents a monotone modulus of uniform continuity. The meaning of the third axiom is that η_X is a function having the first argument a rational number on the level of codes; c is the canonical representation for rational numbers defined by (3.1). It is easy to see, using the representation of real numbers in $\mathcal{A}^\omega$, that all the three axioms are universal. Moreover, the constant η_X of degree 1 is majorizable.

The notion that a sentence of $\mathcal{L}(\mathcal{A}^\omega[X, d, UCW, \eta]_{-b})$ holds in a nonempty UCW-hyperbolic space (X, d, W) with monotone modulus of uniform convexity η is defined as above, by interpreting the new constant η_X as $\eta_X(r, k) := \eta(c(r), k)$.

Since $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ results from $\mathcal{A}^\omega[X, d, W]_{-b}$ by adding a majorizable constant and three $\forall$-axioms, we get that the logical metatheorem and its corollaries hold in this setting too.

Theorem 3.2.1. *[163] Theorem 3.1.5 holds for $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ and nonempty UCW-hyperbolic spaces (X, d, W) with monotone modulus of uniform convexity η, with the bound Φ depending additionally on η.*

Corollary 3.2.2. *Corollary 3.1.8 holds also for for the theory* $\mathcal{A}^\omega[X,d,UCW,\eta]_{-b}$ *and nonempty* UCW*-hyperbolic spaces* (X,d,W) *with monotone modulus of uniform convexity* η*, with the bound* Φ *depending additionally on* η*.*

Corollary 3.2.3. *Let* P *be* $\mathbb{N}$ *or* $\mathbb{N}^\mathbb{N}$*,* K *be a* $\mathcal{A}^\omega$*-definable compact metric space,* $B_\forall(\underline{u},\underline{y},x,T,n)$ *be a* $\forall$*-formula and* $C_\exists(\underline{u},\underline{y},x,T,N)$ *a* $\exists$*-formula. Assume that* $\mathcal{A}^\omega[X,d,UCW,\eta]_{-b}$ *proves that*

$$\forall\,\underline{u}\in P\,\forall\,\Omega:\mathbb{N}\to\mathbb{N}\,\forall\,\underline{y}\in K\,\forall\,x\in X\,\forall\,T:X\to X$$
$$\Big(T\,x\text{-majorizable with modulus }\Omega\,\wedge\,Fix(T)\neq\emptyset\wedge\forall n\in\mathbb{N}\,B_\forall\,\to\,\exists N\in\mathbb{N}\,C_\exists\Big).$$

Then one can extract a computable functional Φ *such that for all* $b\in\mathbb{N}$*,*

$$\forall\,\underline{u}\in P\,\forall\,\Omega:\mathbb{N}\to\mathbb{N}\,\forall\,\underline{y}\in K\,\forall\,x\in X\,\forall\,T:X\to X$$
$$\Big(T\,x\text{-majorizable with modulus }\Omega\,\wedge\,\forall\delta>0\big(Fix_\delta(T,x,b)\neq\emptyset\big)\wedge$$
$$\forall n\leq\Phi(\underline{u},b,\eta,\Omega)B_\forall\,\to\,\exists N\leq\Phi(\underline{u},b,\eta,\Omega)\,C_\exists\Big).$$

holds in any nonempty UCW*-hyperbolic space* (X,d,W) *with monotone modulus of uniform convexity* η*. We recall that*

$$Fix_\delta(T,x,b):=\{y\in X\mid d(y,x)\leq b\text{ and }d(y,Ty)<\delta\}.$$

Proof. The statement proved in $\mathcal{A}^\omega[X,d,UCW,\eta]_{-b}$ can be written as

$$\forall\,\underline{u}\in P\,\forall\,\Omega:\mathbb{N}\to\mathbb{N}\,\forall\,\underline{y}\in K\,\forall\,x,p\in X\,\forall\,T:X\to X$$
$$\Big(Tx\text{-maj. mod.}\Omega\,\wedge\,\forall k\in\mathbb{N}\left(d(p,Tp)\leq 2^{-k}\right)\wedge\forall n\in\mathbb{N}\,B_\forall\,\to\,\exists N\in\mathbb{N}\,C_\exists\Big).$$

We have used the fact that $Fix(T)\neq\emptyset$ is equivalent with $\exists p\in X(Tp=_X p)$ that is further equivalent with $\exists p\in X\,\forall\,k\in\mathbb{N}\big(d(p,Tp)\leq 2^{-k}\big)$, by using the definition of $=_X$ and $=_\mathbb{R}$ in our system. As all the premises are $\forall$-formulas, we can apply Corollary 3.2.2 to extract a functional Φ such that for all $b\in\mathbb{N}$,

$$\forall\,\underline{u}\in P\,\forall\,\Omega:\mathbb{N}\to\mathbb{N}\,\forall\,\underline{y}\in K\,\forall\,x,p\in X\,\forall\,T:X\to X$$
$$\Big(T\,x\text{-maj. mod. }\Omega\,\wedge\,d(x,p)\leq b\,\wedge\,\forall k\leq\Phi(\underline{u},b,\eta,\Omega)\big(d(p,Tp)\leq 2^{-k}\big)$$
$$\wedge\,\forall n\leq\Phi(\underline{u},b,\eta,\Omega)\,B_\forall\,\to\,\exists N\leq\Phi(\underline{u},b,\eta,\Omega)\,C_\exists\Big),$$

that is

$$\forall\,\underline{u}\in P\,\forall\,\Omega:\mathbb{N}\to\mathbb{N}\,\forall\,\underline{y}\in K\,\forall\,x\in X\,\forall\,T:X\to X$$
$$\Big(T\,x\text{-maj. mod. }\Omega\,\wedge\,\exists p\in X\Big(d(x,p)\leq b\,\wedge\,\forall k\leq\Phi(\underline{u},b,\eta,\Omega)\big(d(p,Tp)\leq 2^{-k}\big)\Big)$$
$$\wedge\,\forall n\leq\Phi(\underline{u},b,\eta,\Omega)\,B_\forall\,\to\,\exists N\leq\Phi(\underline{u},b,\eta,\Omega)\,C_\exists\Big).$$

Use the fact that the existence of $p\in X$ such that $d(x,p)\leq b$ and $\forall k\leq\Phi\big(d(p,Tp)\leq 2^{-k}\big)$ is equivalent with the existence of $p\in X$ such that $d(x,p)\leq b$ and $d(p,Tp)\leq 2^{-\Phi}$ which is obviously implied by $\forall\delta>0\,(Fix_\delta(T,x,b)\neq\emptyset)$. $\square$

We shall apply the above corollary twice. The first application will be in Section 4.4 for nonexpansive mappings T. As we have already discussed, if T is nonexpansive, then its modulus of majorizability at x is simply $\Omega(n) = n + \tilde{b}$ with $\tilde{b} \geq d(x, Tx)$.

For all $\delta > 0$ there exists $y \in X$ such that $Fix_\delta(T, x, b) \neq \emptyset$, hence

$$d(x, Tx) \leq d(x, y) + d(y, Ty) + d(Ty, Tx) \leq 2d(x, y) + d(y, Ty) \leq 2b + \delta$$

for all $\delta > 0$. It follows that $d(x, Tx) \leq 2b$, so we can take $\tilde{b} := 2b$. As a consequence, the bound Φ will depend only on $\underline{u}, b$ and η.

The second application will be in Chapter 7, this time for asymptotically nonexpansive mappings, introduced by Goebel and Kirk [80]. An asymptotically nonexpansive mapping $T : X \to X$ with sequence (k_n) is a $(1 + K)$-Lipschitz mapping, where $K \in \mathbb{N}$ is such that $k_1 \leq K$. By Lemma 3.1.4, we get that T is majorizable with modulus at x given by $\Omega(n) := n + (1 + K)\tilde{b}$, where again $\tilde{b} \geq d(x, Tx)$. Reasoning as above, it is easy to see that if b is such that $Fix_\delta(T, x, b) \neq \emptyset$ for all $\delta > 0$, then we can take $\tilde{b} := (2 + K)b$. Thus, the bound Φ depends on $\underline{u}, b, \eta$ and on $K \in \mathbb{N}$ with $K \geq k_1$.

Part I

Proof mining in nonlinear analysis

Chapter 4

Effective results on the Krasnoselski-Mann iterations

A fundamental theorem in the fixed point theory of nonexpansive mappings is the following result due to Krasnoselski, which shows that, under an additional compactness condition, a fixed point of T can be approximated by a special iteration technique.

Theorem 4.0.4. *[152] Let C be a closed convex subset of a uniformly convex Banach space X, T be a nonexpansive mapping, and suppose that $T(C)$ is contained into a compact subset of C. Then for every $x \in C$, the sequence (x_n) defined by*

$$x_0 := x, \quad x_{n+1} := \frac{1}{2}(x_n + Tx_n) \tag{4.1}$$

converges to a fixed point of T.

Schaefer [213] remarked that Krasnoselski Theorem holds for iterations of the form

$$x_0 := x, \quad x_{n+1} := (1-\lambda)x_n + \lambda Tx_n, \tag{4.2}$$

where $\lambda \in (0,1)$. Moreover, Edelstein [57] proved that strict convexity of X suffices. The iteration (4.2) is today known as the *Krasnoselski iteration*.

For any $\lambda \in (0,1)$, the *averaged mapping* T_λ is defined by

$$T_\lambda : C \to C, \quad T_\lambda(x) = (1-\lambda)x + \lambda Tx.$$

It is easy to see that T_λ is also nonexpansive and that $Fix(T) = Fix(T_\lambda)$. Moreover, the Krasnoselski iteration (x_n) starting with $x \in C$ is the Picard iteration $\big(T_\lambda^n(x)\big)$ of T_λ.

A vast extension of Krasnoselski Theorem was obtained by Ishikawa in his seminal paper [105]. He showed that Krasnoselski Theorem holds without the assumption of X being uniformly convex and for much more general iterations, defined as follows:

$$x_0 := x, \quad x_{n+1} := (1-\lambda_n)x_n + \lambda_n Tx_n, \tag{4.3}$$

where (λ_n) is a sequence in $[0,1]$ and $x \in C$ is the starting point. This iteration is a special case of the generalized iteration method introduced by Mann [179]. Following [21], we call the iteration (4.3) the *Krasnoselski-Mann iteration*. We remark that it is often said to be a *segmenting Mann iteration* [187, 91, 96].

Theorem 4.0.5. *[105] Let C be a closed convex subset of a Banach space X, T be a nonexpansive mapping, and suppose that $T(C)$ is contained into a compact subset of C. Assume that (λ_n) is a sequence in $[0, 1]$, divergent in sum and bounded away from 1.*

Then for every $x \in C$, the Krasnoselski-Mann iteration converges to a fixed point of T.

Independently, Edelstein and O'Brien [60] obtained a similar result for constant $\lambda_n = \lambda \in (0, 1)$.

The question whether we obtain strong convergence of the Krasnoselski-Mann iterations if the assumption that $T(C)$ is contained into a compact subset of C is exchanged for nicer behaviour of X is very natural. The answer to this question is no, and it was given by Genel and Lindenstrauss [72]. They constructed an example of a bounded closed convex subset C in the Hilbert space ℓ_2 and a nonexpansive mapping $T : C \to C$ with the property that even the original Krasnoselski iteration (4.1) fails to converge to a fixed point of T for some $x \in C$.

A classical weak convergence result is the following theorem due to Reich [196].

Theorem 4.0.6. *Let C be a closed convex subset of a uniformly convex Banach space X with a Fréchet differentiable norm and $T : C \to C$ a nonexpansive mapping with a fixed point. Assume that (λ_n) is a sequence in $[0, 1]$ satisfying the following condition*

$$\sum_{k=0}^{\infty} \lambda_k(1 - \lambda_k) = \infty. \tag{4.4}$$

Then for every $x \in C$, the Krasnoselski-Mann iteration converges weakly to a fixed point of T.

We end this short presentation of Krasnoselski-Mann iterations by emphasizing that a wide variety of iterative procedures used in signal processing and image reconstruction and elsewhere are special cases of the Krasnoselski-Mann iterative procedure, for particular choices of the nonexpansive mapping T. We refer to [42, 12] for nice surveys.

4.1 Asymptotic regularity

Asymptotic regularity is a fundamentally important concept in metric fixed-point theory. Asymptotic regularity was already implicit in [152, 213, 57], but it was formally introduced by Browder and Petryshyn in [33]. A mapping T of a metric space (X, d) into itself is said to be asymptotically regular if for all $x \in C$,

$$\lim_{n\to\infty} d(T^n(x), T^{n+1}(x)) = 0.$$

Let X be a Banach space, $C \subseteq X$ and $T : C \to C$. Then the asymptotic regularity of the averaged mapping $T_\lambda := (1 - \lambda)I + \lambda T$ is equivalent with the fact that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ for all $x \in C$, where (x_n) is the Krasnoselski iteration (4.2).

Following [21], we say that the nonexpansive mapping T is λ_n-*asymptotically regular* (for general $\lambda_n \in [0,1]$) if for all $x \in C$,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

where (x_n) is the general Krasnoselski-Mann iteration (4.3).

The most general assumptions on the sequence (λ_n) for which asymptotic regularity has been proved for arbitrary normed spaces are the following, made in Ishikawa's paper [105]:

$$\sum_{n=0}^{\infty} \lambda_n = \infty \quad \text{and} \quad \limsup \lambda_n < 1. \tag{4.5}$$

Note that if $\lambda_n \in [a,b]$ for all $n \in \mathbb{N}$ and $0 < a \le b < 1$, then (λ_n) satisfies (4.5).

Ishikawa proved the following result, which was the intermediate step in obtaining Theorem 4.0.5.

Theorem 4.1.1. *[105] Let C be a convex subset of a Banach space X and $T : C \to C$ be a nonexpansive mapping. Assume that (λ_n) satisfies (4.5). If (x_n) is bounded for some $x \in C$, then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Thus, if C is bounded, T is λ_n-asymptotically regular.*

As observed in [21], we obtain asymptotic regularity under the weaker assumption that C contains a point x with the property that the Krasnoselski-Mann iteration (x_n) starting with x is bounded. In fact, it is easy to see that if for some $x \in C$, the Krasnoselski-Mann iteration (x_n) starting with x is bounded, then this is true for all $x \in C$.

Theorem 4.1.2. *Let C be a convex subset of a Banach space X and $T : C \to C$ a nonexpansive mapping. Assume that (λ_n) satisfies (4.5) and that (x_n) is bounded for some (each) $x \in C$.*

Then T is λ_n-asymptotically regular.

Using an embedding theorem due to Banach and Mazur [8], Edelstein and O'Brien [60] also proved the asymptotic regularity for constant $\lambda_n = \lambda \in (0,1)$, and noted that it is uniform for $x \in C$. In [81], Goebel and Kirk unified Ishikawa's and Edelstein/O'Brien's results, obtaining uniformity with respect to x and to the family of all nonexpansive mappings $T : C \to C$.

Theorem 4.1.3. *[81] Let C be a bounded convex subset of a Banach space X and (λ_n) satisfying (4.5). Then for every $\varepsilon > 0$ there exists a positive integer N such that for all $x \in C$ and all $T : C \to C$ nonexpansive,*

$$\forall n \ge N\big(\|x_n - Tx_n\| < \varepsilon\big). \tag{4.6}$$

In 2000, Kirk [113] generalized Theorems 4.1.3 and 4.1.2 to directionally nonexpansive mappings, but only for constant $\lambda_n = \lambda \in (0,1)$. A mapping $T : C \to C$ is said to be *directionally nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x \in C$ and all $y \in seg[x, Tx]$.

Theorem 4.1.4. *[113] Let C be a convex subset of a Banach space X, $T : C \to C$ be directionally nonexpansive and $\lambda \in (0,1)$.*

(i) If (x_n) is bounded for each $x \in C$, then the averaged mapping T_λ is asymptotically regular.

(ii) If C is bounded, then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $x \in C$ and all $T : C \to C$ directionally nonexpansive,

$$\forall n \geq N\big(\|T_\lambda^{n+1}(x) - T_\lambda^n(x)\| < \varepsilon\big). \tag{4.7}$$

A very important result is the following theorem due to Borwein, Reich and Shafrir, extending Ishikawa Theorem 4.1.2 to unbounded C.

Theorem 4.1.5. *[21] Let C be a closed convex subset of a Banach space X and $T : C \to C$ a nonexpansive mapping. Assume that (λ_n) satisfies (4.5). Then for all $x \in C$,*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = r_C(T), \tag{4.8}$$

where $r_C(T)$ is the minimal displacement of T, defined by (2.21).

Thus, convergence of $(\|x_n - Tx_n\|)$ towards $r_C(T)$ is obtained for (λ_n) divergent in sum and bounded away from 1, while in [203, 205] (λ_n) was required also to be bounded away from 0. In this way, the case of Cesaro and other summability methods is covered [55, 91, 179].

As an immediate consequence of Borwein-Reich-Shafrir Theorem, we get that any approximately fixed nonexpansive mapping is λ_n-asymptotically regular for (λ_n) satisfying (4.5). A straightforward application of Theorems 4.1.2 and 4.1.5 is the fact that $r_C(T) = 0$ whenever (x_n) is bounded for some (each) $x \in C$, in particular for bounded C. Let us remark that for unbounded C, $r_C(T)$ can be very well strict positive: for example, if $T : \mathbb{R} \to \mathbb{R}$, $Tx = x + 1$, then $r_{\mathbb{R}}(T) = 1$ although T is nonexpansive.

In [7], it is conjectured that Ishikawa's Theorem 4.1.1 holds true if (4.5) is replaced by the weaker condition (4.4), which is symmetric in $\lambda_n, 1-\lambda_n$. For the case of uniformly convex Banach spaces, this has been proved by Groetsch [91] (see also [196]).

Theorem 4.1.6. *Let C be a convex subset of a uniformly convex Banach space and $T : C \to C$ be a nonexpansive mapping such that T has at least one fixed point. Assume that (λ_n) satisfies the following condition:*

$$\sum_{k=0}^{\infty} \lambda_k(1-\lambda_k) = \infty. \tag{4.9}$$

Then T is λ_n-asymptotically regular.

4.2 Krasnoselski-Mann iterations in W-hyperbolic spaces

In the sequel, (X, d, W) is a W-hyperbolic space, $C \subseteq X$ a convex subset of X, and $T : C \to C$ a nonexpansive mapping.

As in the case of normed spaces, we can define the *Krasnoselski-Mann iteration* starting from $x \in C$ by

$$x_0 := x, \quad x_{n+1} := (1-\lambda_n)x_n \oplus \lambda_n T x_n, \tag{4.10}$$

where (λ_n) is a sequence in $[0,1]$. For constant $\lambda_n = \lambda \in (0,1)$, we get the *Krasnoselski iteration*, which can be also defined as the Picard iteration $\big(T_\lambda^n(x)\big)$ of

$$T_\lambda : C \to C, \quad T_\lambda(x) = (1-\lambda)x \oplus \lambda T x.$$

The *averaged mapping* T_λ is also nonexpansive and $Fix(T) = Fix(T_\lambda)$.

The following proposition collects some useful properties of Krasnoselski-Mann iterates in W-hyperbolic spaces. We refer to [142, 166] for the proofs.

Proposition 4.2.1. *Let $(x_n), (x_n^*)$ be the Krasnoselski-Mann iterations starting with $x, x^* \in C$. Then*

(i) $(d(x_n, x_n^*))$ is nonincreasing;

(ii) $(d(x_n, Tx_n))$ is nonincreasing;

(iii) $(d(x_n, p))$ is nonincreasing for any fixed point p of T.

The following very useful result was proved in [81] for *spaces of hyperbolic type*, thus holds for W-hyperbolic spaces too.

Theorem 4.2.2. *Let (X,d,W) be a W-hyperbolic space and (λ_n) be a sequence in $[0,1]$ which is divergent in sum and bounded away from 1. Assume that $(u_n), (v_n)$ are sequences in X satisfying for all $n \in \mathbb{N}$,*

$$u_{n+1} = (1-\lambda_n)u_n \oplus \lambda_n v_n \quad \text{and} \quad d(v_n, v_{n+1}) \le d(u_n, u_{n+1}). \tag{4.11}$$

Then $(d(u_n, v_n))$ is nonincreasing and $\lim\limits_{n\to\infty} d(u_n, v_n) = 0$ whenever (u_n) is bounded.

As an immediate consequence of the above theorem, we get the generalization of Theorem 4.1.2 to W-hyperbolic spaces.

Theorem 4.2.3. *Let C be a convex subset of a W-hyperbolic space (X,d,W) and $T : C \to C$ a nonexpansive mapping. Assume that (λ_n) is divergent in sum and bounded away from 1.*

If there exists $x^* \in C$ such that (x_n^*) is bounded, then T is λ_n-asymptotically regular, that is $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$ for all $x \in C$.

Proof. Since $(d(x_n, x_n^*))$ is nonincreasing, we get that (x_n) is bounded for all $x \in C$. Apply now Theorem 4.2.2 with $u_n := x_n$ and $v_n := Tx_n$. $\square$

It follows that for bounded convex C, any nonexpansive self-mapping of C is approximately fixed.

Corollary 4.2.4. *Bounded convex subsets of W-hyperbolic spaces have the AFPP for nonexpansive mappings.*

We finish this section by remarking that Goebel-Kirk Theorem 4.1.3 is valid in spaces of hyperbolic type, hence in W-hyperbolic spaces too.

4.3 A quantitative version of Borwein-Reich-Shafrir Theorem

Our first application of proof mining is an effective quantitative version of the generalization to W-hyperbolic spaces of the Borwein-Reich-Shafrir Theorem 4.1.5. Let us recall it.

Theorem 4.3.1. *[21] Let C be a convex subset of a W-hyperbolic space (X, d, W) and $T : C \to C$ a nonexpansive mapping. Assume that (λ_n) is divergent in sum and bounded away from 1.*

Then for all $x \in C$,

$$\lim d(x_n, Tx_n) = r_C(T). \tag{4.12}$$

In the following, we give an explicit quantitative version of the above theorem, generalizing to W-hyperbolic spaces and directionally nonexpansive mappings the logical analysis made by Kohlenbach [127, 128] for normed spaces and nonexpansive functions. Our Theorem 4.3.12 extends Kohlenbach's results (even with the same numerical bounds) to W-hyperbolic spaces and directionally nonexpansive mappings and contains all previously known results of this kind as special cases. In this way, we obtain significantly stronger and much more general forms of Kirk's Theorem 4.1.4 with explicit bounds. As a special feature of our approach, which is based on logical analysis instead of functional analysis, no functional analytic embeddings are needed to obtain our uniformity results.

The main application of the quantitative version of the Borwein-Reich-Shafrir Theorem is a *uniform* effective rate of λ_n-asymptotic regularity in the case of bounded C for general (λ_n) divergent in sum and bounded away from 1 (see Theorem 4.3.12). Thus, the rate of asymptotic regularity is uniform in the nonexpansive mapping $T : C \to C$ and in the starting point $x \in C$ of the Krasnoselski-Mann iteration (x_n) and in the bounded convex subset C (by this we mean that it depends on C only via its diameter d_C).

As we have already discussed in Section 4.1, uniformity in $x \in C$ for Banach spaces and constant $\lambda_n = \lambda$ was first established by Edelstein and O'Brien in [60]. Subsequently, in [81], Goebel and Kirk obtained uniformity in x and T for general (λ_n), but no uniformity in C. In 2000 [113], Kirk established uniformity in x, T for Banach spaces and directionally nonexpansive mappings only in the case of constant $\lambda_n = \lambda$. In 2001 [127], by using methods of proof mining, Kohlenbach obtained for the first time uniformity in x, T, C for nonexpansive mappings and general (λ_n) in the case of Banach spaces with explicit rates of asymptotic regularity.

None of the papers [105, 60, 81, 21, 113] contain any bounds and in fact [60, 81, 113] use non-trivial functional theoretic embeddings to get the uniformities. Kirk and Martinez-Yanez [116, p.191] explicitly mention the non-effectivity of all these results and state that *"it seems unlikely that such estimates would be easy to obtain in a general setting"* and, therefore, only study the *tractable* case of uniformly convex Banach spaces.

Not even the ineffective existence of bounds uniform in C was known for general (λ_n) and still in 1990, Goebel and Kirk conjecture [82, p. 101] as *"unlikely" to be true.* Only for Banach spaces and constant $\lambda_n = \lambda$, uniformity with respect to C

has been established by Baillon and Bruck in [7], where for this special case an optimal quadratic bound was obtained.

4.3.1 Logical discussion

The proof of Theorem 4.3.1 is prima facie ineffective and does not provide any rate of convergence of $(d(x_n, Tx_n))$. Moreover, its statement does not have the required logical form for the logical metatheorems from Chapter 3 to apply, due to the two implicative assumptions on (λ_n) and, more seriously, to the existence of $r_C(T)$, which can not be formed in the theory $\mathcal{A}^\omega[X, d, W]_{-b}$ of W-hyperbolic spaces.

However, we show in the sequel that it can be reformulated in such a way that the logical metatheorems apply (more precisely Corollary 3.1.8). Firstly, let us remark that any convex subset C of a W-hyperbolic space is also a W-hyperbolic space, so it suffices to consider only the case $C = X$, and hence only nonexpansive functions $T : X \to X$. For simplicity, we shall denote $r_X(T)$ with $r(T)$.

Let us consider the conclusion of Theorem 4.3.1.

Proposition 4.3.2. *The following are equivalent for all $x \in X$.*

(i) $\lim\limits_{n\to\infty} d(x_n, Tx_n) = r(T)$;

(ii) $\forall \varepsilon > 0\, \exists N \in \mathbb{N}\, \forall m \geq N \big(d(x_m, Tx_m) < r(T) + \varepsilon\big)$;

(iii) $\forall \varepsilon > 0\, \exists N \in \mathbb{N}\, \forall m \geq N\, \forall x^* \in X \big(d(x_m, Tx_m) < d(x^*, Tx^*) + \varepsilon\big)$;

(iv) $\forall \varepsilon > 0\, \exists N \in \mathbb{N}\, \forall x^* \in X \big(d(x_N, Tx_N) < d(x^*, Tx^*) + \varepsilon\big)$;

(v) $\forall \varepsilon > 0\, \forall x^* \in X\, \exists N \in \mathbb{N} \big(d(x_N, Tx_N) < d(x^*, Tx^*) + \varepsilon\big)$.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ are obvious, by the definition of $r(T)$.
$(iii) \Leftrightarrow (iv)$ follows immediately from the fact that $(d(x_n, Tx_n))$ is nonincreasing, hence the quantifier $\forall m \geq N$ in (iii) is superfluous.
$(iv) \Rightarrow (v)$ is obvious, so it remains to prove $(v) \Rightarrow (iv)$ Since $r(T) = \inf\{d(x^*, Tx^*) : x^* \in X\}$, there exists $y^* \in X$ such that $d(y^*, Ty^*) < r(T) + \varepsilon/2$. Applying (v) with $\varepsilon/2$ and y^*, we get $N \in \mathbb{N}$ such that $d(x_N, Tx_N) < d(y^*, Ty^*) + \varepsilon/2 < r(T) + \varepsilon \leq d(x^*, Tx^*) + \varepsilon$ for all $x^* \in X$. Thus, (iv) is satisfied with this N. $\square$

Thus, the conclusion $\forall x \in X \left(\lim\limits_{n\to\infty} d(x_n, Tx_n) = r(T)\right)$ of Borwein-Reich-Shafrir Theorem can be reformulated as

$$\forall x \in X\, \forall \varepsilon > 0\, \forall x^* \in X\, \exists N \in \mathbb{N} \bigg(d(x_N, Tx_N) < d(x^*, Tx^*) + \varepsilon \bigg),$$

that has the $\forall\exists$-form required by the logical metatheorems.

Let us now examine the hypotheses on (λ_n): $\limsup \lambda_n < 1$ and $\sum\limits_{n=0}^{\infty} \lambda_n = \infty$.

The first one, $\limsup \lambda_n < 1$, states the existence of a $K \in \mathbb{N}^*$ such that $\lambda_n \leq 1 - \dfrac{1}{K}$ for all n from some index N_0 on. Since N_0 only contributes an additive constant to our bound, we may assume for simplicity that $N_0 = 0$, which is anyway the case if (λ_n) is a sequence in $[0,1)$. Hence, we may replace the hypothesis $\limsup \lambda_n < 1$ with

$$\exists K \in \mathbb{N}\, \forall n \in \mathbb{N} \left(\lambda_n \leq 1 - \frac{1}{K}\right). \tag{4.13}$$

The second one, $\sum\limits_{n=0}^{\infty} \lambda_n = \infty$, is (ineffectively, using countable axiom of choice) equivalent with

$$\exists\, \theta : \mathbb{N} \to \mathbb{N}\, \forall n \in \mathbb{N} \left(\sum_{s=0}^{\theta(n)} \lambda_s \geq n \right), \tag{4.14}$$

that is with the existence of a rate of divergence $\theta : \mathbb{N} \to \mathbb{N}$.

It is easy to see that $\mathcal{A}^\omega[X,d,W]_{-b}$ proves the following formalized version of Theorem 4.3.1:

$$\begin{aligned}
&\forall\, (\lambda_n) \in [0,1]^{\mathbb{N}}\, \forall x \in X\, \forall\, T : X \to X \\
&\Bigg(T \text{ n.e. } \wedge \exists K \in \mathbb{N}\, \forall n \in \mathbb{N} \left(\lambda_n \leq 1 - \frac{1}{K}\right) \wedge \exists\, \theta : \mathbb{N} \to \mathbb{N}\, \forall n \in \mathbb{N} \left(\sum_{s=0}^{\theta(n)} \lambda_s \geq n \right) \\
&\qquad\qquad \to \forall \varepsilon > 0\, \forall x^* \in X\, \exists N \in \mathbb{N} \Big(d(x_N, Tx_N) < d(x^*, Tx^*) + \varepsilon \Big) \Bigg),
\end{aligned}$$

hence,

$$\begin{aligned}
&\forall\, K \in \mathbb{N}\, \forall \varepsilon > 0\, \forall \theta : \mathbb{N} \to \mathbb{N}\, \forall\, (\lambda_n) \in [0,1]^{\mathbb{N}}\, \forall x, x^* \in X\, \forall T : X \to X \\
&\qquad \Bigg(T \text{ n.e. } \wedge \forall n \in \mathbb{N} \left(\lambda_n \leq 1 - \frac{1}{K}\right) \wedge \forall n \in \mathbb{N} \left(\sum_{s=0}^{\theta(n)} \lambda_s \geq n \right) \\
&\qquad\qquad \to \exists N \in \mathbb{N} \Big(d(x_N, Tx_N) < d(x^*, Tx^*) + \varepsilon \Big) \Bigg),
\end{aligned}$$

The Hilbert cube $[0,1]^{\mathbb{N}}$ is a compact metric space which is $\mathcal{A}^\omega$-definable and we can let $\varepsilon = 2^{-p}$ with $p \in \mathbb{N}$, hence the above formalization of the statement of Borwein-Reich-Shafrir Theorem has the required logical form.

Corollary 3.1.8 yields the existence of a computable functional Φ such that for all $b \in \mathbb{N}$,

$$\begin{aligned}
&\forall\, K \in \mathbb{N}\, \forall \varepsilon > 0\, \forall \theta : \mathbb{N} \to \mathbb{N}\, \forall (\lambda_n) \in [0,1]^{\mathbb{N}}\, \forall x, x^* \in X\, \forall T : X \to X \\
&\Bigg(T \text{ n.e. } \wedge d(x, Tx) \leq b \wedge d(x, x^*) \leq b \wedge \forall n \in \mathbb{N} \left(\lambda_n \leq 1 - \frac{1}{K}\right) \wedge \\
&\forall n \in \mathbb{N} \left(\sum_{s=0}^{\theta(n)} \lambda_s \geq n \right) \to \exists N \leq \Phi(\varepsilon, b, K, \theta) \Big(d(x_N, Tx_N) < d(x^*, Tx^*) + \varepsilon \Big) \Bigg).
\end{aligned}$$

holds in any W-hyperbolic space (X,d,W); "n.e." abbreviates "nonexpansive". Using again the fact that $(d(x_n, Tx_n))$ is nonincreasing, we get in fact that

$$\forall n \geq \Phi(\varepsilon, b, K, \theta) \Big(d(x_n, Tx_n) < d(x^*, Tx^*) + \varepsilon \Big).$$

In fact, a slight reformulation of (4.14) is better suited for the proof of our theorem.

Lemma 4.3.3. *The following are equivalent:*

(i) there exists $\theta : \mathbb{N} \to \mathbb{N}$ *such that* $\sum_{s=0}^{\theta(n)} \lambda_s \geq n$ *for all* $n \in \mathbb{N}$*;*

(ii) there exists $\gamma : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *such that* $\sum_{s=i}^{i+\gamma(i,n)-1} \lambda_s \geq n$ *for all* $n, i \in \mathbb{N}$*;*

(iii) there exists $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *such that for all* $n, i \in \mathbb{N}$,

$$\alpha(i,n) \leq \alpha(i+1,n) \quad \text{and} \quad \sum_{s=i}^{i+\alpha(i,n)-1} \lambda_s \geq n. \tag{4.15}$$

Proof. $(i) \Rightarrow (ii)$ Define $\gamma(i,n) = \theta(n+i) - i + 1 \geq 0$, since $n + i \leq \sum_{s=0}^{\theta(n+i)} \lambda_s \leq \theta(n+i) + 1$. Furthermore,

$$\sum_{s=i}^{i+\gamma(i,n)-1} \lambda_s = \sum_{s=i}^{\theta(n+i)} \lambda_s = \sum_{s=0}^{\theta(n+i)} \lambda_s - \sum_{s=0}^{i-1} \lambda_s \geq n + i - i = n, \quad \text{as} \sum_{s=0}^{i-1} \lambda_s \leq i.$$

$(ii) \Rightarrow (iii)$ Define $\alpha(i,n) = \max_{j \leq i}\{\gamma(j,n)\}$. Then α is increasing in i, $\alpha(i,n) \geq \gamma(i,n)$, so $\sum_{s=i}^{i+\alpha(i,n)-1} \lambda_s \geq \sum_{s=i}^{i+\gamma(i,n)-1} \lambda_s \geq n$.

$(iii) \Rightarrow (i)$ Applying (iii) with $i = 0$, we get that $n \leq \sum_{s=0}^{\alpha(0,n)-1} \lambda_s \leq \alpha(0,n)$, so $\alpha(0,n) - 1 \geq n - 1 \geq 0$ for all $n \geq 1$. We can define then $\theta(n) = \alpha(0,n) - 1$ for $n \geq 1$ and $\theta(0)$ arbitrary. $\square$

Hence, Corollary 3.1.8 guarantees the extractability of a computable functional Φ such that for all $b \in \mathbb{N}$,

$$\begin{aligned}
&\forall K \in \mathbb{N} \,\forall \varepsilon > 0 \,\forall \alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \,\forall (\lambda_n) \in [0,1]^{\mathbb{N}} \,\forall x, x^* \in X \,\forall T : X \to X \\
&\Bigg(T \text{ n.e.} \wedge d(x,Tx) \leq b \wedge d(x,x^*) \leq b \wedge \forall n \in \mathbb{N} \left(\lambda_n \leq 1 - \frac{1}{K}\right) \wedge \\
&\alpha \text{ satisfies (4.15)} \to \forall n \geq \Phi(\varepsilon, b, K, \alpha)\Big(d(x_n, Tx_n) < d(x^*, Tx^*) + \varepsilon\Big)\Bigg).
\end{aligned}$$

An explicit such bound Φ has been extracted by Kohlenbach and the author in [142] and will be given in the next subsection.

4.3.2 Main results

We present now the quantitative version of the Borwein-Reich-Shafrir Theorem.

Theorem 4.3.4. *[142] Let $K \in \mathbb{N}, K \geq 1$, $\alpha : \mathbb{N} \times \mathbb{N} \to N$ and $b > 0$. Then for all W-hyperbolic spaces (X, d, W), for all convex subsets $C \subseteq X$, for all sequences (λ_n) in $[0, 1 - 1/K]$ satisfying*

$$\forall\, i, n \in \mathbb{N} \left((\alpha(i,n) \leq \alpha(i+1,n)) \quad \text{and} \quad n \leq \sum_{s=i}^{i+\alpha(i,n)-1} \lambda_s \right), \tag{4.16}$$

for all $x, x^* \in C$ and for all nonexpansive mappings $T : C \to C$ such that

$$d(x, Tx) \leq b \quad \text{and} \quad d(x, x^*) \leq b, \tag{4.17}$$

the following holds

$$\forall \varepsilon > 0 \forall n \geq \Phi(\varepsilon, b, K, \alpha) \Big(d(x_n, Tx_n) < d(x^*, Tx^*) + \varepsilon \Big), \tag{4.18}$$

where $\Phi(\varepsilon, b, K, \alpha) = \widehat{\alpha}(\lceil 2b \cdot \exp(K(M+1)) \rceil \dot{-} 1, M)$, with

$$n \dot{-} 1 = \max\{0, n-1\}, \quad M = \left\lceil \frac{1+2b}{\varepsilon} \right\rceil, \quad \widehat{\alpha}, \tilde{\alpha} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$
$$\widehat{\alpha}(0,n) = \tilde{\alpha}(0,n), \quad \widehat{\alpha}(i+1,n) = \tilde{\alpha}(\widehat{\alpha}(i,n),n), \quad \tilde{\alpha}(i,n) = i + \alpha(i,n).$$

Remark 4.3.5. *As we have seen in Lemma 4.3.3, we could have started with a rate of divergence $\theta : \mathbb{N} \to \mathbb{N}$ for $\sum_{n=0}^{\infty} \lambda_n$ and then define $\alpha(i,n) = \max_{j \leq i} \{\theta(n+j) - j + 1\}$. Starting with θ would in general give less good bounds than when working with α directly, as it can be seen from [142, Remark 3.19].*

The above theorem was proved for normed spaces and nonexpansive mappings by Kohlenbach [127]. For W-hyperbolic spaces, it was obtained by Kohlenbach and the author in [142] as a consequence of an extension to the more general class of directionally nonexpansive mappings

As we have seen in Section 4.1, the notion of directionally nonexpansive mapping was introduced by Kirk [113] in the context of normed spaces, but W-hyperbolic spaces in our sense suffice:

$T : C \to C$ is *directionally nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x \in C$ and all $y \in [x, Tx]$.

Obviously, any nonexpansive mapping is directionally nonexpansive, but the converse fails as directionally nonexpansive mappings not even need to be continuous on the whole space, as it can be seen from the following example.

Example 4.3.6. (simplified by Paulo Oliva): Consider the normed space $(\mathbb{R}^2, \|\cdot\|_{\max})$ and the mapping

$$T : [0,1]^2 \to [0,1]^2, \; T(x,y) = \begin{cases} (1,y), & \text{if } y > 0 \\ (0,y), & \text{if } y = 0. \end{cases}$$

Clearly, T is directionally nonexpansive, but discontinuous at $(0,0)$, hence T is not nonexpansive.

Since $x_{n+1} \in [x_n, Tx_n]$, we have that $d(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1})$ for directionally nonexpansive mappings too, so we can apply Goebel-Kirk Theorem 4.2.2 to get that $(d(x_n, Tx_n))$ is nonincreasing and to obtain the following generalization of Ishikawa Theorem 4.1.1.

Theorem 4.3.7. *Let C be a convex subset of a W-hyperbolic space (X, d, W) and $T : C \to C$ a directionally nonexpansive mapping. Assume that (λ_n) is divergent in sum and bounded away from 1.*

If there exists $x \in C$ such that (x_n) is bounded, then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ for all $x \in C$.

As a consequence

Corollary 4.3.8. *Bounded convex subsets of W-hyperbolic spaces have the AFPP for directionally nonexpansive mappings.*

We remark that in the case of directionally nonexpansive mappings, the sequence $(d(x_n, x_n^*))$ is not necessarily nonincreasing, so we do not have an analogue of Theorem 4.2.3.

The following is the main result of [142].

Theorem 4.3.9. *Theorem 4.3.4 holds for directionally nonexpansive mappings too, if the hypothesis $d(x, x^*) \leq b$ is strengthened to $d(x_n, x_n^*) \leq b$ for all $n \in \mathbb{N}$.*

As we have already remarked, $(d(x_n, x_n^*))$ is not necessarily nonincreasing for directionally nonexpansive mappings and that's why we need the stronger assumption that $d(x_n, x_n^*) \leq b$ for all $n \in \mathbb{N}$, which is equivalent to $d(x, x^*) \leq b$ in the nonexpansive case, since $(d(x_n, x_n^*))$ is nonincreasing. Thus, Theorem 4.3.4 is an immediate consequence of Theorem 4.3.9.

Let us note also that as a corollary to Theorem 4.3.9 we get the following (nonquantitative) generalization of Borwein-Reich-Shafrir Theorem to directionally nonexpansive mappings.

Corollary 4.3.10. *Let C be a convex subset of a W-hyperbolic space (X, d, W), $T : C \to C$ be a directionally nonexpansive mapping, and (λ_n) be divergent in sum and bounded away from 1.*
Assume $x \in C$ is such that for all $\varepsilon > 0$ there exists $x^* \in C$ satisfying

$$d(x_n, x_n^*) \text{ is bounded} \quad \text{and} \quad d(x^*, Tx^*) \leq r_C(T) + \varepsilon. \tag{4.19}$$

Then $\lim_{n\to\infty} d(x_n, Tx_n) = r_C(T)$.

Combining Corollaries 4.3.8 and 4.3.10 we get asymptotic regularity for bounded C.

Theorem 4.3.11. *Let C be a bounded convex subset of a W-hyperbolic space (X, d, W) and $T : C \to C$ a directionally nonexpansive mapping. Assume that (λ_n) is divergent in sum and bounded away from 1.*

Then T is λ_n-asymptotically regular.

From Theorem 4.3.9, various strong effective uniformity results for the case of bounded C can be derived, as well as for the more general case of bounded (x_n) for some $x \in C$.

In the case of bounded C with finite diameter d_C, the assumptions $d(x, Tx) \leq d_C$ and $d(x_n, x_n^*) \leq d_C$ hold trivially for all $x, x^* \in C$ and all $n \in \mathbb{N}$. The following result is a consequence of Theorema 4.3.9 and 4.3.11.

Theorem 4.3.12. *Let (X, d, W) be a W-hyperbolic space, $C \subseteq X$ be a* bounded *convex subset with diameter d_C, and $T : C \to C$ be directionally nonexpansive. Assume that $K, \alpha, (\lambda_n)$ are as in the hypothesis of Theorem 4.3.4.*

Then $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$ for all $x \in C$ and, moreover,

$$\forall \varepsilon > 0 \forall n \geq \Phi(\varepsilon, d_C, K, \alpha)\bigg(d(x_n, Tx_n) < \varepsilon\bigg), \tag{4.20}$$

where $\Phi(\varepsilon, d_C, K, \alpha)$ is defined as in Theorem 4.3.4 by replacing b with d_C.

For bounded C, we derive an explicit rate of asymptotic regularity $\Phi(\varepsilon, d_C, K, \alpha)$ depending only on the error ε, on the diameter d_C of C, and on (λ_n) via K and α, but which does not depend on the nonexpansive mapping T, the starting point $x \in C$ of the Krasnoselski-Mann iteration or other data related with C and X.

We can simplify the rate of asymptotic regularity further, if we assume that (λ_n) is a sequence in $[1/K, 1 - 1/K]$. In this case, it is easy to see that

$$\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad \alpha(i, n) = Kn$$

satisfies (4.16).

Corollary 4.3.13. *Let (X, d, W) be a W-hyperbolic space, $C \subseteq X$ be a* bounded *convex subset with diameter d_C, and $T : C \to C$ be directionally nonexpansive. Let $K \in \mathbb{N}, K \geq 2$ and assume that $\lambda_n \in [1/K, 1 - 1/K]$ for all $n \in \mathbb{N}$. Then*

$$\forall \varepsilon > 0 \forall n \geq \Phi(\varepsilon, d_C, K)\bigg(d(x_n, Tx_n) < \varepsilon\bigg), \tag{4.21}$$

where $\Phi(\varepsilon, d_C, K) = K \cdot M \cdot \lceil 2d_C \cdot \exp(K(M + 1))\rceil$, with $M = \left\lceil \dfrac{1 + 2d_C}{\varepsilon} \right\rceil$.

Thus, we obtain an exponential (in $1/\varepsilon$) rate of asymptotic regularity. The above corollary is significantly stronger and more general than Kirk Theorem 4.1.4.(ii).

As another consequence of our quantitative version of Borwein-Reich-Shafrir Theorem, we extend, for the case of nonexpansive mappings, Theorem 4.3.12 to the situation where C no longer is required to be bounded but only the existence of a point $x^* \in C$ whose iteration sequence (x_n^*) is bounded. In this way, we obtain a quantitative version of Theorem 4.2.3. This is of interest, since the functional analytic embedding techniques from [81, 113] seem to require that C is bounded, while our proof is a straightforward generalization of Kohlenbach's proof of the corresponding result for normed spaces [128].

Theorem 4.3.14. *Assume that* $(X,d,W), C, (\lambda_n), K, \alpha$ *are as in the hypothesis of Theorem 4.3.4 and let* $T : C \to C$ *be nonexpansive. Suppose* $x, x^* \in C$ *and* $b > 0$ *satisfy*

$$d(x,x^*) \leq b \quad \text{and} \quad \forall n,m \in \mathbb{N}(d(x^*_n, x^*_m) \leq b). \tag{4.22}$$

Then the following holds

$$\forall \varepsilon > 0 \forall n \geq \Phi(\varepsilon, b, K, \alpha)\Big(d(x_n, Tx_n) < \varepsilon\Big), \tag{4.23}$$

where $\Phi(\varepsilon, b, K, \alpha) = \widehat{\alpha}\left(\lceil 12b \cdot \exp(K(M+1))\rceil \dot{-} 1, M\right)$, *with* $M = \left\lceil \dfrac{1+6b}{\varepsilon} \right\rceil$ *and* $\widehat{\alpha}$ *as in Theorem 4.3.4.*

For the case of directionally nonexpansive mappings, however, the additional assumption in Theorem 4.3.9 causes various problems and significant changes in the proofs. In the following, we will only consider the case where (x_n) itself is bounded (i.e. $x = x^*$).

For any $k \in \mathbb{N}$, we define the sequence $((x_k)_m)_{m\in\mathbb{N}}$ by:

$$(x_k)_0 := x_k, \qquad (x_k)_{m+1} := (1-\lambda_m)(x_k)_m \oplus \lambda_m T((x_k)_m).$$

Hence, for any $k \in \mathbb{N}$, $((x_k)_m)_{m\in\mathbb{N}}$ is the Krasnoselski-Mann iteration starting with x_k. Let us remark that $((x_k)_m)_{m\in\mathbb{N}}$ is not in general a subsequence of (x_n).

The following result is the quantitative version of Theorem 4.3.7.

Theorem 4.3.15. *[142] Let* $(X,d,W), C, (\lambda_n), K, \alpha$ *be as in the hypothesis of Theorem 4.3.4 and* $T : C \to C$ *be directionally nonexpansive.*
Assume that $x \in C, b > 0$ *are such that*

$$\forall\, n,k,m \in \mathbb{N}\,\big(d(x_n, (x_k)_m) \leq b\big). \tag{4.24}$$

Then

$$\forall \varepsilon > 0 \forall n \geq \Phi(\varepsilon, b, K, \alpha)\Big(d(x_n, Tx_n) < \varepsilon\Big), \tag{4.25}$$

where $\Phi(\varepsilon, b, K, \alpha) = \alpha(0,1) + \widehat{\beta}(\lceil 2b \cdot \alpha(0,1) \cdot \exp(K(M+1))\rceil - 1, M)$, *with*

$$M = \left\lceil \frac{1+2b}{\varepsilon} \right\rceil, \quad \beta, \widehat{\beta}, \tilde{\beta} : \mathbb{N}\times\mathbb{N} \to \mathbb{N}, \quad \beta(i,n) = \alpha(i + \alpha(0,1), n)$$
$$\tilde{\beta}(i,n) = i + \beta(i,n), \quad \widehat{\beta}(0,n) = \tilde{\beta}(0,n), \quad \widehat{\beta}(i+1,n) = \tilde{\beta}(\widehat{\beta}(i,n), n).$$

Thus, in the case of directionally nonexpansive mappings, we need the stronger requirement (4.24). Note that for constant $\lambda_n = \lambda$, $(x_k)_m = x_{k+m}$ for all $m, k \in \mathbb{N}$, so $((x_k)_m)_{m\in\mathbb{N}}$ is a subsequence of (x_n). In this case, the assumption $d(x_n, x_m) \leq b$ for all $m, n \in \mathbb{N}$ suffices.

Corollary 4.3.16. *Let* $(X,d,W), C, T, K$ *be as before. Assume that* $\lambda_n = \lambda$ *for all* $n \in \mathbb{N}$, *where* $\lambda \in [1/K, 1-1/K]$. *Let* $x \in C, b > 0$ *be such that* $d(x_n, x_m) \leq b$ *for all* $m, n \in \mathbb{N}$. *Then*

$$\forall \varepsilon > 0 \forall n \geq \Phi(\varepsilon, b, K)\Big(d(x_n, Tx_n) < \varepsilon\Big), \tag{4.26}$$

where $\Phi(\varepsilon, b, K) = K + K \cdot M \cdot \lceil 2b \cdot K \cdot \exp(K(M+1))\rceil$, *with* M *as above.*

Hence, we obtain the a strong uniform version of Kirk Theorem 4.1.4.(i), which does not state any uniformity of the convergence at all.

4.4 A quadratic rate of asymptotic regularity for $CAT(0)$ spaces

If $T : C \to C$ is a nonexpansive self-mapping of a bounded convex subset C of a W-hyperbolic space and (λ_n) is a sequence in $[1/K, 1-1/K]$ for some $K \in \mathbb{N}, K \geq 2$ (in particular, $\lambda_n = \lambda \in (0,1)$), then, as we have seen in the previous section, Corollary 4.3.13 gives an exponential (in $1/\varepsilon$) rate of asymptotic regularity for the Krasnoselski-Mann iteration.

For normed spaces and the special case of constant $\lambda_n = \lambda \in (0,1)$, this exponential bound is not optimal. In this case, a uniform and optimal quadratic bound was obtained by Baillon and Bruck [7] using an extremely complicated computer aided proof, and only for $\lambda_n = 1/2$ a classical proof of a result of this type was given [37]. However, the questions whether the methods of proof used by them hold for non-constant sequences (λ_n) or for W-hyperbolic spaces are left as open problems in [7], and as far as we know they received no positive answer until now. Hence, the bound from Corollary 4.3.13 is the only effective bound known at all for non-constant sequences (λ_n) (even for normed spaces).

Our result guarantees only an exponential rate of asymptotic regularity in the case of $CAT(0)$ spaces, and as we have already remarked, it seems that Baillon and Bruck's approach does not extend to this more general setting.

In this section we show that we can still get a quadratic rate of asymptotic regularity for $CAT(0)$ spaces, but following a completely different approach, inspired by the results on asymptotic regularity obtained before Ishikawa and Edelstein-O'Brien theorems, in the setting of uniformly convex Banach spaces. The method we use is to find explicit uniform bounds on the rate of asymptotic regularity in the general setting of UCW-hyperbolic spaces and then to specialize them to $CAT(0)$ spaces. As we have seen in Chapter 2, $CAT(0)$ spaces are UCW-hyperbolic spaces with a nice modulus of uniform convexity. More specifically, our point of departure is the following theorem due to Groetsch.

Theorem 4.4.1. *[91] Let C be a convex subset of a UCW-hyperbolic space and $T : C \to C$ be a nonexpansive mapping such that T has at least one fixed point. Assume that (λ_n) is a sequence in $[0,1]$ satisfying*

$$\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n) = \infty. \tag{4.27}$$

Then $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$ for all $x \in C$.

The above theorem was proved by Groetsch for uniformly convex Banach spaces (see Theorem 4.1.6), but it is easy to see that its proof extends to UCW-hyperbolic spaces. By proof mining, Kohlenbach [128] obtained a quantitative version of Groetsch Theorem 4.1.6 for uniformly convex Banach spaces, generalizing previous results obtained by Kirk and Martinez-Yanez [116] for constant $\lambda_n = \lambda \in (0,1)$.

In [164], we extend Kohlenbach's results to the more general setting of UCW-hyperbolic spaces. The most important consequence of our results is that for $CAT(0)$ spaces we obtain a quadratic rate of asymptotic regularity (see Corollary 4.4.7).

The following table presents a general picture of the cases where effective bounds for asymptotic regularity were obtained.

	$\lambda_n = \lambda$	non-constant λ_n
Hilbert spaces	*quadratic* in $1/\varepsilon$: Browder and Petryshyn [34]	$\theta\left(1/\varepsilon^2\right)$: Kohlenbach [128]
ℓ_p, $2 \leq p < \infty$	*quadratic* in $1/\varepsilon$: Kirk and Martinez-Yanez[116] Kohlenbach [128]	$\theta\left(1/\varepsilon^p\right)$: Kohlenbach [128]
uniformly convex Banach spaces	Kirk and Martinez-Yanez[116] Kohlenbach [128]	Kohlenbach [128]
Banach	*quadratic* in $1/\varepsilon$: Baillon and Bruck [7]	Kohlenbach [127]
$CAT(0)$ spaces	*quadratic* in $1/\varepsilon$: Corollary 4.4.7	$\theta\left(1/\varepsilon^2\right)$: Corollary 4.4.6
UCW-hyperbolic spaces	Corollary 4.4.5	Corollary 4.4.4
W-hyperbolic spaces	*exponential* in $1/\varepsilon$: Corollary 4.3.13	Theorem 4.3.12

4.4.1 Logical discussion

As in the case of the logical analysis of Borwein-Reich-Shafrir Theorem, it suffices to consider nonexpansive mappings $T : X \to X$. Moreover, it is easy to see that the proof of Groetsch Theorem can be formalized in the theory $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ of UCW-hyperbolic spaces with a monotone modulus of uniform convexity η.

The assumption on (λ_n) in Theorem 4.4.1 is equivalent with the existence of a rate of divergence $\theta : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{\theta(n)} \lambda_i(1-\lambda_i) \geq n.$$

Since $(d(x_n, Tx_n)$ is nonincreasing, it follows that $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ proves the following formalized version of Theorem 4.4.1:

$$\begin{aligned}&\forall \varepsilon > 0 \,\forall \theta : \mathbb{N} \to \mathbb{N} \,\forall (\lambda_n) \in [0,1]^{\mathbb{N}} \,\forall x \in X \,\forall T : X \to X \\ &\Bigg(T \text{ nonexpansive } \wedge Fix(T) \neq \emptyset \wedge \forall n \in \mathbb{N} \left(\sum_{i=0}^{\theta(n)} \lambda_i(1-\lambda_i) \geq n\right) \\ &\qquad\qquad \to \exists N \in \mathbb{N}\Big(d(x_N, Tx_N) < \varepsilon\Big)\Bigg)\end{aligned}$$

Since we can let $\varepsilon = 2^{-p}$ with $p \in \mathbb{N}$, the above formalization of the statement of Theorem 4.4.1 has the required logical form for applying Corollary 3.2.3. It follows that we can extract a computable functional Φ such that for all $\varepsilon > 0, b \in \mathbb{N}, \theta : \mathbb{N} \to \mathbb{N}$,

$$\begin{aligned}&\forall (\lambda_n) \in [0,1]^{\mathbb{N}} \,\forall x \in X \,\forall T : X \to X \\ &\Bigg(T \text{ nonexpansive } \wedge \forall \delta > 0 (Fix_\delta(T, x, b) \neq \emptyset) \wedge \forall n \in \mathbb{N} \left(\sum_{i=0}^{\theta(n)} \lambda_i(1-\lambda_i) \geq n\right) \\ &\qquad\qquad \to \exists N \leq \Phi(\varepsilon, \eta, b, \theta)\Big(d(x_N, Tx_N) < \varepsilon\Big)\Bigg)\end{aligned}$$

holds in any UCW-hyperbolic space with monotone modulus η. We recall that

$$Fix_\delta(T, x, b) = \{y \in X \mid d(x, y) \leq b \wedge d(y, Ty) < \delta\}.$$

Using again that $(d(x_n, Tx_n))$ is nonincreasing, it follows that $\Phi(\varepsilon, \eta, b, \theta)$ is in fact a rate convergence of $(d(x_n, Tx_n))$ towards 0.

4.4.2 Main results

The following quantitative version of Groetsch Theorem is the main result of [164].

Theorem 4.4.2. *Let C be a convex subset of a UCW-hyperbolic space (X, d, W) and $T : C \to C$ be a nonexpansive mapping.*

Assume that (λ_n) is a sequence in $[0,1]$ and $\theta : \mathbb{N} \to \mathbb{N}$ satisfies for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{\theta(n)} \lambda_k(1-\lambda_k) \geq n. \tag{4.28}$$

Let $x \in C, b > 0$ be such that T has approximate fixed points in a b-neighborhood of x.

Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ *and, moreover,*

$$\forall \varepsilon > 0\, \forall n \geq \Phi(\varepsilon, \eta, b, \theta) \left(d(x_n, Tx_n) < \varepsilon \right), \tag{4.29}$$

where η *is a monotone modulus of uniform convexity and*

$$\Phi(\varepsilon, \eta, b, \theta) = \begin{cases} \theta\left(\left\lceil \dfrac{b+1}{\varepsilon \cdot \eta\left(b+1, \dfrac{\varepsilon}{b+1}\right)} \right\rceil\right) & \text{for } \varepsilon < 2b \\ 0 & \text{otherwise.} \end{cases}$$

If we assume moreover that η *can be written as* $\eta(r, \varepsilon) = \varepsilon \cdot \tilde{\eta}(r, \varepsilon)$ *such that* $\tilde{\eta}$ *increases with* ε *(for a fixed* r*), then the bound* $\Phi(\varepsilon, \eta, b, \theta)$ *can be replaced for* $\varepsilon < 2b$ *by*

$$\tilde{\Phi}(\varepsilon, \eta, b, \theta) = \theta\left(\left\lceil \frac{b+1}{2\varepsilon \cdot \tilde{\eta}\left(b+1, \dfrac{\varepsilon}{b+1}\right)} \right\rceil\right).$$

As an immediate consequence of our main theorem, we obtain a slight strengthening of Groetsch Theorem.

Corollary 4.4.3. *Let* C *be a convex subset of a* UCW*-hyperbolic space* (X, d, W) *and* $T : C \to C$ *be nonexpansive. Assume that* (λ_n) *is a sequence in* $[0, 1]$ *satisfying* $\sum_{n=0}^{\infty} \lambda_n(1 - \lambda_n) = \infty$.
Let $x \in C, b > 0$ *be such that* T *has approximate fixed points in a* b*-neighborhood of* x.

Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Thus, we assume that T has approximate fixed points in a b-neighborhood of some $x \in C$ instead of having fixed points. However, by Proposition 2.3.5, for *closed* convex subsets C of *complete* UCW-hyperbolic spaces, T has fixed points is equivalent with T having approximate fixed points in a b-neighborhood of x.

If C is bounded with diameter d_C, then C has the AFPP for nonexpansive mappings by Proposition 4.2.4, so we can apply Theorem 4.4.2 for all $x \in C$ with d_C instead of b.

Corollary 4.4.4. *Let* $(X, d, W), \eta, C, T, (\lambda_n), \theta$ *be as in the hypothesis of Theorem 4.4.2. Assume moreover that* C *is bounded with diameter* d_C.

Then T *is* λ_n*-asymptotically regular and the following holds for all* $x \in C$:

$$\forall \varepsilon > 0\, \forall n \geq \Phi(\varepsilon, \eta, d_C, \theta) \left(d(x_n, Tx_n) < \varepsilon \right),$$

where $\Phi(\varepsilon, \eta, d_C, \theta)$ *is defined as in Theorem 4.4.2 by replacing* b *with* d_C.

For bounded C, we get λ_n-asymptotic regularity for general (λ_n) satisfying (4.27) and we also obtain an effective rate $\Phi(\varepsilon, \eta, d_C, \theta)$ of asymptotic regularity that depends only on the error ε, on the modulus of uniform convexity η, on the diameter d_C of C, and on (λ_n) via θ, but not on the nonexpansive mapping T, the starting point $x \in C$ of the iteration or other data related with C and X.

The rate of asymptotic regularity can be further simplified for constant $\lambda_n = \lambda \in (0,1)$. In this case, it is easy to see that $\theta : \mathbb{N} \to \mathbb{N} \quad \theta(n) = n \cdot \left\lceil \frac{1}{\lambda(1-\lambda)} \right\rceil$ satisfies (4.28).

Corollary 4.4.5. *Let $(X, d, W), \eta, C, d_C, T$ be as in the hypothesis of Corollary 4.4.4. Assume moreover that $\lambda_n = \lambda \in (0,1)$ for all $n \in \mathbb{N}$.*

Then T is λ-asymptotically regular and for all $x \in C$,

$$\forall \varepsilon > 0 \, \forall n \geq \Phi(\varepsilon, \eta, d_C, \lambda) \left(d(x_n, Tx_n) < \varepsilon \right), \tag{4.30}$$

where

$$\Phi(\varepsilon, \eta, d_C, \lambda) = \begin{cases} \left\lceil \frac{1}{\lambda(1-\lambda)} \right\rceil \cdot \left\lceil \frac{d_C + 1}{\varepsilon \cdot \eta\left(d_C + 1, \frac{\varepsilon}{d_C+1}\right)} \right\rceil & \text{for } \varepsilon < 2d_C \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if $\eta(r, \varepsilon)$ can be written as $\eta(r, \varepsilon) = \varepsilon \cdot \tilde{\eta}(r, \varepsilon)$ such that $\tilde{\eta}$ increases with ε (for fixed r), then the bound $\Phi(\varepsilon, \eta, d_C, \lambda)$ can be replaced for $\varepsilon < 2d_C$ with

$$\tilde{\Phi}(\varepsilon, \eta, d_C, \lambda) = \left\lceil \frac{1}{\lambda(1-\lambda)} \right\rceil \cdot \left\lceil \frac{d_C + 1}{2\varepsilon \cdot \tilde{\eta}\left(d_C + 1, \frac{\varepsilon}{d_C+1}\right)} \right\rceil.$$

As we have seen in Chapter 2, $CAT(0)$ spaces are UCW-hyperbolic spaces with a modulus of uniform convexity $\eta(r, \varepsilon) = \frac{\varepsilon^2}{8} = \varepsilon \cdot \tilde{\eta}(r, \varepsilon)$, where $\tilde{\eta}(r, \varepsilon) = \frac{\varepsilon}{8}$ increases with ε. It follows that the above results can be applied to $CAT(0)$ spaces.

Corollary 4.4.6. *Let X be a $CAT(0)$ space, and $C, d_C, T, (\lambda_n), \theta$ be as in the hypothesis of Corollary 4.4.4.*

Then T is λ_n-asymptotically regular and for all $x \in C$,

$$\forall \varepsilon > 0 \, \forall n \geq \Psi(\varepsilon, d_C, \theta) \left(d(x_n, Tx_n) < \varepsilon \right), \tag{4.31}$$

where

$$\Psi(\varepsilon, d_C, \theta) = \begin{cases} \theta\left(\left\lceil \frac{4(d_C+1)^2}{\varepsilon^2} \right\rceil \right) & \text{for } \varepsilon < 2d_C \\ 0 & \text{otherwise.} \end{cases}$$

For general (λ_n), the rate of asymptotic regularity is of order $\theta\left(\frac{1}{\varepsilon^2}\right)$, where θ is a rate of divergence for $\sum_{n=1}^{\infty}\lambda_n(1-\lambda_n)$.

Corollary 4.4.7. *Let X be a $CAT(0)$ space, $C \subseteq X$ be a bounded convex subset with diameter d_C, and $T : C \to C$ be nonexpansive. Assume that $\lambda_n = \lambda \in (0,1)$.*

Then T is λ-asymptotically regular, and for all $x \in C$,

$$\forall\varepsilon > 0\,\forall n \geq \Psi(\varepsilon, d_C, \lambda)\left(d(x_n, Tx_n) < \varepsilon\right), \tag{4.32}$$

where

$$\Psi(\varepsilon, d_C, \lambda) = \begin{cases} \left\lceil\dfrac{1}{\lambda(1-\lambda)}\right\rceil \cdot \left\lceil\dfrac{4(d_C+1)^2}{\varepsilon^2}\right\rceil & \text{for } \ \varepsilon < 2d_C \\ 0 & \text{otherwise.}\end{cases}$$

Hence, for bounded convex subsets of $CAT(0)$ spaces and constant $\lambda_n = \lambda$, we get a quadratic (in $1/\varepsilon$) rate of asymptotic regularity.

4.5 Uniform approximate fixed point property

Inspired by Theorem 4.3.4, our quantitative version of Borwein-Reich-Shafrir Theorem, we introduced in [143] the notions of uniform approximate fixed point property and uniform asymptotic regularity property. The idea is to forget about the quantitative features of Theorem 4.3.4 and to look only at the uniformities.

Let (X, d) be a metric space, $C \subseteq X$ and $\mathcal{F}$ be a class of mappings $T : C \to C$. We say that C has the *uniform approximate fixed point property (UAFPP)* for $\mathcal{F}$ if for all $\varepsilon > 0$ and $b > 0$ there exists $D > 0$ such that for each point $x \in C$ and for each mapping $T \in \mathcal{F}$,

$$d(x, Tx) \leq b \text{ implies } T \text{ has } \varepsilon\text{-fixed points in a } D\text{-neighborhood of } x. \tag{4.33}$$

Formally, $d(x, Tx) \leq b \ \Rightarrow\ \exists x^* \in C\big(d(x, x^*) \leq D \wedge d(x^*, Tx^*) < \varepsilon\big)$.

Using the same ideas, we can define the notion of C having the uniform fixed point property. Thus, C has the *uniform fixed point property (UFPP)* for $\mathcal{F}$ if for all $b > 0$ there exists $D > 0$ such that for each point $x \in C$ and for each mapping $T \in \mathcal{F}$,

$$d(x, Tx) \leq b \text{ implies } T \text{ has fixed points in a } D\text{-neighborhood of } x. \tag{4.34}$$

That is, $d(x, Tx) \leq b \ \Rightarrow\ \exists x^* \in C\big(d(x, x^*) \leq D \wedge Tx^* = x^*\big)$. As an immediate application of Banach's Contraction Mapping Principle, we get the following.

Proposition 4.5.1. *Assume that (X, d) is a complete metric space and let $\mathcal{F}$ be the class of contractions with a common contraction constant $k \in (0, 1)$. Then each closed subset C of X has the UFPP for $\mathcal{F}$.*

Proof. By Banach's Contraction Mapping Principle we know that each mapping $T \in \mathcal{F}$ has a unique fixed point x_0 and, moreover, for each $x \in C$,

$$d(T^n x, x_0) \leq \frac{k^n}{1-k} d(x, Tx) \quad \text{for all } n \in \mathbb{N}. \tag{4.35}$$

For $n = 0$, this yields $d(x, x_0) \leq \dfrac{d(x,Tx)}{1-k}$, so $d(x, Tx) \leq b$ implies $d(x, x_0) \leq \dfrac{b}{1-k}$. Hence, (4.33) holds with $D = \dfrac{b}{1-k} b$. □

Let (X, d, W) be a W-hyperbolic space, and $C \subseteq X$ be a convex subset and assume that (λ_n) is a sequence in $[0, 1]$. We say that C has the *λ_n-uniform asymptotic regularity property* for $\mathcal{F}$ if for all $\varepsilon > 0$ and $b > 0$ there exists $N \in \mathbb{N}$ such that for each point $x \in C$ and for each mapping $T \in \mathcal{F}$,

$$d(x, Tx) \leq b \quad \Rightarrow \quad \forall n \geq N\big(d(x_n, Tx_n) < \varepsilon\big), \tag{4.36}$$

where (x_n) is the Krasnoselski-Mann iteration.

As an immediate consequence of Theorem 4.3.12, bounded convex subsets of W-hyperbolic spaces have the λ_n-uniform asymptotic regularity property for directionally nonexpansive mappings for all (λ_n) divergent in sum and bounded away from 1.

Theorem 4.3.4 is used to prove the following equivalent characterizations.

Proposition 4.5.2. *[143]*
Let C be a convex subset of a W-hyperbolic space (X, d, W). The following are equivalent.

(i) C has the UAFPP for nonexpansive mappings;

(ii) there exists (λ_n) in $[0, 1]$ such that C has the λ_n-uniform asymptotic regularity property for nonexpansive mappings;

(iii) for all (λ_n) in $[0, 1]$ which are divergent in sum and bounded away from 1, C has the λ_n-uniform asymptotic regularity property for nonexpansive mappings.

Proof. We give only the proof of $(i) \Rightarrow (iii)$, for which the main ingredient is our quantitative Borwein-Reich-Shafrir Theorem 4.3.4. We refer to [143] for the complete proof.

Let $\varepsilon > 0, b > 0$, and $D > 0$ be such that (4.33) holds with $\mathcal{F}$ being the class of nonexpansive mappings. If (λ_n) in $[0, 1]$ is divergent in sum and bounded away from 1, then, as we have already discussed in Subsection 4.3.1, there exist $K \in \mathbb{N}$ and $\alpha : \mathbb{N} \to \mathbb{N}$ satisfying the corresponding hypothesis of Theorem 4.3.4. Let $x \in C$ and $T : C \to C$ nonexpansive be such that $d(x, Tx) \leq b$. By (4.33), there exists $x^* \in C$ satisfying $d(x, x^*) \leq D$, and $d(x^*, Tx^*) < \varepsilon$. By taking $b^* = \max\{b, D\}$, it follows that

$$d(x, Tx) \leq b^* \quad \text{and} \quad d(x, x^*) \leq b^*,$$

so the hypothesis (4.17) is also satisfied. It follows that we can apply Theorem 4.3.4 to get $N = \Phi(\varepsilon, b^*, K, \alpha)$ such that

$$\forall n \geq N\big(d(x_n, Tx_n) < d(x^*, Tx^*) + \varepsilon < 2\varepsilon\big).$$

□

Let us remark the following fact. A first attempt to define the property that C has the uniform approximate fixed point property for nonexpansive mappings is in the line of Goebel-Kirk Theorem 4.1.3, that is: for all $\varepsilon > 0$ there exists $D > 0$ such that for all $x \in C$ and for all $T \in \mathcal{F}$

$$\exists x^* \in C\big(d(x, x^*) \leq D \wedge d(x^*, Tx^*) < \varepsilon\big). \tag{4.37}$$

In this case, it follows that, even if we consider only constant mappings T, the only subsets C satisfying (4.37) are the bounded ones. If C is bounded, then C satisfies (4.37) by Goebel-Kirk Theorem 4.1.3. Conversely, assume that C satisfies (4.37) for all constant mappings T. Then for $\varepsilon = 1$ we get $D_1 \in \mathbb{N}$ such that for all $x \in C$, and for all constant mappings $T : C \to C$, there is $x^* \in C$ with $d(x, x^*) \leq D_1$ and $d(x^*, Tx^*) < 1$. It follows that

$$d(x, Tx) \leq d(x, x^*) + d(x^*, Tx^*) + d(Tx^*, Tx) \leq 2D_1 + 1 \tag{4.38}$$

Now, if we assume that C is unbounded, there are $x, y \in C$ such that $d(x, y) > 2D_1 + 1$. Define $T : C \to C$, $T(z) = y$ for all $z \in C$. Then $d(x, Tx) = d(x, y) > 2D_1 + 1$ which contradicts (4.38).

Chapter 5

Approximate fixed points in product spaces

If (X, ρ) and (Y, d) are metric spaces, then the metric d_∞ on $X \times Y$ is defined in the usual way:

$$d_\infty((x, u), (y, v)) = \max\{\rho(x, y), d(u, v)\}$$

for $(x, u), (y, v) \in X \times Y$. We denote by $(X \times Y)_\infty$ the metric space thus obtained.

The following theorem was proved first by Espínola and Kirk [62] for Banach spaces and then by Kirk [115] for $CAT(0)$ spaces.

Theorem 5.0.3. *Assume that X is a Banach space or a $CAT(0)$ space and $C \subseteq X$ is a bounded closed convex subset of X. If (M, d) is a metric space with the AFPP for nonexpansive mappings, then*

$$H := (C \times M)_\infty$$

has the AFPP for nonexpansive mappings.

The proof of this result uses essentially Goebel-Kirk Theorem 4.1.3.

In the following, we generalize Theorem 5.0.3 to *unbounded* convex subsets C of W-hyperbolic spaces. We extend the results further, to families $(C_u)_{u\in M}$ of unbounded convex subsets of a W-hyperbolic space. The key ingredient in obtaining these generalizations is Theorem 4.3.4, our uniform quantitative version of Borwein-Reich-Shafrir Theorem. The results presented in the sequel were obtained by Kohlenbach and the author in [143].

5.1 The case of one convex subset C

In the sequel, $C \subseteq X$ is a convex subset of a W-hyperbolic space (X, ρ, W), (M, d) is a metric space which has the AFPP for nonexpansive mappings and $H := (C \times M)_\infty$ and (λ_n) is a sequence in $[0, 1]$.

Let us denote with $P_1 : H \to C$, $P_2 : H \to M$ the coordinate projections and define for each nonexpansive mapping $T : H \to H$ and for each $u \in M$,

$$T_u : C \to C, \quad T_u(x) = (P_1 \circ T)(x, u).$$

Since T_u is nonexpansive, we can associate with T_u the Krasnoselski-Mann iteration (x_n^u) starting with an arbitrary $x \in C$.

In the sequel, $\delta : M \to C$ is a nonexpansive mapping that *selects* for each $u \in M$ an element $\delta(u) \in C$. Trivial examples of such nonexpansive selection mappings are the constant ones. For simplicity, we shall denote the Krasnoselski-Mann iteration starting from $\delta(u)$ and associated with T_u by $(\delta_n(u))$:

$$\delta_0(u) := \delta(u), \quad \delta_{n+1}(u) := (1-\lambda_n)\delta_n(u) \oplus \lambda_n T_u(\delta_n(u)).$$

For each $n \in \mathbb{N}$, let us define

$$\varphi_n : M \to M, \;\; \varphi_n(u) = (P_2 \circ T)(\delta_n(u), u).$$

Theorem 5.1.1. *[143] Assume that*

$$\sup_{u\in M} r_C(T_u) < \infty,$$

and $\varphi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is such that for each $\varepsilon > 0$ and $v \in M$ there exists $x^* \in C$ satisfying

$$\rho(\delta(v), x^*) \le \varphi(\varepsilon) \quad \textit{and} \quad \rho(x^*, T_v(x^*)) \le \sup_{u\in M} r_C(T_u) + \varepsilon. \tag{5.1}$$

Then $r_H(T) \le \sup_{u\in M} r_C(T_u)$.

As an immediate consequence, we get the following result.

Corollary 5.1.2. *Assume that $\varphi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is such that*

$$\forall \varepsilon > 0 \forall u \in M \exists x^* \in C \Big(\rho(\delta(u), x^*) \le \varphi(\varepsilon) \quad \textit{and} \quad \rho(x^*, T_u(x^*)) \le \varepsilon \Big). \tag{5.2}$$

Then $r_H(T) = 0$.

Proof. From the hypothesis, it follows immediately that $r_C(T_u) = 0$ for all $u \in M$. ∎

The next theorem is obtained by applying Theorem 4.3.14 to the family $(T_u)_{u\in M}$.

Theorem 5.1.3. *[143] Assume that (λ_n) is divergent in sum and bounded away from 1 and that there exists $b > 0$ such that*

$$\forall u \in M \exists y \in C \Big(\rho(\delta(u), y) \le b \quad \textit{and} \quad \forall m, p \in \mathbb{N} \big(\rho(y_m^u, y_p^u) \le b\big)\Big), \tag{5.3}$$

where (y_n^u) is the Krasnoselski-Mann iteration associated with T_u, starting with y:

$$y_0^u := y, \quad y_{n+1}^u = (1-\lambda_n)y_n^u \oplus \lambda_n T_u(y_n^u).$$

Then $r_H(T) = 0$.

Applying the above theorem with $y := \delta(u)$, we get the following generalization of Theorem 5.0.3.

Corollary 5.1.4. *Assume that for all $u \in M$, the Krasnoselski-Mann iteration $\delta_n(u)$ is bounded. Then $r_H(T) = 0$.*

Theorem 5.0.3 is an immediate consequence of Corollary 5.1.4, since if C is bounded, $\delta_n(u)$ is bounded for each $u \in M$.

5.2 Families of unbounded convex sets

In this subsection we indicate that all the above results can be generalized to families $(C_u)_{u\in M}$ of *unbounded* convex subsets of the W-hyperbolic space (X, ρ, W).

Let $(C_u)_{u\in M}$ be a family of convex subsets of X with the property that there exists a nonexpansive *selection* mapping $\delta : M \to \bigcup_{u\in M} C_u$, that is a nonexpansive mapping satisfying

$$\forall u \in M\big(\delta(u) \in C_u\big). \tag{5.4}$$

We consider the following subspace of $(X \times M)_\infty$:

$$H := \{(x, u) : u \in M, x \in C_u\}$$

and let $P_1 : H \to \bigcup\limits_{u\in M} C_u$, $P_2 : H \to M$ be the projections.

In the following, we consider nonexpansive mappings $T : H \to H$ satisfying

$$\forall (x, u) \in H \left((P_1 \circ T)(x, u) \in C_u\right). \tag{5.5}$$

It is easy to see that we can define a nonexpansive mapping

$$T_u : C_u \to C_u, \quad T_u(x) = (P_1 \circ T)(x, u)$$

for each $u \in M$. We denote the Krasnoselski-Mann iteration starting from $x \in C_u$ and associated with T_u by (x_n^u).

For each $n \in \mathbb{N}$, we define

$$\varphi_n : M \to M, \quad \varphi_n(u) = (P_2 \circ T)(\delta_n(u), u).$$

The following results can be proved in a similar manner with Theorems 5.1.1, 5.1.3.

Theorem 5.2.1. *Assume that*

$$\sup_{u\in M} r_{C_u}(T_u) < \infty$$

and that $\varphi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ *is such that for each* $\varepsilon > 0$ *and* $v \in M$ *there exists* $x^* \in C_v$ *satisfying*

$$\rho(\delta(v), x^*) \le \varphi(\varepsilon) \quad \textit{and} \quad \rho(x^*, T_v(x^*)) \le \sup_{u\in M} r_{C_u}(T_u) + \varepsilon.$$

Then $r_H(T) \le \sup\limits_{u\in M} r_{C_u}(T_u)$.

Theorem 5.2.2. *Let* (λ_n) *divergent in sum and bounded away from* 1. *Assume that there is* $b > 0$ *such that*

$$\forall u \in M \exists y \in C_u\big(\rho(\delta(u), y) \le b \quad \textit{and} \quad \forall m, p \in \mathbb{N}(\rho(y_m^u, y_p^u) \le b)\big).$$

Then $r_H(T) = 0$.

We get also the following corollary.

Corollary 5.2.3. *Assume that* $(C_u)_{u\in M}$ *is a family of* bounded *convex subsets of* X *such that* $\sup\limits_{u\in M} diam(C_u) < \infty$.

Then H *has the AFPP for nonexpansive mappings* $T : H \to H$ *satisfying (5.5).*

Proof. The hypothesis of Theorem 5.2.2 is satisfied with $y := \delta(u)$. □

5.3 Partial answer to an open problem of Kirk

In the following, we use our notion of uniform approximate fixed point property, introduced in Section 4.5, to give some partial answers to the following problem of Kirk [115, Problem 27]:

Let C be a closed convex subset of a complete $CAT(0)$ space X (having the geodesic line extension property) and M be a metric space. If both C and M have the AFPP for nonexpansive mappings, is it true that the product $H := (C \times M)_\infty$ again has the AFPP?

We show that this is true if C has the UAFPP (even in the case where X is just a W-hyperbolic space) and a technical condition is satisfied which, in particular, holds if M is bounded.

Theorem 5.3.1. *Let C be a convex subset of a W-hyperbolic space (X, ρ, W) and (M, d) be a metric space with the AFPP for nonexpansive mappings. Assume that C has the UAFPP for nonexpansive mappings.*
Let $\delta : M \to C$ be a nonexpansive selection mapping and $T : H \to H$ be a nonexpansive mapping such that $\sup\limits_{u\in M} \rho(T_u(\delta(u)), \delta(u)) < \infty$.

Then $r_H(T) = 0$.

Proof. Let $\varepsilon > 0$ and $b > 0$ be such that $\rho(T_u(\delta(u)), \delta(u)) \leq b$ for all $u \in M$. Since C has the UAFPP for nonexpansive mappings, there exists $D > 0$ (depending on ε and b) such that (4.33) holds for each nonexpansive self-mapping of C and each $x \in C$. For each $u \in M$, we can apply (4.33) for $x := \delta(u)$ and T_u to get $x^* \in C$ such that $\rho(\delta(u), x^*) \leq D$ and $\rho(x^*, T_u(x^*)) \leq \varepsilon$. Hence, the hypothesis of Corollary 5.1.2 is satisfied with $\varphi(\varepsilon) = D$, so $r_H(T) = 0$ follows. $\square$

Corollary 5.3.2. *Let C be a convex subset of a W-hyperbolic space (X, ρ, W) and (M, d) be a* bounded *metric space. Assume that C has the UAFPP and that (M, d) has the AFPP for nonexpansive mappings.*

Then $H := (C \times M)_\infty$ has the AFPP for nonexpansive mappings.

Proof. Let $x \in C$ be arbitrary, and define $\delta : M \to C$ by $\delta(u) = x$. Let $T : H \to H$ be a nonexpansive mapping. Fix some $u_0 \in M$, and define $b := \rho(x, T_{u_0}(x)) + diam(M)$. Then $\rho(x, T_u(x)) \leq \rho(x, T_{u_0}(x)) + d(u_0, u) \leq b$ for each $u \in M$, so we can apply Theorem 5.3.1 to conclude that $r_H(T) = 0$. $\square$

Chapter 6

Effective asymptotic regularity for Ishikawa iterations

Let C be a convex subset of a normed space X and $T : C \to C$ be nonexpansive. The *Ishikawa iteration* [104] starting with $x \in C$ is defined by

$$x_0 := x, \quad x_{n+1} = (1-\lambda_n)x_n + \lambda_n T\big((1-s_n)x_n + s_n T x_n\big), \tag{6.1}$$

where $(\lambda_n), (s_n)$ are sequences in $[0,1]$. By letting $s_n = 0$ for all $n \in \mathbb{N}$, we get the Krasnoselski-Mann iteration as a special case.

An extension of Ishikawa Theorems 4.1.1 and 4.0.5 to these iterations was proved by Deng [53].

Theorem 6.0.3. *[53] Let C be a convex subset of a Banach space X and $T : C \to C$ be a nonexpansive mapping. Assume that (λ_n) satisfies (4.5) and that $\sum_{n=0}^{\infty} s_n$ converges.*

(i) If (x_n) is bounded for some $x \in C$, then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

(ii) Assume furthermore that C is closed and $T(C)$ is contained into a compact subset of C. Then (x_n) converges to a fixed point of T.

Tan and Xu [230] obtained a weak convergence result for Ishikawa iterates that generalizes Reich Theorem 4.0.6.

Theorem 6.0.4. *Let C be a bounded closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition or has a Fréchet differentiable norm and $T : C \to C$ be a nonexpansive mapping. Assume that $(\lambda_n), (s_n)$ satisfy*

$$\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n) \text{ diverges}, \quad \limsup_n s_n < 1 \quad \text{and} \sum_{n=0}^{\infty} s_n(1-\lambda_n) \text{ converges.} \tag{6.2}$$

(i) For every $x \in C$, the Ishikawa iteration (x_n) converges weakly to a fixed point of T.

(ii) If, moreover, $T(C)$ is contained into a compact subset of C, then the convergence is strong.

As in the case of Krasnoselski-Mann iterations, the first step towards getting weak or strong convergence is proving asymptotic regularity (with respect to Ishikawa iterates), and this was done by Tan and Xu [230] for uniformly convex Banach spaces and, recently, by Dhompongsa and Panyanak [54] for $CAT(0)$ spaces.

Theorem 6.0.5. *Let X be a uniformly convex Banach space or a $CAT(0)$ space, $C \subseteq X$ a bounded closed convex subset and $T : C \to C$ nonexpansive. Assume that $(\lambda_n), (s_n)$ satisfy (6.2).*

Then $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$ for every $x \in C$.

Using proof mining methods we obtained [166] a quantitative version (Theorem 6.1.5) of a two-fold generalization of the above result:

- firstly, we consider UCW-hyperbolic spaces;
- secondly, we assume that $Fix(T) \neq \emptyset$ instead of assuming the boundedness of C.

The idea is to combine methods used in [164] to obtain effective rates of asymptotic regularity for Krasnoselski-Mann iterates with the ones used in [165] to get rates of asymptotic regularity for Halpern iterates.

In this way, we provide for the first time (even for the normed case) effective rates of asymptotic regularity for the Ishikawa iterates, that is rates of convergence of $(d(x_n, Tx_n))$ towards 0.
If C is a convex subset of a W-hyperbolic space (X, d, W) and $T : C \to C$ is nonexpansive, then, as in the case of normed spaces, we can define the *Ishikawa iteration* starting with $x \in C$ by

$$x_0 := x, \quad x_{n+1} = (1-\lambda_n)x_n \oplus \lambda_n T((1-s_n)x_n \oplus s_n Tx_n), \tag{6.3}$$

where $(\lambda_n), (s_n)$ are sequences in $[0, 1]$. By letting $s_n = 0$ for all $n \in \mathbb{N}$, we get the Krasnoselski-Mann iteration as a special case.

We shall use the following notations

$$y_n := (1-s_n)x_n \oplus s_n Tx_n$$

and

$$T_n : C \to C, \quad T_n(x) = (1-\lambda_n)x \oplus \lambda_n T((1-s_n)x \oplus s_n Tx).$$

Then

$$x_{n+1} = (1-\lambda_n)x_n \oplus \lambda_n Ty_n = T_n x_n$$

and it is easy to see that $Fix(T) \subseteq Fix(T_n)$ for all $n \in \mathbb{N}$.

The following lemma collects some basic properties of Ishikawa iterations.

Lemma 6.0.6. *[166]*

(i) $d(x_{n+1}, Tx_{n+1}) \leq (1 + 2s_n(1-\lambda_n))d(x_n, Tx_n)$ for all $n \in \mathbb{N}$;

(ii) T_n is nonexpansive for all $n \in \mathbb{N}$;

(iii) For all $p \in Fix(T)$, the sequence $(d(x_n, p))$ is nonincreasing and for all $n \in \mathbb{N}$,

$$d(y_n, p) \leq d(x_n, p) \quad \textit{and} \quad d(x_n, Ty_n), d(x_n, Tx_n) \leq 2d(x_n, p).$$

6.1 Main results

Proposition 6.1.1. *[166]*
Let C be a convex subset of a UCW-hyperbolic space (X, d, W) and $T : C \to C$ nonexpansive with $Fix(T) \neq \emptyset$. Assume that $\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n)$ is divergent.

Then $\liminf_n d(x_n, Ty_n) = 0$ for all $x \in C$.
Furthermore, if η is a monotone modulus of uniform convexity and $\theta : \mathbb{N} \to \mathbb{N}$ is a rate of divergence for $\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n)$, then

for all $x \in C, \varepsilon > 0, k \in \mathbb{N}$ there exists $N \in \mathbb{N}$ satisfying

$$k \leq N \leq h(\varepsilon, k, \eta, b, \theta) \quad \text{and} \quad d(x_N, Ty_N) < \varepsilon, \tag{6.4}$$

where

$$h(\varepsilon, k, \eta, b, \theta) = \begin{cases} \theta\left(\left\lceil \dfrac{b+1}{\varepsilon \cdot \eta\left(b, \dfrac{\varepsilon}{b}\right)} \right\rceil + k\right) & \text{for } \varepsilon \leq 2b, \\ k & \text{otherwise,} \end{cases}$$

with $b > 0$ such that $b \geq d(x, p)$ for some $p \in Fix(T)$.

As an immediate consequence of the above proposition, we get a rate of asymptotic regularity for the Krasnoselski-Mann iterates that is basically the same with the one obtained in Theorem 4.4.2 .

Corollary 6.1.2. *Let $(X, d, W), \eta, C, T, b, (\lambda_n), \theta$ be as in the hypotheses of Proposition 6.1.1 and assume that (x_n) is the Krasnoselski-Mann iteration starting with x, defined by (4.10).*

Then $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$ for all $x \in C$ and

$$\forall \varepsilon > 0 \, \forall n \geq \Phi(\varepsilon, \eta, b, \theta) \Big(d(x_n, Tx_n) < \varepsilon \Big), \tag{6.5}$$

where $\Phi(\varepsilon, \eta, b, \theta) = h(\varepsilon, 0, \eta, b, \theta)$, with h defined as above.

Proposition 6.1.3. *In the hypotheses of Proposition 6.1.1, assume moreover that* $\limsup_n s_n < 1$.

Then $\liminf_n d(x_n, Tx_n) = 0$ for all $x \in C$.

Furthermore, if $L, N_0 \in \mathbb{N}$ are such that $s_n \leq 1 - \dfrac{1}{L}$ for all $n \geq N_0$, then

for all $x \in C, \varepsilon > 0, k \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that

$$k \leq N \leq \Psi(\varepsilon, k, \eta, b, \theta, L, N_0) \quad \text{and} \quad d(x_N, Tx_N) < \varepsilon, \tag{6.6}$$

where $\Psi(\varepsilon, k, \eta, b, \theta, L, N_0) = h\left(\dfrac{\varepsilon}{L}, k + N_0, \eta, b, \theta\right)$, with h defined as in Proposition 6.1.1.

As a corollary, we obtain an approximate fixed point bound for the nonexpansive mapping T.

Corollary 6.1.4. *In the hypotheses of Proposition 6.1.3,*

$$\forall \varepsilon > 0 \, \exists N \leq \Phi(\varepsilon, \eta, b, \theta, L, N_0) \Big(d(x_N, Tx_N) < \varepsilon \Big), \tag{6.7}$$

where $\Phi(\varepsilon, \eta, b, \theta, L, N_0) = \Psi(\varepsilon, 0, \eta, b, \theta, L, N_0)$, *with* Ψ *defined as above.*

The following theorem is the main result of this section.

Theorem 6.1.5. *[166] Let* (X, d, W) *be a UCW-hyperbolic space,* $C \subsetneq X$ *a convex subset and* $T : C \to C$ *nonexpansive with* $Fix(T) \neq \emptyset$. *Assume that* $\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n)$ *diverges,* $\limsup_n s_n < 1$ *and* $\sum_{n=0}^{\infty} s_n(1-\lambda_n)$ *converges.*

Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ *for all* $x \in C$.

Furthermore, if η *is a monotone modulus of uniform convexity,* θ *is a rate of divergence for* $\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n)$, L, N_0 *are such that* $s_n \leq 1 - \frac{1}{L}$ *for all* $n \geq N_0$ *and* γ *is a Cauchy modulus for* $\sum_{n=0}^{\infty} s_n(1-\lambda_n)$, *then for all* $x \in C$,

$$\forall \varepsilon > 0 \forall n \geq \Phi(\varepsilon, \eta, b, \theta, L, N_0, \gamma) \Big(d(x_n, Tx_n) < \varepsilon \Big), \tag{6.8}$$

where

$$\Phi(\varepsilon, \eta, b, \theta, L, N_0, \gamma) = \begin{cases} \theta\left(\left\lceil \dfrac{2L(b+1)}{\varepsilon \cdot \eta\left(b, \dfrac{\varepsilon}{2Lb}\right)} \right\rceil + \gamma\left(\dfrac{\varepsilon}{8b}\right) + N_0 + 1\right) & \text{for } \varepsilon \leq 4Lb, \\ \gamma\left(\dfrac{\varepsilon}{8b}\right) + N_0 + 1 & \text{otherwise,} \end{cases}$$

with $b > 0$ *such that* $b \geq d(x, p)$ *for some* $p \in Fix(T)$.

Remark 6.1.6. *Assume, moreover, that* $\eta(r, \varepsilon) = \varepsilon \cdot \tilde{\eta}(r, \varepsilon)$ *such that* $\tilde{\eta}$ *increases with* ε *(for a fixed* r*). Then for* $\varepsilon \leq 4Lb$ *the bound* $\Phi(\varepsilon, \eta, b, \theta, L, N_0, \gamma)$ *can be replaced with*

$$\tilde{\Phi}(\varepsilon, \eta, b, \theta, L, N_0, \gamma) = \theta\left(\left\lceil \frac{L(b+1)}{\varepsilon \cdot \tilde{\eta}\left(b, \dfrac{\varepsilon}{2Lb}\right)} \right\rceil + \gamma\left(\frac{\varepsilon}{8b}\right) + N_0 + 1\right).$$

For bounded C, we get an effective rate of asymptotic regularity which depends on the error ε, on the modulus of uniform convexity η, on the diameter d_C of C, on $(\lambda_n), (s_n)$ via θ, L, N_0, γ, but does not depend on the nonexpansive mapping T, the starting point $x \in C$ of the iteration or other data related with C and X.

Corollary 6.1.7. *Let (X,d,W) be a complete UCW-hyperbolic space, $C \subseteq X$ a* bounded *convex closed subset with diameter d_C and $T: C \to C$ nonexpansive. Assume that $\eta, (\lambda_n), (s_n), \theta, L, N_0, \gamma$ are as in the hypotheses of Theorem 6.1.5.*

Then $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$ for all $x \in C$ and, moreover,

$$\forall \varepsilon > 0\, \forall n \geq \Phi(\varepsilon, \eta, d_C, \theta, L, N_0, \gamma) \left(d(x_n, Tx_n) < \varepsilon \right),$$

where $\Phi(\varepsilon, \eta, d_C, \theta, L, N_0, \gamma)$ is defined as in Theorem 6.1.5 by replacing b with d_C.

Proof. We can apply Corollary 2.3.6, the generalization of Browder-Goehde-Kirk Theorem to complete UCW-hyperbolic spaces, to get that $Fix(T) \neq \emptyset$. Moreover, $d(x,p) \leq d_C$ for any $x \in C$ and any $p \in Fix(T)$, hence we can take $b := d_C$ in Theorem 6.1.5. $\square$

The rate of asymptotic regularity can be further simplified for constant $\lambda_n = \lambda \in (0,1)$.

Corollary 6.1.8. *Let $(X,d,W), \eta, C, d_C, T$ be as in the hypotheses of Corollary 6.1.7. Assume that $\lambda_n = \lambda \in (0,1)$ for all $n \in \mathbb{N}$.*
Furthermore, let L, N_0 be such that $s_n \leq 1 - \dfrac{1}{L}$ for all $n \geq N_0$ and assume that the series $\sum\limits_{n=0}^{\infty} s_n$ converges with Cauchy modulus δ.

Then for all $x \in C$,

$$\forall \varepsilon > 0 \forall n \geq \Phi(\varepsilon, \eta, d_C, \lambda, L, N_0, \delta) \left(d(x_n, Tx_n) < \varepsilon \right), \tag{6.9}$$

where

$$\Phi(\varepsilon, \eta, d_C, \lambda, L, N_0, \delta) = \begin{cases} \left\lceil \dfrac{1}{\lambda(1-\lambda)} \cdot \dfrac{2L(d_C+1)}{\varepsilon \cdot \eta\left(d_C, \dfrac{\varepsilon}{2Ld_C}\right)} \right\rceil + M & \text{for } \varepsilon \leq 4Ld_C, \\ M & \text{otherwise,} \end{cases}$$

with $M = \delta\left(\dfrac{\varepsilon}{8d_C(1-\lambda)}\right) + N_0 + 1$.

Moreover, if $\eta(r,\varepsilon)$ can be written as $\eta(r,\varepsilon) = \varepsilon \cdot \tilde{\eta}(r,\varepsilon)$ such that $\tilde{\eta}$ increases with ε (for a fixed r), then the bound $\Phi(\varepsilon, \eta, d_C, \lambda, L, N_0, \delta)$ can be replaced for $\varepsilon \leq 4Ld_C$ with

$$\Phi(\varepsilon, \eta, d_C, \lambda, L, N_0, \delta) = \left\lceil \frac{1}{\lambda(1-\lambda)} \cdot \frac{L(d_C+1)}{\varepsilon \cdot \tilde{\eta}\left(d_C, \dfrac{\varepsilon}{2Ld_C}\right)} \right\rceil + M.$$

As we have already seen, $CAT(0)$ spaces are UCW-hyperbolic spaces with a modulus of uniform convexity $\eta(r,\varepsilon) = \dfrac{\varepsilon^2}{8}$, which has the form required in Remark 6.1.6. Thus, the above result can be applied to $CAT(0)$ spaces.

Corollary 6.1.9. *Let X be a $CAT(0)$ space, $C \subseteq X$ a* bounded *convex closed subset with diameter d_C and $T : C \to C$ nonexpansive. Assume that $\lambda_n = \lambda \in (0,1)$ for all $n \in \mathbb{N}$ and $L, N_0, (s_n), \delta$ are as in the hypotheses of Corollary 6.1.8*

Then $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$ for all $x \in C$ and, moreover

$$\forall \varepsilon > 0 \forall n \geq \Phi(\varepsilon, d_C, \lambda, L, N_0, \delta)\bigg(d(x_n, Tx_n) < \varepsilon\bigg), \tag{6.10}$$

where

$$\Phi(\varepsilon, d_C, \lambda, L, N_0, \delta) = \begin{cases} \left\lceil \dfrac{D}{\varepsilon^2} \right\rceil + M, & \text{for } \varepsilon \leq 4Ld_C, \\ M & \text{otherwise,} \end{cases}$$

with $M = \delta\left(\dfrac{\varepsilon}{8d_C(1-\lambda)}\right) + N_0 + 1,\ D = \dfrac{16L^2 d_C(d_C+1)}{\lambda(1-\lambda)}$.

Chapter 7

Asymptotically nonexpansive mappings in UCW-hyperbolic spaces

Asymptotically nonexpansive mappings were introduced by Goebel and Kirk [80] as a generalization of the nonexpansive ones. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive with sequence* (k_n) in $[0, \infty)$ if $\lim\limits_{n\to\infty} k_n = 0$ and

$$d(T^n x, T^n y) \leq (1 + k_n)d(x, y), \text{ forall } n \in \mathbb{N} \text{ and all } x, y \in C.$$

It is obvious that an asymptotically nonexpansive mapping with sequence (k_n) is $(1+k_1)$-Lipschitz. Examples showing that the class of asymptotically nonexpansive mappings is wider than the class of nonexpansive mappings are given in [80, 117].

Goebel and Kirk [80] extended the Browder-Göhde-Kirk Theorem to this class of mappings.

Theorem 7.0.10. *[80] Bounded closed convex subsets of uniformly convex Banach spaces have the FPP for asymptotically nonexpansive mappings.*

Recently [115], Kirk proved the same result for $CAT(0)$ spaces.

Theorem 7.0.11. *[115] Bounded closed convex subsets of complete $CAT(0)$ spaces have the FPP for asymptotically nonexpansive mappings.*

Kirk proved Theorem 7.0.11 using nonstandard methods, inspired by Khamsi's proof that bounded hyperconvex metric spaces have the AFPP for asymptotically nonexpansive mappings [110].

For these mappings, the *Krasnoselski-Mann iteration* starting from $x \in C$ is defined by

$$x_0 := x, \quad x_{n+1} := (1 - \lambda_n)x_n + \lambda_n T^n x_n, \tag{7.1}$$

where (λ_n) is a sequence in $[0, 1]$. The above iteration was introduced by Schu [214]; it is called *modified Mann iteration* in [231].

Asymptotically nonexpansive mappings have been studied mostly in the context of uniformly convex Banach spaces. In fact, for general Banach spaces it is not known whether bounded closed convex subsets have the AFPP (see [117] for a discussion).

In the setting of uniformly convex Banach spaces, the following weak convergence result was proved by Schu [215] with the assumption that Opial's condition is satisfied and by Tan and Xu [231] in the hypothesis that the space has a Fréchet differentiable norm.

Theorem 7.0.12. *[215, 231] Let X be a uniformly convex Banach space which satisfies Opial's condition or has a Fréchet differentiable norm, C be a bounded closed convex subset of X and $T : C \to C$ an asymptotically nonexpansive mapping with sequence (k_n) satisfying $\sum\limits_{i=0}^{\infty} k_i < \infty$. Assume that (λ_n) is bounded away from 0 and 1.*

Then for all $x \in C$, the Krasnoselski-Mann iteration (x_n) starting with x converges weakly to a fixed point of T.

As in the case of nonexpansive mappings, if $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$ for all $x \in C$, T is said to be λ_n-*asymptotically regular.* The following asymptotic regularity result is essentially contained in [214, 215].

Theorem 7.0.13. *Let C be a convex subset of a uniformly convex Banach space X and $T : C \to C$ an asymptotically nonexpansive mapping with sequence (k_n) in $[0, \infty)$ satisfying $\sum\limits_{i=0}^{\infty} k_i < \infty$. Let (λ_n) be a sequence in $[a, b]$ for $0 < a < b < 1$.*

If T has a fixed point, then T is λ_n-asymptotically regular.

We present in the sequel results on the fixed point theory and the asymptotic behaviour of asymptotically nonexpansive mappings in the very general setting of UCW-hyperbolic spaces. These results were obtained by Kohlenbach and the author in [145].

In the following, (X, d, W) is a UCW-hyperbolic space and $C \subseteq X$ a convex subset of X. Let us recall that a mapping $T : C \to C$ is said to be asymptotically nonexpansive with sequence (k_n) in $[0, \infty)$ if $\lim\limits_{n\to\infty} k_n = 0$ and

$$d(T^n x, T^n y) \leq (1 + k_n) d(x, y) \quad \text{for all } n \in \mathbb{N}, x, y \in C.$$

The first main result is a generalization to UCW-spaces of Goebel-Kirk Theorem 7.0.10 and Kirk Theorem 7.0.11.

Theorem 7.0.14. *[145]*
Closed convex and bounded subsets of complete UCW-hyperbolic spaces have the FPP for asymptotically nonexpansive mappings.

Our proof generalizes Goebel and Kirk's proof of Theorem 7.0.10 and, as a consequence, we obtain also an elementary proof of Theorem 7.0.11.

In fact, as it was already pointed out for uniformly convex normed spaces in [141], the proof of the FPP can be transformed into an elementary proof of the AFPP, which does not need the completeness of X or the closedness of C.

Proposition 7.0.15. *Bounded convex subsets of UCW-hyperbolic spaces have the AFPP for asymptotically nonexpansive mappings.*

The main part of this chapter is devoted to getting a quantitative version of an asymptotic regularity theorem for the Krasnoselski-Mann iterations of asymptotically nonexpansive mappings.

As in the case of normed spaces, the *Krasnoselski-Mann iteration* starting from $x \in C$ is defined by:

$$x_0 := x, \quad x_{n+1} := (1-\lambda_n)x_n \oplus \lambda_n T^n x_n, \tag{7.2}$$

where (λ_n) is a sequence in $[0,1]$.

We apply proof mining techniques to the following generalization to UCW-hyperbolic spaces of Theorem 7.0.13.

Theorem 7.0.16. *Let C be a convex subset of a UCW-hyperbolic space (X,d,W) and $T: C \to C$ be asymptotically nonexpansive with sequence $(k_n) \in [0,\infty)$ satisfying $\sum\limits_{i=0}^{\infty} k_i < \infty$. Assume that (λ_n) be a sequence in $[a,b]$ for $0 < a < b < 1$.*

If $Fix(T) \neq \emptyset$, then T is λ_n-asymptotically regular.

There does not seem to exist a computable rate of asymptotic regularity in this case; in [141] it is shown that the proof even holds for asymptotically weakly-quasi nonexpansive functions for which one can prove that no uniform effective rate does exist. Anyway, the general logical metatheorems from Chapter 3 guarantee (see also the logical discussion below) effective uniform bounds on the so-called *Herbrand normal form* or *no-counterexample interpretation* of the convergence i.e. on

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \in \mathbb{N}\, \forall m \in [N, N+g(N)]\big(d(x_m, Tx_m) < \varepsilon\big), \tag{7.3}$$

which (ineffectively) is equivalent to the fact that $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$. Here $[n, n+m] := \{n, n+1, n+2, \ldots, n+m\}$.

This coincides with what recently has been advocated under the name *metastability* or *finite convergence* in an essay posted by Terence Tao [234] (see also [232, 236]). Thus, in Tao's terminology, the logical metatheorems guarantee an effective uniform bound on the *metastability of* $(d(x_n, Tx_n))$.

In the sequel, we give a quantitative version of the above theorem, generalizing to UCW-hyperbolic spaces the logical analysis and the results of Kohlenbach and Lambov [141]. As a consequence, for $CAT(0)$ spaces we get a quadratic bound on the approximate fixed point property of (x_n) (see Corollary 7.2.5). We recall that for nonexpansive mappings, a quadratic rate of asymptotic regularity for the Krasnoselski-Mann iterations was obtained in Corollary 4.4.7.

7.1 Logical discussion

It is easy to see that the proof of the above theorem can be formalized in the theory of UCW-hyperbolic spaces $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$, defined in Section 3.2. Unfortunately, the conclusion of the above theorem, that for all $x \in C$, $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$, i.e.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall p \in \mathbb{N}\big(d(x_{N+p}, Tx_{N+p}) < \varepsilon\big), \tag{7.4}$$

is a $\forall\exists\forall$-formula, so it has a too complicated logical form for the logical metatheorems to apply. In the case of nonexpansive mappings, due to the fact that $(d(x_n, Tx_n))$ is nonincreasing, (7.4) could be rewritten as

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \big(d(x_N, Tx_N) < \varepsilon \big), \tag{7.5}$$

which has the required $\forall\exists$-form. This is no longer possible for asymptotically nonexpansive mappings, since for this class of mappings the sequence $(d(x_n, Tx_n))$ is not necessarily nonincreasing.

Lemma 7.1.1. *The following are equivalent*

(1) $\forall \varepsilon > 0\, \exists N \in \mathbb{N}\, \forall p \in \mathbb{N} \big(d(x_{N+p}, Tx_{N+p}) < \varepsilon \big)$;

(2) $\forall \varepsilon > 0\, \exists N \in \mathbb{N}\, \forall m \in \mathbb{N}\, \forall i \in [N, N+m] \big(d(x_i, Tx_i) < \varepsilon \big)$;

(2^H) $\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \in \mathbb{N}\, \forall i \in [N, N+g(N)] \big(d(x_i, Tx_i) < \varepsilon \big)$.

Proof. (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (2^H) are obvious. Assume that (2^H) is true. If (2) would be false, then for some $\varepsilon > 0$

$$\forall n \in \mathbb{N}\, \exists m_n \in \mathbb{N}\, \exists i \in [n, n+m_n]\, (d(x_i, Tx_i) \geq \varepsilon).$$

Define $g(n) := m_n$. Then (2^H) applied to g leads to a contradiction. $\square$

The transformed version (2^H) is the *Herbrand normal form* of (2) or the *no-counterexample interpretation* [153, 154] of (2), well-known in mathematical logic. The good news is that (2^H) has the $\forall\exists$-form, as the universal quantifier over i is bounded. Obviously, since the above argument is ineffective, a bound on $\exists N \in \mathbb{N}$ in (2^H) cannot be converted effectively into a bound on $\exists N \in \mathbb{N}$ in (2).

As it suffices to consider only mappings $T : X \to X$, it is easy to see that $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ proves the following formalized version of Theorem 7.0.16:

$$\begin{aligned}
&\forall g : \mathbb{N} \to \mathbb{N}\, \forall \varepsilon > 0\, \forall K, L \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \forall (\lambda_n) \in [0,1]^{\mathbb{N}}\, \forall (k_n) \in [0, K]^{\mathbb{N}} \\
&\forall x \in X\, \forall T : X \to X \left(Fix(T) \neq \emptyset \wedge L \geq 2 \wedge \forall n \in \mathbb{N} \left(\frac{1}{L} \leq \lambda_n \leq 1 - \frac{1}{L} \right) \wedge \right. \\
&\forall n \in \mathbb{N}\, \forall y, z \in X \left(d(T^n y, T^n z) \leq \quad (1 + k_n) d(y, z) \right) \wedge \forall n \in \mathbb{N} \left(\sum_{i=0}^{n} k_i \leq K \right) \\
&\qquad \qquad \left. \to \exists N \in \mathbb{N}\, \forall i \in [N, N+g(N)] \Big(d(x_i, Tx_i) < \varepsilon \Big) \right).
\end{aligned}$$

Moreover, the asymptotic nonexpansivity of T and the fact that $k_1 \leq K$ imply that T is $(1+K)$-Lipschitz. Thus, we can apply Corollary 3.2.3 which guarantees the extractability of a computable bound Φ on $\exists N \in \mathbb{N}$ in the conclusion

$$\begin{aligned}
&\forall b \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \forall \varepsilon > 0\, \forall K, L \in \mathbb{N}\, \forall (\lambda_n) \in [0,1]^{\mathbb{N}}\, \forall (k_n) \in [0, K]^{\mathbb{N}} \\
&\forall x \in X\, \forall T : X \to X \left(\forall \delta > 0 \Big(Fix_\delta(T, x, b) \neq \emptyset \Big) \wedge \forall n \in \mathbb{N} \left(\frac{1}{L} \leq \lambda_n \leq 1 - \frac{1}{L} \right) \right. \\
&\wedge \forall n \in \mathbb{N}\, \forall y, z \in X \Big(d(T^n y, T^n z) \leq (1 + k_n) d(y, z) \Big) \wedge \forall n \in \mathbb{N} \left(\sum_{i=0}^{n} k_i \leq K \right) \wedge \\
&\left. L \geq 2 \to \exists N \leq \Phi(\varepsilon, K, L, b, \eta, g)\, \forall i \in [N, N+g(N)] \Big(d(x_i, Tx_i) < \varepsilon \Big) \right).
\end{aligned}$$

Thus, the premise that T has fixed points is weakened to T having approximate fixed points in a b-neighborhood of x and the bound Φ depends, in addition to ε, K, L, η, on $b \in \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$. By taking $g(n) \equiv 0$, we get an approximate fixed point bound for T.

We refer to Section [145, Section 5] for details on the above logical discussion.

7.2 Main results on asymptotic regularity

The following quantitative version of Theorem 7.0.16 is the second main result of the paper [145].

Theorem 7.2.1. *Let C be a convex subset of a UCW-hyperbolic space (X, d, W) and $T : C \to C$ be asymptotically nonexpansive with sequence (k_n).*
Assume that η is a monotone modulus of uniform convexity η, $K \in \mathbb{N}$ is such that $\sum_{n=0}^{\infty} k_n \le K$ and $L \in \mathbb{N}, L \ge 2$ satisfies $\frac{1}{L} \le \lambda_n \le 1 - \frac{1}{L}$ for all $n \in \mathbb{N}$.
Let $x \in C$ and $b > 0$ be such that T has approximate fixed points in a b-neighborhood of x.

Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and, moreover, for all $\varepsilon \in (0, 1]$ and all $g : \mathbb{N} \to \mathbb{N}$,

$$\exists N \le \Phi(K, L, b, \eta, \varepsilon, g)\forall m \in [N, N + g(N)]\Big(d(x_m, Tx_m) < \varepsilon\Big), \tag{7.6}$$

where $\Phi(K, L, b, \eta, \varepsilon, g) = h^M(0)$, with

$$h(n) = g(n + 1) + n + 2, \quad M = \left\lceil \frac{3\left(5KD + D + \frac{11}{2}\right)}{\delta} \right\rceil, \quad D = e^K (b + 2),$$

$$\delta = \frac{\varepsilon}{L^2 f(K)} \cdot \eta\left((1 + K)D + 1, \frac{\varepsilon}{f(K)((1 + K)D + 1)}\right),$$

$$f(K) = 2(1 + (1 + K)^2(2 + K)).$$

Moreover, $N = h^i(0) + 1$ for some $i < M$.

Remark 7.2.2. *Assume, moreover, that $\eta(r, \varepsilon)$ can be written as $\eta(r, \varepsilon) = \varepsilon \cdot \tilde{\eta}(r, \varepsilon)$ such that $\tilde{\eta}$ increases with ε (for a fixed r). Then we can replace η with $\tilde{\eta}$ in the bound $\Phi(K, L, b, \eta, \varepsilon, g)$.*

We give now some consequences. By taking $g(n) \equiv 0$, we obtain an approximate fixed point bound for the asymptotically nonexpansive mapping T.

Corollary 7.2.3. *Assume $(X, d, W), \eta, C, T, (k_n), K, (\lambda_n), L$ are as in the hypotheses of Theorem 7.2.1. Let $x \in C$ and $b > 0$ be such that T has approximate fixed points in a b-neighborhood of x.*

Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and, moreover,

$$\forall \varepsilon \in (0, 1]\, \exists N \le \Phi(K, L, b, \eta, \varepsilon)\Big(d(x_N, Tx_N) < \varepsilon\Big), \tag{7.7}$$

where $\Phi(K, L, b, \eta, \varepsilon) = 2M$ and $M, D, \theta, f(K)$ are as in Theorem 7.2.1.

Furthermore, if C is bounded with diameter d_C, C has the AFPP for asymptotically nonexpansive mappings by Proposition 7.0.15, so T has approximate fixed points in a d_C-neighborhood of x for all $x \in C$. Hence, we get asymptotic regularity and an explicit approximate fixed point bound.

Corollary 7.2.4. *Let $(X, d, W), \eta, C, T, (k_n), K, (\lambda_n), L$ be as in the hypotheses of Theorem 7.2.1. Assume moreover that C is bounded with diameter d_C.*

Then T is λ_n-asymptotically regular and the following holds for all $x \in C$:

$$\forall \varepsilon \in (0,1] \exists N \le \Phi(K, L, d_C, \eta, \varepsilon) \Big(d(x_N, Tx_N) < \varepsilon \Big), \tag{7.8}$$

where $\Phi(K, L, d_C, \eta, \varepsilon)$ is defined as in Theorem 7.2.3 by replacing b with d_C.

Finally, in the case of convex bounded subsets of $CAT(0)$ spaces, we get a quadratic (in $1/\varepsilon$) approximate fixed point bound.

Corollary 7.2.5. *Let X be a $CAT(0)$ space, C be a convex bounded subset of X with diameter d_C and $T : C \to C$ be asymptotically nonexpansive with sequence (k_n).*

Assume that $K \in \mathbb{N}, L \in \mathbb{N}, L \ge 2$ are such that $\sum\limits_{n=0}^{\infty} k_n \le K$ and $\frac{1}{L} \le \lambda_n \le 1 - \frac{1}{L}$ for all $n \in \mathbb{N}$.

Then T is λ_n-asymptotically regular and the following holds for all $x \in C$:

$$\forall \varepsilon \in (0,1] \, \exists N \le \Phi(K, L, d_C, \varepsilon) \Big(d(x_N, Tx_N) < \varepsilon \Big), \tag{7.9}$$

where $\Phi(K, L, d_C, \varepsilon) = 2M$, with

$$\begin{aligned} M &= \left\lceil \frac{1}{\varepsilon^2} \cdot 24L^2 \left(5KD + D + \frac{11}{2} \right) (f(K))^3((1+K)D+1)^2 \right\rceil, \\ D &= e^K \left(d_C + 2 \right), \quad f(K) = 2(1 + (1+K)^2(2+K)). \end{aligned}$$

Chapter 8

Firmly nonexpansive mappings in geodesic spaces

Firmly nonexpansive mappings were introduced by Bruck [35] in the context of Banach spaces and by Browder [31], under the name of *firmly contractive*, in the setting of Hilbert spaces.

Given a closed convex subset C of a Hilbert space, a mapping $T : C \subseteq H \to H$ is said to be *firmly contractive* [32] if the following inequality is satisfied for all $x, y \in C$:

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle. \tag{8.1}$$

As Browder points out, these mappings play an important role in the study of (weak) convergence for sequences of nonlinear operators. An example of a firmly contractive mapping is the metric projection on C. One can easily see that any firmly contractive mapping T is nonexpansive. The converse is not true, as one can see by taking $T = -Id$.

In his study of nonexpansive projections on subsets of Banach spaces, Bruck [35] defined a *firmly nonexpansive* mapping $T : C \to E$, where C is a closed convex subset of a Banach space E, to be a mapping with the property that for all $x, y \in C$ and $t \geq 0$,

$$\|Tx - Ty\| \leq \|(1 - t)(Tx - Ty) + t(x - y)\|. \tag{8.2}$$

In Hilbert spaces these mappings coincide with the firmly contractive ones introduced by Browder. As Bruck shows, to any nonexpansive selfmapping $T : C \to C$ that has fixed points, one can associate a 'large' family of firmly nonexpansive mappings having the same fixed point set with T. Hence, from the point of view of the existence of fixed points on convex closed sets, firmly nonexpansive mappings exhibit a similar behaviour with the nonexpansive ones. However, this is not anymore true if we consider non-convex domains [220].

Firmly nonexpansive mappings in Banach spaces have also been studied in [36, 194]. Furthermore, firmly nonexpansive mappings in the Hilbert ball and, more generally, in hyperbolic spaces in the sense of Reich and Shafrir, have already been studied in [84, 204, 205] and, more recently, in the paper by Kopecká and Reich [150].

If T is firmly nonexpansive and has fixed points, it is well known [32] that the Picard iterate $(T^n x)$ converges weakly to a fixed point of T for any starting point

x, while this is not true for nonexpansive mappings (take again $T = -Id$). This is a first reason for the importance of firmly nonexpansive mappings.

A second reason for the importance of this class of mappings is their correspondence with maximal monotone operators, due to Minty [181]. We refer to [13] for a very nice presentation of this correspondence. The resolvent of a monotone operator was introduced by Minty [181] in Hilbert spaces and by Brézis, Crandall and Pazy [23] in Banach spaces. Given a maximal monotone operator $A : H \to 2^H$ and $\mu > 0$, its associated *resolvent* of order μ, defined by $J_\mu^A := (Id + \mu A)^{-1}$, is a firmly nonexpansive mapping from H to H and the set of fixed points of J_μ^A coincides with the set of zeros of A. Rockafellar's [209] proximal point algorithm uses the resolvent to approximate the zeros of maximal monotone operators.

The goals of this chapter are twofold. First we generalize known results on firmly nonexpansive mappings in Hilbert or Banach spaces to suitable classes of geodesic spaces. Second we give effective results on the asymptotic behaviour of Picard iterations, using proof mining techniques.

The results presented in the sequel were obtained in a joint paper by Ariza-Ruiz, the author and López-Acedo [2].

8.1 Examples of firmly nonexpansive mappings

Bruck's definition of a firmly nonexpansive mapping has a natural extension to W-hyperbolic spaces. Let (X, d, W) be a W-hyperbolic space, $C \subseteq X$ and $T : C \to X$. Given $\lambda \in (0, 1)$, we say that T is *λ-firmly nonexpansive* if for all $x, y \in C$,

$$d(Tx, Ty) \leq d((1-\lambda)x \oplus \lambda Tx, (1-\lambda)y \oplus \lambda Ty) \quad \text{for all } x, y \in C. \tag{8.3}$$

If (8.3) holds for all $\lambda \in (0, 1)$, then T is said to be *firmly nonexpansive*. Applying (W4) one gets that any λ-firmly nonexpansive mapping is nonexpansive.

The first example of a firmly nonexpansive mapping is the metric projection $P_C : X \to C$ on a closed convex set of a CAT(0) space. By Proposition 2.1.5, we know that P_C is well-defined even for closed convex subsets C of UCW-hyperbolic spaces X. By [24, Proposition II.2.4], P_C is nonexpansive and $P_C((1-\lambda)x \oplus \lambda P_C x) = P_C(x)$ for all $x \in X$ and all $\lambda \in (0, 1)$. It is well known that in the setting of Hilbert spaces the metric projection is firmly nonexpansive. We remark that for the Hilbert ball this was proved in [84, p. 111]. The following result shows that the same holds in general CAT(0) spaces.

Proposition 8.1.1. *Let C be a nonempty closed convex subset of a CAT(0) space (X, d). The metric projection P_C onto C is a firmly nonexpansive mapping.*

Proof. Let $x, y \in X$ and $\lambda \in (0, 1)$. One gets that

$$\begin{aligned} d(P_C x, P_C y) &= d\big(P_C((1-\lambda)x \oplus \lambda P_C x), P_C((1-\lambda)y \oplus \lambda P_C y)\big) \\ &\leq d((1-\lambda)x \oplus \lambda P_C x, (1-\lambda)y \oplus \lambda P_C y). \end{aligned}$$

□

As we have already pointed out, Bruck [35] showed for Banach spaces that one can associate to any nonexpansive mapping a family of firmly nonexpansive mappings having the same fixed points. Goebel and Reich [84] obtained the same result for the Hilbert ball. We show in the sequel that Bruck's construction can be adapted also to Busemann spaces.

Let C be a nonempty closed convex subset of a complete Busemann space X and $T : C \to C$ be nonexpansive. For $t \in (0,1)$ and $x \in C$ define

$$T_t^x : C \to C, \quad T_t^x(y) = (1-t)x \oplus tT(y). \tag{8.4}$$

Using (W4), one can easily see that T_t^x is a contraction, so it has a unique fixed point $z_t^x \in C$, by Banach's Contraction Mapping Principle. Let

$$U_t : C \to C, \quad U_t(x) = z_t^x. \tag{8.5}$$

Then $U_t(x) = (1-t)x \oplus tT(U_t(x))$ for all $x \in C$.

Proposition 8.1.2. *[2] U_t is a firmly nonexpansive mapping having the same set of fixed points as T.*

A third example of a firmly nonexpansive mapping is the resolvent of a proper, convex and lower semicontinuous mapping $F : X \to (-\infty, \infty]$ in a CAT(0) space X. Given $\mu > 0$, following Jost [108], the *Moreau-Yosida approximation* F^μ of F is defined by

$$F^\mu(x) := \inf_{y \in X} \left\{\mu\, F(y) + d(x,y)^2\right\}. \tag{8.6}$$

We refer to [6, 224] for applications of the Moreau-Yosida approximation in CAT(0) spaces.

Jost proved [108, Lemma 2] that if $F : X \to (-\infty, \infty]$ is proper, convex and lower semicontinuous, then for every $x \in X$ and $\mu > 0$, there exists a unique $y_\mu \in X$ such that

$$F^\mu(x) = \mu\, F(y_\mu) + d(x, y_\mu)^2.$$

We denote this y_μ with $J_\mu(x)$ and call J_μ the *resolvent* of F of order μ.

In the same paper, Jost shows that for all $\mu > 0$ the resolvent J_μ is nonexpansive [108, Lemma 4] and, furthermore, that for all $\lambda \in [0,1]$,

$$J_{(1-\lambda)\mu}\big((1-\lambda)x \oplus \lambda J_\mu(x)\big) = J_\mu(x) \quad \text{(see [108, Corollary 1])} \tag{8.7}$$

One gets than easily that

Proposition 8.1.3. *Let $F : X \to (-\infty, \infty]$ be proper, convex and lower semicontinuous. Then for every $\mu > 0$, its resolvent J_μ is a firmly nonexpansive mapping.*

We remark that another example of a firmly nonexpansive mapping, given by Kopecká and Reich [150, Lemma 2.2], is the resolvent of a coaccretive operator in the Hilbert ball.

8.2 A fixed point theorem

Given a subset C of a metric space (X, d), a nonexpansive mapping $T : C \to C$ and $x \in C$, the *orbit* $\mathcal{O}(x)$ of x under T is defined by $\mathcal{O}(x) = \{T^n x \mid n = 0, 1, 2, \ldots\}$.

As an immediate consequence of the nonexpansiveness of T, if $\mathcal{O}(x)$ is bounded for some $x \in C$, then all other orbits $\mathcal{O}(y)$, $y \in C$ are bounded. If this is the case, we say that T *has bounded orbits*. Obviously, if T has fixed points, then T has bounded orbits.

The following fixed point theorem is one of the main results of [2].

Theorem 8.2.1. *Let (X, d, W) be a complete UCW-hyperbolic space, $C = \bigcup_{k=1}^{p} C_k$ be a union of nonempty closed convex subsets C_k of X, and $T : C \to C$ be λ-firmly nonexpansive for some $\lambda \in (0,1)$. The following two statements are equivalent:*

(i) T has bounded orbits.

(ii) T has fixed points.

Let us remark that fixed points are not guaranteed if T is merely nonexpansive, as the following trivial example shows. Let $x \neq y \in X$, take $C_1 = \{x\}$, $C_2 = \{y\}$, $C = C_1 \cup C_2$ and $T : C \to C$, $T(x) = y, T(y) = x$. Then T is fixed point free and nonexpansive. If T were λ-firmly nonexpansive for some $\lambda \in (0,1)$, we would get

$$\begin{aligned} 0 < d(x,y) &= d(Tx, Ty) \le d((1-\lambda)x \oplus \lambda Tx, (1-\lambda)y \oplus \lambda Ty) \\ &= d((1-\lambda)x \oplus \lambda y, \lambda x \oplus (1-\lambda)y) = |2\lambda - 1| d(x,y) \quad \text{by (W2)} \\ &< d(x,y), \end{aligned}$$

that is a contradiction.

As an immediate consequence, we get a strengthening of Smarzewski's fixed point theorem for uniformly convex Banach spaces [220], obtained by weakening the hypothesis of C_k being bounded for all $k = 1, \ldots, p$ to T having bounded orbits.

Corollary 8.2.2. *Let X be a uniformly convex Banach space, $C = \bigcup_{k=1}^{p} C_k$ be a union of nonempty closed convex subsets C_k of X, and $T : C \to C$ be λ-firmly nonexpansive for some $\lambda \in (0,1)$.*

Then T has fixed points if and only if T has bounded orbits.

Theorem 8.2.1 follows from the following Propositions 8.2.4 and 8.2.3, which are themselves of independent interest.

Proposition 8.2.3. *Let X be a Busemann space, $C \subseteq X$ be nonempty and $T : C \to C$ be λ-firmly nonexpansive for some $\lambda \in (0,1)$. Then any periodic point of T is a fixed point of T.*

We remark that Proposition 8.2.3 holds for strictly convex Banach spaces too, as they are Busemann spaces.

Proposition 8.2.4. *Let (X, d, W) be a complete UCW-hyperbolic space, $C = \bigcup_{k=1}^{p} C_k$ be a union of nonempty closed convex subsets C_k of X, and $T : C \to C$ be a nonexpansive mapping having bounded orbits.*

Then T has periodic points.

We refer to [2, Section 4] for the proofs.

8.3 Asymptotic behaviour of Picard iterations

The second main result of this section is a theorem on the asymptotic behaviour of Picard iterations of λ-firmly nonexpansive mappings, which generalizes results obtained by Reich and Shafrir [204] for firmly nonexpansive mappings in Banach spaces and the Hilbert ball.

Theorem 8.3.1. *[2] Let C be a subset of a W-hyperbolic space X and $T : C \to C$ be a λ-firmly nonexpansive mapping with $\lambda \in (0,1)$. Then for all $x \in X$ and $k \in \mathbb{Z}_+$,*

$$\lim_{n\to\infty} d(T^{n+1}x, T^n x) = \frac{1}{k} \lim_{n\to\infty} d(T^{n+k}x, T^n x) = \lim_{n\to\infty} \frac{d(T^n x, x)}{n} = r_C(T).$$

The next results are immediate consequences of Theorem 8.3.1.

Corollary 8.3.2. *The following statements are equivalent:*

(i) T is asymptotically regular at some $x \in C$.

(ii) $r_C(T) = 0$.

(iii) T is asymptotically regular.

Corollary 8.3.3. *If T has bounded orbits, then T is asymptotically regular.*

As Adriana Nicolae pointed out to us in a private communication, one can easily see that Proposition 8.2.3 is an immediate consequence of the above corollary. However, the proof given by us in [2, Section 4] holds (with small adaptations) also in more general spaces like geodesic spaces with the betweenness property [184], for which it is not known whether Corollary 8.3.3 is true.

8.3.1 Effective rates of asymptotic regularity

In the sequel we give, for UCW-hyperbolic spaces, a rate of asymptotic regularity for the Picard iteration of T. The methods of proof are inspired by those used for Krasnoselski-Mann iterations of nonexpansive mappings by Kohlenbach [128] in Banach spaces and the author [164] in UCW-hyperbolic spaces (see Section 4.4). We point out that our results are new even for uniformly convex Banach spaces.

Theorem 8.3.4. *[2] Let $b > 0, \lambda \in (0,1)$ and $\eta : (0,\infty) \times (0,2] \to (0,1]$ be a mapping that decreases with r for fixed ε. Then for all UCW-hyperbolic spaces (X, d, W, η), nonempty subsets $C \subseteq X$, λ-firmly nonexpansive mappings $T : C \to C$ and all $x \in C$ such that T has approximate fixed points in a b-neighborhood of x, the following holds:*

$$\forall \varepsilon > 0\, \forall n \geq \Phi(\varepsilon, \eta, \lambda, b) \left(d(T^n x, T^{n+1}x) \leq \varepsilon \right), \tag{8.8}$$

where

$$\Phi(\varepsilon,\eta,\lambda,b) := \begin{cases} \left\lceil \dfrac{b+1}{\varepsilon\,\lambda\,(1-\lambda)\,\eta\left(b+1,\dfrac{\varepsilon}{b+1}\right)} \right\rceil & \text{for } \varepsilon < 2b, \\ 0 & \text{otherwise.} \end{cases} \tag{8.9}$$

Remark 8.3.5. *If, moreover, $\eta(r,\varepsilon)$ can be written as $\eta(r,\varepsilon) = \varepsilon \cdot \tilde{\eta}(r,\varepsilon)$ such that $\tilde{\eta}$ increases with ε (for a fixed r), then the bound $\Phi(\varepsilon,\eta,\lambda,b)$ can be replaced for $\varepsilon < 2b$ by*

$$\tilde{\Phi}(\varepsilon,\eta,\lambda,b) = \left\lceil \frac{b+1}{\varepsilon\,\lambda\,(1-\lambda)\,\tilde{\eta}\left(b+1,\dfrac{\varepsilon}{b+1}\right)} \right\rceil \tag{8.10}$$

For bounded C, we get that T is asymptotically regular with a rate $\Phi(\varepsilon,\eta,\lambda,b)$ that only depends on ε, on X via the monotone modulus of uniform convexity η, on C via an upper bound b on its diameter d_C and on the mapping T via λ. The rate of asymptotic regularity is uniform in the starting point $x \in C$ of the iteration and other data related with X, C and T.

Corollary 8.3.6. *Let b, λ, η be as in the hypothesis of Theorem 8.3.4. Then for all UCW-hyperbolic spaces (X,d,W,η), bounded subsets $C \subseteq X$ with diameter $d_C \le b$, λ-firmly nonexpansive mappings $T : C \to C$ and all $x \in C$,*

$$\forall \varepsilon > 0\, \forall n \ge \Phi(\varepsilon,\eta,\lambda,b) \left(d(T^n x, T^{n+1}x) \le \varepsilon \right),$$

where $\Phi(\varepsilon,\eta,\lambda,b)$ is given by (8.9).

Proof. If C is bounded, then T is asymptotically regular by Corollary 8.3.3. Hence, for all $b \ge d_C$, T has approximate fixed points in a b-neighborhood of x for all $x \in C$. □

As in the case of Krasnoselski-Mann iterations for nonexpansive mappings, we get for CAT(0) spaces a quadratic (in $1/\varepsilon$) rate of asymptotic regularity.

Corollary 8.3.7. *Let $b > 0$ and $\lambda \in (0,1)$. Then for all CAT(0) spaces X, bounded subsets $C \subseteq X$ with diameter $d_C \le b$, λ-firmly nonexpansive mappings $T : C \to C$ and $x \in C$, the following holds*

$$\forall \varepsilon > 0\, \forall n \ge \Psi(\varepsilon,\lambda,b) \left(d(T^n x, T^{n+1}x) \le \varepsilon \right),$$

where

$$\Psi(\varepsilon,\lambda,b) := \begin{cases} \left\lceil \dfrac{8(b+1)}{\lambda\,(1-\lambda)} \cdot \dfrac{1}{\varepsilon^2} \right\rceil & \text{for } \varepsilon < 2b, \\ 0 & \text{otherwise.} \end{cases}$$

8.4 Δ-convergence of Picard iterates

In 1976, Lim [168] introduced a concept of convergence in the general setting of metric spaces, which is known as Δ-convergence. Kuczumow [159] introduced an identical notion of convergence in Banach spaces, which he called *almost convergence*. As shown in [118], Δ-convergence could be regarded, at least for CAT(0) spaces, as an analogue to the usual weak convergence in Banach spaces. Jost [107] introduced a notion of weak convergence in CAT(0) spaces, which was rediscovered by Espínola and Fernández-León [61], who also proved that it is equivalent to Δ-convergence. We refer to [221] for other notions of weak convergence in geodesic spaces.

Let (x_n) be a bounded sequence of a metric space (X, d). We say that (x_n) Δ-*converges* to x if x is the unique asymptotic center of (u_n) for every subsequence (u_n) of (x_n). In this case, we write $x_n \xrightarrow{\Delta} x$ or $\Delta - \lim\limits_{n\to\infty} x_n = x$ and we call x the Δ-limit of (x_n).

Let (X, d) be a metric space and $F \subseteq X$ be a nonempty subset. A sequence (x_n) in X is said to be *Fejér monotone* with respect to F if

$$d(p, x_{n+1}) \le d(p, x_n) \quad \text{for all } p \in F \text{ and } n \ge 0. \tag{8.11}$$

Thus each point in the sequence is not further from any point in F than its predecessor. Obviously, any Fejér monotone sequence (x_n) is bounded and moreover $(d(x_n, p))$ converges for every $p \in F$.

The following lemma is very easy to prove.

Lemma 8.4.1. *Let (X, d) be a metric space, $F \subseteq X$ be a nonempty subset and (x_n) be Fejér monotone with respect to F. Then*

(i) For all $p \in F$, $(d(p, x_n))$ converges and $r(p,(x_n)) = \lim\limits_{n\to\infty} d(p, x_n)$.

(ii) Every subsequence (u_n) of (x_n) is Fejér monotone with respect to F and for all $p \in F$, $r(p,(u_n)) = r(p,(x_n))$. Hence, $r(F,(u_n)) = r(F,(x_n))$ and $A(F,(u_n)) = A(F,(x_n))$.

(iii) If $A(F,(x_n)) = \{x\}$ and $A((u_n)) \subseteq F$ for every subsequence (u_n) of (x_n), then (x_n) Δ-converges to $x \in F$.

This lemma is one of the main tools in obtaining Δ-convergence results, as the following one.

Proposition 8.4.2. *[2] Let (X, d, W) be a complete UCW-hyperbolic space, $C \subseteq X$ be nonempty closed convex and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If T is asymptotically regular at $x \in C$, then the Picard iterate $(T^n x)$ Δ-converges to a fixed point of T.*

By [166, Theorem 3.5] one can replace the assumption that T has fixed points with the equivalent one that T has bounded orbits.

We get finally the following Δ-convergence result for the Picard iteration of a firmly nonexpansive mapping.

Theorem 8.4.3. *Let (X, d, W) be a complete UCW-hyperbolic space, $C \subseteq X$ be nonempty closed convex and $T : C \to C$ be a λ-firmly nonexpansive mapping for some $\lambda \in (0, 1)$. Assume that $Fix(T) \neq \emptyset$. Then for all x in C, $(T^n x)$ Δ-converges to a fixed point of T.*

Proof. Since $Fix(T) \neq \emptyset$, it follows that $r_C(T) = 0$, so, by Corollary 8.3.2, that T is asymptotically regular. Apply now Proposition 8.4.2. $\square$

Part II

Proof mining in (nonlinear) ergodic theory

Chapter 9

A quantitative mean ergodic theorem

In this chapter, we apply proof mining techniques to obtain an explicit uniform rate of metastability of ergodic averages in uniformly convex Banach spaces. This result was obtained by Kohlenbach and the author in [144]. Our result can also be viewed as a finitary version in the sense of Tao of the von Neumann mean ergodic theorem for such spaces and so generalizes similar results obtained for Hilbert spaces by Avigad, Gerhardy and Towsner [5] and Tao [236].

In the following $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let X be a Banach space and $T : X \to X$ be a self-mapping of X. The *ergodic average* starting with $x \in X$ is the sequence $(x_n)_{n\geq 1}$ defined by

$$x_n := \frac{1}{n}\sum_{i=0}^{n-1} T^i x.$$

Uniformly convex Banach spaces were introduced in 1936 by Clarkson in his seminal paper [46]. A Banach space X is called *uniformly convex* if for all $\varepsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $x, y \in X$,

$$\|x\| \leq 1, \quad \|y\| \leq 1 \;\text{ and }\; \|x - y\| \geq \varepsilon \;\text{ imply }\; \left\|\frac{1}{2}(x + y)\right\| \leq 1 - \delta. \tag{9.1}$$

A mapping $\eta : (0, 2] \to (0, 1]$ providing such a $\delta := \eta(\varepsilon)$ for given $\varepsilon \in (0, 2]$ is called a *modulus of uniform convexity*. An example of a modulus of uniform convexity is Clarkson's *modulus of convexity* [46], defined for any Banach space X as the function $\delta_X : [0, 2] \to [0, 1]$ given by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\right\}. \tag{9.2}$$

It is easy to see that $\delta_X(0) = 0$ and that δ_X is nondecreasing. A well-known result is the fact that a Banach space X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ for $\varepsilon \in (0, 2]$. Note that for uniformly convex Banach spaces X, δ_X is the largest modulus of uniform convexity.

In 1939, Garrett Birkhoff proved the following generalization of the von Neumann mean ergodic theorem.

Theorem 9.0.4. *[19] Let X be a uniformly convex Banach space and $T : X \to X$ be a linear nonexpansive mapping. Then for any $x \in X$, the ergodic average (x_n) is convergent.*

In [5], Avigad, Gerhardy and Towsner addressed the issue of finding an effective rate of convergence for (x_n) in Hilbert spaces. They showed that even for the separable Hilbert space L_2 there are simple computable such operators T and computable points $x \in L_2$ such that there is no computable rate of convergence of (x_n). In such a situation, the best one can hope for is an effective bound on the Herbrand normal form of the Cauchy property of (x_n):

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \in \mathbb{N}\, \forall i, j \in [N, N + g(N)]\, \big(\|x_i - x_j\| < \varepsilon\big). \tag{9.3}$$

In [129] (see also [134, Section 17.3]), Kohlenbach obtained general logical metatheorems for (uniformly convex) normed spaces, similar with the ones for metric or W-hyperbolic spaces presented in Chapter 3. These metatheorems guarantee, given a proof of (9.3), the extractability of an effective bound $\Phi(\varepsilon, g, b)$ on $\exists N$ in (9.3) that is highly uniform in the sense that it only depends on g, ε and an upper bound $b \geq \|x\|$ but otherwise is independent from x, X and T. In fact, by a simple renorming argument one can always achieve to have the bound to depend on b, ε only via b/ε.

Guided by this approach, Avigad, Gerhardy and Towsner [5] extracted such a bound from a standard textbook proof of the von Neumann mean ergodic theorem. A less direct proof for the existence of a bound with the above mentioned uniformity features is - for a particular finitary dynamical system - also given by Tao [236] as part of his proof of a generalization of the von Neumann mean ergodic theorem to commuting families of invertible measure preserving transformations $T_1, \ldots, T_l$.

In [144], we apply the same methodology to Birkhoff's proof of Theorem 9.0.4 and extract an even easier to state bound for the more general case of uniformly convex Banach spaces. In this setting, the bound additionally depends on a given modulus of uniform convexity η for X. Despite of our result being significantly more general then the Hilbert space case treated in [5], the extraction of our bound is considerably easier compared to [5] and even numerically better.

9.1 Logical discussion

The proof of the above theorem can be formalized in the theory $\mathcal{A}^\omega[X, \|\cdot\|, \eta]$ of uniformly convex normed spaces, defined in [129]. We refer to [134, Section 17.3] for details on this theory and the corresponding logical metatheorems.

The conclusion of the above theorem is that (x_n) converges for all $x \in C$, that is

$$\exists l \in X \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall p \in \mathbb{N}\big(\|x_{N+p} - l\| < \varepsilon\big), \tag{9.4}$$

which is a $\exists\forall\exists\forall$-formula, so it has a too complicated logical form. One can cut down the complexity a little bit by considering the equivalent (for Banach spaces) conclusion that for all $x \in C$, (x_n) is Cauchy, i.e.:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall p \in \mathbb{N}\big(\|x_{N+p} - x_N\| < \varepsilon\big). \tag{9.5}$$

The Cauchy property of (x_n) is a $\forall\exists\forall$-formula, still too complicated. We are in a situation similar with the one in Section 7. The idea is again to consider the Herbrand normal form of the Cauchy property of (x_n). As in the proof of Lemma 7.1.1, one can easily see that for all $x \in X$, the fact that (x_n) is Cauchy is equivalent to

$$\forall \varepsilon > 0\, \exists N \in \mathbb{N}\, \forall m \in \mathbb{N}\, \forall i, j \in [N, N+m] \big(\|x_i - x_j\| < \varepsilon\big), \tag{9.6}$$

which in turn is equivalent with its Herbrand normal form, given by (9.3):

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \in \mathbb{N}\, \forall i, j \in [N, N+g(N)] \big(\|x_i - x_j\| < \varepsilon\big).$$

As we have discussed above, the logical metatheorems guarantee the extractability of an effective bound $\Phi(\varepsilon, g, b, \eta)$ on $\exists N$, where $b \geq \|x\|$ and η is a modulus of uniform convexity of X.

The only ineffective principle used in Birkhoff's original proof is the fact that any sequence (a_n) of nonnegative real numbers has an infimum. We denote it with (GLB).

In our analysis we first replace this analytical existential statement by a purely arithmetical one, namely

$$(GLB_{ar}): \quad \forall \varepsilon > 0\, \exists N \in \mathbb{N}\, \forall m \in \mathbb{N}\ (a_N \leq a_m + \varepsilon).$$

For the general underlying facts from logic that guarantee this to be possible, we refer to [126] or to [134, Chapter 13]. The principle (GLB_{ar}) is still ineffective as, in general, there is no computable bound on $\exists N$, even for computable (a_n). As above, we consider the equivalent reformulation

$$\forall \varepsilon > 0\, \exists N \in \mathbb{N}\, \forall m \in \mathbb{N}\, \forall i \leq m (a_N \leq a_i + \varepsilon).$$

and then we take its Herbrand normal form

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \in \mathbb{N}\, \forall i \leq g(N) (a_N \leq a_i + \varepsilon).$$

We carry out informally monotone functional interpretation, by which (GLB_{ar}) gets replaced in the proof by the quantitative form provided in Lemma 9.1.1.

Lemma 9.1.1. *[144] Let $(a_n)_{n\geq 0}$ be a sequence of nonnegative real numbers. Then*

(i) $\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \leq \Theta(b, \varepsilon, g)\, \big(a_N \leq a_{g(N)} + \varepsilon\big)$,
where $\Theta(b, \varepsilon, g) = \max\limits_{i \leq K} g^i(1),\ b \geq a_0,\ K = \left\lceil \dfrac{b}{\varepsilon} \right\rceil$.
Moreover, $N = g^i(1)$ *for some* $i < K$.

(ii) $\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \leq h^K(1)\, \forall m \leq g(N) \big(a_N \leq a_m + \varepsilon\big)$,
where $h(n) = \max\limits_{i \leq n} g(i)$ *and* b, K *are as above.*

In the above lemma, h^K is the K-th iterative of $h : \mathbb{N} \to \mathbb{N}$.

9.2 Main results

The main result of the paper [144] is the following quantitative version of Birkhoff's generalization to uniformly convex Banach spaces of the von Neumann mean ergodic theorem.

Theorem 9.2.1. *Assume that X is a uniformly convex Banach space, η is a modulus of uniform convexity and $T : X \to X$ is a linear nonexpansive mapping. Let $b > 0$. Then for all $x \in X$ with $\|x\| \leq b$,*

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \leq \Phi(\varepsilon, g, b, \eta)\, \forall i, j \in [N, N + g(N)] \left(\|x_i - x_j\| < \varepsilon\right). \tag{9.7}$$

where $\Phi(\varepsilon, g, b, \eta) = M \cdot \tilde{h}^K(1)$, with

$$M = \left\lceil \frac{16b}{\varepsilon} \right\rceil, \qquad \gamma = \frac{\varepsilon}{16}\eta\left(\frac{\varepsilon}{8b}\right), \qquad K = \left\lceil \frac{b}{\gamma} \right\rceil,$$
$$h,\, \tilde{h} : \mathbb{N} \to \mathbb{N},\ h(n) = 2(Mn + g(Mn)), \quad \tilde{h}(n) = \max_{i \leq n} h(i).$$

If $\eta(\varepsilon)$ can be written as $\varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $0 < \varepsilon_1 \leq \varepsilon_2 \to \tilde{\eta}(\varepsilon_1) \leq \tilde{\eta}(\varepsilon_2)$, then we can replace η by $\tilde{\eta}$ and the constant '16' by '8' in the definition of γ in the bound above.

Note that our bound Φ is independent from T and depends on the space X and the starting point $x \in X$ only via the modulus of convexity η and the norm upper bound $b \geq \|x\|$. Moreover, it is easy to see that the bound depends on b and ε only via b/ε.

It is well-known that as a modulus of uniform convexity of a Hilbert space X one can take $\eta(\varepsilon) = \varepsilon^2/8$ with $\tilde{\eta}(\varepsilon) = \varepsilon/8$ satisfying the requirements in the last claim of our main theorem. As an immediate consequence, we get the following quantitative version of the von Neumann mean ergodic theorem.

Corollary 9.2.2. *Assume that X is a Hilbert space and $T : X \to X$ is a $T : X \to X$ is a linear nonexpansive mapping. Let $b > 0$. Then for all $x \in X$ with $\|x\| \leq b$,*

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \leq \Phi(\varepsilon, g, b)\, \forall i, j \in [N, N + g(N)] \left(\|x_i - x_j\| < \varepsilon\right). \tag{9.8}$$

where (x_n), Φ are defined as above, but with $K = \left\lceil \frac{512b^2}{\varepsilon^2} \right\rceil$.

We get a similar result for L_p-spaces ($2 < p < \infty$), using the fact that $\eta(\varepsilon) = \dfrac{\varepsilon^p}{p\, 2^p}$ is a modulus of uniform convexity for L_p (see e.g. [128]). Note that $\dfrac{\varepsilon^p}{p\, 2^p} = \varepsilon \cdot \tilde{\eta}_p(\varepsilon)$ with $\tilde{\eta}_p(\varepsilon) = \dfrac{\varepsilon^{p-1}}{p\, 2^p}$ satisfying the monotonicity condition in Theorem 9.2.1.

The bound extracted by Avigad et al. [5] for Hilbert spaces is the following one:

$$\Phi(\varepsilon, g, b) = h^K(1),$$

where $h(n) = n + 2^{13}\rho^4\tilde{g}((n+1)\tilde{g}(2n\rho)\rho^2)$, $\rho = \left\lceil \frac{b}{\varepsilon} \right\rceil$, $K = 512\rho^2$ and $\tilde{g}(n) = \max\limits_{i \leq n}(i + g(i))$. Note that, disregarding the different placement of '$\lceil \cdot \rceil$', the number of iterations K in both this bound and in our bound in Corollary 9.2.2 coincide, whereas the function h being iterated in our bound is much simper than that occurring in the above bound from [5].

Avigad et al. [5] have an improved bound (roughly corresponding to our bound for T being linear nonexpansive) only in the special case when the linear mapping T is an isometry. For this case, they show that one can take h as

$$h(n) = n + 2^{13}\rho^4\tilde{g}\big((n+1)\tilde{g}(1)\rho^2\big),$$

which still is somewhat more complicated than the function h in our bound for the general case of T being nonexpansive. From this, Avigad et al. [5] obtain in the isometric case that $\Phi(\varepsilon, g, b) = 2^{O(\rho^2 \log \rho)}$ for linear functions g, i.e. $g = O(n)$.

Our bound in Corollary 9.2.2 generalizes this complexity upper bound on Φ to T being nonexpansive rather than being an isometry.

Chapter 10

Asymptotic behaviour of Halpern iterations

Let C be a convex subset of a normed space X and $T : C \to C$ nonexpansive. The so-called *Halpern iteration* is defined as follows:

$$x_0 := x, \quad x_{n+1} := \lambda_{n+1} u + (1 - \lambda_{n+1}) T x_n, \tag{10.1}$$

where $(\lambda_n)_{n \geq 1}$ is a sequence in $[0, 1]$, $x \in C$ is the starting point and $u \in C$ is the anchor.

If T is positively homogeneous (i.e. $T(tx) = tT(x)$ for all $t \geq 0$ and all $x \in C$), $\lambda_n = \dfrac{1}{n+1}$ and $u = x$, then

$$x_n = \frac{1}{n+1} S_n x, \quad \text{where} \quad S_0 x = x, \;\; S_{n+1} x = x + T(S_n x). \tag{10.2}$$

Furthermore, if T is linear, then $x_n = \dfrac{1}{n+1} \displaystyle\sum_{i=0}^{n} T^i x$, so the Halpern iteration could be regarded as a nonlinear generalization of ergodic averages. We refer to [244, 170] for a systematic study of the behaviour of iterations given by (10.2).

The following problem was formulated by Reich [199] (see also [193]) and it is still open in its full generality.

Problem 10.0.3. *[199, Problem 6]*
Let X be a Banach space. Is there a sequence (λ_n) such that whenever a weakly compact convex subset C of X possesses the fixed point property for nonexpansive mappings, then (x_n) converges to a fixed point of T for all $x \in C$ and all nonexpansive mappings $T : C \to C$?

Different conditions on (λ_n) were considered in the literature (see also [227] for even more conditions):

$$
\begin{aligned}
&(C1) \qquad \lim_{n\to\infty} \lambda_n = 0,\\
&(C2) \qquad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| \text{ converges},\\
&(C3) \qquad \sum_{n=1}^{\infty} \lambda_n = \infty,\\
&(C4) \qquad \prod_{n=1}^{\infty} (1-\lambda_n) = 0,
\end{aligned}
$$

and, in the case $\lambda_n > 0$ for all $n \geq 1$,

$$
\begin{aligned}
&(C5) \quad \lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0,\\
&(C6) \quad \lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}} = 0.
\end{aligned}
$$

For sequences λ_n in $(0,1)$, conditions (C3) and (C4) are equivalent.

Halpern [97] initiated the study in the Hilbert space setting of the convergence of a particular case of the scheme (10.1). He proved that the sequence (x_n), obtained by taking $u = 0$ in (10.1), converges to a fixed point of T for (λ_n) satisfying certain conditions, two of which are (C1) and (C3). P.-L. Lions [171] improved Halpern's result by showing the convergence of the general (x_n) if (λ_n) satisfies (C1), (C3) and (C5). However, both Halpern's and Lions' conditions exclude the natural choice $\lambda_n = \dfrac{1}{n+1}$.

This was overcome by Wittmann [245], who obtained the most important result on the convergence of Halpern iterations in Hilbert spaces. Wittmann's result, given below, is a nonlinear generalization of the mean ergodic theorem.

Theorem 10.0.4. *[245] Let C be a closed convex subset of a Hilbert space X and $T : C \to C$ a nonexpansive mapping such that the set $Fix(T)$ of fixed points of T is nonempty. Assume that (λ_n) satisfies (C1), (C2) and (C3). Then for any $x \in C$, the Halpern iteration (x_n) converges to the projection Px of x on $Fix(T)$.*

All the partial answers to Reich's problem require that the sequence (λ_n) satisfies (C1) and (C3). Halpern [97] showed in fact that conditions (C1) and (C3) are necessary in the sense that if, for every closed convex subset C of a Hilbert space X and every nonexpansive mappings $T : C \to C$ such that $Fix(T) \neq \emptyset$, the Halpern iteration (x_n) converges to a fixed point of T, then (λ_n) must satisfy (C1) and (C3). That (C1) and (C3) alone are not sufficient to guarantee the convergence of (x_n) was shown in [227]. Recently, Chidume and Chidume [43] and Suzuki [226] proved that if the nonexpansive mapping T in (10.1) is averaged, then (C1) and (C3) suffice for obtaining the convergence of (x_n).

Halpern obtained his result by applying a limit theorem for the resolvent, first shown by Browder [31]. This approach has the advantage that the result can be immediately generalized, once the limit theorem for the resolvent is generalized. This was done by Reich [197].

For $t \in (0,1)$ and $u \in C$, define $T_t^u : C \to C$ by $T_t^u(y) = tu + (1-t)Ty$. Since T_t^u is a contraction, we apply Banach's Contraction Mapping Principle to get a unique fixed point $z_t^u \in C$:

$$z_t^u = tu + (1-t)Tz_t^u. \tag{10.3}$$

Theorem 10.0.5. *[197] Let C be a closed convex subset of a uniformly smooth Banach space X, and let $T : C \to C$ be nonexpansive such that $Fix(T) \neq \emptyset$. Then $\lim\limits_{t \to 0^+} z_t^u$ exists and is a fixed point of T.*

A similar result was obtained by Kirk [114] for CAT(0) spaces (for the Hilbert ball, which is an example of a CAT(0) space, this is already due to [84]).

As a consequence of Theorem 10.0.5, a partial positive answer to Problem 10.0.3 was obtained [197] for uniformly smooth Banach spaces and $\lambda_n = \dfrac{1}{(n+1)^\alpha}$ with $0 < \alpha < 1$. Furthermore, Reich [202] proved the strong convergence of (x_n) in the setting of uniformly smooth Banach spaces that have a weakly sequentially continuous duality mapping for general (λ_n) satisfying (C1), (C3) and being decreasing. Another partial answer in the case of uniformly smooth Banach spaces was obtained by Xu [246] for (λ_n) satisfying (C1), (C3) and (C6) (which is weaker than Lions' (C5)).

In [218], Shioji and Takahashi extended Wittmann's result to Banach spaces with uniformly Gâteaux differentiable norm and with the property that $\lim\limits_{t \to 0^+} z_t^u$ exists and is a fixed point of T. Since Wittmann's theorem does not refer to any linearity but only to a convexity structure of the underlying space X (in order to make sense of the Halpern iteration) it can be formulated in the context of W-hyperbolic spaces and was established by Saejung [210] for CAT(0) spaces. A similar result for the Hilbert ball had already been proved in [151].

Kohlenbach extracted in [135] – making use of a rate of asymptotic regularity due to the author [165] – a rate of metastability for Wittmann's theorem 10.0.4. Wittmann's proof is based on weak compactness which, though covered by the existing proof mining machinery, in general can cause bounds of extremely poor quality. In the case at hand that could be avoided as during the logical extraction procedure the use of weak compactness turned out to be eliminable.

In this chapter we present effective results on Saejung's [210] and Shioji and Takahashi [218] generalizations of Wittmann's nonlinear ergodic theorem, obtained by Kohlenbach and the author [147, 146]. These results are of broader relevance in the proof mining program as they open up new frontiers for its applicability, namely to proofs that prima facie use some substantial amount of the axiom of choice. This stems from the use of Banach limits made in the proofs in [210] and [218]. The existence of Banach limits is either proved by applying the Hahn-Banach theorem to l^∞, which due to the nonseparability of that space needs the axiom of choice, or via ultralimits which, again, needs choice. While weak compactness as used in Wittmann's proof at least was in principle covered by existing logical metatheorems, this is not the case for Banach limits.

In these convergence proofs, Banach limits are used to establish the almost convergence in the sense of Lorentz of some sequence (a_n) of reals towards a which –

together with $\limsup_{n\to\infty}(a_{n+1}-a_n) \le 0$ – in turn implies that $\limsup_{n\to\infty} a_n \le a$. This line of reasoning goes back to Lorentz' classical paper [174], whose relevance in nonlinear ergodic theory was first realized by Reich [195]. Other relevant papers using Banach limits in the context of nonlinear ergodic theory are [38, 206, 151].

We develop in [147] a method to replace the use of Banach limits in Saejung's proof by a direct arithmetical reasoning and apply it again to Shioji and Takahashi proof. As the way Banach limits are used in these proofs seems to be rather typical for other proofs in nonlinear ergodic theory, our method may also be seen as providing a blueprint for doing similar unwindings in those cases as well.

10.1 Effective results on Halpern iterations in CAT(0) spaces

Let $T : C \to C$ be a nonexpansive selfmapping of a convex subset C of a W-hyperbolic space (X, d, W). We can define the Halpern iteration in this setting too:

$$x_0 := x, \quad x_{n+1} := \lambda_{n+1}u \oplus (1-\lambda_{n+1})Tx_n, \tag{10.4}$$

where $x, u \in C$ and $(\lambda_n)_{n\ge1}$ is a sequence in $[0, 1]$.

The following theorem generalizes Wittmann's theorem to CAT(0) spaces and was proved recently by Saejung [210].

Theorem 10.1.1. *[210] Let C be a closed convex subset of a complete* CAT(0) *space X and $T : C \to C$ a nonexpansive mapping such that the set $Fix(T)$ of fixed points of T is nonempty. Assume that (λ_n) satisfies (C1), (C2) and (C3). Then for any $u, x \in C$, the iteration (x_n) converges to the projection Pu of u on $Fix(T)$.*

By [114, Theorem 18], $Fix(T) \ne \emptyset$ is guaranteed to hold if C is bounded. In [147] we only consider this case and our bounds will depend on an upper bound M on the diameter d_C of C. However, similar to [135], it is not hard to adopt our bounds to the case where the condition $M \ge d_C$ is being replaced by $M \ge d(u, p), d(x, p)$ for some fixed point $p \in C$ of T.

The main results of this section are effective versions of Theorem 10.1.1, obtained by Kohlenbach and the author [147] by applying proof mining techniques to Saejung's proof. As this proof is essentially ineffective – as we discussed above – a computable rate of convergence does not exist, while an effective and highly uniform rate of metastability (depending only on the input data displayed in Theorems 10.1.8, 10.1.9) is guaranteed to exist (via our elimination of Banach limits from the proof) by Theorem 3.1.11. Note that the conditions on α, β, θ as well as T are all purely universal, while the conclusion $\exists N \forall m, n \in [N, N + g(N)]\ (d(x_n, x_m) < \varepsilon)$ can be written as a purely existential formula and that quantification over all (λ_n) in $[0, 1]$ can be represented as $\forall y \le s$ for some simple function $s : \mathbb{N}^2 \to \mathbb{N}$).

10.1.1 Effective rates of asymptotic regularity

The first step towards proving the convergence of Halpern iterations is to obtain the asymptotic regularity and this can be done in the very general setting of W-hyperbolic spaces.

The following two propositions provide effective rates of asymptotic regularity for the Halpern iteration. Proposition 10.1.2 generalizes to W-hyperbolic spaces a result obtained by the author for Banach spaces [164]. Similar methods were used in [48] to obtain rates of asymptotic regularity for alternative iterative methods of nonexpansive mappings. Proposition 10.1.3, proved by Kohlenbach and the author in [146], is new even for the case of Banach spaces.

Let (X,d,W) be a W-hyperbolic space, $C\subseteq X$ be a bounded convex subset with diameter d_C, $T:C\to C$ be nonexpansive and (x_n) given by (10.4).

Proposition 10.1.2. *[147] Assume that (λ_n) satisfies (C1), (C2) and (C3). Then (x_n) is asymptotically regular and $\lim\limits_{n\to\infty} d(x_n,x_{n+1})=0$.*
Furthermore, if α is a rate of convergence of (λ_n), β is a Cauchy modulus of $s_n := \sum\limits_{i=1}^{n}|\lambda_{i+1}-\lambda_i|$ and θ is a rate of divergence of $\sum\limits_{n=1}^{\infty}\lambda_{n+1}$, then for all $\varepsilon\in(0,2)$,

$$\forall n\geq \tilde{\Phi}\ \left(d(x_n,x_{n+1})\leq\varepsilon\right) \quad \textit{and} \quad \forall n\geq \Phi\ \left(d(x_n,Tx_n)\leq\varepsilon\right),$$

where

$$\begin{aligned}\tilde{\Phi}:=\tilde{\Phi}(\varepsilon,M,\theta,\beta) &= \theta\left(\beta\left(\frac{\varepsilon}{4M}\right)+1+\left\lceil\ln\left(\frac{2M}{\varepsilon}\right)\right\rceil\right)+1,\\ \Phi:=\Phi(\varepsilon,M,\theta,\alpha,\beta) &= \max\left\{\theta\left(\beta\left(\frac{\varepsilon}{8M}\right)+1+\left\lceil\ln\left(\frac{4M}{\varepsilon}\right)\right\rceil\right)+1,\alpha\left(\frac{\varepsilon}{4M}\right)\right\},\end{aligned}$$

with $M\in\mathbb{Z}_+$ such that $M\geq d_C$.

Thus, we obtain an effective rate of asymptotic regularity $\Phi(\varepsilon,M,\theta,\alpha,\beta)$ which depends only on the error ε, on an upper bound M on the diameter d_C of C, and on (λ_n) via α,β,θ. In particular, the rate Φ does not depend on u,x or T, so Proposition 10.1.2 provides a quantitative version of the main theorem in [3].

Proposition 10.1.3. *[147] Assume that $\lambda_n\in(0,1)$ for all $n\geq 2$ and that (λ_n) satisfies (C1), (C2) and (C4). Then $\lim\limits_{n\to\infty} d(x_n,x_{n+1})=0$ and (x_n) is asymptotically regular.*
Furthermore, if α is a rate of convergence of (λ_n), β is a Cauchy modulus of $s_n := \sum\limits_{i=1}^{n}|\lambda_{i+1}-\lambda_i|$ and θ is a rate of convergence of $\prod\limits_{n=1}^{\infty}(1-\lambda_{n+1})=0$ towards 0, then for all $\varepsilon\in(0,2)$,

$$\forall n\geq \tilde{\Phi}\ \left(d(x_n,x_{n+1})\leq\varepsilon\right) \quad \textit{and} \quad \forall n\geq \Phi\ \left(d(x_n,Tx_n)\leq\varepsilon\right),$$

where

$$\begin{aligned}\tilde{\Phi}(\varepsilon,M,\theta,\beta,D) &= \theta\left(\frac{D\varepsilon}{2M}\right)+1,\\ \Phi(\varepsilon,M,\theta,\alpha,\beta,D) &= \max\left\{\theta\left(\frac{D\varepsilon}{4M}\right)+1,\alpha\left(\frac{\varepsilon}{4M}\right)\right\},\end{aligned}$$

with $M\in\mathbb{Z}_+$ such that $M\geq d_C$ and $0<D\leq \prod\limits_{n=1}^{\beta(\varepsilon/4M)}(1-\lambda_{n+1})$.

As an immediate consequence of Proposition 10.1.3, for $\lambda_n = \dfrac{1}{n+1}$ we get a quadratic (in $1/\varepsilon$) rate of asymptotic regularity. For Banach spaces, this rate of asymptotic regularity was obtained by Kohlenbach in [135]. In [164], the author obtained an exponential rate of asymptotic regularity due to the fact that he used the version for Banach spaces of Proposition 10.1.2, which needs a rate of divergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1}$.

Corollary 10.1.4. *Assume that $\lambda_n = \dfrac{1}{n+1}$ for all $n \geq 1$. Then for all $\varepsilon \in (0,1)$,*

$$\forall n \geq \tilde{\Psi}(\varepsilon, M) \ \left(d(x_n, x_{n+1}) \leq \varepsilon\right) \quad \textit{and} \quad \forall n \geq \Psi(\varepsilon, M) \ \left(d(x_n, Tx_n) \leq \varepsilon\right), \tag{10.5}$$

where

$$\tilde{\Psi}(\varepsilon, M) = \left\lceil \frac{2M}{\varepsilon} + \frac{8M^2}{\varepsilon^2} \right\rceil - 1 \quad \textit{and} \quad \Psi(\varepsilon, M) = \left\lceil \frac{4M}{\varepsilon} + \frac{16M^2}{\varepsilon^2} \right\rceil - 1,$$

with $M \in \mathbb{Z}_+$ such that $M \geq d_C$.

Proof. Obviously, $\displaystyle\lim_{n\to\infty} \frac{1}{n+1} = 0$ with a rate of convergence $\alpha(\varepsilon) = \left\lceil \frac{1}{\varepsilon} \right\rceil - 1 \geq 1$. Furthermore, $\theta(\varepsilon) := \left\lceil \dfrac{2}{\varepsilon} \right\rceil - 2$ is a rate of convergence of $\displaystyle\prod_{n=1}^{\infty} \left(1 - \frac{1}{n+2}\right)$ towards 0. Since

$$s_n := \sum_{k=1}^{n} \left| \frac{1}{k+2} - \frac{1}{k+1} \right| = \frac{1}{2} - \frac{1}{n+2},$$

we get that $\displaystyle\lim_{n\to\infty} s_n = 1/2$ with Cauchy modulus $\beta(\varepsilon) := \begin{cases} \lceil 1/\varepsilon \rceil - 1 & \text{if } \varepsilon \geq 1/2 \\ \lceil 1/\varepsilon \rceil - 2 & \text{if } \varepsilon < 1/2 \end{cases}$.

Finally, $\displaystyle\prod_{n=1}^{\beta(\varepsilon/4M)} \left(1 - \frac{1}{n+2}\right) = \frac{2}{\lceil 4M/\varepsilon \rceil}$, as $\dfrac{\varepsilon}{4M} < \dfrac{1}{2}$, so we can take $D := \dfrac{2}{\lceil 4M/\varepsilon \rceil}$. Apply now Proposition 10.1.3 and use the fact that $\lceil x \rceil \leq x + 1$ to get the result. $\square$

Logical discussion

That we even get full rates of convergence in Propositions 10.1.2, 10.1.3 is due to the fact that the original proof of asymptotic regularity is essentially constructive. For such proofs, the requirement of the statement to be proved to have the form $\forall x \exists y\, A_{qf}(x, y)$ with quantifier-free A_{qf}, which is crucial for ineffective proofs, is not needed (note that the Cauchy property is a $\forall\exists\forall$-statement). This is because we do not have to preprocess the proof using some negative translation (which maps proofs with classical logic into ones with constructive logic only) and can directly apply proof-theoretic techniques such as (an appropriate monotone form of) Kreisel's so-called modified realizability interpretation. Logical metatheorems covering such situations are proved in [75]. As a consequence of getting full rates of convergence in Propositions 10.1.2, 10.1.3 one then also has to strengthen the premises on the convergence of (λ_n) and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|$ by full rates of convergence α, β. If we would interpret the proof as an ineffective one using the metatheorems from [129], then one would only get a rate of metastability in the conclusion but also would only need rates of metastability for these premises (note that $\sum_{n=1}^{\infty} \lambda_n = \infty$ is a $\forall\exists$-statement so that there is no difference here between a full rate and a rate of metastability).

10.1.2 Elimination of Banach limits

Let us recall that a *Banach limit* [8] is a linear functional $\mu : \ell^\infty \to \mathbb{R}$ satisfying the following properties:

(i) $\mu((x_n)) \geq 0$ if $x_n \geq 0$ for all $n \geq 0$;

(ii) $\mu(\mathbf{1}) = 1$;

(iii) $\mu((x_n)) = \mu((x_{n+1}))$.

Here $\mathbf{1}$ is the sequence $(1, 1, \ldots)$ and (x_{n+1}) is the sequence $(x_1, x_2, \ldots)$.

As we have already said, to prove the existence of Banach limits one needs the axiom of choice (see, e.g., [225]). Banach limits are mainly used in Saejung's convergence proof to get the following.

Lemma 10.1.5. *[218] Let $(a_k) \in \ell^\infty$ and $a \in \mathbb{R}$ be such that $\mu((a_k)) \leq a$ for all Banach limits μ and $\limsup_{k\to\infty}(a_{k+1} - a_k) \leq 0$. Then $\limsup_{k\to\infty} a_k \leq a$.*

Given a sequence $(a_k)_{k\geq 1}$, consider for all $n, p \geq 1$ the following average

$$C_{n,p}((a_k)) = \frac{1}{p} \sum_{i=n}^{n+p-1} a_i. \tag{10.6}$$

For simplicity we shall write $C_{n,p}(a_k)$.

Lemma 10.1.5 is proved using a result that goes back to Lorentz [174].

Lemma 10.1.6. *Let $(a_k) \in \ell^\infty$ and $a \in \mathbb{R}$. The following are equivalent:*

(i) $\mu((a_k)) \leq a$ for all Banach limits μ.

(ii) For all $\varepsilon > 0$ there exists $P \geq 1$ such that $C_{n,p}(a_k) \leq a + \varepsilon$ for all $p \geq P$ and $n \geq 1$.

In fact, one only needs the implication '(i) $\Rightarrow$ (ii)' which is established in [218] using the following sublinear functional

$$q : l^\infty \to \mathbb{R}, \quad q((a_k)) := \limsup_{p\to\infty} \sup_{n\geq 1} \frac{1}{p} \sum_{i=n}^{n+p-1} a_i = \limsup_{p\to\infty} \sup_{n\geq 1} C_{n,p}(a_k).$$

Now fix $(a_k) \in l^\infty$ and use the Hahn-Banach theorem to show the existence of a linear functional $\mu : l^\infty \to \mathbb{R}$ such that $\mu \leq q$ and $\mu((a_k)) = q((a_k))$. Then μ is a Banach limit and so – by (i) – $q((a_k)) = \mu((a_k)) \leq a$ which gives (ii).

Our elimination of the use of the Banach limit μ was obtained in two steps.

First, we modified the proof that, for the sequence in question in the proof from [210], the fact $\mu((a_k)) \leq a$ holds for all Banach limits μ by directly showing this for q instead of μ. This already establishes the actual elimination of the use of the axiom of choice hidden in the application of the Hahn-Banach theorem, since the existence of q follows by just using uniform arithmetical comprehension in the form of an operator $E : \mathbb{N}^\mathbb{N} \to \{0, 1\}$ defined by

$$E(f) = 0 \leftrightarrow \forall n \in \mathbb{N}(f(n) = 0),$$

that is needed (and sufficient) to form both the 'sup' as well as the '*limsup*' in the definition of q (as a function in (a_k)). Using an argument due to Feferman [63], the use of E can be eliminated in favor of ordinary (non-uniform) arithmetic comprehension

$$\forall f : \mathbb{N}^2 \to \mathbb{N}\, \exists g : \mathbb{N} \to \mathbb{N}\, \forall k \in \mathbb{N}\, \big(g(k) = 0 \leftrightarrow \forall n \in \mathbb{N}\, (f(k, n) = 0)\big),$$

which is covered (as a very special case of general comprehension over numbers) by the existing logical metatheorems and results in extractable bounds of restricted complexity, namely bounds that are definable by primitive recursive functionals in the extended sense of Gödel's calculus T [87] (which, however, contains the famous so-called Ackermann function), though in general not of ordinarily primitive recursive type.

In a second step, we also eliminated the use of q in favor of just elementary lemmas on the finitary objects $C_{n,p}$. As a consequence, we get a bound having a much more restricted complexity.

Finally, we got the following effective version of Lemma 10.1.6.

Lemma 10.1.7. *[147] Let (a_k) be a real sequence, $a \in \mathbb{R}$ and $P : (0, \infty) \to \mathbb{Z}_+$ be such that*

$$\forall \varepsilon > 0\, \forall n \geq 1\, \big(C_{n,P(\varepsilon)}(a_k) \leq a + \varepsilon\big). \tag{10.7}$$

Assume that $\limsup\limits_{k\to\infty}(a_{k+1} - a_k) \leq 0$ with effective rate θ.

Then $\limsup\limits_{k\to\infty} a_k \leq a$ with effective rate ψ, given by

$$\psi(\varepsilon, P, \theta) = \theta\left(\frac{\varepsilon}{\tilde{P} + 1}\right) + \tilde{P}, \qquad \text{where } \tilde{P} := P\left(\frac{\varepsilon}{2}\right). \tag{10.8}$$

10.1.3 Effective rates of metastability

These are the effective versions of Theorem 10.1.1.

Theorem 10.1.8. *[147] Assume that X is a complete $\mathrm{CAT}(0)$ space, $C \subseteq X$ is a closed bounded convex subset with diameter d_C and $T : C \to C$ is nonexpansive. Let (λ_n) satisfy (C1), (C2) and (C3).*

Then the Halpern iteration (x_n) is Cauchy.
Furthermore, let α be a rate of convergence of (λ_n), β be a Cauchy modulus of $s_n := \sum_{i=1}^{n} |\lambda_{i+1} - \lambda_i|$ and θ be a rate of divergence of $\sum_{n=1}^{\infty} \lambda_{n+1}$.

Then for all $\varepsilon \in (0,2)$ and $g : \mathbb{N} \to \mathbb{N}$,

$$\exists N \le \Sigma(\varepsilon, g, M, \theta, \alpha, \beta)\ \forall m, n \in [N, N+g(N)]\ (d(x_n, x_m) \le \varepsilon),$$

where

$$\Sigma(\varepsilon, g, M, \theta, \alpha, \beta) = \theta^+\left(\Gamma - 1 + \left\lceil \ln\left(\frac{12M^2}{\varepsilon^2}\right)\right\rceil\right) + 1 \tag{10.9}$$

with $M \in \mathbb{Z}_+$ such that $M \ge d_C$,

$$\begin{aligned}
\Gamma &= \max\left\{\chi_k^*(\varepsilon^2/12) \mid \left\lceil \frac{1}{\varepsilon_0}\right\rceil \le k \le \widetilde{f^*}^{(\lceil M^2/\varepsilon_0^2 \rceil)}(0) + \left\lceil \frac{1}{\varepsilon_0}\right\rceil\right\},\\
\chi_k^*(\varepsilon) &= \tilde{\Phi}\left(\frac{\varepsilon}{4M(\tilde{P}_k(\varepsilon/2)+1)}\right) + \tilde{P}_k(\varepsilon/2),\\
\tilde{P}_k(\varepsilon) &= \left\lceil \frac{12M^2(k+1)}{\varepsilon}\Phi\left(\frac{\varepsilon}{12M(k+1)}\right)\right\rceil,\\
\tilde{\Phi}(\varepsilon, M, \theta, \beta) &= \theta\left(\beta\left(\frac{\varepsilon}{4M}\right) + 1 + \left\lceil \ln\left(\frac{2M}{\varepsilon}\right)\right\rceil\right) + 1,\\
\Phi(\varepsilon, M, \theta, \alpha, \beta) &= \max\left\{\tilde{\Phi}\left(\frac{\varepsilon}{2}, M, \theta, \beta\right), \alpha\left(\frac{\varepsilon}{4M}\right)\right\},
\end{aligned}$$

$$\Delta_k^*(\varepsilon, g) = \frac{\varepsilon}{3g_{\varepsilon,k}(\Theta_k(\varepsilon) - \chi_k^*(\varepsilon/3))}, \qquad \varepsilon_0 = \frac{\varepsilon^2}{24(M+1)^2},$$

$$\Theta_k(\varepsilon) = \theta\left(\chi_k^*\left(\frac{\varepsilon}{3}\right) - 1 + \left\lceil \ln\left(\frac{3M^2}{\varepsilon}\right)\right\rceil\right) + 1, \quad g_{\varepsilon,k}(n) = n + g\left(n + \chi_k^*\left(\frac{\varepsilon}{3}\right)\right),$$

$$f(k) = \max\left\{\left\lceil \frac{M^2}{\Delta_k^*(\varepsilon^2/4, g)}\right\rceil, k\right\} - k, \qquad \theta^+(n) = \max\{\theta(i) \mid i \le n\},$$

$$f^*(k) = f\left(k + \left\lceil \frac{1}{\varepsilon_0}\right\rceil\right) + \left\lceil \frac{1}{\varepsilon_0}\right\rceil, \qquad \widetilde{f^*}(k) = k + f^*(k).$$

A similar result can be obtained by assuming that (λ_n) satisfies (C1), (C2) and (C4) with corresponding rates.

Theorem 10.1.9. *[147] Assume that X is a complete* CAT(0) *space, $C \subseteq X$ is a closed bounded convex subset with diameter d_C and $T : C \to C$ is nonexpansive. Let (λ_n) satisfy (C1), (C2), (C4) and $\lambda_n \in (0,1)$ for all $n \ge 2$.*

Then the Halpern iteration (x_n) is Cauchy.
Furthermore, if α is a rate of convergence of (λ_n), β is a Cauchy modulus of $s_n := \sum_{i=1}^{n} |\lambda_{i+1} - \lambda_i|$ and θ is a rate of convergence of $\prod_{n=1}^{\infty}(1 - \lambda_{n+1})$ towards 0, then for all $\varepsilon \in (0,2)$ and $g : \mathbb{N} \to \mathbb{N}$,

$$\exists N \le \Sigma(\varepsilon, g, M, \theta, \alpha, \beta, (\lambda_n))\ \forall m, n \in [N, N+g(N)]\ (d(x_n, x_m) \le \varepsilon),$$

where

$$\Sigma(\varepsilon, g, M, \theta, \alpha, \beta, (\lambda_n)) = max \left\{ \Theta_k(\varepsilon^2/4) \mid \left\lceil \frac{1}{\varepsilon_0} \right\rceil \leq k \leq \widetilde{f^*}^{(\lceil M^2/\varepsilon_0^2 \rceil)}(0) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil \right\},$$

with $M \in \mathbb{Z}_+$ *such that* $M \geq d_C$,

$$0 < D \leq \prod_{n=1}^{\beta(\varepsilon/4M)} (1 - \lambda_{n+1}),$$

$$\begin{aligned}
\tilde{\Phi}(\varepsilon, M, \theta, \beta, D) &= \theta\left(\frac{D\varepsilon}{2M}\right) + 1, \\
\Phi(\varepsilon, M, \theta, \alpha, \beta, D) &= \max\left\{\theta\left(\frac{D\varepsilon}{4M}\right) + 1, \alpha\left(\frac{\varepsilon}{4M}\right)\right\}, \\
\Theta_k(\varepsilon) &= \theta\left(\frac{D_k\varepsilon}{3M^2}\right) + 1, \\
0 < D_k &\leq \prod_{n=1}^{\chi_k^*(\varepsilon/3)-1} (1 - \lambda_{n+1}),
\end{aligned}$$

and the other constants and functionals being defined as in Theorem 10.1.8.

One can modify Theorems 10.1.8, 10.1.9 so that only metastable versions of α, β and θ are needed. However, we refrain from doing so as the result would be rather unreadable and in the practical cases at hand – such as $\lambda_n = \dfrac{1}{n+1}$ – full rates α, β, θ are easy to compute.

Corollary 10.1.10. *Assume that* $\lambda_n = \dfrac{1}{n+1}$ *for all* $n \geq 1$. *Then for all* $\varepsilon \in (0, 1)$ *and* $g : \mathbb{N} \to \mathbb{N}$,

$$\exists N \leq \Sigma(\varepsilon, g, M)\ \forall m, n \in [N, N + g(N)]\ (d(x_n, x_m) \leq \varepsilon),$$

where

$$\Sigma(\varepsilon, g, M) = \left\lceil \frac{12M^2(\chi_L^*(\varepsilon^2/12) + 1)}{\varepsilon^2} \right\rceil - 1$$

with

$$\begin{aligned}
L &= \widetilde{f^*}^{(\lceil M^2/\varepsilon_0^2 \rceil)}(0) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil, \\
\tilde{P}_k(\varepsilon) &= \left\lceil \frac{12M^2(k+1)}{\varepsilon} \cdot \left(\left\lceil \frac{48M(k+1)}{\varepsilon} + \frac{2304M^4(k+1)^2}{\varepsilon^2} \right\rceil - 1 \right) \right\rceil, \\
\chi_k^*(\varepsilon) &= \left\lceil \frac{8M^2(\tilde{P}_k(\varepsilon/2) + 1)}{\varepsilon} + \frac{128M^4(\tilde{P}_k(\varepsilon/2) + 1)^2}{\varepsilon^2} \right\rceil - 1 + \tilde{P}_k(\varepsilon/2), \\
\Theta_k(\varepsilon) &= \left\lceil \frac{3M^2(\chi_k^*(\varepsilon/3) + 1)}{\varepsilon} \right\rceil - 1,
\end{aligned}$$

while the other constants and functionals are defined as in Theorem 10.1.8.

Proof. Since $\prod_{k=1}^{n}\left(1-\frac{1}{k+2}\right) = \frac{2}{n+2}$, we get that $\theta(\varepsilon) := \left\lceil \frac{2}{\varepsilon} \right\rceil - 2$ is a rate of convergence of $\prod_{n=1}^{\infty}\left(1-\frac{1}{n+2}\right)$ towards 0. Furthermore, we can take $D_k := \frac{2}{\chi_k^*(\varepsilon/3)+1}$ in Theorem 10.1.9 and – using Corollary 10.1.4 – $\Phi := \Psi, \tilde{\Phi} := \tilde{\Psi}$ from that corollary. We then get $P_k(\varepsilon)$, $\chi_k^*(\varepsilon)$ as above and

$$\Theta_k(\varepsilon) \;=\; \theta\left(\frac{D_k\varepsilon}{3M^2}\right)+1 = \left\lceil \frac{3M^2(\chi_k^*(\varepsilon/3)+1)}{\varepsilon} \right\rceil - 1.$$

The claim now follows by Theorem 10.1.9 using that χ_k^* increases with k. $\square$

Despite its superficially quite different look, the bound in Corollary 10.1.10 has an overall similar structure as the bound extracted for the Hilbert space case in [135]: the bound results from applying a certain function $\Theta_k(\varepsilon)$ to a number $k := L$ which is the result of an iteration of a function $\tilde{f}^*$ (starting at some arbitrary value, e.g. 0), where $\tilde{f}^*(k)$ is – disregarding many details – something close to $\Theta_k(\varepsilon)+g(\Theta_k(\varepsilon))$. This is also the structure of the bound in [135, Theorem 3.3] (where Δ^* plays the role of $\tilde{f}^*$). Note that the number of iterations essentially is M^6/ε^4 while it was roughly M^4/ε^4 in the bound in [135, Theorem 3.3]. The main difference, though, is that now Θ_k is significantly more involved compared to [135], most of its terms stemming from the remains of the original Banach-limit argument.

Remark 10.1.11. *Subsequently, our results have been further generalized in [212] to the case of unbounded C provided that T possesses a fixed point p. Then the above bounds hold with $M \geq d_C$ being replaced by $M \geq 4\max\{d(u,x), d(u,p)\}$. In [212] our method is also adapted to obtain similar bounds for more general schemes of so-called modified Halpern iterations.*

10.2 Halpern iterations in uniformly smooth Banach spaces

The following extension of Wittmann's result was obtained by Shioji and Takahashi [218]. We refer to [202] for an earlier result in this direction.

Theorem 10.2.1. *Let X be a Banach space whose norm is uniformly Gâteaux differentiable, $C \subseteq X$ be closed and convex and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume that*

(i) $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_{n+1}$ diverges and $\sum_{n=1}^{\infty} |\lambda_{n+1}-\lambda_n|$ converges;

(ii) (z_t^u) converges strongly to a fixed point z of T as $t \downarrow 0$.

Then the Halpern iteration converges strongly to z.

In [146] we extracted a rate of metastability for Theorem 10.2.1. The significance of this is twofold:

(i) As the proof again uses Banach limits we further substantiate our claim that the machinery developed in [147] to eliminate arguments based on Banach limits is indeed a general method. In fact, we can literally re-use most of the technical lemmas from [147] showing the striking modularity of the proof mining approach.

(ii) The proof is based on the existence of a uniformly continuous (in a suitable sense) duality mapping J which also plays an important role in numerous other proofs in nonlinear analysis. In the next section we indicate how this structure can be nicely incorporated into the framework of the logical metatheorems presented in Chapter 3.

10.2.1 A logical metatheorem

Theorem 10.2.1 uses a smoothness property of Banach spaces, namely that the norm is uniformly Gâteaux differentiable. It turns out that monotone functional interpretation requires that the space even has a uniformly Fréchet differentiable norm, i.e. it is uniformly smooth.

The uniform smoothness of a space X can be universally axiomatized once a constant $\tau_X : \mathbb{N} \to \mathbb{N}$ representing a suitable notion of a modulus of uniform smoothness is given. Then the corresponding logical metatheorem will guarantee the extractability of an effective uniform bound that, in addition to its usual input data, will only depend on τ_X.

A very important property for the concrete application given in this section is the norm-to-norm uniform continuity on bounded sets of the normalized duality map J of X which holds in uniformly smooth spaces, whereas uniform Gâteaux differentiability only implies the norm-to-weak* uniform continuity of J.

Definition 10.2.2. *Let X be a Banach space and X^* its dual space. Then the mapping*

$$J : X \to 2^{X^*}, \quad Jx := \{y \in X^* : \langle x, y\rangle = \|x\|^2 = \|y\|^2\}$$

is called the (normalized) duality mapping of X. Here $\langle x, y\rangle$ denotes $y(x)$.

By the Hahn-Banach theorem it follows that Jx is always nonempty. If X is smooth (i.e. has a Gâteaux differentiable norm), then Jx is always single-valued and also the converse holds (see e.g. [229, Theorems 4.3.1 and 4.3.2]). This single-valued mapping is norm-to-norm uniformly continuous on bounded subsets provided that X is uniformly smooth and a modulus of uniform continuity can be obtained from a modulus of uniform smoothness for X (see Proposition 10.2.5 below). For more general information on the duality map and its background we refer to [45, 200].

In our application, we only need a function $J : X \to X^*$ which selects in a uniformly continuous way a point from the duality set. Let us define a *space with a uniformly continuous duality selection map* (X, J) to be a Banach space X together with a mapping $J : X \to X^*$ satisfying

(i) $\langle x, Jx\rangle = \|x\|^2 = \|Jx\|^2$ for all $x \in X$, and

(ii) J is norm-to-norm uniformly continuous on any bounded subsets of X.

Obviously, it suffices to require that J is norm-to-norm uniformly continuous on any open ball $B_d(0)$ (resp. closed ball $\overline{B}_d(0)$), $d > 0$. By a *modulus* for the space with a uniformly continuous duality selection map (X, J) we shall understand a mapping $\omega : (0,\infty) \times (0,\infty) \to (0,\infty)$ such that for all $d > 0$, $\omega(d, \cdot)$ is a modulus of uniform continuity for the restriction of J to $\overline{B}_d(0)$, that is for all $\varepsilon > 0$ and $x, y \in \overline{B}_d(0)$,

$$\|x - y\| \leq \omega(d, \varepsilon) \text{ implies } \|Jx - Jy\| \leq \varepsilon. \tag{10.10}$$

One can easily see that the existence of a modulus ω satisfying (10.10) is equivalent to the existence of $\omega : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for any $d, k \in \mathbb{N}$, and $x, y \in B_d(0)$,

$$\|x - y\| < 2^{-\omega(d,k)} \text{ implies } \|Jx - Jy\| \leq 2^{-k}. \tag{10.11}$$

Rather than having to formalize the proof of the existence of J and its continuity property we directly add constants J_X, ω_X and axioms (J_X) and (J_X, ω_X) to the formal framework expressing that for $x \in X$, $J_X x$ represents a linear operator $X \to \mathbb{R}$ with $\|J_X x\| \leq \|x\|$ and $J_X xx = \|x\|^2$, which - taken together - yields $\|J_X x\| = \|x\|$, i. e. $J_X xx = \|x\|^2 = \|J_X x\|^2$, and that J_X is norm-to-norm uniformly continuous on any bounded ball $B_d(0)$ with modulus of uniform continuity $\omega_X(d, \cdot)$. Instead of using the operator norm and stating $\|J_X x - J_X y\| \leq 2^{-k}$ we express things equivalently in the language of X as $\forall z \in X (|J_X xz - J_X yz| \leq 2^{-k} \cdot \|z\|)$.

In formulating (J_X) and (J_X, ω_X) we rely on the formal framework presented in Chapter 3. J_X then is an object of type $X \to X \to 1$ (where 1 denotes the type $\mathbb{N} \to \mathbb{N}$, that is the type of objects used to represent real numbers) and ω_X has type $\mathbb{N}^2 \to \mathbb{N}$:

$$\begin{aligned}
(J_X) \quad &:\equiv \quad \forall x^X, y^X \big(J_X xx =_{\mathbb{R}} \|x\|_X^2 \wedge |J_X xy|_{\mathbb{R}} \leq_{\mathbb{R}} \|x\|_X \cdot_{\mathbb{R}} \|y\|_X \wedge \\
&\qquad \forall \alpha^1, \beta^1, u^X, v^X (J_X x(\alpha \cdot_X u +_X \beta \cdot_X v) =_{\mathbb{R}} \alpha \cdot_{\mathbb{R}} J_X xu +_{\mathbb{R}} \beta \cdot_{\mathbb{R}} J_X xv) \big) \\
(J_X, \omega_X) \quad &:\equiv \quad \forall x^X, y^X, z^X, k^{\mathbb{N}}, d^{\mathbb{N}} \big(\|x\|_X, \|y\|_X <_{\mathbb{R}} (d)_{\mathbb{R}} \wedge \|x - y\|_X <_{\mathbb{R}} 2^{-\omega_X(k,d)} \\
&\qquad \to |J_X xz -_{\mathbb{R}} J_X yz|_{\mathbb{R}} \leq_{\mathbb{R}} 2^{-k} \cdot_{\mathbb{R}} \|z\|_X \big)
\end{aligned}$$

Let $\mathcal{A}^\omega$ be the system presented in Chapter 3 and $\mathcal{A}^\omega[X, \|\cdot\|, J_X, \omega_X, C, \mathcal{C}]$ its extension by an abstract real normed space with the constants J_X, ω_X together with their above axioms, an abstract nonempty convex subset $C \subseteq X$ and a completeness axiom stating the completeness of X / closedness of C (see [134] for details). Then the logical metatheorems for Banach spaces from [134] hold if the extracted bound is allowed to depend on ω_X. We only formulate here a special instance of these theorems sufficient for our main application:

Theorem 10.2.3. *[146] Let $A_\exists(k^{\mathbb{N}}, g^{\mathbb{N}\to\mathbb{N}}, x^X, T^{X\to X}, n^{\mathbb{N}})$ be a purely existential formula containing only k, g, x, T, n free.*

Assume that $\mathcal{A}^\omega[X, \|\cdot\|, J_X, \omega_X, C, \mathcal{C}]$ proves that of

$$\forall k \in \mathbb{N} \, \forall g : \mathbb{N} \to \mathbb{N} \, \forall x \in C \, \forall T : C \to C \, \big(T \textit{ nonexpansive } \to \exists n \in \mathbb{N} \, A_\exists\big).$$

Then one can extract a computable bound $\Phi : \mathbb{N} \times \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}^2} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N} \, \forall g : \mathbb{N} \to \mathbb{N} \, \forall x \in C \, \forall T : C \to C \, \big(T \textit{ nonexpansive } \to \exists n \leq \Phi(k, b, g, \omega) \, A_\exists\big)$$

holds in any Banach space X with a duality selection map J that has ω as modulus of uniform norm-to-norm continuity and any closed b-bounded convex subset $C \subseteq X$. If C is not bounded, then one has to choose b such that $b \geq \|x\|, \|x - Tx\|$.

Note that Φ does not depend on x, T and depends on X, C only via ω respectively b. The extraction of the bound proceeds by monotone functional interpretation from the proof and its complexity faithfully reflects the complexity of the principles used in the proof. In our case, this will yield a Φ of very restricted complexity.

Some examples

Let us recall that a Banach space X is

(i) *uniformly convex* if for all $\varepsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $x, y \in X$,

$$\|x\| \leq 1, \quad \|y\| \leq 1 \text{ and } \|x - y\| \geq \varepsilon \text{ imply } \left\|\frac{1}{2}(x + y)\right\| \leq 1 - \delta. \tag{10.12}$$

(ii) *uniformly smooth* whenever given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$,

$$\|x\| = 1 \text{ and } \|y\| \leq \delta \text{ imply } \|x + y\| + \|x - y\| < 2 + \varepsilon\|y\|. \tag{10.13}$$

A mapping $\eta : (0, 2] \to (0, 1]$ providing a $\delta := \eta(\varepsilon)$ satisfying (10.12) for given $\varepsilon \in (0, 2]$ will be called a *modulus of uniform convexity*. Similarly, a function $\tau : (0, \infty) \to (0, \infty)$ providing such a $\delta := \tau(\varepsilon)$ satisfying (10.13) is said to be a *modulus of uniform smoothness.*

Remark 10.2.4. *The property of X being a uniformly smooth Banach space with a modulus $\tau_X : \mathbb{N} \to \mathbb{N}$ (formulated with 2^{-k} instead of ε/δ) can be axiomatized by a universal axiom over our framework (so that the logical metatheorems guarantee effective bounds depending additionally only on τ_X) as follows (using again that $\leq_{\mathbb{R}}$ is universal while $<_{\mathbb{R}}$ is existential):*

$$\forall x^X, y^X \forall k^{\mathbb{N}} \big(\|x\|_X >_{\mathbb{R}} 1 \wedge \|y\|_X <_{\mathbb{R}} 2^{-\tau_X(k)} \to \\ \|\tilde{x} +_X y\| +_{\mathbb{R}} \|\tilde{x} -_X y\| \leq_{\mathbb{R}} 2 +_{\mathbb{R}} 2^{-k} \cdot_{\mathbb{R}} \|y\|_X\big),$$

where $\tilde{x} := \dfrac{1}{\max_{\mathbb{R}}\{1, \|x\|_X\}} \cdot_X x$. Note that for x with $\|x\| > 1$ one has $\tilde{x} \in S_1$. Conversely, for $x \in S_1$ and $x' := 2 \cdot x$ one has $\|x'\| = 2 > 1$ and $\widetilde{x'} =_X x$. So in the axiom above we indeed quantify over all vectors $x \in S_1$.

Proposition 10.2.5. *[146]*

(i) If X is uniformly smooth with modulus τ, then X^* is uniformly convex with modulus $\eta(\varepsilon) = \dfrac{\varepsilon}{4} \cdot \tau\left(\dfrac{\varepsilon}{2}\right)$.

(ii) If X^* is uniformly convex with modulus η, then X is a space with a uniformly continuous duality selection map with modulus $\omega(d,\varepsilon) = \frac{\varepsilon}{3} \cdot \eta\left(\frac{\varepsilon}{d}\right)$ for all $\varepsilon \in (0,2]$ and $d \geq 1$. For $d < 1$ one can trivially define $\omega(d,\varepsilon) = \omega(1,\varepsilon)$ for all $\varepsilon > 0$, while for $\varepsilon > 2$, one defines $\omega(d,\varepsilon) = \omega(d,2)$ for all $d > 0$.

Remark 10.2.6. *If $\eta(\varepsilon)$ can be written as $\varepsilon \cdot \tilde{\eta}(\varepsilon)$, where $\varepsilon_1 \leq \varepsilon_2 \rightarrow \tilde{\eta}(\varepsilon_1) \leq \tilde{\eta}(\varepsilon_2)$, then ω can be improved to $\omega(d,\varepsilon) := \frac{2}{3} \cdot \varepsilon \cdot \tilde{\eta}\left(\frac{\varepsilon}{d}\right)$.*

It is well known that the Banach spaces L_p with $1 < p < \infty$ are both uniformly convex and uniformly smooth. A modulus of uniform convexity $\eta_p(\varepsilon)$ is given by

$$\eta_p(\varepsilon) = \varepsilon \cdot \tilde{\eta}_p(\varepsilon) \text{ where } \tilde{\eta}_p(\varepsilon) = \begin{cases} \dfrac{(p-1)}{8} \cdot \varepsilon, & 1 < p < 2, \\ \dfrac{1}{p \cdot 2^p} \cdot \varepsilon^{p-1}, & 2 \leq p < \infty. \end{cases} \quad \text{(see [169, p. 63]).}$$

Since L_p^* is isometrically isomorphic with $L_{p'}$, where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p, we get (using the remark above) that L_p is a space with a uniformly continuous duality selection map with modulus $\omega(d,\varepsilon) = \frac{2\varepsilon}{3} \cdot \tilde{\eta}_{p'}\left(\frac{\varepsilon}{d}\right)$ for all $\varepsilon \in (0,2]$ and $d \geq 1$.

10.2.2 Main results

The main result of this section is the following effective version of Theorem 10.2.1.

Theorem 10.2.7. *[146] Let (X,J) be a space with a uniformly continuous duality selection map with modulus ω, $C \subseteq X$ be a bounded convex closed subset with diameter d_C, $T : C \to C$ a nonexpansive mapping and $x, u \in C$. Let $M \in \mathbb{Z}_+$ be such that $M \geq d_C$.*

Assume that $\lim_{n\to\infty} \lambda_n = 0$ with rate of convergence α, $\sum_{n=1}^{\infty} \lambda_{n+1}$ diverges with rate of divergence θ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|$ converges with β being a Cauchy modulus of $s_n := \sum_{i=1}^{n} |\lambda_{i+1} - \lambda_i|$.

Let $t_k := \frac{1}{k+1}$, $k \geq 1$ and assume that $(z^u_{t_k})$ is Cauchy with rate of metastability K, i.e.

$$\forall \varepsilon > 0 \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists K_1 \leq K(\varepsilon, g) \, \forall i,j \in [K_1, K_1 + g(K_1)] \left(\|z^u_{t_i} - z^u_{t_j}\| \leq \varepsilon\right). \tag{10.14}$$

Then the Halpern iteration (x_n) is Cauchy and for all $\varepsilon \in (0,2)$ and $g : \mathbb{N} \to \mathbb{N}$,

$$\exists N \leq \Sigma(\varepsilon, \omega, g, M, K, \theta, \alpha, \beta) \, \forall m,n \in [N, N + g(N)] \, (\|x_n - x_m\| \leq \varepsilon), \tag{10.15}$$

where

$$\Sigma(\varepsilon, g, \omega, M, \theta, \alpha, \beta, K) = \theta^+\left(\Gamma - 1 + \left\lceil \ln\left(\frac{12M}{\varepsilon^2}\right)\right\rceil\right) + 1, \quad \textit{with}$$

$$
\begin{aligned}
\Gamma &= \max\left\{\chi_k^*(\varepsilon^2/12) \mid \left\lceil\frac{1}{\varepsilon_0}\right\rceil \le k \le K(\varepsilon_0, f^*) + \left\lceil\frac{1}{\varepsilon_0}\right\rceil\right\},, \\
\tilde{P}_k(\varepsilon) &= \left\lceil\frac{12M^2(k+1)}{\varepsilon}\cdot\Phi\left(\frac{\varepsilon}{12M(k+1)}\right)\right\rceil, \\
\tilde{\Phi}(\varepsilon) &= \theta\left(\beta\left(\frac{\varepsilon}{4M}\right)+1+\left\lceil\ln\left(\frac{2M}{\varepsilon}\right)\right\rceil\right)+1, \\
\Phi(\varepsilon) &= \max\left\{\tilde{\Phi}\left(\frac{\varepsilon}{2}\right), \alpha\left(\frac{\varepsilon}{4M}\right)\right\}, \\
\Theta_k(\varepsilon) &= \theta\left(\chi_k^*\left(\frac{\varepsilon}{3},\omega\right)-1+\left\lceil\ln\left(\frac{3M}{\varepsilon}\right)\right\rceil\right)+1,
\end{aligned}
$$

$$
\begin{aligned}
\chi_k(\varepsilon,\omega) &= \tilde{\Phi}\left(\omega\left(M,\frac{\varepsilon_k}{M}\right)\right)+\tilde{P}_k(\varepsilon), & \chi_k^*(\varepsilon,\omega) &= \chi_k\left(\frac{\varepsilon}{2},\omega\right) \\
\Delta_k^*(\varepsilon,g) &= \frac{\varepsilon}{3g_{\varepsilon,k}\left(\Theta_k(\varepsilon)-\chi_k^*\left(\frac{\varepsilon}{3},\omega\right)\right)}, & \varepsilon_k &= \frac{\varepsilon}{\tilde{P}_k(\varepsilon)+1}, \\
f(k) &= \max\left\{\left\lceil\frac{2M^2}{\Delta_k^*(\varepsilon^2/4,g)}\right\rceil, k\right\}-k, & \varepsilon_0 &= \min\{\delta,\omega(M,\delta)\}, \\
f^*(k) &= f\left(k+\left\lceil\frac{1}{\varepsilon_0}\right\rceil\right)+\left\lceil\frac{1}{\varepsilon_0}\right\rceil, & \delta &= \frac{\varepsilon^2}{24M(4+M)} \\
g_{\varepsilon,k}(n) &= n+g\left(n+\chi_k^*\left(\frac{\varepsilon}{3},\omega\right)\right), & \theta^+(n) &= \max\{\theta(i) \mid i\le n\}.
\end{aligned}
$$

For the most important case $\lambda_n := 1/(n+1)$, the moduli θ, α, β are all easily computable. In fact, one can avoid the use of the exponential θ by using $\lim\limits_{n\to\infty}\prod_{n=1}^{\infty}(1-\lambda_{n+1}) = 0$ instead of the divergence of $\sum_{n=1}^{\infty}\lambda_{n+1}$. We refer to [147] for details on this.

Theorem 10.2.8. *[146] Let $\lambda_n := 1/(n+1), n \ge 1$, $t_k := 1/(k+1)$, $k \ge 1$ and denote $z^u_{t_k}$ by z^u_k.*

(i) If $K(\varepsilon)$ is a rate of convergence of (z^u_k), then the bound in Theorem 10.1.8 gives a rate of convergence of (x_n).

(ii) If X is an effective Hilbert space and T, u are computable, then (z^u_k) has a computable rate of convergence iff $\|z-u\|$ is computable, where $z := \lim\limits_{k\to\infty} z^u_k$.

Proof. (i) $K(\varepsilon/2)$ is a witness (not only a bound) of metastability for any function g (i.e. we can take $K_1 := K(\varepsilon/2)$ in (10.14)). Hence we can replace in the bound Σ from Theorem 10.1.8 $K(\varepsilon_0, f^*)$ by $K(\varepsilon_0/2)$ which makes the bound independent of g since g only enters via the definition of f^*. Also note that the maximum in the definition of Γ can be replaced by just taking $k := K(\varepsilon_0/2)$. Then (10.15) holds with $N := \Sigma$ for all g where now Σ is independent of g.

(ii) From [135, p. 2789] it follows that a rate of convergence for (z^u_k) is given by a rate of convergence of the nondecreasing and M-bounded sequence $(\|z^u_k - u\|^2)$ which is computable provided that the limit $\|z-u\|^2 = \lim\limits_{k\to\infty}\|z^u_k-u\|^2$ is.

Conversely, if we have a computable rate of convergence for (z_k^u), then z and hence $\|z - u\|$ is computable.

$\square$

Logical discussion

Theorem 10.2.1 is proved under the hypothesis that the sequence (z_n) of the fixed points of the contractions $T_n(y) := \frac{1}{n+1}x + \left(1 - \frac{1}{n+1}\right)Ty$ strongly converges towards a fixed point of T. This fact is known for bounded, closed and convex C in many cases such as for Hilbert spaces [31, 97], the Hilbert ball (see [84, Sections 24 and 27] and [167]), general CAT(0)-spaces [114] and also uniformly smooth Banach spaces [197].

Under this assumption, the proof of the strong convergence of the Halpern iteration (x_n) that results by our elimination of the use of Banach limits is basically constructive. Hence metatheorems for the constructive case [75] guarantee a uniform effective procedure to transform a rate of convergence for (z_n) into one for (x_n). Theorem 10.2.8 displays this fact. The problem, is that even in very simple cases there is no computable rate of (z_n). In fact, to show that there is no effective operator which would effectively in a computable sequence of operators $(T(l))_l$ produce a rate of convergence for (z_n) is almost trivial and holds already for $X := \mathbb{R}$ and $C := [0, 1]$, similarly to [134, Theorem 18.4]. Only in certain cases, e.g. when, in particular, the norm $\|z\|$ of the limit $z := \lim_{n\to\infty} z_n$ is known, one gets computable rates of (z_n) (but not uniform ones and without any complexity information as the argument is based on unbounded search); see Theorem 10.2.8.

However, one can obtain in many cases a fully uniform rate of metastability for (z_n), which is not only computable but with low complexity. Since such a rate only ineffectively implies the convergence of (z_n), it is this feature which makes the proof of the convergence of (x_n) nonconstructive and forces us to also weaken the conclusion to the metastability of (x_n). Hence, we actually use the logical metatheorem 10.2.3 in the form where we have as an additional input a (majorant of a) rate $K(\varepsilon, g)$ of metastability for (z_n) (or – equivalently – a selfmajorizing such rate) and extract a bound on the metastability of (x_n) that depends in addition to $\varepsilon, g, b, \omega$ also on K. For the Hilbert case a simple primitive recursive such K has been extracted in [135]. We have seen in Section 10.1.3 that the same holds for CAT(0) spaces.

Chapter 11

Future plans and research directions

The author of this thesis has attained in October 2009 the Habilitation in Mathematics at Technische Universität Darmstadt, Germany. Beginning with November 2009, Laurenţiu Leuştean came back to Romania, as a Scientific Researcher at the Simion Stoilow Institute of Mathematics of the Romanian Academy (IMAR).

Since October 2011, Laurenţiu Leuştean is the director of a grant of the Romanian National Authority for Scientific Research, CNCS - UEFISCDI. He intends in the future to apply for other grants or fellowships.

Laurenţiu Leuştean has already begun to create a research group on proof mining at IMAR including Adriana Nicolae (PhD in Mathematics at Babes-Bolyai University of Cluj-Napoca and, in co-tutelle, University of Seville), Mircea Dan Hernest (PhD in Mathematics at École Polytechnique and, in co-tutelle, at Ludwig-Maximilians-Universität München) and Daniel Ivan, who has a Master both in Mathematics and Computer Science. Emanuel Vlad, who is now a graduate student at University of Paris 13, was also a member of the group.

A plan for the future is, of course, to attract new members in this group. These could be both graduate and postdoctoral students, interested in mathematics or computer science.

One way to do this is by giving lectures at the University of Bucharest or Şcoala Normală Superioară Bucureşti (SNSB). In the Summer Semester 2012/2013 L. Leuştean will give a lecture on ergodic theory and additive combinatorics for SNSB Master students. This lecture is a follow-up to a lecture on ergodic Ramsey theory given at SNSB in the Winter Semester 2010.

Another way is to continue to organize scientific seminars at IMAR or at the University of Bucharest. L. Leuştean is, since 1997, one of the organizers of the scientific seminar "Alexandru Brezuleanu" of the Group of Logic and Universal Algebra (GLAU). Since 2011, he organizes together with Marius Buliga the IMAR scientific seminar "Effective methods in metric analysis".

Furthermore, L. Leuştean will continue to have a close collaboration with Ulrich Kohlenbach and his research group at Technische Universität Darmstadt, as well as Genaro López-Acedo and his group at University of Seville. Other collaborations that will be developed or strengthened are with Jeremy Avigad (Carnegie Mellon

University), Fernando Ferreira (University of Lisbon), Paolo Oliva (Queen Mary - University of London), Simeon Reich (Technion Haifa Institute of Technology) or Andreas Weiermann (Ghent University). This will be done by organizing workshops or exchange visits between the members of our groups.

The future research directions are to obtain finitary, stronger versions with effective bounds of some important results in ergodic theory and geometric group theory.

11.1 Proof mining in ergodic theory

Ergodic theory has become a powerful tool in combinatorics and number theory. This development has culminated in the spectacular Green-Tao theorem on the existence of arbitrarily long progressions in the set of prime numbers [90]. As pointed out by Tao [234, 236], an important aspect guiding his research in this area is to systematically replace infinitary analytic arguments by finitary ones. The interplay between infinitary ergodic theoretic methods and their use in finite combinatorics makes our applications of proof mining very promising, as they unwind the combinatorial skeleton of the ergodic theoretic arguments.

11.1.1 Generalizations of the mean ergodic theorem

As we have seen in Chapter 9, Kohlenbach and the author obtained a finitary version of a generalization of the mean ergodic theorem to uniformly convex Banach spaces.

A natural research direction is to obtain finitary versions with effective bounds of different generalizations of the mean ergodic theorem.

One such generalization was obtained by Cohen [47] by considering *weighted averages* $A_n(x) = \sum_{j=1}^{\infty} c_{nj}T^j x$, where $C = (c_{ij})$ is a regular matrix satisfying

$$\lim_k \sum_{j=k}^{\infty} |c_{n,j+1} - c_{nj}| = 0 \qquad \text{uniformly in } n.$$

We shall analyze results on the convergence of the weighted averages for power bounded linear operators T in Hilbert spaces (see, e.g., [157, Chapter 8]).

Another idea is to consider *multiple ergodic averages*, introduced by Furstenberg [69] in his ergodic theoretic proof of Szemerédi's theorem:

$$A_N(x) = \frac{1}{N} \sum_{n=M_N+1}^{M_N+N} f_1\left(T^n x\right) f_2\left(T^{2n} x\right) \dots f_l\left(T^{ln} x\right), \tag{11.1}$$

where $l \geq 1$ is an integer, T is an invertible measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$ and $f_1, \dots, f_l \in L^\infty(X, \mathcal{B}, \mu)$. For general T, the L^2-convergence of these averages was proved for $l = 2$ by Furstenberg [69], for $l = 3$ by Furstenberg and Weiss [71] and by Host and Kra [99], for $l = 4$ by Ziegler [249].

Finally, the L^2-convergence of the multiple ergodic averages (11.1) was obtained by Host and Kra [101], and independently by Ziegler [249].

A further idea is to consider *polynomial* multiple ergodic averages

$$A_N(x) = \frac{1}{N} \sum_{n=M_N+1}^{M_N+N} f_1\left(T^{p_1(n)}x\right) f_2\left(T^{p_2(n)}x\right) \dots f_l\left(T^{p_l(n)}x\right) \tag{11.2}$$

where $p_1, \dots, p_l \in \mathbb{Z}[n]$. The L^2-convergence of these averages was established for weakly mixing T by Bergelson [15]. Furstenberg and Weiss [71] proved the convergence for general T, but for $l = 2$ and the pair of polynomials $p_1(n) = n, p_2(n) = n^2$. For general l, the L^2-convergence of the polynomial multiple ergodic averages (11.2) in the case T is totally ergodic or under some negligible restrictions on the polynomials $p_1, \dots, p_l$ was obtained by Host and Kra [100].

Even more general multiple ergodic averages were considered, for example, by Leibman [162], Frantzikinakis, Johnson, Lesigne and Wierdl [68] or Walsh [241]. We refer to [67] for some open problems on multiple ergodic averages.

11.1.2 Effective rates of metastability for diagonal ergodic averages

A further direction of research is related to the following theorem of Tao, that settles in full generality the norm convergence problem for several commuting transformations:

Theorem 11.1.1. *[236, Theorem 1.1]*
Let $l \geq 1$ be an integer. Assume that $T_1, \dots, T_l : X \to X$ are commuting invertible measure-preserving transformations of a measure space $(X, \mathcal{B}, \mu)$. Then for any $f_1, \dots, f_l \in L^\infty(X, \mathcal{B}, \mu)$, the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T_1^n x) \dots f_l(T_l^n x)$$

are convergent in $L^2(X, \mathcal{B}, \mu)$.

By now, different proofs of this theorem were given by Host [98], Towsner [240] and Austin [4], for example.

Tao deduces Theorem 11.1.1 from the following finitary version, in which the general measure-preserving system $(X, \mathcal{B}, \mu, T_1, \dots, T_l)$ has been replaced by the finite abelian group $\mathbb{Z}_p^l$ for some large integer P, with the discrete σ-algebra, the uniform probability measure, and the standard l commuting shifts $T_i x := x + e_i$. The *multiple average* $A_N(f_1, \dots, f_l)$ is defined by

$$A_N(f_1, \dots, f_l) : \mathbb{Z}_P^l \to \mathbb{R}, \quad A_N(f_1, \dots, f_l)(a) = \mathbf{E}_{n \in [N]} \prod_{i=1}^{l} f_i(a + ne_i),$$

where $[N] := \{0, 1, \dots, N-1\}$ and for every $f : [N] \to \mathbb{R}$, $\mathbf{E}_{n \in [N]} f(n) = \dfrac{1}{N} \displaystyle\sum_{n=0}^{N-1} f(n)$.

Theorem 11.1.2 (Finitary norm convergence). *[236, Theorem 1.6]*
Let $l \geq 1$ be an integer, let $F : \mathbb{N} \to \mathbb{N}$ be a function, and let $\varepsilon > 0$. Then there exists an integer $M^* > 0$ with the following property: if $P \geq 1$ and $f_1, \ldots, f_l : \mathbb{Z}_P^l \to [-1, 1]$ are arbitrary functions on $\mathbb{Z}_P^l$, then there exists an integer $1 \leq M \leq M^*$ such that we have the 'L^2 metastability'

$$\|A_N(f_1, \ldots, f_l) - A'_N(f_1, \ldots, f_l)\|_{L^2(\mathbb{Z}_P^l)} \leq \varepsilon \tag{11.3}$$

for all $M \leq N, N' \leq F(M)$.

Tao's approach was recently used by Walsh [241] to show the L^2-convergence of multiple polynomial ergodic averages arising from nilpotent groups of measure-preserving transformations.

For the $l = 1$ version of Theorem 11.1.2, effective bounds are given by Theorem 9.2.1, our finitary version of the mean ergodic theorem.

We think that by applying proof mining we can obtain for general $l \geq 1$ an explicit uniform bound on M^*, whose ineffective existence is proved by Tao in Theorem 11.1.2.

11.1.3 Ergodic Ramsey Theory

Ergodic Ramsey theory was initiated in 1977 when Furstenberg [69] proved a far reaching extension of the classical Poincaré recurrence theorem and derived from it the celebrated Szemerédi's theorem, which states that any subset of integers of positive upper density must necessarily contain arbitrarily long arithmetic progressions. Since then, Furstenberg's ergodic approach was used to establish many more types of recurrence theorems, which (via the Furstenberg correspondence principle) yield a number of highly non-trivial combinatorial theorems. Many of the results obtained by these ergodic techniques are not known, even today, to have any "elementary" proof, thus testifying to the power of this method.

An early use of proof mining in this context was Girard's [78] logical analysis of the topological dynamics proof of van der Waerden's theorem, given by Furstenberg and Weiss [70]. Recently, this has been further developed in [74], where a quantitative version of the multiple Birkhoff recurrence theorem is extracted.

We plan to analyze the topological dynamics proofs and obtain explicit bounds for refinements of van der Waerden's theorem, like the multidimensional version due to Gallai, the IP-sets version [70] or the polynomial version, due to Bergelson and Leibman [16].

Furthermore, we shall consider finitary versions of Szemerédi's theorem and its generalizations:

Theorem 11.1.3 (Finitary Szemerédi Theorem).
For any $k \geq 1$ and real number $0 < \delta \leq 1$, there exists an integer $N_{SZ}(k, \delta) \geq 1$ such that for every $N \geq N_{SZ}(k, \delta)$, every set $A \subseteq \{1, \ldots, N\}$ of cardinality $|A| \geq \delta N$ contains at least one arithmetic progression of length k.

The best bounds on $N_{SZ}(k, \delta)$, essentially of double exponential growth in δ, were obtained by Gowers [89], using combinatorics and Fourier analysis. In 2006, Tao [233] gave a new, quantitative, ergodic theoretic proof of the finitary Szemerédi theorem. As he points out, an explicit bound on $N_{SZ}(k,\delta)$ can be in principle obtained, but it is very poor, perhaps even worse than Ackermann growth. In a footnote [233, p. 4], Tao writes

"It may be also possible in principle to extract some bound for $N_{SZ}(k,\delta)$ directly from the original Furstenberg argument via proof theory ...".

That is one of our future directions of research.

11.2 Proof mining in geometric group theory

This is a completely new direction of research.

11.2.1 Effective bounds for the finitary Gromov Theorem

Given a finitely generated group G and a finite generating set S, we have the *word length* $\ell_S : G \to \mathbb{N}$, for which $\ell_S(g)$ is the smallest integer n such that there exist $s_1, \ldots, s_n \in S \cup S^{-1}$ with $g = s_1 \ldots s_n$ and the *growth function* with respect to S, $\gamma = \gamma_G^S : \mathbb{N} \to \mathbb{N}$, for which $\gamma_G^S(n)$ is the number of elements $g \in G$ with $\ell_S(G) \leq n$.

One says that G has *polynomial growth* if there are positive numbers K, d such that $\gamma(n) \leq Kn^d$ for all $n \in \mathbb{N}$. One can easily see that this definition does not depend on the particular choice of the generating set S and so this notion is correctly defined for finitely generated groups.

Groups of polynomial growth were classified by a well-known theorem of Gromov:

Theorem 11.2.1. *[92] Let G be a finitely generated group of polynomial growth. Then G is virtually nilpotent (i.e. it has a finite index subgroup that is nilpotent).*

As observed in Gromov's original paper (see also [237]), this result is equivalent to a finitary version:

Theorem 11.2.2. *For every integers s, K, d, there exists R such that any finitely generated group with s generators, such that $\gamma(n) \leq Kn^d$ for all $1 \leq n \leq R$, is virtually nilpotent.*

In the essay [237], Tao uses a correspondence principle to show the equivalence of the two formulations, remarks that no explicit bound for R in terms of s, K, d is currently known and adds:

"though presumably some such bound can eventually be extracted from the above argument and the existing proofs of Gromov's theorem by proof mining techniques."

We think, as Tao in his remark, that this can be obtained by methods of proof mining, based also on the fact that there exist already proofs of results related to Gromov's theorem using methods from logic, like Hrushovski [103] or van den Dries and Wilkie [56].

A first idea is to apply proof mining to a recent proof of Shalom and Tao [217] of the finitary version of Gromov's theorem. They refine Kleiner's [120] new proof of

Gromov's theorem to obtain a strenghtening of it that only requires the polynomial growth condition at one (sufficiently large, yet explicit) scale. An effective bound on the finite index is provided by Shalom and Tao only if *nilpotent* is replaced by *polycyclic*.

A second proof of Gromov's theorem that we shall consider is the one of van den Dries and Wilkie [56]. They formulated the geometric construction used by Gromov using nonstandard analysis, as what is now called the asymptotic cone.

A third proof proof of a strengthening of Gromov's theorem was obtained recently by Hrushovsky [103] using again methods from nonstandard analysis. Hrushovsky's fundamental work was one of the main tools used by Breuillard, Green and Tao [22] to give a qualitative description of approximate groups as being essentially finite-by-nilpotent.

We intend to develop proof mining techniques that will allow us to eliminate these ultrafilter arguments, as suggested in [22, p. 20, footnote]. A first step in this direction was made recently by Kreuzer [158], who provides a method to extract bounds from proofs using the fact that non-principal ultrafilters on $\mathbb{N}$ exist.

In this way, we aim to get uniform finitary versions of (strengthenings) of Gromov's theorem. These methods can be applied to finitize other results from this field, having the logical form:

$$G \text{ has polynomial growth } \Rightarrow A, \text{ where } A \text{ is a } \forall\exists\text{-statement.}$$

11.2.2 Uniform exponential growth

The second direction is related with *uniform* exponential growth, which is a uniform version of the notion of exponential growth (see [95] for a good survey). We say that G has *exponential growth* if $\omega(G,S) := \lim_{n\to\infty} \sqrt[n]{\gamma_G^S(n)} > 1$ for some (hence any) finite generating set S. Furthermore, G has *uniform exponential growth* if $\inf_S \omega(G,S) > 1$, where S varies over all finite generating sets of G.

In 1981, Gromov [93, Remark 4.2] asked whether groups of exponential growth necessarily have uniform exponential growth. This question remained open until 2004, when Wilson [243] gave a negative answer.

However, there are important classes of groups for which exponential growth implies uniform exponential growth, like, for example, free groups, hyperbolic groups, free products with amalgamation, solvable groups or polyclic groups.

Since the property of having uniform exponential growth is a uniform existence statement, proof mining techniques can make a contribution. By developing metatheorems for the structures involved (finitely-generated groups with length functions), and by defining an appropriate notion of majorization, we think that we can encapsulate in a very precise way "how much" of the exponential growth is needed to get uniform exponential growth, and in this way to get more general classes of groups for which exponential growth implies uniform exponential growth.

11.2.3 New logical metatheorems

As we have seen, logical metatheorems were obtained in [129, 163] for important classes of spaces in geometric group theory, like (Gromov) hyperbolic spaces, CAT(0)-spaces and $\mathbb{R}$-trees.

We intend to develop logical metatheorems for hyperbolic groups, introduced by Gromov in the seminal paper [94] as the finitely generated groups whose Cayley graph is a hyperbolic space. In a sense that can be precisely formulated, Gromov [94] observed that among all finitely presented groups, hyperbolic groups occur with probability 1.

Furthermore, hyperbolic group have very nice algorithmic properties: they have solvable word and conjugacy problems and the isomorphism problem is solvable for torsion-free hyperbolic groups. Moreover, there is a simple algorithmic procedure for recognizing geodesics in hyperbolic groups and the growth of a hyperbolic group with respect to any finite generating set is rational. The fact that hyperbolic groups have very nice algorithmic properties makes them a possible candidate for applying our proof mining techniques.

Another direction of research is to obtain logical metatheorems for Λ-*trees*, where Λ is a totally ordered abelian group [1]. These structures generalize $\mathbb{R}$-trees and they made their appearance as an essential tool in the study of groups acting on hyperbolic manifolds in the work of Morgan and Shalen [182]. In fact, a natural extension of Λ-trees, known from the 50's to lattice theorists under the name *median algebra* (see, e.g. [20]), and rediscovered by Basarab (see, e.g. [9]) under the name of *generalized tree* can be considered. Furthermore, interested in applying the theory of generalized trees and motivated by problems concerning the model theory of free groups and free profinite groups, Basarab considers [10, 11] a class of groups called *median* or *arboreal*, that seems also suitable for proof mining techniques.

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