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A quadratic rate of asymptotic regularity for CAT(0)-spaces

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Abstract

In this paper we obtain a quadratic bound on the rate of asymptotic regularity for the Krasnoselski–Mann iterations of nonexpansive mappings in CAT(0)-spaces, whereas previous results guarantee only exponential bounds. The method we use is to extend to the more general setting of uniformly convex hyperbolic spaces a quantitative version of a strengthening of Groetsch's theorem obtained by Kohlenbach using methods from mathematical logic (so-called "proof mining").

Keywords: Proof mining; Metric fixed point theory; Nonexpansive functions; Asymptotic regularity; CAT(0)-spaces; Uniformly convex hyperbolic spaces

1. Introduction

In this paper we present another case study in the general project of *proof mining* in functional analysis, developed by Kohlenbach (see [20] for details). By "proof mining" we mean the logical analysis of mathematical proofs with the aim of extracting new numerically relevant information hidden in the proofs.

Thus, we obtain a quadratic bound on the rate of asymptotic regularity for the Krasnoselski–Mann iterations of nonexpansive self-mappings of nonempty convex, bounded sets C in CAT(0)-

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spaces in the sense of Gromov (see [3] for a detailed treatment). Moreover, the bound we get is uniform in the sense that does not depend on the nonexpansive mapping T or on the starting point $x \in C$, and depends on C only via its diameter.

The method we use to get this bound is to find explicit uniform bounds on the rate of asymptotic regularity in the general setting of uniformly convex hyperbolic spaces, and then to specialize them to CAT(0)-spaces.

The notion of nonexpansive mapping can be introduced in the very general setting of metric spaces. Thus, if (X, ρ) is a metric space, and $C \subseteq X$ a nonempty subset, than a mapping $T : C \to C$ is called *nonexpansive* if for all $x, y \in C$,

$$\rho(Tx, Ty) \leqslant \rho(x, y).$$

Different notions of "hyperbolic space" [10,11,14,26] can be found in the literature. We work in the setting of hyperbolic spaces as introduced by Kohlenbach [19], which are slightly more restrictive than the spaces of hyperbolic type [10] by (W4), but more general then the concept of hyperbolic space from [26]. See [19,22] for detailed discussion of this and related notions.

A *hyperbolic space* is a triple (X, ρ, W) where (X, ρ) is a metric space and $W : X \times X \times [0, 1] \to X$ is such that

(W1) $\rho(z, W(x, y, \lambda)) \leqslant (1 - \lambda)\rho(z, x) + \lambda\rho(z, y)$;
(W2) $\rho(W(x, y, \lambda), W(x, y, \tilde{\lambda})) = |\lambda - \tilde{\lambda}| \cdot \rho(x, y)$;
(W3) $W(x, y, \lambda) = W(y, x, 1 - \lambda)$;
(W4) $\rho(W(x, z, \lambda), W(y, w, \lambda)) \leqslant (1 - \lambda)\rho(x, y) + \lambda\rho(z, w)$

for all $x, y, z, w \in X$, $\lambda, \tilde{\lambda} \in [0, 1]$.

If $x, y \in X$, and $\lambda \in [0, 1]$ then we use the notation $(1 - \lambda)x \oplus \lambda y$ for $W(x, y, \lambda)$. It is easy to see that for any $x, y \in X$, and $\lambda \in [0, 1]$,

$$\rho\big(x, (1 - \lambda)x \oplus \lambda y\big) = \lambda\rho(x, y) \quad \text{and} \quad \rho\big(y, (1 - \lambda)x \oplus \lambda y\big) = (1 - \lambda)\rho(x, y). \tag{1}$$

We shall denote by $[x, y]$ the set $\{(1 - \lambda)x \oplus \lambda y\colon \lambda \in [0, 1]\}$. A nonempty subset $C \subseteq X$ is *convex* if $[x, y] \in C$ for all $x, y \in C$.

We remark that any normed space $(X, \|\cdot\|)$ is a hyperbolic space, with $(1 - \lambda)x \oplus \lambda y := (1 - \lambda)x + \lambda y$.

The notion of uniformly convex hyperbolic space (X, ρ, W) with a modulus of uniform convexity η is defined in Section 2 following the normed space case.

A very important class of hyperbolic spaces are the CAT(0)-spaces. Thus, a hyperbolic space is a CAT(0)-space if and only if it satisfies the so-called CN-inequality of Bruhat–Tits [7]: For all $x, y, z \in X$,

$$\rho\left(z, \frac{1}{2}x \oplus \frac{1}{2}y\right)^2 \leqslant \frac{1}{2}\rho(z, x)^2 + \frac{1}{2}\rho(z, y)^2 - \frac{1}{4}\rho(x, y)^2 \tag{2}$$

(see [3, p. 163] and [19, p. 98] for details).

Moreover, it will turn out that CAT(0)-spaces are uniformly convex hyperbolic spaces with a quadratic modulus of uniform convexity.

In the sequel, (X, ρ, W) is a hyperbolic space, $C \subseteq X$ a nonempty convex subset of X, and $T : C \to C$ a nonexpansive mapping.

As in the case of normed spaces [23,24], we can define the *Krasnoselski–Mann iteration* starting from $x \in C$ by:

$$x_0 := x, \qquad x_{n+1} := (1-\lambda_n)x_n \oplus \lambda_n T x_n, \tag{3}$$

where (λ_n) is a sequence in $[0, 1]$.

Asymptotic regularity was already implicit in [8,23,27], but only in 1966 Browder and Petryshyn [4] defined it for normed spaces $(X, \|\cdot\|)$. In our setting, the mapping $T : C \to C$ is called *asymptotically regular* if for all $x \in C$,

$$\lim_{n\to\infty} \rho\big(T^n(x), T^{n+1}(x)\big) = 0.$$

For constant $\lambda_n = \lambda \in [0, 1]$, the fact that $\lim_{n\to\infty} \rho(x_n, Tx_n) = 0$ for all $x \in C$ is equivalent to the asymptotic regularity of the averaged mapping

$$T_\lambda := (1-\lambda)I \oplus \lambda T.$$

Therefore, for general $\lambda_n \in [0, 1]$, following [2], we say that the nonexpansive mapping T is λ_n*-asymptotically regular* if for all $x \in C$,

$$\lim_{n\to\infty} \rho(x_n, Tx_n) = 0.$$

The most general assumptions on the sequence (λ_n) for which asymptotic regularity has been proved for arbitrary normed spaces and bounded sets C are the following:

$$\sum_{n=0}^{\infty} \lambda_n = +\infty, \quad \text{and} \tag{4}$$

$$\limsup_{n\to\infty} \lambda_n < 1. \tag{5}$$

Thus, Ishikawa proved in his seminal paper [13] one of the most important results in the fixed point theory of nonexpansive mappings.

Theorem 1. *Let $(X, \|\cdot\|)$ be a normed space, $C \subseteq X$ a nonempty convex bounded subset, and $T : C \to C$ a nonexpansive mapping. Assume that (λ_n) satisfies* (4), (5).
Then T is λ_n-asymptotically regular.

Independently, Edelstein and O'Brien [9] also proved the asymptotic regularity for constant $\lambda_n = \lambda \in [0, 1]$, and noted that it is uniform in x.

By a logical analysis of the proof of a theorem due to Borwein, Reich and Shafrir [2] (which generalizes Ishikawa's result to unbounded C), Kohlenbach [17] obtained for the first time explicit bounds on the asymptotic regularity when general sequences (λ_n) satisfying (4), (5) are considered.

Subsequently, Kohlenbach and the author [21] extended these results to the very general setting of hyperbolic spaces (and even to the more general class of directional nonexpansive mappings as introduced in [15]).

The following result (which is a corollary of the main theorem in [21]) is proved there for hyperbolic spaces in the sense of [26], but the proof goes through for the setting used in the present paper.

Theorem 2. *Let (X, ρ, W) be a hyperbolic space, $C \subseteq X$ a nonempty convex bounded subset with diameter d_C, and $T : C \to C$ a nonexpansive mapping. Assume that $K \in \mathbb{N}$, $K \geqslant 2$ and (λ_n) is a sequence in $[\frac{1}{K}, 1 - \frac{1}{K}]$.*

Then T is λ_n-asymptotically regular, and the following holds for all $x \in C$:

$$\forall \varepsilon > 0\ \forall n \geqslant h(\varepsilon, d, K) \quad \big(\rho(x_n, Tx_n) \leqslant \varepsilon\big),$$

where

$$h(\varepsilon, d, K) := K \cdot M \cdot \big\lceil 2d \cdot \exp\big(K(M+1)\big)\big\rceil, \quad \text{with}$$
$$d \in \mathbb{R}, \quad d \geqslant d_C, \quad \text{and}$$
$$M \in \mathbb{N}, \quad M \geqslant \frac{1+2d}{\varepsilon}.$$

For normed spaces and the special case of constant $\lambda_n = \lambda \in (0, 1)$ the exponential bound in the above theorem is not optimal. In this case, a uniform and optimal quadratic bound was obtained by Baillon and Bruck [1] using an extremely complicated computer aided proof, and only for $\lambda_n = 1/2$ a classical proof of a result of this type was given [6]. However, the questions whether the methods of proof used by them hold for nonconstant sequences λ_n or for hyperbolic spaces are left as open problems in [1], and as far as we know they received no positive answer until now. Hence, the bound from Theorem 2 is the only effective bound known at all for nonconstant sequences λ_n (even for normed spaces).

Our result guarantees only an exponential bound for the asymptotic regularity in the case of CAT(0)-spaces, and as we have already remarked, it seems that Baillon and Bruck's approach does not extend to the more general setting of hyperbolic spaces.

In this paper we show that we can still get a quadratic rate of asymptotic regularity for CAT(0)-spaces, but following a completely different approach, inspired by the results on asymptotic regularity obtained before Ishikawa, and Edelstein and O'Brien theorems, in the setting of uniformly convex normed spaces.

More specifically, our point of departure is the following result, proved by Groetsch [12] (see also [25]):

Theorem 3. *Let $(X, \|\cdot\|)$ be a uniformly convex normed space, $C \subseteq X$ a nonempty convex subset, and $T : C \to C$ a nonexpansive mapping such that T has at least one fixed point.*

Assume that (λ_n) satisfies the following condition:

$$\sum_{k=0}^{\infty} \lambda_k(1-\lambda_k) = \infty. \tag{6}$$

Then T is λ_n-asymptotically regular.

Assumption (6) is equivalent with the existence of a witness $\theta : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{\theta(n)} \lambda_k(1-\lambda_k) \geqslant n. \tag{7}$$

By "proof mining," Kohlenbach [18] obtained a quantitative version of a strengthening of Groetsch's theorem which only assumes the existence of approximate fixed points in some neighborhood of x, and generalizing previous results obtained by Kirk and Martinez-Yanez [16] for constant $\lambda_n = \lambda \in (0, 1)$.

Table 1

	$\lambda_n = \lambda$	Nonconstant λ_n
Hilbert	Quadratic: [5]	$\theta(\frac{1}{\varepsilon^2})$: [18]
$\ell_p, 2 \leqslant p < \infty$	Quadratic: [16,18]	$\theta(\frac{1}{\varepsilon^p})$: [18]
UC normed	[16,18]	[18]
Normed	Quadratic: [1]	[17]
CAT(0)	Quadratic: present paper	$\theta(\frac{1}{\varepsilon^2})$: present paper
UC hyperbolic	Present paper	Present paper
Hyperbolic	[21]	[21]

In this paper, we extend Kohlenbach's results to the more general setting of uniformly convex hyperbolic spaces. In this way, for bounded C, we get λ_n-asymptotic regularity for general λ_n satisfying (6), and we also obtain explicit bounds on the rate of asymptotic regularity, which are uniform in x, T, and depend on the uniformly convex hyperbolic space (X, ρ, W) only via a modulus η, on C only weakly via its diameter d_C, and on λ_n only via the witness θ.

The most important consequence of our results is that for CAT(0)-spaces, which are uniformly convex hyperbolic spaces with a nice quadratic modulus η, we obtain a quadratic rate of asymptotic regularity (see Corollary 20).

Table 1 presents a general picture of the cases where effective bounds for asymptotic regularity were obtained (UC means uniformly convex).

2. Uniformly convex hyperbolic spaces

In the following, (X, ρ, W) is a hyperbolic space.

Following [28], (X, ρ, W) is called *strictly convex* if for any $x, y \in X$, and $\lambda \in [0, 1]$ there exists a unique element $z \in X$ such that

$$\rho(z, x) = \lambda\rho(x, y) \quad \text{and} \quad \rho(z, y) = (1 - \lambda)\rho(x, y).$$

We define uniform convexity following [11, p. 105].

Definition 4. The hyperbolic space (X, ρ, W) is called *uniformly convex* if for any $r > 0$, and $\varepsilon \in (0, 2]$ there exists a $\delta \in (0, 1]$ such that for all $a, x, y \in X$,

$$\left.\begin{array}{l} \rho(x, a) \leqslant r \\ \rho(y, a) \leqslant r \\ \rho(x, y) \geqslant \varepsilon r \end{array}\right\} \quad \Rightarrow \quad \rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leqslant (1 - \delta)r. \tag{8}$$

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is called a *modulus of uniform convexity*.

Proposition 5. *Any uniformly convex hyperbolic space is strictly convex.*

Proof. Let (X, ρ, W) be uniformly convex with modulus of uniform convexity η. We proof that (X, ρ, W) is strictly convex by contradiction. Thus, assume that there are $x, y \in X$, and $\lambda \in [0, 1]$ such that there exist two distinct points $z, w \in X$ with

$$\rho(z, x) = \rho(w, x) = \lambda\rho(x, y), \qquad \rho(z, y) = \rho(w, y) = (1 - \lambda)\rho(x, y).$$

It follows immediately that $x \neq y$ and $\lambda \in (0,1)$. Let $r_1 := \lambda\rho(x,y) > 0$, $r_2 := (1-\lambda)\rho(x,y) > 0$, $\varepsilon_1 := \frac{\rho(z,w)}{r_1}$, and $\varepsilon_2 := \frac{\rho(z,w)}{r_2}$. It is easy to see that $\varepsilon_1, \varepsilon_2 \in (0,2]$, so we can apply (8) to get

$$\rho\left(\frac{1}{2}z \oplus \frac{1}{2}w, x\right) \leqslant \big(1-\eta(r_1,\varepsilon_1)\big)r_1, \qquad \rho\left(\frac{1}{2}z \oplus \frac{1}{2}w, y\right) \leqslant \big(1-\eta(r_2,\varepsilon_2)\big)r_2.$$

Since $x \neq y$, we must have $\eta(r_1,\varepsilon_1) < 1$ or $\eta(r_2,\varepsilon_2) < 1$. It follows that

$$\begin{aligned}\rho(x,y) &\leqslant \rho\left(\frac{1}{2}z \oplus \frac{1}{2}w, x\right) + \rho\left(\frac{1}{2}z \oplus \frac{1}{2}w, y\right)\\ &\leqslant \big(1-\eta(r_1,\varepsilon_1)\big)r_1 + \big(1-\eta(r_2,\varepsilon_2)\big)r_2 < r_1 + r_2 = \rho(x,y),\end{aligned}$$

that is a contradiction. □

The proof of the following proposition is similar to the one of the corresponding result for uniformly convex Banach spaces.

Proposition 6. *Let $\eta : (0,\infty) \times (0,2] \to (0,1]$. The following are equivalent*:

(1) *(X,ρ,W) is uniformly convex with modulus of uniform convexity η*;
(2) *for any $r > 0$, $\varepsilon \in (0,2]$, $\lambda \in [0,1]$, and $a,x,y \in X$,*

$$\left.\begin{array}{l}\rho(x,a) \leqslant r\\ \rho(y,a) \leqslant r\\ \rho(x,y) \geqslant \varepsilon r\end{array}\right\} \quad \Rightarrow \quad \rho\big((1-\lambda)x \oplus \lambda y, a\big) \leqslant \big(1-\gamma(r,\varepsilon,\lambda)\big)r, \tag{9}$$

where

$$\gamma(r,\varepsilon,\lambda) = \begin{cases} 2\lambda\eta(r,\varepsilon), & \text{if } \lambda \leqslant \frac{1}{2},\\ 2(1-\lambda)\eta(r,\varepsilon), & \text{otherwise}.\end{cases}$$

Lemma 7. *Let (X,ρ,W) be a uniformly convex hyperbolic space with modulus of uniform convexity η. For any $r > 0$, $\varepsilon \in (0,2]$, $\lambda \in [0,1]$, and $a,x,y \in X$,*

$$\left.\begin{array}{l}\rho(x,a) \leqslant r\\ \rho(y,a) \leqslant r\\ \rho(x,y) \geqslant \varepsilon r\end{array}\right\} \quad \Rightarrow \quad \rho\big((1-\lambda)x \oplus \lambda y, a\big) \leqslant \big(1-2\lambda(1-\lambda)\eta(r,\varepsilon)\big)r. \tag{10}$$

Proof. Apply (9) and the fact that $2\lambda, 2(1-\lambda) \geqslant 2\lambda(1-\lambda)$. □

Proposition 8. *Assume that X is a CAT(0)-space. Then X is uniformly convex, and*

$$\eta(r,\varepsilon) := \frac{\varepsilon^2}{8} \tag{11}$$

is a modulus of uniform convexity.

Proof. Let $r > 0$, $\varepsilon \in (0, 2]$, $a, x, y \in X$ be such that $\rho(x, a) \leqslant r$, $\rho(y, a) \leqslant r$, $\rho(x, y) \geqslant \varepsilon r$. Applying (2) we get that

$$\rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leqslant \sqrt{\frac{1}{2}\rho(x, a)^2 + \frac{1}{2}\rho(y, a)^2 - \frac{1}{4}\rho(x, y)^2}$$
$$\leqslant \sqrt{\frac{1}{2}r^2 + \frac{1}{2}r^2 - \frac{1}{4}\varepsilon^2 r^2} = \sqrt{1 - \frac{\varepsilon^2}{4}} \cdot r \leqslant \left(1 - \frac{\varepsilon^2}{8}\right)r.$$

Hence, X is uniformly convex, and $\eta(r, \varepsilon)$ defined by (11) is a modulus of uniform convexity for X. □

3. Technical results

In the sequel, (X, ρ, W) is a uniformly convex hyperbolic space with modulus of uniform convexity η, $C \subseteq X$ a nonempty convex subset, and $T : C \to C$ nonexpansive.

Lemma 9. *Let $x, y \in X$, and (x_n) be the Krasnoselski–Mann iteration starting with x. Then*

(1) $(\rho(x_n, Tx_n))$ *is nonincreasing*;
(2) *for any* $n \in \mathbb{N}$,

$$\rho(x_{n+1}, y) \leqslant \rho(x_n, y) + \lambda_n \rho(y, Ty); \tag{12}$$

(3) *for any* $n \in \mathbb{N}$,

$$\rho(x_n, y) \leqslant \rho(x, y) + \left(\sum_{i=0}^{n-1} \lambda_i\right)\rho(y, Ty). \tag{13}$$

Proof.

(1) See [21, Proposition 3.4], whose proof immediately generalizes to the notion of hyperbolic space used in this paper.
(2)

$$\begin{aligned}\rho(x_{n+1}, y) &\leqslant (1 - \lambda_n)\rho(x_n, y) + \lambda_n \rho(Tx_n, y)\\ &\leqslant (1 - \lambda_n)\rho(x_n, y) + \lambda_n \rho(Tx_n, Ty) + \lambda_n \rho(Ty, y)\\ &\leqslant \rho(x_n, y) + \lambda_n \rho(Ty, y).\end{aligned}$$

(3) (13) follows from (12) by an easy induction. □

For any $x, y \in X$, $n \in \mathbb{N}$, let us denote

$$\alpha(x, n, y) := \rho(x_n, y) + \rho(y, Ty). \tag{14}$$

Lemma 10 *(Main technical lemma). Assume that η decreases with r (for a fixed ε).*
Let $x, y \in X$, $n \in \mathbb{N}$, and $\gamma, \beta, \tilde{\beta}, a > 0$ be such that

$$\gamma \leqslant \alpha(x, n, y) \leqslant \beta, \tilde{\beta} \quad \textit{and} \quad a \leqslant \rho(x_n, Tx_n). \tag{15}$$

Then the following inequality is satisfied:

$$\rho(x_{n+1}, y) \leqslant \rho(x_n, y) + \rho(y, Ty) - 2\gamma\lambda_n(1-\lambda_n)\eta\left(\tilde{\beta}, \frac{a}{\beta}\right). \tag{16}$$

Proof. First, let us remark that

$$\begin{aligned}
&\rho(Tx_n, y) \leqslant \rho(Tx_n, Ty) + \rho(Ty, y) \leqslant \alpha(x,n,y) \leqslant \beta,\\
&\rho(x_n, y) \leqslant \alpha(x,n,y) \leqslant \beta,\\
&\rho(x_n, Tx_n) \geqslant a = \left(\frac{a}{\beta}\right)\cdot\beta.
\end{aligned}$$

Moreover,

$$\begin{aligned}
0 < a &\leqslant \rho(x_n, Tx_n) \leqslant \rho(x_n, y) + \rho(y, Ty) + \rho(Ty, Tx_n)\\
&\leqslant 2\rho(x_n, y) + \rho(y, Ty) \leqslant 2\alpha(x,n,y) \leqslant 2\beta,
\end{aligned}$$

so $\frac{a}{\beta} \in (0, 2]$. Thus, we can apply (10) to obtain

$$\begin{aligned}
\rho(x_{n+1}, y) &= \rho\big((1-\lambda_n)x_n \oplus \lambda_n Tx_n, y\big)\\
&\leqslant \left(1 - 2\lambda_n(1-\lambda_n)\eta\left(\alpha(x,n,y), \frac{a}{\beta}\right)\right)\alpha(x,n,y)\\
&= \rho(x_n, y) + \rho(y, Ty) - 2\lambda_n(1-\lambda_n)\eta\left(\alpha(x,n,y), \frac{a}{\beta}\right)\alpha(x,n,y).
\end{aligned}$$

Since $\alpha(x,n,y) \leqslant \tilde{\beta}$, and η decreases with r, we get that

$$\eta\left(\alpha(x,n,y), \frac{a}{\beta}\right) \geqslant \eta\left(\tilde{\beta}, \frac{a}{\beta}\right),$$

hence

$$\begin{aligned}
\rho(x_{n+1}, y) &\leqslant \rho(x_n, y) + \rho(y, Ty) - 2\lambda_n(1-\lambda_n)\eta\left(\tilde{\beta}, \frac{a}{\beta}\right)\alpha(x,n,y)\\
&\leqslant \rho(x_n, y) + \rho(y, Ty) - 2\gamma\lambda_n(1-\lambda_n)\eta\left(\tilde{\beta}, \frac{a}{\beta}\right),
\end{aligned}$$

since $\alpha(x,n,y) \geqslant \gamma$ by the hypothesis. □

Corollary 11. *Assume that η decreases with r (for a fixed ε), and moreover that η can be written as $\eta(r,\varepsilon) = \varepsilon \cdot \tilde{\eta}(r,\varepsilon)$ such that $\tilde{\eta}$ increases with ε (for a fixed r).*

Let $x, y \in X$, $n \in \mathbb{N}$, and $\delta, a > 0$ be such that

$$\alpha(x,n,y) \leqslant \delta \quad \text{and} \quad a \leqslant \rho(x_n, Tx_n). \tag{17}$$

Then

$$\rho(x_{n+1}, y) \leqslant \rho(x_n, y) + \rho(y, Ty) - 2a\lambda_n(1-\lambda_n)\tilde{\eta}\left(\delta, \frac{a}{\delta}\right). \tag{18}$$

Proof. Applying Lemma 10 with $\gamma := \beta := \alpha(x, n, y)$, $\tilde{\beta} := \delta$, we get that

$$\begin{aligned}\rho(x_{n+1}, y) &\leqslant \rho(x_n, y) + \rho(y, Ty) - 2\alpha(x, n, y)\lambda_n(1-\lambda_n)\eta\left(\delta, \frac{a}{\alpha(x, n, y)}\right)\\ &= \rho(x_n, y) + \rho(y, Ty) - 2a\lambda_n(1-\lambda_n)\tilde{\eta}\left(\delta, \frac{a}{\alpha(x, n, y)}\right)\\ &\leqslant \rho(x_n, y) + \rho(y, Ty) - 2a\lambda_n(1-\lambda_n)\tilde{\eta}\left(\delta, \frac{a}{\delta}\right),\end{aligned}$$

since $\frac{a}{\delta} \leqslant \frac{a}{\alpha(x,n,y)}$, and $\tilde{\eta}$ increases with ε. □

Corollary 12. *Assume that η decreases with r (for a fixed ε).*
Let $x, y \in X$, $N \in \mathbb{N}$, and $b, c, \gamma, a > 0$ be such that

$$\rho(x, y) \leqslant b \quad and \quad \rho(y, Ty) \leqslant c, \tag{19}$$

and for all $n = \overline{0, N}$,

$$\gamma \leqslant \alpha(x, n, y) \quad and \quad a \leqslant \rho(x_n, Tx_n). \tag{20}$$

Let $d \geqslant (N + 1)c$. Then

$$\rho(x_{N+1}, y) \leqslant b + (N+1)c - 2\gamma\eta\left(b + d, \frac{a}{b+d}\right)\sum_{n=0}^{N}\lambda_n(1-\lambda_n). \tag{21}$$

Proof. Using (13), we get for $n = \overline{0, N}$,

$$\begin{aligned}\alpha(x, n, y) = \rho(x_n, y) + \rho(y, Ty) &\leqslant \rho(x, y) + \left(\sum_{i=0}^{n-1}\lambda_i\right)\rho(y, Ty) + \rho(y, Ty)\\ &\leqslant b + (n+1)\rho(y, Ty) \leqslant b + (N+1)c \leqslant b + d.\end{aligned}$$

Applying Lemma 10 with $\beta := \tilde{\beta} := b + d$, we get

$$\rho(x_{n+1}, y) \leqslant \rho(x_n, y) + \rho(y, Ty) - 2\gamma\lambda_n(1-\lambda_n)\eta\left(b + d, \frac{a}{b+d}\right). \tag{22}$$

Adding (22) for $n = \overline{0, N}$, it follows that

$$\begin{aligned}\rho(x_{N+1}, y) &\leqslant \rho(x, y) + (N+1)\rho(y, Ty) - 2\gamma\eta\left(b + d, \frac{a}{b+d}\right)\sum_{n=0}^{N}\lambda_n(1-\lambda_n)\\ &\leqslant b + (N+1)c - 2\gamma\eta\left(b + d, \frac{a}{b+d}\right)\sum_{n=0}^{N}\lambda_n(1-\lambda_n). \quad \square\end{aligned}$$

Corollary 13. *In the hypothesis of Corollary 12, assume moreover that $\eta(r, \varepsilon)$ can be written as $\eta(r, \varepsilon) = \varepsilon \cdot \tilde{\eta}(r, \varepsilon)$ such that $\tilde{\eta}$ increases with ε (for a fixed r). Then*

$$\rho(x_{N+1}, y) \leqslant b + (N+1)c - 2a\tilde{\eta}\left(b + d, \frac{a}{b+d}\right)\sum_{n=0}^{N}\lambda_n(1-\lambda_n). \tag{23}$$

Proof. Follow the proof of Corollary 12, using this time Corollary 11 instead of Lemma 10. □

4. Main theorem

Theorem 14. *Let (X, ρ, W) be a uniformly convex hyperbolic space with modulus of uniform convexity η such that η decreases with r (for a fixed ε).*

Let $C \subseteq X$ be a nonempty convex subset, and $T : C \to C$ nonexpansive.

Assume that (λ_n) is a sequence in $[0,1]$ and $\theta : \mathbb{N} \to \mathbb{N}$ is such that for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{\theta(n)} \lambda_k(1-\lambda_k) \geqslant n. \tag{24}$$

Let $x \in C$, $b > 0$ be such that for any $\delta > 0$ there is $y \in C$ with

$$\rho(x,y) \leqslant b \quad \text{and} \quad \rho(y,Ty) < \delta. \tag{25}$$

Then

$$\lim_{n\to\infty} \rho(x_n, Tx_n) = 0,$$

and, moreover,

$$\forall \varepsilon > 0\ \forall n \geqslant h(\varepsilon, \eta, b, \theta) \quad \big(\rho(x_n, Tx_n) \leqslant \varepsilon\big), \tag{26}$$

where

$$h(\varepsilon, \eta, b, \theta) := \begin{cases} \theta\big(\big\lceil \frac{b+1}{\varepsilon\cdot\eta(b+1,\frac{\varepsilon}{b+1})} \big\rceil\big) & \text{for } \varepsilon < 2b, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. First, let us remark that for any $n \in \mathbb{N}$, and $\delta > 0$, by the fact that $(\rho(x_n, Tx_n))$ is nonincreasing, and (25) we get

$$\rho(x_n, Tx_n) \leqslant \rho(x, Tx) \leqslant 2\rho(x,y) + \rho(y,Ty) < 2b + \delta.$$

It follows that $\rho(x_n, Tx_n) \leqslant 2b$ for any $n \in \mathbb{N}$, hence the case $\varepsilon \geqslant 2b$ is obvious. Let us consider $\varepsilon < 2b$ and denote

$$K := \left\lceil \frac{b+1}{\varepsilon \cdot \eta\big(b+1, \frac{\varepsilon}{b+1}\big)} \right\rceil, \qquad N := \theta(K) = h(\varepsilon, \eta, b, \theta).$$

Assume that for all $n \leqslant N$ we have that $\rho(x_n, Tx_n) > \varepsilon$. Let $\delta > 0$ be such that

$$\delta < \frac{1}{2(N+1)}, \quad \text{so } (N+1)\delta < \frac{1}{2} < 1.$$

Let $y \in C$ satisfying (25) for this δ. It follows that for all $n = \overline{0, N}$,

$$\alpha(x, n, y) \geqslant \frac{\rho(x_n, Tx_n)}{2} > \frac{\varepsilon}{2}.$$

Applying Corollary 12 with $a := \varepsilon$, $c := \delta$, $\gamma := \frac{\varepsilon}{2}$, $d := 1$, we get

$$\begin{aligned} \rho(x_{N+1}, y) &\leqslant b + (N+1)\delta - \varepsilon\eta\left(b+1, \frac{\varepsilon}{b+1}\right)\sum_{n=0}^{N} \lambda_n(1-\lambda_n) \\ &\leqslant b + \frac{1}{2} - \varepsilon\eta\left(b+1, \frac{\varepsilon}{b+1}\right)\sum_{n=0}^{\theta(K)} \lambda_n(1-\lambda_n) \end{aligned}$$

$$\leqslant b + \frac{1}{2} - \varepsilon\eta\left(b+1, \frac{\varepsilon}{b+1}\right)K, \quad \text{by (24)}$$
$$\leqslant b + \frac{1}{2} - (b+1) < 0,$$

that is a contradiction. Thus, there is $n \leqslant h(\varepsilon, \eta, b, \theta)$ such that $\rho(x_n, Tx_n) \leqslant \varepsilon$. Since $(\rho(x_n, Tx_n))$ is nonincreasing, (26) follows. □

Remark 15. In the hypothesis of the above theorem, assume moreover that $\eta(r, \varepsilon)$ can be written as $\eta(r, \varepsilon) = \varepsilon \cdot \tilde{\eta}(r, \varepsilon)$ such that $\tilde{\eta}$ increases with ε (for a fixed r). Then the bound $h(\varepsilon, \eta, b, \theta)$ can be replaced for $\varepsilon < 2b$ by

$$\tilde{h}(\varepsilon, \eta, b, \theta) := \theta\left(\left\lceil \frac{b+1}{2\varepsilon \cdot \tilde{\eta}\left(b+1, \frac{\varepsilon}{b+1}\right)} \right\rceil\right).$$

Proof. Define

$$K := \left\lceil \frac{b+1}{2\varepsilon \cdot \tilde{\eta}\left(b+1, \frac{\varepsilon}{b+1}\right)} \right\rceil,$$

and follow the proof of the theorem using Corollary 13 instead of Corollary 12. □

As an immediate consequence of our main theorem, we obtain Groetsch's theorem for hyperbolic spaces.

Corollary 16. *Let (X, ρ, W) be a uniformly convex hyperbolic space with modulus of uniform convexity η such that η decreases with r (for a fixed ε), $C \subseteq X$ be a nonempty convex subset, and $T : C \to C$ nonexpansive such that T has at least one fixed point. Assume that (λ_n) is a sequence in* $[0, 1]$ *such that*

$$\sum_{n=0}^{\infty} \lambda_n(1 - \lambda_n) = \infty.$$

Then for any $x \in C$,

$$\lim_{n\to\infty} \rho(x_n, Tx_n) = 0.$$

Proof. Since $\sum_{n=0}^{\infty} \lambda_n(1 - \lambda_n)$ diverges, for any $n \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that $\sum_{k=0}^{N} \lambda_k \times (1 - \lambda_k) \geqslant n$, so by defining $\theta(n)$ as the least N with this property, (24) is satisfied. Let p be a fixed point of T. Then for any $x \in C$, if we take $b := \rho(x, p)$, (25) is satisfied with $y := p$. Hence, we can apply Theorem 14 to get $\lim_{n\to\infty} \rho(x_n, Tx_n) = 0$. □

Corollary 17. *Let (X, ρ, W), η, C, T, (λ_n), θ be as in the hypothesis of Theorem* 14. *Assume moreover that C is bounded with finite diameter d_C. Then T is λ_n-asymptotically regular, and the following holds for all $x \in C$:*

$$\forall\varepsilon > 0\ \forall n \geqslant h(\varepsilon, \eta, d_C, \theta) \quad \big(\rho(x_n, Tx_n) \leqslant \varepsilon\big),$$

where $h(\varepsilon, \eta, d_C, \theta)$ is defined as in Theorem 14 *by replacing b with d_C.*

Proof. If C is bounded, then C has approximate fixed point property, as a consequence of [10, Theorem 1], which was proved for spaces of hyperbolic type, but, as we have already remarked, any hyperbolic space in our sense is a space of hyperbolic type. It follows that condition (25) holds for all $x \in C$ with d_C instead of b. Hence, we can apply Theorem 14 for all $x \in C$. $\square$

Thus, for bounded C, we get asymptotic regularity for general (λ_n), and an explicit bound $h(\varepsilon, \eta, d_c, \theta)$ on the rate of asymptotic regularity, which depends only on the error ε, on the modulus of uniform convexity η, on the diameter d_C of C, and on (λ_n) only via θ, but not on the nonexpansive mapping T, the starting point $x \in C$ of the iteration or other data related with C and X.

The bound $h(\varepsilon, \eta, d_C, \theta)$ on the rate of asymptotic regularity can be further simplified in the case of constant $\lambda_n := \lambda \in (0, 1)$.

Corollary 18. *Let (X, ρ, W), η, C, d_C, T be as in the hypothesis of Corollary* 17. *Assume moreover that $\lambda_n := \lambda \in (0, 1)$ for all $n \in \mathbb{N}$.*

Then T is λ-asymptotically regular, and for all $x \in C$,

$$\forall \varepsilon > 0\, \forall n \geqslant h(\varepsilon, \eta, d_C, \lambda) \quad \big(\rho(x_n, Tx_n) \leqslant \varepsilon\big), \tag{27}$$

where

$$h(\varepsilon, \eta, d_C, \lambda) := \begin{cases} \frac{1}{\lambda(1-\lambda)} \left\lceil \frac{d_C+1}{\varepsilon \cdot \eta(d_C+1, \frac{\varepsilon}{d_C+1})} \right\rceil & \text{for } \varepsilon < 2d_C, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if $\eta(r, \varepsilon)$ can be written as $\eta(r, \varepsilon) = \varepsilon \cdot \tilde{\eta}(r, \varepsilon)$ such that $\tilde{\eta}$ increases with ε (for fixed r), then the bound $h(\varepsilon, \eta, d_C, \lambda)$ can be replaced for $\varepsilon < 2d_C$ with

$$\tilde{h}(\varepsilon, \eta, d_C, \lambda) = \frac{1}{\lambda(1-\lambda)} \left\lceil \frac{d_C+1}{2\varepsilon \tilde{\eta}\big(d_C+1, \frac{\varepsilon}{d_C+1}\big)} \right\rceil.$$

Proof. (24) is satisfied with

$$\theta : \mathbb{N} \to \mathbb{N}, \qquad \theta(n) := \frac{n}{\lambda(1-\lambda)}.$$

Hence, we can apply Corollary 17, and Remark 15. In this case, for $\varepsilon < 2d_C$, we get that

$$\begin{aligned} h(\varepsilon, \eta, d_C, \theta) &= \theta\left(\left\lceil \frac{d_C+1}{\varepsilon \cdot \eta\big(d_C+1, \frac{\varepsilon}{d_C+1}\big)} \right\rceil \right) \\ &= \frac{1}{\lambda(1-\lambda)} \left\lceil \frac{d_C+1}{\varepsilon \cdot \eta\big(d_C+1, \frac{\varepsilon}{d_C+1}\big)} \right\rceil. \qquad \square \end{aligned}$$

As we have proved in Section 2, CAT(0)-spaces are uniformly convex hyperbolic spaces with a "nice" modulus of uniform convexity $\eta(r, \varepsilon) := \frac{\varepsilon^2}{8}$, which has the form required in Remark 15. Thus, the above results can be applied to CAT(0)-spaces.

Corollary 19. *Let X be a CAT*(0)*-space, and C, d_C, T, (λ_n), θ be as in the hypothesis of Theorem* 14.

Then T is λ_n-asymptotically regular, and for all $x \in C$,

$$\forall \varepsilon > 0\, \forall n \geqslant g(\varepsilon, d_C, \theta) \quad \big(\rho(x_n, Tx_n) \leqslant \varepsilon\big), \tag{28}$$

where

$$g(\varepsilon, d_C, \theta) := \begin{cases} \theta\big(\lceil \frac{4(d_C+1)^2}{\varepsilon^2} \rceil\big) & \textit{for } \varepsilon < 2d_C, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Apply Corollaries 8, 17, and Remark 15. □

Corollary 20. *Let X be a CAT(0)-space, and C, d_C, T be as in the hypothesis of Corollary* 19. *Assume that $(\lambda_n) := \lambda \in (0,1)$. Then T is λ-asymptotically regular, and for all $x \in C$,*

$$\forall \varepsilon > 0\ \forall n \geqslant \tilde{g}(\varepsilon, d_C, \lambda) \quad \big(\rho(x_n, Tx_n) \leqslant \varepsilon\big), \tag{29}$$

where

$$\tilde{g}(\varepsilon, d_C, \lambda) := \begin{cases} \frac{1}{\lambda(1-\lambda)} \lceil \frac{4(d_C+1)^2}{\varepsilon^2} \rceil & \textit{for } \varepsilon < 2d_C, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Apply Corollaries 8, 18, and Remark 15. □

Hence, in the case of CAT(0)-spaces, we get a quadratic (in $1/\varepsilon$) rate of asymptotic regularity.

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