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A splitting method for the nonlinear Schrödinger equation

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ABSTRACT

We introduce a splitting method for the semilinear Schrödinger equation and prove its convergence for those nonlinearities which can be handled by the classical well-posedness $L^2(\mathbb{R}^d)$-theory. More precisely, we prove that the scheme is of first order in the $L^2(\mathbb{R}^d)$-norm for $H^2(\mathbb{R}^d)$-initial data.

1. Introduction

Let us consider the nonlinear Schrödinger equation (NSE):

$$\begin{cases} \dfrac{du}{dt} = i\Delta u + i\lambda |u|^p u, & x \in \mathbb{R}^d,\ t \neq 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

For any $0 \leqslant p < 4/d$, $\lambda \in \mathbb{R}$ and $\varphi \in L^2(\mathbb{R}^d)$, Eq. (1.1) has a unique global solution $u \in C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^q_{loc}(\mathbb{R}, L^r(\mathbb{R}^d))$ for some suitable pairs (q, r). This has been proved by Tsutsumi in [16] by using a fix point argument and the so-called Strichartz estimates [15]. These estimates show that the semigroup generated by the linear Schrödinger equation (LSE), $S(t) = \exp(it\Delta)$, satisfies

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$$\big\|S(\cdot)\varphi\big\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \leqslant C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)} \quad \text{for all } \varphi \in L^2\big(\mathbb{R}^d\big), \tag{1.2}$$

for the so-called admissible pairs (q,r) (cf. [10]): $2 \leqslant q, r \leqslant \infty$, $(q,r,d) \neq (2,\infty,2)$ and

$$\frac{1}{q} = \frac{d}{2}\left(\frac{1}{2} - \frac{1}{r}\right). \tag{1.3}$$

In addition, in [16] the stability of solutions under perturbation of the initial data has been proved. In fact there exists a time T, depending on the $L^2(\mathbb{R}^d)$-norm of the initial data, such that on the interval $(0,T)$ the difference between two solutions of Eq. (1.1) is controlled by the error made in the linear part $S(t)(\varphi_1 - \varphi_2)$ in a certain $L^q(0,T,L^r(\mathbb{R}^d))$-norm. Thus, Strichartz's estimate (1.2) shows that, locally, the error between two solutions u_1 and u_2 can be estimated in terms of the $L^2(\mathbb{R}^d)$-norm of the difference of the initial data $\varphi_1 - \varphi_2$. Using the global well-posedness of system (1.1) the same procedure can be extended to any bounded time interval. We will adapt this idea to the numerical context in order to estimate the error committed when approximating the solutions of (1.1) by a splitting method.

A splitting method consists in decomposing the flow (1.1) in two flows, which in principle should be computed easily. To be more precise, we define the flow $N(t)$ for the differential equation:

$$\begin{cases} \dfrac{du}{dt} = i\lambda|u|^p u, & x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.4}$$

i.e.

$$N(t)\varphi = \exp\big(it\lambda|\varphi|^p\big)\varphi. \tag{1.5}$$

The idea of splitting methods is to approximate the solutions of (1.1) by combining the two flows $S(t)$ and $N(t)$. For a fixed time interval $[0,T]$ we can choose a small positive time step τ and consider either the Lie approximation:

$$Z(n\tau) = \big(S(\tau)N(\tau)\big)^n\varphi, \quad 0 \leqslant n\tau \leqslant T, \tag{1.6}$$

or Strang approximation

$$Z(n\tau) = \big(S(\tau/2)N(\tau)S(\tau/2)\big)^n\varphi, \quad 0 \leqslant n\tau \leqslant T. \tag{1.7}$$

In the two-dimensional case, Besse et al. [1] have analyzed the convergence of the above methods for globally Lipschitz-continuous nonlinearities. Also Lubich [11] analyzed the Strang method for the Schrödinger–Poisson equation and the cubic NSE in the case of $H^4(\mathbb{R}^3)$-initial data. There, the $H^4(\mathbb{R}^3)$-regularity was imposed to guarantee that the approximate solution Z remains bounded in the $H^2(\mathbb{R}^3)$-norm.

In this paper we introduce a splitting method for the NSE with $1 \leqslant p < 4/d$ and prove the convergence in the $L^2(\mathbb{R}^d)$-norm for $H^2(\mathbb{R}^d)$-initial data. The scheme we analyse is based on an approximation $S_\tau(t)$ of the linear semigroup $S(t)$ which admits Strichartz-like estimates in some time discrete spaces. We make use of these new estimates to establish uniform bounds on the numerical solution in the auxiliary spaces $l^q_{loc}(\tau\mathbb{Z}, L^r(\mathbb{R}^d))$ without assuming more than $L^2(\mathbb{R}^d)$-regularity on the initial data. Once these bounds are obtained we will need the $H^2(\mathbb{R}^d)$ regularity in order to obtain the order of error.

The idea behind the numerical schemes for the LSE which admit uniform (with respect to discretization parameters) estimates of Strichartz type is that when they are applied in the context of

NSE, the error committed is controlled by the error committed in approximating the LSE. The application of these numerical schemes for NSE has been previously used in the case of semidiscrete space approximations [7–9] and in the fully discrete case in [6].

In this paper we will concentrate on Lie's approximation method. We remark that Z defined by (1.6) satisfies

$$Z(n\tau) = S(n\tau)\varphi + \tau \sum_{k=0}^{n-1} S(n\tau - k\tau)\frac{N(\tau)-\mathcal{I}}{\tau} Z(k\tau), \quad n \geqslant 1. \tag{1.8}$$

Since Z is defined on a discrete set of points we need to evaluate Z in some discrete time norms $l^q(\tau\mathbb{Z}, L^r(\mathbb{R}^d))$. We emphasize that for $(q,r) \neq (\infty, 2)$ even the linear part $S(n\tau)\varphi$ does not satisfy Strichartz-like estimates:

$$\big\| S(n\tau)\varphi \big\|_{l^q(\tau\mathbb{Z}, L^r(\mathbb{R}^d))} \leqslant C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)} \quad \text{for all } \varphi \in L^2\big(\mathbb{R}^d\big),$$

where

$$\|u\|_{l^q(\tau\mathbb{Z}, L^r(\mathbb{R}^d))} = \left(\tau \sum_{n\in\mathbb{Z}} \big\| u(k\tau)\big\|^q_{L^r(\mathbb{R}^d)}\right)^{1/q}.$$

Indeed, in contrast with the classical estimate (1.2), the above inequality implies that

$$\tau^{1/q}\big\| S(\tau)\varphi \big\|_{L^r(\mathbb{R}^d)} \leqslant C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)},$$

inequality which does not hold for all $\varphi \in L^2(\mathbb{R}^d)$ (choose $\varphi = S(-\tau)\psi$ with $\psi \in L^2(\mathbb{R}^d)\backslash L^r(\mathbb{R}^d)$ for $r \neq 2$). This implies that we have to choose an approximation $S_\tau(t)$ of the linear semigroup $S(t)$ such that $S_\tau(t)$ admits Strichartz-like estimates which are discrete in time and moreover, these estimates are uniform with respect to the time parameter τ:

$$\big\| S_\tau(n\tau)\varphi \big\|_{l^q(\tau\mathbb{Z}, L^r(\mathbb{R}^d))} \leqslant C\|\varphi\|_{L^2(\mathbb{R}^d)}, \quad \forall \varphi \in L^2\big(\mathbb{R}^d\big).$$

One of the possible choices is the filtered operator

$$S_\tau(t)\varphi = S(t)\Pi_\tau\varphi$$

where Π_τ filters the high frequencies as follows

$$\widehat{\Pi_\tau\varphi}(\xi) = \widehat{\varphi}(\xi)\mathbf{1}_{\{|\xi|\leqslant \tau^{-1/2}\}}(\xi), \quad \xi \in \mathbb{R}^d. \tag{1.9}$$

For other possible choices of the operator S_τ we refer to the previous work on dispersive methods for LSE [7–9]. Also as initial data we have to choose a filtration of φ, $\Pi_\tau\varphi$, since otherwise $Z_\tau(0)\varphi = \varphi$ does not belong to $L^r(\mathbb{R}^d)$ and we cannot evaluate the $l^q(0 \leqslant n\tau \leqslant T, L^r(\mathbb{R}^d))$-norm of the approximation Z_τ.

The splitting scheme we propose is the following one:

$$Z_\tau(n\tau) = \big(S_\tau(\tau)N(\tau)\big)^n \Pi_\tau\varphi, \quad n \geqslant 0. \tag{1.10}$$

Observe that in this scheme only the linear equation is filtered while the nonlinear one is solved exactly.

In the following, for any interval I with $|I| \geqslant \tau$, the space $l^q(n\tau \in I, L^r(\mathbb{R}^d))$ contains all functions defined on $\tau\mathbb{Z} \cap I$ with values in $L^r(\mathbb{R}^d)$ and the norm on this space is defined by

$$\|u\|_{l^q(n\tau\in I, L^r(\mathbb{R}^d))} = \left(\tau \sum_{n\in\mathbb{Z}} \|u(k\tau)\|^q_{L^r(\mathbb{R}^d)}\right)^{1/q}.$$

Along the paper we always assume that τ is a small parameter, in the sense that there exists $\tau_0 = \tau_0(\|\varphi\|_{L^2(\mathbb{R}^d)})$ such that all the results hold for $\tau \leqslant \tau_0$.

The main results of this paper are the following.

Theorem 1.1 *(Stability). Let $0 < p < 4/d$. For any $\varphi \in L^2(\mathbb{R}^d)$ the approximation Z_τ introduced in* (1.10) *satisfies*:

(i) *a uniform $L^2(\mathbb{R}^d)$-bound*

$$\max_{n\geqslant 0} \|Z_\tau(n\tau)\|_{L^2(\mathbb{R}^d)} \leqslant \|\varphi\|_{L^2(\mathbb{R}^d)}, \tag{1.11}$$

(ii) *there exists $T_0 \simeq \|\varphi\|^{-\frac{4p}{4-dp}}$ such that for any interval I with $|I| \leqslant T_0$ and for any admissible pair (q,r) the following*

$$\|Z_\tau(n\tau)\|_{l^q(n\tau\in I, L^r(\mathbb{R}^d))} \leqslant C(d,p,q)\|\varphi\|_{L^2(\mathbb{R}^d)} \tag{1.12}$$

holds for some constant $C(d,p,q)$ independent of the time step τ,

(iii) *for any $T > 0$ and (q,r) admissible-pair the following*

$$\|Z_\tau(n\tau)\|_{l^q(0\leqslant n\tau\leqslant T; L^r(\mathbb{R}^d))} \leqslant C(T,d,p,q)\|\varphi\|_{L^2(\mathbb{R}^d)} \tag{1.13}$$

holds for some constant $C(T,d,p,q)$ independent of the time step τ.

Theorem 1.2 *(Convergence). Let $d \leqslant 3$, $p \in [1, 4/d)$ and $\varphi \in H^2(\mathbb{R}^d)$. The numerical solution Z_τ has a first-order error bound in $L^2(\mathbb{R}^d)$*:

$$\max_{0\leqslant n\tau\leqslant T} \|Z_\tau(n\tau) - u(n\tau)\|_{L^2(\mathbb{R}^d)} \leqslant \tau C\big(T, d, p, \|\varphi\|_{H^2(\mathbb{R}^d)}\big).$$

We point out that Theorem 1.2 works in the case $d \leqslant 3$ which is quite restrictive. The restriction $p \geqslant 1$ comes from the fact that in our proof we need to guarantee that u solution of (1.1) belongs to $C(0, T, H^2(\mathbb{R}^d))$ (see [2, Chapter 5.3]).

We now comment on the possible analysis of the order of error in the case of less regularity or other nonlinearities. It is convenient to write u in the semigroup formulation:

$$u(t) = S(t)\varphi + i\lambda \int_0^t S(t-s)|u|^p u(s)\,ds, \quad t \geqslant 0. \tag{1.14}$$

Looking at (1.8), we observe that Z (or Z_τ) defined by (1.6) (or (1.10)), differs from u in two important facts: the integral in (1.14) is replaced by a sum in (1.8) and the nonlinear term $f(u) = \lambda|u|^p u$ is replaced by $\tau^{-1}(N(\tau) - \mathcal{I})Z$. In view of this, it seems to be reasonable that Z better approximates the solution of the following NSE:

$$\begin{cases} \dfrac{dv}{dt} = i\Delta v + \dfrac{\exp(i\lambda\tau|v|^p)-1}{\tau}v, & x\in\mathbb{R}^d,\ t>0,\\ v(x,0)=\varphi(x), & x\in\mathbb{R}^d, \end{cases} \tag{1.15}$$

whose solution satisfies

$$v(t) = S(t)\varphi + \int_0^t S(t-s)\frac{N(\tau)-\mathcal{I}}{\tau}v(s)\,ds, \quad t\geqslant 0. \tag{1.16}$$

When $0\leqslant p<4/d$ and $\varphi\in H^1(\mathbb{R}^d)$, Eq. (1.15) has a global $H^1(\mathbb{R}^d)$-solution (see [2, Theorem 5.2.1]). We conjecture that in this case similar results to those obtained in this paper could be obtained once the results of Lemma 4.6 are obtained with less regularity assumptions.

In what concerns the range $4/d<p<4/(d-2)$, $d\geqslant 3$ ($4/d<p<\infty$ if $d\in\{1,2\}$) Eq. (1.1) entries in the subcritical H^1-case and there are instances where the solution is global (see [2, Chapter 6] for a precise statement) since we have the following conservation of energy:

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2 - \frac{\lambda}{p+1}\int_{\mathbb{R}^d}|u|^{p+1}.$$

However, in this range of p's we cannot guarantee that system (1.15) has a global H^1-solution since it is not obvious what is the *energy* which is preserved. This suggests that the $H^1(\mathbb{R}^d)$-stability for large time intervals for the splitting methods (1.6)–(1.7) will be very difficult to prove, or even impossible, even though the solutions of (1.1) are global and belong to $H^1(\mathbb{R}^d)$ at any positive time. It has been proved in [11] that the $H^1(\mathbb{R}^3)$-stability of the numerical scheme can be established assuming more regularity on the initial data, for example $H^3(\mathbb{R}^3)$ in the case $p=2$.

Since in the case $4/d<p<4/(d-2)$, $d\geqslant 3$ ($4/d<p<\infty$ if $d\in\{1,2\}$) the global existence of an H^1-solution for (1.15) is not an easy task we can only guarantee the existence of a local solution v in some time interval $[0,T_0]$ with $T_0=T_0(\|\varphi\|_{H^1(\mathbb{R}^d)})$. In what concerns the splitting method we conjecture that there exists a positive time $T_1\simeq T_0$ such that the solution $\{Z(n\tau)\}_{0\leqslant n\tau\leqslant T_1}$ is uniformly bounded with respect to the time parameter τ in the $H^1(\mathbb{R}^d)$-norm. This smallness on the time interval has been also previously imposed by Fröhlich in [4] where the order of error has been obtained in the case of the Schrödinger–Poisson equation. The error analysis for small intervals of time remains to be analysed in a future work.

The paper is organised as follows. In Section 2 we obtain discrete in time Strichartz estimates which are similar to the classical ones in [10]. Once these estimates are obtained we prove Theorem 1.1. Section 3 is devoted to presenting some classical results about the NSE and to estimating the error between u, solution of system (1.1), and v solution of system (1.15). In Section 5 we first measure the error between Z_τ and v and then apply it to prove Theorem 1.2. The last section contains some auxiliary results that are used throughout the paper.

The analysis in this paper can be extended to splitting methods in fully discrete framework by using the schemes introduced and analyzed in [6]. This will be the object of a future work.

2. Discrete time Strichartz estimates and stability

In this section we prove discrete in time Strichartz-like estimates for the operator S_τ introduced in previous section. Similar estimates for space semidiscretizations and fully discrete schemes have been obtained in [7,8,6]. Once the Strichartz estimates are obtained we apply them to obtain uniform bounds on the discrete solution Z_τ.

Theorem 2.1. *The semigroup* $\{S_\tau(t)\}_{t\in\mathbb{R}}$ *satisfies*

$$\big\|S_\tau(t)\varphi\big\|_{L^2(\mathbb{R}^d)} \leqslant \|\varphi\|_{L^2(\mathbb{R}^d)}, \quad \forall t \in \mathbb{R}, \tag{2.1}$$

and

$$\big\|S_\tau(t)\varphi\big\|_{L^\infty(\mathbb{R}^d)} \leqslant \frac{C(d)}{\tau^{d/2}+|t|^{d/2}}\|\varphi\|_{L^1(\mathbb{R}^d)}, \quad \forall t \in \mathbb{R}. \tag{2.2}$$

Moreover, for any admissible pairs (q,r) *and* $(\tilde{q},\tilde{r})$ *the following hold*:

(i) *Continuous in time estimates*:

$$\big\|S_\tau(\cdot)\varphi\big\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \leqslant C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)}, \tag{2.3}$$

$$\bigg\|\int_{\mathbb{R}} S_\tau(s)^* f(s)\,ds\bigg\|_{L^2(\mathbb{R}^d)} \leqslant C(d,\tilde{q})\|f\|_{L^{\tilde{q}'}(\mathbb{R},L^{\tilde{r}'}(\mathbb{R}^d))}, \tag{2.4}$$

and

$$\bigg\|\int_{s<t} S_\tau(t-s)f(s)\,ds\bigg\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \leqslant C(d,q,\tilde{q})\|f\|_{L^{\tilde{q}'}(\mathbb{R},L^{\tilde{r}'}(\mathbb{R}^d))}. \tag{2.5}$$

(ii) *Discrete in time estimates*:

$$\big\|S_\tau(\cdot)\varphi\big\|_{l^q(\tau\mathbb{Z},L^r(\mathbb{R}^d))} \leqslant C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)}, \tag{2.6}$$

$$\bigg\|\tau\sum_{n\in\mathbb{Z}} S_\tau(n\tau)^* f(n\tau)\bigg\|_{L^2(\mathbb{R}^d)} \leqslant C(d,\tilde{q})\|f\|_{l^{\tilde{q}'}(\tau\mathbb{Z},L^{\tilde{r}'}(\mathbb{R}^d))}, \tag{2.7}$$

and

$$\bigg\|\tau\sum_{k=-\infty}^{n-1} S_\tau\big((n-k)\tau\big)f(k\tau)\bigg\|_{l^q(\tau\mathbb{Z},L^r(\mathbb{R}^d))} \leqslant C(d,q,\tilde{q})\|f\|_{l^{\tilde{q}'}(\tau\mathbb{Z},L^{\tilde{r}'}(\mathbb{R}^d))}. \tag{2.8}$$

Remark 2.1. A useful consequence of (2.8) is given by the following estimate

$$\bigg\|\tau\sum_{k=0}^{n-1} S_\tau\big((n-k)\tau\big)g(k\tau)\bigg\|_{l^q(\tau\leqslant n\tau\leqslant(N+1)\tau,L^r(\mathbb{R}^d))} \leqslant C(d,q,\tilde{q})\|g\|_{l^{\tilde{q}'}(0\leqslant n\tau\leqslant N\tau,L^{\tilde{r}'}(\mathbb{R}^d))}, \tag{2.9}$$

which holds for any positive integer N. It is a consequence of (2.8) applied to the function $f(n\tau) = g(n\tau)\mathbf{1}_{\{0\leqslant n\tau\leqslant N\tau\}}(n\tau)$.

Remark 2.2. Inequalities (2.1) and (2.2) give us estimates for S_τ in norms which are discrete in time. When considering continuous in time norms $L^q(\mathbb{R},L^r(\mathbb{R}^d))$ we obtain similar results since (2.2) implies that

$$\left\|S_\tau(t)S_\tau(s)^*\varphi\right\|_{L^\infty(\mathbb{R}^d)} \leqslant \frac{C}{|t-s|^{d/2}}\|\varphi\|_{L^1(\mathbb{R}^d)}, \quad \forall t \neq s,$$

and we apply the results of Keel and Tao [10, Theorem 1.2].

Proof of Theorem 2.1. A scaling argument reduces all the estimates to the case $\tau = 1$ since

$$\big(S_\tau(t)\varphi\big)(x) = \big(S_1(t/\tau)\varphi\big(\tau^{1/2}\cdot\big)\big)\big(\tau^{-1/2}x\big).$$

Inequality (2.1) is obvious. In the case of (2.2) observe that S_1 satisfies $S_1(t)\varphi = K_t * \varphi$ where K_t is given by

$$K_t(x) = \int_{|\xi|\leqslant 1} e^{ix\xi}e^{-it\xi^2}\,d\xi.$$

We obviously have

$$\|K_t\|_{L^\infty(\mathbb{R}^d)} \leqslant c(d).$$

Using the stationary phase method (see [13, Theorem 1.1.4, p. 45]) we also obtain

$$\|K_t\|_{L^\infty(\mathbb{R}^d)} \leqslant c(d)|t|^{-d/2}.$$

Both inequalities prove that for some constant $C(d)$ the kernel K_t satisfies

$$\|K_t\|_{L^\infty(\mathbb{R}^d)} \leqslant \frac{C(d)}{1+|t|^{d/2}}.$$

Applying Young's inequality we prove (2.2). Observe that (2.2) implies

$$\left\|S_\tau(t)S_\tau(s)^*\varphi\right\|_{L^\infty(\mathbb{R}^d)} \leqslant \frac{C}{|t-s|^{d/2}}\|\varphi\|_{L^1(\mathbb{R}^d)}, \quad \forall t \neq s.$$

Applying the classical results of Keel and Tao [10, Theorem 1.2] we obtain estimates (2.3)–(2.5).

Let us now concentrate on the discrete estimates (2.6)–(2.8). We first point out that the argument of Christ and Kiselev (see [3, Theorem 1.1]) reduces estimate (2.8) to the following one

$$\left\|\sum_{k=-\infty}^{\infty} S_1(n-k)f(k)\right\|_{l^q(\mathbb{Z},L^r(\mathbb{R}^d))} \leqslant C(d,q,\tilde{q})\|f\|_{l^{\tilde{q}'}(\mathbb{Z},L^{\tilde{r}'}(\mathbb{R}^d))}. \tag{2.10}$$

The TT^* argument shows that (2.6), (2.7) and (2.10) are equivalent. In the following we prove (2.7). By duality (2.7) is equivalent with the bilinear estimate:

$$\left|\left\langle \sum_{n\in\mathbb{Z}} S_1(n)^*f(n), \sum_{n\in\mathbb{Z}} S_1(n)^*g(n)\right\rangle\right| \leqslant C(d,\tilde{q})\|f\|_{l^{\tilde{q}'}(\mathbb{Z},L^{\tilde{r}'}(\mathbb{R}^d))}\|g\|_{l^{\tilde{q}'}(\mathbb{Z},L^{\tilde{r}'}(\mathbb{R}^d))}$$

where $\langle\, ,\rangle$ is the $L^2(\mathbb{R}^d)$ inner product. In fact we prove the stronger inequality:

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\left|\left\langle S_1(n)^* f(n), S_1(m)^* g(m)\right\rangle\right| \leqslant C(d,\tilde{q})\|f\|_{l^{\tilde{q}'}(\mathbb{Z},L^{\tilde{r}'}(\mathbb{R}^d))}\|g\|_{l^{\tilde{q}'}(\mathbb{Z},L^{\tilde{r}'}(\mathbb{R}^d))}.$$

Observe that

$$\begin{aligned}\left|\left\langle S_1(n)^* f(n), S_1(m)^* g(m)\right\rangle\right| &= \left|\left\langle f(n), S_1(n)S_1(m)^* g(m)\right\rangle\right| = \left|\left\langle f(n), S_1(n-m)g(m)\right\rangle\right| \\ &\leqslant \left\|f(n)\right\|_{L^{r'}(\mathbb{R}^d)}\left\|S_1(n-m)g\right\|_{L^r(\mathbb{R}^d)} \leqslant \left\|f(n)\right\|_{L^{r'}(\mathbb{R}^d)}\frac{\|g(m)\|_{L^{r'}(\mathbb{R}^d)}}{1+|n-m|^{2/q}}.\end{aligned}$$

It implies that

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\left|\left\langle S_1(n)^* f(n), S_1(m)^* g(m)\right\rangle\right| \leqslant \|f\|_{l^{q'}(\mathbb{Z},L^{r'}(\mathbb{R}^d))}\left\|\sum_{m\in\mathbb{Z}}\frac{\|g(m)\|_{L^{r'}(\mathbb{R}^d)}}{1+|n-m|^{2/q}}\right\|_{l^q(\mathbb{Z})}.$$

At this point we make use of the following lemma (see [12,14]) which is a discrete version of the well-known Hardy–Litlewood–Sobolev inequality:

Lemma 2.1. *Let $0<\alpha<1$ and k be a sequence such that*

$$\left|k(n)\right| \leqslant \frac{1}{1+|n|^\alpha}, \quad \forall n\in\mathbb{Z}.$$

Then the operator $\mathbb{T}$ defined by $\mathbb{T}(f)=f*k$ maps continuously $l^p(\mathbb{Z})$ into $l^q(\mathbb{Z})$ for any p and q satisfying

$$1<p<q<\infty \quad \text{and} \quad \frac{1}{q}=\frac{1}{p}-1+\alpha.$$

Applying this lemma we obtain that

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\left|\left\langle S_1(n)^* f(n), S_1(m)^* g(m)\right\rangle\right| \leqslant \|f\|_{l^{q'}(\mathbb{Z},L^{r'}(\mathbb{R}^d))}\|g\|_{l^{q'}(\mathbb{Z},L^{r'}(\mathbb{R}^d))}$$

which finishes the proof. □

We now prove that Z_τ introduced in (1.10) is uniformly bounded in the auxiliary norms $l^q_{loc}(\tau\mathbb{N}, L^r(\mathbb{R}^d))$.

Throughout the paper we will denote by (q_0,r_0) the admissible pair with $r_0=p+2$. The relevance of this pair comes from the fact that $f(u)=|u|^p u$ maps $L^{r_0}(\mathbb{R}^d)$ to $L^{r_0'}(\mathbb{R}^d)$. In order to simplify the presentation we consider in what follows we consider the case $\lambda=1$.

Proof of Theorem 1.1. The uniform boundedness of the $L^2(\mathbb{R}^d)$-norm follows from the following properties of the two operators S_τ and $N(\tau)$:

$$\left\|S_\tau(\tau)\varphi\right\|_{L^2(\mathbb{R}^d)} \leqslant \|\varphi\|_{L^2(\mathbb{R}^d)}$$

and

$$\big\|N(\tau)\varphi\big\|_{L^2(\mathbb{R}^d)}=\big\|\exp\big(i\tau|\varphi|^p\big)\varphi\big\|_{L^2(\mathbb{R}^d)}=\|\varphi\|_{L^2(\mathbb{R}^d)}.$$

The definition of Z_τ,

$$Z_\tau(n\tau)=\big(S_\tau(\tau)+S_\tau(\tau)\big(N(\tau)-\mathcal{I}\big)\big)^n\Pi_\tau\varphi,\quad n\geqslant 0,$$

gives us that

$$Z_\tau(n\tau)=S_\tau(n\tau)\varphi+\Psi(Z_\tau)(n\tau),\quad n\geqslant 0, \tag{2.11}$$

where

$$\Psi(Z_\tau)(n\tau)=\begin{cases}0, & n=0,\\ \sum_{k=0}^{n-1}S_\tau(n\tau-k\tau)(N(\tau)-\mathcal{I})Z_\tau(k\tau), & n\geqslant 1.\end{cases}$$

Estimate (2.6) of Theorem 2.1 applied to (q_0,r_0) shows that

$$C(d,p)=\sup_{\tau>0}\sup_{\varphi\in L^2(\mathbb{R}^d)}\frac{\|S_\tau(\cdot\tau)\varphi\|_{l^{q_0}(\tau\mathbb{Z};\,L^{r_0}(\mathbb{R}^d))}}{\|\varphi\|_{L^2(\mathbb{R}^d)}}<\infty.$$

We consider the following set of integers:

$$\Lambda=\left\{N\in\mathbb{Z},\ N\geqslant 0,\ \left(\tau\sum_{k=0}^{N}\big\|Z_\tau(k\tau)\big\|^{q_0}_{L^{r_0}(\mathbb{R}^d)}\right)^{1/q_0}\leqslant 2C(d,p)\|\varphi\|_{L^2(\mathbb{R}^d)}\right\}.$$

First we show that the set Λ is not empty by showing that $0\in\Lambda$:

$$\begin{aligned}\tau^{1/q_0}\big\|Z_\tau(0)\varphi\big\|_{L^{r_0}(\mathbb{R}^d)}&=\tau^{1/q_0}\big\|S_\tau(0)\varphi\big\|_{L^{r_0}(\mathbb{R}^d)}\leqslant\big\|S_\tau(\tau\cdot)\varphi\big\|_{l^{q_0}(\tau\mathbb{Z},L^{r_0}(\mathbb{R}^d))}\\&\leqslant C(d,p)\|\varphi\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

If $\sup\Lambda=\infty$ then (1.13) holds for the admissible pair (q_0,r_0). Otherwise, let N_* be the largest element of Λ, i.e. $N_*+1\notin\Lambda$. Using representation (2.11) and estimate (2.6) given by Theorem 2.1, we obtain that

$$\begin{aligned}&\left(\tau\sum_{n=0}^{N_*+1}\big\|Z_\tau(n\tau)\big\|^{q}_{L^{r_0}(\mathbb{R}^d)}\right)^{1/q_0}\\&\quad\leqslant\big\|S_\tau(n\tau)\varphi\big\|_{l^{q_0}(0\leqslant n\tau\leqslant(N_*+1)\tau;L^{r_0}(\mathbb{R}^d))}+\big\|\Psi(Z_\tau)(n\tau)\big\|_{l^{q_0}(0\leqslant n\tau\leqslant(N_*+1)\tau;L^{r_0}(\mathbb{R}^d))}\\&\quad\leqslant C(d,p)\|\varphi\|_{L^2(\mathbb{R}^d)}+\left\|\tau\sum_{k=0}^{n-1}S_\tau\big((n-k)\tau\big)\frac{N(\tau)-\mathcal{I}}{\tau}Z_\tau(k\tau)\right\|_{l^{q_0}(\tau\leqslant n\tau\leqslant(N_*+1)\tau;L^{r_0}(\mathbb{R}^d))}.\end{aligned}$$

Applying estimate (2.9) with $g(n\tau)=\tau^{-1}(N(\tau)-\mathcal{I})Z_\tau(n\tau)$ we obtain

$$\left(\tau\sum_{n=0}^{N_*+1}\left\|Z_\tau(n\tau)\right\|_{L^{r_0}(\mathbb{R}^d)}^{q_0}\right)^{1/q_0}$$
$$\leqslant C(d,p)\|\varphi\|_{L^2(\mathbb{R}^d)}+C_1(d,p)\left\|\frac{N(\tau)-\mathcal{I}}{\tau}Z_\tau(n\tau)\right\|_{l^{q_0'}(0\leqslant n\tau\leqslant N_*\tau;L^{r_0'}(\mathbb{R}^d))}. \tag{2.12}$$

We now use that the operator $N(\tau)-\mathcal{I}$ satisfies

$$\left|\frac{N(\tau)-\mathcal{I}}{\tau}\psi\right|=\left|\frac{\exp(i\tau|\psi|^p)-1}{\tau}\psi\right|\leqslant|\psi|^{p+1}.$$

We introduce this inequality in (2.12) to obtain

$$\left\|Z_\tau(n\tau)\right\|_{l^{q_0}(0\leqslant n\tau\leqslant(N_*+1)\tau;\,L^{r_0}(\mathbb{R}^d))}$$
$$\leqslant C(d,p)\|\varphi\|_{L^2(\mathbb{R}^d)}+C_1(d,p)\left\|\left|Z_\tau(n\tau)\right|^{p+1}\right\|_{l^{q_0'}(0\leqslant n\tau\leqslant N_*\tau;L^{r_0'}(\mathbb{R}^d))}. \tag{2.13}$$

Using that $N_*\in\Lambda$ and Hölder's inequality in time variable (see Lemma 4.1) we get

$$\begin{aligned}&\left\|Z_\tau(n\tau)\right\|_{l^{q_0}(0\leqslant n\tau\leqslant(N_*+1)\tau;\,L^{r_0}(\mathbb{R}^d))}\\&\quad\leqslant C(d,p)\|\varphi\|_{L^2(\mathbb{R}^d)}+C_2(d,p)(N_*\tau)^{1-\frac{dp}{4}}\left\|Z_\tau(\cdot\tau)\right\|_{l^{q_0}(0\leqslant n\tau\leqslant N_*\tau;L^{r_0}(\mathbb{R}^d))}^{p+1}\\&\quad\leqslant C(d,p)\|\varphi\|_{L^2(\mathbb{R}^d)}+C_2(d,p)(N_*\tau)^{1-\frac{dp}{4}}\left(C(d,p)\|\varphi\|_{L^2(\mathbb{R}^d)}\right)^{p+1}\\&\quad\leqslant 2C(d,p)\|\varphi\|_{L^2(\mathbb{R}^d)}\end{aligned}$$

as long as

$$C_2(d,p)(N_*\tau)^{1-\frac{dp}{4}}\left(C(d,p)\|\varphi\|_{L^2(\mathbb{R}^d)}\right)^{p+1}\leqslant C(d,p)\|\varphi\|_{L^2(\mathbb{R}^d)}.$$

This means that if the following inequality holds

$$N_*\tau\leqslant T_0:=\left[\frac{(C(d,p)\|\varphi\|_{L^2(\mathbb{R}^d)})^{-p}}{C_2(d,p)}\right]^{4/(4-dp)}, \tag{2.14}$$

then $N_*+1\in\Lambda$, which contradicts the assumption on the maximality of N_*. Thus (2.14) is false and

$$N_*\tau>T_0.$$

Thus (1.13) holds for $T=T_0$ and the admissible pair (q_0,r_0).

Let us choose (q_1,r_1) another admissible pair. Using representation formula (2.11) and a similar argument as the one above we obtain the following estimate:

$$\begin{aligned}
&\left\|Z_\tau(n\tau)\right\|_{l^{q_1}(0\leqslant n\tau\leqslant T_0;L^{r_1}(\mathbb{R}^d))}\\
&\quad\leqslant C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)}+c(d,q,p)T_0^{1-\frac{dp}{4}}\left\|Z_\tau(n\tau)\right\|^{p+1}_{l^{q_0}(0\leqslant n\tau\leqslant T_0;L^{r_0}(\mathbb{R}^d))}\\
&\quad\leqslant C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)}+T_0^{1-\frac{dp}{4}}c(d,p,q)\big(C(d,p)\|\varphi\|_{L^2(\mathbb{R}^d)}\big)^{p+1}\\
&\quad\leqslant C(d,q,p)\|\varphi\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

This proves estimates (1.13) for $T=T_0$.

Let us now choose any integer N with $N\tau\leqslant T_0$. Definition (1.10) gives us that Z_τ satisfies

$$Z_\tau(N\tau+n\tau)=S_\tau(n\tau)Z_\tau(N\tau)+\tau\sum_{k=0}^{n-1}S_\tau(n\tau-k\tau)\frac{N(\tau)-\mathcal{I}}{\tau}Z_\tau(N\tau+k\tau),\quad n\geqslant 1.$$

With the same argument as above we obtain

$$\left\|Z_\tau(\cdot)\right\|_{l^q(N\tau\leqslant n\tau\leqslant N\tau+T_1,L^r(\mathbb{R}^d))}\leqslant C(d,q,p)\left\|Z_\tau(N\tau)\right\|_{L^2(\mathbb{R}^d)}\leqslant C(d,q,p)\|\varphi\|_{L^2(\mathbb{R}^d)},$$

where

$$T_1=\left[\frac{(C(d,p)\|Z(N\tau)\|_{L^2(\mathbb{R}^d)})^{-p}}{C_2(d,p)}\right]^{4/(4-dp)}.$$

Taking into account that the $L^2(\mathbb{R}^d)$-norm of Z_τ does not increase we get

$$\left\|Z(N\tau)\right\|_{L^2(\mathbb{R}^d)}\leqslant\|\varphi\|_{L^2(\mathbb{R}^d)}$$

and $T_1\geqslant T_0$. This proves (1.13) for the interval $[0,2T_0]$.

The proof is now finished by iterating the same argument on any interval $[0,kT_0]$ with $k\geqslant 1$. $\square$

3. Nonlinear Schrödinger equations

In this section we present some classical results on NSE and use them to estimate the difference between u and v solutions of Eqs. (1.1) and (1.15). In the sequel $\Re(z)$ denotes the real part of the complex number z.

We first state the global existence result for NSE, cf. [2, Theorem 4.6.1, p. 109].

Theorem 3.1. *Let $0<p<4/d$ and $f:\mathbb{C}\to\mathbb{C}$ such that $f(0)=0$ and*

$$\big|f(z_1)-f(z_2)\big|\leqslant C\big(1+|z_1|+|z_2|\big)^p|z_1-z_2|.\tag{3.1}$$

Also assume that

$$\Re\bigg(\int_{\mathbb{R}^d}f\big(z(x)\big)\bar{z}(x)\,dx\bigg)\leqslant 0,\quad\forall z\in L^2\big(\mathbb{R}^d\big)\cap L^{p+1}\big(\mathbb{R}^d\big).\tag{3.2}$$

For every $\varphi \in L^2(\mathbb{R})$, the equation

$$\begin{cases} \dfrac{du}{dt} = i\Delta u + f(u), & x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.3}$$

has a unique global solution $u \in C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^{q_0}_{loc}(\mathbb{R}, L^{r_0}(\mathbb{R}^d))$.

Moreover, the following properties hold:

(i) *$u \in L^q_{loc}(\mathbb{R}, L^r(\mathbb{R}^d))$ for every admissible pair (q,r).*
(ii) *$\|u(t)\|_{L^2(\mathbb{R})} \leqslant \|\varphi\|_{L^2(\mathbb{R})}$ for all $t \geqslant 0$.*
(iii) *For any admissible pair (q,r) there exists $T_0 = T_0(d, p, q, \|\varphi\|_{L^2(\mathbb{R}^d)})$ such that for any interval I with $|I| < T_0$,*

$$\|u\|_{L^q(I,L^r(\mathbb{R}^d))} \leqslant C(d,p,q)\|\varphi\|_{L^2(\mathbb{R}^d)}.$$

(iv) (*Regularity [2, Theorem 5.3.4, p. 154]) If $p \geqslant 1$ and $\varphi \in H^2(\mathbb{R}^d)$ then*

$$u \in C\big(\mathbb{R}, H^2\big(\mathbb{R}^d\big)\big) \cap L^q_{loc}\big(\mathbb{R}, W^{2,r}\big(\mathbb{R}^d\big)\big) \cap W^{1,q}_{loc}\big(\mathbb{R}, L^r\big(\mathbb{R}^d\big)\big)$$

and

$$\begin{aligned} |||u|||_T &:= \|u\|_{L^\infty(0,T,H^2(\mathbb{R}^d))} + \|u\|_{L^{q_0}(0,T,W^{2,r_0}(\mathbb{R}^d))} + \|u_t\|_{L^{q_0}(0,T,L^{r_0}(\mathbb{R}^d))} \\ &\leqslant C\big(T, d, p, \|\varphi\|_{H^2(\mathbb{R}^d)}\big). \end{aligned}$$

Remark 3.1. The $H^1(\mathbb{R}^d)$-regularity of the solutions holds for any $p \in (0, 4/d)$, see [2, Theorem 5.2.1, p. 149]. However, we cannot exploit this fact since in the proof of Theorem 5.1 when we apply Lemma 4.6 we need to assume $H^2(\mathbb{R}^d)$-regularity on the initial data.

We now apply this theorem to prove the existence of a global solution v of Eq. (1.15).

Theorem 3.2. *Let $1 \leqslant p < 4/d$ and $\varphi \in H^2(\mathbb{R}^d)$. There exists a unique global solution of Eq.* (1.15) *which satisfies properties* (i)–(iv) *of Theorem* 3.1.

Proof. In order to apply Theorem 3.1 it is sufficient to check that

$$f(z) = \frac{\exp(i\tau|z|^p) - 1}{\tau} z$$

satisfies hypotheses (3.1) and (3.2). The first one is a consequence of Lemma 4.2 and the second one holds since for any function $z \in L^2(\mathbb{R}^d) \cap L^{p+2}(\mathbb{R}^d)$ the following holds

$$\Re\bigg(\int_{\mathbb{R}^d} f(z)\bar{z}\,dx\bigg) = \Re\bigg(\int_{\mathbb{R}^d} \frac{\exp(i\tau|z|^p) - 1}{\tau}|z|^2\,dx\bigg) = \int_{\mathbb{R}^d} \frac{\cos(\tau|z|^p) - 1}{\tau}|z|^2\,dx \leqslant 0.$$

The proof is now complete. □

With the above theorem we are able to estimate the distance between u and v.

Theorem 3.3. *Let $0 \leqslant p < 4/d$, $\varphi \in L^2(\mathbb{R}^d)$ and u and v solutions of* (1.1) *and* (1.15)*. Assume the existence of an admissible pair $(\tilde{q}, \tilde{r})$ such that $u \in L^{(2p+1)\tilde{q}'}_{loc}(\mathbb{R}, L^{(2p+1)\tilde{r}'}(\mathbb{R}^d))$.*

For any $T > 0$ there exists $C = C(T, p, \tilde{q}, \|\varphi\|_{L^2(\mathbb{R}^d)})$ such that

$$\|u - v\|_{L^\infty(0,T:L^2(\mathbb{R}^d))} \leqslant C\tau \|u\|^{2p+1}_{L^{(2p+1)\tilde{q}'}(0,T,\,L^{(2p+1)\tilde{r}'}(\mathbb{R}^d))}. \tag{3.4}$$

Moreover, if $1 \leqslant p < 4/d$ and $\varphi \in H^2(\mathbb{R}^d)$ then

$$\|u - v\|_{L^\infty(0,T:L^2(\mathbb{R}^d))} \leqslant C\tau \|u\|^{2p+1}_{L^\infty(0,T,\,H^2(\mathbb{R}^d))}. \tag{3.5}$$

Remark 3.2. For any $p < 2/d$ and $\varphi \in L^2(\mathbb{R}^d)$ we can find a pair $(\tilde{q}, \tilde{r})$ such that $u \in L^{(2p+1)\tilde{q}'}_{loc}(\mathbb{R}, L^{(2p+1)\tilde{r}'}(\mathbb{R}^d))$. Indeed, we can find (q, r) an admissible pair with $(2p+1)\tilde{r}' = r$ and $q < (2p+1)\tilde{q}'$ and use that $u \in L^q_{loc}(\mathbb{R}, L^r(\mathbb{R}^d))$. Also for any $\varphi \in H^s(\mathbb{R}^d)$, $s > 0$, we can find a range of exponents p such that the norm of u in the right-hand side of (3.4) is finite.

Proof of Theorem 3.3. In the following, the constants C's occurring in the proof could change from line to line.

Let us choose an admissible pair $(q, r) \in \{(\infty, 2), (q_0, r_0)\}$. Writing u and v in the semigroup formulation

$$u(t) = S(t)\varphi + i\int_0^t S(t-s)|u|^p u(s)\,ds, \quad t \geqslant 0$$

and

$$v(t) = S(t)\varphi + \int_0^t S(t-s)\frac{N(\tau) - I}{\tau} v(s)\,ds, \quad t \geqslant 0,$$

we obtain that

$$\|u - v\|_{L^q(0,T,L^r(\mathbb{R}^d))} \leqslant \left\| \int_0^t S(t-s)\big(g_1\big(u(s), v(s)\big) + g_2\big(u(s)\big)\big)\,ds \right\|_{L^q(0,T,L^r(\mathbb{R}^d))},$$

where

$$g_1(u, v) = \frac{\exp(i\tau|v|^p) - \exp(i\tau|u|^p)}{\tau} v + \frac{\exp(i\tau|u|^p) - 1}{\tau}(v - u)$$

and

$$g_2(u) = \left(\frac{\exp(i\tau|u|^p) - 1}{\tau} - i|u|^p\right)u.$$

Applying classical Strichartz's estimates (see [10, Theorem 1.2]) with $(\tilde{q}, \tilde{r})$ an admissible pair we get

$$\|u - v\|_{L^q(0,T,L^r(\mathbb{R}^d))} \leqslant C(d, q, p)\big\|g_1(u, v)\big\|_{L^{q_0'}(0,T,L^{r_0'}(\mathbb{R}^d))} + C(d, q, \tilde{q})\big\|g_2(u)\big\|_{L^{\tilde{q}'}(0,T,L^{\tilde{r}'}(\mathbb{R}^d))}.$$

Using that g_1 and g_2 satisfy

$$\left|g_1(u,v)\right| \leqslant \left||v|^p - |u|^p\right||v| + |u|^p|v-u| \leqslant c(p)|v-u|\left(|v|^p + |u|^p\right)$$

and

$$\left|g_2(v)\right| \leqslant \tau c(p)|u|^{2p+1}$$

we obtain by Lemma 4.1 that

$$\begin{aligned}
&\|u-v\|_{L^q(0,T,L^r(\mathbb{R}^d))} \\
&\quad \leqslant C(d,q,p)T^{1-dp/4}\|u-v\|_{L^{q_0}(0,T,L^{r_0}(\mathbb{R}^d))}\left(\|u\|_{L^{q_0}(0,T,L^{r_0}(\mathbb{R}^d))} + \|v\|_{L^{q_0}(0,T,L^{r_0}(\mathbb{R}^d))}\right)^p \\
&\qquad + \tau C(d,q,\tilde{q},p)\|u\|^{2p+1}_{L^{(2p+1)\tilde{q}'}(0,T,L^{(2p+1)\tilde{r}'}(\mathbb{R}^d))}.
\end{aligned}$$

For any $T < T_0$ with T_0 given by Theorem 3.1 and Theorem 3.2 we get

$$\begin{aligned}
\|u-v\|_{L^q(0,T,L^r(\mathbb{R}^d))} &\leqslant C(d,q,p)T^{1-dp/4}\|u-v\|_{L^{q_0}(0,T,L^{p+2}(\mathbb{R}^d))}\|\varphi\|^p_{L^2(\mathbb{R}^d)} \\
&\quad + \tau C(d,q,\tilde{q},p)\|u\|^{2p+1}_{L^{(2p+1)\tilde{q}'}(0,T,L^{(2p+1)\tilde{r}'}(\mathbb{R}^d))}.
\end{aligned}$$

Choosing $T_1 < T_0$ but still depending on the $L^2(\mathbb{R}^d)$-norm of φ we obtain

$$\|u-v\|_{L^q(0,T_1,L^r(\mathbb{R}^d))} \leqslant \tau C(d,q,\tilde{q},p)\|u\|^{2p+1}_{L^{(2p+1)\tilde{q}'}(0,T_1,L^{(2p+1)\tilde{r}'}(\mathbb{R}^d))}, \tag{3.6}$$

which proves estimate (3.4) for the interval $(0,T_1)$.

Applying the same argument on the interval $(T_1,2T_1)$ we obtain

$$\|u-v\|_{L^q(T_1,2T_1,L^r(\mathbb{R}^d))} \leqslant c(d,q)\left\|u(T_1)-v(T_1)\right\|_{L^2(\mathbb{R}^d)} + \tau C(d,q,\tilde{q},p)\|u\|^{2p+1}_{L^{(2p+1)\tilde{q}'}(T_1,2T_1,L^{(2p+1)\tilde{r}'})}.$$

Using estimate (3.6) with $(q,r)=(\infty,2)$ we obtain

$$\|u-v\|_{L^q(T_1,2T_1,L^r(\mathbb{R}^d))} \leqslant 2C(d,q,\tilde{q},p)\tau\|u\|^{2p+1}_{L^{(2p+1)\tilde{q}'}(0,2T_1,L^{(2p+1)\tilde{r}'})}.$$

An induction step allows us to prove the same inequality on any interval $(kT_1,(k+1)T_1)$ and then for any interval $(0,T)$

$$\|u-v\|_{L^q(0,T,L^r(\mathbb{R}^d))} \leqslant C(T,d,p,q,\tilde{q})\tau\|u\|^{2p+1}_{L^{(2p+1)\tilde{q}'}(0,T,L^{(2p+1)\tilde{r}'})}.$$

The proof of estimate (3.4) is now finished.

In the particular case of $\varphi \in H^2(\mathbb{R}^d)$ Theorem 3.1 shows that $u \in C(\mathbb{R},H^2(\mathbb{R}^d))$. Thus, using the embedding $H^2(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, $d \leqslant 3$, and estimate (3.4) with $(\tilde{q},\tilde{r})=(\infty,2)$ we obtain estimate (3.5).

The proof is now complete. □

4. Preliminary estimates

In this section we prove some results that will be used in the proof of the main result.

Lemma 4.1. *Let* $0 \leqslant p \leqslant 4/d$ *and* $f : \mathbb{C} \to \mathbb{C}$ *satisfying* $f(0) = 0$ *and*

$$\left|f(z_1) - f(z_2)\right| \leqslant C|z_1 - z_2|\left(|z_1|^p + |z_2|^p\right).$$

Then

$$\left\|f(u)\right\|_{L^{q_0'}(I, L^{r_0'}(\mathbb{R}^d))} \leqslant C(p)|I|^{1-\frac{dp}{4}} \|u\|^{p+1}_{L^{q_0}(I, L^{r_0}(\mathbb{R}^d))} \tag{4.1}$$

and

$$\begin{aligned} &\left\|f(u) - f(v)\right\|_{L^{q_0'}(I, L^{r_0'}(\mathbb{R}^d))} \\ &\quad \leqslant C(p)|I|^{1-\frac{dp}{4}} \|u - v\|_{L^{q_0}(I, L^{r_0}(\mathbb{R}^d))} \left(\|u\|^p_{L^{q_0}(I, L^{r_0}(\mathbb{R}^d))} + \|v\|^p_{L^{q_0}(I, L^{r_0}(\mathbb{R}^d))}\right). \end{aligned} \tag{4.2}$$

Also, for any interval I *with* $|I| \geqslant \tau$ *similar inequalities hold in the discrete time spaces*:

$$\left\|f(u)\right\|_{l^{q_0'}(I, L^{r_0'}(\mathbb{R}^d))} \leqslant C(p)|I|^{1-\frac{dp}{4}} \|u\|^{p+1}_{l^{q_0}(I, L^{r_0}(\mathbb{R}^d))} \tag{4.3}$$

and

$$\begin{aligned} &\left\|f(u) - f(v)\right\|_{l^{q_0'}(I, L^{r_0'}(\mathbb{R}^d))} \\ &\quad \leqslant C(p)|I|^{1-\frac{dp}{4}} \|u - v\|_{l^{q_0}(I, L^{r_0}(\mathbb{R}^d))} \left(\|u\|^p_{l^{q_0}(I, L^{r_0}(\mathbb{R}^d))} + \|v\|^p_{l^{q_0}(I, L^{r_0}(\mathbb{R}^d))}\right). \end{aligned} \tag{4.4}$$

Proof. Let us first consider the case of continuous in time norms. Using that $r_0' = (p+2)/(p+1)$ we get

$$\begin{aligned} \left\|f(u)\right\|_{L^{q_0'}(I, L^{r_0'}(\mathbb{R}^d))} &\leqslant c(p)\left\||u|^{p+1}\right\|_{L^{q_0'}(I, L^{r_0'}(\mathbb{R}^d))} \leqslant c(p)\|u\|^{p+1}_{L^{(p+1)q_0'}(I, L^{(p+1)r_0'}(\mathbb{R}^d))} \\ &= c(p)\|u\|^{p+1}_{L^{(p+1)q_0'}(I, L^{r_0}(\mathbb{R}^d))}. \end{aligned}$$

Hölder's inequality shows that for any $1 \leqslant a \leqslant b \leqslant \infty$ the following holds

$$\|v\|_{L^a(I)} \leqslant \|v\|_{L^b(I)} |I|^{\frac{1}{a} - \frac{1}{b}}.$$

Thus

$$\left\|f(u)\right\|_{L^{q_0'}(I, L^{r_0'}(\mathbb{R}^d))} \leqslant c(p)|I|^{\frac{1}{(p+1)q_0'} - \frac{1}{q_0}} \|u\|^{p+1}_{L^{q_0}(I, L^{r_0}(\mathbb{R}^d))} = c(p)|I|^{1-\frac{dp}{4}} \|u\|^{p+1}_{L^{q_0}(I, L^{r_0}(\mathbb{R}^d))}.$$

The second inequality can be treated in a similar way and we leave it to the reader.

The case of discrete norms can be treated similarly once we observe that

$$\|v\|_{l^a(n\tau\in I)} \leqslant \|v\|_{l^b(n\tau\in I)}\left(\tau\left[\frac{|I|}{\tau}\right]\right)^{\frac{1}{a}-\frac{1}{b}} \leqslant \|v\|_{l^b(n\tau\in I)}|I|^{\frac{1}{a}-\frac{1}{b}},$$

where $[\cdot]$ is the floor function. □

Lemma 4.2. *For any $p>0$ there exists a positive constant $c(p)$ such that*

$$\left|\frac{N(\tau)-\mathcal{I}}{\tau}u-\frac{N(\tau)-\mathcal{I}}{\tau}v\right| \leqslant c(p)|u-v|\big(|u|^p+|v|^p\big) \tag{4.5}$$

holds for all complex numbers u and v. Moreover if $p\leqslant 4/d$ and $|I|\geqslant\tau$ then

$$\begin{aligned}&\left\|\frac{N(\tau)-\mathcal{I}}{\tau}u-\frac{N(\tau)-\mathcal{I}}{\tau}v\right\|_{l^{q_0}(I,L^{r_0}(\mathbb{R}^d))}\\&\quad\leqslant c(p)|I|^{1-dp/4}\|u-v\|_{l^{q_0}(I,L^{r_0}(\mathbb{R}^d))}\big(\|u\|^p_{l^{q_0}(I,L^{r_0}(\mathbb{R}^d))}+\|v\|^p_{l^{q_0}(I,L^{r_0}(\mathbb{R}^d))}\big).\end{aligned} \tag{4.6}$$

Proof. Using the definition of the nonlinear operator $N(\tau)$ we get

$$\begin{aligned}\left|\frac{N(\tau)-\mathcal{I}}{\tau}u-\frac{N(\tau)-\mathcal{I}}{\tau}v\right| &= \left|\frac{\exp(i\tau|u|^p-1)}{\tau}(u-v)+\frac{\exp(i\tau|u|^p)-\exp(i\tau|v|^p)}{\tau}v\right|\\&\leqslant |u|^p|u-v|+\big||u|^p-|v|^p\big||v|\leqslant c(p)|u-v|\big(|u|^p+|v|^p\big).\end{aligned}$$

The second inequality is obtained by applying Lemma 4.1. □

Lemma 4.3. *Let $d\leqslant 3$ and $1\leqslant p\leqslant 4/d$. Then the function*

$$f(u)=\frac{N(\tau)-\mathcal{I}}{\tau}u$$

satisfies

$$\big\|\partial_t\big(f(u)\big)\big\|_{L^{q_0'}(I,L^{r_0'}(\mathbb{R}^d))} \leqslant |I|^{1-dp/4}\|u\|^{p+1}_{W^{1,q_0}(I,L^{r_0}(\mathbb{R}^d))} \tag{4.7}$$

and

$$\big\|\partial_{xx}\big(f(u)\big)\big\|_{L^{q_0'}(I,L^{r_0'}(\mathbb{R}^d))} \leqslant |I|^{1-dp/4}\|u\|^{p+1}_{L^{q_0}(I,W^{2,r_0}(\mathbb{R}^d))}\big(1+\tau\|u\|^{p-1}_{L^\infty(0,T,H^2(\mathbb{R}^d))}\big). \tag{4.8}$$

Proof. The first inequality follows from Hölder's inequality in time variable and the following inequality

$$\big|\partial_t\big(f(u)\big)\big|\leqslant C|u|^p|\partial_t u|.$$

For the second one, after an elementary calculus we get

$$\begin{aligned}\left|\partial_{xx}\bigl(f(u)\bigr)\right| &\leqslant C\bigl(|u_{xx}||u|^p + |u_x|^2|u|^{p-1} + \tau|u_x|^2|u|^{2(p-1)}\bigr)\\ &\leqslant C\bigl(|u_{xx}||u|^p + |u_x|^2|u|^{p-1}\bigr) + \tau\|u\|^{p-1}_{L^\infty(\mathbb{R}^d)}|u_x|^2|u|^{(p-1)}.\end{aligned}$$

Thus

$$\begin{aligned}\bigl\|\partial_{xx}\bigl(f(u)\bigr)\bigr\|_{L^{q_0'}(I,L^{r_0'}(\mathbb{R}^d))} &\leqslant |I|^{1-dp/4}\bigl(\|u\|^{p+1}_{L^{q_0}(I,W^{2,r_0}(\mathbb{R}^d))} + \tau\|u\|^{p-1}_{L^\infty(I\times\mathbb{R}^d)}\|u\|^{p+1}_{L^{q_0}(I,W^{1,r_0}(\mathbb{R}^d))}\bigr)\\ &\leqslant |I|^{1-dp/4}\|u\|^{p+1}_{L^{q_0}(I,W^{2,r_0}(\mathbb{R}^d))}\bigl(1+\tau\|u\|^{p-1}_{L^\infty(I,H^2(\mathbb{R}^d))}\bigr),\end{aligned}$$

since $H^2(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $d \leqslant 3$. □

Lemma 4.4. *Let $s > 0$ and $r \in (1,\infty)$. Then*

$$\|\Pi_\tau v - v\|_{L^r(\mathbb{R}^d)} \leqslant \tau^{s/2}\bigl\|(-\Delta)^{s/2}v\bigr\|_{L^r(\mathbb{R}^d)} \tag{4.9}$$

and

$$\|\Pi_\tau v\|_{L^r(\mathbb{R}^d)} \leqslant \|v\|_{L^r(\mathbb{R}^d)}. \tag{4.10}$$

Proof. Using that

$$(\Pi_\tau v)(x) = \bigl(\Pi_1\bigl(v\bigl(\tau^{1/2}\cdot\bigr)\bigr)\bigr)\bigl(\tau^{-1/2}x\bigr)$$

the proof is reduced to the case $\tau = 1$. To prove (4.9) it is sufficient to show that the operator T defined by $\widehat{Tv}(\xi) = m_s(\xi)\widehat{v}(\xi)$ with $m_s(\xi) = |\xi|^{-s}\mathbf{1}_{\{|\xi|>1\}}(\xi)$ is continuous from $L^r(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$. Since $1 < r < \infty$, inequality (4.9) follows from [5, Theorem 5.2.2, p. 356]. In the case of inequality (4.10) we apply the same argument to the multiplier $m(\xi) = \mathbf{1}_{\{|\xi|<1\}}(\xi)$. □

Lemma 4.5. *For any admissible pairs (q,r) and $(\tilde{q},\tilde{r})$ the operator Λ defined by*

$$\Lambda f(n\tau) = \int\limits_{s<n\tau} S_\tau(n\tau - s)f(s)\,dt,$$

satisfies

$$\|\Lambda f\|_{l^q(\tau\mathbb{Z},\,L^r(\mathbb{R}^d))} \leqslant C(d,q,\tilde{q})\|f\|_{L^{\tilde{q}'}(\mathbb{R},\,L^{\tilde{r}'}(\mathbb{R}^d))}. \tag{4.11}$$

Remark 4.1. Choosing in (4.11) functions f supported in some interval I we get

$$\bigl\|\Lambda f(n\tau)\bigr\|_{l^q(n\tau\in I,\,L^r(\mathbb{R}^d))} \leqslant \|\Lambda f\|_{l^q(\tau\mathbb{Z},\,L^r(\mathbb{R}^d))} \leqslant C(d,q,\tilde{q})\|f\|_{L^{\tilde{q}'}(I,L^{\tilde{r}'}(\mathbb{R}^d))}. \tag{4.12}$$

Proof of Lemma 4.5. We consider the linear operator $\tilde{\Lambda}$ defined by

$$\tilde{\Lambda} f(n\tau) = \int\limits_{-\infty}^{\infty} S_\tau(n\tau - s)f(s)\,ds = S_\tau(n\tau)\int\limits_{-\infty}^{\infty} S_\tau(s)^* f(s)\,ds.$$

We now use the argument of Christ and Kiselev (see [3, Theorem 1.1]) which reduces estimate (4.11) on Λ to the one on the operator $\tilde{\Lambda}$:

$$\|\tilde{\Lambda} f\|_{l^q(\tau\mathbb{Z},L^r(\mathbb{R}^d))} \leqslant C(d,q,\tilde{q})\|f\|_{L^{\tilde{q}'}(\tau\mathbb{Z},L^{\tilde{r}'}(\mathbb{R}^d))}. \tag{4.13}$$

Using the discrete-time estimate (2.6) on the operator S_τ we obtain

$$\|\tilde{\Lambda} f(n)\|_{l^q(\tau\mathbb{Z},L^r(\mathbb{R}^d))} \leqslant C(d,q)\left\|\int_{-\infty}^{\infty} S_\tau(t)^* f(t)\,dt\right\|_{L^2(\mathbb{R}^d)}. \tag{4.14}$$

Applying the continuous in time estimate (2.4) we get

$$\left\|\int_{-\infty}^{\infty} S_\tau(t)^* f(t)\right\|_{L^2(\mathbb{R}^d)} \leqslant C(d,\tilde{q})\|f\|_{L^{\tilde{q}'}(\mathbb{R},L^{\tilde{r}'}(\mathbb{R}^d))}$$

which proves (4.13) and finishes the proof. □

Lemma 4.6. *Let T be defined by*

$$T\eta(n\tau,\cdot) = \int_{s<n\tau} S_\tau(n\tau - s)\eta(s) - \tau\sum_{k=-\infty}^{n-1} S_\tau(n\tau - k\tau)\eta(k\tau).$$

For any (q,r) and $(\tilde{q},\tilde{r})$ admissible pairs the following holds

$$\|T\eta\|_{l^q(\tau\mathbb{Z},L^r(\mathbb{R}))} \leqslant \tau C(d,q,\tilde{q})\big(\|\eta_{xx}\|_{L^{\tilde{q}'}(\mathbb{R},L^{\tilde{r}'}(\mathbb{R}^d))} + \|\eta_t\|_{L^{\tilde{q}'}(\mathbb{R},L^{\tilde{r}'}(\mathbb{R}^d))}\big).$$

Remark 4.2. In particular, for any admissible pair (q,r) we obtain that

$$\|T\eta\|_{l^q(|n|\tau\leqslant T,L^r(\mathbb{R}))} \leqslant \tau C(d,q,\tilde{q})T\big(\|\eta\|_{L^\infty(|n|\tau\leqslant T,H^2(\mathbb{R}^d))} + \|\eta_t\|_{L^\infty(|n|\tau\leqslant T,L^2(\mathbb{R}^d))}\big).$$

This is a consequence of the previous estimate with $(\tilde{q},\tilde{r}) = (\infty,2)$.

Proof of Lemma 4.6. We write $T\eta$ as follows

$$\begin{aligned}
T\eta(n\tau) &= \sum_{k=-\infty}^{n-1}\int_{k\tau}^{(k+1)\tau}\big[S_\tau(n\tau - s)\eta(s) - S_\tau(n\tau - k\tau)\eta(k\tau)\big]\,ds \\
&= \sum_{k=-\infty}^{n-1}\int_{k\tau}^{(k+1)\tau}\int_{k\tau}^{s}\frac{d}{dt}\big(S_\tau(n\tau - t)\eta(t)\big)\,dt\,ds \\
&= \sum_{k=-\infty}^{n-1}\int_{k\tau}^{(k+1)\tau}\int_{k\tau}^{s}\big[-iS_\tau(n\tau - t)\eta_{xx}(t) + S_\tau(n\tau - t)\eta_t(t)\big]\,dt\,ds
\end{aligned}$$

$$= \sum_{k=-\infty}^{n-1} \int_{k\tau}^{(k+1)\tau} \int_{t}^{(k+1)\tau} \big[-iS_\tau(n\tau - t)\eta_{xx}(t) + S_\tau(n\tau - t)\eta_t(t)\big]\, ds\, dt$$

$$= \sum_{k=-\infty}^{n-1} \int_{k\tau}^{(k+1)\tau} \big[(k+1)\tau - t\big] S_\tau(n\tau - t)\big(-i\eta_{xx}(t) + \eta_t(t)\big)\, dt$$

$$= \sum_{k=-\infty}^{n-1} \int_{k\tau}^{(k+1)\tau} S_\tau(n\tau - t)\big[(k+1)\tau - t\big]\big(-i\eta_{xx}(t) + \eta_t(t)\big)\, dt.$$

With Λ as in Lemma 4.5 we write

$$T\eta = \Lambda(-i\eta_1) + \Lambda(\eta_2)$$

where

$$\eta_1(t) = \sum_{k\in\mathbb{Z}} \big[(k+1)\tau - t\big]\eta_{xx}(t)\mathbf{1}_{(k\tau,(k+1)\tau)}(t)$$

and

$$\eta_2(t) = \sum_{k\in\mathbb{Z}} \big[(k+1)\tau - t\big]\eta_t(t)\mathbf{1}_{(k\tau,(k+1)\tau)}(t).$$

Using Lemma 4.5 we obtain

$$\begin{aligned}\|T\eta\|_{l^q(\tau\mathbb{Z},\, L^r(\mathbb{R}^d))} &\leqslant C(d,q,\tilde{q})\big(\|\eta_1\|_{L^{\tilde{q}'}(\mathbb{R},\, L^{\tilde{r}'}(\mathbb{R}^d))} + \|\eta_2\|_{L^{\tilde{q}'}(\mathbb{R},\, L^{\tilde{r}'}(\mathbb{R}^d))}\big) \\ &\leqslant C(d,q,\tilde{q})\tau\big(\|\eta_{xx}\|_{L^{\tilde{q}'}(\mathbb{R},\, L^{\tilde{r}'}(\mathbb{R}^d))} + \|\eta_t\|_{L^{\tilde{q}'}(\mathbb{R},\, L^{\tilde{r}'}(\mathbb{R}^d))}\big),\end{aligned}$$

which finishes the proof. □

Lemma 4.7. *Let $s > 0$, $0 \leqslant p \leqslant 4/d$ and (q,r) an admissible pair. Then*

$$R_\tau(n\tau) = \int_0^{n\tau} S_\tau(n\tau - s)\left(\frac{N(\tau) - \mathcal{I}}{\tau}\Pi_\tau v(s) - \frac{N(\tau) - \mathcal{I}}{\tau} v(s)\right) ds$$

satisfies

$$\|R_\tau v\|_{l^q(I, L^r(\mathbb{R}^d))} \leqslant C(d,q,p)\tau^{s/2}|I|^{1-dp/4}\|v\|^{p+1}_{L^{q_0}(I, W^{s,r_0}(\mathbb{R}^d))}. \tag{4.15}$$

Proof. We use estimate (4.12) and Lemma 4.2 to obtain

$$\begin{aligned}
&\|R_\tau v\|_{l^q(I,L^r(\mathbb{R}^d))}\\
&\quad\leqslant C(d,q,p)\left\|\left(\frac{N(\tau)-I}{\tau}\Pi_\tau v-\frac{N(\tau)-I}{\tau}v\right)\right\|_{L^{q_0'}(I,L^{r_0'}(\mathbb{R}^d))}\\
&\quad\leqslant C(d,q,p)|I|^{1-dp/4}\big(\|\Pi_\tau v\|^p_{L^{q_0}(I,L^{r_0}(\mathbb{R}^d))}+\|v\|^p_{L^{q_0}(I,L^{r_0}(\mathbb{R}^d))}\big)\|\Pi_\tau v-v\|_{L^{q_0}(I,L^{r_0}(\mathbb{R}^d))}.
\end{aligned}$$

Estimates (4.9) and (4.10) give us

$$\begin{aligned}
\|R_\tau v\|_{L^q(I,L^r(\mathbb{R}^d))}&\leqslant \tau^{s/2}C(d,q,p)|I|^{1-dp/4}\|v\|^p_{L^{q_0}(I,L^{r_0}(\mathbb{R}^d))}\big\|(-\Delta)^{s/2}v\big\|_{L^{q_0}(I,L^{r_0}(\mathbb{R}^d))}\\
&\leqslant \tau^{s/2}C(d,q,p)|I|^{1-dp/4}\|v\|^{p+1}_{L^{q_0}(I,W^{s,r_0}(\mathbb{R}^d))},
\end{aligned}$$

which finishes the proof. □

5. Error estimates

In this section we prove the main result of this paper, namely Theorem 1.2. Using Theorem 3.3 it is sufficient to estimate the difference between Z_τ and v in the $L^2(\mathbb{R}^d)$-norm. This is done in the following theorem.

Theorem 5.1. *Let $p\in[1,4/d)$ and $\varphi\in H^2(\mathbb{R}^d)$. Then for any $T>0$ the following holds*

$$\|Z-v\|_{L^\infty(0,T,L^2(\mathbb{R}^d))}\leqslant \tau C\big(T,d,p,\|\!|v|\!\|_T\big). \tag{5.1}$$

Proof. Using that

$$\|v-\Pi_\tau v\|_{l^\infty(0,T,L^2(\mathbb{R}^d))}\leqslant \|v-\Pi_\tau v\|_{L^\infty(0,T,L^2(\mathbb{R}^d))}\leqslant \tau\|\!|v|\!\|_{L^\infty(0,T,H^2(\mathbb{R}^d))}\leqslant \tau\|\!|v|\!\|_T$$

it is sufficient to estimate the difference between Z and $\Pi_\tau v$ in the $L^2(\mathbb{R}^d)$-norm.

We write Z and $\Pi_\tau v$ as follows

$$Z_\tau(n\tau)=S_\tau(n\tau)\varphi+\tau\sum_{k=0}^{n-1}S_\tau(n\tau-k\tau)\frac{N(\tau)-I}{\tau}Z_\tau(k\tau),\quad n\geqslant 1$$

and

$$\begin{aligned}
\Pi_\tau v(t)&=S_\tau(t)\varphi+\int_0^t S_\tau(t-s)\frac{N(\tau)-I}{\tau}v(s)\,ds\\
&=S_\tau(t)\varphi+\int_0^t S_\tau(t-s)\frac{N(\tau)-I}{\tau}\Pi_\tau v(s)\,ds+R_\tau v(t)
\end{aligned}$$

where

$$R_\tau v(t)=\int_0^t S_\tau(t-s)\left(\frac{N(\tau)-I}{\tau}\Pi_\tau v(s)-\frac{N(\tau)-I}{\tau}v(s)\right)ds. \tag{5.2}$$

In order to proceed we need the following estimate on $\Pi_\tau v$ which we will prove later.

Lemma 5.1. *Let (q,r) be an admissible pair. There exist $T_1=T_1(d,q,p,\|\varphi\|_{L^2(\mathbb{R}^d)})$ and a constant $C(q,p)$ such that*

$$\|\Pi_\tau v\|_{l^q(I,L^r(\mathbb{R}^d))}\leqslant C(q,p)\|\varphi\|_{L^2(\mathbb{R}^d)}$$

holds for all intervals $|I|\leqslant T_1$.

To simplify the presentation we get rid of all the constants which depend by p, q and d.

Step I. Local error estimate. Let $T>0$ and $(q,r)\in\{(q_0,r_0),(\infty,2)\}$. We make use of the Strichartz estimate (2.8), Lemma 4.6 and Lemma 4.7 to obtain

$$\begin{aligned}
&\|Z_\tau-\Pi_\tau v\|_{l^q(0,T;L^r(\mathbb{R}^d))}\\
&\quad\leqslant\left\|\tau\sum_{k=0}^{n-1}S_\tau(n\tau-k\tau)\left(\frac{N(\tau)-I}{\tau}Z_\tau(k\tau)-\frac{N(\tau)-I}{\tau}\Pi_\tau v(k\tau)\right)\right\|_{l^q(0,T;L^r(\mathbb{R}^d))}\\
&\qquad+\left\|\tau\sum_{k=0}^{n-1}S_\tau(n\tau-k\tau)\frac{N(\tau)-I}{\tau}\Pi_\tau v(k\tau)-\int_0^t S_\tau(t-s)\frac{N(\tau)-I}{\tau}\Pi_\tau v(s)\,ds\right\|_{l^q(0,T;L^r(\mathbb{R}^d))}\\
&\qquad+\|R_\tau v\|_{l^q(0,T;L^r(\mathbb{R}^d))}\\
&\quad\leqslant\left\|\frac{N(\tau)-I}{\tau}Z_\tau-\frac{N(\tau)-I}{\tau}\Pi_\tau v\right\|_{l^{q_0'}(0,T;L^{r_0'}(\mathbb{R}^d))}\\
&\qquad+\tau\left\|\frac{N(\tau)-I}{\tau}\Pi_\tau v\right\|_{L^{q_0}(0,T,W^{2,r_0}(\mathbb{R}^d))}+\tau\left\|\frac{N(\tau)-I}{\tau}\Pi_\tau v\right\|_{W^{1,q_0}(0,T,L^{r_0}(\mathbb{R}^d))}\\
&\qquad+C(I)\tau\|v\|^{p+1}_{L^{q_0}(0,T;W^{2,r_0}(\mathbb{R}^d))}.
\end{aligned}$$

We now estimate the first two terms in the last inequality. Lemma 4.2 gives us that

$$\begin{aligned}
&\left\|\frac{N(\tau)-I}{\tau}Z_\tau-\frac{N(\tau)-I}{\tau}\Pi_\tau v\right\|_{l^{q_0'}(0,T;L^{r_0'}(\mathbb{R}^d))}\\
&\quad\leqslant T^{1-dp/4}\|Z_\tau-\Pi_\tau v\|_{l^{q_0}(0,T;L^{r_0}(\mathbb{R}^d))}\left(\|Z_\tau\|^p_{l^{q_0}(0,T;L^{r_0}(\mathbb{R}^d))}+\|\Pi_\tau v\|^p_{l^{q_0}(0,T;L^{r_0}(\mathbb{R}^d))}\right).
\end{aligned}$$

The estimates on Z_τ and $\Pi_\tau v$ obtained in Theorem 1.1 and Lemma 5.1 give us the existence of a time $T_0=T_0(\|\varphi\|_{L^2(\mathbb{R}^d)})$ such that for all intervals I with $|I|<T_0$ the following hold

$$\|Z_\tau\|_{l^{q_0}(I;L^{r_0}(\mathbb{R}^d))}\leqslant\|\varphi\|_{L^2(\mathbb{R}^d)},\qquad\|\Pi_\tau v\|_{l^{q_0'}(I;L^{r_0'}(\mathbb{R}^d))}\leqslant\|\varphi\|_{L^2(\mathbb{R}^d)}. \tag{5.3}$$

Thus

$$\left\| \frac{N(\tau)-I}{\tau} Z_\tau - \frac{N(\tau)-I}{\tau} \Pi_\tau v \right\|_{l^{q_0'}(0,T;L^{r_0'}(\mathbb{R}^d))} \leqslant T^{1-dp/4} \| Z_\tau - \Pi_\tau v \|_{l^{q_0}(0,T;L^{r_0}(\mathbb{R}^d))} \|\varphi\|^p_{L^2(\mathbb{R}^d)}. \tag{5.4}$$

Applying Lemma 4.3 and estimate (4.10) of Lemma 4.4 we obtain

$$\left\| \frac{N(\tau)-I}{\tau} \Pi_\tau v \right\|_{L^{q_0}(0,T,W^{2,r_0}(\mathbb{R}^d))} + \left\| \frac{N(\tau)-I}{\tau} \Pi_\tau v \right\|_{W^{1,q_0}(0,T,L^{r_0}(\mathbb{R}^d))} \leqslant C\big(T, |||v|||_T\big). \tag{5.5}$$

Using estimates (5.4) and (5.5) we get

$$\| Z_\tau - \Pi_\tau v \|_{l^q(0,T;L^r(\mathbb{R}^d))} \leqslant T^{1-dp/4} \| Z_\tau - \Pi_\tau v \|_{l^{q_0}(0,T;L^{r_0}(\mathbb{R}^d))} \|\varphi\|^p_{L^2(\mathbb{R}^d)} + \tau C\big(T, |||v|||_T\big). \tag{5.6}$$

We now choose $T_1 < T_0$ with $T_1 \in \tau\mathbb{Z}$ such that $T_1^{1-dp/4} \|\varphi\|^p_{L^2(\mathbb{R}^d)} < 1/4$. We emphasize that T_1 depends only on the size of the $L^2(\mathbb{R}^d)$-norm of φ and is independent of the size of τ.

Using inequality (5.6) with $(q,r) \in \{(\infty,2),(q_0,r_0)\}$ we obtain that

$$\| Z_\tau - \Pi_\tau v \|_{l^\infty(0,T_1;L^2(\mathbb{R}^d))} + \| Z_\tau - \Pi_\tau v \|_{l^{q_0}(0,T_1;L^{r_0}(\mathbb{R}^d))} \leqslant \tau C\big(T_1, |||v|||_{T_1}\big).$$

Step II. Global error estimate. Using that v satisfies (1.15) we have for any positive T and t that v verifies the following integral equation

$$\Pi_\tau v(T+t) = S_\tau(t)v(T) + \int_0^t S_\tau(t-s) \frac{\exp(i\tau|v|^p)-1}{\tau} \Pi_\tau v(T+s)\,ds + R_\tau(T+t).$$

Also, for any positive integers N and n, Z_τ satisfies

$$Z_\tau\big((N+n)\tau\big) = \big(S_\tau(\tau)N(\tau)\big)^{N+n} Z(N\tau)$$

and consequently

$$Z_\tau(N\tau+n\tau) = S_\tau(n\tau)Z_\tau(N\tau) + \tau \sum_{k=0}^{n-1} S_\tau(n\tau-k\tau) \frac{N(\tau)-I}{\tau} Z_\tau(N\tau+k\tau), \quad n \geqslant 1.$$

We apply the same argument as in Step I on any interval on $I_k = [kT_1,(k+1)T_1]$ with the same admissible pairs $(q,r) \in \{(\infty,2),(q_0,r_0)\}$:

$$\begin{aligned} &\| Z_\tau - \Pi_\tau v \|_{l^q(I_k;L^r(\mathbb{R}^d))} \\ &\quad \leqslant \big\| S_\tau\big(Z_\tau(kT_1) - \Pi_\tau v(kT_0)\big)\big\|_{l^q(0,T_1;L^r(\mathbb{R}^d))} + \left\| \frac{N(\tau)-I}{\tau} Z_\tau - \frac{N(\tau)-I}{\tau} \Pi_\tau v \right\|_{l^{\tilde q_0'}(I_k;L^{\tilde r_0'}(\mathbb{R}^d))} \\ &\qquad + \tau \left\| \frac{N(\tau)-I}{\tau} \Pi_\tau v \right\|_{L^q(I_k,W^{2,r}(\mathbb{R}^d))} + \tau \left\| \frac{N(\tau)-I}{\tau} \Pi_\tau v \right\|_{W^{1,q}(I_k,L^r(\mathbb{R}^d))}. \end{aligned}$$

Let us denote

$$\mathrm{err}_k = \|Z_\tau - \Pi_\tau v\|_{l^\infty(I_k, L^2(\mathbb{R}^d))} + \|Z_\tau - \Pi_\tau v\|_{l^{q_0}(I_k, L^{r_0}(\mathbb{R}^d))}.$$

Using estimates (2.6) and (5.6) we obtain

$$\begin{aligned}
&\|Z_\tau - \Pi_\tau v\|_{l^q(I_k; L^r(\mathbb{R}^d))} \\
&\quad \leqslant \big\|Z(kT_1) - \Pi_\tau v(kT_1)\big\|_{L^2(\mathbb{R}^d)} + \tau C\big(T_1, |||v|||_{I_k}\big) \\
&\qquad + T_1^{1-dp/4}\|Z - \Pi_\tau v\|_{l^{q_0}(I_k; L^{r_0}(\mathbb{R}^d))}\big(\|Z\|^p_{l^{q_0}(I_k; L^{r_0}(\mathbb{R}^d))} + \|\Pi_\tau v\|^p_{l^{q_0}(I_k; L^{r_0}(\mathbb{R}^d))}\big) \\
&\quad \leqslant \mathrm{err}_{k-1} + T^{1-dp/4}\|\varphi\|^p_{L^2(\mathbb{R}^d)}\|Z - \Pi_\tau v\|_{l^{q_0}(I_k; L^{r_0}(\mathbb{R}^d))} + \tau C\big(T_1, |||v|||_{I_k}\big) \\
&\quad \leqslant \mathrm{err}_{k-1} + \frac{\|Z - \Pi_\tau v\|_{l^{q_0}(I_k; L^{r_0}(\mathbb{R}^d))}}{4} + \tau C\big(T_1, |||v|||_{I_k}\big) \\
&\quad \leqslant \mathrm{err}_{k-1} + \frac{\mathrm{err}_k}{4} + \tau C\big(T_1, |||v|||_{I_k}\big).
\end{aligned}$$

Summing the above inequality for the two pairs $(q, r) \in \{(\infty, 2), (q_0, r_0)\}$ we obtain that

$$\mathrm{err}_k \leqslant 4\big(\mathrm{err}_{k-1} + \tau C\big(T_1, |||v|||_{I_k}\big)\big), \quad k \geqslant 1.$$

Moreover, by Step I, $\mathrm{err}_0 \leqslant \tau$. These imply that

$$\mathrm{err}_k \leqslant \tau c\big(kT_1, |||v|||_{kT_1}\big), \quad \text{for all } k \geqslant 1.$$

This means that for any interval $(0, T)$ the following holds

$$\|Z - \Pi_\tau v\|_{l^\infty(0,T,L^2(\mathbb{R}^d))} \leqslant \tau C\big(T, |||v|||_T\big).$$

The proof is now finished. □

Proof of Lemma 5.1. By Theorem 3.2 we know the existence of a $T_0 = T_0(d, p, q, \|\varphi\|_{L^2(\mathbb{R}^d)})$ such that

$$\|v\|_{L^q(I, L^r(\mathbb{R}^d))} \leqslant C(q, p)\|\varphi\|_{L^2(\mathbb{R}^d)}$$

holds for all intervals I with $|I| \leqslant T_0$.

We use that for any T and t positive $\Pi_\tau v$ satisfies

$$\Pi_\tau v(T + t) = S_\tau(t)v(T) + \int_0^t S_\tau(t - s)\frac{N(\tau) - I}{\tau}v(T + s)\,ds.$$

We apply Theorem 2.1 and Lemma 4.5 to obtain

$$\|\Pi_\tau v\|_{l^q(T,T+T_1,L^r(\mathbb{R}^d))} \leqslant c(d, q)\big\|v(T)\big\|_{L^2(\mathbb{R}^d)} + c(d, p, q)\left\|\frac{N(\tau) - I}{\tau}v\right\|_{L^{q_0'}(T,T+T_1,L^{r_0'}(\mathbb{R}^d))}.$$

Lemma 4.1 and Lemma 4.2 give now

$$\|\Pi_\tau v\|_{l^q(T,T+T_1,L^r(\mathbb{R}^d))} \leqslant c(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)} + c(d,p,q)T_1^{1-dp/4}\|v\|^{p+1}_{L^{q_0'}(T,T+T_1,L^{r_0'}(\mathbb{R}^d))}.$$

Thus, for any interval $I=(T,T+T_1)$ with $T_1<T_0$ we get

$$\begin{aligned}\|\Pi_\tau v\|_{l^q(I,L^r(\mathbb{R}^d))} &\leqslant c(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)} + c(d,p,q)T_1^{1-dp/4}\big(C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)}\big)^{p+1}\\ &\leqslant 2c(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)}\end{aligned}$$

provided that

$$c(d,p,q)T_1^{1-p/4}\big(C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)}\big)^{p+1} \leqslant c(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)}.$$

The lemma is now proved. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. Using the previous results of Theorem 5.1 and Theorem 3.3 we obtain

$$\max_{0\leqslant n\tau\leqslant T}\big\|Z_\tau(n\tau)-v(n\tau)\big\|_{L^2(\mathbb{R}^d)} \leqslant \tau C\big(T,|||v|||_T\big)$$

and

$$\max_{0\leqslant n\tau\leqslant T}\big\|u(n\tau)-v(n\tau)\big\|_{L^2(\mathbb{R}^d)} \leqslant \tau C\big(T,|||u|||_T\big).$$

This implies that

$$\max_{0\leqslant n\tau\leqslant T}\big\|Z_\tau(n\tau)-u(n\tau)\big\|_{L^2(\mathbb{R}^d)} \leqslant \tau C\big(T,|||v|||_T,|||u|||_T\big) \leqslant \tau C\big(T,\|\varphi\|_{H^2(\mathbb{R}^d)}\big).$$

The proof is now finished. □

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