Dispersion for the Schrödinger equation on networks

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In this paper, we consider the Schrödinger equation on a network formed by a tree with the last generation of edges formed by infinite strips. We give an explicit description of the solution of the linear Schrödinger equation with constant coefficients. This allows us to prove dispersive estimates, which in turn are useful for solving the nonlinear Schrödinger equation. The proof extends also to the laminar case of positive step-function coefficients having a finite number of discontinuities. © *2011 American Institute of Physics.* [doi:10.1063/1.3629474]

I. INTRODUCTION

Let us first consider the linear Schrödinger equation (LSE) on $\mathbb{R}$:

$$\begin{cases} iu_t + u_{xx} = 0, & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

The linear semigroup $e^{it\Delta}$ has two important properties, which can be easily seen via the Fourier transform. First, the conservation of the L^2-norm:

$$\|e^{it\Delta}u_0\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}, \tag{2}$$

and a dispersive estimate of the form

$$\|e^{it\Delta}u_0\|_{L^\infty(\mathbb{R})} \le \frac{C}{\sqrt{|t|}}\|u_0\|_{L^1(\mathbb{R})}, \quad t \ne 0. \tag{3}$$

From these two inequalities, by using the classical TT^* argument, space-time estimates follow, known as Strichartz estimates (see Refs. 17 and 30):

$$\|e^{it\Delta}u_0\|_{L_t^q(\mathbb{R}, L_x^r(\mathbb{R}))} \le C\|u_0\|_{L^2(\mathbb{R})}, \tag{4}$$

where (q, r) are so-called admissible pairs:

$$\frac{2}{q} + \frac{1}{r} = \frac{1}{2}, \quad 2 \le q, r \le \infty. \tag{5}$$

These dispersive estimates have been successfully applied to obtain well-posedness results for the nonlinear Schrödinger equation (see, Refs. 12 and 31, and the reference therein).

In this article, we prove the dispersion inequality for the linear Schrödinger operator defined on a tree (bounded, connected graph without closed paths) with the external edges infinite. We assume

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that the tree does not contain vertices of multiplicity two, since they are irrelevant for our model. Let us notice that in this context we cannot use Fourier analysis as done on $\mathbb{R}$ for getting the dispersion inequality.

The presentation of the Laplace operator will be given in full details in Sec. II. Let us just say here that the Laplacian operator, Δ_Γ, acts as the usual Laplacian on $\mathbb{R}$ on each edge and that at vertices the Kirchhoff conditions must be fulfilled: continuity condition for the functions on the graph and transmission condition at the level of their first derivative. So our analysis will be a 1D ramified analysis. More general coupling conditions are discussed in Sec. IV.

In Ref. 19 the second author proved the same result in the case of regular trees. This means some restrictions on the shape of the trees: all the vertices of the same generation have the same number of descendants and all the edges of the same generation are of the same length. These restrictions allow to define some average functions on the edges of the same generation and to analyze some 1D laminar Schrödinger equation (depending on the shape of the tree), where dispersion estimates were available from the first author's paper.[5] The strategy used in Ref. 19 cannot be applied in the case of a general tree and the scope of this article is to extend the class of trees where the dispersion estimate holds. In the case of a graph with a closed path, in general there exist compact supported eigenfunctions for the considered Laplace operator and then the dispersion estimate fails.

The motivation for studying thin structures comes from mesoscopic physics and nanotechnology. Mesoscopic systems are those that have some dimensions which are too small to be treated using classical physics while they are too large to be considered on the quantum level only. The *quantum wires* are physical systems with two dimensions reduced to a few nanometers. We refer to Ref. 24 and references therein for more details on such type of structures.

The simplest model describing conduction in quantum wires is a Hamiltonian on a planar graph, i.e., a one-dimensional object. Throughout the paper we consider a class of idealized quantum wires, where the configuration space is a planar graph and the Hamiltonian is minus the Laplacian with Kirchhoff's boundary conditions at the vertices of the graph. This condition makes the Hamiltonian to be a self-adjoint operator. More general coupling conditions that guarantee the self-adjointness are given in Ref. 21.

The problems addressed here enter in the framework of metric graphs or networks. Those are metric spaces which can be written as the union of finitely many intervals, which are compact or $[0, \infty)$ and any two of these intervals are either disjoint or intersect only in one or both of their endpoints. Differential operators on metric graphs arise in a variety of applications. We mention some of them: carbon nano-structures,[27] photonic crystals,[15] high-temperature granular superconductors,[3] quantum waveguides,[9] free-electron theory of conjugated molecules in chemistry, quantum chaos, etc. For more details we refer the reader to review papers.[14,18,24,25]

The linear and cubic Schrödinger equation on simple networks with Kirchhoff connection conditions and particular type of data has been analyzed in Ref. 10. The symmetry imposed on the initial data and the shape of the networks allow to reduce the problem to a Schrödinger equation on the half-line with appropriate boundary conditions, for which a detailed study is done by inverse scattering. Some numerical experiments are also presented in Ref. 10. The propagation of solitons for the cubic Schrödinger equation on simple networks but with connection conditions in link with the mass and energy conservation is analyzed in Ref. 29. On star graphs, the cubic Schrödinger equation was considered in Refs. 1 and 2 and well-posedness was proved at the H^1 energy level, as well as the existence of stationary states and the description of the propagation of solitons for boundary conditions of δ type, including Kirchhoff vertex.

The main result is the following, where by $\{I_e\}_{e \in E}$ we shall denote the edges of the tree.

Theorem 1.1: *The solution of the linear Schrödinger equation on a tree is of the form*

$$e^{it\Delta_\Gamma} \mathbf{u}_0(x) = \sum_{\lambda \in \mathbb{R}} \frac{a_\lambda}{\sqrt{|t|}} \int_{I_\lambda} e^{i\frac{\phi_\lambda(x,y)}{t}} \mathbf{u}_0(y)\, dy, \tag{6}$$

with $\phi_\lambda(x, y) \in \mathbb{R}$, $I_\lambda \in \{I_e\}_{e \in E}$, $\sum_{\lambda \in \mathbb{R}} |a_\lambda| < \infty$, and it satisfies the dispersion inequality

$$\|e^{it\Delta_\Gamma} \mathbf{u}_0\|_{L^\infty(\Gamma)} \le \frac{C}{\sqrt{|t|}} \|\mathbf{u}_0\|_{L^1(\Gamma)}, \quad t \ne 0. \tag{7}$$

The proof uses the method in Ref. 5 in an appropriate way related to the ramified analysis on the tree, by recursion on the number of vertices. It consists in writing the solution in terms of the resolvent of the Laplacian, which in turn is computed in the framework of almost periodic functions.

As mentioned before, Strichartz estimates (4) can be derived from the dispersion inequality and have been used intensively to obtain well-posedness results for the nonlinear Schrödinger equation (NSE). The arguments used in the context of NSE on $\mathbb{R}$ can also be used here to obtain the following as a typical result.

Corollary 1.2: Let $p \in (0, 4)$. For any $\mathbf{u}_0 \in L^2(\Gamma)$ there exists a unique solution

$$\mathbf{u} \in C(\mathbb{R}, L^2(\Gamma)) \cap \bigcap_{(q,r)\,admissible} L^q_{loc}(\mathbb{R}, L^r(\Gamma)),$$

of the nonlinear Schrödinger equation

$$\begin{cases} i\mathbf{u}_t + \Delta_\Gamma \mathbf{u} \pm |\mathbf{u}|^p \mathbf{u} = 0, & t \neq 0, \\ \mathbf{u}(0) = \mathbf{u}_0, & t = 0. \end{cases} \tag{8}$$

Moreover, the $L^2(\Gamma)$-norm of $\mathbf{u}$ is conserved along the time

$$\|\mathbf{u}(t)\|_{L^2(\Gamma)} = \|\mathbf{u}_0\|_{L^2(\Gamma)}.$$

The proof is standard once the dispersion property is obtained and it follows as in [Ref. 12, p. 109, Theorem 4.6.1].

With the same method we obtain the same results in the case of the Laplacian on the graph with laminar coefficients (piecewise constants, bounded between two positive constants—the details on the laminar Laplacian are given in Sec. III). This might be of physical interest when the wire on a edge is composed of different pieces. Equations with variable coefficients on networks have been previously analyzed in Ref. 33 for the heat equation and in Ref. 4 for the wave equations. For clearness we give in details the proof in the constant coefficient case because it contains the core ideas. We then very briefly describe the way to extend it to laminar coefficients.

The paper is organized as follows. In Sec. II, we introduce the Laplacian on a graph and write the systems we want to analyze. In Sec. III, we present the proof of Theorem 1.1. Some open problems are discussed in Sec. IV.

II. NOTATIONS AND PRELIMINARIES

In this section, we present some generalities about metric graphs and introduce the Laplace operator on such structure. Let $\Gamma = (V, E)$ be a graph where V is a set of vertices and E is the set of edges. For each $v \in V$, we denote $E_v = \{e \in E : v \in e\}$. We assume that Γ is a countable connected locally finite graph, i.e., the degree of each vertex v of Γ is finite: $d(v) = |E_v| < \infty$. The edges could be of finite length and then their ends are vertices of V or they have infinite length and then we assume that each infinite edge is a ray with a single vertex belonging to V (see, Ref. 26 for more details on graphs with infinite edges).

We fix an orientation of Γ and for each oriented edge e, we denote by $I(e)$ the initial vertex and by $T(e)$ the terminal one. Of course, in the case of infinite edges we have only initial vertices.

We identify every edge e of Γ with an interval I_e, where $I_e = [0, l_e]$ if the edge is finite and $I_e = [0, \infty)$ if the edge is infinite. This identification introduces a coordinate x_e along the edge e. In this way Γ is a metric space and is often named metric graph.[26]

Let v be a vertex of V and e be an edge in E_v. We set for finite edges e

$$j(v, e) = \begin{cases} 0 & \text{if} \quad v = I(e) \\ l_e & \text{if} \quad v = T(e) \end{cases}$$

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and

$$j(v, e) = 0, \quad \text{if } v = I(e),$$

for infinite edges.

We identify any function $\mathbf{u}$ on Γ with a collection $\{u^e\}_{e \in E}$ of functions u^e defined on the edges e of Γ. Each u^e can be considered as a function on the interval I_e. In fact, we use the same notation u^e for both the function on the edge e and the function on the interval I_e identified with e. For a function $\mathbf{u} : \Gamma \to \mathbb{C}$, $\mathbf{u} = \{u^e\}_{e \in E}$, we denote by $f(\mathbf{u}) : \Gamma \to \mathbb{C}$ the family $\{f(u^e)\}_{e \in E}$, where $f(u^e) : e \to \mathbb{C}$.

A function $\mathbf{u} = \{u^e\}_{e \in E}$ it is continuous if and only if u^e is continuous on I_e for every $e \in E$, and moreover, is continuous at the vertices of Γ:

$$u^e(j(v, e)) = u^{e'}(j(v, e')), \quad \forall \, e, e' \in E_v.$$

The space $L^p(\Gamma)$, $1 \leq p < \infty$ consists of all functions $\mathbf{u} = \{u_e\}_{e \in E}$ on Γ that belong to $L^p(I_e)$ for each edge $e \in E$ and

$$\|\mathbf{u}\|_{L^p(\Gamma)}^p = \sum_{e \in E} \|u^e\|_{L^p(I_e)}^p < \infty.$$

Similarly, the space $L^\infty(\Gamma)$ consists of all functions that belong to $L^\infty(I_e)$ for each edge $e \in E$ and

$$\|\mathbf{u}\|_{L^\infty(\Gamma)} = \sup_{e \in E} \|u^e\|_{L^\infty(I_e)} < \infty.$$

The Sobolev space $H^m(\Gamma)$, $m \geq 1$ an integer, consists in all continuous functions on Γ that belong to $H^m(I_e)$ for each $e \in E$ and

$$\|\mathbf{u}\|_{H^m(\Gamma)}^2 = \sum_{e \in E} \|u^e\|_{H^m(e)}^2 < \infty.$$

The above spaces are Hilbert spaces with the inner products

$$(\mathbf{u}, \mathbf{v})_{L^2(\Gamma)} = \sum_{e \in E} (u^e, v^e)_{L^2(I_e)} = \sum_{e \in E} \int_{I_e} u^e(x) \overline{v^e}(x) dx$$

and

$$(\mathbf{u}, \mathbf{v})_{H^m(\Gamma)} = \sum_{e \in E} (u^e, v^e)_{H^m(I_e)} = \sum_{e \in E} \sum_{k=0}^{m} \int_{I_e} \frac{d^k u^e}{dx^k} \overline{\frac{d^k v^e}{dx^k}} dx.$$

We now introduce the Laplace operator Δ_Γ on the graph Γ. Even if it is a standard procedure we prefer for the sake of completeness to follow Ref. 11. Consider the sesquilinear continuous form φ on $H^1(\Gamma)$ defined by

$$\varphi(\mathbf{u}, \mathbf{v}) = (\mathbf{u}_x, \mathbf{v}_x)_{L^2(\Gamma)} = \sum_{e \in E} \int_{I_e} u_x^e(x) \overline{v_x^e}(x) dx.$$

We denote by $D(\Delta_\Gamma)$ the set of all the functions $\mathbf{u} \in H^1(\Gamma)$ such that the linear map $\mathbf{v} \in H^1(\Gamma) \to \varphi_{\mathbf{u}}(\mathbf{v}) = \varphi(\mathbf{u}, \mathbf{v})$ satisfies

$$|\varphi(\mathbf{u}, \mathbf{v})| \leq C \|\mathbf{v}\|_{L^2(\Gamma)} \quad \text{for all} \quad \mathbf{v} \in H^1(\Gamma).$$

For $\mathbf{u} \in D(\Delta_\Gamma)$, we can extend $\varphi_{\mathbf{u}}$ to a linear continuous mapping on $L^2(\Gamma)$. There is a unique element in $L^2(\Gamma)$ denoted by $\Delta_\Gamma \mathbf{u}$, such that

$$\varphi(\mathbf{u}, \mathbf{v}) = -(\Delta_\Gamma \mathbf{u}, \mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in H^1(\Gamma).$$

We now define the normal exterior derivative of a function $\mathbf{u} = \{u^e\}_{e \in E}$ at the endpoints of the edges. For each $e \in E$ and v an endpoint of e we consider the normal derivative of the restriction of $\mathbf{u}$ to the edge e of E_v evaluated at $i(v, e)$ to be defined by

$$\frac{\partial u^e}{\partial n_e}(j(v, e)) = \begin{cases} -u_x^e(0+) & \text{if } j(v, e) = 0, \\ u_x^e(l_e-) & \text{if } j(v, e) = l_e. \end{cases}$$

With this notation it is easy to characterise $D(\Delta_\Gamma)$ (see, Ref. 11):

$$D(\Delta_\Gamma) = \left\{ \mathbf{u} = \{u^e\}_{e \in E} \in H^2(\Gamma) : \sum_{e \in E_v} \frac{\partial u^e}{\partial n_e}(j(v,e)) = 0 \quad \text{for all} \quad v \in V \right\}$$

and

$$(\Delta_\Gamma \mathbf{u})^e = (u^e)_{xx} \quad \text{for all} \quad e \in E, \mathbf{u} \in D(\Delta_\Gamma).$$

In other words, $D(\Delta_\Gamma)$ is the space of all continuous functions on Γ, $\mathbf{u} = \{u^e\}_{e \in E}$, such that for every edge $e \in E$, $u^e \in H^2(I_e)$ and satisfying the following Kirchhoff-type condition:

$$\sum_{e \in E : T(e) = v} u_x^e(l_e-) - \sum_{e \in E : I(e) = v} u_x^e(0+) = 0 \quad \text{for all} \quad v \in V.$$

It is easy to verify that $(\Delta_\Gamma, D(\Delta_\Gamma))$ is a linear, unbounded, self-adjoint, dissipative operator on $L^2(\Gamma)$, i.e., $\Re(\Delta_\Gamma \mathbf{u}, \mathbf{u})_{L^2(\Gamma)} \leq 0$ for all $\mathbf{u} \in D(\Delta_\Gamma)$.

Let us consider the LSE on Γ:

$$\begin{cases} i\mathbf{u}_t(t,x) + \Delta_\Gamma \mathbf{u}(t,x) = 0, \ x \in \Gamma, t \neq 0, \\ \mathbf{u}(0) = \mathbf{u}_0, \qquad\qquad\qquad x \in \Gamma. \end{cases} \tag{9}$$

Using the properties of the operator $i\Delta_\Gamma$ we obtain as a consequence of the Hille-Yosida theorem the following well-posedness result:

Theorem 2.1: *For any $\mathbf{u}_0 \in D(\Delta_\Gamma)$ there exists a unique solution $\mathbf{u}(t)$ of system (9) that satisfies*

$$\mathbf{u} \in C(\mathbb{R}, D(\Delta_\Gamma)) \cap C^1(\mathbb{R}, L^2(\Gamma)).$$

Moreover, for any $\mathbf{u}_0 \in L^2(\Gamma)$, there exists a unique solution $\mathbf{u} \in C(\mathbb{R}, L^2(\Gamma))$ that satisfies

$$\|\mathbf{u}(t)\|_{L^2(\Gamma)} = \|\mathbf{u}_0\|_{L^2(\Gamma)} \quad \text{for all} \quad t \in \mathbb{R}.$$

The $L^2(\Gamma)$-isometry property is a consequence of the fact that the operator $i\Delta_\Gamma$ satisfies $\Re(i\Delta_\Gamma \mathbf{u}, \mathbf{u})_{L^2(\Gamma)} = 0$ for all $\mathbf{u} \in D(\Delta_\Gamma)$.

For any $\mathbf{u}_0 \in D(\Delta_\Gamma)$ system (9) can be written in an explicit way as follows:

$$\begin{cases} u^e \in C(\mathbb{R}, H^2(I_e)) \cap C^1(\mathbb{R}, L^2(I_e)), & e \in E, \\ iu_t^e(t,x) + \Delta u^e(t,x) = 0, & x \in I_e, t \neq 0, \\ \text{for all } v \in V, \ u^e(t, j(v,e)) = u^{e'}(t, j(v,e')), & \forall e, e' \in E_v, t \neq 0, \\ \displaystyle\sum_{e \in E : T(e) = v} u_x^e(t, l_e-) - \sum_{e \in E : I(e) = v} u_x^e(t, 0+) = 0 & \text{for all} \quad v \in V. \end{cases} \tag{10}$$

Let us now consider the laminar case. We consider σ as a piecewise constant function on each edge of the tree such that there exist two positive constants σ_1 and σ_2 such that

$$0 < \sigma_1 < \sigma(x) < \sigma_2, \quad \forall x \in I_e, \forall \ e \in E.$$

With a similar argument as before we introduce the operator $\Delta_{\sigma,\Gamma}$ as follows:

$$D(\Delta_{\sigma,\Gamma}) = \left\{ \mathbf{u} = \{u^e\}_{e \in E} \in H^2(\Gamma) : \sum_{e \in E_v} \sigma(j(v,e)) \frac{\partial u^e}{\partial n_e}(j(v,e)) = 0 \quad \text{for all} \quad v \in V \right\}$$

and

$$(\Delta_{\sigma,\Gamma} \mathbf{u})^e = \partial_x(\sigma(\cdot)\partial_x(u^e)) \quad \text{for all} \quad e \in E, \mathbf{u} \in D(\partial_x(\sigma \partial_x)).$$

It follows that for any $\mathbf{u}_0 \in D(\Delta_{\sigma,\Gamma})$ the following system is well-posed

$$
\begin{cases}
u^e \in C(\mathbb{R}, H^2(I_e)) \cap C^1(\mathbb{R}, L^2(I_e)), & e \in E, \\
iu_t^e(t,x) + \partial_x(\sigma(x)\partial_x u^e)(t,x) = 0, & x \in I_e, t \neq 0, \\
\text{for all } v \in V, \ u^e(t, j(v,e)) = u^{e'}(t, j(v,e')), & \forall e, e' \in E_v, t \neq 0, \\
\displaystyle\sum_{e \in E: T(e)=v} \sigma(l_e-)u_x^e(t, l_e-) = \sum_{e \in E: I(e)=v} \sigma(0+)u_x^e(t, 0+) & \text{for all} \quad v \in V.
\end{cases} \tag{11}
$$

We remark that when function σ is identically equal to one we obtain the previous system (10).

III. PROOF OF THEOREM 1.1

A. The description of the solution

For $\omega \geq 0$, let R_ω be the resolvent of the Laplacian on a tree

$$
R_\omega \mathbf{f} = (-\Delta_\Gamma + \omega^2 I)^{-1}\mathbf{f}.
$$

We shall prove in Lemma 3.3 that $\omega R_\omega \mathbf{f}(x)$ can be analytically continued in a region containing the imaginary axis. Therefore, we can use a spectral calculus argument to write the solution of the Schrödinger equation with initial data $\mathbf{u}_0$ as

$$
e^{it\Delta_\Gamma}\mathbf{u}_0(x) = \int_{-\infty}^{\infty} e^{it\tau^2}\tau R_{i\tau}\mathbf{u}_0(x)\frac{d\tau}{\pi}. \tag{12}
$$

We shall also obtain in Lemma 3.4 that the following decomposition holds

$$
\tau R_{i\tau}\mathbf{u}_0(x) = \sum_{\lambda \in \mathbb{R}} b_\lambda e^{i\tau\psi_\lambda(x)} \int_{I_\lambda} \mathbf{u}_0(y)e^{i\tau\beta_\lambda y}dy, \tag{13}
$$

with $\psi_\lambda(x)$, $\beta_\lambda \in \mathbb{R}$, $I_\lambda \in \{I_e\}_{e \in E}$, and $\sum_{\lambda \in \mathbb{R}} |b_\lambda| < \infty$. Then decomposition (6) is implied by (12), (13), and the fact that for $t > 0$ and $r \in \mathbb{R}$

$$
\int_{-\infty}^{\infty} e^{it\tau^2}e^{i\tau r}d\tau = e^{i\frac{\pi}{4}}\sqrt{\pi}\frac{e^{-\frac{r^2}{4t}}}{\sqrt{t}}.
$$

From (6) the dispersion estimate (7) of Theorem 1.1 follows immediately since $\sum_{\lambda \in \mathbb{R}} |\alpha_\lambda| < \infty$.

Above and in what follows the integration of function $\mathbf{f} = (f^e)_{e \in E}$ on interval I_e means the integral of f^e on the considered interval.

Remark 3.1: As in Ref. 5 we notice that since we can express the solution of the wave equation $\mathbf{v}_{tt} - \Delta_\Gamma \mathbf{v} = 0$ *with initial data* $(\mathbf{v}_0, 0)$ *as*

$$
\mathbf{v}(t,x) = \int_{-\infty}^{\infty} e^{it\tau}R_{i\tau}\mathbf{v}_0(x)i\tau\frac{d\tau}{2\pi},
$$

the property

$$
\sup_{x \in \Gamma}\int_{-\infty}^{\infty} |\mathbf{v}(t,x)|dt \leq C\|\mathbf{v}_0\|_{L^1(\Gamma)},
$$

follows similarly. Let us mention here that the well-posedness of a class of nonlinear dispersive waves on trees, the Benjamin-Bona-Mahony equation, has been investigated in Ref. 6.

B. Structure of the resolvent

In order to obtain the expression of the resolvent second-order equations

$$
(R_\omega \mathbf{f})'' = \omega^2 R_\omega \mathbf{f} - \mathbf{f},
$$

must be solved on each edge of the tree together with coupling conditions at each vertex. Then, on each edge parametrized by I_e,

$$R_\omega \mathbf{f}(x) = c e^{\omega x} + \tilde{c} e^{-\omega x} + \frac{1}{2\omega} \int_{I_e} \mathbf{f}(y) e^{-\omega|x-y|} dy, \quad x \in I_e.$$

Since $R_\omega \mathbf{f}$ belongs to $L^2(\Gamma)$ the coefficients c's are zero on the infinite edges $e \in \mathcal{E}$, parametrized by $[0, \infty)$. If we denote by $\mathcal{I}$ the set of internal edges, we have $2|\mathcal{I}| + |\mathcal{E}|$ coefficients. The Kirchhoff conditions of continuity of $R_\omega \mathbf{f}$ and of transmission of $\partial_x R_\omega \mathbf{f}$ at the vertices of the tree give the system of equations on the coefficients. We have the same number of equations as the number of unknowns. We denote D_Γ the matrix of the system, whose elements are real powers of e^ω.

Therefore, the resolvent $R_\omega \mathbf{f}(x)$ is a finite sum of terms

$$R_\omega \mathbf{f}(x) = \frac{1}{\omega \det D_\Gamma(\omega)} \sum_{\lambda=1}^{N(\Gamma)} c_\lambda e^{\pm \omega \Phi_\lambda(x)} \int_{I_\lambda} \mathbf{f}(y) e^{\pm \omega y} dy + \frac{1}{2\omega} \int_{I_e} \mathbf{f}(y) e^{-\omega|x-y|} dy, \quad (14)$$

where $x \in I_e$, $\Phi_\lambda(x) \in \mathbb{R}$, $I_\lambda \in \{I_e\}_{e \in E,}$ and $|N(\Gamma)| < \infty$. We shall prove the following proposition that will imply Lemma 3.3 and 3.4 needed for obtaining Theorem 1.1.

Proposition 3.2: Function $\det D_\Gamma(\omega)$ *is lower bounded by a positive constant on a strip containing the imaginary axis:*

$$\exists c_\Gamma, \epsilon_\Gamma > 0, \ |\det D_\Gamma(\omega)| > c_\Gamma, \forall \omega \in \mathbb{C}, |\Re\omega| < \epsilon_\Gamma.$$

Lemma 3.3: Function $\omega R_\omega \mathbf{f}(x)$ *can be analytically continued in a region containing the imaginary axis.*

Proof: The proof is an immediate consequence of decomposition (14) and of Proposition 3.2. □

Lemma 3.4: The following decomposition holds

$$\tau R_{i\tau} \mathbf{u}_0(x) = \sum_{\lambda \in \mathbb{R}} b_\lambda e^{i\tau \psi_\lambda(x)} \int_{I_\lambda} \mathbf{u}_0(y) e^{i\tau \beta_\lambda y} dy,$$

with $\psi_\lambda(x), \beta_\lambda \in \mathbb{R}$, $I_\lambda \in \{I_e\}_{e \in E,}$ and $\sum_{\beta \in \mathbb{R}} |b_\lambda| < \infty$.

Proof: We notice that for $\tau \in \mathbb{R}$, $\det D_\Gamma(i\tau)$ is a finite sum of powers of $e^{i\tau}$. Then, by Proposition 3.2 we are in the framework of a classical theorem in representation theory (Sec. 29, Corollary 1 of Ref. 16) that asserts that the inverse of $\det D_\Gamma(i\tau)$ is $\sum_{\lambda \in \mathbb{R}} d_\lambda e^{i\tau\lambda}$ with $\sum_{\lambda \in \mathbb{R}} |d_\lambda| < \infty$ and from (14) the Lemma follows. □

The rest of this section is the proof of Proposition 3.2. We shall show by recursion on the number of vertices the following stronger "double" property:

$$P(n): \quad \text{If } \Gamma \text{ has } n \text{ vertices, we have the property } \mathcal{P},$$

$$\mathcal{P}: \exists c_\Gamma, \epsilon_\Gamma > 0, \exists 0 < r_\Gamma < 1, \ |\det D_\Gamma(\omega)| > c_\Gamma, \left| \frac{\det \tilde{D}_\Gamma(\omega)}{\det D_\Gamma(\omega)} \right| < r_\Gamma, \ \forall \omega \in \mathbb{C}, |\Re\omega| < \epsilon_\Gamma.$$

We have denoted by $\tilde{D}_\Gamma(\omega)$, the matrix of the system verified by the coefficients, if we impose that on one of the last infinite edges $l \in \mathcal{E}$ we replace in the expression of the resolvent $\tilde{c} e^{-\omega x}$ by $c e^{\omega x}$.

C. Proof of *P*(1)

In this case, we have a star-shaped tree with $m \geq 3$ of edges. All the edges are parametrized by $[0, \infty)$. In particular, $D_\Gamma(\omega) = D_\Gamma$. We shall actually prove a stronger property, which implies the
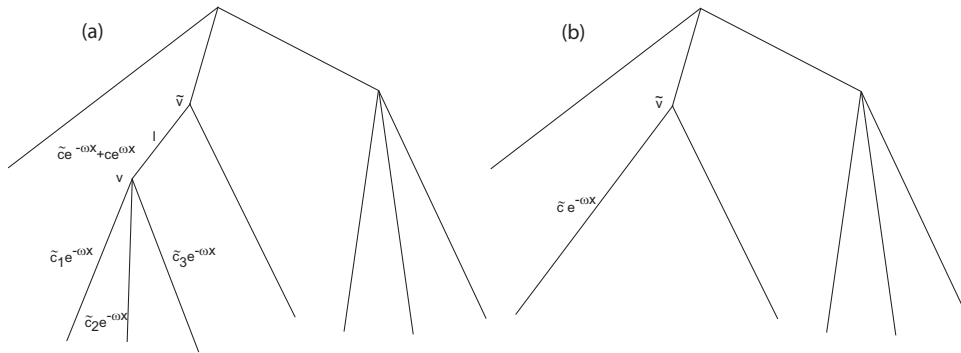
property $\mathcal{P}$ for any $\epsilon_\Gamma > 0$:

$P(1, m)$:   If Γ has 1 vertex and m edges, $\det D_\Gamma(\omega) = m$ and $\det \tilde{D}_\Gamma(\omega) = m - 2$.

The resolvent contains $\tilde{c}_j e^{-\omega x}$, $1 \le j \le m$, on each of the external edges. We write matrix D_{Γ_m} such that the last line is coming from Kirchhoff derivative condition and that the other lines are coming from Kirchhoff continuity condition. So matrix D_{Γ_m} can be written such that it has components 1 on the last line, and on the principal diagonal, $d_{i,i+1} = -1$ for $1 \le j \le m - 1$, and zeros elsewhere

$$
D_{\Gamma^m} = \begin{pmatrix}
1 & -1 & & & & & & \\
 & 1 & -1 & & & & & \\
 & & \cdot & \cdot & & & & \\
 & & & \cdot & \cdot & & & \\
 & & & & \cdot & \cdot & & \\
 & & & & & 1 & -1 & \\
 & & & & & & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
$$

By development with respect to the last column and $P(1, m - 1)$,

$$
\det D_{\Gamma^m} = 1 + \det D_{\Gamma^{m-1}} = m.
$$

Similarly,

$$
\det \tilde{D}_{\Gamma^m} = \begin{vmatrix}
1 & -1 & & & & & & \\
 & 1 & -1 & & & & & \\
 & & \cdot & \cdot & & & & \\
 & & & \cdot & \cdot & & & \\
 & & & & \cdot & \cdot & & \\
 & & & & & 1 & -1 & \\
 & & & & & & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1
\end{vmatrix} = 1 + \det \tilde{D}_{\Gamma^{m-1}} = m - 2,
$$

so $P(1, m)$ is proven for any $m \ge 3$ and implicitly $P(1)$.

D. Proof of $P(n - 1) \Rightarrow P(n)$.

Any tree Γ_n with n vertices, $n \ge 2$, can be seen as a tree Γ_{n-1} with $n - 1$ vertices on which we add an extra-vertex. More precisely, let us consider a vertex v from which there start $m \ge 2$ external infinite edges and one internal edge connecting it to the rest of the tree (see, Fig. 1). Let us notice that such a choice is possible since the graph has no cycles. In particular, the edge whose lower extremity is this vertex v is an internal edge l, whose length should be denoted by a and whose upper vertex is denoted by $\tilde{v}$. Now we remove this vertex and transform the internal edge l into an external infinite one. The new graph Γ_{n-1} has $n - 1$ vertices.

With respect to the problem on Γ_{n-1}, the resolvent on Γ_n involves a new term $c e^{\omega x}$ aside from $\tilde{c} e^{-\omega x}$ on the interval edge l, and on the external edges emerging from the vertex v it involves terms $\tilde{c}_j e^{-\omega x}$, $1 \le j \le m$. We also have the Kirchhoff conditions at the vertex v, which give $m + 1$ equations on the coefficients.

We write the square $N \times N$ matrix D_{Γ_n} such that the last $m + 2$ column corresponds to the unknowns $\tilde{c}, c, \tilde{c}_1, ..., \tilde{c}_m$. On the last line we write the Kirchhoff derivative condition at the vertex v, and on the $N - j$ lines, $1 \le j \le m$ the Kirchhoff continuity conditions at the vertex v. Also, on

FIG. 1. With vertex v we obtain tree Γ_4 (left) from Γ_3 (right).

the $N - m - 1$ line we write the derivative condition in the vertex $\tilde{v}$ and on the $N - m - 2$ line the continuity condition in $\tilde{v}$ relating $\tilde{c}$, and now also c, to the others coefficients. So D_{Γ_n} is a matrix obtained from the $(N - m - 1) \times (N - m - 1)$ matrix $D_{\Gamma_{n-1}}$ (whose last column corresponds to the unknown $\tilde{c}$) and from D_{Γ^m} (see Sec. III C) in the following way:

$$(D_{\Gamma_n})_{i,j} = \begin{cases} (D_{\Gamma_{n-1}})_{i,j}, \ 1 \leq i, j \leq N - m - 1, \\ -1, (i, j) \in \{(N - m - 2, N - m), (N - m - 1, N - m), (N - m, N - m + 1)\}, \\ e^{\omega a}, (i, j) \in \{(N - m, N - m), (N, N - m)\}, \\ e^{-\omega a}, (i, j) = (N - m, N - m - 1), \\ -e^{-\omega a}, (i, j) = (N, N - m - 1), \\ (D_{\Gamma^m})_{i,j}, \ N - m + 1 \leq i, j \leq N, \end{cases}$$

$$D_{\Gamma_n} = \begin{pmatrix} D_{\Gamma_{n-1}} & & & & & & & & & \\ & & -1 & & & & & & & \\ & & -1 & & & & & & & \\ e^{-\omega a} & e^{\omega a} & -1 & & & & & & & \\ & & 1 & -1 & & & & & & \\ & & & 1 & -1 & & & & & \\ & & & & \cdot & \cdot & & & & \\ & & & & \cdot & \cdot & & & & \\ & & & & & & \cdot & \cdot & & \\ & & & & & & & 1 & -1 & \\ & & & & & & & & 1 & -1 \\ -e^{-\omega a} & e^{\omega a} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

In the case $n = 1$, we have that $D_{\Gamma_1} = D_{\Gamma^{\tilde{m}}}$ for some $\tilde{m} \geq 3$. Also, we emphasize that with the above recursion, matrix $\tilde{D}_{\Gamma_n}$ is obtained from D_{Γ_n} by replacing its $N \times N$ element with -1.

We develop $\det D_{\Gamma_n}$ with respect to the last $m + 1$ lines, that is, as an alternated sum of determinants of $m + 1 \times m + 1$ minors composed from the last $m + 1$ lines of D_{Γ_n} times the determinant of D_{Γ_n} without the lines and columns the minor is made of. The only possibility to obtain a $m + 1 \times m + 1$ minor composed from the last $m + 1$ lines of D_{Γ_n} different from zero is to

choose one of the columns $N - m - 1$ and $N - m$, together with all last m columns. This follows from the fact that if we eliminate from $\det D_{\Gamma_n}$ both columns $N - m - 1$ and $N - m$, together with $m - 1$ columns among the last m columns, we obtain a block-diagonal type matrix, with first diagonal block $D_{\Gamma_{n-1}}$ with its last column replaced by zeros, so its determinant vanishes. Therefore,

$$
\det D_{\Gamma_n} = \det D_{\Gamma_{n-1}}
\begin{vmatrix}
e^{\omega a} & -1 & & & & & & & \\
 & 1 & -1 & & & & & & \\
 & & 1 & -1 & & & & & \\
 & & & \ddots & \ddots & & & & \\
 & & & & \ddots & \ddots & & & \\
 & & & & & \ddots & \ddots & & \\
 & & & & & & 1 & -1 & \\
 & & & & & & & 1 & -1 \\
e^{\omega a} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{vmatrix}
$$

$$
- \det \tilde{D}_{\Gamma_{n-1}}
\begin{vmatrix}
e^{-\omega a} & -1 & & & & & & & \\
 & 1 & -1 & & & & & & \\
 & & 1 & -1 & & & & & \\
 & & & \ddots & \ddots & & & & \\
 & & & & \ddots & \ddots & & & \\
 & & & & & \ddots & \ddots & & \\
 & & & & & & 1 & -1 & \\
 & & & & & & & 1 & -1 \\
-e^{-\omega a} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{vmatrix} .
$$

By developing with respect to the first column the $m + 1 \times m + 1$ minors,

$$
\det D_{\Gamma_n} = \det D_{\Gamma_{n-1}} (e^{\omega a} \det D_{\Gamma^m} + (-1)^{m+2} e^{\omega a} (-1)^m)
$$

$$
- \det \tilde{D}_{\Gamma_{n-1}} (e^{-\omega a} \det D_{\Gamma^m} - (-1)^{m+2} e^{-\omega a} (-1)^m),
$$

so using from Sec. III C that $\det D_{\Gamma^m} = m$, $\det \tilde{D}_{\Gamma^m}(\omega) = m - 2$, we find

$$
\det D_{\Gamma_n}(\omega) = (m + 1) e^{\omega a} \det D_{\Gamma_{n-1}}(\omega) - (m - 1) e^{-\omega a} \det \tilde{D}_{\Gamma_{n-1}}(\omega)
$$

$$
= (m + 1) e^{\omega a} \det D_{\Gamma_{n-1}}(\omega) \left(1 - e^{-2\omega a} \frac{m - 1}{m + 1} \frac{\det \tilde{D}_{\Gamma_{n-1}}(\omega)}{\det D_{\Gamma_{n-1}}(\omega)} \right).
$$

Now, from $P(n - 1)$ we have for $|\Re \omega|$ small enough

$$
1 - e^{-2\omega a} \frac{m - 1}{m + 1} \frac{\det \tilde{D}_{\Gamma_{n-1}}(\omega)}{\det D_{\Gamma_{n-1}}(\omega)} > c_0 > 0.
$$

Also, $P(n - 1)$ gives us the existence of two positive constants $c_{\Gamma_{n-1}}$ and $\epsilon_{\Gamma_{n-1}}$ such that $|\det D_{\Gamma_{n-1}}(\omega)| > c_{\Gamma_{n-1}}$, $\forall \omega \in \mathbb{C}$, $|\Re \omega| < \epsilon_{\Gamma_{n-1}}$, so eventually, we get

$$
\exists c_{\Gamma_n}, \epsilon_{\Gamma_n} > 0, \ |\det D_{\Gamma_n}(\omega)| > c_{\Gamma_n}, \ \forall \omega \in \mathbb{C}, |\Re \omega| < \epsilon_{\Gamma_n},
$$

and the first part of property $\mathcal{P}$ is proved for $P(n)$.

In a similar way, we get

$$\det \tilde{D}_{\Gamma_n}(\omega) = (m-1)e^{\omega a} \det D_{\Gamma_{n-1}}(\omega) - (m-3)e^{-\omega a} \det \tilde{D}_{\Gamma_{n-1}}(\omega),$$

so

$$\frac{\det \tilde{D}_{\Gamma_n}(\omega)}{\det D_{\Gamma_n}(\omega)} = \frac{\frac{m-1}{m+1} - \frac{m-3}{m+1}e^{-2\omega a}\frac{\det \tilde{D}_{\Gamma_{n-1}}(\omega)}{\det D_{\Gamma_{n-1}}(\omega)}}{1 - \frac{m-1}{m+1}e^{-2\omega a}\frac{\det \tilde{D}_{\Gamma_{n-1}}(\omega)}{\det D_{\Gamma_{n-1}}(\omega)}}.$$

Thus, we also get the second part of $\mathcal{P}$ for $P(n)$ since

$$\left| \frac{\frac{m-1}{m+1} - \frac{m-3}{m+1}z}{1 - \frac{m-1}{m+1}z} \right| < 1 \iff 0 < (m-2)(|z|^2 - 1) + 2(m-1)(1 - \Re z).$$

E. Extension to the laminar coefficient case

For $\omega \geq 0$ let R_ω be the resolvent of the operator $\Delta_{\sigma,\Gamma}$ defined in Secs. III A and III B. We shall proceed as in Sec. III A and III B, and the main point will be the proof of Proposition 3.2 in the laminar coefficient case. On each side of the edge, parametrized by $x \in I \subset \mathbb{R}$, where the coefficient in the laminar Laplacian is $\sigma(x) = \frac{1}{b^2}$, the resolvent writes

$$R_\omega \mathbf{f}(x) = ce^{\omega bx} + \tilde{c}e^{-\omega bx} + \frac{1}{2\omega}\int_I \mathbf{f}(y)\, e^{-\omega b|x-y|}dy, \ x \in I.$$

As in Secs. III C and III D and using the same notations, we shall show by recursion on the number of vertices property $P(n)$ which leads to the dispersion estimate.

We prove property $P(1)$ on a star-shaped tree by recursion on the number of discontinuities in the laminar structure:

$P(1, p):$ If Γ has 1 vertex and p discontinuities along its edges we have property $\mathcal{P}$.

We denote by $m \geq 3$ the number of edges. We start with $P(1, 0)$. We denote by $\frac{1}{b_j^2}$, $1 \leq j \leq m$ the coefficients of the laminar Laplacian on each edge. The resolvent contains the terms $\tilde{c}_j e^{-\omega b_j x}$, $1 \leq j \leq m$ (and $c_m e^{\omega b_3 x}$ on the last edge for the computation of $\tilde{D}_\Gamma$) on each edge. The matrix $D_{\Gamma^{m,0}}$ is the same as D_{Γ^m} in Sec. III C but with $\frac{1}{b_i}$ on the last line. By developing with respect to the last column,

$$\det D_{\Gamma^{m,0}} = \sum_{j=1}^{m}\frac{1}{b_j}, \quad \det \tilde{D}_{\Gamma^{m,0}} = \sum_{j=1}^{m-1}\frac{1}{b_j} - \frac{1}{b_m},$$

so the property $\mathcal{P}$ follows immediately. Now, we shall prove that $P(1, p-1)$ implies $P(1, p)$. Without loss of generality we can suppose that on the last mth edge of $\Gamma^{m,p}$ there is at least one discontinuity. We denote by x_f the last discontinuity point on this edge, and by x_i the previous discontinuity if there is one, or $x_i = 0$ otherwise. We denote the length $x_f - x_i$ by a, the coefficient of the laminar Laplacian on (x_i, x_f) by $\frac{1}{b_f^2}$, and the coefficient of the laminar Laplacian on (x_f, ∞) by $\frac{1}{b_i^2}$. We call $\Gamma^{m,p-1}$ the graph obtained from $\Gamma^{m,p}$ by removing the last discontinuity x_f on the last edge, and we extend the laminar Laplacian on it on $[x_f, \infty)$ by $\frac{1}{b_i^2}$. With respect to the problem on $\Gamma^{m,p-1}$, in the expression of the resolvent we have on $[x_i, x_f]$ aside from the term $\tilde{c}_i e^{-\omega b_i x}$, the extra term $c_i e^{\omega b_i x}$, and on $[x_f, \infty)$ a term $\tilde{c}_f e^{-\omega b_f x}$. Also, there are two connection conditions at the new discontinuity point x_f. We write the matrix $D_{\Gamma^{m,p}}$ such that the last three columns correspond to the unknowns $(\tilde{c}_i, c_i, \tilde{c}_f)$, the last two lines come from the connection condition at x_f, and the previous last two lines come from the connection condition at x_i, if x_i is a discontinuity point, and

from the Kirchhoff conditions concerning c_i, $\tilde{c}_i$ if it is the vertex. We have

$$
D_{\Gamma^{m,p}} = \begin{pmatrix} D_{\Gamma^{m,p-1}} & & & \\ & & -e^{\omega b_i x_i} & \\ & & -\frac{e^{\omega b_i x_i}}{b_i} & \\ \hline e^{-\omega b_i x_f} & e^{\omega b_i x_f} & -e^{-\omega b_f x_f} & \\ -\frac{e^{-\omega b_i x_f}}{b_i} & \frac{e^{\omega b_i x_f}}{b_i} & -\frac{e^{-\omega b_f x_f}}{b_f} \end{pmatrix}.
$$

By developing the determinants of $D_{\Gamma^{m,p}}$ and $\tilde{D}_{\Gamma^{m,p}}$ with respect to the last two lines, we obtain

$$
\det D_{\Gamma^{m,p}} = e^{\omega b_i x_f} e^{-\omega b_f x_f} \left(\frac{1}{b_f} + \frac{1}{b_i} \right) \det D_{\Gamma^{m,p-1}} - e^{-\omega b_i x_f} e^{-\omega b_f x_f} \left(\frac{1}{b_f} - \frac{1}{b_i} \right) \det \tilde{D}_{\Gamma^{m,p-1}},
$$

$$
\det \tilde{D}_{\Gamma^{m,p}} = e^{\omega b_i x_f} e^{\omega b_f x_f} \left(-\frac{1}{b_f} + \frac{1}{b_i} \right) \det D_{\Gamma^{m,p-1}} - e^{-\omega b_i x_f} e^{\omega b_f x_f} \left(-\frac{1}{b_f} - \frac{1}{b_i} \right) \det \tilde{D}_{\Gamma^{m,p-1}}.
$$

Thus, $P(1, p)$ follows from $P(1, p-1)$ as in the constant coefficient case.

Finally, we shall prove that $P(n-1) \Rightarrow P(n)$. Let us consider a vertex v from which there start $m \geq 2$ external infinite edges and one internal edge connecting it to the rest of the graph. The proof of $P(1, p-1) \Rightarrow P(1, p)$, of eliminating discontinuities on infinite edges also works for trees with $n \geq 1$ vertices. So it is enough to prove $P(n)$ for a graph Γ_n with no discontinuities on the infinite edges emanating from vertex v. We call $\frac{1}{b_j^2}$, $1 \leq j \leq m$ the coefficients of the laminar Laplacian on the m edges emerging from v. With the previous notations and from Sec. III D we have

$$
\det D_{\Gamma_n}(\omega) = e^{\omega b_i x_f} \left(\sum_{j=1}^{m} \frac{1}{b_j} + \frac{1}{b_i} \right) \det D_{\Gamma_{n-1}} - e^{-\omega b_i x_f} \left(\sum_{j=1}^{m} \frac{1}{b_j} - \frac{1}{b_i} \right) \det \tilde{D}_{\Gamma_{n-1}},
$$

$$
\det \tilde{D}_{\Gamma_n}(\omega) = e^{\omega b_i x_f} \left(\sum_{j=1}^{m-1} \frac{1}{b_j} - \frac{1}{b_m} + \frac{1}{b_i} \right) \det D_{\Gamma_{n-1}} - e^{-\omega b_i x_f} \left(\sum_{j=1}^{m-1} \frac{1}{b_j} - \frac{1}{b_m} - \frac{1}{b_i} \right) \det \tilde{D}_{\Gamma_{n-1}}.
$$

Using these recursion relations, we find that $P(n-1)$ implies $P(n)$ completing the proof in the laminar case.

IV. OPEN PROBLEMS

In this paper, we have analyzed the dispersive properties for the linear Schrödinger equation on trees. We have assumed that the coupling is given by the classical Kirchhoff's conditions. However, there are other coupling conditions (see Ref. 21) which allow to define an operator $\Delta(A, B)$ on a metric graph. Under some assumptions on the coupling conditions (A, B) it has been proved in Ref. 21 that the considered operator, denoted by $\Delta(A, B)$, is self-adjoint.

The existence of the dispersive properties for the solutions of the Schrödinger on a graph under general coupling conditions on the vertices $iu_t + \Delta_\Gamma(A, B)u = 0$ is mainly an open problem. The resolvent formula obtained in Refs. 20 and 23 in terms of the coupling matrices A and B might help to understand the general problem. In the same papers, there are also some combinatorial formulations of the resolvent in terms of walks on graphs. Such combinational aspects could clarify if the dispersion is possible only on trees or there are graphs (with some of the edges infinite) with suitable couplings where the dispersion is still true.

It is expected that other results on the Schrödinger equation on $\mathbb{R}$ are still valid on networks. For instance, the smoothing estimate for the linear equation with constant coefficients is still valid. Although its classical proof on $\mathbb{R}$ relies on Fourier analysis, one may easily adapt the proof in Ref. 7 which uses only integrations by parts and Besovs spaces that can still be defined on a tree using the

heat operator. Strichartz estimates has been used previously to treat controllability issues for NSE in Ref. 28. The possible applications of the present results in the control context remains to be analyzed. We mention here some previous works on the controllability/stabilization of the wave equation on networks.[13,32] Finally, another problem of interest is the study of the dispersion properties for the magnetic operators analyzed in Refs. 22 and 24. The analysis in this case is more difficult since in the presence of an external magnetic field the effect of the topology of the graph becomes more pronounced. In contrast with the analysis done here, in the case of magnetic operators the graphs are viewed as structures in the three dimensional Euclidean space $\mathbb{R}^3$ and the orientation of the edges becomes important.

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[1] Adami, R., Cacciapuoti, C., Finco, D., and Noja, D., "Fast solitons on star graphs," Rev. Math. Phys. **23**(4), 409–451 (2011).

[2] Adami, R., Cacciapuoti, C., Finco, D., and Noja, D., "Stationary states of NLS on Star Graphs," e-print arXiv: 1104.3839.

[3] Alexander, S., "Superconductivity of networks. A percolation approach to the effects of disorder," Phys. Rev. B **27**(3), 1541–1557 (1983).

[4] Ali Mehmeti, F., *Nonlinear Waves in Networks*, Mathematical Research, Vol. 80 (Akademie-Verlag, Berlin, 1994).

[5] Banica, V., "Dispersion and Strichartz inequalities for Schrödinger equations with singular coefficients," SIAM J. Math. Anal. **35**(4), 868–883 (2003).

[6] Bona, J. L. and Cascaval, R. C., "Nonlinear dispersive waves on trees," Can. Appl. Math. Q. **16**(1), 1–18 (2008).

[7] Burq, N. and Planchon, F., "Smoothing and dispersive estimates for 1d Schrödinger equations with BV coefficients and applications," J. Funct. Anal. **236**(1), 265–298 (2006).

[8] Cameron, R. H., "Analytic functions of absolutely convergent generalized trigonometric sums," Duke Math. J. **3**, 682–688 (1937).

[9] Carini, J. P., Londergan, J. T., Murdock, D. P., Trinkle, D., Young, C. S., "Bound states in waveguides and bent quantum wires. I. Applications to waveguide systems," Phys. Rev. B **55**, 9842–9851 (1997).

[10] Cascaval, R. C. and Hunter, C. T., "Linear and nonlinear Schrödinger equations on simple networks," Libertas Math. **30**, 85–98 (2010).

[11] Cattaneo, C., "The spectrum of the continuous Laplacian on a graph," Monatsh. Math. **124**(3), 215–235 (1997).

[12] Cazenave, T., *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics Vol. 10 (American Mathematical Society (AMS)/Courant Institute of Mathematical Sciences, Providence, RI/New York, 2003), xiii.

[13] Dáger, R. and Zuazua, E., *Wave Propagation, Observation and Control in 1-d Flexible Multi-Structures*, Mathématiques & Applications [Mathematics & Applications] Vol. 50 (Springer-Verlag, Berlin, 2006).

[14] Exner, P., Vertex coupling in quantum graphs: approximations by scaled Schroedinger operators, in *Proceedings of the ICM satellite conference "Mathematics in Science and Technology"* New Delhi, 15–17 August , 2010.

[15] Figotin, A. and Godin, Y. A., "Spectral properties of thin-film photonic crystals," SIAM J. Appl. Math. **61**(6), 1959–1979 (2001).

[16] Gelfand, I. M., Raikov, D. A., and Chilov, G. E., *Les Anneaux Normés Commutatifs, Monographies Internationales de Mathématiques Modernes* (Gauthier-Villars, Paris, 1964).

[17] Ginibre, J. and Velo, G., "The global Cauchy problem for the nonlinear Schrödinger equation revisited," Ann. Inst. H. Poincaré Anal. Non Linéaire **2**(4), 309–327 (1985).

[18] Gnutzmann, S. and Smilansky, U., "Quantum graphs: Applications to quantum chaos and universal spectral statistics," Adv. Phys. **55**(5–6), 527–625 (2006).

[19] Ignat, L. I., "Strichartz estimates for the Schrödinger equation on a tree and applications," SIAM J. Math. Anal. **42**(5), 2041–2057 (2010).

[20] Kostrykin, V., Potthoff, J., and Schrader, R., "Contraction semigroups on metric graphs," in *Analysis on graphs and its applications*, Proc. Symp. Pure Math. (Amer. Math. Soc., Providence, RI, 2008) Vol. 77, pp. 423–458.

[21] Kostrykin, V. and Schrader, R., "Kirchhoff's rule for quantum wires," J. Phys. A **32**(4), 595–630 (1999).

[22] Kostrykin, V. and Schrader, R., "Quantum wires with magnetic fluxes," Commun. Math. Phys. **237**(1–2), 161–179 (2003).

[23] Kostrykin, V. and Schrader, R. "Laplacians on metric graphs: eigenvalues, resolvents and semigroups," in *Quantum graphs and their applications*, Contemp. Math. **415**, 201–225 (Amer. Math. Soc., Providence, RI, 2006).

[24] Kuchment, P. "Graph models for waves in thin structures," Waves Random Media **12**(4), R1–R24 (2002).

[25] Kuchment, P., "Quantum graphs. I. Some basic structures," Waves Random Media **14**(1), S107–S128 (2004) (Special section on quantum graphs).

[26] Kuchment, P. "Quantum graphs: an introduction and a brief survey," in *Analysis on graphs and its applications*, Proc. Symp. Pure Math. (Amer. Math. Soc., Providence, RI, 2008) Vol. 77, pp. 291–312.

[27] Kuchment, P. and Post, O., "On the spectra of carbon nano-structures," Commun. Math. Phys. **275**(3), 805–826 (2007).

[28] Rosier, L. and Zhang, B.Y., "Exact boundary controllability of the nonlinear Schrödinger equation," J. Differ. Equations **246**(10), 4129–4153 (2009).

[29] Sobirov, Z., Matrasulov, D., Sabirov, K., Sawada, S., and Nakamura, K., "Integrable nonlinear Schrödinger equation on simple networks: Connection formula at vertices," Phys. Rev. E **81**, 066602 (2010).

[30] Strichartz, R.S., "Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations," Duke Math. J. **44**(3), 705–714 (1977).

[31] Tao, T., *Nonlinear dispersive equations*, CBMS Regional Conference Series in Mathematics, (Conference Board of the Mathematical Sciences, Washington, DC, 2006) (Local and global analysis).

[32] Valein, J. and Zuazua, E., "Stabilization of the wave equation on 1D networks," SIAM J. Control Optim. **48**(4), 2771–2797 (2009).

[33] von Below, J., "Classical solvability of linear parabolic equations on networks," J. Differ. Equations **72**(2), 316–337 (1988).