STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER EQUATION ON A TREE AND APPLICATIONS*

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Abstract. In this paper, we consider the Schrödinger equation on a network formed by a tree with the last generation of edges formed by infinite strips. We obtain Strichartz-like estimates for the linear semigroup and apply them to obtain well-posedness results for a nonlinear problem.

Key words. Schrödinger equation, Strichartz estimates, quantum graphs

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1. Introduction. Let us consider the linear Schrödinger equation (LSE):

$$(1.1) \qquad \begin{cases} iu_t + u_{xx} = 0, & x \in \mathbb{R}, \, t \neq 0, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}. \end{cases}$$

The linear equation (1.1) is solved by $u(t,x) = S(t)\varphi$, where $S(t) = e^{it\Delta}$ is the free Schrödinger operator. The linear semigroup has two important properties. First, there is the conservation of the L^2-norm:

$$(1.2) \qquad \|S(t)\varphi\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}.$$

And there is a dispersive estimate of the form

$$(1.3) \qquad |(S(t)\varphi)(x)| \leq \frac{1}{(4\pi|t|)^{1/2}}\|\varphi\|_{L^1(\mathbb{R})}, \quad x \in \mathbb{R}, \, t \neq 0.$$

The space-time estimate

$$(1.4) \qquad \|S(\cdot)\varphi\|_{L^6(\mathbb{R}, \, L^6(\mathbb{R}))} \leq C\|\varphi\|_{L^2(\mathbb{R})},$$

due to Strichartz [16], is deeper. It guarantees that the solutions of system (1.1) decay as t becomes large and that they gain some spatial integrability. Inequality (1.4) was generalized by Ginibre and Velo [9]. They proved the mixed space-time estimates, well known as Strichartz estimates:

$$(1.5) \qquad \|S(\cdot)\varphi\|_{L^q(\mathbb{R}, \, L^r(\mathbb{R}))} \leq C(q,r)\|\varphi\|_{L^2(\mathbb{R})}$$

for the so-called *admissible pairs* (q,r):

$$(1.6) \qquad \frac{2}{q} + \frac{1}{r} = \frac{1}{2}, \quad 2 \leq q, r \leq \infty.$$

Similar results can be stated in any space dimension, but it is beyond the scope of this article. These estimates have been successfully applied to obtain well-posedness results for the nonlinear Schrödinger equation (see [6], [17] and the references therein).

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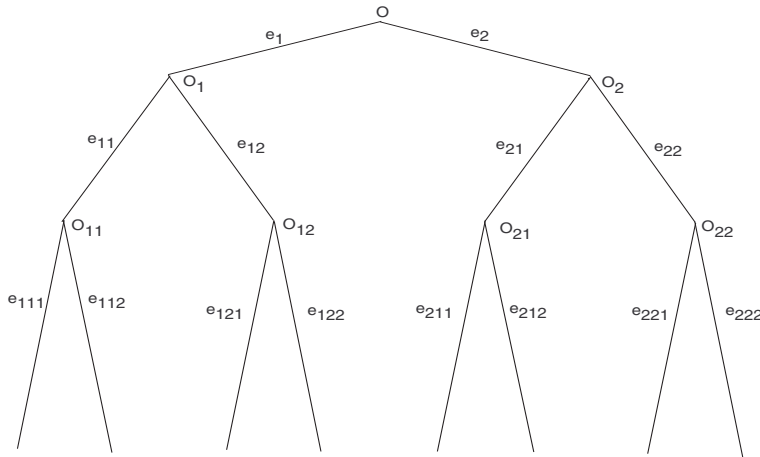
FIG. 1. *A tree with the third generation formed by infinite edges.*

In this paper, we consider the Schrödinger equation on a network formed by the edges of a tree, Γ, as in Figure 1. We assume that each internal node of the tree has two children nodes, the edges at the same generation have the same length, and the last generation of edges is formed by infinite strips. We prove that estimates similar to those mentioned above (1.5) hold in this case. The main idea behind our result is that solving the linear Schrödinger equation on such a structure could be reduced to solving an equation with finitely many piecewise constant coefficients on the whole real line.

The result obtained in this paper can be extended to regular trees Γ, i.e., trees having the property that all the vertices of the same generation have the same number of descendants and all the edges of the same generation are of the same length (see [15] for more details on regular trees), assuming that the last generation of edges is formed by infinite strips.

The problems we address here enter the framework of quantum graphs. The name "quantum graph" is used for a graph considered as a one-dimensional singular variety and equipped with a differential operator. These quantum graphs arise as simplified models in mathematics, physics, chemistry, and engineering (e.g., nanotechnology and microelectronics), when one considers propagation of waves through a quasi–one-dimensional system that looks like a thin neighborhood of a graph. We can mention, in particular, the quantum wires and thin waveguides. We refer the reader to the survey paper [11] and the references therein for more information on this topic.

The paper is organized as follows. In section 2, for the sake of completeness, we introduce some notation, and basic facts about graphs and trees, even if these are classical (see, for example, [7], [5]). We define the spaces $L^2(\Gamma)$, $H^m(\Gamma)$ and introduce the Laplace operator on a graph network. Finally, we write the Schrödinger equation on a graph network.

In section 3, we analyze the dispersive properties of the LSE on the considered tree and apply them to the semilinear Schrödinger equation. In the last section, we state some open problems.

2. Notation and preliminaries. Let $\Gamma = (V, E)$ be a graph, where V is a set of vertices and E the set of edges. For each $v \in V$, we denote $E_v = \{e \in E : v \in e\}$. We assume that Γ is a countable connected locally finite graph; i.e., the degree of

each vertex v of Γ is finite: $|E_v| < \infty$. The edges could be of finite length, and then their ends are vertices of V or they have infinite length; and then we assume that each infinite edge is a ray with a single vertex belonging to V (see [11] for more details on graphs with infinite edges).

We fix an orientation of Γ, and, for each oriented edge e, we denote by $I(e)$ the initial vertex and by $T(e)$ the terminal one. Of course in the case of infinite edges we have only initial vertices.

We identify every edge e of Γ with an interval I_e, where $I_e = [0, l_e]$ if the edge is finite and $I_e = [0, \infty)$ if the edge is infinite. This identification introduces a coordinate x_e along the edge e. In this way, Γ is a metric space and is often called a metric graph [11].

Let v be a vertex of V and e be an edge in E_v. We set for finite edges e

$$i(v, e) = \begin{cases} 0 & \text{if} \quad v = I(e), \\ l_e & \text{if} \quad v = T(e) \end{cases}$$

and

$$i(v, e) = 0 \quad \text{if} \quad v = I(e)$$

for infinite edges.

We identify any function $\mathbf{u}$ on Γ with a collection $\{u^e\}_{e \in E}$ of functions u^e defined on the edges e of Γ. Each u^e can be considered as a function on the interval I_e. In fact, we use the same notation u^e for both the function on the edge e and the function on the interval I_e identified with e. For a function $\mathbf{u} : \Gamma \to \mathbb{C}$, $\mathbf{u} = \{u^e\}_{e \in E}$, we denote by $f(\mathbf{u}) : \Gamma \to \mathbb{C}$ the family $\{f(u^e)\}_{e \in E}$, where $f(u^e) : e \to \mathbb{C}$.

A function $\mathbf{u} = \{u^e\}_{e \in E}$ is continuous if and only if u^e is continuous on I_e for every $e \in E$ and, moreover, it is continuous at the vertices of Γ:

$$u^e(i(v, e)) = u^{e'}(i(v, e')) \quad \text{for all } e, e' \in E_v.$$

The space $L^p(\Gamma)$, $1 \leq p < \infty$, consists of all functions $\mathbf{u} = \{u_e\}_{e \in E}$ on Γ that belong to $L^p(I_e)$ for each edge $e \in E$ and

$$\|\mathbf{u}\|_{L^p(\Gamma)}^p = \sum_{e \in E} \|u^e\|_{L^p(I_e)}^p < \infty.$$

Similarly, the space $L^\infty(\Gamma)$ consists of all functions that belong to $L^\infty(I_e)$ for each edge $e \in E$ and

$$\|\mathbf{u}\|_{L^\infty(\Gamma)} = \sup_{e \in E} \|u^e\|_{L^\infty(I_e)} < \infty.$$

The Sobolev space $H^m(\Gamma)$, $m \geq 1$, an integer, consists of all continuous functions on Γ that belong to $H^m(I_e)$ for each $e \in E$ and

$$\|\mathbf{u}\|_{H^m(\Gamma)}^2 = \sum_{e \in E} \|u^e\|_{H^m(e)}^2 < \infty.$$

The above spaces are Hilbert spaces with the inner products

$$(\mathbf{u}, \mathbf{v})_{L^2(\Gamma)} = \sum_{e \in E} (u^e, v^e)_{L^2(I_e)} = \sum_{e \in E} \int_{I_e} u^e(x) \overline{v^e}(x) dx$$

and

$$(\mathbf{u}, \mathbf{v})_{H^m(\Gamma)} = \sum_{e \in E} (u^e, v^e)_{H^m(I_e)} = \sum_{e \in E} \sum_{k=0}^{m} \int_{I_e} \frac{d^k u^e}{dx^k} \frac{\overline{d^k v^e}}{dx^k} dx.$$

We now introduce the Laplace operator Δ_Γ on the graph Γ. Even if it is a standard procedure, we prefer for the sake of completeness to follow [4]. Consider the sesquilinear continuous form φ on $H^1(\Gamma)$ defined by

$$\varphi(\mathbf{u}, \mathbf{v}) = (\mathbf{u}_x, \mathbf{v}_x)_{L^2(\Gamma)} = \sum_{e \in E} \int_{I_e} u_x^e(x) \overline{v_x^e}(x) dx.$$

We denote by $D(\Delta_\Gamma)$ the set of all the functions $\mathbf{u} \in H^1(\Gamma)$ such that the linear map $\mathbf{v} \in H^1(\Gamma) \to \varphi_{\mathbf{u}}(\mathbf{v}) = \varphi(\mathbf{u}, \mathbf{v})$ satisfies

$$|\varphi(\mathbf{u}, \mathbf{v})| \leq C \|\mathbf{v}\|_{L^2(\Gamma)} \quad \text{for all } \mathbf{v} \in H^1(\Gamma).$$

For $\mathbf{u} \in D(\Delta_\Gamma)$, we can extend $\varphi_{\mathbf{u}}$ to a linear continuous mapping on $L^2(\Gamma)$. There is a unique element in $L^2(\Gamma)$, denoted by $\Delta_\Gamma \mathbf{u}$, such that

$$\varphi(\mathbf{u}, \mathbf{v}) = -(\Delta_\Gamma \mathbf{u}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in H^1(\Gamma).$$

We now define the normal exterior derivative of a function $\mathbf{u} = \{u^e\}_{e \in E}$ at the endpoints of the edges. For each $e \in E$ and v an endpoint of e, we consider the normal derivative of the restriction of $\mathbf{u}$ to the edge e of E_v evaluated at $i(v, e)$ to be defined by

$$\frac{\partial u^e}{\partial n_e}(i(v, e)) = \begin{cases} -u_x^e(0+) & \text{if } i(v, e) = 0, \\ u_x^e(l_e-) & \text{if } i(v, e) = l_e. \end{cases}$$

With this notation, it is easy to characterize $D(\Delta_\Gamma)$ (see [4]):

$$D(\Delta_\Gamma) = \left\{ \mathbf{u} = \{u^e\}_{e \in E} \in H^2(\Gamma) : \sum_{e \in E_v} \frac{\partial u^e}{\partial n_e}(i(v, e)) = 0 \quad \text{for all } v \in V \right\}$$

and

$$(\Delta_\Gamma \mathbf{u})^e = (u^e)_{xx} \quad \text{for all } e \in E, \mathbf{u} \in D(\Delta_\Gamma).$$

In other words, $D(\Delta_\Gamma)$ is the space of all continuous functions on Γ, $\mathbf{u} = \{u^e\}_{e \in E}$, such that for every edge $e \in E$, $u^e \in H^2(I_e)$, and it satisfies the following Kirchhoff-type condition:

$$\sum_{e \in E: T(e) = v} u_x^e(l_e-) - \sum_{e \in E: I(e) = v} u_x^e(0+) = 0 \quad \text{for all } v \in V.$$

It is easy to verify that $(\Delta_\Gamma, D(\Delta_\Gamma))$ is a linear, unbounded, self-adjoint, dissipative operator on $L^2(\Gamma)$, i.e., $\Re(\Delta_\Gamma \mathbf{u}, \mathbf{u})_{L^2(\Gamma)} \leq 0$ for all $\mathbf{u} \in D(\Delta_\Gamma)$.

Let us consider the LSE on Γ:

$$(2.1) \qquad \begin{cases} i\mathbf{u}_t(t, x) + \Delta_\Gamma \mathbf{u}(t, x) = 0, & x \in \Gamma, \quad t \neq 0, \\ \mathbf{u}(0) = \mathbf{u}_0, & x \in \Gamma. \end{cases}$$

Using the properties of the operator $i\Delta_\Gamma$, we obtain as a consequence of the Hille–Yosida theorem the following well-posedness result.

THEOREM 2.1. *For any $\mathbf{u}_0 \in D(\Delta_\Gamma)$, there exists a unique solution $\mathbf{u}(t)$ of system (2.1) that satisfies*

$$\mathbf{u} \in C(\mathbb{R}, D(\Delta_\Gamma)) \cap C^1(\mathbb{R}, L^2(\Gamma)).$$

Moreover, for any $\mathbf{u}_0 \in L^2(\Gamma)$, there exists a unique solution $\mathbf{u} \in C(\mathbb{R}, L^2(\Gamma))$ that satisfies

$$\|\mathbf{u}(t)\|_{L^2(\Gamma)} = \|\mathbf{u}_0\|_{L^2(\Gamma)} \quad \text{for all } t \in \mathbb{R}.$$

The $L^2(\Gamma)$-isometry property is a consequence of the fact that the operator $i\Delta_\Gamma$ satisfies $\Re(i\Delta_\Gamma \mathbf{u}, \mathbf{u})_{L^2(\Gamma)} = 0$ for all $\mathbf{u} \in D(\Delta_\Gamma)$.

3. Strichartz estimates for the Schrödinger equation on a tree. In what follows, we consider a tree (planar, connected graph without closed paths) $\Gamma = (E, V)$. All the vertices, except the root O that has multiplicity two, have multiplicity three; i.e., the number of edges that branch out from each vertex is three (see Figure 1). Also the edges of the same generation have the same length (infinite in the case of the last generation of edges).

In this section, we obtain Strichartz estimates for the solutions of the LSE on the network formed by the edges of the tree Γ. Following the same arguments, we can extend the results presented here to the case of a regular tree.

Using the notation introduced in section 2, we consider system (2.1) on the tree Γ and denote its solution by $S_{\Delta_\Gamma}(t)\mathbf{u}_0 = \exp(it\Delta_\Gamma)\mathbf{u}_0$.

In the following theorem, we state the dispersive property of the linear semigroup $S_{\Delta_\Gamma}(t)$.

THEOREM 3.1. *For $p \in [2, \infty]$ and $t \neq 0$, $S_{\Delta_\Gamma}(t)$ maps $L^{p'}(\Gamma)$ continuously to $L^p(\Gamma)$ and*

$$(3.1) \qquad \|S_{\Delta_\Gamma}(t)\mathbf{u}_0\|_{L^p(\Gamma)} \leq C(\Gamma)|t|^{-(\frac{1}{2}-\frac{1}{p})}\|\mathbf{u}_0\|_{L^{p'}(\Gamma)} \quad \text{for all } \mathbf{u}_0 \in L^{p'}(\Gamma).$$

The above result permits us to obtain more general estimates on the linear semigroup.

THEOREM 3.2 (Strichartz estimates). *The linear semigroup $S_{\Delta_\Gamma}(t)$ satisfies the following properties:*

$$(3.2) \qquad\qquad \|S_{\Delta_\Gamma}(t)\mathbf{u}_0\|_{L^q(\mathbb{R}, L^r(\Gamma))} \leq C(\Gamma, q)\|\mathbf{u}_0\|_{L^2(\Gamma)},$$

$$(3.3) \qquad \left\|\int_{\mathbb{R}} S_{\Delta_\Gamma}(s)^*\mathbf{F}(s, \cdot)ds\right\|_{L^2(\Gamma)} \leq C(\Gamma, q)\|\mathbf{F}\|_{L^{q'}(\mathbb{R}, L^{r'}(\Gamma))},$$

$$(3.4) \qquad \left\|\int_0^t S_{\Delta_\Gamma}(t-s)\mathbf{F}(s)ds\right\|_{L^q(\mathbb{R}, L^r(\Gamma))} \leq C(\Gamma, q, \tilde{q})\|\mathbf{F}\|_{L^{\tilde{q}'}(\mathbb{R}, L^{\tilde{r}'}(\Gamma))}$$

for all admissible pairs (q, r) and $(\tilde{q}, \tilde{r})$ satisfying (1.6).

Remark 3.3. All the constants in Theorems 3.1 and 3.2 depend on the number of generations of the tree Γ and not on the length of the edges. In the case of a regular tree with all the vertices at the generation k each having d_{k+1} descendants,

$0 \leq k \leq n$, the constants will also depend on the sequence $\{d_k\}_{k=1}^{n+1}$. In the considered case, $d_k = 2$ for all $1 \leq k \leq n+1$. At the end of the proof of Theorem 3.1, we will sketch how our argument can be adapted to the case of a regular tree.

As we said in the introduction, the Strichartz estimates have been used in the context of the nonlinear Schrödinger equation (NSE) to obtain well-posedness results. The arguments used in the context of the NSE on the whole space can be used here since we have similar Strichartz-like estimates on the linear semigroup generated by $i\Delta_\Gamma$, $S_{\Delta_\Gamma}(t)$.

THEOREM 3.4. *Let $p \in (0,4)$. For any $\mathbf{u}_0 \in L^2(\Gamma)$, there exists a unique solution*

$$\mathbf{u} \in C(\mathbb{R}, L^2(\Gamma)) \cap \bigcap_{(q,r)\,admissible} L^q_{loc}(\mathbb{R}, L^r(\Gamma))$$

of the equation

(3.5)
$$\begin{cases} i\mathbf{u}_t + \Delta_\Gamma \mathbf{u} \pm |\mathbf{u}|^p \mathbf{u} = 0, & t \neq 0, \\ \mathbf{u}(0) = \mathbf{u}_0, & t = 0. \end{cases}$$

Moreover, the $L^2(\Gamma)$-norm of $\mathbf{u}$ is conserved along the time

$$\|\mathbf{u}(t)\|_{L^2(\Gamma)} = \|\mathbf{u}_0\|_{L^2(\Gamma)}.$$

The proof is standard once Theorem 3.2 is obtained, and it follows the same lines as in [6, Theorem 4.6.1, p. 109]. We leave it to the reader.

In order to proceed to the proofs of Theorems 3.1 and 3.2, we now describe the procedure of indexing the edges and vertices of the tree (see Figure 1). For each index $\overline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \{1,2\}^k$, we denote by $|\overline{\alpha}|$ the number of its components: $|\overline{\alpha}| = k$.

The root of the tree is denoted by O. The remaining vertices and edges will be denoted by $O_{\overline{\alpha}}$, $|\overline{\alpha}| \leq n$ and $e_{\overline{\alpha}}$, $|\overline{\alpha}| \leq n+1$, respectively. Here n and $n+1$ represent the number of generations of vertices (respectively, of edges).

The vertices and edges are defined by recurrence in the following way. For each vertex $O_{\overline{\alpha}}$ with the index $|\overline{\alpha}| \leq n$ (possibly empty in the case of the root O), there are two edges that branch out from it $e_{\overline{\alpha\beta}}$, where $\overline{\alpha\beta} = (\alpha_1, \alpha_2, \ldots, \alpha_k, \beta)$ and $\beta \in \{1,2\}$. If $|\overline{\alpha}| \leq n-1$, the other endpoint of $e_{\overline{\alpha\beta}}$ will be denoted by $O_{\overline{\alpha\beta}}$. In the case when $|\overline{\alpha}| = n$, the edges that branch out from these vertices are infinite strips.

With our notation, $E = \{e_{\overline{\alpha}} : \overline{\alpha} \in \{1,2\}^k, 1 \leq k \leq n+1\}$. A function $\mathbf{u} : \Gamma \to \mathbb{C}$ is a collection of functions $\{u_{\overline{\alpha}}\}_{e_{\overline{\alpha}} \in E}$, where each component $u_{\overline{\alpha}}$ is a function defined on the corresponding edge $e_{\overline{\alpha}}$, $u_{\overline{\alpha}} : e_{\overline{\alpha}} \to \mathbb{C}$. Each edge $e_{\overline{\alpha}}$ will be identified with the interval $[0, l_{|\overline{\alpha}|}]$ if $|\overline{\alpha}| \leq n$ and with $[0, \infty)$ if $|\overline{\alpha}| = n+1$.

Before starting the proof, since it is quite technical, let us point out its main steps. Equation (2.1) gives us a system of LSEs on intervals of the type $(0, l_k)$ or $(0, \infty)$ coupled by Kirchhoff's law. Using a translation in the space variable, we transform our problem to a system of coupled LSEs on some intervals $(a_{|\overline{\alpha}|-1}, a_{|\overline{\alpha}|})$ or (a_n, ∞), where the sequence $\{a_k\}_{k=0}^n$ depends on the length of the edges. We then define the functions $Z^{\overline{\alpha}}$ that equal the average of the functions defined on the edges emanating from the vertex $O_{|\alpha|}$ and its descendants. Using an inductive argument on $|\overline{\alpha}|$, we prove dispersive estimates for $Z^{\overline{\alpha}}$ which give us the same ones for $\mathbf{u}$.

Proof of Theorem 3.1. We recall that Theorem 2.1 shows that $S_{\Delta_\Gamma}(t)$ is an $L^2(\Gamma)$-isometry. Thus it is sufficient to prove that, for any $t \neq 0$, $S_{\Delta_\Gamma}(t)$ maps continuously

$L^1(\Gamma)$ to $L^\infty(\Gamma)$ with a norm less than $C(\Gamma)|t|^{-1/2}$. By density we can consider $\mathbf{u}_0 \in D(\Delta_\Gamma)$ and prove the following estimate:

$$\|S_{\Delta_\Gamma}(t)\mathbf{u}_0\|_{L^\infty(\Gamma)} \le C(\Gamma)|t|^{-\frac{1}{2}}\|\mathbf{u}_0\|_{L^1(\Gamma)} \quad \text{for all } t \ne 0.$$

In the following, we prove that the above estimate holds with a constant $C(\Gamma) = C(n)$.

Using Theorem 2.1 in the particular case of the tree considered here, we have that the solution of system (2.1) satisfies

$$S_{\Delta_\Gamma}(t)\mathbf{u}_0 = (u^{\overline{\alpha}}(t))_{|\overline{\alpha}| \le n+1} \in C(\mathbb{R}, D(\Delta_\Gamma)) \cap C^1(\mathbb{R}, L^2(\Gamma)).$$

It means that for all $t \in \mathbb{R}$ the functions $u^{\overline{\alpha}}$, $1 \le |\overline{\alpha}| \le n+1$, satisfy

$$u^{\overline{\alpha}}(t) \in \begin{cases} H^2(0, l_{|\alpha|}), & |\overline{\alpha}| \le n, \\ H^2(0, \infty), & |\overline{\alpha}| = n+1. \end{cases}$$

Moreover, the family $\{u^{\overline{\alpha}}\}_{1 \le |\overline{\alpha}| \le n+1}$ solves the following system:

(3.6)
$$\begin{cases} iu_t^{\overline{\alpha}}(t, x) + u_{xx}^{\overline{\alpha}}(t, x) = 0, & x \in (0, l_{|\alpha|}),\ 1 \le |\overline{\alpha}| \le n, \\[2mm] iu_t^{\overline{\alpha}}(t, x) + u_{xx}^{\overline{\alpha}}(t, x) = 0, & x \in (0, \infty),\ |\overline{\alpha}| = n+1, \\[2mm] \begin{cases} u^{\overline{\alpha}}(t, l_{|\overline{\alpha}|}) = u^{\overline{\alpha\beta}}(t, 0), & \beta \in \{1, 2\},\ 1 \le |\overline{\alpha}| \le n, \\[2mm] u^1(0, t) = u^2(0, t), \end{cases} \\[5mm] \begin{cases} u_x^{\overline{\alpha}}(t, l_{|\overline{\alpha}|}) = \sum\limits_{\beta=1}^{2} u_x^{\overline{\alpha\beta}}(t, 0), & 1 \le |\overline{\alpha}| \le n, \\[2mm] u_x^1(0, t) + u_x^2(0, t) = 0, \end{cases} \\[5mm] u^{\overline{\alpha}}(0, x) = u_0^{\overline{\alpha}}(x). \end{cases}$$

The first two equations represent the LSE satisfied by each $u^{\overline{\alpha}}$. The second type of properties gives the continuity at the ends of the vertices, and the third one is the Kirchhoff-type condition on the normal derivatives.

However, the above system is not very useful in order to reduce it to an LSE with discontinuous coefficients, as we announced in the introduction. We will rewrite the above system in a convenient manner that will allow us to apply previous results on the dispersive properties of the Schrödinger equation $iu_t + (\sigma u_x)_x = 0$ (see [1]), where σ is a step function taking a finite number of values.

We consider the intervals

$$I_k = \begin{cases} (a_{k-1}, a_k) & \text{if } 1 \le k \le n, \\ (a_n, \infty) & \text{if } k = n+1, \end{cases}$$

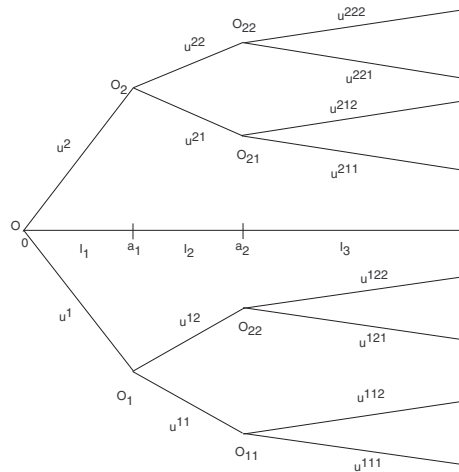where $a_0 = 0$ and $a_{k+1} = a_k + l_{k+1}$ for $k = 0, \ldots, n-1$.

FIG. 2. *The functions situated above each interval are defined on that interval; for example, u^1 and u^2 are defined on I_1, etc.*

With the new notation, by applying translations in the space variable, system (3.6) can be written in an equivalent form:

$$(3.7) \quad \begin{cases} iu_t^{\overline{\alpha}}(t,x) + u_{xx}^{\overline{\alpha}}(t,x) = 0, \quad x \in I_{|\overline{\alpha}|}, \ 1 \leq |\overline{\alpha}| \leq n+1, \\[2mm] \begin{cases} u^{\overline{\alpha}}(t, a_{|\overline{\alpha}|}) = u^{\overline{\alpha\beta}}(t, a_{|\overline{\alpha}|}), \quad \beta \in \{1,2\}, \ 1 \leq |\overline{\alpha}| \leq n, \\[2mm] u^1(t,0) = u^2(t,0), \end{cases} \\[2mm] \begin{cases} u_x^{\overline{\alpha}}(t, a_{|\overline{\alpha}|}) = \sum\limits_{\beta=1}^{2} u_x^{\overline{\alpha\beta}}(t, a_{|\overline{\alpha}|}), \quad 1 \leq |\overline{\alpha}| \leq n, \\[2mm] u_x^1(t,0) + u_x^2(t,0) = 0, \end{cases} \\[2mm] u^{\overline{\alpha}}(0,x) = u_0^{\overline{\alpha}}(x), \quad x \in I_{|\alpha|}, \ 1 \leq |\overline{\alpha}| \leq n. \end{cases}$$

In Figure 2, we can visualize where each function $u^{\overline{\alpha}}$ is defined after the translation. We point out that once the dispersive properties are obtained for the second system (3.7) they also hold for the first one (3.6). In what follows, we will concentrate on system (3.7) and prove that for each index $\overline{\alpha}$ with $|\overline{\alpha}| = k$, $1 \leq k \leq n+1$, the following holds for all $t \neq 0$:

$$(3.8) \quad \max_{|\overline{\alpha}|=k} \|u^{\overline{\alpha}}(t)\|_{L^{\infty}(I_k)} \leq C(n)|t|^{-1/2} \sum_{1 \leq |\overline{\beta}| \leq n+1} \|u_0^{\overline{\beta}}\|_{L^1(I_{|\overline{\beta}|})}.$$

For any $\overline{\alpha}$ with $1 \leq |\overline{\alpha}| \leq n+1$, we define the functions

$$Z^{\overline{\alpha}} : J_{|\overline{\alpha}|} = \bigcup_{k=0}^{n+1-|\overline{\alpha}|} I_{|\overline{\alpha}|+k} \to \mathbb{C}$$

as follows: for each $0 \leq k \leq n+1-|\alpha|$ and $x \in I_{|\alpha|+k}$, we set

$$(3.9) \quad Z^{\overline{\alpha}}(t,x) = \frac{\sum_{|\overline{\beta}|=k} u^{\overline{\alpha\beta}}(t,x)}{2^{|\overline{\beta}|}}.$$

The domain where each function $Z^{\overline{\alpha}}$ is defined can be viewed in Figure 3.
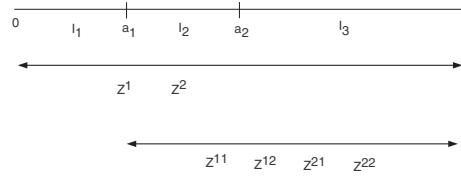
FIG. 3. *The domain where the functions $Z^{\overline{\alpha}}$ are defined.*

The definition of the functions $Z^{\overline{\alpha}}$ shows that it coincides with $u^{\overline{\alpha}}$ on the interval $I_{|\overline{\alpha}|}$. Thus it is sufficient to prove that

$$(3.10) \qquad \|Z^{\overline{\alpha}}(t)\|_{L^\infty(J_{\overline{\alpha}})} \leq c(n)|t|^{-1/2} \sum_{1 \leq |\overline{\beta}| \leq n+1} \|u_0^{\overline{\beta}}\|_{L^1(I_{\overline{\beta}})}.$$

In what follows, it will be convenient to write $Z^{\overline{\alpha}}$, $1 \leq |\overline{\alpha}| \leq n+1$, in a compact form:

$$(3.11) \qquad Z^{\overline{\alpha}}(t,x) = \sum_{k=0}^{n+1-|\overline{\alpha}|} \chi_{\{I_{|\overline{\alpha}|+k}\}}(x)\frac{1}{2^k} \sum_{|\overline{\beta}|=k} u^{\overline{\alpha\beta}}(t,x).$$

Moreover, the above identity shows that for each index $\overline{\alpha}$ with $1 \leq |\overline{\alpha}| \leq n+1$ we have the following inequality:

$$(3.12) \qquad \|Z^{\overline{\alpha}}(0)\|_{L^1(J_{|\overline{\alpha}|})} \leq \sum_{|\overline{\alpha}| \leq |\overline{\beta}| \leq n+1} \|u_0^{\overline{\beta}}\|_{L^1(I_{|\overline{\beta}|})}.$$

To prove inequality (3.10), we need the following result, which we will prove later.

LEMMA 3.5. *Let $n \geq 0$, $-\infty \leq a_0 < a_1 < \cdots < a_n < a_{n+1} = \infty$, and the function*

$$v \in C(\mathbb{R}, H^2((a_k, a_{k+1}))) \cap C^1(\mathbb{R}, L^2((a_k, a_{k+1}))), \quad k = 0, \ldots, n,$$

which solves the following system:

$$(3.13) \quad \begin{cases} iv_t(t,x) + v_{xx}(t,x) = 0, & x \in (a_k, a_{k+1}), \ 0 \leq k \leq n, \ t \neq 0, \\[2mm] v(t,a_k-) = v(t,a_k+), & 1 \leq k \leq n, \ t \neq 0, \\[2mm] v_x(t,a_k-) = c_k v_x(t,a_k+), & 1 \leq k \leq n, \ t \neq 0, \\[2mm] v(t,a_0) = 0, & \text{if } a_0 > -\infty, \ t \neq 0, \\[2mm] v(0,x) = v_0(x), & x \in (a_k, a_{k+1}), \ 0 \leq k \leq n, \end{cases}$$

for some positive constants $\{c_k\}_{k=1}^n$.

If $v_0 \in L^1(a_0, \infty)$, then there exists a positive constant $c(n, \{c_k\}_{k=1}^n)$ such that

$$(3.14) \qquad \|v(t)\|_{L^\infty((a_0,\infty))} \leq c(n, \{c_k\}_{k=1}^n)|t|^{-1/2}\|v_0\|_{L^1((a_0,\infty))} \quad \text{for all } t \neq 0.$$

Remark 3.6. The case $n = 0$, $a_0 = -\infty$ corresponds to system (1.1). The result in this case is classical [16].

Remark 3.7. The constant in the right-hand side of (3.14) does not depend on the sequence $\{a_k\}_{k=0}^{n+1}$.

We now prove inequality (3.10) by using an inductive argument following the length of the index $\overline{\alpha}$. We first consider the case when $|\overline{\alpha}| = 1$.

Step I. *The first generation of Z's:* $|\overline{\alpha}| = 1$. We consider the functions

$$Z(t,x) = \frac{Z^1(t,x) + Z^2(t,x)}{2}, \quad x \in (0,\infty), \ t \in \mathbb{R},$$

and

$$\tilde{Z}^1 = Z - Z^1, \quad \tilde{Z}^2 = Z - Z^2.$$

We claim that

$$(3.15) \qquad \|Z(t)\|_{L^\infty((0,\infty))} \leq C(n)|t|^{-1/2}\|Z^1(0) + Z^2(0)\|_{L^1((0,\infty))}$$

and

$$(3.16) \qquad \|\tilde{Z}^1(t)\|_{L^\infty((0,\infty))} \leq C(n)|t|^{-1/2}\|Z^1(0) - Z^2(0)\|_{L^1((0,\infty))}.$$

A similar estimate will also hold for $\tilde{Z}^2(t)$.

In view of the definition of the functions Z and $\tilde{Z}^1$, estimates (3.15) and (3.16) imply that

$$\|Z^1(t)\|_{L^\infty((0,\infty))} \leq C(n)|t|^{-1/2}(\|Z^1(0)\|_{L^1((0,\infty))} + \|Z^2(0)\|_{L^1((0,\infty))}).$$

Thus, inequality (3.12) gives us that

$$\|Z^1(t)\|_{L^\infty((0,\infty))} \leq C(n)|t|^{-1/2} \sum_{1 \leq |\overline{\beta}| \leq n+1} \|u_0^{\overline{\beta}}\|_{L^1(I_{|\overline{\beta}|})},$$

which proves estimate (3.10) in the considered case $|\overline{\alpha}| = 1$.

We now prove estimates (3.15) and (3.16).

For the first one, we observe that Z satisfies the system

$$(3.17) \quad \begin{cases} iZ_t(t,x) + Z_{xx}(t,x) = 0, & x \in \mathbb{R}\backslash\{a_k, 1 \leq k \leq n\}, \\[2mm] Z(t,a_k-) = Z(t,a_k+), & 1 \leq k \leq n, \\[2mm] Z_x(t,a_k-) = \dfrac{1}{2}Z_x(t,a_k+), & 1 \leq k \leq n, \\[2mm] Z_x(t,0) = 0, & t \in \mathbb{R}, \\[2mm] Z(0,x) = \dfrac{Z^1(0,x) + Z^2(0,x)}{2}, & x \in \mathbb{R}\backslash\{a_k, 1 \leq k \leq n\}. \end{cases}$$

Making an even extension of the function Z to the whole real line, we enter the framework of Lemma 3.5, which gives us the following estimates for the function Z:

$$\|Z(t)\|_{L^\infty((0,\infty))} \leq C(n)|t|^{-1/2}\|Z(0)\|_{L^1((0,\infty))} \leq C(n)|t|^{-1/2}\|Z^1(0) + Z^2(0)\|_{L^1((0,\infty))}.$$

This proves (3.15).

We now prove (3.16). The function $\tilde{Z}^1$ satisfies the following system:

$$(3.18) \quad \begin{cases} i\tilde{Z}^1_t(t,x) + \tilde{Z}^1_{xx}(t,x) = 0, & x \in \mathbb{R}\backslash\{a_k, 1 \leq k \leq n\}, \\[2mm] \tilde{Z}^1(t,a_k-) = \tilde{Z}^1(t,a_k+), & 1 \leq k \leq n, \\[2mm] \tilde{Z}^1(t,0) = 0, & t \in \mathbb{R}, \\[2mm] \tilde{Z}^1_x(t,a_k-) = \dfrac{1}{2}\tilde{Z}^1_x(t,a_k+), & 1 \leq k \leq n, \\[2mm] \tilde{Z}^1(0,x) = \dfrac{Z^2(0,x) - Z^1(0,x)}{2}, & x \in \mathbb{R}\backslash\{a_k, 1 \leq k \leq n\}. \end{cases}$$

We now apply Lemma 3.5, and we obtain that $\tilde{Z}^1$ satisfies

$$\|\tilde{Z}^1(t)\|_{L^\infty((0,\infty))} \leq C(n)|t|^{-1/2}\|\tilde{Z}^1(0)\|_{L^1((0,\infty))} \leq C(n)|t|^{-1/2}\|Z^1(0) - Z^2(0)\|_{L^1((0,\infty))}.$$

The proof of (3.10) in the case $|\overline{\alpha}| = 1$ is now finished.

Step II. *The next generations of Z's.* We assume that we previously proved (3.10) for all indices $\overline{\alpha}$ with its length satisfying $|\overline{\alpha}| = k \leq n$. Let us choose an arbitrary index $\overline{\alpha}$ with $|\overline{\alpha}| = k$. We prove that (3.10) also holds for the index $\overline{\alpha\beta}$ with $\beta \in \{1,2\}$.

We consider the function

$$\tilde{Z}^{\overline{\alpha\beta}}(t,x) = Z^{\overline{\alpha\beta}}(t,x) - Z^{\overline{\alpha}}(t,x), \quad x \in J_{k+1} = \bigcup_{m=k+1}^{n+1} I_m.$$

In the case $k \leq n-1$, the new function $\tilde{Z}^{\overline{\alpha\beta}}$ satisfies the following system:

$$(3.19) \quad \begin{cases} i\tilde{Z}^{\overline{\alpha\beta}}_t(t,x) + \tilde{Z}^{\overline{\alpha\beta}}_{xx}(t,x) = 0, & t \neq 0, \; x \in \bigcup_{m=k+1}^{n+1} I_m, \\[2mm] \tilde{Z}^{\overline{\alpha\beta}}(t,a_k) = 0, & t \neq 0, \\[2mm] \tilde{Z}^{\overline{\alpha\beta}}(t,a_m-) = \tilde{Z}^{\overline{\alpha\beta}}(t,a_m+), & k+1 \leq m \leq n, \\[2mm] \tilde{Z}^{\overline{\alpha\beta}}_x(t,a_m-) = \dfrac{1}{2}\tilde{Z}^{\overline{\alpha\beta}}_x(t,a_m+), & k+1 \leq m \leq n, \\[2mm] \tilde{Z}^{\overline{\alpha\beta}}(0,x) = \tilde{Z}^{\overline{\alpha\beta}}_0(x), & x \in \bigcup_{m=k+1}^{n+1} I_m. \end{cases}$$

In the other case, $k = n$, we are dealing with an equation on the last generation of edges of the tree, and we get the following system:

$$(3.20) \quad \begin{cases} i\tilde{Z}^{\overline{\alpha\beta}}_t(t,x) + \tilde{Z}^{\overline{\alpha\beta}}_{xx}(t,x) = 0, & t \neq 0, \; x \in I_{n+1}, \\[2mm] \tilde{Z}^{\overline{\alpha\beta}}(t,a_n) = 0, & t \neq 0, \\[2mm] \tilde{Z}^{\overline{\alpha\beta}}(0,x) = \tilde{Z}^{\overline{\alpha\beta}}_0(x), & x \in I_{n+1}. \end{cases}$$

In both systems, the second property is a consequence of the fact that $u^{\overline{\alpha}}(t,a_k) = u^{\overline{\alpha\beta}}(t,a_k)$, and thus $Z^{\overline{\alpha}}(t,a_k) = Z^{\overline{\alpha\beta}}(t,a_k)$. The third and fourth properties in system (3.19) are given by the Kirchhoff-type conditions imposed in system (3.7).

Systems (3.19) and (3.20) enter the framework of Lemma 3.5. Thus, we have that

$$\|\tilde{Z}^{\overline{\alpha\beta}}(t)\|_{L^\infty(J_{k+1})} \le C(n)|t|^{-1/2}\|\tilde{Z}_0^{\overline{\alpha\beta}}\|_{L^1(J_{k+1})}.$$

It follows that the same property holds for $Z^{\overline{\alpha\beta}}$ since $Z^{\overline{\alpha\beta}} = \tilde{Z}^{\overline{\alpha\beta}} + Z^{\overline{\alpha}}$. Indeed, for any $t \ne 0$, we have

$$\|Z^{\overline{\alpha\beta}}(t)\|_{L^\infty(J_{k+1})} \le \|\tilde{Z}^{\overline{\alpha\beta}}(t)\|_{L^\infty(J_{k+1})} + \|Z^{\overline{\alpha}}(t)\|_{L^\infty(J_{k+1})}$$

$$\le \|\tilde{Z}^{\overline{\alpha\beta}}(t)\|_{L^\infty(J_{k+1})} + \|Z^{\overline{\alpha}}(t)\|_{L^\infty(J_k)},$$

and using the inductive assumption on $Z^{\overline{\alpha}}$ we obtain that

$$\|Z^{\overline{\alpha\beta}}(t)\|_{L^\infty(I_k)} \le C(n)|t|^{-1/2}(\|Z_0^{\overline{\alpha\beta}} - Z_0^{\overline{\alpha}}\|_{L^1(J_{k+1})} + \|Z_0^{\overline{\alpha}}\|_{L^1(J_k)})$$

(3.21)
$$\le C(n)|t|^{-1/2} \sum_{|\overline{\beta}| \le n+1} \|u_0^{\overline{\beta}}\|_{L^1(I_{\overline{\beta}})}.$$

This implies that (3.10) holds for all indices with length $k + 1$.

The proof of (3.10) is now finished. ☐

Let us now comment about how the above proof can be adapted to obtain similar results in the case of a regular tree. Assume that all the vertices at the generation k have d_{k+1} descendants, $0 \le k \le n$, and all the edges of the same generation have the same length.

In this framework, we have to modify the functions $Z^{\overline{\alpha}}$ in (3.9) in the following way:

$$Z^{\overline{\alpha}}(t,x) = \begin{cases} u^{\overline{\alpha}}, & x \in I_{|\overline{\alpha}|}, \\ \dfrac{\sum_{|\overline{\beta}|=k} u^{\overline{\alpha\beta}}(t,x)}{d_{|\overline{\alpha}|+1} \dots d_{|\overline{\alpha}|+k}}, & x \in I_{|\overline{\alpha}|+k}, \ 1 \le k \le n+1-|\overline{\alpha}|. \end{cases}$$

In Step I, we replace Z by

$$Z = \frac{Z^1 + \dots + Z^{d_1}}{d_1}, \quad \tilde{Z}^j = Z - Z^j, \ j = 1, \dots, d_1,$$

the constant $1/2$ in coupling the derivatives at the points a_k in systems (3.17) and (3.18) by $1/d_{k+1}$, and the initial data in the two systems in agreement with the new definition of the functions $Z^{\overline{\alpha}}$. In Step II, we replace in a similar manner the constant $1/2$ in systems (3.19) and (3.20) with $1/d_{m+1}$ and the initial data.

The assumption on the geometry of the tree and the definition of the functions $Z^{\overline{\alpha}}$ as the average of the functions defined on the edges emanating from the vertex $O_{|\alpha|}$ and its descendants allow us to obtain the continuity property at the points $\{a_k\}_{k=1}^n$ in systems (3.17)–(3.20).

With the above changes, we can extend the results of Theorem 3.1, Theorem 3.2, and Theorem 3.4 to regular trees.

Before proceeding to prove Lemma 3.5, we recall the following result of Banica [1].

THEOREM 3.8 (see [1]). *Consider a partition of the real axis*

$$-\infty = x_0 < x_1 < \dots < x_{n+1} = \infty$$

and a step function

$$\sigma(x) = \sigma_i \quad for \quad x \in (x_i, x_{i+1}),$$

where σ_i are positive numbers.

The solution u of the Schrödinger equation

$$\begin{cases} iu_t(t,x) + (\sigma(x)u_x)_x(t,x) = 0 & for \quad x \in \mathbb{R}, \ t \neq 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

satisfies the dispersion inequality

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C|t|^{-1/2}\|u_0\|_{L^1(\mathbb{R})}, \quad t \neq 0,$$

where the constant C depends on n and on the sequence $\{\sigma_i\}_{i=0}^n$.

Proof of Lemma 3.5. We first show how we can reduce the proof to the case $a_0 = -\infty$. If $a_0 > -\infty$, we can make a translation in the space variable and consider the case $a_0 = 0$. In this case, we make an odd extension of the function v and prove the desired properties for the extended function. The above argument allows us to consider only the case $a_0 = -\infty$. Also the case $n = 0$ is classical. Thus we will consider the case $n \geq 1$.

In the space variable, we can assume that $a_1 = 0$. In this case, we reduce the problem to case when the system holds in the whole real line since we can make an odd extension of the function v and prove the properties for the extended function.

In the following, we consider $a_0 = -\infty$ and let v be the solution of the following system:

(3.22)
$$\begin{cases} iv_t(t,x) + v_{xx}(t,x) = 0, & x \in (a_k, a_{k+1}), \ 0 \leq k \leq n, \ t \neq 0, \\ v(t, a_k-) = v(t, a_k+), & 1 \leq k \leq n, \\ v_x(t, a_k-) = c_k v_x(t, a_k+), & 1 \leq k \leq n, \\ v(0,x) = v_0(x), & x \in (a_k, a_{k+1}), \ 0 \leq k \leq n. \end{cases}$$

For each $k = 0, \ldots, n$, we define T_k to be the linear function that maps the interval (a_k, a_{k+1}) to (b_k, b_{k+1}), where $\{b_k\}_{k=0}^n$ is an increasing sequence which will be defined later. More precisely, T_k satisfies

(3.23)
$$\begin{cases} T_k : (a_k, a_{k+1}) \to (b_k, b_{k+1}), & k = 0, \ldots, n, \\ T_k(a_k) = b_k, \ T_k(a_{k+1}) = b_{k+1}, & k = 0, \ldots, n, \\ (T_k)_x(x) = \mu_k, & x \in (a_k, a_{k+1}), \ k = 0, \ldots, n. \end{cases}$$

The constants $\{\mu_k\}_{k=0}^n$ will be chosen later in order to satisfy some compatibility conditions. We choose $b_0 = a_0$ and $b_{n+1} = a_{n+1}$. The other constants b_k, $1 \leq k \leq n$, are uniquely determined by the choice of the sequence $\{\mu_k\}_{k=1}^n$.

We introduce a new function w depending on v and show that it solves an LSE with a finite number of positive piecewise constant coefficients. Set

(3.24) $$w(T_k(x)) = v(x), \quad x \in (a_k, a_{k+1}), \ 0 \leq k \leq n.$$

We claim that w satisfies the following system:

$$(3.25) \quad \begin{cases} iw_t(t, x) + \mu_k^2 w_{xx}(t, x) = 0, & x \in (b_k, b_{k+1}), \ 0 \leq k \leq n, \\ w(t, b_k-) = w(t, b_k+), & 1 \leq k \leq n, \\ w_x(t, b_k-)\mu_{k-1} = c_k w_x(t, b_k+)\mu_k, & 1 \leq k \leq n, \\ w(x, 0) = w_0(x), & x \in (b_k, b_{k+1}), \ 0 \leq k \leq n. \end{cases}$$

We observe that, for any $x \in (a_k, a_{k+1})$, $0 \leq k \leq n$, we have

$$iv_t(t, x) + v_{xx}(t, x) = iw_t(t, T_k(x)) + \mu_k^2 w_{xx}(t, T_k(x)) = 0, \quad x \in (a_k, a_{k+1}).$$

This proves the first property in system (3.25). The second property holds since v is continuous at each point a_k with $1 \leq k \leq n$. For the third property, we use that for each $1 \leq k \leq n$ the following hold:

$$v_x(a_k-) = w_x(b_k-)\mu_{k-1}$$

and

$$v_x(a_k+) = w_x(b_k+)\mu_k.$$

We now choose the sequence $\{\mu_k\}_{k=0}^n$ such that the function w defined in (3.24) satisfies the following equation on the whole real line:

$$(3.26) \quad \begin{cases} iw_t + (\sigma(x)w_x)_x = 0, & x \in \mathbb{R}, \ t \neq 0, \\ w(0, x) = w_0(x), & x \in \mathbb{R}, \end{cases}$$

where

$$(3.27) \quad \sigma(x) = \mu_k^2, \quad x \in (b_k, b_{k+1}), \ 0 \leq k \leq n.$$

Observe that for (3.26) to hold with σ defined by (3.27) we have to impose that

$$\sigma(b_k+)w_x(t, b_k+) = \sigma(b_k-)w_x(t, b_k-), \quad k = 1, \ldots, n.$$

It means that

$$\mu_k^2 w_x(t, b_k+) = \mu_{k-1}^2 w_x(t, b_k-), \quad k = 1, \ldots, n.$$

The third property in (3.25) requires that

$$\mu_k = c_k \mu_{k-1} \quad \text{for} \quad k = 1, \ldots, n.$$

Choosing $\mu_0 = 1$ and $\mu_k = c_1 \ldots c_k$, $k \geq 1$, the above property is satisfied, and w solves system (3.26) with σ defined by (3.27).

We are now able to obtain the $L^1(\mathbb{R}) - L^\infty(\mathbb{R})$ property for v and w. Using Theorem 3.8, we obtain that w satisfies

$$(3.28) \quad \|w(t)\|_{L^\infty(\mathbb{R})} \leq c(\sigma)|t|^{-1/2}\|w_0\|_{L^1(\mathbb{R})},$$

where $c(\sigma) = c(\{n, \{\mu_k\}_{k=0}^n\}) = c(\{n, \{c_k\}_{k=0}^n\})$.

It is then obvious that v satisfies property (3.14).

The proof is now finished. ∎

Using Theorem 3.1, we can prove Theorem 3.2. To proceed, we need the following result of Keel and Tao [10].

PROPOSITION 3.9 (see [10, Theorem 1.2]). *Let H be a Hilbert space, (X, dx) be a measure space, and $U(t) : H \to L^2(X)$ be a one parameter family of mappings with $t \in \mathbb{R}$, which obey the energy estimate*

$$(3.29) \qquad \|U(t)f\|_{L^2(X)} \leq C\|f\|_H$$

and the decay estimate

$$(3.30) \qquad \|U(t)U(s)^*g\|_{L^\infty(X)} \leq C|t - s|^{-1/2}\|g\|_{L^1(X)}.$$

Then

$$(3.31) \qquad \|U(t)f\|_{L^q(\mathbb{R},\, L^r(X))} \leq C\|f\|_H,$$

$$(3.32) \qquad \left\|\int_{\mathbb{R}} (U(s))^* F(s, \cdot)ds\right\|_H \leq C\|F\|_{L^{q'}(\mathbb{R},\, L^{r'}(X))},$$

$$(3.33) \qquad \left\|\int_0^t U(t-s)F(s)ds\right\|_{L^q(\mathbb{R},\, L^r(X))} \leq C\|F\|_{L^{\tilde{q}'}(\mathbb{R},\, L^{\tilde{r}'}(X))}$$

for all admissible pairs (q, r) and $(\tilde{q}, \tilde{r})$ satisfying (1.6).

Proof of Theorem 3.2. We apply Proposition 3.9 to the operators $\{S_{\Delta_\Gamma}(t)\}_{t\in\mathbb{R}}$, where $X = \Gamma$ and $H = L^2(\Gamma)$. The first hypothesis (3.29) is satisfied since $S_{\Delta_\Gamma}(t)$ is an $L^2(\Gamma)$-isometry.

In order to check the second hypothesis, we point out that since $i\Delta_\Gamma$ is skew-adjoint the operator $S_{\Delta_\Gamma}(t)$ satisfies

$$S_{\Delta_\Gamma}(t)^* = S_{\Delta_\Gamma}(-t).$$

Thus by Theorem 3.1 we get

$$\|S_{\Delta_\Gamma}(t)S_{\Delta_\Gamma}(s)^*g\|_{L^\infty(\Gamma)} = \|S_{\Delta_\Gamma}(t-s)g\|_{L^\infty(\Gamma)} \leq C(\Gamma)|t-s|^{-1/2}\|g\|_{L^1(\Gamma)},$$

and hence hypothesis (3.30) is satisfied.

Applying Proposition 3.9, we obtain the desired result. ∎

4. Some open problems. In this paper, we have considered the case of a tree such that each internal vertex has two descendants and the results can be extended to the case of a regular tree as defined in [15]. Our proof does not cover the case when the edges of the tree have arbitrary lengths. The fact that all the edges at the same generation have the same length allowed us to consider the averages of the functions $u^{\overline{\alpha}}$ defined at the same generation of edges by introducing the functions $Z^{\overline{\alpha}}$ in the proof of Theorem 3.1. In the case when the edges of a generation have different lengths, we cannot define the functions $Z^{\overline{\alpha}}$, and our argument cannot be applied. This is the case, for example, of the tree in Figure 4, where the functions $u^{\overline{\alpha}}$, $\overline{\alpha} \in \{1,2\}^2$, are not defined on the same interval.
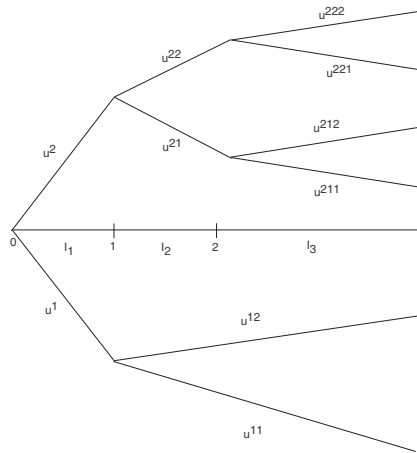
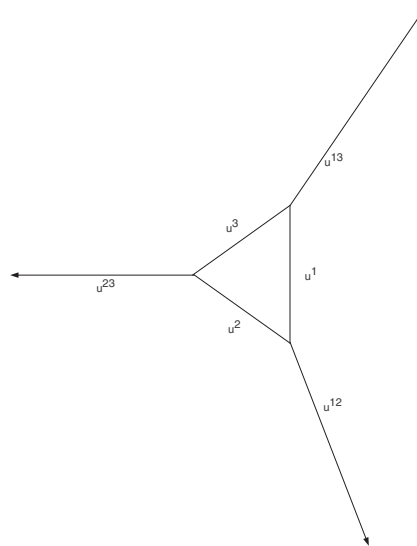FIG. 4. *A tree where the infinite strips occur at different levels.*

FIG. 5. *A tree where two kinds of structures occur: a periodic one given by the triangle and the infinite strips.*

On the other hand, the proof of Theorem 3.1 cannot be applied to the case when at some generation we have two vertices with a different number of descendants. The fact that at the same generation the number of descendants is the same helps us obtain the continuity property at the points $\{a_k\}_{k=1}^n$ of the functions involved in systems (3.17)–(3.20) and then to use Lemma 3.5.

We conjecture that the results we obtained in the present paper can be extended to trees with the last generation of edges formed by infinite strips. In this context, new methods have to be developed. We mention the works [12, 8, 13, 14] in the context of the asymptotic behavior of the heat kernels on trees and graph networks.

Another interesting problem consists of the analysis of the same properties on some graphs which combine the periodic structure with the infinite strips. This is the case in Figure 5. We recall that for an LSE on the one-dimensional torus Bourgain [2]

has analyzed the existence of Strichartz estimates. In the same framework, we also mention the work of Burq, Gérard, and Tzvetkov [3].

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