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Inverse problem for the heat equation and the Schrödinger equation on a tree

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Abstract

In this paper, we establish global Carleman estimates for the heat and Schrödinger equations on a network. The heat equation is considered on a general tree and the Schrödinger equation on a star-shaped tree. The Carleman inequalities are used to prove the Lipschitz stability for an inverse problem consisting in retrieving a stationary potential in the heat (resp. Schrödinger) equation from boundary measurements.

1. Introduction

In this paper, we consider two inverse problems on a network formed by the edges of a tree. The problems we address here enter in the framework of quantum graphs. The name *quantum graph* is used for a graph considered as a one-dimensional singular variety and equipped with a differential operator. Those quantum graphs are metric spaces which can be written as the union of finitely many intervals, which are compact or $[0, \infty)$ and any two of these intervals are either disjoint or intersect only at one of their endpoints.

Quantum graphs arise as simplified models in mathematics, physics, chemistry and engineering (e.g., nanotechnology and microelectronics), when one considers propagation of waves through a quasi-one-dimensional system that looks like a thin neighborhood of a graph. We can mention in particular the quantum wires and thin waveguides. Differential operators on metric graphs arise in a variety of applications, to quote a few: carbon nano-structures [31], photonic crystals [19], high-temperature granular superconductors [1], quantum waveguides [15], free-electron theory of conjugated molecules in chemistry, quantum chaos, etc. For more details, we refer the reader to the review papers [28–30, 18] and the references therein for more information on this topic.

To be more precise, we consider the heat equation on a 1D network Γ given by the edges of a general tree and the Schrödinger equation on a star-shaped tree.

The first system that we consider is the following:

$$\begin{cases} \mathbf{u}_t - \Delta_\Gamma \mathbf{u} + \mathbf{p}\mathbf{u} = 0, & \text{in } \Gamma \times (0,T), \\ \mathbf{u} = \mathbf{h}, & \text{on } \partial\Gamma \times (0,T), \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0, & \text{in } \Gamma, \end{cases} \tag{1.1}$$

where Δ_Γ is the Laplace operator on the network Γ. The system is closed with the coupling conditions at the internal nodes of the tree, namely the continuity and the Kirchhoff's law on the flux at all internal vertices of Γ. Here, $\mathbf{u}$ is a collection of functions $u^{\overline{\alpha}}$, each of them satisfying a heat equation on some edge of the network.

Simultaneously, with problem (1.1), we consider the following problem:

$$\begin{cases} \mathrm{i}\mathbf{u}_t + \Delta_\Gamma \mathbf{u} + \mathbf{p}\mathbf{u} = 0, & \text{in } \Gamma \times (0,T), \\ \mathbf{u} = \mathbf{h}, & \text{on } \partial\Gamma \times (0,T), \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0, & \text{in } \Gamma, \end{cases} \tag{1.2}$$

under similar coupling conditions as in the previous model.

In both cases, we are interested in determining the potential $\mathbf{p}$, a collection of functions defined on the edges of Γ, from boundary measurements. In [2, 3], it was proved that the connectivity, the lengths of the edges and the potential can be retrieved from the knowledge of the dynamic Dirichlet-to-Neumann map for the wave equation. Here, the potential is determined from only a *finite number of measurements* (more explicitly, the measurements at $k \leqslant N$ external vertices of the normal derivatives for the solution of the Cauchy problem corresponding to the initial data).

In the case of the first system, we are able to prove that we can recover $\mathbf{p}$ using only $N-1$ measurements, where N is the total number of exterior vertices of the network Γ. However, in the case of the second system, besides the fact that we need to deal with a star-shaped network, we can only recover the potential $\mathbf{p}$ from measurements performed at *all* the exterior vertices of Γ.

The use of Carleman estimates to achieve uniqueness and stability results in inverse problems is well known. Some authors use local Carleman inequalities and deduce uniqueness and Hölder estimates. Others make use of global Carleman inequalities and deduce Lipschitz stability results and hence uniqueness results. We shall follow the second approach.

Inverse problems with a finite number of measurements have been widely studied by Bukhgeim and Klibanov (see [11], [24] and [25]) by means of Carleman estimates (see also [23] and the references therein). For a wide class of partial differential equations, their method provides the stability in the inverse problem, whenever a suitable Carleman estimate is available. Since [11], there have been many works based on their methodology.

The theory of global Carleman estimates for a parabolic operator has been largely developed since the work by Fursikov and Imanuvilov [20] and it has been applied to many situations (e.g., to prove the controllability along the trajectories or the stability in inverse problems). Since a complete list of references is too long, we refer the reader to [37] for a quite complete review of the state of the art.

Concerning the Schrödinger equation, we refer to [6, 8, 10, 12, 13, 33, 38], where the Carleman estimates are proved and used to establish the stability for some inverse problems (see also [22, 32, 39] for some other Carleman estimates for the Schrödinger equation).

The same approach has given many results for the wave equation. Since a complete list is too long, we quote only some of them, related to the same inverse problem consisting in retrieving a stationary potential in the wave equation: [34] and [36] for Dirichlet boundary data and a Neumann measurement, and [21] for the Neumann boundary data and a Dirichlet measurement. These works are based on the use of local or global Carleman estimates. In the framework of Carleman inequalities on networks, we mention the recent paper [5] where the authors establish a global Carleman estimate for the wave equation on a star-shaped tree and used it to derive the Lipschitz stability in an inverse problem. The Carleman estimate in [5] involves some positive-definite matrix introduced in [9] to derive a Carleman estimate for the one-dimensional heat equation with discontinuous coefficients.

As far as we know, the determination of a time-independent potential for the heat or the Schrödinger equation in a network-like structure has not been addressed in the literature yet. This type of problem has been studied, e.g., for membranes or elastic strings (see for instance [4] and the references therein).

Let us now state the main results of the paper. For the given initial data $\mathbf{u}_0$ and boundary data $\mathbf{h}$, we denote by $\mathbf{u}(\mathbf{p})$ the solution of the above systems associated with the potential $\mathbf{p} \in L^\infty(\Gamma, \mathbb{R})$. We introduce the space

$$H^{2,1}(\Gamma \times (0, T)) := L^2(0, T; H^2(\Gamma)) \cap H^1(0, T; L^2(\Gamma)).$$

(See section 2 for the definition of $H^2(\Gamma)$.) We also introduce the ball $B_m(0) := \{\mathbf{q} \in L^\infty(\Gamma, \mathbb{R});\ ||\mathbf{q}||_{L^\infty(\Gamma)} \leqslant m\}$. Then, the following stability results hold.

Theorem 1.1. *Assume that* $\mathbf{p} \in L^\infty(\Gamma)$, $\mathbf{u}_0 = \mathbf{u}_0(x)$, $h = h(x, t)$ *and* $r > 0$ *are such that the solution* $\mathbf{u}(\mathbf{p})$ *of (1.1) fulfils* $\mathbf{u}(\mathbf{p}) \in H^{2,1}(\Gamma \times (0, T))$, $\partial_t \mathbf{u}(\mathbf{p}) \in H^{2,1}(\Gamma \times (0, T))$, *and such that for some* $t_0 \in (0, T)$, *the following holds:*

$$|\mathbf{u}(\mathbf{p})(\cdot, t_0)| \geqslant r \text{ a.e. on } \Gamma.$$

Then, for any $m > 0$, *there exists a constant* $C = C(m, ||\partial_t \mathbf{u}(\mathbf{p})||_{L^\infty(\Gamma\times(0,T))}, r)$ *such that for any* $\mathbf{q} \in B_m(0)$ *satisfying*

$$\partial_x[\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})](v, \cdot) \in H^1(0, T) \quad \textit{for all exterior nodes } v,$$

we have

$$\|\mathbf{p} - \mathbf{q}\|_{L^2(\Gamma)} \leqslant C\Big(\|[\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})](\cdot, t_0)\|_{H^2(\Gamma)} + \sum_{v\in\mathcal{E}} \|\partial_x[\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})](v, \cdot)\|_{H^1(0,T)}\Big),$$

where $\mathcal{E}$ *denotes the set of all the exterior vertices of* Γ *except one.*

For the second system, under the assumption that the network is a star-shaped tree, we can prove a similar stability result.

Theorem 1.2. *Assume that* $\mathbf{p} \in L^\infty(\Gamma; \mathbb{R})$, $\mathbf{u}_0 = \mathbf{u}_0(x)$, $h = h(x, t)$ *and* $r > 0$ *are such that the solution of (1.2) satisfies*

- $\mathbf{u}_0(x) \in \mathbb{R}$ *or* $i\mathbf{u}_0(x) \in \mathbb{R}$ *a.e. in* Γ,
- $|\mathbf{u}_0(x)| \geqslant r > 0$ *a.e. in* Γ *and*
- $\partial_t \mathbf{u}(\mathbf{p}) \in H^{2,1}(\Gamma \times (0, T))$.

Then, for any $m \geqslant 0$, *there exists a constant* $C = C(m, ||\partial_t \mathbf{u}(\mathbf{p})||_{H^{2,1}(\Gamma\times(0,T))}, r) > 0$ *such that for any* $\mathbf{q} \in B_m(0)$ *satisfying*

$$\partial_t \mathbf{u}(\mathbf{q}) \in H^{2,1}(\Gamma \times (0, T)),$$

we have

$$||\mathbf{p} - \mathbf{q}||_{L^2(\Gamma)} \leqslant C \sum_{v\in\partial\Gamma} ||\partial_x[\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})](v, .)||_{H^1(0,T)}.$$

The above theorems extend to network classical results on inverse problems. To prove these results, we need to establish the (new) global Carleman estimates for the heat (resp. the Schrödinger) equation on trees. Note that if we impose Kirchhoff-type conditions to the weight function at the internal vertices, the Carleman estimate cannot be derived. In our Carleman estimates, the weight function has to fulfil some *nonlinear* flux condition at each internal vertex. Note that we shall write the Carleman estimates with two parameters (namely s and λ), although a single parameter (s) is usually sufficient in dimension 1. This is just to simplify the construction of the weight function. On the other hand, for the Schrödinger equation posed on a star-shaped tree with N external vertices, we consider a combination of N weight functions in order to cancel some 'bad' terms at the internal vertices involving time derivatives (unfortunately, the extension of our result to a tree with two internal nodes cannot be done by that approach, see below). That strategy was used in [7], with two different weight functions, in order to improve the observation region for the wave equation.

This paper is organized as follows. In section 2, we introduce the notation and some classical facts about the heat and Schrödinger equations on trees. In section 3, we present the analysis in the case of the heat equation. The Schrödinger equation is considered in section 4. Finally, we discuss some open problems in section 5.

2. Notation and preliminaries

Let $\Gamma = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. The edges are assumed to be of finite length and their ends are the vertices of V. For each $v \in V$, we denote $E_v = \{e \in E : v \in e\}$. The *multiplicity* of a vertex of Γ is equal to the number of edges that branch out from it. If the multiplicity is equal to 1, the vertex is said to be *exterior*, otherwise it is said to be *interior*. We assume that Γ does not contain vertices with multiplicity 2, since they are irrelevant for our models.

From now on, we assume that Γ is a tree, i.e. Γ is a planar finite connected graph without circuit (closed path). We fix an orientation of Γ and for each oriented edge e, we denote by $I(e)$ its initial vertex and by $T(e)$ its terminal one.

We identify every edge e of Γ with an interval I_e, where $I_e = [0, l_e]$, with l_e being the length of e. This identification introduces a coordinate x_e along the edge e.

Let v be a vertex of V and e be an edge in E_v. We set

$$i(v, e) = \begin{cases} 0 & \text{if} \quad v = I(e), \\ l_e & \text{if} \quad v = T(e). \end{cases}$$

We identify any function $\mathbf{u}$ on Γ with a collection $\{u^e\}_{e\in E}$ of functions u^e defined on the edges e of Γ. Each u^e can be considered as a function on the interval I_e. In fact, we use the same notation u^e for both the function on the edge e and the function on the interval I_e identified with e. For a function $\mathbf{u} : \Gamma \to \mathbb{C}$, $\mathbf{u} = \{u^e\}_{e\in E}$, we denote by $f(\mathbf{u}) : \Gamma \to \mathbb{C}$ the family $\{f(u^e)\}_{e\in E}$, where $f(u^e) : e \to \mathbb{C}$.

A function $\mathbf{u} = \{u^e\}_{e\in E}$ is continuous if and only if u^e is continuous on I_e for every $e \in E$, and $\mathbf{u}$ is continuous at the vertices of Γ:

$$u^e(i(v, e)) = u^{e'}(i(v, e')) \quad \forall\, e, e' \in E_v.$$

The space $L^p(\Gamma)$, $1 \leqslant p < \infty$, consists of all the functions $\mathbf{u} = \{u^e\}_{e\in E}$ on Γ that belong to $L^p(I_e)$ for each edge $e \in E$. This space is endowed with the norm

$$\|\mathbf{u}\|^p_{L^p(\Gamma)} = \sum_{e\in E} \|u^e\|^p_{L^p(I_e)} < \infty.$$

Similarly, the space $L^\infty(\Gamma)$ consists of all the functions $\mathbf{u} = \{u_e\}_{e\in E}$ that belong to $L^\infty(I_e)$ for each edge $e \in E$. The corresponding norm is

$$\|\mathbf{u}\|_{L^\infty(\Gamma)} = \sup_{e\in E} \|u^e\|_{L^\infty(I_e)} < \infty.$$

The Sobolev space $H^m(\Gamma)$, with $m \in \mathbb{N}^*$, consists of all the continuous functions on Γ (viewed as a closed subset of $\mathbb{R}^2$) that belong to $H^m(I_e)$ for each $e \in E$. It is endowed with the norm

$$\|\mathbf{u}\|^2_{H^m(\Gamma)} = \sum_{e\in E} \|u^e\|^2_{H^m(e)} < \infty.$$

The spaces $L^2(\Gamma)$ and $H^m(\Gamma)$ are the Hilbert spaces when endowed with the inner products

$$(\mathbf{u}, \mathbf{v})_{L^2(\Gamma)} = \sum_{e\in E} (u^e, v^e)_{L^2(I_e)} = \sum_{e\in E} \int_{I_e} u^e(x)\overline{v^e}(x)\,\mathrm{d}x$$

and

$$(\mathbf{u}, \mathbf{v})_{H^m(\Gamma)} = \sum_{e\in E} (u^e, v^e)_{H^m(I_e)} = \sum_{e\in E}\sum_{k=0}^{m} \int_{I_e} \frac{\mathrm{d}^k u^e}{\mathrm{d}x^k}\overline{\frac{\mathrm{d}^k v^e}{\mathrm{d}x^k}}\,\mathrm{d}x.$$

$H^1_0(\Gamma)$ denotes the set of functions in $H^1(\Gamma)$ that vanish at the exterior vertices. We now introduce the Laplace operator Δ_Γ on the tree Γ. Even if it is a standard procedure, we prefer to recall it following [14], for the sake of completeness. Consider the sesquilinear continuous form φ on $H^1_0(\Gamma)$ defined by

$$\varphi(\mathbf{u}, \mathbf{v}) = (\mathbf{u}_x, \mathbf{v}_x)_{L^2(\Gamma)} = \sum_{e\in E} \int_{I_e} u^e_x(x)\overline{v^e_x}(x)\,\mathrm{d}x.$$

We denote by $D(\Delta_\Gamma)$ the set of all the functions $\mathbf{u} \in H^1_0(\Gamma)$ such that the linear map $\mathbf{v} \in H^1_0(\Gamma) \to \varphi_{\mathbf{u}}(\mathbf{v}) := \varphi(\mathbf{u}, \mathbf{v})$ satisfies

$$|\varphi_{\mathbf{u}}(\mathbf{v})| \leqslant C\|\mathbf{v}\|_{L^2(\Gamma)} \quad \text{for all } \mathbf{v} \in H^1_0(\Gamma).$$

For $\mathbf{u} \in D(\Delta_\Gamma)$, we can extend $\varphi_{\mathbf{u}}$ to a linear continuous mapping on $L^2(\Gamma)$. There is a unique element in $L^2(\Gamma)$, denoted by $\Delta_\Gamma \mathbf{u}$, such that

$$\varphi(\mathbf{u}, \mathbf{v}) = -(\Delta_\Gamma \mathbf{u}, \mathbf{v})_{L^2(\Gamma)} \quad \text{for all } \mathbf{v} \in H^1_0(\Gamma).$$

We now define the normal exterior derivative of a function $\mathbf{u} = \{u^e\}_{e\in E}$ at the endpoints of the edges. For each $e \in E$ and v, i.e. an endpoint of e, we consider the normal derivative of the restriction of $\mathbf{u}$ to the edge e of E_v evaluated at $i(v, e)$ to be defined by

$$\frac{\partial u^e}{\partial n_e}(i(v, e)) = \begin{cases} -u^e_x(0^+) & \text{if} \quad i(v, e) = 0, \\ u^e_x(l_e^-) & \text{if} \quad i(v, e) = l_e. \end{cases}$$

With this notation, it is easy to characterize $D(\Delta_\Gamma)$ (see [14]):

$$D(\Delta_\Gamma) = \left\{\mathbf{u} = \{u^e\}_{e\in E} \in H^2(\Gamma) \cap H^1_0(\Gamma); \sum_{e\in E_v} \frac{\partial u^e}{\partial n_e}(i(v, e)) = 0 \quad \text{for any interior vertex } v\right\}$$

and

$$(\Delta_\Gamma \mathbf{u})^e = (u^e)_{xx} \quad \text{for all } e \in E,\ \mathbf{u} \in D(\Delta_\Gamma).$$

In other words, $D(\Delta_\Gamma)$ is the space of all the continuous functions $\mathbf{u} = \{u^e\}_{e\in E}$ on Γ, such that for each edge $e \in E$, $u^e \in H^2(I_e)$, and which vanish at each exterior node and fulfil the following Kirchhoff-type condition:

$$\sum_{e\in E;\ T(e)=v} u^e_x(l_e^-) - \sum_{e\in E;\ I(e)=v} u^e_x(0^+) = 0$$

at each interior node v. It is easy to verify that $(\Delta_\Gamma, D(\Delta_\Gamma))$ is a linear, unbounded, self-adjoint, dissipative operator on $L^2(\Gamma)$, i.e. $\mathrm{Re}(\Delta_\Gamma \mathbf{u}, \mathbf{u})_{L^2(\Gamma)} \leqslant 0$ for all $\mathbf{u} \in D(\Delta_\Gamma)$.

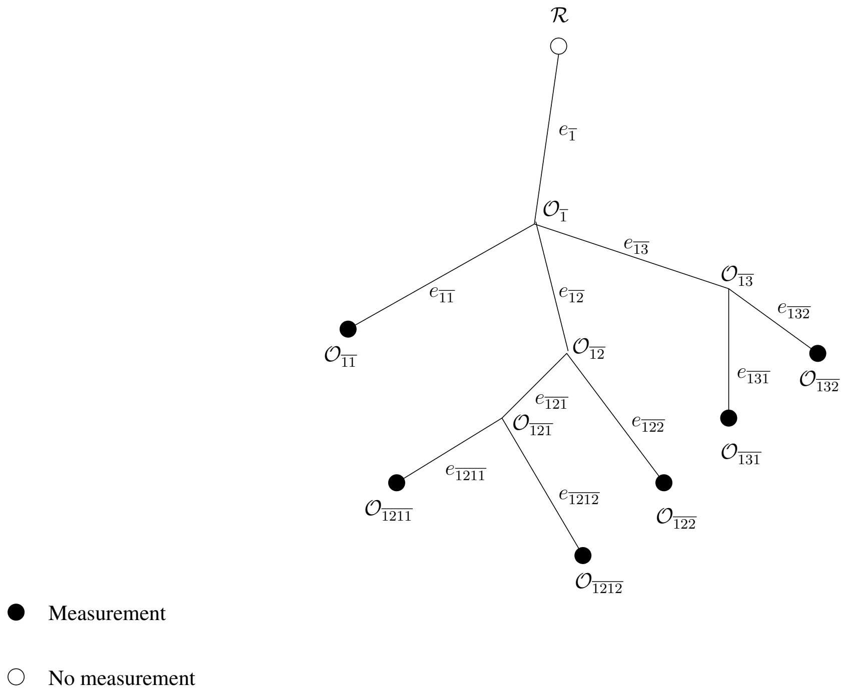

Figure 1. A tree with ten edges.

3. The heat equation

3.1. Preliminaries and notation

In this section, we introduce the notation for the elements of the considered tree. We mainly follow the notation of [17].

We first describe the procedure to index the edges and vertices of the tree. We first choose an exterior vertex, called the *root* of the tree and denoted by $\mathcal{R}$. The remaining edges and vertices will be denoted by $e_{\overline{\alpha}}$ and $\mathcal{O}_{\overline{\alpha}}$, respectively, where $\overline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ is a multi-index (taking value in $\{1\} \cup \bigcup_{k\geqslant 2} \mathbb{N}^k$). The multi-indices are defined by induction in the following way. For the edge containing the root $\mathcal{R}$, we choose the index 1. This edge is denoted by $e_{\overline{1}}$ and its second end is denoted by $\mathcal{O}_{\overline{1}}$. Assume now that the interior vertex $\mathcal{O}_{\overline{\alpha}}$, which is the end of the edge $e_{\overline{\alpha}}$, has multiplicity equal to $m_{\overline{\alpha}} + 1$. The $m_{\overline{\alpha}}$ edges, different from $e_{\overline{\alpha}}$, that branch out from $\mathcal{O}_{\overline{\alpha}}$ are denoted by $e_{\overline{\alpha\beta}}$ with $\beta \in \{1, \ldots, m_{\overline{\alpha}}\}$. (See figure 1.)

Let now $\mathcal{I}$ be the set of the interior vertices of Γ and $\mathcal{E}$ be the set of the exterior vertices of Γ, with $\mathcal{R}$ being excepted. We denote by

$$I_{\mathcal{I}} = \{\overline{\alpha}, \mathcal{O}_{\overline{\alpha}} \in \mathcal{I}\}, \qquad I_{\mathcal{E}} = \{\overline{\alpha}, \mathcal{O}_{\overline{\alpha}} \in \mathcal{E}\}$$

the sets of the indices for the interior and exterior vertices (except the root $\mathcal{R}$). With this notation, $I = I_{\mathcal{I}} \cup I_{\mathcal{E}}$ is the set of the indices of all the vertices except the root $\mathcal{R}$.

The length of the edge $e_{\overline{\alpha}}$ will be denoted by $l_{\overline{\alpha}}$. Each $e_{\overline{\alpha}}$ is parametrized by the interval $[0, l_{\overline{\alpha}}]$ so that the end $\mathcal{O}_{\overline{\alpha}}$ of $e_{\overline{\alpha}}$ corresponds to $x = l_{\overline{\alpha}}$, while the origin of $e_{\overline{\alpha}}$ corresponds to $x = 0$.

3.2. Carleman estimate for the heat equation

In this section, we derive a Carleman estimate for the heat equation on a tree. The following properties for a function $\mathbf{u} = \{u^{\overline{\alpha}}\}_{\overline{\alpha}\in I} : \Gamma \to \mathbb{R}$ will be relevant for our work.

(C1) The continuity condition at the internal vertices: $u^{\overline{\alpha}}(l_{\overline{\alpha}}) = u^{\overline{\alpha\beta}}(0)$ for all $\overline{\alpha} \in I_{\mathcal{I}}$ and $\beta \in [[1, m_{\overline{\alpha}}]]$.

(C2) The flux condition at the internal vertices: $u_x^{\overline{\alpha}}(l_{\overline{\alpha}}) = \sum_{\beta=1}^{m_{\overline{\alpha}}} u_x^{\overline{\alpha\beta}}(0)$ for all $\overline{\alpha} \in I_{\mathcal{I}}$.

(C3) The vanishing condition at the root $\mathcal{R}$ and at the external vertices: $\mathbf{u}(v) = 0$ for all $v \in \{\mathcal{R}\} \cup \mathcal{E}$.

Throughout the paper, we shall use the notation $[[1, m]] := [1, m] \cap \mathbb{N} = \{1, ..., m\}$.

We introduce the set

$$\mathcal{Z} = \{\mathbf{u} = \{u^{\overline{\alpha}}\}_{\overline{\alpha}\in I} : \Gamma \times [0, T] \to \mathbb{R};\ u^{\overline{\alpha}} \in C^{2,1}([0, l_{\overline{\alpha}}][0, T]), \mathbf{u}(\cdot, t) \text{ satisfies}(\text{C1}) - (\text{C3})\}.$$

Note that $\mathbf{u}(\cdot, t) \in D(\Delta_\Gamma)$ for $\mathbf{u} \in \mathcal{Z}$ and $t \in [0, T]$. The aim of this section is to define a continuous weight function $\boldsymbol{\psi} = \{\psi^{\overline{\alpha}}\}_{\overline{\alpha}\in I} : \Gamma \to (0, \infty)$ and a constant $C_{\boldsymbol{\psi}} > 0$ such that if we set

$$\boldsymbol{\theta}(x, t) = \frac{\mathrm{e}^{\lambda\boldsymbol{\psi}(x)}}{t(T - t)}, \qquad \boldsymbol{\varphi}(x, t) = \frac{\mathrm{e}^{\lambda C_{\boldsymbol{\psi}}} - \mathrm{e}^{\lambda\boldsymbol{\psi}(x)}}{t(T - t)}, \qquad x \in \Gamma, \qquad t \in (0, T),$$

we have the following Carleman estimate.

Proposition 3.1. *There exist a continuous function* $\boldsymbol{\psi} : \Gamma \to (0, +\infty)$ *and some positive constants* λ_0, s_0, C *such that for all* $\lambda \geqslant \lambda_0$, $s \geqslant s_0$ *and* $q \in \mathcal{Z}$, *the following holds:*

$$\begin{aligned}\int_0^T \int_\Gamma \big((s\boldsymbol{\theta})^{-1}(|\mathbf{q}_t|^2 + |\Delta_\Gamma \mathbf{q}|^2) + \lambda^2(s\boldsymbol{\theta})|\mathbf{q}_x|^2 + \lambda^4(s\boldsymbol{\theta})^3|\mathbf{q}|^2\big)\, \mathrm{e}^{-2s\boldsymbol{\varphi}}\, \mathrm{d}x\, \mathrm{d}t \\ + \int_0^T \lambda(s\boldsymbol{\theta})(|\mathbf{q}_x|^2 \mathrm{e}^{-2s\boldsymbol{\varphi}})(\mathcal{R}, t)\, \mathrm{d}t \\ \leqslant C\left(\int_0^T \int_\Gamma |\mathbf{q}_t + \Delta_\Gamma \mathbf{q}|^2 \mathrm{e}^{-2s\boldsymbol{\varphi}}\, \mathrm{d}x\, \mathrm{d}t + \sum_{v\in\mathcal{E}} \int_0^T \lambda(s\boldsymbol{\theta})(|\mathbf{q}_x|^2 \mathrm{e}^{-2s\boldsymbol{\varphi}})(v, t)\, \mathrm{d}t\right).\end{aligned} \tag{3.1}$$

In the above proposition, we have used the following notation: $|\mathbf{q}|^2 = \{|q^{\overline{\alpha}}|^2\}_{\overline{\alpha}\in I}$, $|\mathbf{q}_t|^2 = \{|q_t^{\overline{\alpha}}|^2\}_{\overline{\alpha}\in I}$, $|\mathbf{q}_x|^2 = \{|q_x^{\overline{\alpha}}|^2\}_{\overline{\alpha}\in I}$, etc, and

$$\int_\Gamma \mathbf{u}\, \mathrm{d}x = \sum_{\overline{\alpha}\in I} \int_{l_{\overline{\alpha}}} u^{\overline{\alpha}}\, \mathrm{d}x.$$

In what follows, $\varphi_t = \partial\varphi/\partial t$, $\varphi_x = \partial\varphi/\partial x$, $\varphi_{nx} = \partial^n\varphi/\partial x^n$ for $n \geqslant 3$, etc.

Remark 3.2.

(1) The same inequality holds for the operator $\partial_t - \Delta_\Gamma$ instead of $\partial_t + \Delta_\Gamma$ just by changing t into $T - t$.

(2) In the definition of $\mathcal{Z}$, we can replace $C^{2,1}$ by $H^{2,1}$ as well.

(3) We note that the result is false with only $N-2$ measurements for a star-shaped tree with N edges of length 1. Indeed, a nontrivial solution of the heat equation that vanishes on $N-2$

edges, at the internal vertex and at all the external vertices, does exist. Indeed, consider the function $\mathbf{q} = \{q^{\overline{\alpha}}\}_{\overline{\alpha}\in I}$, where

$$
\begin{aligned}
&q^{\overline{1}} \equiv 0,\\
&q^{\overline{11}}(x,t) = \mathrm{e}^{-\pi^2 t}\sin(\pi x), \qquad x\in(0,1), \qquad t\geqslant 0,\\
&q^{\overline{12}}(x,t) = -\mathrm{e}^{-\pi^2 t}\sin(\pi x), \qquad x\in(0,1), \qquad t\geqslant 0,\\
&q^{\overline{1j}} \equiv 0, \qquad 3\leqslant j\leqslant N-1.
\end{aligned}
$$

Then, $\mathbf{q}$ is a nontrivial solution of the heat equation on the star-shaped tree with null measurements at the external nodes $\mathcal{R}$ and $\mathcal{O}_{\overline{1j}}$ for $3\leqslant j\leqslant N-1$.

Proof. Let us consider the operator $P = \partial_t + \Delta_\Gamma$. Set $\mathbf{u} = \mathrm{e}^{-s\varphi}\mathbf{q}$ and $\mathbf{w} = \mathrm{e}^{-s\varphi}P(\mathrm{e}^{s\varphi}\mathbf{u})$. Following [35], we obtain

$$\mathbf{w} = M\mathbf{u} = \mathbf{u}_t + s\varphi_t\mathbf{u} + (\Delta_\Gamma\mathbf{u} + 2s\varphi_x\mathbf{u}_x + s(\Delta_\Gamma\boldsymbol{\varphi})\mathbf{u} + s^2|\varphi_x|^2\mathbf{u}) = M_1\mathbf{u} + M_2\mathbf{u},$$

where

$$M_1\mathbf{u} = \Delta_\Gamma\mathbf{u} + s\varphi_t\mathbf{u} + s^2|\varphi_x|^2\mathbf{u} \tag{3.2}$$

and

$$M_2\mathbf{u} = \mathbf{u}_t + 2s\varphi_x\mathbf{u}_x + s(\Delta_\Gamma\boldsymbol{\varphi})\mathbf{u} \tag{3.3}$$

are the (formal) self-adjoint and skew-adjoint parts of M, respectively. Then,

$$\|\mathbf{w}\|^2 = \|M_1\mathbf{u} + M_2\mathbf{u}\|^2 = \|M_1\mathbf{u}\|^2 + \|M_2\mathbf{u}\|^2 + 2(M_1\mathbf{u}, M_2\mathbf{u}),$$

where $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and the inner product of $L^2(\Gamma\times(0,T))$, respectively.

Step 1. *Exact computation of* $(M_1\mathbf{u}, M_2\mathbf{u})$.

Recall that

$$(M_1\mathbf{u}, M_2\mathbf{u}) = \sum_{\overline{\alpha}\in I}\int_0^T\int_0^{l_{\overline{\alpha}}}(M_1\mathbf{u})^{\overline{\alpha}}(M_2\mathbf{u})^{\overline{\alpha}}\,\mathrm{d}x\,\mathrm{d}t.$$

We compute the integral term on the rhs of the above identity only for one (arbitrary) edge $e_{\overline{\alpha}}$ that we denote by e for simplicity. We assume that e is parametrized by $x\in[0,l]$. Also, where there is no confusion, we use the symbols $\int\int$ and $\int$ to denote $\int_0^T\int_0^l$ and $\int_0^T$, respectively.

We write

$$\int_0^T\int_0^l (M_1u)(M_2u)\,\mathrm{d}x\,\mathrm{d}t = I_1 + I_2 + I_3 + I_4$$

with

$$
\begin{aligned}
I_1 &= \int_0^T\int_0^l u_{xx}u_t,\\
I_2 &= \int_0^T\int_0^l u_{xx}(2s\varphi_x u_x + s\varphi_{xx}u),\\
I_3 &= \int_0^T\int_0^l (s\varphi_t u + s^2\varphi_x^2 u)(u_t + 2s\varphi_x u_x),\\
I_4 &= \int_0^T\int_0^l (s\varphi_t u + s^2\varphi_x^2 u)(s\varphi_{xx}u).
\end{aligned}
$$

For I_1, we have that

$$I_1 = -\int\int u_x u_{xt} + \int u_x u_t\Big|_0^l = \int u_x u_t\Big|_0^l.$$

We write the second term as $I_2 = I_2^1 + I_2^2$, where

$$I_2^1 = 2s\int\int u_{xx}\varphi_x u_x \qquad \text{and} \qquad I_2^2 = s\int\int u_{xx}\varphi_{xx}u.$$

Thus,

$$I_2^1 = -s\int\int \varphi_{xx}|u_x|^2 + s\int \varphi_x|u_x|^2\Big|_0^l$$

and

$$\begin{aligned} I_2^2 &= -s\int\int u_x(\varphi_{3x}u + \varphi_{xx}u_x) + s\int u_x\varphi_{xx}u\Big|_0^l \\ &= \frac{s}{2}\int\int \varphi_{4x}u^2 - s\int \varphi_{3x}\frac{u^2}{2}\Big|_0^l - s\int\int \varphi_{xx}|u_x|^2 + s\int \varphi_{xx}uu_x\Big|_0^l. \end{aligned}$$

I_3 is decomposed as $I_3 = I_3^1 + I_3^2$, where

$$\begin{aligned} I_3^1 &= \int\int (s\varphi_t u + s^2|\varphi_x|^2 u)u_t, \\ I_3^2 &= \int\int (s\varphi_t u + s^2|\varphi_x|^2 u)(2s\varphi_x u_x). \end{aligned}$$

Then,

$$I_3^1 = -\int\int (s\varphi_{tt} + 2s^2\varphi_x\varphi_{xt})\frac{|u|^2}{2},$$

$$\begin{aligned} I_3^2 &= \int\int s\varphi_x(s\varphi_t + s^2|\varphi_x|^2)\partial_x(|u|^2) \\ &= -s^2\int\int \big((\varphi_x\varphi_t)_x + s\big(\varphi_x^3\big)_x\big)|u|^2 + \int s\varphi_x(s\varphi_t + s^2|\varphi_x|^2)|u|^2\Big|_0^l. \end{aligned}$$

Finally,

$$I_4 = \int\int (s^2\varphi_t + s^3|\varphi_x|^2)\varphi_{xx}|u|^2.$$

We conclude that for the edge e,

$$\begin{aligned} \int_0^T\int_0^l M_1uM_2u\,\mathrm{d}x\,\mathrm{d}t &= -2s\int_0^T\int_0^l \varphi_{xx}|u_x|^2 \\ &+ \int_0^T\int_0^l |u|^2\left[\frac{s}{2}(\varphi_{4x} - \varphi_{tt}) - s^2(|\varphi_x|^2)_t - s^3\varphi_x(|\varphi_x|^2)_x\right] \\ &+ \int_0^T\left[u_xu_t + s\varphi_{xx}uu_x + s|u_x|^2\varphi_x + |u|^2\left(-\frac{s}{2}\varphi_{3x} + s^2\varphi_x\varphi_t + s^3(\varphi_x)^3\right)\right]\Bigg|_0^l. \end{aligned} \tag{3.4}$$

Summing now the above identity over all the edges $\{e_{\overline{\alpha}}\}_{\overline{\alpha}\in I}$, we obtain the exact expression of the scalar product $(M_1\mathbf{u}, M_2\mathbf{u})$:

$$\begin{aligned} (M_1\mathbf{u}, M_2\mathbf{u}) &= -2s\int_0^T\int_\Gamma (\Delta_\Gamma\boldsymbol{\varphi})|\mathbf{u}_x|^2 \\ &+ \int_0^T\int_\Gamma |\mathbf{u}|^2\left[\frac{s}{2}(\boldsymbol{\varphi}_{4x} - \boldsymbol{\varphi}_{tt}) - s^2(|\boldsymbol{\varphi}_x|^2)_t - s^3\boldsymbol{\varphi}_x(|\boldsymbol{\varphi}_x|^2)_x\right] \\ &+ \sum_{\overline{\alpha}\in I}\int_0^T\Big[u_x^{\overline{\alpha}}u_t^{\overline{\alpha}} + s\varphi_{xx}^{\overline{\alpha}}u^{\overline{\alpha}}u_x^{\overline{\alpha}} + s|u_x^{\overline{\alpha}}|^2\varphi_x^{\overline{\alpha}} + |u^{\overline{\alpha}}|^2\Big(-\frac{s}{2}\varphi_{3x}^{\overline{\alpha}} + s^2\varphi_x^{\overline{\alpha}}\varphi_t^{\overline{\alpha}} \\ &+ s^3(\varphi_x^{\overline{\alpha}})^3\Big)\Big]\Bigg|_0^{l_{\overline{\alpha}}}. \end{aligned} \tag{3.5}$$

Step 2. *Terms in the inner product related to the internal nodes.*
Let us consider an internal node $\mathcal{O}_{\overline{\alpha}}$. Using our previous notation, its parent edge is $e_{\overline{\alpha}}$ and its children edges are denoted by $e_{\overline{\alpha\beta}}$ with $\beta \in [[1, m_{\overline{\alpha}}]]$. Let us denote by $X^{\overline{\alpha}}$ the sum of the boundary terms involving this internal node $\mathcal{O}_{\overline{\alpha}}$ on the rhs of (3.5). Thus,

$$X^{\overline{\alpha}} = \int_0^T \left[u_x^{\overline{\alpha}} u_t^{\overline{\alpha}} + s\varphi_{xx}^{\overline{\alpha}} u^{\overline{\alpha}} u_x^{\overline{\alpha}} + s\left|u_x^{\overline{\alpha}}\right|^2 \varphi_x^{\overline{\alpha}} + \left|u^{\overline{\alpha}}\right|^2 \Big(- \frac{s}{2}\varphi_{3x}^{\overline{\alpha}} + s^2 \varphi_x^{\overline{\alpha}} \varphi_t^{\overline{\alpha}} + s^3 \big(\varphi_x^{\overline{\alpha}}\big)^3 \Big)\right] (l_{\overline{\alpha}}, t)\, \mathrm{d}t$$
$$- \int_0^T \sum_{\beta \in [[1, m_{\overline{\alpha}}]]} \left[u_x^{\overline{\alpha\beta}} u_t^{\overline{\alpha\beta}} + s\varphi_{xx}^{\overline{\alpha\beta}} u^{\overline{\alpha\beta}} u_x^{\overline{\alpha\beta}} + s\left|u_x^{\overline{\alpha\beta}}\right|^2 \varphi_x^{\overline{\alpha\beta}} \right.$$
$$\left. + \left|u^{\overline{\alpha\beta}}\right|^2 \left(- \frac{s}{2}\varphi_{3x}^{\overline{\alpha\beta}} + s^2 \varphi_x^{\overline{\alpha\beta}} \varphi_t^{\overline{\alpha\beta}} + s^3 \big(\varphi_x^{\overline{\alpha\beta}}\big)^3 \right)\right](0, t)\, \mathrm{d}t.$$

Moreover, in (3.5), we also have contributions from the exterior nodes in $\mathcal{E}$ and from the root $\mathcal{R}$. These contributions are given by

$$Y = -s \int_0^T |u_x^{\overline{1}}|^2 \varphi_x^{\overline{1}}(0, t)\, \mathrm{d}t + s \sum_{\overline{\alpha} \in I_{\mathcal{E}}} \int_0^T |u_x^{\overline{\alpha}}|^2 \varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}}, t)\, \mathrm{d}t. \tag{3.6}$$

Let us now define the weight function $\boldsymbol{\psi} = \{\psi^{\overline{\alpha}}\}_{\overline{\alpha} \in I}$ on the tree as follows. The components $\psi^{\overline{\alpha}} : [0, l_{\overline{\alpha}}] \to \mathbb{R}$ are chosen in such a way that $\psi^{\overline{\alpha}} \in C^\infty([0, l_{\overline{\alpha}}])$ and

(B1) $|\psi_x^{\overline{\alpha}}(x)|^2 + \psi_{xx}^{\overline{\alpha}}(x) \geqslant 0$ on $[0, l_{\overline{\alpha}}]$,
(B2) $\psi_x^{\overline{\alpha}} > 0$ on $[0, l_{\overline{\alpha}}]$,
(B3) $\frac{3}{4} C_{\boldsymbol{\psi}} \geqslant \psi^{\overline{\alpha}} > \frac{2}{3} C_{\boldsymbol{\psi}}$ on $[0, l_{\overline{\alpha}}]$ for some positive constant $C_{\boldsymbol{\psi}}$,
(B4) $|\psi_{xx}^{\overline{\alpha}}| \leqslant K \psi_x^{\overline{\alpha}}$ on $[0, l_{\overline{\alpha}}]$ for some positive constant K,
(B5) $\psi^{\overline{\alpha}}(l_{\overline{\alpha}}) = \psi^{\overline{\alpha\beta}}(0)$ for all $\overline{\alpha} \in I_{\mathcal{I}}$, $\beta \in [[1, m_{\overline{\alpha}}]]$,
(B6) $\psi_x^{\overline{\alpha\beta}}(0) - (m_{\overline{\alpha}} + 1)\psi_x^{\overline{\alpha}}(l_{\overline{\alpha}}) > 0$ for all $\overline{\alpha} \in I_{\mathcal{I}}$, $\beta \in [[1, m_{\overline{\alpha}}]]$,
(B7) $\sum_{\beta \in [[1, m_{\overline{\alpha}}]]} (\psi_x^{\overline{\alpha\beta}}(0))^3 - (\psi_x^{\overline{\alpha}}(l_{\overline{\alpha}}))^3 - (m_{\overline{\alpha}} + 1)\psi_x^{\overline{\alpha}}(l_{\overline{\alpha}}) \Big|\psi_x^{\overline{\alpha}}(l_{\overline{\alpha}}) - \sum_{\beta \in [[1, m_{\overline{\alpha}}]]} \psi_x^{\overline{\alpha\beta}}(0)\Big|^2 > 0$

for all $\overline{\alpha} \in I_{\mathcal{I}}$.

Finding a set of functions as above is easy. We can even take $\psi^{\overline{\alpha}}$ to be affine, $\psi^{\overline{\alpha}}(x) = a_{\overline{\alpha}} x + b_{\overline{\alpha}}$. The coefficients $a_{\overline{\alpha}}$ and $b_{\overline{\alpha}}$ are the positive numbers that satisfy

(P1) $\frac{3}{4} C_\psi \geqslant a_{\overline{\alpha}} l_{\overline{\alpha}} + b_{\overline{\alpha}} > b_{\overline{\alpha}} > \frac{2}{3} C_\psi$ for all $\overline{\alpha} \in I$,
(P2) $a_{\overline{\alpha}} l_{\overline{\alpha}} + b_{\overline{\alpha}} = b_{\overline{\alpha\beta}}$ for all $\overline{\alpha} \in I_{\mathcal{I}}$ and $\beta \in [[1, m_{\overline{\alpha}}]]$,
(P3) $a_{\overline{\alpha\beta}} - (m_{\overline{\alpha}} + 1) a_{\overline{\alpha}} > 0$ for all $\overline{\alpha} \in I_{\mathcal{I}}$ and $\beta \in [[1, m_{\overline{\alpha}}]]$,
(P4) $\sum_{\beta \in [[1, m_{\overline{\alpha}}]]} (a_{\overline{\alpha\beta}})^3 - (a_{\overline{\alpha}})^3 - (m_{\overline{\alpha}} + 1) a_{\overline{\alpha}} \Big| a_{\overline{\alpha}} - \sum_{\beta \in [[1, m_{\overline{\alpha}}]]} a_{\overline{\alpha\beta}} \Big|^2 > 0$ for all $\overline{\alpha} \in I_{\mathcal{I}}$.

Let us first deal with conditions (P2)–(P4). We define the constants corresponding to the edge $e_{\overline{1}}$ by $a_{\overline{1}} = 2$ and $b_{\overline{1}} = 1$. Assuming that we have already constructed $a_{\overline{\alpha}}$ and $b_{\overline{\alpha}}$ for some multi-index $\overline{\alpha}$, then $b_{\overline{\alpha\beta}}$ is given by (P2). Next, we have to find $a_{\overline{\alpha\beta}}$ large enough to satisfy (P3)–(P4). Let us choose $a_{\overline{\alpha\beta}} = r_{\overline{\alpha}} a_{\overline{\alpha}}$. Obviously, for large enough $r_{\overline{\alpha}}$, depending on $m_{\overline{\alpha}}$, conditions (P3) and (P4) are satisfied. Finally, assume that all the coefficients $a_{\overline{\alpha}}$ and $b_{\overline{\alpha}}$ have been defined to satisfy (P2)–(P4). Adding $\frac{2}{3} C_{\boldsymbol{\psi}}$ to all the $b_{\overline{\alpha\beta}}$, we see that (P1) is fulfilled for $C_{\boldsymbol{\psi}}$ large enough, while (P2)–(P4) still hold true.

Let us split $X^{\overline{\alpha}}$ into $X^{\overline{\alpha}} = X_1^{\overline{\alpha}} + X_2^{\overline{\alpha}} + X_3^{\overline{\alpha}} + X_4^{\overline{\alpha}}$, where

$$X_1^{\overline{\alpha}} := \int_0^T \Big[\big[u_x^{\overline{\alpha}} u_t^{\overline{\alpha}}\big](l_{\overline{\alpha}}, t) - \sum_{\beta \in [[1, m_{\overline{\alpha}}]]} \big[u_x^{\overline{\alpha\beta}} u_t^{\overline{\alpha\beta}}\big](0, t)\Big]\, \mathrm{d}t,$$

$$X_2^{\overline{\alpha}} := \int_0^T \Big[\big[s\varphi_{xx}^{\overline{\alpha}} u^{\overline{\alpha}} u_x^{\overline{\alpha}}\big](l_{\overline{\alpha}}, t) - \sum_{\beta \in [[1, m_{\overline{\alpha}}]]} \big[s\varphi_{xx}^{\overline{\alpha\beta}} u^{\overline{\alpha\beta}} u_x^{\overline{\alpha\beta}}\big](0, t)\Big]\, \mathrm{d}t,$$

$$X_3^{\overline{\alpha}} := \int_0^T \Big[\big[s|u_x^{\overline{\alpha}}|^2 \varphi_x^{\overline{\alpha}}\big](l_{\overline{\alpha}}, t) - \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} \big[s|u_x^{\overline{\alpha\beta}}|^2 \varphi_x^{\overline{\alpha\beta}}\big](0,t)\Big]\, \mathrm{d}t,$$

$$X_4^{\overline{\alpha}} := \int_0^T \Big\{ \Big[|u^{\overline{\alpha}}|^2 \Big(-\frac{s}{2}\varphi_{3x}^{\overline{\alpha}} + s^2 \varphi_x^{\overline{\alpha}}\varphi_t^{\overline{\alpha}} + s^3 (\varphi_x^{\overline{\alpha}})^3 \Big)\Big](l_{\overline{\alpha}}, t) - \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} \Big[|u^{\overline{\alpha\beta}}|^2 \Big(-\frac{s}{2}\varphi_{3x}^{\overline{\alpha\beta}} + s^2 \varphi_x^{\overline{\alpha\beta}}\varphi_t^{\overline{\alpha\beta}} + s^3 (\varphi_x^{\overline{\alpha\beta}})^3 \Big)\Big](0,t)\Big\}\, \mathrm{d}t.$$

We now estimate each term $X_i^{\overline{\alpha}}$, $i = 1, ..., 4$. Using the definition of the function $\mathbf{u}$, we have for any index $\overline{\alpha} \in I$ the following identities:

$$u_x^{\overline{\alpha}} = \mathrm{e}^{-s\varphi^{\overline{\alpha}}}(-s\varphi_x^{\overline{\alpha}} q^{\overline{\alpha}} + q_x^{\overline{\alpha}}), \quad u_t^{\overline{\alpha}} = \mathrm{e}^{-s\varphi^{\overline{\alpha}}}(-s\varphi_t^{\overline{\alpha}} q^{\overline{\alpha}} + q_t^{\overline{\alpha}}).$$

Let us set $u(\mathcal{O}_{\overline{\alpha}}, t) = u^{\overline{\alpha}}(l_{\overline{\alpha}}, t) = u^{\overline{\alpha\beta}}(0,t)$ and $\varphi(\mathcal{O}_{\overline{\alpha}}, t) = \varphi^{\overline{\alpha}}(l_{\overline{\alpha}}, t) = \varphi^{\overline{\alpha\beta}}(0,t)$ for any $\overline{\alpha} \in I_{\mathcal{I}}$ and $\beta \in [[1, m_{\overline{\alpha}}]]$. With this notation, we have

$$\begin{aligned}
X_1^{\overline{\alpha}} &= \int_0^T \Big(u_x^{\overline{\alpha}} u_t^{\overline{\alpha}}(l_{\overline{\alpha}}, t) - \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} u_x^{\overline{\alpha\beta}} u_t^{\overline{\alpha\beta}}(0,t)\Big)\, \mathrm{d}t \\
&= \int_0^T u_t(\mathcal{O}_{\overline{\alpha}}, t)\, \mathrm{e}^{-s\varphi(\mathcal{O}_{\overline{\alpha}}, t)} \Big(-s\varphi_x^{\overline{\alpha}} q^{\overline{\alpha}}(l_{\overline{\alpha}}) + s \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} \varphi_x^{\overline{\alpha\beta}} q^{\overline{\alpha\beta}}(0) + q_x^{\overline{\alpha}}(l_{\overline{\alpha}}) \\
&\quad - \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} q_x^{\overline{\alpha\beta}}(0)\Big)\, \mathrm{d}t \\
&= \int_0^T u_t(\mathcal{O}_{\overline{\alpha}}, t)\, \mathrm{e}^{-s\varphi(\mathcal{O}_{\overline{\alpha}}, t)} \Big(-s\varphi_x^{\overline{\alpha}} q^{\overline{\alpha}}(l_{\overline{\alpha}}) + s \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} \varphi_x^{\overline{\alpha\beta}} q^{\overline{\alpha\beta}}(0)\Big)\, \mathrm{d}t \\
&= \int_0^T u_t(\mathcal{O}_{\overline{\alpha}}, t) u(\mathcal{O}_{\overline{\alpha}}, t) \Big(-s\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}}) + s \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} \varphi_x^{\overline{\alpha\beta}}(0)\Big)\, \mathrm{d}t \\
&= -\int_0^T \Big(-s\varphi_{xt}^{\overline{\alpha}}(l_{\overline{\alpha}}, t) + s \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} \varphi_{xt}^{\overline{\alpha\beta}}(0,t)\Big) \frac{|u(\mathcal{O}_{\overline{\alpha}}, t)|^2}{2}\, \mathrm{d}t. \qquad (3.7)
\end{aligned}$$

Let us estimate $X_2^{\overline{\alpha}}$. Using property (B4), we infer that

$$|\varphi_{xx}^{\overline{\alpha}}| = \frac{\lambda\, \mathrm{e}^{\lambda\psi^{\overline{\alpha}}}}{t(T-t)} |\lambda(\psi_x^{\overline{\alpha}})^2 + \psi_{xx}^{\overline{\alpha}}| \leqslant \frac{T^2}{4}|\varphi_x^{\overline{\alpha}}|^2 + K|\varphi_x^{\overline{\alpha}}|.$$

This gives that

$$\begin{aligned}
|X_2^{\overline{\alpha}}| &= \Big| \int_0^T \Big(s\varphi_{xx}^{\overline{\alpha}} u^{\overline{\alpha}} u_x^{\overline{\alpha}}(l_{\overline{\alpha}}, t) - s \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} \varphi_{xx}^{\overline{\alpha\beta}} u^{\overline{\alpha\beta}} u_x^{\overline{\alpha\beta}}(0,t)\Big)\, \mathrm{d}t \Big| \\
&\leqslant \frac{s^2}{2} \int_0^T |u^{\overline{\alpha}}|^2 \Big(\frac{T^2}{4}|\varphi_x^{\overline{\alpha}}| + K\Big)^2 |\varphi_x^{\overline{\alpha}}|(l_{\overline{\alpha}}, t)\, \mathrm{d}t + \frac{1}{2}\int_0^T |\varphi_x^{\overline{\alpha}}||u_x^{\overline{\alpha}}|^2(l_{\overline{\alpha}}, t)\, \mathrm{d}t \\
&\quad + \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} \Big\{ \frac{s^2}{2}\int_0^T |u^{\overline{\alpha\beta}}|^2 \Big(\frac{T^2}{4}|\varphi_x^{\overline{\alpha\beta}}| + K\Big)^2 |\varphi_x^{\overline{\alpha\beta}}|(0,t)\, \mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^T |\varphi_x^{\overline{\alpha\beta}}||u_x^{\overline{\alpha\beta}}|^2(0,t)\, \mathrm{d}t\Big\} \\
&= \frac{s^2}{2}\int_0^T |u(\mathcal{O}_{\overline{\alpha}}, t)|^2 \Big[\Big(\frac{T^2}{4}|\varphi_x^{\overline{\alpha}}| + K\Big)^2 |\varphi_x^{\overline{\alpha}}|(l_{\overline{\alpha}}, t)
\end{aligned}$$

$$+\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}\Big(\frac{T^2}{4}|\varphi_x^{\overline{\alpha\beta}}|+K\Big)^2|\varphi_x^{\overline{\alpha\beta}}|(0,t)\Big]\,dt$$

$$+\frac{1}{2}\int_0^T\Big[|\varphi_x^{\overline{\alpha}}||u_x^{\overline{\alpha}}|^2(l_{\overline{\alpha}},t)\,dt+\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}|\varphi_x^{\overline{\alpha\beta}}||u_x^{\overline{\alpha\beta}}|^2(0,t)\Big]\,dt. \tag{3.8}$$

To estimate the term $\int_0^T|\varphi_x^{\overline{\alpha}}||u_x^{\overline{\alpha}}|^2(l_{\overline{\alpha}},t)\,dt$, which occurs in (3.8) and in $X_3^{\overline{\alpha}}$, we note that

$$u_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)-\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}u_x^{\overline{\alpha\beta}}(0,t)=e^{-s\varphi(\mathcal{O}_{\overline{\alpha}},t)}(-s)\big(\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)-\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}\varphi_x^{\overline{\alpha\beta}}(0,t)\big)q(\mathcal{O}_{\overline{\alpha}},t).$$

Hence,

$$|u_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)|^2\leqslant(m_{\overline{\alpha}}+1)\Bigg(\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}|u_x^{\overline{\alpha\beta}}(0,t)|^2+s^2|u(\mathcal{O}_{\overline{\alpha}},t)|^2\Big|\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)-\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}\varphi_x^{\overline{\alpha\beta}}(0,t)\Big|^2\Bigg). \tag{3.9}$$

We infer that

$$\int_0^T|\varphi_x^{\overline{\alpha}}||u_x^{\overline{\alpha}}|^2(l_{\overline{\alpha}},t)\,dt\leqslant(m_{\overline{\alpha}}+1)\int_0^T|\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)|\Bigg(\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}|u_x^{\overline{\alpha\beta}}(0,t)|^2+s^2|u(\mathcal{O}_{\overline{\alpha}},t)|^2\times\Big|\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)-\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}\varphi_x^{\overline{\alpha\beta}}(0,t)\Big|^2\Bigg). \tag{3.10}$$

Combined to (3.8) and to the fact that $\varphi_x^{\overline{\alpha\beta}}(0,t)\leqslant 0$ by (B2), this yields

$$\begin{aligned}X_2^{\overline{\alpha}}+X_3^{\overline{\alpha}}\geqslant&-\frac{s^2}{2}\int_0^T|u(\mathcal{O}_{\overline{\alpha}},t)|^2\Big[\Big(\frac{T^2}{4}|\varphi_x^{\overline{\alpha}}|+K\Big)^2|\varphi_x^{\overline{\alpha}}|(l_{\overline{\alpha}},t)\\&+\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}\Big(\frac{T^2}{4}|\varphi_x^{\overline{\alpha\beta}}|+K\Big)^2|\varphi_x^{\overline{\alpha\beta}}|(0,t)\Big]\,dt\\&+\Big(s-\frac{1}{2}\Big)\int_0^T\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}|\varphi_x^{\overline{\alpha\beta}}||u_x^{\overline{\alpha\beta}}|^2(0,t)\,dt-\Big(s+\frac{1}{2}\Big)(m_{\overline{\alpha}}+1)\int_0^T|\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)|\\&\times\Bigg(\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}|u_x^{\overline{\alpha\beta}}(0,t)|^2+s^2|u(\mathcal{O}_{\overline{\alpha}},t)|^2\Big|\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)-\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}\varphi_x^{\overline{\alpha\beta}}(0,t)\Big|^2\Bigg).\end{aligned} \tag{3.11}$$

Using the definition of $X^{\overline{\alpha}}$ and estimates (3.7) and (3.11), we obtain that

$$X^{\overline{\alpha}}\geqslant Z_1^{\overline{\alpha}}+Z_2^{\overline{\alpha}},$$

where

$$\begin{aligned}Z_1^{\overline{\alpha}}=&-\int_0^T\Big(-s\varphi_{xt}^{\overline{\alpha}}(l_{\overline{\alpha}},t)+s\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}\varphi_{xt}^{\overline{\alpha\beta}}(0,t)\Big)\frac{|u(\mathcal{O}_{\overline{\alpha}},t)|^2}{2}\,dt\\&-\frac{s^2}{2}\int_0^T|u(\mathcal{O}_{\overline{\alpha}},t)|^2\Big[\Big(\frac{T^2}{4}|\varphi_x^{\overline{\alpha}}|+K\Big)^2|\varphi_x^{\overline{\alpha}}|(l_{\overline{\alpha}},t)\\&+\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}\Big(\frac{T^2}{4}|\varphi_x^{\overline{\alpha\beta}}|+K\Big)^2|\varphi_x^{\overline{\alpha\beta}}|(0,t)\Big]\,dt\end{aligned}$$

$$-\Big(s^3+\frac{1}{2}s^2\Big)(m_{\overline{\alpha}}+1)\int_0^T|u(\mathcal{O}_{\overline{\alpha}},t)|^2|\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)|\Big|\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)$$
$$-\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}\varphi_x^{\overline{\alpha\beta}}(0,t)\Big|^2\,\mathrm{d}t$$
$$+\int_0^T|u(\mathcal{O}_{\overline{\alpha}},t)|^2\Big[\Big(-\frac{s}{2}\varphi_{3x}^{\overline{\alpha}}+s^2\varphi_x^{\overline{\alpha}}\varphi_t^{\overline{\alpha}}+s^3(\varphi_x^{\overline{\alpha}})^3\Big)\Big](l_{\overline{\alpha}},t)\,\mathrm{d}t$$
$$-\int_0^T|u(\mathcal{O}_{\overline{\alpha}},t)|^2\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}\Big(-\frac{s}{2}\varphi_{3x}^{\overline{\alpha\beta}}+s^2\varphi_x^{\overline{\alpha\beta}}\varphi_t^{\overline{\alpha\beta}}+s^3(\varphi_x^{\overline{\alpha\beta}})^3\Big)(0,t)\,\mathrm{d}t$$

and

$$Z_2^{\overline{\alpha}}=\Big(s-\frac{1}{2}\Big)\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}\int_0^T\big[|\varphi_x^{\overline{\alpha\beta}}||u_x^{\overline{\alpha\beta}}|^2\big](0,t)\,\mathrm{d}t-\Big(s+\frac{1}{2}\Big)(m_{\overline{\alpha}}+1)$$
$$\times\int_0^T\Big[|\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)|\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}|u_x^{\overline{\alpha\beta}}(0,t)|^2\Big]\,\mathrm{d}t.$$

We note that $|\varphi_x^{\overline{\alpha}}|^3\geqslant c(\lambda\theta)^3$, while, with (B3),

$$|\varphi_x^{\overline{\alpha}}|+|\varphi_x^{\overline{\alpha}}|^2+|\varphi_{xt}^{\overline{\alpha}}|+|\varphi_x^{\overline{\alpha}}\varphi_t^{\overline{\alpha}}|+|\varphi_{3x}^{\overline{\alpha}}|\leqslant c(\lambda\theta)^3.$$

It follows that

$$Z_1^{\overline{\alpha}}=\int_0^T|u(\mathcal{O}_{\overline{\alpha}},t)|^2\Big[s^3\Big((\varphi_x^{\overline{\alpha}})^3(l_{\overline{\alpha}},t)$$
$$-\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}(\varphi_x^{\overline{\alpha\beta}})^3(0,t)-(m_{\overline{\alpha}}+1)|\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)|\Big|\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)$$
$$-\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}\varphi_x^{\overline{\alpha\beta}}(0,t)\Big|^2\Big)+\cdots\Big]$$
$$=\int_0^T|u(\mathcal{O}_{\overline{\alpha}},t)|^2\Big[(s\lambda\theta)^3\Big(-(\psi_x^{\overline{\alpha}})^3(l_{\overline{\alpha}})+\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}(\psi_x^{\overline{\alpha\beta}})^3(0)$$
$$-(m_{\overline{\alpha}}+1)|\psi_x^{\overline{\alpha}}(l_{\overline{\alpha}})|\Big|\psi_x^{\overline{\alpha}}(l_{\overline{\alpha}})-\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}\psi_x^{\overline{\alpha\beta}}(0)\Big|^2\Big)+O(s^2\lambda^3\theta^3)\Big]$$

and

$$Z_2^{\overline{\alpha}}=\int_0^T\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}|u_x^{\overline{\alpha\beta}}(0,t)|^2\Big[s\lambda\theta\Big(\psi_x^{\overline{\alpha\beta}}(0)-(m_{\overline{\alpha}}+1)|\psi_x^{\overline{\alpha}}(l_{\overline{\alpha}})|\Big)+O(\lambda\theta)\Big]\,\mathrm{d}t.$$

Looking at the coefficient of s^3 in $Z_1^{\overline{\alpha}}$ and of s in $Z_2^{\overline{\alpha}}$ and using (B6), (B7) and (3.10), we obtain that for $s\geqslant s_0$ and $\lambda\geqslant\lambda_0$ (with $s_0$andλ_0 large enough)

$$X^{\overline{\alpha}}\geqslant Z_1^{\overline{\alpha}}+Z_2^{\overline{\alpha}}$$
$$\geqslant C\int_0^T s^3\lambda^3\theta^3|u(\mathcal{O}_{\overline{\alpha}},t)|^2\,\mathrm{d}t$$
$$+C\int_0^T s\lambda\theta\Bigg(|u_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)|^2+\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}|u_x^{\overline{\alpha\beta}}(0,t)|^2\Bigg)\,\mathrm{d}t. \qquad (3.12)$$

In particular, $X^{\overline{\alpha}}>0$.

Step 3. *Estimation of the integrals along the edges.*

We need the following lemma.

Lemma 3.3 ([35, claim 1]). *There exist $\lambda_1 \geqslant \lambda_0$, $s_1 \geqslant s_0$ and $A > 0$ such that for all $\lambda \geqslant \lambda_1$, $s \geqslant s_1$, the following holds:*

$$\int_0^T \int_\Gamma |\mathbf{u}|^2 \left[\frac{s}{2}(\varphi_{4x} - \varphi_{tt}) - s^2(|\varphi_x|^2)_t - s^3\varphi_x(|\varphi_x|^2)_x\right] \mathrm{d}x\, \mathrm{d}t \geqslant A\lambda s^3 \int_0^T \int_\Gamma |\mathbf{u}|^2|\varphi_x|^3 \,\mathrm{d}x\, \mathrm{d}t. \tag{3.13}$$

As the proof of [35, claim 1] does not involve any integration by parts in x, it is still valid in our context.

The following lemma is inspired by [35, claim 2].

Lemma 3.4. *There exist $s_2 \geqslant s_1$, $\lambda_2 \geqslant \lambda_1$, and a positive constant C such that for all $\lambda \geqslant \lambda_2$ and $s \geqslant s_2$*

$$\begin{aligned}
&\lambda s \int_0^T \int_\Gamma |\varphi_x||\mathbf{u}_x|^2 + \lambda s^{-1} \int_0^T \int_\Gamma |\varphi_x|^{-1}|\Delta_\Gamma \mathbf{u}|^2 \\
&\quad\leqslant C\Bigg(s^{-1}\|M_1\mathbf{u}\|^2 + \lambda s^3 \int_0^T \int_\Gamma |\varphi_x|^3|\mathbf{u}|^2 + \lambda s \int_0^T \sum_{\overline{\alpha}\in I} |\varphi_x^{\overline{\alpha}} u^{\overline{\alpha}} u_x^{\overline{\alpha}}|(0) + |\varphi_x^{\overline{\alpha}} u^{\overline{\alpha}} u_x^{\overline{\alpha}}|(l_{\overline{\alpha}}) \\
&\quad + \lambda s \int_0^T \sum_{\overline{\alpha}\in I} |(|\varphi_x^{\overline{\alpha}}|)_x||u^{\overline{\alpha}}|^2(0) + |(|\varphi_x^{\overline{\alpha}}|)_x||u^{\overline{\alpha}}|^2(l_{\overline{\alpha}})\Bigg).
\end{aligned} \tag{3.14}$$

Proof. Let us consider any edge $e_{\overline{\alpha}}$. To simplify the writing, we remove the index $\overline{\alpha}$ in our computations. Using the definition of $M_1\mathbf{u}$ (see (3.2)), we have that

$$\begin{aligned}
s^{-1} \int_0^T \int_0^l |\varphi_x|^{-1}|u_{xx}|^2 &= s^{-1} \int_0^T \int_0^l |\varphi_x|^{-1}|M_1 u - s\varphi_t u - s^2\varphi_x^2 u|^2 \\
&\leqslant Cs^{-1} \int_0^T \int_0^l |\varphi_x|^{-1}[|M_1 u|^2 + s^2|\varphi_t|^2 u^2 + s^4|\varphi_x|^4 u^2].
\end{aligned}$$

Using property (B3), we obtain that $|\varphi_t| \leqslant C|\varphi_x|^2$. Therefore, we have for some constant $A > 0$

$$s^{-1} \int_0^T \int_0^l |\varphi_x|^{-1}|u_{xx}|^2 \leqslant A\left(\frac{\|M_1 u\|^2}{\lambda s} + s^3 \int_0^T \int_0^l |\varphi_x|^3 u^2\right). \tag{3.15}$$

The first term on the lhs of (3.14) satisfies

$$\begin{aligned}
\lambda s \int\int |\varphi_x||u_x|^2 &= \lambda s\left(\int\int |\varphi_x|(-u_{xx})u - \int\int (|\varphi_x|)_x u_x u + \int_0^T |\varphi_x| u u_x\Big|_0^l\right) \\
&\leqslant \frac{\lambda}{2s}\int\int |\varphi_x|^{-1}|u_{xx}|^2 + \frac{\lambda s^3}{2}\int\int |\varphi_x|^3|u|^2 + \frac{\lambda s}{2}\int\int (|\varphi_x|)_{xx}|u|^2 \\
&\quad + \lambda s \int_0^T \left(|\varphi_x| u u_x - (|\varphi_x|)_x \frac{|u|^2}{2}\right)\Big|_0^l \leqslant \frac{\lambda}{2}A\left(\frac{\|M_1 u\|^2}{\lambda s} + s^3 \int\int |\varphi_x|^3 u^2\right) \\
&\quad + \frac{\lambda s^3}{2}\int\int |\varphi_x|^3|u|^2 + \frac{\lambda s}{2}\int\int (|\varphi_x|)_{xx}|u|^2 \\
&\quad + \lambda s \int_0^T \left(|\varphi_x| u u_x - (|\varphi_x|)_x \frac{|u|^2}{2}\right)\Big|_0^l
\end{aligned}$$

$$\leqslant \frac{A}{2}\left(s^{-1}\|M_1u\|^2 + s^3\lambda \int\int |\varphi_x|^3 u^2\right) + \frac{\lambda s^3}{2}\int\int |\varphi_x|^3|u|^2$$
$$+ \frac{\lambda s}{2}\int\int (|\varphi_x|)_{xx}|u|^2 + \lambda s\int_0^T \left(|\varphi_x||u||u_x| + |(|\varphi_x|)_x|\frac{|u|^2}{2}\right)(0)$$
$$+ \lambda s\int_0^T \left(|\varphi_x||u||u_x| + |(|\varphi_x|)_x|\frac{|u|^2}{2}\right)(l). \tag{3.16}$$

The claim follows by summing (3.15) and (3.16) over all the edges. □

Step 4. *Conclusion.* □

By (3.5), (3.13) and (B1), we obtain for $\lambda \geqslant 1$

$$\|\mathbf{w}\|^2 = \|M_1\mathbf{u} + M_2\mathbf{u}\|^2$$
$$= \|M_1\mathbf{u}\|^2 + \|M_2\mathbf{u}\|^2 + 2(M_1\mathbf{u}, M_2\mathbf{u})$$
$$= \|M_1\mathbf{u}\|^2 + \|M_2\mathbf{u}\|^2 + 2\Big\{\sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}} + Y - 2s\int_0^T\int_\Gamma \boldsymbol{\varphi}_{xx}|\mathbf{u}_x|^2$$
$$+ \int_0^T\int_\Gamma |\mathbf{u}|^2\left[\frac{s}{2}(\boldsymbol{\varphi}_{4x} - \boldsymbol{\varphi}_{tt}) - s^2(|\boldsymbol{\varphi}_x^2|)_t - s^3\boldsymbol{\varphi}_x(|\boldsymbol{\varphi}_x|^2)_x\right]\Big\}$$
$$\geqslant \|M_1\mathbf{u}\|^2 + \|M_2\mathbf{u}\|^2 + 2\Big\{\sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}} + Y$$
$$+ 2s\int_0^T\int_\Gamma (\lambda^2\boldsymbol{\psi}_x^2 + \lambda\boldsymbol{\psi}_{xx})\theta|\mathbf{u}_x|^2 + A\lambda s^3\int_0^T\int_\Gamma |\mathbf{u}|^2|\boldsymbol{\varphi}_x|^3\Big\}$$
$$\geqslant \|M_1\mathbf{u}\|^2 + \|M_2\mathbf{u}\|^2 + \sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}} + Y + A\lambda s^3\int_0^T\int_\Gamma |\mathbf{u}|^2|\boldsymbol{\varphi}_x|^3. \tag{3.17}$$

Multiplying (3.14) by $A/2C$ and adding it to (3.17), we obtain

$$\|M_2\mathbf{u}\|^2 + \|M_1\mathbf{u}\|^2\left(1 - \frac{A}{2s}\right) + \frac{A\lambda s^3}{2}\int\int |\mathbf{u}|^2|\boldsymbol{\varphi}_x|^3 + \frac{A\lambda s}{2C}\int\int |\boldsymbol{\varphi}_x||\mathbf{u}_x|^2$$
$$+ \frac{A\lambda}{2sC}\int\int |\boldsymbol{\varphi}_x|^{-1}|\Delta_\Gamma\mathbf{u}|^2 + \sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}} + Y \leqslant \|\mathbf{w}\|^2 + \frac{A}{2}B, \tag{3.18}$$

where

$$B = B_1 + B_2$$
$$= \lambda s\int_0^T \sum_{\overline{\alpha}\in I} |\varphi_x^{\overline{\alpha}} u^{\overline{\alpha}} u_x^{\overline{\alpha}}|(0) + |\varphi_x^{\overline{\alpha}} u^{\overline{\alpha}} u_x^{\overline{\alpha}}|(l_{\overline{\alpha}})$$
$$+ \lambda s\int_0^T \sum_{\overline{\alpha}\in I} |(|\varphi_x^{\overline{\alpha}}|)_x||u^{\overline{\alpha}}|^2(0) + |(|\varphi_x^{\overline{\alpha}}|)_x||u^{\overline{\alpha}}|^2(l_{\overline{\alpha}})\Bigg).$$

We now prove that for s large enough, the term B is small compared to $\sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}}$ so that B can be absorbed by the lhs of (3.18). Using (3.12) and the fact that $\mathbf{u}$ vanishes at the vertices of $\mathcal{E}\cup\mathcal{R} = \partial\Gamma$, we see that

$$B_2 \leqslant C\lambda^3 s\int_0^T \sum_{\overline{\alpha}\in I} \theta|u(\mathcal{O}_{\overline{\alpha}}, t)|^2\mathrm{d}t = C\lambda^3 s\int_0^T \sum_{\overline{\alpha}\in I_{\mathcal{I}}} \theta|u(\mathcal{O}_{\overline{\alpha}}, t)|^2\mathrm{d}t \leqslant \frac{C}{s^2}\sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}}. \tag{3.19}$$

Using again the fact that $\mathbf{u}$ vanishes at the vertices of $\partial\Gamma$, we obtain with (3.12) that

$$B_1 \leqslant C\lambda s \int_0^T \sum_{\overline{\alpha}\in I_{\mathcal{I}}} |\varphi_x^{\overline{\alpha}} u^{\overline{\alpha}}|(\mathcal{O}_{\overline{\alpha}}, t)\left(|u_x^{\overline{\alpha}}(l_{\overline{\alpha}}, t)| + \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} |u_x^{\overline{\alpha\beta}}(0,t)|\right)$$
$$\leqslant C \int_0^T \sum_{\overline{\alpha}\in I_{\mathcal{I}}} \left((s\lambda)^2 |\varphi_x^{\overline{\alpha}}||u^{\overline{\alpha}}|^2(\mathcal{O}_{\overline{\alpha}}, t) + |\varphi_x^{\overline{\alpha}}(\mathcal{O}_{\overline{\alpha}}, t)|(|u_x^{\overline{\alpha}}(l_{\overline{\alpha}}, t)|^2 + \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} |u_x^{\overline{\alpha\beta}}(0,t)|^2)\right) \mathrm{d}t$$
$$\leqslant \frac{C}{s} \sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}}. \tag{3.20}$$

Gathering together (3.18), (3.19) and (3.20), we obtain

$$\|M_2\mathbf{u}\|^2 + \|M_1\mathbf{u}\|^2\left(1 - \frac{A}{2s}\right) + \frac{A\lambda s^3}{2}\int_0^T\int_\Gamma |\mathbf{u}|^2|\boldsymbol{\varphi}_x|^3 + \frac{A\lambda s}{2C}\int\int |\boldsymbol{\varphi}_x||\mathbf{u}_x|^2$$
$$+ \frac{A\lambda}{2sC}\int_0^T\int_\Gamma |\boldsymbol{\varphi}_x|^{-1}|\Delta_\Gamma \mathbf{u}|^2 + \left(1 - \frac{C}{s}\right)\sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}} + Y \leqslant \|\mathbf{w}\|^2.$$

Writing explicitly the term Y and tacking into account the sign of the functions $\psi_x^{\overline{\alpha}}$ occurring in Y, we obtain for s and λ large enough the following:

$$\|M_1\mathbf{u}\|^2 + \|M_2\mathbf{u}\|^2 + \lambda s^3\int_0^T\int_\Gamma |\mathbf{u}|^2|\boldsymbol{\varphi}_x|^3 + \lambda s\int_0^T\int_\Gamma |\boldsymbol{\varphi}_x||\mathbf{u}_x|^2 + \lambda s^{-1}\int_0^T\int_\Gamma |\boldsymbol{\varphi}_x|^{-1}|\Delta_\Gamma\mathbf{u}|^2$$
$$+ \sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}} + \int_0^T \lambda s\theta^{\overline{1}}|u_x|^2(\mathcal{R}, t)\,\mathrm{d}t \leqslant C\left(\|\mathbf{w}\|^2 + \int_0^T \sum_{\overline{\alpha}\in I_{\mathcal{E}}} \lambda s\theta^{\overline{\alpha}}|u_x^{\overline{\alpha}}|^2(l_{\overline{\alpha}}, t)\,\mathrm{d}t\right). \tag{3.21}$$

Finally, using the definition of M_2, we obtain

$$\lambda s^{-1}\int_0^T\int_\Gamma |\boldsymbol{\varphi}_x|^{-1}|\mathbf{u}_t|^2 \mathrm{d}x\,\mathrm{d}t \leqslant C\lambda s^{-1}\int_0^T\int_\Gamma |\boldsymbol{\varphi}_x|^{-1}(|M_2\mathbf{u}|^2 + s^2|\boldsymbol{\varphi}_x|^2|\mathbf{u}_x|^2 + s^2|\boldsymbol{\varphi}_{xx}|^2|\mathbf{u}|^2)$$
$$\leqslant C\int_0^T\int_\Gamma (s^{-1}|M_2\mathbf{u}|^2 + \lambda s|\boldsymbol{\varphi}_x||\mathbf{u}_x|^2 + \lambda s|\boldsymbol{\varphi}_x|^{-1}|\boldsymbol{\varphi}_{xx}|^2|\mathbf{u}|^2). \tag{3.22}$$

From (3.21) and (3.22), we infer that for $s \geqslant s_3$ and $\lambda \geqslant \lambda_3$ (with s_3, λ_3 large enough) we have that

$$\|M_1\mathbf{u}\|^2 + \|M_2\mathbf{u}\|^2 + \lambda s^3\int_0^T\int_\Gamma |\mathbf{u}|^2|\boldsymbol{\varphi}_x|^3 + \lambda s\int_0^T\int_\Gamma |\boldsymbol{\varphi}_x||\mathbf{u}_x|^2$$
$$+ \lambda s^{-1}\int_0^T\int_\Gamma |\boldsymbol{\varphi}_x|^{-1}(|\Delta_\Gamma\mathbf{u}|^2 + |\mathbf{u}_t|^2) + \sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}} + \int_0^T \lambda s\theta^{\overline{1}}|u_x|^2(\mathcal{R}, t)\,\mathrm{d}t$$
$$\leqslant C\left(\|\mathbf{w}\|^2 + \int_0^T \sum_{\overline{\alpha}\in I_{\mathcal{E}}} \lambda s\theta^{\overline{\alpha}}|u_x^{\overline{\alpha}}|^2(l_{\overline{\alpha}}, t)\,\mathrm{d}t\right).$$

Replacing $\mathbf{u}$ by $\mathrm{e}^{-s\boldsymbol{\varphi}}\mathbf{q}$ in the last inequality, we readily obtain (3.1). □

3.3. Inverse problem

Before proving the stability result in theorem 1.1, we need to analyze the following system:

$$\begin{cases} u_t^{\overline{\alpha}}(x,t) = u_{xx}^{\overline{\alpha}}(x,t) + b^{\overline{\alpha}}(x)u^{\overline{\alpha}}(x,t) & \\ +R^{\overline{\alpha}}(x,t)f^{\overline{\alpha}}(x), & (x,t)\in(0,l_{\overline{\alpha}})\times(0,T), \quad \overline{\alpha}\in I, \\ u^{\overline{\alpha}}(l_{\overline{\alpha}},t) = 0, & t\in(0,T), \quad \overline{\alpha}\in I_{\mathcal{E}}, \\ u^{\overline{1}}(0,t) = 0, & t\in(0,T), \\ u^{\overline{\alpha}}(l_{\overline{\alpha}},t) = u^{\overline{\alpha\beta}}(0,t), & t\in(0,T), \quad \overline{\alpha}\in I_{\mathcal{I}}, \quad \beta\in[[1,m_{\overline{\alpha}}]], \\ u_x^{\overline{\alpha}}(l_{\overline{\alpha}},t) = \sum_{\beta=1}^{m_{\overline{\alpha}}} u_x^{\overline{\alpha\beta}}(0,t), & t\in(0,T), \quad \overline{\alpha}\in I_{\mathcal{I}}, \end{cases} \tag{3.23}$$

where $\mathbf{b} = \{b^{\overline{\alpha}}\}_{\overline{\alpha}\in I} \in L^\infty(\Gamma)$.

Proposition 3.5. *Assume that* $\mathbf{u} = \{u^{\overline{\alpha}}\}_{\overline{\alpha}\in I}$ *is a solution of (3.23), which satisfies* $\mathbf{u}_t \in H^{2,1}(\Gamma\times(0,T))$. *If* $\mathbf{R} = \{R^{\overline{\alpha}}(x,t)\}_{\overline{\alpha}\in I}$ *is such that* $\mathbf{R}_t \in L^\infty(\Gamma\times(0,T))$ *and that*

$$|\mathbf{R}(x,t_0)| \geqslant r > 0, \quad \text{for a.e. } x\in\Gamma \text{ and some } t_0\in(0,T), \tag{3.24}$$

then there exists a positive constant $C = C(||\mathbf{R}_t||_{L^\infty(\Gamma\times(0,T))}, ||\mathbf{b}||_{L^\infty(\Gamma)}, r)$ *such that*

$$\|\mathbf{f}\|_{L^2(\Gamma)} \leqslant C\left(\|\mathbf{u}(\cdot,t_0)\|_{H^2(\Gamma)} + \sum_{v\in\mathcal{E}} \|\partial_{xt}\mathbf{u}(v,\cdot)\|_{L^2(0,T)}\right) \tag{3.25}$$

for any $\mathbf{f}\in L^2(\Gamma)$.

Proof. We proceed as in [37]. Set $\mathbf{z} = \partial_t\mathbf{u}$. Then, $\mathbf{z} = \{z^{\overline{\alpha}}\}_{\overline{\alpha}\in I}$ satisfies

$$\begin{cases} z_t^{\overline{\alpha}}(x,t) = z_{xx}^{\overline{\alpha}}(x,t) + b^{\overline{\alpha}}(x)z^{\overline{\alpha}}(x,t) & \\ +R_t^{\overline{\alpha}}(x,t)f^{\overline{\alpha}}(x), & (x,t)\in(0,l_{\overline{\alpha}})\times(0,T), \quad \overline{\alpha}\in I, \\ z^{\overline{\alpha}}(l_{\overline{\alpha}},t) = 0, & t\in(0,T), \quad \overline{\alpha}\in I_{\mathcal{E}}, \\ z^{\overline{1}}(0,t) = 0, & t\in(0,T), \\ z^{\overline{\alpha}}(l_{\overline{\alpha}},t) = z^{\overline{\alpha\beta}}(0,t), & t\in(0,T), \quad \overline{\alpha}\in I_{\mathcal{I}}, \quad \beta\in[[1,m_{\overline{\alpha}}]], \\ z_x^{\overline{\alpha}}(l_{\overline{\alpha}},t) = \sum_{\beta=1}^{m_{\overline{\alpha}}} z_x^{\overline{\alpha\beta}}(0,t), & t\in(0,T), \quad \overline{\alpha}\in I_{\mathcal{I}}. \end{cases} \tag{3.26}$$

On the other hand,

$$\mathbf{R}(x,t)\mathbf{f}(x) = \partial_t\mathbf{u}(x,t) - (\Delta_\Gamma\mathbf{u})(x,t) - \mathbf{bu}(x,t). \tag{3.27}$$

Using a change of variables, it is sufficient to prove (3.25) when $t_0 = T/2$.

We now apply the Carleman estimate (3.1) with $\mathbf{q} = \mathbf{z} = \partial_t\mathbf{u}$ (and some fixed $\lambda > 0$)

$$\begin{aligned} \int_0^T \int_\Gamma [(s\boldsymbol{\theta})^{-1}|\partial_{tt}\mathbf{u}|^2 + s\boldsymbol{\theta}|\partial_{xt}\mathbf{u}|^2 + (s\boldsymbol{\theta})^3|\partial_t\mathbf{u}|^2]\,\mathrm{e}^{-2s\varphi} + \int_0^T (s\boldsymbol{\theta})(|\partial_{xt}\mathbf{u}|^2\,\mathrm{e}^{-2s\varphi})(\mathcal{R},t)\,\mathrm{d}t \\ \leqslant C\left(\int_0^T\int_\Gamma |\partial_{tt}\mathbf{u} - \Delta_\Gamma\partial_t\mathbf{u}|^2\,\mathrm{e}^{-2s\varphi} + \sum_{v\in\mathcal{E}}\int_0^T s\boldsymbol{\theta}(|\partial_{xt}\mathbf{u}|^2\,\mathrm{e}^{-2s\varphi})(v,t)\,\mathrm{d}t\right) \\ \leqslant C\left(\int_0^T\int_\Gamma \left(|(\partial_t\mathbf{R})\mathbf{f}|^2 + |\partial_t\mathbf{u}|^2\right)\mathrm{e}^{-2s\varphi} + \sum_{v\in\mathcal{E}}\int_0^T s\boldsymbol{\theta}(|\partial_{xt}\mathbf{u}|^2\,\mathrm{e}^{-2s\varphi})(v,t)\,\mathrm{d}t\right), \end{aligned}$$

which gives, for s large enough,

$$\int_0^T \int_\Gamma [(s\theta)^{-1}|\partial_{tt}\mathbf{u}|^2 + s\theta|\partial_{xt}\mathbf{u}|^2 + (s\theta)^3|\partial_t\mathbf{u}|^2]\,e^{-2s\varphi} + \int_0^T (s\theta)(|\partial_{xt}\mathbf{u}|^2\,e^{-2s\varphi})(\mathcal{R},t)\,dt \leqslant C\left(\int_0^T \int_\Gamma |(\partial_t\mathbf{R})\mathbf{f}|^2\,e^{-2s\varphi} + \sum_{v\in\mathcal{E}}\int_0^T s\theta(|\partial_{xt}\mathbf{u}|^2\,e^{-2s\varphi})(v,t)\,dt\right).$$

Since $\lim_{t\to 0} e^{-2s\varphi(x,t)} = 0$ for $x \in \Gamma$ and $|\varphi_t(x,t)| \leqslant C\theta^2(x,t)$ for all $x \in \Gamma$ and $t > 0$, we obtain

$$\begin{aligned}\int_\Gamma |\partial_t\mathbf{u}\left(x,\frac{T}{2}\right)|^2\,e^{-2s\varphi(x,\frac{T}{2})}\,dx &= \int_0^{T/2} \frac{\partial}{\partial t}\left(\int_\Gamma |\partial_t\mathbf{u}(x,t)|^2\,e^{-2s\varphi(x,t)}\,dx\right)dt\\ &= \int_0^{T/2}\int_\Gamma \left[2\partial_t\mathbf{u}\partial_{tt}\mathbf{u} - 2s\partial_t\varphi|\partial_t\mathbf{u}|^2\right]e^{-2s\varphi(x,t)}\,dx\,dt\\ &\leqslant \int_0^{T/2}\int_\Gamma \left(2|\partial_t\mathbf{u}||\partial_{tt}\mathbf{u}| + Cs\theta^2|\partial_t\mathbf{u}|^2\right)e^{-2s\varphi(x,t)}\,dx\,dt\\ &\leqslant C\int_0^{T/2}\int_\Gamma ((s^2\theta)^{-1}|\partial_{tt}\mathbf{u}|^2 + (s\theta)^2|\partial_t\mathbf{u}|^2)\,e^{-2s\varphi(x,t)}\,dx\,dt\\ &\leqslant \frac{C}{s}\int_0^T\int_\Gamma |(\partial_t\mathbf{R})\mathbf{f}|^2\,e^{-2s\varphi}\,dx\,dt + C\,e^{-Cs}\sum_{v\in\mathcal{E}}\int_0^T |\partial_{xt}\mathbf{u}|^2(v,t)\,dt. \end{aligned} \tag{3.28}$$

Using (3.24), (3.27) and (3.28), we obtain that

$$\begin{aligned}\int_\Gamma |\mathbf{f}(x)|^2\,e^{-2s\varphi(x,T/2)}\,dx &\leqslant C\int_\Gamma |\mathbf{R}(x,T/2)\mathbf{f}(x)|^2\,e^{-2s\varphi(x,T/2)}\,dx\\ &\leqslant C\int_\Gamma \left(\left|\partial_t\mathbf{u}\left(x,\frac{T}{2}\right)\right|^2 + \left|\Delta_\Gamma\mathbf{u}\left(x,\frac{T}{2}\right)\right|^2 + \left|\mathbf{u}\left(x,\frac{T}{2}\right)\right|^2\right)e^{-2s\varphi(x,T/2)}\,dx\\ &\leqslant \frac{C}{s}\int_0^T\int_\Gamma |(\partial_t\mathbf{R})\mathbf{f}|^2\,e^{-2s\varphi}\,dx\,dt + C\,e^{-Cs}\sum_{v\in\mathcal{E}}\int_0^T |\partial_{xt}\mathbf{u}|^2(v,t)\,dt\\ &\quad + C\left\|\mathbf{u}\left(\cdot,\frac{T}{2}\right)\right\|^2_{H^2(\Gamma)}.\end{aligned}$$

Since $\partial_t\mathbf{R} \in L^\infty(\Gamma \times (0,T))$, the following holds:

$$\int_\Gamma |\mathbf{f}(x)|^2\,e^{-2s\varphi(x,T/2)}\,dx \leqslant \frac{C}{s}\int_0^T\int_\Gamma |\mathbf{f}|^2\,e^{-2s\varphi}\,dx\,dt + C\,e^{-Cs}\sum_{v\in\mathcal{E}}\int_0^T |\partial_{xt}\mathbf{u}|^2(v,t)\,dt + C\left\|\mathbf{u}\left(\cdot,\frac{T}{2}\right)\right\|^2_{H^2(\Gamma)}.$$

It follows from the definition of φ that

$$\varphi\left(x,\frac{T}{2}\right) \leqslant \varphi(x,t) \quad \text{for all } (x,t) \in \Gamma \times (0,T)$$

so that

$$\int_0^T\int_\Gamma |\mathbf{f}(x)|^2\,e^{-2s\varphi(x,t)}\,dx\,dt \leqslant T\int_\Gamma |\mathbf{f}(x)|^2\,e^{-2s\varphi(x,T/2)}\,dt.$$

Therefore,

$$\left(1 - \frac{CT}{s}\right)\int_\Gamma |\mathbf{f}(x)|^2\,e^{-2s\varphi(x,T/2)}\,dt \leqslant C\sum_{v\in\mathcal{E}}\int_0^T |\partial_{xt}\mathbf{u}|^2(v,t)\,dt + C\left\|\mathbf{u}\left(\cdot,\frac{T}{2}\right)\right\|^2_{H^2(\Gamma)}.$$

The desired inequality follows for s large enough. □

We are now able to prove the stability result for the system (1.1).

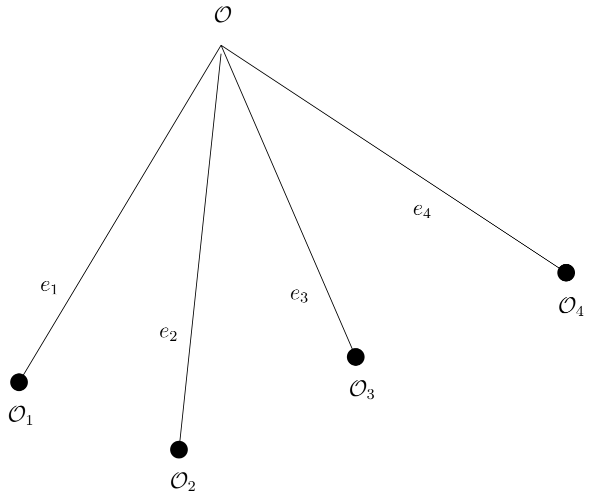

Figure 2. A star-shaped tree with four edges.

Proof of theorem 1.1. Let us denote

$$\mathbf{w} = \mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q}).$$

It satisfies the following system:

$$\begin{cases} \mathbf{w}_t = \Delta_\Gamma \mathbf{w} - \mathbf{q}\mathbf{w} + \mathbf{R}\mathbf{f} \text{ in } \Gamma \times (0, T), \\ \mathbf{w}(x, t) = 0, \text{ on } \partial\Gamma \times (0, T), \end{cases}$$

where $\mathbf{f} = \mathbf{q} - \mathbf{p}$, $\mathbf{R} = \mathbf{u}(\mathbf{p})$. Note that $\mathbf{R} \in C([0, T]; H^1(\Gamma)) \subset C(\overline{\Gamma} \times [0, T])$, for $u \in L^2(0, T; H^2(\Gamma))$ and $u_t \in L^2(0, T; L^2(\Gamma))$. Using our hypothesis, we see that $|\mathbf{R}(\cdot, t_0)| \geqslant r > 0$ on Γ; hence, we can apply proposition 3.5 to obtain

$$\|\mathbf{p} - \mathbf{q}\|_{L^2(\Gamma)} \leqslant C\Big(\|[\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})](\cdot, t_0)\|_{H^2(\Gamma))} + \sum_{v\in\mathcal{E}} \|\partial_x[\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})](v, \cdot)\|_{H^1(0,T)}\Big),$$

where $C = C(\|\partial_t \mathbf{u}(\mathbf{p})\|_{L^\infty(\Gamma\times(0,T))}, \|\mathbf{q}\|_{L^\infty(\Gamma)}, r)$. The proof is now complete. □

4. The Schrödinger equation on a star-shaped tree

In this section, we consider a network Γ, which is a star-shaped tree constituted by N edges e_j (with $N \geqslant 3$) connected at the internal node $\mathcal{O}$ (see figure 2). Here, the parametrization of the edge e_j is chosen so that the origin $\mathcal{O}$ of e_j corresponds to $x = 0$, while the endpoint $\mathcal{O}_j$ of e_j corresponds to $x = l_j$, for all $j \in [[1, N]]$.

We consider the following Cauchy problem:

$$\mathrm{i}y_{j,t} + y_{j,xx} + p_j(x)y_j = f_j(x, t), \qquad x \in (0, l_j), \quad j \in [[1, N]], \quad t \in (0, T), \tag{4.1}$$

$$y_j(0, t) = y_l(0, t), \qquad t \in (0, T), \quad j, l \in [[1, N]], \tag{4.2}$$

$$\sum_{1\leqslant j\leqslant N} y_{j,x}(0, t) = 0, \qquad t \in (0, T), \tag{4.3}$$

$$y(l_j, t) = 0, \qquad j \in [[1, N]], \quad t \in (0, T), \tag{4.4}$$

$$y(x, 0) = y_0(x), \qquad x \in \Gamma, \tag{4.5}$$

where $\mathbf{p} = \{p_j\}_{j=1,N} \in L^\infty(\Gamma)$ is some given potential function. Our main aim is to prove the stability for the inverse problem consisting in retrieving the potential $\mathbf{p}$ from the measurement of $y_x(l_j, t)$ for $j \in [[1, N]]$. This is done thanks to some Carleman estimate in following the classical Bukhgeim–Klibanov method.

The first step will be the proof of a Carleman inequality on Γ. The key point is that choosing only one weight function $\boldsymbol{\psi} = \{\psi_j\}_{j=1,N} : \Gamma \to \mathbb{R}$ as in the case of the heat equation is not convenient since we fail to control some boundary terms. Instead, we consider a family of weights $\{\boldsymbol{\psi}^k\}_{k=1,N}$ allowing us to get rid of some *bad* boundary terms at the internal node.

4.1. Carleman estimate

Let $(\psi_j^k)_{1\leqslant j,k\leqslant N}$ be a family of functions fulfilling the following properties:

$$\psi_j^k : [0, l_j] \to \mathbb{R} \text{ is of class } C^2 \quad \forall j, k \in [[1, N]], \tag{4.6}$$

$$\psi_{j_1}^{k_1}(0) = \psi_{j_2}^{k_2}(0) \quad \forall j_1, k_1, j_2, k_2 \in [[1, N]], \tag{4.7}$$

$$|(\psi_j^k)'(x)|^2 + (\psi_j^k)''(x) \geqslant 0 \quad \forall x \in [0, l_j] \quad \forall j, k \in [[1, N]], \tag{4.8}$$

$$(\psi_j^k)'(x) \neq 0 \quad \forall x \in [0, l_j] \quad \forall j, k \in [[1, N]], \tag{4.9}$$

$$\frac{C}{2} \geqslant \psi_j^k(x) > \frac{C}{3} \quad \forall x \in [0, l_j] \quad \forall j, k \in [[1, N]], \tag{4.10}$$

where $C > 0$ is some positive constant. We also assume that the following flux conditions at $x = 0$ are satisfied:

$$\sum_{1\leqslant j\leqslant N} (\psi_j^k)'(0) = 0 \quad \forall k \in [[1, N]], \tag{4.11}$$

$$\sum_{1\leqslant k\leqslant N} (\psi_j^k)'(0) = 0 \quad \forall j \in [[1, N]], \tag{4.12}$$

$$\sum_{1\leqslant k\leqslant N} |(\psi_j^k)'(0)|^2 = C_1 \quad \forall j \in [[1, N]], \tag{4.13}$$

$$\sum_{1\leqslant k\leqslant N} (\psi_j^k)''(0) = C_2 \quad \forall j \in [[1, N]], \tag{4.14}$$

$$\sum_{1\leqslant k\leqslant N} [(\psi_j^k)'(0)]^3 > 0 \quad \forall j \in [[1, N]], \tag{4.15}$$

for some constants $C_1 > 0$ and $C_2 \in \mathbb{R}$. Such a family of weight functions $(\psi_j^k)_{1\leqslant j,k\leqslant N}$ exists. It is sufficient to consider (affine) functions of the form $\psi_j^k(x) = a_j^k x + \frac{5}{12}C$ with $C \ll 1$ and

$$a_j^k := \begin{cases} N-1 & \text{if } j = k, \\ -1 & \text{if } j \neq k. \end{cases}$$

Note that if another internal node would be present, condition (4.15) (with the opposite sign due to the orientation) would not be satisfied. Let us introduce the families of weights

$$\theta_j^k(x,t) = \frac{e^{\lambda\psi_j^k(x)}}{t(T-t)}, \qquad \varphi_j^k(x,t) = \frac{e^{\lambda C} - e^{\lambda\psi_j^k(x)}}{t(T-t)},$$

and the class of functions

$$\mathcal{Z} = \{\mathbf{q} = (q_j)_{j=1,N} \in C(\Gamma \times [0,T]);\\ q_j \in C^{2,1}([0,l_j]\times[0,T])\ \forall j \in [[1,N]],\ \text{and (4.2)–(4.4) hold}\}.$$

Proposition 4.1. *Assume that the family of weights* (ψ_k^j) *fulfils (4.6)–(4.15). Then, there exist some constants* $\lambda_0 \geqslant 1$, $s_0 \geqslant 1$ *and* $C_0 > 0$ *such that for all* $\lambda \geqslant \lambda_0$, *all* $s \geqslant s_0$, *and all* $\mathbf{q} \in \mathcal{Z}$, *the following holds:*

$$\sum_{1\leqslant j,k\leqslant N} \int_0^T\int_0^{l_j} [\lambda^2 s\theta_j^k |q_{j,x}|^2 + \lambda^4 (s\theta_j^k)^3 |q_j|^2 + |(\widetilde{M}_1^k\mathbf{q})_j|^2 + |(\widetilde{M}_2^k\mathbf{q})_j|^2]\, e^{-2s\varphi_j^k}\,dx\,dt$$
$$\leqslant C_0 \sum_{1\leqslant j,k\leqslant N}\left(\int_0^T\int_0^{l_j} |q_{j,t} + iq_{j,xx}|^2\, e^{-2s\varphi_j^k}\,dx\,dt \right.$$
$$\left. + \int_0^T \lambda s\theta_j^k(l_j)|q_{j,x}(l_j)|^2\, e^{-2s\varphi_j^k}\,dt\right), \tag{4.16}$$

where $i = \sqrt{-1}$ *and* $\widetilde{M}_1^k$ *and* $\widetilde{M}_2^k$ *denote the operators*

$$(\widetilde{M}_1^k\mathbf{q})_j := [s(\varphi_{j,t}^k + i\varphi_{j,xx}^k) - 2is^2|\varphi_{j,x}^k|^2]q_j + 2is\varphi_{j,x}^k q_{j,x}, \tag{4.17}$$

$$(\widetilde{M}_2^k\mathbf{q})_j := [-s(\varphi_{j,t}^k + i\varphi_{j,xx}^k) + 2is^2|\varphi_{j,x}^k|^2]q_j + q_{j,t} - 2is\varphi_{j,x}^k q_{j,x} + iq_{j,xx}. \tag{4.18}$$

Proof. In what follows, the letter c will denote a constant (independent of s, λ, $\mathbf{q}$, j, k) which may vary from line to line. Let $\mathbf{q} \in \mathcal{Z}$ be given, and for $j, k \in [[1,N]]$, let

$$u_j^k = e^{-s\varphi_j^k} q_j, \quad w_j^k = e^{-s\varphi_j^k} L(e^{s\varphi_j^k} u_j^k),$$

where L denotes the operator

$$L = \partial_t + i\partial_x^2.$$

Straightforward computations show that $\mathbf{w}^k = M^k\mathbf{u}^k$ with

$$w_j^k = (M^k\mathbf{u}^k)_j := u_{j,t}^k + s\varphi_{j,t}^k u_j^k + i(u_{j,xx}^k + 2s\varphi_{j,x}^k u_{j,x}^k + s\varphi_{j,xx}^k u_j^k + s^2|\varphi_{j,x}^k|^2 u_j^k),$$

with the operator M^k acting simply on the components of $\mathbf{u}^k$ along the different edges. Let M_1^k and M_2^k denote, respectively, the (formal) adjoint and skew-adjoint parts of the operator M^k. We readily obtain that

$$\left(M_1^k\mathbf{u}^k\right)_j = i\left(2s\varphi_{j,x}^k u_{j,x}^k + s\varphi_{j,xx}^k u_j^k\right) + s\varphi_{j,t}^k u_j^k, \tag{4.19}$$

$$\left(M_2^k\mathbf{u}^k\right)_j = u_{j,t}^k + i\left(u_{j,xx}^k + s^2|\varphi_{j,x}^k|^2 u_j^k\right). \tag{4.20}$$

Letting $(\widetilde{M}_1^k\mathbf{q})_j := e^{s\varphi_j^k}(M_1^k\mathbf{u}^k)_j$ and $(\widetilde{M}_2^k\mathbf{q})_j := e^{s\varphi_j^k}(M_2^k\mathbf{u}^k)_j$, we easily check that (4.17) and (4.18) hold. On the other hand,

$$||\mathbf{w}^k||^2 = ||M_1^k\mathbf{u}^k + M_2^k\mathbf{u}^k||^2 = ||M_1^k\mathbf{u}^k||^2 + ||M_2^k\mathbf{u}^k||^2 + 2\mathrm{Re}(M_1^k\mathbf{u}^k, M_2^k\mathbf{u}^k),$$

where $(\mathbf{u}, \mathbf{v}) := \sum_{1\leqslant j\leqslant N} \int_0^T\int_0^{l_j} u_j(x,t)\overline{v_j(x,t)}\,\mathrm{d}x\,\mathrm{d}t$ and $||\mathbf{w}||^2 = (\mathbf{w}, \mathbf{w})$. The proof of the Carleman estimate is inspired by those of [33, proposition 2.1]. In the first step, we compute precisely $\mathrm{Re}(M_1^k\mathbf{u}^k, M_2^k\mathbf{u}^k)$. In the second step, we check that the boundary terms related to the internal node $\mathcal{O}$ give positive contributions. The third step is completely similar to the second step in the proof of [33, proposition 2.1].

Step 1. *Exact computation of Re* $(M_1^k\mathbf{u}^k, M_2^k\mathbf{u}^k)$.

We intend to compute

$$\mathrm{Re}(M_1^k\mathbf{u}^k, M_2^k\mathbf{u}^k) = \mathrm{Re}\sum_{1\leqslant j\leqslant N}\int_0^T\int_0^{l_j}(M_1^ku^k)_j\overline{(M_2^ku^k)_j}\,\mathrm{d}x\,\mathrm{d}t.$$

Let us fix any pair (j,k) of indices in $[[1,N]]$ and let us compute $\int_0^T\int_0^{l_j}(M_1^ku^k)_j\,\overline{(M_2^ku^k)_j}\mathrm{d}x\,\mathrm{d}t$. For the sake of simplicity, we shall drop the indices j and k during the computations, and we shall write $\iint u$ for $\int_0^T\int_0^{l_j} u(x,t)\,\mathrm{d}x\,\mathrm{d}t$, and $\int h$ for $\int_0^T h(t)\,\mathrm{d}t$. Then,

$$\begin{aligned}2\mathrm{Re}\int_0^T\int_0^{l_j}(M_1^ku^k)_j\overline{(M_2^ku^k)_j}\mathrm{d}x\,\mathrm{d}t &= 2\mathrm{Re}\iint[\mathrm{i}(2s\varphi_xu_x + s\varphi_{xx}u) + s\varphi_tu][\overline{u_t} - \mathrm{i}(\overline{u_{xx}} + s^2|\varphi_x|^2\overline{u})]\\ &= I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{aligned}I_1 &= 2\mathrm{Re}\iint \mathrm{i}(2s\varphi_xu_x + s\varphi_{xx}u)(\overline{u_t} - \mathrm{i}(\overline{u_{xx}} + s^2|\varphi_x|^2\overline{u})),\\ I_2 &= 2\mathrm{Re}\iint s\varphi_tu(\overline{u_t} - \mathrm{i}\overline{u_{xx}}),\\ I_3 &= 2\mathrm{Re}\iint s\varphi_tu(-\mathrm{i}s^2|\varphi_x|^2\overline{u}).\end{aligned}$$

Obviously, $I_3 = 0$. Let us begin with the computation of I_1:

$$I_1 = 2\mathrm{Re}\iint(2s\varphi_xu_x + s\varphi_{xx}u)(\overline{u_{xx}} + s^2|\varphi_x|^2\overline{u}) + 2\mathrm{Re}\iint \mathrm{i}(2s\varphi_xu_x + s\varphi_{xx}u)\overline{u_t} = I_1^1 + I_1^2.$$

To calculate I_1^1, we need to evaluate the real part of the integral term $J := \iint \overline{u_{xx}}\varphi_xu_x$. Integrating by part yields

$$J = -\iint\overline{u_x}(\varphi_{xx}u_x + \varphi_xu_{xx}) + \int\varphi_x|u_x|^2\Big|_0^l,$$

where l stands for l_j. On the other hand,

$$\begin{aligned}2\mathrm{Re}\iint\overline{u_x}\varphi_xu_{xx} &= \iint\varphi_x(\overline{u_x}u_{xx} + u_x\overline{u_{xx}}) = \iint\varphi_x\partial_x|u_x|^2\\ &= -\iint\varphi_{xx}|u_x|^2 + \int\varphi_x|u_x|^2\Big|_0^l.\end{aligned}$$

Therefore,

$$\begin{aligned}2\mathrm{Re}J &= -2\iint\varphi_{xx}|u_x|^2 + \iint\varphi_{xx}|u_x|^2 - \int\varphi_x|u_x|^2\Big|_0^l + 2\int\varphi_x|u_x|^2\Big|_0^l\\ &= -\iint\varphi_{xx}|u_x|^2 + \int\varphi_x|u_x|^2\Big|_0^l.\end{aligned}$$

It follows that

$$I_1^1 = 2\mathrm{Re}\left\{2sJ + s\iint \varphi_{xx}u\overline{u_{xx}} + 2s^3\iint(\varphi_x)^3u_x\overline{u} + s^3\iint \varphi_{xx}|\varphi_x|^2|u|^2\right\}$$

$$= 4s\mathrm{Re}J - 2s\mathrm{Re}\iint(\varphi_{3x}u + \varphi_{xx}u_x)\overline{u_x} + 2s\mathrm{Re}\int \varphi_{xx}u\overline{u_x}\Big|_0^l$$

$$+ 2s^3\iint(\varphi_x)^3\partial_x|u|^2 + 2s^3\iint\varphi_{xx}|\varphi_x|^2|u|^2$$

$$= 2s\left\{-\iint\varphi_{xx}|u_x|^2 + \int\varphi_x|u_x|^2\Big|_0^l\right\} + s\left(\iint\varphi_{4x}|u|^2 - \int[\varphi_{3x}|u|^2]_0^l\right.$$

$$\left. - 2\iint\varphi_{xx}|u_x|^2\right) + 2s\mathrm{Re}\int\varphi_{xx}u\overline{u_x}\Big|_0^l - 6s^3\iint(\varphi_x)^2\varphi_{xx}|u|^2$$

$$+2s^3\int(\varphi_x)^3|u|^2\Big|_0^l + 2s^3\iint\varphi_{xx}|\varphi_x|^2|u|^2$$

$$= -4s\iint\varphi_{xx}|u_x|^2 + s\iint\varphi_{4x}|u|^2 - 4s^3\iint(\varphi_x)^2\varphi_{xx}|u|^2$$

$$+ \int[2s\varphi_x|u_x|^2 + (-s\varphi_{3x} + 2s^3(\varphi_x)^3)|u|^2 + 2s\varphi_{xx}\mathrm{Re}(u\overline{u_x})]\Big|_0^l.$$

On the other hand,

$$I_1^2 = \mathrm{i}\iint(2s\varphi_xu_x + s\varphi_{xx}u)\overline{u_t} - \mathrm{i}\iint(2s\varphi_x\overline{u_x} + s\varphi_{xx}\overline{u})u_t$$

$$= -\mathrm{i}\iint(2s\varphi_{xt}u_x + 2s\varphi_xu_{xt} + s\varphi_{xxt}u + s\varphi_{xx}u_t)\overline{u}$$

$$+\mathrm{i}\iint 2s(\varphi_{xx}u_t + \varphi_xu_{xt})\overline{u} - \mathrm{i}\int 2s\varphi_x\overline{u}u_t\Big|_0^l - \mathrm{i}\iint s\varphi_{xx}\overline{u}u_t$$

$$= -\mathrm{i}\iint(s\varphi_{xxt}|u|^2 + 2s\varphi_{xt}u_x\overline{u}) - \mathrm{i}\int 2s\varphi_x\overline{u}u_t\Big|_0^l$$

$$= \mathrm{i}\iint s\varphi_{xt}(u\overline{u_x} + u_x\overline{u}) - \mathrm{i}\int s\varphi_{xt}|u|^2\Big|_0^l - 2\mathrm{i}s\iint\varphi_{xt}u_x\overline{u} - \mathrm{i}\int 2s\varphi_x\overline{u}u_t\Big|_0^l$$

$$= \mathrm{i}\iint s\varphi_{xt}(u\overline{u_x} - u_x\overline{u}) + \mathrm{i}\int[s\varphi_x(u\overline{u_t} - u_t\overline{u})]\Big|_0^l.$$

It remains to estimate I_2:

$$I_2 = \iint s\varphi_t(u\overline{u_t} + \overline{u}u_t) + \iint s\varphi_t(-\mathrm{i}u\overline{u_{xx}} + \mathrm{i}\overline{u}u_{xx}) =: I_2^1 + I_2^2.$$

We find that

$$I_2^1 = -\iint s\varphi_{tt}|u|^2,$$

$$I_2^2 = \mathrm{i}s\iint(\varphi_{xt}u + \varphi_tu_x)\overline{u_x} - \mathrm{i}s\int\varphi_tu\overline{u_x}\Big|_0^l$$

$$-\mathrm{i}s\iint(\varphi_{tx}\overline{u} + \varphi_t\overline{u_x})u_x + \mathrm{i}s\int\varphi_t\overline{u}u_x\Big|_0^l$$

$$= 2\mathrm{Re}\iint(\mathrm{i}s)\varphi_{xt}u\overline{u_x} + 2\mathrm{Re}\int(-\mathrm{i}s)\varphi_tu\overline{u_x}\Big|_0^l.$$

Thus,

$$I_1 + I_2 + I_3 = -4s\iint\varphi_{xx}|u_x|^2 + s\iint\varphi_{4x}|u|^2 - 4s^3\iint|\varphi_x|^2\varphi_{xx}|u|^2$$

$$+\,\mathrm{i}\iint s\varphi_{xt}(u\overline{u_x}-u_x\overline{u})-s\iint\varphi_{tt}|u|^2+2\mathrm{Re}\iint(\mathrm{i}s)\varphi_{tx}u\overline{u_x}$$
$$+\int[2s\varphi_x|u_x|^2+(-s\varphi_{3x}+2s^3(\varphi_x)^3)|u|^2+2s\varphi_{xx}\mathrm{Re}(u\overline{u_x})$$
$$+\,\mathrm{i}\,s\varphi_x(u\overline{u_t}-u_t\overline{u})+2\mathrm{Re}\{(-\mathrm{i}s)\varphi_t u\overline{u_x}\}]\Big|_0^l.$$

We conclude that (with the indices written again)

$$\sum_{1\leqslant k\leqslant N}||\mathbf{w}^k||^2=\sum_{1\leqslant k\leqslant N}\left[||M_1^k\mathbf{u}^k||^2+||M_2^k\mathbf{u}^k||^2\right]$$
$$+\sum_{1\leqslant j,k\leqslant N}\Big\{-4s\int_0^T\int_0^{l_j}\varphi^k_{j,xx}|u^k_{j,x}|^2-4s\mathrm{Im}\int_0^T\int_0^{l_j}\varphi^k_{j,xt}u^k_j\overline{u^k_{j,x}}$$
$$+\int_0^T\int_0^{l_j}|u^k_j|^2[s(\varphi^k_{j,4x}-\varphi^k_{j,tt})-4s^3(\varphi^k_{j,x})^2\varphi^k_{j,xx}]$$
$$+\int_0^T[2s\varphi^k_{j,x}|u^k_{j,x}|^2+(-s\varphi^k_{j,3x}+2s^3(\varphi^k_{j,x})^3)|u^k_j|^2+2s\varphi^k_{j,xx}\mathrm{Re}(u^k_j\overline{u^k_{j,x}})$$
$$+2s\varphi^k_{j,t}\mathrm{Re}(-\mathrm{i}u^k_j\overline{u^k_{j,x}})+\mathrm{i}s\varphi^k_{j,x}(u^k_j\overline{u^k_{j,t}}-u^k_{j,t}\overline{u^k_j})]\Big|_0^l\Big\}. \tag{4.21}$$

Step 2. *Estimation of the boundary terms at the internal node $\mathcal{O}$.*

We estimate each term in

$$\sum_{j,k}(-2s)\int_0^T\varphi^k_{j,x}(0)|u^k_{j,x}(0)|^2+\sum_{j,k}\int_0^T(s\varphi^k_{j,3x}(0)-2s^3(\varphi^k_{j,x}(0))^3)|\mathbf{u}(0)|^2$$
$$+\sum_{j,k}(-2s)\int_0^T\varphi^k_{j,xx}(0)\mathrm{Re}(\mathbf{u}(0)\overline{u^k_{j,x}(0)})+\sum_{j,k}(-2s)\int_0^T\varphi^k_{j,t}(0)\mathrm{Re}(-\mathrm{i}\mathbf{u}(0)\overline{u^k_{j,x}(0)})$$
$$+\sum_{j,k}\int_0^T(-\mathrm{i}s)\varphi^k_{j,x}(0)(\mathbf{u}(0)\overline{\mathbf{u}_t(0)}-\mathbf{u}_t(0)\overline{\mathbf{u}(0)})=:J_1+J_2+J_3+J_4+J_5.$$

In the above equation and in the following ones, we write merely

$$\mathbf{u}(0):=u^k_j(0,t),\quad \boldsymbol{\varphi}(0)=\varphi^k_j(0,t),\quad \text{etc.}$$

Using (4.12), (4.13) and (4.3) (for q_j), we see that

$$J_1=(-2s)\sum_j\int_0^T\sum_k\varphi^k_{j,x}(0)|\,\mathrm{e}^{-s\boldsymbol{\varphi}(0)}(-s\varphi^k_{j,x}q_j+q_{j,x})|^2$$
$$=-2s^3\int_0^T\sum_{j,k}(\varphi^k_{j,x}(0))^3|\mathbf{u}(0)|^2-2s\sum_j\int_0^T\Big(\sum_k\varphi^k_{j,x}(0)\Big)\mathrm{e}^{-2s\boldsymbol{\varphi}(0)}|q_{j,x}(0)|^2$$
$$+4s^2\mathrm{Re}\sum_j\int_0^T\Big(\sum_k[\varphi^k_{j,x}(0)]^2\Big)\mathbf{u}(0)\overline{q_{j,x}(0)}$$
$$=-2s^3\int_0^T\sum_{j,k}(\varphi^k_{j,x}(0))^3|\mathbf{u}(0)|^2.$$

Therefore,

$$J_1+J_2=\sum_{j,k}\int_0^T(s\varphi^k_{j,3x}(0)-4s^3(\varphi^k_{j,x}(0))^3)|\mathbf{u}(0)|^2. \tag{4.22}$$

On the other hand, using (4.13), (4.14) and (4.3), we have that

$$J_3 = -2s\sum_{j,k}\int_0^T \varphi^k_{j,xx}(0)\mathrm{Re}[\mathbf{u}(0)\,\mathrm{e}^{-s\varphi(0)}(-s\varphi^k_{j,x}(0)\overline{q_j(0)}+\overline{q_{j,x}(0)})]$$

$$= 2s^2\,\mathrm{Re}\int_0^T\left(\sum_{j,k}\varphi^k_{j,xx}(0)\varphi^k_{j,x}(0)\right)|\mathbf{u}(0)|^2$$

$$+\,2s\lambda\,\mathrm{Re}\int_0^T \frac{\mathbf{u}(0)\,\mathrm{e}^{-s\varphi(0)}\,\mathrm{e}^{\lambda\psi(0)}}{t(T-t)}\sum_j\left(\sum_k(\psi^k_j)''(0)+\lambda\sum_k[(\psi^k_j)'(0)]^2\right)\overline{q_{j,x}(0)}$$

$$= 2s^2\,\mathrm{Re}\int_0^T\left(\sum_{j,k}\varphi^k_{j,xx}(0)\varphi^k_{j,x}(0)\right)|\mathbf{u}(0)|^2. \tag{4.23}$$

Combining (4.22), (4.23) and (4.15), we obtain that for $s \geqslant s_1$ and $\lambda \geqslant \lambda_1$,

$$J_1+J_2+J_3 \geqslant cs^3\lambda^3\sum_{j,k}\int_0^T\left(\frac{\mathrm{e}^{\lambda\psi^k_j(0)}}{t(T-t)}\right)^3|\mathbf{u}(0)|^2. \tag{4.24}$$

Finally, we claim that $J_4 = J_5 = 0$. Indeed, using (4.3), we obtain that

$$J_4 = -2s\mathrm{Im}\Big(\int_0^T \varphi_t(0)u(0)\sum_{j,k}\overline{u^k_{j,x}(0)}\mathrm{d}t\Big)$$

$$= -2s\mathrm{Im}\int_0^T \varphi_t(0)u(0)\sum_{j,k}(-s\varphi^k_{j,x}(0)\overline{q(0)}+\overline{q_{j,x}(0)})\,\mathrm{e}^{-s\varphi(0)}\mathrm{d}t$$

$$= 0,$$

while $J_5 = 0$ by (4.11). Thus, we conclude that

$$J_1+J_2+J_3+J_4+J_5 \geqslant cs^3\lambda^3\int_0^T\left(\frac{\mathrm{e}^{\lambda\psi(0)}}{t(T-t)}\right)^3|\mathbf{u}(0)|^2, \tag{4.25}$$

for $s \geqslant s_1$, $\lambda \geqslant \lambda_1$.

Step 3. *Estimation of the integrals along the edges.*

Direct estimations as in [33, proposition 2.1] (without any integration by parts) yield that for some constant $A > 0$

$$\sum_{j,k}\Bigg\{(-4s)\int_0^T\int_0^{l_j}\varphi^k_{j,xx}|u^k_{j,x}|^2 - 4s\mathrm{Im}\int_0^T\int_0^{l_j}\varphi^k_{j,xt}u^k_j\overline{u^k_{j,x}}$$

$$+\int_0^T\int_0^{l_j}|u^k_j|^2[s(\varphi^k_{j,4x}-\varphi^k_{j,tt})-4s^3(\varphi^k_{j,x})^2\varphi^k_{j,xx}]\Bigg\}$$

$$\geqslant A\sum_{j,k}\left\{\lambda^2 s\int_0^T\int_0^{l_j}\frac{\mathrm{e}^{\lambda\psi^k_j}}{t(T-t)}|(\psi^k_j)'u^k_{j,x}|^2+\lambda s^3\int_0^T\int_0^{l_j}|\varphi^k_{j,x}|^3|u^k_j|^2\right\} \tag{4.26}$$

provided that $s \geqslant s_2$, $\lambda \geqslant \lambda_2$. Combining (4.21), (4.25) and (4.26), we infer that

$$\sum_{j,k}\Bigg\{\int_0^T\int_0^{l_j}[|(M^k_1\mathbf{u}^k)_j|^2+|(M^k_2\mathbf{u}^k)_j|^2]+\lambda^2 s\int_0^T\int_0^{l_j}\frac{\mathrm{e}^{\lambda\psi^k_j}}{t(T-t)}|(\psi^k_j)'u^k_{j,x}|^2$$

$$+\lambda s^3\int_0^T\int_0^{l_j}|\varphi^k_{j,x}|^3|u^k_j|^2+cs^3\lambda^3\int_0^T\left(\frac{\mathrm{e}^{\lambda\psi(0)}}{t(T-t)}\right)^3|u(0)|^2\Bigg\}$$

$$\leqslant c\sum_{j,k}\left(\int_0^T\int_0^{l_j}|w^k_j|^2+s\int_0^T|\varphi^k_{j,x}(l_j)|\,|u^k_{j,x}(l_j)|^2\mathrm{d}t\right). \tag{4.27}$$

Replacing u_j^k by $e^{-s\varphi_j^k} q_j$ in (4.27) gives (4.16). □

Remark 4.2. Note that (4.16) is still valid if, in the definition of $\mathcal{Z}$, one replaces

$$q_j \in C^{2,1}([0, l_j] \times [0, T]) \quad \forall j \in [[1, N]]$$

by

$$\mathbf{q} \in H^{2,1}(\Gamma \times (0, T)).$$

4.2. The boundary problem

We consider the following boundary initial-value problem:

$$\begin{cases} \mathrm{i}u_{j,t} + u_{j,xx} + p_j(x)u_j = 0, & x \in (0, l_j), \quad j \in [[1, N]], \quad t \in (0, T), \\ u_j(0, t) = u_l(0, t), & j, k \in [[1, N]], \quad t \in (0, T), \\ \sum_{1 \leqslant j \leqslant N} u_{j,x}(0, t) = 0, & t \in (0, T), \\ u_j(l_j, t) = h_j(t), & j \in [[1, N]], \quad t \in (0, T), \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x), & x \in \Gamma. \end{cases} \tag{4.28}$$

In what follows, we fix the initial data $\mathbf{u}_0$ and the boundary data $\mathbf{h} = \{h_j\}_{j=1,N}$, and we denote by $\mathbf{u}(\mathbf{p})$ the solution of the system (4.28) associated with the potential $\mathbf{p} \in L^\infty(\Gamma)$.

Theorem 4.3. *Assume that* $\mathbf{p} \in L^\infty(\Gamma; \mathbb{R})$, $\mathbf{u}_0 \in L^\infty(\Gamma)$ *and* $r > 0$ *are such that*

- $\mathbf{u}_0(x) \in \mathbb{R}$ *or* $\mathrm{i}\mathbf{u}_0(x) \in \mathbb{R}$ *a.e. in* Γ,
- $|\mathbf{u}_0(x)| \geqslant r > 0$ *a.e. in* Γ *and*
- $\partial_t \mathbf{u}(\mathbf{p}) \in H^{2,1}(\Gamma \times (0, T))$.

Then, for any $m \geqslant 0$, *there exists a constant* $C = C(m, ||\partial_t \mathbf{u}(\mathbf{p})||_{H^{2,1}(\Gamma \times (0,T))}, r) > 0$ *such that for any* $\mathbf{q} \in B_m(0) \subset L^\infty(\Gamma; \mathbb{R})$ *satisfying*

$$\partial_t \mathbf{u}(\mathbf{q}) \in H^{2,1}(\Gamma \times (0, T)),$$

we have that

$$||\mathbf{p} - \mathbf{q}||_{L^2(\Gamma)} \leqslant C \sum_{1 \leqslant j \leqslant N} ||\partial_x[\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})]_j(l_j, \cdot)||_{H^1(0,T)}.$$

Proof. Consider any $\mathbf{p}, \mathbf{q}$ as in the statement of the theorem, and introduce the difference $\mathbf{y} := \mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})$ of the corresponding solutions of (4.28). Then, y fulfils the system

$$\begin{cases} \mathrm{i}y_{j,t} + y_{j,xx} + q_j(x)y_j = f_j(x)R_j(x, t), & x \in (0, l_j), \quad j \in [[1, N]], \quad t \in (0, T), \\ y_j(0, t) = y_l(0, t), & j, k \in [[1, N]], \quad t \in (0, T), \\ \sum_{1 \leqslant j \leqslant N} y_{j,x}(0, t) = 0, & t \in (0, T), \\ y_j(l_j, t) = 0, & j \in [[1, N]], \quad t \in (0, T), \\ y(x, 0) = 0, & x \in \Gamma, \end{cases} \tag{4.29}$$

with $f_j = q_j - p_j$ (real valued) and $R_j := (\mathbf{u}(\mathbf{p}))_j$. To complete the proof of theorem 4.3, we need the following result.

Proposition 4.4. *Suppose that* $\mathbf{R} = \{R_j\}_{j=1,N}$ *satisfies*

- $\mathbf{R}(x, 0) \in \mathbb{R}$ *or* $\mathrm{i}\mathbf{R}(x, 0) \in \mathbb{R}$ *a.e. in* Γ,
- $|\mathbf{R}(x, 0)| \geqslant r > 0$ *a.e. in* Γ,
- $\mathbf{R} \in H^1(0, T; L^\infty(\Gamma))$ *and*
- $\partial_t \mathbf{y} \in H^{2,1}(\Gamma \times (0, T))$.

Then, for any $m \geqslant 0$*, there exists a constant* $C = C(m, ||\mathbf{R}_t||_{L^2(0,T;L^\infty(\Gamma))}, r)$ *such that for any* $\mathbf{q} \in L^\infty(\Gamma, \mathbb{R})$ *with* $||\mathbf{q}||_{L^\infty(\Gamma)} \leqslant m$ *and for all* $\mathbf{f} \in L^2(\Gamma, \mathbb{R})$*, the solution* $\mathbf{y}$ *of (4.29) satisfies*

$$||\mathbf{f}||_{L^2(\Gamma)} \leqslant C \sum_{1\leqslant j\leqslant N} ||y_{j,x}(l_j, \cdot)||_{H^1(0,T)} \,. \tag{4.30}$$

Proof of proposition 4.4. Let $\mathbf{f} \in L^2(\Gamma; \mathbb{R})$ and $\mathbf{R} \in H^1(0, T; L^\infty(\Gamma))$ be such that $\mathbf{R}(x, 0) \in \mathbb{R}$ a.e. in Γ, and let $\mathbf{y}$ be the solution of (4.29). We take the even-conjugate extensions of $\mathbf{y}$ and $\mathbf{R}$ to the interval $(-T, T)$, i.e. we set $\mathbf{y}(x, t) = \overline{\mathbf{y}(x, -t)}$ for $t \in (-T, 0)$ and similarly for $\mathbf{R}$. Since $\mathbf{R}(x, 0) \in \mathbb{R}$ a.e. in Γ, we have that $\mathbf{R} \in H^1(-T, T; L^\infty(\Gamma))$, and $\mathbf{y}$ satisfies the system (4.29) in $\Gamma \times (-T, T)$. In the case when $\mathbf{R}(x, 0) \in \mathrm{i}\mathbb{R}$ a.e. in Γ, the proof is still valid by taking odd-conjugate extensions.

Let $\mathbf{z}(x, t) = \mathbf{y}_t(x, -t)$. Then, $\mathbf{z}$ satisfies the following system:

$$\begin{cases} z_{j,t} + \mathrm{i}z_{j,xx} + \mathrm{i}q_j(x)z_j = \mathrm{i}f_j(x)R_{j,t}(x, t), & x \in (0, l_j), \quad j \in [[1, N]], \quad t \in (-T, T), \\ z_j(0, t) = z_l(0, t), & j, k \in [[1, N]], \quad t \in (-T, T), \\ \sum_{1\leqslant j\leqslant N} z_{j,x}(0, t) = 0, & t \in (-T, T), \\ z_j(l_j, t) = 0, & j \in [[1, N]], \quad t \in (-T, T), \\ \mathbf{z}(x, 0) = -\mathrm{i}\,\mathbf{f}(x)\mathbf{R}(x, 0), & x \in \Gamma. \end{cases} \tag{4.31}$$

We apply proposition 4.1, but on the time interval $(-T, T)$ instead of $(0, T)$. Therefore, here we consider

$$\theta_j^k(x, t) = \frac{\mathrm{e}^{\lambda\psi_j^k(x)}}{(T + t)(T - t)}, \qquad \varphi_j^k(x, t) = \frac{\mathrm{e}^{\lambda C} - \mathrm{e}^{\lambda\psi_j^k(x)}}{(T + t)(T - t)} \qquad \forall (x, t) \in \Gamma \times (-T, T).$$

As in the proof of proposition 4.1, we introduce $w_j^k = \mathrm{e}^{-s\varphi_j^k} z_j$, $(\widetilde{M_2^k}\mathbf{z})_j = \mathrm{e}^{s\varphi_j^k}(M_2^k\mathbf{w}^k)_j$ and $(M_2^k\mathbf{w}^k)_j = w_{j,t}^k + \mathrm{i}(w_{j,xx}^k + s^2|\varphi_{j,x}^k|^2 w_j^k)$. Next, we set

$$J = \sum_{1\leqslant j,k\leqslant N} \int_{-T}^{0} \int_0^{l_j} \mathrm{e}^{-2s\varphi_j^k} (\widetilde{M_2^k}\mathbf{z})_j \overline{z_j} \mathrm{d}x\, \mathrm{d}t.$$

Then, we have

$$\begin{aligned} J &= \sum_{j,k} \int_{-T}^{0}\int_0^{l_j} (M_2^k\mathbf{w}^k)_j \overline{w_j^k} \mathrm{d}x\, \mathrm{d}t \\ &= \sum_{j,k} \left\{ \int_{-T}^{0}\int_0^{l_j} w_{j,t}^k \overline{w_j^k} \mathrm{d}x\, \mathrm{d}t + \mathrm{i} \int_{-T}^{0}\int_0^{l_j} (-|w_{j,x}^k|^2 + s^2|\varphi_{j,x}^k|^2|w_j^k|^2)\, \mathrm{d}x\, \mathrm{d}t \right. \\ &\quad \left. + \mathrm{i} \int_{-T}^{0} w_{j,x}^k \overline{w_j^k}\Big|_0^{l_j} \mathrm{d}t \right\}. \end{aligned}$$

Note that, by (4.31) and (4.11),

$$\sum_j w_{j,x}^k(0)\overline{w_j^k(0)} = \sum_j (z_{j,x}(0) - s\varphi_{j,x}^k(0)z(0))\, \mathrm{e}^{-2s\varphi(0)}\overline{\mathbf{z}(0)} = 0.$$

Therefore,

$$\mathrm{Re}(J) = \frac{1}{2} \sum_{j,k} \int_0^{l_j} |w_j^k(x, 0)|^2 \mathrm{d}x\, \mathrm{d}t = \frac{1}{2} \sum_{j,k} \int_0^{l_j} \mathrm{e}^{-2s\varphi_j^k(x,0)} |\mathbf{f}(x)|^2 |\mathbf{R}(x, 0)|^2 \mathrm{d}x.$$

Using the hypothesis on $\mathbf{R}(x, 0)$, we infer that

$$\operatorname{Re}(J) \geqslant \frac{r^2}{2} \sum_{j,k} \int_0^{l_j} \mathrm{e}^{-2s\varphi_j^k(x,0)} |\mathbf{f}(x)|^2 \mathrm{d}x. \tag{4.32}$$

On the other hand, we have that

$$\begin{aligned}
|J| &\leqslant \sum_{j,k} \left\{ \left(\int_{-T}^0 \int_0^{l_j} \mathrm{e}^{-2s\varphi_j^k} |(\widetilde{M_2^k}\mathbf{z})_j)|^2 \mathrm{d}x\, \mathrm{d}t \right)^{\frac{1}{2}} \left(\int_{-T}^0 \int_0^{l_j} \mathrm{e}^{-2s\varphi_j^k} |z_j|^2 \mathrm{d}x\, \mathrm{d}t \right)^{\frac{1}{2}} \right\} \\
&\leqslant \frac{1}{2} \sum_{j,k} \left\{ \lambda^{-2} s^{-\frac{3}{2}} \int_{-T}^0 \int_0^{l_j} \mathrm{e}^{-2s\varphi_j^k} |(\widetilde{M_2^k}\mathbf{z})_j|^2 \mathrm{d}x\, \mathrm{d}t \right. \\
&\quad \left. + \lambda^2 s^{\frac{3}{2}} \int_{-T}^0 \int_0^{l_j} \mathrm{e}^{-2s\varphi_j^k} |z_j|^2 \mathrm{d}x\, \mathrm{d}t \right\} \\
&\leqslant c\lambda^{-2} s^{-\frac{3}{2}} \sum_{j,k} \left\{ \int_{-T}^0 \int_0^{l_j} \mathrm{e}^{-2s\varphi_j^k} |(\widetilde{M_2^k}\mathbf{z})_j|^2 \mathrm{d}x\, \mathrm{d}t \right. \\
&\quad \left. + \lambda^4 s^3 \int_{-T}^0 \int_0^{l_j} (\theta_j^k)^3 \mathrm{e}^{-2s\varphi_j^k} |z_j|^2 \mathrm{d}x\, \mathrm{d}t \right\},
\end{aligned} \tag{4.33}$$

where we used the fact that

$$\theta_j^k \geqslant T^{-2}.$$

From (4.33), the Carleman estimate (4.16) (applied on the interval $(-T, T)$ instead of $(0, T)$), and the fact that $\varphi_j^k(x, 0) \leqslant \varphi_j^k(x, t)$ for all $(x, t) \in (0, l_j) \times (-T, T)$, that $\theta_j^k \mathrm{e}^{-2s\varphi_j^k}$ is bounded from above in $(0, l_j) \times (-T, T)$, that $\mathbf{q} \in L^\infty(\Gamma)$, and that $\mathbf{R}_t \in L^2(-T, T; L^\infty(\Gamma))$, we infer that for s and λ large enough

$$\begin{aligned}
|J| &\leqslant c\lambda^{-2} s^{-\frac{3}{2}} \sum_{j,k} \left\{ \int_{-T}^T \int_0^{l_j} \mathrm{e}^{-2s\varphi_j^k} |\mathbf{f}\, \mathbf{R}_t|^2 \mathrm{d}x\, \mathrm{d}t + \lambda s \int_{-T}^T \theta_j^k \mathrm{e}^{-2s\varphi_j^k} |z_{j,x}(l_j, t)|^2 \mathrm{d}t \right\} \\
&\leqslant c\lambda^{-2} s^{-\frac{3}{2}} \sum_{j,k} \int_0^{l_j} \mathrm{e}^{-2s\varphi_j^k(x,0)} |\mathbf{f}|^2 \mathrm{d}x + c\lambda^{-1} s^{-\frac{1}{2}} \sum_j \int_{-T}^T |z_{j,x}(l_j, t)|^2 \mathrm{d}t.
\end{aligned} \tag{4.34}$$

It follows from (4.32), (4.34), and the fact that $\mathbf{z}(x, t) = -\overline{\mathbf{z}(x, -t)}$ for $(x, t) \in \Gamma \times (-T, 0)$ that for s and λ large enough

$$\sum_{j,k} \int_0^{l_j} \mathrm{e}^{-2s\varphi_j^k(x,0)} |\mathbf{f}(x)|^2 \mathrm{d}x \leqslant c \sum_j \int_{-T}^0 |z_{j,x}(l_j, t)|^2 \mathrm{d}t. \tag{4.35}$$

Then, (4.30) follows from (4.35) since

$$\mathrm{e}^{-2s\varphi_j^k(x,0)} \geqslant \mathrm{e}^{-2sT^{-2}(\mathrm{e}^{\lambda C}-1)}.$$

This completes the proof of proposition 4.4 and of theorem 4.3. □

5. Open problems

We now mention a few open problems related to our work. One of them is whether it is possible to reduce the number of measurements at the boundaries. It could be interesting to combine the ideas of the paper with those appearing in [16] and [17] where fewer measurements on the boundary are needed but some rationality assumptions on the lengths of the edges have to be made. For the Schrödinger equation, the question whether a Carleman estimate on a tree with N exterior vertices can be written with only one weight function and $N-1$ boundary observations seems to be challenging.

The extension of this work to more general graphs with other kind of coupling is also an open problem. We recall here the works of Kostrykin and Schrader [26, 27], where self-adjoint Laplace operators with general coupling conditions are introduced.

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