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Decay estimates for nonlocal problems via energy methods

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Abstract

In this paper we study the applicability of energy methods to obtain bounds for the asymptotic decay of solutions to nonlocal diffusion problems. With these energy methods we can deal with nonlocal problems that not necessarily involve a convolution, that is, of the form $u_t(x,t) = \int_{\mathbb{R}^d} G(x-y)(u(y,t)-u(x,t))\,dy$. For example, we will consider equations like,

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)\big(u(y,t)-u(x,t)\big)\,dy + f(u)(x,t),$$

and a nonlocal analogous to the p-Laplacian,

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)\big|u(y,t)-u(x,t)\big|^{p-2}\big(u(y,t)-u(x,t)\big)\,dy.$$

The energy method developed here allows us to obtain decay rates of the form,

$$\big\|u(\cdot,t)\big\|_{L^q(\mathbb{R}^d)} \leqslant Ct^{-\alpha},$$

for some explicit exponent α that depends on the parameters, d, q and p, according to the problem under consideration.

Résumé

Dans cet article, nous étudions les applications des méthodes d'énergie à l'obtention de bornes pour la décroissance asymptotique des solutions de problèmes diffusifs non locaux. Avec ces méthodes d'énergie, nous pouvons considérer des problèmes non locaux qui ne sont pas nécessairement des convolutions, i.e. de la forme $u_t(x,t) = \int_{\mathbb{R}^d} G(x-y)(u(y,t)-u(x,t))\,dy$. Par exemple, nous pouvons traiter le cas des équations,

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)\big(u(y,t)-u(x,t)\big)\,dy + f(u)(x,t),$$

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et de l'analogue non local du p-Laplacien,

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)\big|u(y,t)-u(x,t)\big|^{p-2}\big(u(y,t)-u(x,t)\big)\,dy.$$

Les méthodes d'énergie développées ici nous permettent d'obtenir des taux de décroissance de la forme,

$$\big\|u(\cdot,t)\big\|_{L^q(\mathbb{R}^d)} \leqslant Ct^{-\alpha},$$

pour des exposants explicites α qui dépendent des paramètres d, q et p, selon le problème considéré.

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1. Introduction

In this paper our main aim is to apply energy methods to obtain decay estimates for solutions to nonlocal evolution equations.

First, let us introduce the prototype of nonlocal equation that we have in mind. Let $G:\mathbb{R}^d \to \mathbb{R}$ be a nonnegative, compactly supported, radial, continuous function with $\int_{\mathbb{R}^d} G(z)\,dz = 1$. Nonlocal evolution equations of the form:

$$u_t(x,t) = (G*u-u)(x,t) = \int_{\mathbb{R}^d} G(x-y)u(y,t)\,dy - u(x,t), \tag{1.1}$$

and variations of it, have been recently widely used to model diffusion processes. Eq. (1.1) is called nonlocal diffusion equation since the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$, but on all the values of u in a neighborhood of x through the convolution term $G*u$. As stated in [20], if $u(x,t)$ is thought of as a density at the point x at time t and $G(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^d} G(y-x)u(y,t)\,dy = (G*u)(x,t)$ is the rate at which individuals are arriving at position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^d} G(y-x)u(x,t)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies Eq. (1.1). For recent references on nonlocal diffusion see [1–9,11,13–22,26] and references therein. This equation shares many properties with the classical heat equation, $u_t = \Delta u$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and, even if G is compactly supported, perturbations propagate with infinite speed [20]. However, there is no regularizing effect in general.

The asymptotic behavior as $t \to \infty$ for the nonlocal model (1.1) was studied in [12], see also [21] and [22], where the authors prove that every solution to (1.1) with an initial condition u_0 such that $u_0, \widehat{u_0} \in L^1(\mathbb{R}^d)$ has an asymptotic behavior given by $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leqslant Ct^{-d/2}$.

The proof of this fact is based on a explicit representation formula for the solution in Fourier variables. In fact, from Eq. (1.1) we obtain $\widehat{u}_t(\xi,t) = (\widehat{G}(\xi)-1)\widehat{u}(\xi,t)$, and hence the solution is given by, $\widehat{u}(\xi,t) = e^{(\widehat{J}(\xi)-1)t}\widehat{u_0}(\xi)$. From this explicit formula it can be obtained the decay in $L^\infty(\mathbb{R}^d)$ of the solutions, see [12] and [21]. This decay, together with the conservation of mass, gives the decay of the $L^q(\mathbb{R}^d)$-norms by interpolation. It holds, $\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \leqslant C\,t^{-d/2(1-1/q)}$. Note that the asymptotic behavior is the same as the one for solutions of the heat equation and, as happens for the heat equation, the asymptotic profile is a Gaussian [12].

As we have mentioned, our main task here is to develop an energy method to obtain decay estimates. Our motivation to introduce energy methods to deal with nonlocal problems is twofold, first we want to see how energy methods can be applied to equations possibly without any regularization effect and moreover we want to deal with nonlinear problems for which there are no explicit representation formula for the solution (in general, Fourier methods are not applicable to nonlinear problems).

To begin our analysis, we first deal with a linear nonlocal diffusion operator with a nonlinear source, that is, we consider the following evolution problem:

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)\big(u(y,t) - u(x,t)\big)\,dy + f(u)(x,t), \tag{1.2}$$

with f a locally Lipshitz function satisfying the sign condition $f(s)s \leqslant 0$ and $J(x,y)$ a symmetric nonnegative kernel.

We generalize the previous results in two ways, we allow a nonlinear term $f(u)$ imposing only a dissipativity condition, $f(s)s \leqslant 0$, and, what is even more relevant, we can consider equations in which the nonlocal part is not given by a convolution but for a general operator of the form $\int_{\mathbb{R}^d} J(x,y)(u(y)-u(x))\,dy$.

Our first result reads as follows: *under adequate hypothesis on J (see Theorem* 2.1 *in Section* 2) *and f a locally Lipshitz function satisfying the sign condition $f(s)s \leqslant 0$, consider an initial condition $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with $d \geqslant 3$. Then, for any $1 \leqslant q < \infty$ the solution to* (1.2) *verifies the following decay bound*:

$$\big\|u(\cdot,t)\big\|_{L^q(\mathbb{R}^d)} \leqslant C\,t^{-\frac{d}{2}(1-\frac{1}{q})}.$$

Our main hypotheses on J can be summarized as follows: $J(x,y)$ is strictly positive ($\geqslant c_1 > 0$) for $|y-a(x)| \leqslant c_2$, where a is a function with bounded derivatives.

We remark that this decay bound need not be optimal, in the final section we present examples of functions J that give exponential decay in $L^2(\mathbb{R})$. To obtain a complete classification of all possible decay rates seems a very difficult but challenging problem.

Our energy approach not only simplifies the proof of the asymptotic decay in the linear case but also can de applied to handle nonlinear operators, like a nonlocal analogous to the p-Laplacian. Let $p > 2$, and consider

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)\big|u(y,t) - u(x,t)\big|^{p-2}\big(u(y,t) - u(x,t)\big)\,dy. \tag{1.3}$$

This problem, with a convolution kernel, $J(x,y) = G(x-y)$ was considered in [3] and [2] where the authors found existence, uniqueness and the convergence of the solutions to solutions of the local p-Laplacian evolution problem, $v_t = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$ when a rescaling parameter (that measures the size of the support of the convolution kernel G) goes to zero.

In this case the asymptotic decay is described as follows: *given $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a unique solution to* (1.3). *Moreover, under adequate hypothesis on J (see Theorem* 2.1 *in Section* 2) *and $2 \leqslant p < d$, its asymptotic decay is bounded by*:

$$\big\|u(\cdot,t)\big\|_{L^q(\mathbb{R}^d)} \leqslant C\,t^{-(\frac{d}{d(p-2)+p})(1-\frac{1}{q})},$$

for $1 \leqslant q < \infty$.

This asymptotic decay is the same one that holds for solutions to the local p-Laplacian, $v_t = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$, see Chapter 11 in [28].

The assumption on the initial data, $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, is imposed since, in general, nonlocal evolution equations have no regularizing $L^1(\mathbb{R}^d) - L^q(\mathbb{R}^d)$ effect. In the particular case of a convolution kernel $J(x,y) = G(x-y)$, i.e. Eq. (1.1), in [12] it is proved that solutions u can be written as $u(t) = e^{-t}u_0 + K_t * u_0$, where K_t is a smooth function. As a consequence at any time $t > 0$, the solution u is as regular as the initial datum u_0 is. Thus, it is hopeless to guarantee that at any time $t > 0$, the solution $u(t)$ belongs to $L^q(\mathbb{R}^d)$ without assuming that $u_0 \in L^q(\mathbb{R}^d)$.

We also have to mention that we are assuming the following hypothesis on the kernel $J(x;\cdot) \in L^1(\mathbb{R}^d)$. This excludes the analysis of the possibility of a faster decay for u if for example J has fat tails, as happens for equations involving generators of Levy processes.

The rest of the paper is organized as follows: In Section 2 we collect some preliminaries and prove a decomposition theorem that will be used to apply energy methods; in Section 3 we deal with the decay of solutions with linear nonlocal diffusion and a nonlinear dissipative source and in Section 4 we prove the decay for the nonlocal p-Laplacian. Finally in Section 5 we present examples of J for which we can prove exponential decay bounds for the linear problem.

2. Preliminaries

In this section we collect some preliminaries and state and prove a crucial decomposition theorem. In what follows we denote by:

$$p^* = \frac{pd}{(d-p)}$$

the usual Sobolev exponent, while

$$p' = \frac{p}{p-1}$$

denotes the usual conjugate exponent.

First, let us describe briefly how the energy method can be applied to obtain decay estimates for local problems. Let us begin with the simpler case of the estimate for solutions to the heat equation in $L^2(\mathbb{R}^d)$-norm,

$$u_t = \Delta u.$$

If we multiply by u and integrate in $\mathbb{R}^d$, we obtain:

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)\,dx = -\int_{\mathbb{R}^d} \left|\nabla u(x,t)\right|^2 dx.$$

Now we use Sobolev's inequality,

$$\int_{\mathbb{R}^d} |\nabla u|^2(x,t)\,dx \geqslant C\left(\int_{\mathbb{R}^d} |u|^{2^*}(x,t)\,dx\right)^{2/2^*},$$

to obtain:

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)\,dx \leqslant -C\left(\int_{\mathbb{R}^d} |u|^{2^*}(x,t)\,dx\right)^{2/2^*}.$$

If we use interpolation and conservation of mass, that implies $\|u(t)\|_{L^1(\mathbb{R}^d)} \leqslant C$ for any $t > 0$, we have:

$$\left\|u(t)\right\|_{L^2(\mathbb{R}^d)} \leqslant \left\|u(t)\right\|^{\alpha}_{L^1(\mathbb{R}^d)} \left\|u(t)\right\|^{1-\alpha}_{L^{2^*}(\mathbb{R}^d)} \leqslant C\left\|u(t)\right\|^{1-\alpha}_{L^{2^*}(\mathbb{R}^d)},$$

with α determined by

$$\frac{1}{2} = \alpha + \frac{1-\alpha}{2^*}, \quad \text{that is,} \quad \alpha = \frac{2^*-2}{2(2^*-1)}.$$

Hence we get:

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)\,dx \leqslant -C\left(\int_{\mathbb{R}^d} u^2(x,t)\,dx\right)^{1/(1-\alpha)},$$

from where the decay estimate

$$\left\|u(t)\right\|_{L^2(\mathbb{R}^d)} \leqslant C\,t^{-\frac{d}{2}(1-\frac{1}{2})}, \quad t > 0,$$

follows.

In the case of the p-Laplacian in the whole space,

$$u_t = \Delta_p u,$$

the argument is similar, we multiply by u, integrate in $\mathbb{R}^d$ and use Sobolev inequality, that in this case reads,

$$\int_{\mathbb{R}^d} |\nabla u|^p(x,t)\,dx \geqslant C\left(\int_{\mathbb{R}^d} |u|^{p^*}(x,t)\,dx\right)^{p/p^*},$$

and interpolation to get a similar inequality for the L^2-norm of a solution,

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)\,dx \leqslant -C\biggl(\int_{\mathbb{R}^d} u^2(x,t)\,dx\biggr)^{\theta},$$

for an explicit $\theta < 1$ that depends on p and d. As before this inequality implies a decay bound for the L^2-norm.

We want to mimic the steps for the nonlocal evolution problem:

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)\bigl(u(y,t) - u(x,t)\bigr)\,dy.$$

Hence, we multiply by u and integrate in $\mathbb{R}^d$ to obtain:

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)\,dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\bigl(u(y,t) - u(x,t)\bigr)\,dy\,u(x,t)\,dx. \tag{2.1}$$

Now, we need to "integrate by parts". Therefore, let us begin by a simple algebraic identity (whose proof is immediate) that plays the role of an integration by parts formula for nonlocal operators.

Lemma 2.1. *If J is symmetric, $J(x,y) = J(y,x)$ then it holds*:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\bigl(\varphi(y) - \varphi(x)\bigr)\psi(x)\,dy\,dx = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\bigl(\varphi(y) - \varphi(x)\bigr)\bigl(\psi(y) - \psi(x)\bigr)\,dy\,dx.$$

If we apply this lemma to (2.1) we get:

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)\,dx = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\bigl(u(y,t) - u(x,t)\bigr)^2\,dy\,dx,$$

but now we run into troubles since there is no analogous to Sobolev inequality. In fact, an inequality of the form,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\bigl(u(y,t) - u(x,t)\bigr)^2\,dy\,dx \geqslant C\biggl(\int_{\mathbb{R}^d} u^q(x,t)\,dx\biggr)^{2/q},$$

cannot hold for any $q > 2$.

Now the idea is to split the function u as the sum of two functions $u = v + w$, where on the function v (the "smooth" part of the solution) the nonlocal operator acts as a gradient and on the function w (the "rough" part) it does not increase its norm significatively.

Therefore, we need to obtain estimates for the $L^p(\mathbb{R}^d)$-norm of the nonlocal operators. The main result of this section is the following.

Theorem 2.1. *Let $p \in [1,\infty)$ and $J(\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$ be a symmetric nonnegative function satisfying*:

(HJ1) *There exists a positive constant $C < \infty$ such that*

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\,dx \leqslant C.$$

(HJ2) *There exist positive constants c_1, c_2 and a function $a \in C^1(\mathbb{R}^d,\mathbb{R}^d)$ satisfying,*

$$\sup_{x\in\mathbb{R}^d}\bigl|\nabla a(x)\bigr| < \infty, \tag{2.2}$$

such that the set

$$B_x = \bigl\{y \in \mathbb{R}^d\colon\ \bigl|y - a(x)\bigr| \leqslant c_2\bigr\} \tag{2.3}$$

verifies

$$B_x \subset \{ y \in \mathbb{R}^d:\ J(x,y) > c_1 \}.$$

Then, for any function $u \in L^p(\mathbb{R}^d)$ there exist two functions v and w such that $u = v + w$, and

$$\|\nabla v\|^p_{L^p(\mathbb{R}^d)} + \|w\|^p_{L^p(\mathbb{R}^d)} \leqslant C(J,p) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big|u(x)-u(y)\big|^p \, dx\, dy. \tag{2.4}$$

Moreover, if $u \in L^q(\mathbb{R}^d)$ with $q \in [1,\infty]$ then the functions v and w satisfy:

$$\|v\|_{L^q(\mathbb{R}^d)} \leqslant C(J,q)\|u\|_{L^q(\mathbb{R}^d)}, \tag{2.5}$$

and

$$\|w\|_{L^q(\mathbb{R}^d)} \leqslant C(J,q)\|u\|_{L^q(\mathbb{R}^d)}. \tag{2.6}$$

Before the proof we collect some remarks and a prove a corollary.

Remark 2.1. The above result says that there exists a decomposition of u in a smooth part, v, and a rough part, w, such that the action of the nonlocal operator is like a gradient on the smooth part and as the identity on the rough part.

Remark 2.2. The constant $C(J;q)$ in the theorem depends only on the constants of (HJ1) and (HJ2) and not on any other characteristic of the kernel J.

Remark 2.3. We note that in the case $1 \leqslant p < d$ using the classical Sobolev's inequality $\|v\|_{L^{p^*}(\mathbb{R}^d)} \leqslant \|\nabla v\|_{L^p(\mathbb{R}^d)}$ we get that (2.4) implies:

$$\|v\|^p_{L^{p*}(\mathbb{R}^d)} + \|w\|^p_{L^p(\mathbb{R}^d)} \leqslant C(J,p) \int_{\mathbb{R}^r}\int_{\mathbb{R}^d} J(x,y)\big|u(x)-u(y)\big|^p \, dx\, dy.$$

Remark 2.4. In particular, we can consider $a(x) = x$, that is, the case of a convolution kernel, $J(x,y) = G(x-y)$, with $G(0) > 0$. In fact, it is reasonable to assume that $J(x,x) > 0$ since in biological models this means that the probability that some individuals that are in x at time t remain at the same position is positive.

To simplify the notation let us note by $\langle A_p u, u\rangle$ the following quantity:

$$\langle A_p u, u\rangle := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big|u(x)-u(y)\big|^p \, dx\, dy.$$

Observe that, in order that the above quantity to be finite, we have to assume a priori that u belongs to $L^p(\mathbb{R}^d)$.

Note that our main result of this section, Theorem 2.1 gives estimates from below for $\langle A_p u, u\rangle$. A corollary of this result is the following.

Corollary 2.1. *Let $J(\cdot,\cdot): \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a symmetric nonnegative function satisfying hypotheses* (HJ1) *and* (HJ2) *in Theorem* 2.1 *and $p \in [1,d)$. There exist two positive constants $C_1 = C_1(J,p)$ and $C_2 = C_2(J,p)$ such that for any $u \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ the following holds*:

$$\|u\|^p_{L^p(\mathbb{R}^d)} \leqslant C_1 \|u\|^{p(1-\alpha(p))}_{L^1(\mathbb{R}^d)} \langle A_p u, u\rangle^{\alpha(p)} + C_2 \langle A_p u, u\rangle, \tag{2.7}$$

where $\alpha(p)$ satisfies:

$$\frac{1}{p} = \frac{\alpha(p)}{p^*} + 1 - \alpha(p).$$

Remark 2.5. The explicit value of $\alpha(p)$ is given by:

$$\alpha(p) = \frac{p^*}{p'(p^*-1)} = \frac{d(p-1)}{d(p-1)+p}. \tag{2.8}$$

Remark 2.6. In the case of the local operator $B_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, using Sobolev's inequality and interpolation inequalities we have the following estimate:

$$\|u\|^p_{L^p(\mathbb{R}^d)} \leqslant C_1 \|u\|^{p(1-\alpha(p))}_{L^1(\mathbb{R}^d)} \langle B_p u, u\rangle^{\alpha(p)}.$$

In the nonlocal case an extra term involving $\langle A_p u, u\rangle$ occurs, see (2.7).

Proof of Corollary 2.1. We use the decomposition $u = v + w$ given by Theorem 2.1 to obtain:

$$\|u\|^p_{L^p(\mathbb{R}^d)} \leqslant \|v\|^p_{L^p(\mathbb{R}^d)} + \|w\|^p_{L^p(\mathbb{R}^d)}.$$

Also, by (2.4), we have:

$$\|\nabla v\|^p_{L^p(\mathbb{R}^d)} \leqslant C(J,p)\langle A_p u, u\rangle,$$

and

$$\|w\|^p_{L^p(\mathbb{R}^d)} \leqslant C(J,p)\langle A_p u, u\rangle.$$

Then, from the interpolation inequality,

$$\|v\|_{L^p(\mathbb{R}^d)} \leqslant \|v\|^{\alpha(p)}_{L^{p*}(\mathbb{R}^d)} \|v\|^{1-\alpha(p)}_{L^1(\mathbb{R}^d)},$$

we obtain that the $L^p(\mathbb{R}^d)$-norm of u satisfies:

$$\begin{aligned}
\|u\|^p_{L^p(\mathbb{R}^d)} &\leqslant \|v\|^{\alpha(p)p}_{L^{p*}(\mathbb{R}^d)} \|v\|^{(1-\alpha(p))p}_{L^1(\mathbb{R}^d)} + \|w\|^p_{L^p(\mathbb{R}^d)} \\
&\leqslant \|\nabla v\|^{\alpha(p)p}_{L^p(\mathbb{R}^d)} \|u\|^{(1-\alpha(p))p}_{L^1(\mathbb{R}^d)} + C(J,p)\langle A_p u, u\rangle \\
&\leqslant C_1 \|u\|^{(1-\alpha(p))p}_{L^1(\mathbb{R}^d)} \langle A_p u, u\rangle^{\alpha(p)} + C_2 \langle A_p u, u\rangle,
\end{aligned}$$

as we wanted to prove. □

Now we proceed with the proof of the decomposition theorem.

Proof of Theorem 2.1. We divide the proof in two steps. First of all, we prove under the assumptions (HJ1)–(HJ2) the existence of a function $\rho(\cdot,\cdot)$ satisfying:

(H1) $\rho(x,\cdot) \in C_c^\infty(\mathbb{R}^d)$ for a.e. $x \in \mathbb{R}^d$,
(H2) $\int_{\mathbb{R}^d} \rho(x,y)\,dy = 1$ for a.e. $x \in \mathbb{R}^d$,
(H3) $\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x,y)\,dx \leqslant M < \infty$,
(H4) $\operatorname{supp} \rho(x,\cdot) \subset B_x$ for a.e. $x \in \mathbb{R}^d$,
(H5) $\sup_{x\in\mathbb{R}^d} \|\rho(x,\cdot)\|_{L^{p'}(\mathbb{R}^d)} \leqslant M < \infty$,
(H6) $\sum_{k=1}^d \sup_{x\in\mathbb{R}^d} \|\partial_{x_k} \rho(x,\cdot)\|_{L^{p'}(\mathbb{R}^d)} \leqslant M < \infty$.

Next, we define:

$$v(x) = \int_{\mathbb{R}^d} \rho(x,y)u(y)\,dy, \quad \text{and} \quad w = u - v,$$

and prove (2.4), (2.5) and (2.6).

Step I. Construction of ρ. With c_2 given by (HJ2) we consider a smooth function $\psi \in C_c^\infty(\mathbb{R}^d)$ supported in the ball $B_{c_2}(0)$, $0 \leqslant \psi \leqslant C$ and having mass one:

$$\int_{B_{c_2}(0)} \psi(x)\,dx = 1.$$

For any $x \in \mathbb{R}^d$ we consider the function $a(x)$ and the set B_x as in (2.3), see (HJ2). We then define $\rho(x,y)$ by:

$$\rho(x,y) = \psi\big(y - a(x)\big). \tag{2.9}$$

We will prove properties (H3) and (H6) since the others easily follow with a constant $M(J)$. We point out that the assumption on the existence of a ball B_x centered at $a(x)$ with radius c_2 is necessary in proving (H5). Otherwise, $\inf_{x\in\mathbb{R}^d} |B_x| = 0$ and by Hölder inequality, we get:

$$\big\|\rho(x,\cdot)\big\|_{L^{p'}(\mathbb{R}^d)} \geqslant \frac{\int_{\mathbb{R}^d} \rho(x,y)\,dy}{|B_x|^{1/p}} = \frac{1}{|B_x|^{1/p}},$$

and then

$$\sup_{x\in\mathbb{R}^d} \big\|\rho(x,\cdot)\big\|_{L^{p'}(\mathbb{R}^d)} \geqslant \frac{1}{\inf_{x\in\mathbb{R}^d} |B_x|^{1/p}} = \infty.$$

Therefore, we cannot obtain property (H5).

We now prove property (H3). Observe that, by definition (2.9) of the function $\rho(\cdot,\cdot)$ and the fact that $\psi \leqslant C$ we have:

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x,y)\,dx = \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \psi\big(y-a(x)\big)\,dx = \sup_{y\in\mathbb{R}^d} \int_{|y-a(x)|\leqslant c_2} \psi\big(y-a(x)\big)\,dx$$
$$\leqslant C \sup_{y\in\mathbb{R}^d} \big|\big\{x\colon \big|y-a(x)\big| \leqslant c_2\big\}\big|.$$

It remains to show that the last term in the right-hand side is finite. Indeed, given y, we have:

$$\big|\big\{x\colon \big|y-a(x)\big| \leqslant c_2\big\}\big| \leqslant \int_{\{x\colon |y_n-a(x)|\leqslant c_2\}} \frac{J(x,y)}{c_1}\,dx \leqslant \frac{1}{c_1} \int_{\mathbb{R}^d} J(x,y)\,dx \leqslant C.$$

We now prove (HJ6). By definition (2.9) for any $x \in \mathbb{R}^d$ we have:

$$\big\|\partial_{x_k} \rho(x,\cdot)\big\|_{L^{p'}(\mathbb{R}^d)} = \big\|\nabla\psi\big(\cdot - a(x)\big)\cdot \partial_{x_k} a(x)\big\|_{L^{p'}(\mathbb{R}^d)} \leqslant \big|\partial_{x_k} a(x)\big| \|\nabla\psi\|_{L^{p'}(\mathbb{R}^d)}.$$

Using (2.2) and the construction of ψ we obtain (HJ6).

Step II. Proof of the estimates on u, v and w. We have proved that there exists a function ρ satisfying hypotheses (H1)–(H6). Let us take

$$v(x) = \int_{\mathbb{R}^d} \rho(x,y)u(y)\,dy, \quad \text{and} \quad w = u - v.$$

First we prove (2.5) and (2.6). Hölder's inequality applied to the function v and (H2) guarantee that

$$\big|v(x)\big|^q \leqslant \int_{\mathbb{R}^d} \rho(x,y)\big|u(y)\big|^q\,dy \bigg(\int_{\mathbb{R}^d} \rho(x,y)\,dy\bigg)^{\frac{q}{q'}} = \int_{\mathbb{R}^d} \rho(x,y)\big|u(y)\big|^q\,dy.$$

Then, property (H3) gives us,

$$\int_{\mathbb{R}^d} \big|v(x)\big|^q\,dx \leqslant \int_{\mathbb{R}^d} \big|u(y)\big|^q \int_{\mathbb{R}^d} \rho(x,y)\,dx\,dy \leqslant \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x,y)\,dx \int_{\mathbb{R}^d} \big|u(y)\big|^q\,dy$$
$$\leqslant M \int_{\mathbb{R}^d} \big|u(y)\big|^q\,dy,$$

which proves (2.5).

Also, we obviously have:

$$\|w\|_{L^q(\mathbb{R}^d)} \leqslant \|u\|_{L^q(\mathbb{R}^d)} + \|v\|_{L^q(\mathbb{R}^d)} \leqslant \big(1 + M^{1/q}\big)\|u\|_{L^q(\mathbb{R}^d)}.$$

We now proceed to prove (2.4). To do that we prove the following inequalities:

$$\|w\|^p_{L^p(\mathbb{R}^d)} \leqslant c_1^{-1} \sup_{x\in\mathbb{R}^d} \big\|\rho(x,\cdot)\big\|^p_{L^{p'}(\mathbb{R}^d)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big|u(x)-u(y)\big|^p\,dx\,dy, \tag{2.10}$$

and

$$\|\nabla v\|^p_{L^p(\mathbb{R}^d)} \leqslant \sum_{k=1}^d c_1^{-1} \sup_{x\in\mathbb{R}^d} \big\|\partial_{x_k}\rho(x,\cdot)\big\|^p_{L^{p'}(\mathbb{R}^d)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big|u(x)-u(y)\big|^p\,dx\,dy. \tag{2.11}$$

The fact that for any $x \in \mathbb{R}^d$, $\rho(x,\cdot)$ is supported in the set B_x and has mass one gives the following:

$$\begin{aligned} w(x) &= u(x) - \int_{\mathbb{R}^d} \rho(x,y)u(y)\,dy = \int_{\mathbb{R}^d} \rho(x,y)\big(u(x)-u(y)\big)\,dy \\ &= \int_{B_x} \rho(x,y)\big(u(x)-u(y)\big)\,dy. \end{aligned}$$

Then by Hölder's inequality we get:

$$\begin{aligned} \|w\|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \left|\int_{B_x} \rho(x,y)\big(u(x)-u(y)\big)\,dy\right|^p dx \\ &\leqslant \int_{\mathbb{R}^d}\int_{B_x} \big|u(x)-u(y)\big|^p\,dy \left(\int_{B_x} \rho(x,y)^{p'}\,dy\right)^{\frac{p}{p'}} dx \\ &\leqslant \sup_{x\in\mathbb{R}^d} \left(\int_{B_x} \rho(x,y)^{p'}\,dy\right)^{\frac{p}{p'}} \int_{\mathbb{R}^d}\int_{B_x} \big|u(x)-u(y)\big|^p\,dy\,dx \\ &\leqslant \sup_{x\in\mathbb{R}^d} \big\|\rho(x,\cdot)\big\|^p_{L^{p'}(\mathbb{R}^d)} \int_{\mathbb{R}^d}\int_{B_x} \big|u(x)-u(y)\big|^p\,dy\,dx. \end{aligned}$$

Using now that for any $x \in \mathbb{R}^d$ and $y \in B_x$ we have $J(x,y) > c_1$ we obtain:

$$\begin{aligned} \|w\|^p_{L^p(\mathbb{R}^d)} &\leqslant c_1^{-1} \sup_{x\in\mathbb{R}^d} \big\|\rho(x,\cdot)\big\|^p_{L^{p'}(\mathbb{R}^d)} \int_{\mathbb{R}^d}\int_{B_x} J(x,y)\big|u(x)-u(y)\big|^p\,dy\,dx \\ &\leqslant c_1^{-1} \sup_{x\in\mathbb{R}^d} \big\|\rho(x,\cdot)\big\|^p_{L^{p'}(\mathbb{R}^d)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big|u(x)-u(y)\big|^p\,dy\,dx, \end{aligned}$$

which proves (2.10).

In the case of v we proceed in a similar manner, by tacking into account that for any $x \in \mathbb{R}$ the mass of $\partial_{x_k}\rho(x,y)$, $k = 1,\dots,d$ vanishes:

$$\int_{\mathbb{R}^d} \partial_{x_k}\rho(x,y)\,dy = \partial_{x_k}\left(\int_{\mathbb{R}^d} \rho(x,y)\,dy\right) = 0.$$

The definition of v and this mass property gives,

$$\partial_{x_k} v(x) = \int_{\mathbb{R}^d} \partial_{x_k}\rho(x,y)\big(u(y)-u(x)\big)\,dy = \int_{B_x} \partial_{x_k}\rho(x,y)\big(u(y)-u(x)\big)\,dy.$$

Thus, by Hölder inequality and the fact that $J(x,y) > c_1$ for all $y \in B_x$ we obtain:

$$\begin{aligned}
\|\partial_{x_k} v\|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \left| \int_{B_x} \partial_{x_k}\rho(x,y)\big(u(y)-u(x)\big)\,dy \right|^p dx \\
&\leqslant \int_{\mathbb{R}^d} \int_{B_x} \big|u(y)-u(x)\big|^p\,dy \bigg(\int_{B_x} \big|\partial_{x_k}\rho(x,y)\big|^{p'}\,dy \bigg)^{\frac{p}{p'}} dx \\
&= \sup_{x\in\mathbb{R}^d} \big\|\partial_{x_k}\rho(x,\cdot)\big\|^p_{L^{p'}(\mathbb{R}^d)} \int_{\mathbb{R}^d}\int_{B_x} \big|u(y)-u(x)\big|^p\,dx\,dy \\
&\leqslant c^{-1} \sup_{x\in\mathbb{R}^d} \big\|\partial_{x_k}\rho(x,\cdot)\big\|^p_{L^{p'}(\mathbb{R}^d)} \int_{\mathbb{R}^d}\int_{B_x} J(x,y)\big|u(y)-u(x)\big|^p\,dx\,dy \\
&\leqslant c^{-1} \sup_{x\in\mathbb{R}^d} \big\|\partial_{x_k}\rho(x,\cdot)\big\|^p_{L^{p'}(\mathbb{R}^d)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big|u(y)-u(x)\big|^p\,dx\,dy.
\end{aligned}$$

Summing the above inequalities for all $k = 1, \dots, d$ we get (2.11).

The proof is now finished since (2.10) and (2.11) imply (2.4). □

Now we present a similar result to Corollary 2.1 which can be used to obtain less accurate bounds (hence we prefer to use the more general result presented above) in the particular case of the nonlocal Laplacian, i.e. $p = 2$, and $J(x,y) = G(x-y)$. The result is no so general as Corollary 2.1, but it is obtained using Fourier analysis tools and has the advantage that the previous decomposition $u = v + w$ can be better understood. We include it here just for this purpose. In fact this decomposition can be viewed as a Fourier splitting of the function u in two parts, the first one, v, corresponding to the low frequencies (the smooth part) of u, and the second one, w, corresponds to the high frequencies component (the rough part) of u.

We will use that in the particular case $p = 2$ and $J(x,y) = G(x-y)$, G with mass one, the operator $\langle A_2 u, u\rangle$ can be represented by means of the Fourier transform of G as follows:

$$\langle A_2 u, u\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)\big|u(x)-u(y)\big|^2\,dx\,dy = \int_{\mathbb{R}^d} \big(1-\widehat{G}(\xi)\big)\big|\widehat{u}(\xi)\big|^2\,d\xi.$$

Lemma 2.2. *Let $d \geqslant 3$ and G be such that its Fourier transform $\widehat{G}(\xi)$ satisfies*:

$$\begin{cases} \widehat{G}(\xi) \leqslant 1 - \frac{|\xi|^2}{2}, & |\xi| \leqslant R, \\ \widehat{G}(\xi) \leqslant 1-\delta, & |\xi| \geqslant R, \end{cases} \tag{2.12}$$

for some positive numbers R and δ. Then, for any $\varepsilon \in (0,1)$ there exists a constant $C = C(\varepsilon, \delta, R, d)$ such that the following:

$$\|u\|^2_{L^2(\mathbb{R})} \leqslant C\|u\|^{2(1-\beta(\varepsilon))}_{L^{1+\varepsilon}(\mathbb{R}^d)} \langle A_2 u, u\rangle^{\beta(\varepsilon)} + \langle A_2 u, u\rangle \tag{2.13}$$

holds for all $u \in L^{1+\varepsilon}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, where

$$\beta(\varepsilon) = \frac{(1-\varepsilon)d}{d+2-\varepsilon(d-2)}.$$

Remark 2.7. The limit case $\varepsilon = 0$ cannot be obtained since an estimate of the type,

$$\big\|(\mathbf{1}_{\{|\xi|\leqslant R\}}\widehat{u})^{\vee}\big\|_{L^1(\mathbb{R}^d)} \leqslant \|u\|_{L^1(\mathbb{R}^d)},$$

does not hold for all functions $u \in L^1(\mathbb{R}^d)$. In dimension one this can be seen by choosing a sequence u_ε with $\|u_\varepsilon\|_{L^1(\mathbb{R}^d)} = 1$ such that $u_\varepsilon \to \delta_0$, the Dirac delta. Then

$$(\mathbf{1}_{\{|\xi|\leqslant R\}}\widehat{u}_\varepsilon)^\vee = u_\varepsilon * \frac{\sin(Rx)}{Rx} \to \frac{\sin Rx}{Rx},$$

and the last function does not belong to $L^1(\mathbb{R}^d)$. Thus $\|(\mathbf{1}_{\{|\xi|\leqslant R\}}\widehat{u}_\varepsilon)^\vee\|_{L^1(\mathbb{R}^d)} \to \infty$ but $\|u_\varepsilon\|_{L^1(\mathbb{R}^d)} = 1$.

Remark 2.8. The same arguments can be used to obtain estimates for any function G which satisfies:

$$\begin{cases} \widehat{G}(\xi) \leqslant 1 - \frac{|\xi|^{2s}}{2}, & |\xi| \leqslant R, \\ \widehat{G}(\xi) \leqslant 1 - \delta, & |\xi| \geqslant R, \end{cases}$$

for some positive numbers R, δ and s.

Proof of Lemma 2.2. For any function $u \in L^2(\mathbb{R}^d)$ we define its projections on the low and high frequencies respectively,

$$v := (\mathbf{1}_{\{|\xi|\leqslant R\}}\widehat{u})^\vee, \qquad w := (\mathbf{1}_{\{|\xi|\geqslant R\}}\widehat{u})^\vee. \tag{2.14}$$

Using that the function $\widehat{G}$ satisfies (2.12) we obtain the following estimate for the operator A_2:

$$\begin{aligned} \langle A_2 u, u\rangle &= \int_{\mathbb{R}^d} \big(1-\widehat{G}(\xi)\big)\big|\widehat{u}(\xi)\big|^2\,d\xi \geqslant \int_{|\xi|\leqslant R} \frac{|\xi|^2}{2}\big|\widehat{u}(\xi)\big|^2\,d\xi + \delta \int_{|\xi|\geqslant R} \big|\widehat{u}(\xi)\big|^2\,d\xi \\ &= \frac{1}{2}\int_{\mathbb{R}^d} |\xi|^2\big|\widehat{v}(\xi)\big|^2\,d\xi + \delta\int_{\mathbb{R}^d}\big|\widehat{w}(\xi)\big|^2\,d\xi \\ &\geqslant c(\delta)\big(\|\nabla v\|^2_{L^2(\mathbb{R}^d)} + \|w\|^2_{L^2(\mathbb{R}^d)}\big) \\ &\geqslant c(\delta)\big(\|v\|^2_{L^{2^*}(\mathbb{R}^d)} + \|w\|^2_{L^2(\mathbb{R}^d)}\big). \end{aligned} \tag{2.15}$$

In order to estimate from above the $L^2(\mathbb{R}^d)$-norm of u as in (2.13), using the orthogonality of v and w it is sufficient to estimate each projection v and w, since

$$\|u\|^2_{L^2(\mathbb{R}^d)} = \|v\|^2_{L^2(\mathbb{R}^d)} + \|w\|^2_{L^2(\mathbb{R}^d)}.$$

In the case of w, using (2.14) and (2.15) we have the rough estimate:

$$\|w\|^2_{L^2(\mathbb{R}^d)} \leqslant \frac{1}{c(\delta)}\langle A_2 u, u\rangle. \tag{2.16}$$

Next we estimate the $L^2(\mathbb{R}^d)$-norm of v. We recall that classical results on Fourier multipliers (see Chapter 4 in [27]) give us that for any $p \in (1,\infty)$ the $L^p(\mathbb{R}^d)$-norm of v, defined by (2.14), can be bounded from above by the $L^p(\mathbb{R}^d)$-norm of u as follows:

$$\|v\|_{L^p(\mathbb{R}^d)} \leqslant C(p,d)\|u\|_{L^p(\mathbb{R}^d)}. \tag{2.17}$$

Using this estimate and interpolation inequalities we obtain that v, the low frequency projection of u, satisfies:

$$\begin{aligned} \|v\|^2_{L^2(\mathbb{R}^d)} &\leqslant \big(\|v\|^{1-\beta(\varepsilon)}_{L^{1+\varepsilon}(\mathbb{R}^d)}\|v\|^{\beta(\varepsilon)}_{L^{2^*}(\mathbb{R}^d)}\big)^2 \leqslant \big(c(\varepsilon,d)\|u\|^{1-\beta(\varepsilon)}_{L^{1+\varepsilon}(\mathbb{R}^d)}\|v\|^{\beta(\varepsilon)}_{L^{2^*}(\mathbb{R}^d)}\big)^2 \\ &\leqslant c^2(\varepsilon,d)c(\delta)^{-\beta(\varepsilon)}\|u\|^{2(1-\beta(\varepsilon))}_{L^{1+\varepsilon}(\mathbb{R}^d)}\langle A_2 u, u\rangle^{\beta(\varepsilon)}, \end{aligned} \tag{2.18}$$

where $c(\varepsilon,d)$ is given by applying (2.17) with $p = 1+\varepsilon$ and $\beta(\varepsilon)$ by:

$$\frac{1}{2} = \frac{1-\beta(\varepsilon)}{1+\varepsilon} + \frac{\beta(\varepsilon)}{2^*},$$

that is,

$$\beta(\varepsilon) = \frac{(1-\varepsilon)d}{d+2-\varepsilon(d-2)}.$$

Combining (2.15), (2.16) and (2.18) we obtain:

$$\|u\|^2_{L^2(\mathbb{R}^d)} \leqslant c(\varepsilon,\delta,d)\|u\|^{2(1-\beta(\varepsilon))}_{L^{1+\beta(\varepsilon)}(\mathbb{R}^d)} \langle A_2u,u\rangle^{\alpha(\varepsilon)} + \langle A_2u,u\rangle. \tag{2.19}$$

The proof is now finished. □

We end this section with a crucial but simple result concerning the decay of solutions to a differential inequality.

Lemma 2.3. *Let $t_0 \geqslant 0$ and $\psi : [0,\infty) \to (0,\infty)$ such that for all $t > t_0$,*

$$\psi_t + \alpha\psi^\beta t^\gamma \leqslant 0, \tag{2.20}$$

holds for some constants $\alpha > 0$, $\beta > 1$ and γ. Then there exists a positive constant $c(\alpha,\beta)$ such that

$$\psi(t) \leqslant c(\alpha,\beta,\gamma)\big(t^{\gamma+1} - t_0^{\gamma+1}\big)^{-\frac{1}{\beta-1}}$$

holds for all $t > t_0$.

Proof. Inequality (2.20) gives us

$$\psi_t\psi^{-\beta} + \alpha t^\gamma \leqslant 0.$$

Integrating on $[t_0,t]$ we find that for any $t \geqslant t_0$,

$$\frac{\psi^{1-\beta}(t)}{1-\beta} - \frac{\psi^{1-\beta}(t_0)}{1-\beta} + \alpha\frac{(t^{\gamma+1} - t_0^{\gamma+1})}{\gamma+1} \leqslant 0.$$

Then

$$\alpha\big(t^{\gamma+1} - t_0^{\gamma+1}\big)(\beta-1)(\gamma+1)^{-1} \leqslant \psi^{1-\beta}(t),$$

and hence

$$\psi(t) \leqslant c(\alpha,\beta,\gamma)\big(t^{\gamma+1} - t_0^{\gamma+1}\big)^{-\frac{1}{\beta-1}},$$

where $c(\alpha,\beta,\gamma)$ is a constant. □

3. Decay estimates for the linear diffusion problem with a nonlinear source

In this section we will obtain the long time behavior of the solutions u to the following equation:

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)\big(u(y,t) - u(x,t)\big)\,dy + f(u)(x,t), \tag{3.1}$$

under suitable assumptions on the kernel J and the nonlinearity f. Our goal is to obtain here a proof of the decay rate of the solution u to (3.1) by using energy methods.

The main result of this section is the following theorem.

Theorem 3.1. *Let $J(x,y)$ be a symmetric nonnegative kernel satisfying* (HJ1) *as in Theorem* 2.1 *and f be a locally Lipshitz function with $f(s)s \leqslant 0$. For any $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a unique solution to Eq.* (3.1) *which satisfies*:

$$\big\|u(t)\big\|_{L^1(\mathbb{R}^d)} \leqslant \|u_0\|_{L^1(\mathbb{R}^d)} \quad \textit{and} \quad \big\|u(t)\big\|_{L^\infty(\mathbb{R}^d)} \leqslant \|u_0\|_{L^\infty(\mathbb{R}^d)}, \tag{3.2}$$

for every $t > 0$.

Moreover, if $d \geqslant 3$ and J also satisfies (HJ2) *then the following holds*:

$$\big\|u(t)\big\|_{L^q(\mathbb{R}^d)} \leqslant C(q,d)\|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{q})}, \tag{3.3}$$

for all $q \in [1,\infty)$ and for all t sufficiently large.

Remark 3.1. The proof uses the results of Theorem 2.1 and Corollary 2.1 obtained in Section 2 in the particular case $p = 2$. In order to apply Corollary 2.1 we need to assume $d > 2$, i.e. $d \geqslant 3$.

The same arguments we use here also work for the convection diffusion equation:

$$\begin{cases} u_t(t,x) = (G_1 * u - u)(t,x) + (G_2 * (|u|^{r-1}u) - |u|^{r-1}u)(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.4}$$

where $r > 1$ and G_1 and G_2 are positive functions with mass one. We have to mention that this time the dissipativity condition on the nonlinear part have to be understood in the following sense:

$$\int_{\mathbb{R}^d} \big(G * \big(|u|^{r-1}u\big) - |u|^{r-1}u\big)|u|^{q-2}u \leqslant 0,$$

for any $q \geqslant 1$.

In the case of Eq. (3.4), the same decay as in (3.3) has been obtained in [21] by means of the so-called *Fourier Splitting method* introduced by Schonbek in [23,24] and [25] in the context of the local convection–diffusion equation. Our method also works if the convolution terms in (3.4) are replaced by integral operators as in (3.1).

The following lemma will be used in the proof of Theorem 3.1.

Lemma 3.1. *Let $d > 2$ and u such that $u(t) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for all $t \geqslant 0$ satisfying*:

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^2(x,t)\,dx + \big\langle A_2 u(t), u(t)\big\rangle \leqslant 0, \quad \textit{for all } t > 0,$$

with J as in Theorem 2.1. *Assuming that*

$$\big\|u(t)\big\|_{L^1(\mathbb{R}^d)} \leqslant \big\|u(0)\big\|_{L^1(\mathbb{R}^d)}, \quad \textit{for all } t > 0, \tag{3.5}$$

there exists a constant $c(d,J)$ such that

$$\big\|u(t)\big\|_{L^2(\mathbb{R}^d)} \leqslant c(d,J)\big\|u(0)\big\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})}$$

holds for all t large enough.

Remark 3.2. Under the same hypotheses we can replace the initial time $t = 0$ with any positive time t_0, the result being the same for large time t,

$$\big\|u(t)\big\|_{L^2(\mathbb{R}^d)} \leqslant c(d)\big\|u(t_0)\big\|_{L^1(\mathbb{R}^d)} (t - t_0)^{-\frac{d}{2}(1-\frac{1}{2})}.$$

Proof of Lemma 3.1. By Corollary 2.1 and property (3.5) we obtain:

$$\begin{aligned} \big\|u(t)\big\|^2_{L^2(\mathbb{R}^d)} &\leqslant C_1(J)\big\|u(t)\big\|^{2(1-\alpha(2))}_{L^1(\mathbb{R}^d)} \big\langle A_2 u(t), u(t)\big\rangle^{\alpha(2)} + C_2(J)\big\langle A_2 u(t), u(t)\big\rangle \\ &\leqslant C_1(J)\big\|u(0)\big\|^{2(1-\alpha(2))}_{L^1(\mathbb{R}^d)} \big\langle A_2 u(t), u(t)\big\rangle^{\alpha(2)} + C_2(J)\big\langle A_2 u(t), u(t)\big\rangle, \end{aligned}$$

where $\alpha(2) = d/(d+2)$ is given by (2.8). To simplify the presentation we will assume without loss of generality that $C_1(J) = C_2(J) = 1$ (otherwise one can track the constants that appear in each step of the proof). Then for any $t > 0$, $\langle A_2 u(t), u(t)\rangle$ satisfies:

$$H^{-1}\big(\big\|u(t)\big\|^2_{L^2(\mathbb{R}^d)}\big) \leqslant \big\langle A_2 u(t), u(t)\big\rangle,$$

where

$$H(x) = \big\|u(0)\big\|^{2(1-\alpha(2))}_{L^1(\mathbb{R}^d)} x^{\alpha(2)} + x.$$

Analyzing the function $H_{a,\beta}(x) = ax^\beta + x$, $a > 0$, $\beta \in (0,1)$, we observe that

$$H_{a,\beta}(x) \leqslant \begin{cases} 2x & x > a^{\frac{1}{1-\beta}}, \\ 2ax^\beta, & x < a^{\frac{1}{1-\beta}}, \end{cases}$$

and then

$$H_{a,\beta}^{-1}(y) \geqslant \begin{cases} \frac{y}{2}, & y > 2a^{\frac{1}{1-\beta}}, \\ (\frac{y}{2a})^{\frac{1}{\beta}}, & y < 2a^{\frac{1}{1-\beta}}. \end{cases} \tag{3.6}$$

Applying this property to $a = \|u(0)\|_{L^1(\mathbb{R}^d)}^{1-\alpha(2)}$, $\beta = \alpha(2)$ we find that $\langle A_2 u(t), u(t)\rangle$ verifies:

$$\langle A_2 u(t), u(t)\rangle \geqslant \begin{cases} \frac{1}{2}\|u(t)\|_{L^2(\mathbb{R}^d)}^2, & \|u(t)\|_{L^2(\mathbb{R}^d)}^2 > 2\,\|u(0)\|_{L^1(\mathbb{R}^d)}^2, \\ \left(\frac{\|u(t)\|_{L^2(\mathbb{R}^d)}^2}{2\|u(0)\|_{L^1(\mathbb{R}^d)}^{2(1-\alpha(2))}}\right)^{\frac{1}{\alpha(2)}}, & \|u(t)\|_{L^2(\mathbb{R}^d)}^2 < 2\|u(0)\|_{L^1(\mathbb{R}^d)}^2. \end{cases}$$

Then, $\phi(t) = \|u(t)\|_{L^2(\mathbb{R})}^2$ satisfies the following differential inequality for all $t \geqslant 0$:

$$\phi_t(t) + \frac{\phi(t)}{2}\chi_{\{\phi(t) > 2\|u(0)\|_{L^1(\mathbb{R}^d)}^2\}} + \left(\frac{\phi(t)}{2\|u(0)\|_{L^1(\mathbb{R}^d)}^{2(1-\alpha(2))}}\right)^{\frac{1}{\alpha(2)}}\chi_{\{\phi(t) < 2\|u(0)\|_{L^1(\mathbb{R}^d)}^2\}} \leqslant 0.$$

This implies the existence of a time t_0 such that $\phi(t_0) < 2\|u(0)\|_{L^1(\mathbb{R}^d)}^2$. If not, then for all time $t \geqslant t_0$ we get $\phi(t) > 2\|u(0)\|_{L^1(\mathbb{R}^d)}^2$, and

$$\phi_t(t) + \frac{1}{2}\phi(t) \leqslant 0.$$

Integrating the above inequality on (t_0, t) we obtain $\phi(t) \leqslant e^{-(t-t_0)/2}\phi(t_0)$ which contradicts our assumption. Thus, there exists t_0 such that $\phi(t_0) < 2\|u(0)\|_{L^1(\mathbb{R}^d)}^2$. Using that $\phi_t(t) \leqslant 0$ we obtain that $\phi(t) < 2\|u(0)\|_{L^1(\mathbb{R}^d)}^2$ holds for all $t \geqslant t_0$ and $\phi(t)$ satisfies the following differential inequality for all $t \geqslant t_0$:

$$\phi_t(t) + \left(\frac{\phi(t)}{2\|u(0)\|_{L^1(\mathbb{R}^d)}^{2(1-\alpha(2))}}\right)^{\frac{1}{\alpha(2)}} \leqslant 0.$$

Integrating it on (t_0, t) we get by Lemma 2.3 with $\gamma = 0$ that ϕ satisfies:

$$\phi(t) \leqslant C\|u(0)\|_{L^1(\mathbb{R}^d)}^2 (t - t_0)^{-d(1-\frac{1}{2})}, \quad t > t_0,$$

in other words,

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \leqslant C\|u(0)\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})},$$

holds for all time t large enough. □

Proof of Theorem 3.1. Step I. Global existence and uniqueness. First, let us prove the existence and uniqueness of a local solution. To this end we use a fixed point argument.

Let us consider the space,

$$X = C^0\big([0,T];\ L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\big),$$

with the norm:

$$\|u\|_X = \max_{t\in[0,T]}\big\{\|u(t)\|_{L^1(\mathbb{R}^d)} + \|u(t)\|_{L^\infty(\mathbb{R}^d)}\big\}.$$

We observe that the operator $A : X \to X$ defined by

$$Au(x) = \int_{\mathbb{R}^d} J(x,y)\big(u(y,t) - u(x,t)\big)\,dy$$

is continuous since using (HJ1) and the symmetry of J we get:

$$\|Au\|_{L^\infty(\mathbb{R}^d)} \leqslant 2\|u\|_{L^\infty(\mathbb{R}^d)} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} J(x,y)\,dy \leqslant 2C\|u\|_{L^\infty(\mathbb{R}^d)},$$

and

$$\|Au\|_{L^1(\mathbb{R}^d)} \leqslant 2\|u\|_{L^1(\mathbb{R}^d)} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} J(x,y)\,dy \leqslant 2C\|u\|_{L^1(\mathbb{R}^d)}.$$

Since the map $u \to f(u)$ is Lipschitz continuous on bounded subsets of X (as a consequence of the properties of f) classical results on semilinear evolution problems (see for example [10], Proposition 4.3.3) guarantees the existence of a unique local solution u.

We now prove (3.2) which guarantee the global existence of solutions to Eq. (3.1). We multiply Eq. (3.1) with $\operatorname{sgn}(u)$ and integrate on $\mathbb{R}^d$:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} \big|u(x,t)\big|\,dx &= \int_{\mathbb{R}^d} u_t(x,t)\operatorname{sgn}\big(u(x,t)\big)\,dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big(u(y,t)-u(x,t)\big)\operatorname{sgn}\big(u(x,t)\big)\,dx\,dy \\ &\quad + \int_{\mathbb{R}^d} f\big(u(x,t)\big)\operatorname{sgn}\big(u(x,t)\big)\,dx.\end{aligned}$$

Using Lemma 2.1 and the fact that $f(s)s \leqslant 0$, $s \in \mathbb{R}$, we get:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} \big|u(x,t)\big|\,dx &\leqslant \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big(u(y,t)-u(x,t)\big)\operatorname{sgn}\big(u(x,t)\big)\,dy\,dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big(u(y,t)-u(x,t)\big)\big(\operatorname{sgn}\big(u(y,t)\big)-\operatorname{sgn}\big(u(x,t)\big)\big)\,dy\,dx \\ &\leqslant 0.\end{aligned}$$

From here it follows that

$$\big\|u(t)\big\|_{L^1(\mathbb{R}^d)} \leqslant \|u_0\|_{L^1(\mathbb{R}^d)}.$$

Now, multiplying the equation by $(u(x,t)-M)_+$, where $M = \|u_0\|_{L^\infty(\mathbb{R}^d)}$, and integrating on $\mathbb{R}^d$ we get:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} \frac{(u(x,t)-M)_+^2}{2}\,dx &= \int_{\mathbb{R}^d} u_t(x,t)\big(u(x,t)-M\big)_+\,dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big(u(y,t)-u(x,t)\big)\big(u(x,t)-M\big)_+\,dy\,dx \\ &\quad + \int_{\mathbb{R}^d} f\big(u(x,t)\big)\big(u(x,t)-M\big)_+\,dx.\end{aligned}$$

Using Lemma 2.1, the sign property of the function f and the fact that for any two real numbers a and b we have:

$$|a_+ - b_+|^2 \leqslant (a-b)(a_+ - b_+),$$

it follows that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d} \frac{(u(x,t)-M)_+^2}{2}\,dx \\
&\quad \leqslant \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big(u(y,t)-u(x,t)\big)\big(u(x,t)-M\big)_+\,dy\,dx \\
&\quad = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big(u(y,t)-u(x,t)\big)\big(\big(u(y,t)-M\big)_+ - \big(u(x,t)-M\big)_+\big)\,dx\,dy \\
&\quad \leqslant -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big|\big(u(y,t)-M\big)_+ - \big(u(x,t)-M\big)_+\big|^2\,dx\,dy.
\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^d} \frac{(u(x,t)-M)_+^2}{2}\,dx = 0,$$

and we obtain that $u(x,t)\leqslant M$ for all $t\geqslant 0$ and a.e. $x\in\mathbb{R}^d$.

In a similar way we get $u(x,t)\geqslant -M$ for all $t\geqslant 0$ and a.e. $x\in\mathbb{R}^d$.

We conclude that $\|u\|_{L^\infty(\mathbb{R}^d)}\leqslant \|u_0\|_{L^\infty(\mathbb{R}^d)}$ and that the solution u is global.

Step II. Proof of the long time behavior. We divide the proof in several steps.

Step II(a). The case $p=2$. Multiplying Eq. (3.1) by $\operatorname{sgn}(u)$ and u we obtain:

$$\frac{d}{dt}\int_{\mathbb{R}^d} \big|u(t,x)\big|\,dx \leqslant 0, \tag{3.7}$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(t)\,dx + \big\langle A_2 u(t), u(t)\big\rangle \leqslant 0. \tag{3.8}$$

Inequality (3.7) implies that (3.5) holds.

Inequalities (3.7) and (3.8) allow us to apply Lemma 3.1. Thus we obtain that

$$\big\|u(t)\big\|_{L^2(\mathbb{R}^d)} \leqslant \|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})}$$

holds for large enough t. This gives us, by interpolation, the long time behavior of the solution u in any $L^q(\mathbb{R}^d)$-norm when $1\leqslant q\leqslant 2$.

Step II(b). The case $p=2^{n+1}$. We use an iterative argument to prove that once the result is assumed for $p=2^n$ we get the result for $p=2^{n+1}$.

Assume that it holds for $p=2^n$. Then

$$\big\|u(t)\big\|_{L^{2^n}(\mathbb{R}^d)} \leqslant \|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2^n})}$$

holds for all t large enough.

Let us fix $r=2^{n+1}$. We multiply Eq. (3.1) with u^{r-1} to obtain:

$$\begin{aligned}
\frac{1}{r}\frac{d}{dt}\int_{\mathbb{R}^d} u^r(x,t)\,dx &\leqslant \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big(u(x,t)-u(y,t)\big)u^{r-1}(x,t)\,dx\,dy \\
&= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big(u(x,t)-u(y,t)\big)\big(u^{r-1}(x,t)-u^{r-1}(y,t)\big)\,dx\,dy \\
&\leqslant -c(r)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big(u^{r/2}(x,t)-u^{r/2}(y,t)\big)^2\,dx\,dy.
\end{aligned}$$

Then $v = u^{r/2}$ verifies:

$$\frac{d}{dt}\int_{\mathbb{R}^d} v^2(x,t)\,dx + c(r)\langle A_2 v(t), v(t)\rangle \leqslant 0, \quad t > 0.$$

By Lemma 3.1 and Remark 3.2 we obtain that for large time t the following holds:

$$\|v(t)\|_{L^2(\mathbb{R}^d)} \leqslant \|v(t/2)\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})}.$$

Then

$$\|u^{r/2}(t)\|_{L^2(\mathbb{R}^d)} \leqslant \|u^{r/2}(t/2)\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})},$$

and using that $r = 2^{n+1}$:

$$\|u(t)\|^{2^n}_{L^{2^{n+1}}(\mathbb{R}^d)} \leqslant C(d,n)\|u(t/2)\|^{2^n}_{L^{2^n}(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})}.$$

Using the hypothesis on the $L^{2^n}(\mathbb{R}^d)$-norm of u we get:

$$\begin{aligned}\|u(t)\|_{L^{2^{n+1}}(\mathbb{R}^d)} &\leqslant C(d,n)\|u(t/2)\|_{L^{2^n}(\mathbb{R}^d)} t^{-\frac{d}{2}(\frac{1}{2^n}-\frac{1}{2^{n+1}})}\\ &\leqslant C(d,n)\|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2^n})} t^{-\frac{d}{2}(\frac{1}{2^n}-\frac{1}{2^{n+1}})}\\ &\leqslant C(d,n)\|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2^{n+1}})}.\end{aligned}$$

The proof is now finished since we can interpolate between the cases $r = 2^n$ and $r = 2^{n+1}$, $n \geqslant 0$, an integer. Indeed, given $q \in (1,\infty)$ we can find a positive integer n such that $2^n \leqslant q < 2^{n+1}$. Then

$$\|u\|_{L^q(\mathbb{R}^d)} \leqslant \|u\|^a_{L^{2^n}(\mathbb{R}^d)} \|u\|^{(1-a)}_{L^{2^{n+1}}(\mathbb{R}^d)},$$

where $a = a(q,n)$ is given by

$$\frac{1}{q} = \frac{a}{2^n} + \frac{1-a}{2^{n+1}},$$

and the general case follows.

The proof of decay property (3.3) is now finished. □

4. Decay estimates for the nonlocal p-Laplacian

In this section we deal with the following nonlocal analogous to the p-Laplacian evolution,

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)\left|u(y,t)-u(x,t)\right|^{p-2}\big(u(y,t)-u(x,t)\big)\,dy. \tag{4.1}$$

Existence and uniqueness of a solution follows from the results in [2] (see also [3] for the Neumann problem). Again for this case we have to note that in those references a convolution kernel was considered $J(x,y) = G(x-y)$ but it can be checked that the same proof gives existence and uniqueness for a general $J(x,y)$.

Theorem 4.1. *(See [2, Proposition 2.4].) Let $1 < p < \infty$. For any initial condition $u_0 \in L^p(\mathbb{R}^d)$ there exists a unique global solution $u \in C([0,\infty) : L^p(\mathbb{R}^d)) \cap W^{1,1}((0,\infty) : L^p(\mathbb{R}^d))$ of Eq.* (4.1).

Concerning the long time behavior of the solutions of Eq. (4.1) we have the following result:

Theorem 4.2. *Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $2 \leqslant p < d$. For any $1 \leqslant q < \infty$ the solution to* (4.1) *verifies*:

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \leqslant Ct^{-(\frac{d}{d(p-2)+p})(1-\frac{1}{q})}, \tag{4.2}$$

for all t sufficiently large.

Remark 4.1. The condition $p \geqslant 2$ is used in the inductive step in our proof. Also $p < d$ is necessary in order to use Corollary 2.1.

Proof. We multiply Eq. (4.1) by $|u|^{r-2}u(x)$, $1 \leqslant r < \infty$, and integrate to obtain, using Lemma 2.1,

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} |u|^r(x,t)\,dx &= \int_{\mathbb{R}^d} |u|^{r-2}uu_t(x,t)\,dx \\
&= C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big|u(y,t)-u(x,t)\big|^{p-2} \\
&\quad \times \big(u(y,t)-u(x,t)\big)|u|^{r-2}u(x,t)\,dy\,dx \\
&= -C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big|u(y,t)-u(x,t)\big|^{p-2}\big(u(y,t)-u(x,t)\big) \\
&\quad \times \big(|u|^{r-2}u(y,t)-|u|^{r-2}u(x,t)\big)\,dy\,dx \\
&\leqslant -C(p,r)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\Big||u|^{\frac{p+r-2}{p}}(y,t)-|u|^{\frac{p+r-2}{p}}(x,t)\Big|^p\,dy\,dx.
\end{aligned}$$

In the last line we have used that for any $p, r > 1$ the following holds:

$$|x-1|^{p-2}(x-1)\big(|x|^{r-2}x-1\big) \geqslant c(p,r)\Big||x|^{\frac{p+r-2}{p}}-1\Big|^p, \quad \forall x \in \mathbb{R}.$$

The above inequality gives us that for any $1 \leqslant r < \infty$, u, the solution to (4.1), satisfies:

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u|^r(t,x)\,dx + C(p,r)\Big\langle A_p\big|u(t)\big|^{\frac{p+r-2}{p}}, \big|u(t)\big|^{\frac{p+r-2}{p}}\Big\rangle \leqslant 0. \tag{4.3}$$

This inequality is crucial to obtain the long time behavior (4.2) of a solution u to (4.1).

Next, we will prove by induction that the sequence $\{p_n\}_{n\geqslant 0}$ defined by:

$$p_0 = 1, \qquad p_{n+1} = pp_n - p + 2, \quad n \geqslant 0,$$

satisfies

$$\big\|u(t)\big\|_{L^{p_n}(\mathbb{R}^d)} \leqslant Ct^{-d_n}, \tag{4.4}$$

where

$$d_n = \frac{d}{d(p-2)+p}\left(1-\frac{1}{p_n}\right).$$

As the sequence p_n verifies $p_n \to \infty$ as $n \to \infty$ the desired inequality (4.2) follows by interpolation. Indeed, given $1 < q < \infty$ there exists n such that $p_n < q \leqslant p_{n+1}$. Then, we conclude applying (4.4) and the standard interpolation inequality:

$$\|u\|_{L^q(\mathbb{R}^d)} \leqslant \|u\|^a_{L^{p_n}(\mathbb{R}^d)}\|u\|^{(1-a)}_{L^{p_{n+1}}(\mathbb{R}^d)},$$

where $a = a(n,q)$ is given by

$$\frac{1}{q} = \frac{a}{p_n} + \frac{1-a}{p_{n+1}}.$$

Now, we proceed with the inductive proof of (4.4).

Case I. $n=0$. Observe that in this case inequality (4.4) holds since the $L^1(\mathbb{R}^d)$-norm of u does not increase:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\big|u(x,t)\big|\,dx &= \int_{\mathbb{R}^d} u_t(x,t)\operatorname{sgn}\big(u(x,t)\big)\,dx\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big|u(y,t)-u(x,t)\big|^{p-2}\big(u(y,t)-u(x,t)\big)\big(\operatorname{sgn}\big(u(x,t)\big)-\operatorname{sgn}\big(u(y,t)\big)\big)\,dy\,dx\\ &\leqslant 0.\end{aligned}$$

Case II. Inductive step. We assume that

$$\big\|u(t)\big\|_{L^{p_n}(\mathbb{R}^d)} \leqslant Ct^{-d_n},$$

and prove that

$$\big\|u(t)\big\|_{L^{p_{n+1}}(\mathbb{R}^d)} \leqslant Ct^{-d_{n+1}}.$$

To this end we will obtain a differential inequality for the $L^{p_{n+1}}(\mathbb{R}^d)$-norm of u.

Step 1. Differential inequality for the $L^{p_{n+1}}(\mathbb{R}^d)$-norm of u. Using inequality (4.3) with $r=p_{n+1}$ we get:

$$\frac{d}{dt}\int_{\mathbb{R}^d}\big|u(x,t)\big|^{p_{n+1}}\,dx + c(p,q)\big\langle A_p\big|u(t)\big|^{p_n},\big|u(t)\big|^{p_n}\big\rangle \leqslant 0. \tag{4.5}$$

We now get an upper bound for $\langle A_p|u|^{p_n},|u|^{p_n}\rangle$ in terms of $\|u\|_{L^{p_{n+1}}(\mathbb{R}^d)}$. This together with (4.5) will allow us to construct a differential inequality for $\|u\|_{L^{p_{n+1}}(\mathbb{R}^d)}$, integrating it we will obtain the desired result.

By the crucial decomposition estimates of Corollary 2.1, for any function $v\in L^1(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$ we have that

$$\|v\|^p_{L^p(\mathbb{R}^d)} \leqslant \|v\|^{p(1-\alpha(p))}_{L^1(\mathbb{R}^d)}\langle A_p v,v\rangle^{\alpha(p)} + \langle A_p v,v\rangle.$$

This implies, taking $v=|u(t)|^{(p+r-2)/p}$, that

$$\begin{aligned}\big\|u(t)\big\|^{p+r-2}_{L^{p+r-2}(\mathbb{R}^d)} \leqslant{}& \big\|u(t)\big\|^{(p+r-2)p(1-\alpha(p))/p}_{L^{(p+r-2)/p}(\mathbb{R}^d)}\big\langle A_p\big|u(t)\big|^{(p+r-2)/p},\big|u(t)\big|^{(p+r-2)/p}\big\rangle^{\alpha(p)}\\ &+\big\langle A_p\big|u(t)\big|^{(p+r-2)/p},\big|u(t)\big|^{(p+r-2)/p}\big\rangle,\end{aligned} \tag{4.6}$$

where $\alpha(p)=p^*/p'(p^*-1)$. Using that $r=p_{n+1}$ we get:

$$p+r-2=p+p_{n+1}-2=pp_n,$$

and then u satisfies

$$\big\|u(t)\big\|^{pp_n}_{L^{pp_n}(\mathbb{R}^d)} \leqslant \big\|u(t)\big\|^{p_n(1-\alpha(p))p}_{L^{p_n}(\mathbb{R}^d)}\big\langle A_p\big|u(t)\big|^{p_n},\big|u(t)\big|^{p_n}\big\rangle^{\alpha(p)} + \big\langle A_p\big|u(t)\big|^{p_n},\big|u(t)\big|^{p_n}\big\rangle.$$

Using that $p\geqslant 2$ we get $p_{n+1}\leqslant pp_n$. Thus we can use now the interpolation inequality:

$$\big\|u(t)\big\|_{L^{p_{n+1}}(\mathbb{R}^d)} \leqslant \big\|u(t)\big\|^{\varepsilon_n}_{L^{pp_n}(\mathbb{R}^d)}\big\|u(t)\big\|^{1-\varepsilon_n}_{L^1(\mathbb{R}^d)} \leqslant C\big\|u(t)\big\|^{\varepsilon_n}_{L^{pp_n}(\mathbb{R}^d)},$$

where ε_n satisfies

$$\frac{1}{p_{n+1}} = \frac{\varepsilon_n}{pp_n} + \frac{1-\varepsilon_n}{1},$$

i.e.

$$\varepsilon_n = \frac{pp_n}{p_{n+1}}\,\frac{p_{n+1}-1}{pp_n-1}.$$

Then for any t large enough by (4.6) we get:

$$\begin{aligned}\big\|u(t)\big\|^{pp_n/\varepsilon_n}_{p_{n+1}} &\leqslant C\big\|u(t)\big\|^{pp_n}_{pp_n}\\ &\leqslant \big\|u(t)\big\|^{p_n(1-\alpha(p))p}_{p_n}\big\langle A_p\big|u(t)\big|^{p_n},\big|u(t)\big|^{p_n}\big\rangle^{\alpha(p)} + \big\langle A_p\big|u(t)\big|^{p_n},\big|u(t)\big|^{p_n}\big\rangle\\ &\leqslant Ct^{-d_n(1-\alpha(p))pp_n}\big\langle A_p\big|u(t)\big|^{p_n},\big|u(t)\big|^{p_n}\big\rangle^{\alpha(p)} + \big\langle A_p\big|u(t)\big|^{p_n},\big|u(t)\big|^{p_n}\big\rangle.\end{aligned}$$

Denoting $\psi(t)=\|u(t)\|_{p_{n+1}}^{p_{n+1}}$ and using (4.5) we get that for t large enough ψ satisfies the following differential inequality:

$$\psi_t + H_n^{-1}\bigl(\psi^{\frac{pp_n-1}{p_{n+1}-1}}\bigr) \leqslant 0, \tag{4.7}$$

where

$$H_n(x)=t^{-d_n(1-\alpha(p))pp_n}x^{\alpha(p)}+x.$$

Using inequality (3.6) for the function H_n with

$$a=t^{-d_n(1-\alpha(p))pp_n},$$

and $\beta=\alpha(p)$ we get:

$$H_n^{-1}(y)\geqslant \begin{cases} \frac{y}{2}, & y>2t^{-d_np_np},\\ \bigl(\frac{y}{2t^{-d_n(1-\alpha(p))pp_n}}\bigr)^{1/\alpha(p)}, & y<2t^{-d_np_np}. \end{cases} \tag{4.8}$$

Thus ψ satisfies the following differential inequality:

$$\begin{aligned}&\psi_t(t)+\psi^{\frac{pp_n-1}{p_{n+1}-1}}(t)\chi\bigl(\psi^{\frac{pp_n-1}{p_{n+1}-1}}(t)>2t^{-d_np_np}\bigr)\\ &\quad+\psi^{\frac{pp_n-1}{p_{n+1}-1}\frac{1}{\alpha(p)}}(t)t^{\frac{d_np_np(1-\alpha(p))}{\alpha(p)}}\chi\bigl(\psi^{\frac{pp_n-1}{p_{n+1}-1}}(t)<2t^{-d_np_np}\bigr)\leqslant 0.\end{aligned} \tag{4.9}$$

Step 2. Decay of the function ψ. First, we show that the function ψ satisfies:

$$\lim_{t\to\infty}\psi(t)=0. \tag{4.10}$$

First observe that (4.7) gives us that ψ is a nonincreasing function. Let us assume that there exists a sequence $t_n\to\infty$ such that

$$\psi(t_n)<2t_n^{-d_np_np}.$$

Using that ψ is a nonincreasing function we get (4.10).

In the case when the above assumption is not satisfied we obtain the existence of a time t_0 such that for all $t>t_0$, $\psi(t)>2t^{-d_np_np}$. Using (4.9) we obtain that for any $t>t_0$, ψ satisfies:

$$\psi_t(t)+\psi^{\frac{pp_n-1}{p_{n+1}-1}}(t)\leqslant 0.$$

The definition of the sequence $(p_n)_{n\geqslant 0}$ guarantees that $pp_n-1>p_{n+1}-1$ and then $\psi(t)$ satisfies (4.10).

Step 3. Sub- and supersolutions for (4.7). We prove that any two functions $\overline{\psi}$ and $\underline{\psi}$ which satisfy

$$\begin{cases}\overline{\psi}_t(t)+H_n^{-1}(\overline{\psi}^{\frac{pp_n-1}{p_{n+1}-1}})>0\geqslant \underline{\psi}_t(t)+H_n^{-1}(\underline{\psi}^{\frac{pp_n-1}{p_{n+1}-1}}) \quad \text{for all } t>t_0,\\ \overline{\psi}(t_0)>\underline{\psi}(t_0),\end{cases}$$

verifies

$$\overline{\psi}(t)>\underline{\psi}(t), \quad \text{for all } t\geqslant t_0. \tag{4.11}$$

To prove the above statement let us assume that (4.11) does not holds for all $t\geqslant t_0$. Then there exists a first $t_1>t_0$ such that $\overline{\psi}(t_1)=\underline{\psi}(t_1)$. Thus

$$0\geqslant(\overline{\psi}-\underline{\psi})_t(t_1)>H_n^{-1}\bigl(\underline{\psi}^{\frac{pp_n-1}{p_{n+1}-1}}(t_1)\bigr)-H_n^{-1}\bigl(\overline{\psi}^{\frac{pp_n-1}{p_{n+1}-1}}(t_1)\bigr)=0.$$

This implies that our assumption does not hold and then (4.11) holds for all $t>t_0$.

Step 4. Construction of a supersolution. We consider supersolutions of the type

$$\overline{\psi}(t)=kt^{-d_{n+1}p_{n+1}}, \tag{4.12}$$

since out final goal is to obtain a bound of the type $\psi(t)\leqslant\overline{\psi}(t)\leqslant Ct^{-d_{n+1}p_{n+1}}$.

We prove the existence of positive constants k and t_0 such that $\overline{\psi}$ is a supersolution for Eq. (4.7):

$$\overline{\psi}^{\frac{pp_n-1}{p_{n+1}-1}}(t) < t^{-d_n p_n p}, \quad \forall t > t_0, \tag{4.13}$$

and

$$\overline{\psi}_t(t) + H_n^{-1}\big(\overline{\psi}^{\frac{pp_n-1}{p_{n+1}-1}}\big) > 0, \quad \forall t > t_0. \tag{4.14}$$

Introducing the explicit form of ψ given by (4.12) in (4.13) we get:

$$k^{\frac{pp_n-1}{p_{n+1}-1}} t^{-\frac{d_{n+1}p_{n+1}}{p_{n+1}-1}(pp_n-1)} < t^{-\frac{d_n p_n}{p_n-1}(p_n-1)p}.$$

Using that for any $n \geqslant 0$,

$$\frac{d_n p_n}{p_n - 1} = c(p,d) = \frac{d}{d(p-2)+p}, \tag{4.15}$$

it remains to impose that t_0 and k satisfy:

$$k^{\frac{pp_n-1}{p_{n+1}-1}} < t^{c(p,d)(p-1)}, \quad \forall t > t_0. \tag{4.16}$$

In what concerns (4.14) we use that (4.13) holds. Thus, (4.8) gives us that

$$H_n^{-1}\big(\overline{\psi}^{\frac{pp_n-1}{p_{n+1}-1}}(t)\big) \geqslant 2^{-1/\alpha(p)}\big(t^{d_n p_n(1-\alpha(p))p}\,\overline{\psi}^{\frac{pp_n-1}{p_{n+1}-1}}(t)\big)^{1/\alpha(p)},$$

and

$$\begin{aligned}&\overline{\psi}_t(t) + H_n^{-1}\big(\overline{\psi}^{\frac{pp_n-1}{p_{n+1}-1}}\big)\\ &\quad\geqslant -kd_{n+1}p_{n+1}t^{-d_{n+1}p_{n+1}} + 2^{-1/\alpha(p)}\big(t^{d_n p_n(1-\alpha(p))p}t^{-d_{n+1}p_{n+1}\frac{pp_n-1}{p_{n+1}-1}}\big)^{1/\alpha(p)}.\end{aligned}$$

Hence, we have to choose k and t_0 such that

$$\big(t^{d_n p_n(1-\alpha(p))p}t^{-d_{n+1}p_{n+1}\frac{pp_n-1}{p_{n+1}-1}}\big)^{1/\alpha(p)} > 2^{1/\alpha(p)}kd_{n+1}p_{n+1}t^{-d_{n+1}p_{n+1}-1}. \tag{4.17}$$

We claim that

$$d_n p_n\big(1-\alpha(p)\big)p - d_{n+1}p_{n+1}\frac{pp_n-1}{p_{n+1}-1} = -\alpha(p)d_{n+1}p_{n+1} - \alpha(p). \tag{4.18}$$

This implies that (4.17) holds for k small enough. Once k is fixed we choose t_0 such that (4.16) also holds. We have constructed a function $\overline{\psi}$ which verifies (4.14) for all $t > t_0$.

We now prove the above claim, (4.18). Using (4.15) we have to check that

$$c(p,d)(p_n-1)\big(1-\alpha(p)\big)p - c(p,d)(pp_n-1) = -\alpha(p)c(p,d)(p_{n+1}-1) - \alpha(p),$$

or equivalently

$$c(p,d)\alpha(p)(p-pp_n) - c(p,d)(p-1) = -\alpha(p)c(p,d)(p_{n+1}-1) - \alpha(p).$$

Using the definition of $p_{n+1} = pp_n - p + 2$ we get:

$$\begin{aligned}c(p,d)\alpha(p)(p-pp_n) - c(p,d)(p-1) &= c(p,d)\alpha(p)(-p_{n+1}+2) - c(p,d)(p-1)\\ &= -c(p,d)\alpha(p)(p_{n+1}-1) + c(p,d)\alpha(p) - c(p,d)(p-1).\end{aligned}$$

It remains to prove that

$$c(p,d)\left[1 - \frac{p-1}{\alpha(p)}\right] = -1.$$

Using the definition of $\alpha(p)$ we easily can prove this fact:

$$1 - \frac{p-1}{\alpha(p)} = 1 - \frac{d(p-1)+p}{d} = -\frac{1}{c(p,d)}.$$

Step 5. Decay of ψ. Let us choose k, t_0 and $\overline{\psi}$ as in Step 4. Using Step 2 we can find $T>0$ such that $\psi(T+t_0)<\overline{\psi}(t_0)$. Thus $\underline{\psi}(t)=\psi(T+t)$ is a subsolution for Eq. (4.7) which satisfies $\underline{\psi}(t_0)<\overline{\psi}(t_0)$. Step 3 gives us that $\underline{\psi}(t)\leqslant\overline{\psi}(t)$ for all $t>t_0$. Then

$$\psi(t)\leqslant k(t-t_0-T)^{-d_{n+1}p_{n+1}},\quad \forall t>t_0.$$

The proof is now finished. □

5. Examples of exponential decay

In this section we present a simple example of $J(x,y)$ for which we obtain exponential decay of the solutions to the linear problem:

$$u_t(x,t)=\int_{\mathbb{R}} J(x,y)\big(u(y,t)-u(x,t)\big)\,dy. \tag{5.1}$$

Note that, to simplify, we restrict ourselves to one space dimension.

Lemma 5.1. *Let $a:\mathbb{R}\to\mathbb{R}$ be a diffeomorphism. Assume that*

$$J(x,y)\geqslant\frac{1}{2}\quad \text{on } \big|y-a(x)\big|\leqslant 1,$$

where the function a satisfies

$$\sup_{\mathbb{R}}\big|\big(a^{-1}\big)_x\big|<1\quad \text{or}\quad \inf_{\mathbb{R}}\big|\big(a^{-1}\big)_x\big|>1,$$

then there exists a positive constant C such that

$$\langle A_2u,u\rangle\geqslant C\|u\|^2_{L^2(\mathbb{R})}.$$

Proof. Using the symmetry of the function J we get:

$$J(x,y)\geqslant\frac{1}{2}\quad \text{on } \big|x-a(y)\big|<1.$$

Thus

$$J(x,y)\geqslant\frac{1}{4}\chi_{\{|x-a(y)|<1\}}+\frac{1}{4}\chi_{\{|y-a(x)|<1\}}. \tag{5.2}$$

Let us consider $\psi:\mathbb{R}\to\mathbb{R}$ a smooth positive function, supported on $(-1,1)$. Then

$$2\|\psi\|_{L^\infty(\mathbb{R})}J(x,y)\geqslant\rho(x,y):=\psi\big(x-a(y)\big)+\psi\big(y-a(x)\big)$$

and

$$2\|\psi\|_{L^\infty(\mathbb{R})}\langle A_2u,u\rangle\geqslant\iint_{\mathbb{R}^2}\rho(x,y)\big(u(x)-u(y)\big)^2\,dx\,dy. \tag{5.3}$$

Let be θ a positive constant which will be fixed latter. Using the elementary inequality

$$(b-c)^2=b^2+c^2-2bc\geqslant b^2+c^2-\theta b^2-\frac{1}{\theta}c^2=(1-\theta)\bigg(b^2-\frac{c^2}{\theta}\bigg),$$

we get:

$$\iint_{\mathbb{R}^2}\rho(x,y)\big(u(x)-u(y)\big)^2\,dx\,dy\geqslant(1-\theta)\iint_{\mathbb{R}^2}\psi\big(y-a(x)\big)\bigg(u^2(x)-\frac{u^2(y)}{\theta}\bigg)\,dx\,dy$$
$$=(1-\theta)\bigg(\int_{\mathbb{R}}u^2(x)\,dx\int_{\mathbb{R}}\psi(y)\,dy-\frac{1}{\theta}\int_{\mathbb{R}}u^2(y)\int_{\mathbb{R}}\psi\big(y-a(x)\big)\,dx\,dy\bigg)$$

$$
\begin{aligned}
&= (1-\theta)\int_{\mathbb{R}} u^2(x)\bigg(\int_{\mathbb{R}} \psi(y)\,dy - \frac{1}{\theta}\int_{\mathbb{R}} \psi\big(x-a(y)\big)\,dy\bigg)\,dx \\
&= (1-\theta)\int_{\mathbb{R}} u^2(x)\bigg(\int_{\mathbb{R}} \psi(y)\,dy - \frac{1}{\theta}\int_{\mathbb{R}} \psi(x-y)\big|\big(a^{-1}\big)_x(y)\big|\,dy\bigg)\,dx \\
&= \frac{1-\theta}{\theta}\int_{\mathbb{R}} \psi(y)\,dy \int_{\mathbb{R}} u^2(x)\bigg(\theta - \frac{(\psi * |(a^{-1})_x|)(x)}{\int_{\mathbb{R}}\psi(y)\,dy}\bigg)\,dx.
\end{aligned}
$$

Then

$$
\iint_{\mathbb{R}^2} \rho(x,y)\big(u(x)-u(y)\big)^2\,dx\,dy
$$

$$
\geqslant \begin{cases}
\dfrac{1-\theta}{\theta}\displaystyle\int_{\mathbb{R}} \psi(y)\,dy \int_{\mathbb{R}} u^2(x)\bigg(\theta - \frac{\sup_{x\in\mathbb{R}} \psi * |(a^{-1})_x|}{\int_{\mathbb{R}}\psi(y)\,dy}\bigg)\,dx, & \theta < 1, \\
\dfrac{1-\theta}{\theta}\displaystyle\int_{\mathbb{R}} \psi(y)\,dy \int_{\mathbb{R}} u^2(x)\bigg(\theta - \frac{\inf \psi * (a^{-1})_x}{\int_{\mathbb{R}}\psi(y)\,dy}\bigg)\,dx, & \theta > 1.
\end{cases}
$$

If

$$
\sup_{x\in\mathbb{R}} \big|\big(a^{-1}\big)_x(x)\big| < 1,
$$

then

$$
\sup_{x\in\mathbb{R}} \big(\psi * \big|\big(a^{-1}\big)_x\big|\big)(x) < \int_{\mathbb{R}} \psi(y)\,dy.
$$

We choose θ satisfying

$$
\frac{\sup_{\mathbb{R}} \psi * |(a^{-1})_x|}{\int_{\mathbb{R}}\psi(y)\,dy} < \theta < 1,
$$

and thus by (5.3)

$$
2\|\psi\|_{L^\infty(\mathbb{R})}\langle A_2 u, u\rangle \geqslant C(\theta,\psi,a)\|u\|^2_{L^2(\mathbb{R})}.
$$

The other case,

$$
\inf_{x\in\mathbb{R}} \big|\big(a^{-1}\big)_x(x)\big| > 1,
$$

can be treated in a similar way. Here we have:

$$
\inf_{x\in\mathbb{R}} \big(\psi * \big|\big(a^{-1}\big)_x\big|\big)(x) > \int_{\mathbb{R}} \psi(y)\,dy,
$$

and then we choose θ satisfying

$$
\frac{\inf_{\mathbb{R}} \psi * |(a^{-1})_x|}{\int_{\mathbb{R}}\psi(y)\,dy} > \theta > 1,
$$

and thus

$$
2\|\psi\|_{L^\infty(\mathbb{R})}\langle A_2 u, u\rangle \geqslant C(\theta,\psi,a)\|u\|^2_{L^2(\mathbb{R})}. \qquad \Box
$$

Theorem 5.1. *Let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then the solution to* (5.1) *verifies*

$$
\big\|u(\cdot,t)\big\|_{L^2(\mathbb{R})} \leqslant Ce^{-Ct},
$$

for all $t > t_0$.

Proof. Multiplying Eq. (5.1) by u we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(t)\,dx + \langle A_2 u(t), u(t)\rangle \leqslant 0,$$

and using our previous estimate (Lemma 5.1) we get:

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(t)\,dx + C\int_{\mathbb{R}^d} u^2(t)\,dx \leqslant 0,$$

from where the result follows. □

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