Institute of Mathematics ,,Simion Stoilow" of the Romanian Academy

HABILITATION THESIS

Qualitative analysis of some differential equations

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To Tatiana and Andrei for their patience and love

To my parents for their support over the years

Abstract

In this thesis we present our recent works on the analysis of qualitative properties of some partial differential equations or integral equations as well as their numerical approximations with emphasis on their well-posedness, asymptotic behaviour, inverse problems and numerical approximation.

The subjects we analyze are grouped in three main thematic parts which are detailed below. The first one, contained in Chapters 1 and 2, is the study of some properties for the heat and Schrödinger equations on trees. The second one is the analysis of dispersive properties for discrete Schrödinger equations and of obtaining convergence rates for some numerical schemes for the nonlinear model. It contains Chapters 3, 4 and 5. The third part is the study of the asymptotic behaviour of some integral equations. It contains Chapter 6 and Chapter 7.

We resume now the main subjects that we address in the three parts of this thesis.

1. Equations on networks. In Chapter 1 we consider the linear Schrödinger (LSE) equation on a network formed by a tree with the last generation of edges formed by infinite strips. We prove dispersive estimates for the linear problem, which in turn are useful for solving the nonlinear Schrödinger equation (NSE). We first consider the case of a regular tree. By using totally different techniques, in a joint work with Valeria Banica, we extend the results to the general case.

In Chapter 2 we establish global Carleman estimates for the heat and Schrödinger equations on a network. The heat equation is considered on a general tree and the Schrödinger equation on a star-shaped tree. The Carleman inequalities are used to prove the Lipschitz stability for an inverse problem consisting in retrieving a stationary potential in the heat (resp. Schrödinger) equation from boundary measurements.

Chapter 1 is based on [55] and [8]. Chapter 2 is based on [57].

2. Discrete equations and numerical approximation. The second part of this thesis contains three works. The first one is a joint work with my former master student Diana Stan (currently doing her PhD thesis in Madrid). The second one is a recent work with Enrique Zuazua. The third one presents a splitting method for the nonlinear Schrödinger equation.

In Chapter 3 we prove dispersive estimates for the system formed by two coupled discrete Schrödinger equations. We obtain estimates for the resolvent of the discrete operator and prove that it satisfies the limiting absorption principle. The decay of the solutions is proved by using classical and new results on oscillatory integrals. The results in this Chapter are contained in [61].

Chapter 4 contains the results in [66] and is devoted to the analysis of the convergence rates of several numerical approximation schemes for linear and nonlinear Schrödinger equations on the real line. Recently, in [65] we introduced viscous and two-grid numerical approximation schemes that mimic at the discrete level the so-called Strichartz dispersive estimates of the continuous Schrödinger equation. This allows one to guarantee the convergence of numerical approximations for initial data in $L^2(\mathbb{R})$, a fact that can not be proved in the nonlinear setting for standard conservative schemes unless more regularity of the initial data is assumed. We obtain explicit convergence rates and prove that dispersive schemes fulfilling the Strichartz estimates are better behaved for $H^s(\mathbb{R})$ data if $0 < s < 1/2$. Indeed, while dispersive schemes ensure a polynomial convergence rate, non-dispersive ones only yield logarithmic ones.

In Chapter 5 we introduce a splitting method for the semilinear Schrödinger equation and prove its convergence for those nonlinearities which can be handled by the classical well-posedness $L^2(\mathbb{R}^d)$-theory. More precisely, we prove that the scheme is of first order in the $L^2(\mathbb{R}^d)$-norm for $H^2(\mathbb{R}^d)$-initial data. The results of this chapter are included in [56].

3. Nonlocal evolution equations In the last part of this thesis we describe two works in collaboration with Julio D. Rossi. It mainly contains the results from [58] and [60].

In Chapter 6 we study a nonlocal equation that takes into account convective and diffusive effects, $u_t = J * u - u + G * (f(u)) - f(u)$ in $\mathbb{R}^d$, with J radially symmetric. First, we prove existence, uniqueness and continuous dependence with respect to the initial condition of solutions. This problem is the nonlocal analogous to the usual local convection-diffusion equation $u_t = \Delta u + b \cdot \nabla(f(u))$. In fact, we prove that solutions of the nonlocal equation converge to the solution of the usual convection-diffusion equation when we rescale the convolution kernels J and G appropriately. Finally we study the asymptotic behaviour of solutions as $t \to \infty$ when $f(u) = |u|^{q-1}u$ with $q > 1$. We find the decay rate and the first order term in the asymptotic regime.

In Chapter 7 we study the applicability of energy methods to obtain bounds for the asymptotic decay of solutions to nonlocal diffusion problems. With these energy methods we can deal with nonlocal problems that do not necessarily involve a convolution. For example, we will consider equations like,

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)(u(y,t) - u(x,t))\, dy + f(u)(x,t),$$

and a nonlocal analogous to the $p-$Laplacian,

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)|u(y,t) - u(x,t)|^{p-2}(u(y,t) - u(x,t))\, dy.$$

The energy method developed here allows us to obtain decay rates of the form

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \le C\, t^{-\alpha}$$

for some explicit exponent α that depends on parameters, d, q and p, according to the problem under consideration.

The last chapter of the thesis presents some ideas regarding the study of some open problems starting from the research presented in this thesis. Some further plans regarding the evolution of the professional and scientific career of the candidate will also be presented.

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Rezumat

În această teză prezentăm o analiză recentă a anumitor proprietaţi calitative ale unor ecuaţii integrale şi cu derivate parţiale. Ne concentrăm pe rezultate de existenţă şi unicitate, comportament asimptotic, probleme inverse şi aproximări numerice.

Subiectele analizate se grupează în trei mari teme ce le vom detalia în continuare. Prima temă este prezentată în Capitolul 1 şi Capitolul 2 şi se referă la studiul anumitor proprietăţi ale ecuaţiei căldurii şi ale ecuaţiei Schrödinger pe reţele. A doua temă analizează proprietăţile dispersive ale unor ecuaţii discrete de tip Schrödinger şi analizează ratele de convergenţă ale anumitor scheme numerice pentru modele neliniare. Rezultatele sunt conţinute în Capitolele 3, 4 şi 5. A treia temă este studiul proprietăţilor asimptotice pentru anumite ecuaţii integrale. Această temă conţine Capitolul 6 şi Capitolul 7.

Vom rezuma subiectele tratate în fiecare din cele trei părţi ale acestei teze.

1. Ecuaţii pe reţele. În Capitolul 1 considerăm ecuaţia Schrödinger pe o reţea formată dintr-un arbore a cărui ultimă generaţie de laturi sunt semidrepte infinite. Demonstrăm proprietăţi dispersive pentru problema liniară ce vor fi folosite pentru a rezolva ecuaţia Schrödinger neliniară. Considerăm mai intâi cazul unui arbore regular. Folosind o tehnică total diferită, în colaborare cu Valeria Bănică, extindem acest rezultat la cazul unui arbore general.

În Capitolul 2 stabilim estimări Carleman globale pentru ecuaţia căldurii şi ecuaţia Schrödinger pe o reţea. Pentru ecuaţia căldurii considerăm un arbore general în timp ce pentru ecuaţia Schrödinger un arbore stelar. Estimările obţinute sunt folosite apoi pentru a obţine stabilitatea Lipschitz pentru o problemă inversă ce constă în determinarea unui potenţial din măsurători pe frontieră.

Capitolul 1 se bazează pe rezultatele din [55] şi [8] iar Capitolul 2 se bazează pe [57].

2. Ecuaţii discrete şi aproximare numerică. A doua parte a acestei teze conţine trei lucrări. Prima este o colaborare cu fosta mea studentă masterandă Diana Stan. A doua conţine un articol recent cu Enrique Zuazua. Cea de-a treia parte prezintă o metodă de *splitting* pentru ecuaţia Schrödinger neliniară.

În Capitolul 3 demonstrăm proprietăţi dispersive pentru un sistem format din două ecuaţii Schrödinger discrete. Obţinem estimări pentru rezolventa operatorului discret şi demonstrăm că satisface principiul absorbţiei limită. Descreşterea soluţiilor este obţinută folosind rezultate clasice şi noi de integrale oscilatorii. Rezultatele sunt conţinute în [61].

Capitolul 4 conţine rezultatele din [66] şi este dedicat analizei ratelor de convergenţă pentru câteva scheme de aproximare numerică pentru ecuaţia Schrödinger liniară şi neliniară. Recent, în [65] au fost introduse două scheme numerice, una vâscoasă şi alta *two-grid*, ce simulează la nivel discret proprietăţile Strichartz ale ecuaţiei continue. Acest lucru garantează convergenţa acestor aproximări numerice pentru date iniţiale în $L^2(\mathbb{R})$ fără a presupune mai multă regularitate asupra datelor iniţiale. Obţinem rate de convergenţă explicite şi demonstrăm că schemele ce satisfac proprietăţi dispersive se comportă mai bine în cazul unor date iniţiale mai puţin regulate, de exemplu în $H^s(\mathbb{R})$ cu $0 < s < 1/2$. Mai mult decât atât, aceste scheme garantează rate de convergenţă polinomiale în timp ce schemele ne-dispersive pot da numai rate de tip logaritmic.

În Capitolul 5 introducem o metodă de descopunere, *splitting*, pentru ecuaţia Schrödinger semi-liniară şi demonstrăm convergenţa acesteia pentru acele neliniarităţi ce pot fi tratate folosind teoria clasică de bine punere pentru date iniţiale în $L^2(\mathbb{R}^d)$. Mai precis, demonstrăm că schema este de ordin unu în norma $L^2(\mathbb{R}^d)$ pentru date iniţiale în $H^2(\mathbb{R}^d)$. Rezultatele acestui capitol sunt conţinute în [56].

3. Ecuaţii de evoluţie nelocale. În această parte a tezei sunt descrise două rezultate în colaborare cu Julio Rossi. Se prezintă în pricipal rezultatele din [58] şi [60].

În Capitolul 6 studiem o ecuaţie nelocală ce conţine efecte convective şi difuzive, $u_t = J * u - u + G * (f(u)) - f(u)$ în $\mathbb{R}^d$, unde J are simetrie radială. Demonstrăm mai întâi existenţa, unicitatea şi dependenţa continuă a soluţiilor în raport cu datele iniţiale. Această problemă este analogul nelocal a mai cunoscutei ecuaţii de convecţie-difuzie $u_t = \Delta u + b \cdot \nabla(f(u))$. Mai precis, demonstrăm că soluţiile problemei nelocale converg la soluţiile problemei clasice de convectie-difuzie atunci când nucleele de convoluţie J şi G sunt rescalate corespunzător. În cele din urmă studiem comportamentul asimptotic al soluţiilor atunci când $t \to \infty$ şi $f(u) = |u|^{q-1}u$ cu $q > 1$.

În Capitolul 7 studiem aplicabilitatea unor metode energetice pentru a obţine estimări ale comportamentului asimptotic pentru probleme de difuzie nelocale. Cu aceste metode energetice putem trata probleme nelocale ce nu conţin termeni în forma de convoluţie. Exemple de acest tip ce le vom analiza sunt următoarele:

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)(u(y,t) - u(x,t))\, dy + f(u)(x,t),$$

şi un analog nelocal al p-laplacianului

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)|u(y,t) - u(x,t)|^{p-2}(u(y,t) - u(x,t))\, dy.$$

Metodele energetice dezvoltate în acest capitol ne permit să obţinem rate de descreştere de forma

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \le C\, t^{-\alpha}$$

pentru anumiţi exponenţi expliciţi α ce depind de parametrii d, q şi p, ce apar în problemele analizate.

Ultimul capitol al tezei prezintă câteva idei cu privire la studiul unor probleme deschise formulate plecând de la cercetările din această teză. Sunt prezentate de asemenea câteva planuri privind evoluţia profesională a candidatului.

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Contents

Part I

Equations on networks

Chapter 1

Dispersion for the Schrödinger equation on networks

In this chapter we consider the Schrödinger equation on a network formed by a tree with the last generation of edges formed by infinite strips. We first consider the case of regular trees and show how it could be reduced to solving an equation with finitely many piecewise constant coefficients on the whole real line. Next we consider the general case and use a totally different method. We give an explicit description of the solution of the linear Schrödinger equation with constant coefficients. This allows us to prove dispersive estimates, which in turn are useful for solving the nonlinear Schrödinger equation. The last method also extends to the laminar case of positive step-function coefficients having a finite number of discontinuities [8].

1.1 Generalities on networks

In this section we present some generalities about metric graphs and introduce the Laplace operator on such structure. Let $\Gamma = (V, E)$ be a graph where V is a set of vertices and E the set of edges. For each $v \in V$ we denote $E_v = \{e \in E : v \in e\}$. We assume that Γ is a connected finite graph, i.e. the degree of each vertex v of Γ is finite: $d(v) = |E_v| < \infty$ and $|V| < \infty$. The edges could be of finite length and then their ends are vertices of V or they have infinite length and then we assume that each infinite edge is a ray with a single vertex belonging to V (see [82] for more details on graphs with infinite edges).

We fix an orientation of Γ and for each oriented edge e, we denote by $I(e)$ the initial vertex and by $T(e)$ the terminal one. Of course in the case of infinite edges we have only initial vertices.

We identify every edge e of Γ with an interval I_e, where $I_e = [0, l_e]$ if the edge is finite and $I_e = [0, \infty)$ if the edge is infinite. This identification introduces a coordinate x_e along the edge e. In this way Γ is a metric space and is often named metric graph [82].

Let v be a vertex of V and e be an edge in E_v. We set for finite edges e

$$j(v,e) = \begin{cases} 0 & \text{if} \quad v = I(e), \\ l_e & \text{if} \quad v = T(e) \end{cases}$$

and

$$j(v,e) = 0, \text{ if } v = I(e)$$

for infinite edges.

We identify any function $\mathbf{u}$ on Γ with a collection $\{u^e\}_{e\in E}$ of functions u^e defined on the edges e of Γ. Each u^e can be considered as a function on the interval I_e. In fact, we use the same notation u^e for both the function on the edge e and the function on the interval I_e identified with

e. For a function $\mathbf{u} : \Gamma \to \mathbb{C}$, $\mathbf{u} = \{u^e\}_{e\in E}$, we denote by $f(\mathbf{u}) : \Gamma \to \mathbb{C}$ the family $\{f(u^e)\}_{e\in E}$, where $f(u^e) : e \to \mathbb{C}$.

A function $\mathbf{u} = \{u^e\}_{e\in E}$ it is continuous if and only if u^e is continuous on I_e for every $e \in E$, and moreover, is continuous at the vertices of Γ:

$$u^e(j(v,e)) = u^{e'}(j(v,e')), \quad \forall\, e, e' \in E_v.$$

The space $L^p(\Gamma)$, $1 \le p < \infty$ consists of all functions $\mathbf{u} = \{u_e\}_{e\in E}$ on Γ that belong to $L^p(I_e)$ for each edge $e \in E$ and

$$\|\mathbf{u}\|^p_{L^p(\Gamma)} = \sum_{e\in E} \|u^e\|^p_{L^p(I_e)} < \infty.$$

Similarly, the space $L^\infty(\Gamma)$ consists of all functions that belong to $L^\infty(I_e)$ for each edge $e \in E$ and

$$\|\mathbf{u}\|_{L^\infty(\Gamma)} = \sup_{e\in E} \|u^e\|_{L^\infty(I_e)} < \infty.$$

The Sobolev space $H^m(\Gamma)$, $m \ge 1$ an integer, consists in all continuous functions on Γ that belong to $H^m(I_e)$ for each $e \in E$ and

$$\|\mathbf{u}\|^2_{H^m(\Gamma)} = \sum_{e\in E} \|u^e\|^2_{H^m(e)} < \infty.$$

The above spaces are Hilbert spaces with the inner products

$$(\mathbf{u},\mathbf{v})_{L^2(\Gamma)} = \sum_{e\in E} (u^e, v^e)_{L^2(I_e)} = \sum_{e\in E} \int_{I_e} u^e(x)\overline{v^e}(x)dx$$

and

$$(\mathbf{u},\mathbf{v})_{H^m(\Gamma)} = \sum_{e\in E} (u^e, v^e)_{H^m(I_e)} = \sum_{e\in E}\sum_{k=0}^{m} \int_{I_e} \frac{d^k u^e}{dx^k}\overline{\frac{d^k v^e}{dx^k}}dx.$$

We now introduce the Laplace operator Δ_Γ on the graph Γ. Even if it is a standard procedure we prefer for the sake of completeness to follow [26]. Consider the sesquilinear continuous form φ on $H^1(\Gamma)$ defined by

$$\varphi(\mathbf{u},\mathbf{v}) = (\mathbf{u}_x,\mathbf{v}_x)_{L^2(\Gamma)} = \sum_{e\in E} \int_{I_e} u^e_x(x)\overline{v^e_x}(x)dx.$$

We denote by $D(\Delta_\Gamma)$ the set of all the functions $\mathbf{u} \in H^1(\Gamma)$ such that the linear map $\mathbf{v} \in H^1(\Gamma) \to \varphi_{\mathbf{u}}(\mathbf{v}) = \varphi(\mathbf{u},\mathbf{v})$ satisfies

$$|\varphi(\mathbf{u},\mathbf{v})| \le C\|\mathbf{v}\|_{L^2(\Gamma)} \quad \text{for all } \mathbf{v} \in H^1(\Gamma).$$

For $\mathbf{u} \in D(\Delta_\Gamma)$, we can extend $\varphi_{\mathbf{u}}$ to a linear continuous mapping on $L^2(\Gamma)$. There is a unique element in $L^2(\Gamma)$ denoted by $\Delta_\Gamma\mathbf{u}$, such that,

$$\varphi(\mathbf{u},\mathbf{v}) = -(\Delta_\Gamma\mathbf{u},\mathbf{v}) \quad \text{for all } \mathbf{v} \in H^1(\Gamma).$$

We now define the normal exterior derivative of a function $\mathbf{u} = \{u^e\}_{e\in E}$ at the endpoints of the edges. For each $e \in E$ and v an endpoint of e we consider the normal derivative of the restriction of $\mathbf{u}$ to the edge e of E_v evaluated at $i(v,e)$ to be defined by:

$$\frac{\partial u^e}{\partial n_e}(j(v,e)) = \begin{cases} -u^e_x(0+) & \text{if} \quad j(v,e) = 0, \\ u^e_x(l_e-) & \text{if} \quad j(v,e) = l_e. \end{cases}$$

With this notation it is easy to characterise $D(\Delta_\Gamma)$ (see [26]):

$$D(\Delta_\Gamma) = \Big\{ \mathbf{u} = \{u^e\}_{e\in E} \in H^2(\Gamma) : \sum_{e\in E_v} \frac{\partial u^e}{\partial n_e}(j(v,e)) = 0 \quad \text{for all } v \in V \Big\}$$

and

$$(\Delta_\Gamma \mathbf{u})^e = (u^e)_{xx} \quad \text{for all } e \in E, \mathbf{u} \in D(\Delta_\Gamma).$$

In other words $D(\Delta_\Gamma)$ is the space of all continuous functions on Γ, $\mathbf{u} = \{u^e\}_{e\in E}$, such that for every edge $e \in E$, $u^e \in H^2(I_e)$, and satisfying the following Kirchhoff-type condition:

$$\sum_{e\in E: T(e)=v} u^e_x(l_e-) - \sum_{e\in E: I(e)=v} u^e_x(0+) = 0 \quad \text{for all } v \in V.$$

It is easy to verify that $(\Delta_\Gamma, D(\Delta_\Gamma))$ is a linear, unbounded, self-adjoint, dissipative operator on $L^2(\Gamma)$, i.e. $\Re(\Delta_\Gamma \mathbf{u}, \mathbf{u})_{L^2(\Gamma)} \le 0$ for all $\mathbf{u} \in D(\Delta_\Gamma)$.

1.2 LSE and NSE on networks

Let us first consider the linear Schrödinger equation (LSE) on $\mathbb{R}$:

$$\begin{cases} iu_t + u_{xx} = 0, \; x \in \mathbb{R}, t \in \mathbb{R}, \\ u(0,x) = u_0(x), \; x \in \mathbb{R}. \end{cases}$$

The linear semigroup $e^{it\Delta}$ has two important properties, that can be easily seen via the Fourier transform. First, the conservation of the L^2-norm:

$$\|e^{it\Delta} u_0\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})} \tag{1.1}$$

and a dispersive estimate of the form:

$$\|e^{it\Delta} u_0\|_{L^\infty(\mathbb{R})} \le \frac{C}{\sqrt{|t|}} \|u_0\|_{L^1(\mathbb{R})}, \quad t \neq 0. \tag{1.2}$$

From these two inequalities, by using the classical TT^* argument, space-time estimates follow, known as Strichartz estimates ([102],[49]):

$$\|e^{it\Delta} u_0\|_{L^q_t(\mathbb{R},\, L^r_x(\mathbb{R}))} \le C\|u_0\|_{L^2(\mathbb{R})},$$

where (q,r) are so-called admissible pairs:

$$\frac{2}{q} + \frac{1}{r} = \frac{1}{2}, \quad 2 \le q, r \le \infty.$$

These dispersive estimates have been successfully applied to obtain well-posedness results for the nonlinear Schrödinger equation (see [28], [105] and the reference therein).

In this chapter we prove the dispersion inequality for the linear Schrödinger operator defined on a tree (bounded, connected graph without closed paths) with the external edges infinite. We assume that the tree does not contain vertices of multiplicity two, since they are irrelevant for our model. Let us notice that in this context we cannot use Fourier analysis as done on $\mathbb{R}$ for getting the dispersion inequality.

The presentation of the Laplace operator will be given in full details in the next section. Let us just say here that the Laplacian operator Δ_Γ acts as the usual Laplacian on $\mathbb{R}$ on each edge, and that at vertices the Kirchhoff conditions must be fulfilled: continuity condition for the functions on the graph and transmission condition at the level of their first derivative. So our analysis will be a 1-D ramified analysis. More general coupling conditions are discussed in Section 1.5.

In Section 1.3, following [55] we consider the case of regular trees. This means some restrictions on the shape of the trees: all the vertices of the same generation have the same number of descendants and all the edges of the same generation are of the same length. These restrictions allow to define some average functions on the edges of the same generation and to analyze some 1-D laminar Schrödinger equation (depending on the shape of the tree), where dispersion estimates were available from Banica's paper [7]. The strategy used in [55] cannot be applied in the case of a general tree. In [8] the dispersion has been proved in the general case. The results of [8] are presented in Section 1.4. In the case of a graph with a closed path, in general there exist compact supported eigenfunctions for the considered Laplace operator and then the dispersion estimate fails.

The motivation for studying thin structures comes from mesoscopic physics and nanotechnology. Mesoscopic systems are those that have some dimensions which are too small to be treated using classical physics while they are too large to be considered on the quantum level only. The *quantum wires* are physical systems with two dimensions reduced to a few nanometers. We refer to [81] and references therein for more details on such type of structures.

The simplest model describing conduction in quantum wires is a Hamiltonian on a planar graph, i.e. a one-dimensional object. Throughtout the paper we consider a class of idealized quantum wires, where the configuration space is a planar graph and the Hamiltonian is minus the Laplacian with Kirchhoff's boundary conditions at the vertices of the graph. This condition makes the Hamiltonian to be a self-adjoint operator. More general coupling conditions that guarantee the self-adjointness are given in [77].

The problems addressed here enter in the framework of metric graphs or networks. Those are metric spaces which can be written as the union of finitely many intervals, which are compact or $[0,\infty)$ and any two of these intervals are either disjoint or intersect only in one or both of their endpoints. Differential operators on metric graphs arise in a variety of applications. We mention some of them: carbon nano-structures [84], photonic crystals [43], high-temperature granular superconductors [1], quantum waveguides [23], free-electron theory of conjugated molecules in chemistry, quantum chaos, etc. For more details we refer the reader to review papers [81], [83], [51] and [40].

The linear and cubic Schrödinger equation on simple networks with Kirchhoff connection conditions and particular type of data has been analyzed in [25]. The symmetry imposed on the initial data and the shape of the networks allow to reduce the problem to a Schrödinger equation on the half-line with appropriate boundary conditions, for which a detailed study is done by inverse scattering. Some numerical experiments are also presented in [25]. The propagation of solitons for the cubic Schrödinger equation on simple networks but with connection conditions in link with the mass and energy conservation is analyzed in [97].

Let us consider the LSE on Γ:

$$\begin{cases} i\mathbf{u}_t(t,x)+\Delta_\Gamma\mathbf{u}(t,x)=0, & x\in\Gamma, t\neq 0,\\ \mathbf{u}(0)=\mathbf{u}_0, & x\in\Gamma. \end{cases} \tag{1.3}$$

Using the properties of the operator $i\Delta_\Gamma$ we obtain as a consequence of the Hille-Yosida theorem the following well-posedness result.

Theorem 1.1. *For any $\mathbf{u}_0\in D(\Delta_\Gamma)$ there exists a unique solution $\mathbf{u}(t)$ of system (1.3) that satisfies*

$$\mathbf{u}\in C(\mathbb{R},D(\Delta_\Gamma))\cap C^1(\mathbb{R},L^2(\Gamma)).$$

Moreover, for any $\mathbf{u}_0\in L^2(\Gamma)$, there exists a unique solution $\mathbf{u}\in C(\mathbb{R},L^2(\Gamma))$ that satisfies

$$\|\mathbf{u}(t)\|_{L^2(\Gamma)}=\|\mathbf{u}_0\|_{L^2(\Gamma)} \quad \text{for all } t\in\mathbb{R}.$$

The $L^2(\Gamma)$-isometry property is a consequence of the fact that the operator $i\Delta_\Gamma$ satisfies $\Re(i\Delta_\Gamma\mathbf{u},\mathbf{u})_{L^2(\Gamma)}=0$ for all $\mathbf{u}\in D(\Delta_\Gamma)$.

The main result is the following, where by $\{I_e\}_{e\in E}$ we shall denote the edges of the tree.

Theorem 1.2. *The solution of the linear Schrödinger equation on a tree is of the form*

$$e^{it\Delta_\Gamma}\mathbf{u}_0(x)=\sum_{\lambda\in\mathbb{R}}\frac{a_\lambda}{\sqrt{|t|}}\int_{I_\lambda}e^{i\frac{\phi_\lambda(x,y)}{t}}\ \mathbf{u}_0(y)\,dy. \tag{1.4}$$

with $\phi_\lambda(x,y)\in\mathbb{R}$, $I_\lambda\in\{I_e\}_{e\in E}$, $\sum_{\lambda\in\mathbb{R}}|a_\lambda|<\infty$, *and it satisfies the dispersion inequality*

$$\|e^{it\Delta_\Gamma}\mathbf{u}_0\|_{L^\infty(\Gamma)}\leq\frac{C}{\sqrt{|t|}}\|\mathbf{u}_0\|_{L^1(\Gamma)},\ \ t\neq 0. \tag{1.5}$$

The proof uses the method in [7] in an appropriate way related to the ramified analysis on the tree, by recursion on the number of vertices. It consists in writing the solution in terms of the resolvent of the Laplacian, which in turn is computed in the framework of almost-periodic functions. With the method used in the proof of Theorem 1.2 we can obtain the same results in the case of the Laplacian on the graph with laminar coefficients (piecewise constants, bounded between two positive constants. This might be of physical interest when the wire on a edge is composed of different pieces. Equations with variable coefficients on networks have been previously analyzed in [109] for the heat equation and in [2] for the wave equations.

Let us recall that Strichartz estimates can be derived from the dispersion inequality and have been used intensively to obtain well-posedness results for the nonlinear Schrödinger equation (NSE). The arguments used in the context of NSE on $\mathbb{R}$ can also be used here to obtain the following as a typical result.

Theorem 1.3. *Let* $p\in(0,4)$. *For any* $\mathbf{u}_0\in L^2(\Gamma)$ *there exists a unique solution*

$$\mathbf{u}\in C(\mathbb{R},L^2(\Gamma))\cap\bigcap_{(q,r)\,admissible}L^q_{loc}(\mathbb{R},L^r(\Gamma)),$$

of the nonlinear Schrödinger equation

$$\begin{cases} i\mathbf{u}_t+\Delta_\Gamma\mathbf{u}\pm|\mathbf{u}|^p\mathbf{u}=0, & t\neq 0,\\ \mathbf{u}(0)=\mathbf{u}_0, & t=0.\end{cases} \tag{1.6}$$

Moreover, the $L^2(\Gamma)$*-norm of* $\mathbf{u}$ *is conserved along the time*

$$\|\mathbf{u}(t)\|_{L^2(\Gamma)}=\|\mathbf{u}_0\|_{L^2(\Gamma)}.$$

The proof is standard once the dispersion property is obtained and it follows as in [28], p. 109, Theorem 4.6.1.

1.3 The case of regular trees

In this section we consider the Schrödinger equation on a network formed by the edges of a tree, Γ, as in Fig. 1.1. All the vortices, except the root O that has multiplicity two, have multiplicity three, i.e. the number of edges that branch out from each vortex is three (see Fig. 1.1). Also the edges of the same generation have the same length (infinite in the case of the last generation of edges). We prove dispersive properties for the Schrödeinger equation in the case of this special tree.

The main idea behind our result is that solving the linear Schrödinger equation on such a structure could be reduced to solving an equation with finitely many piecewise constant coefficients on the whole real line. Using the same method the result obtained here can be extended to regular trees Γ, i.e. trees having the property that all the vertices of the same generation have the same

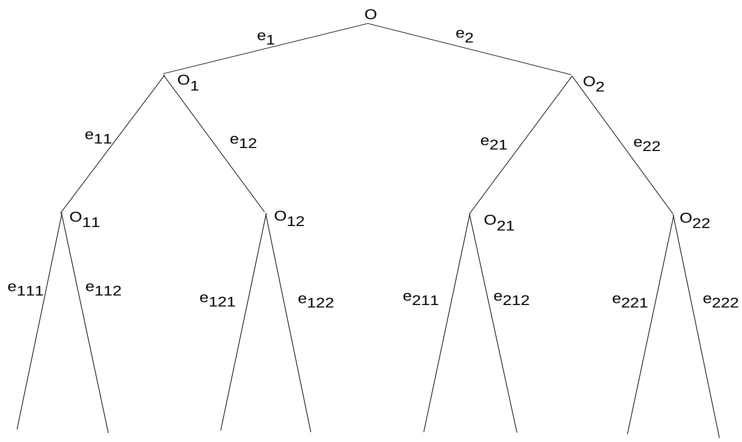

Figure 1.1: A tree with the third generation formed by infinite edges

number of descendants and all the edges of the same generation are of the same length (see [98] for more details on regular trees), assuming that the last generation of edges is formed by infinite strips.

In this section we obtain Strichartz estimates for the solutions of the LSE on the network formed by the edges of the tree Γ. Following the same arguments we can extend the results presented here to the case of a regular tree.

In the following theorem we state the dispersive property of the linear semigroup $e^{it\Delta_\Gamma}$.

Theorem 1.4. *For $p \in [2,\infty]$ and $t \neq 0$, $S_{\Delta_\Gamma}(t)$ maps $L^{p'}(\Gamma)$ continuously to $L^p(\Gamma)$ and*

$$\|e^{it\Delta_\Gamma}\mathbf{u}_0\|_{L^p(\Gamma)} \leq C(\Gamma)|t|^{-(\frac{1}{2}-\frac{1}{p})}\|\mathbf{u}_0\|_{L^{p'}(\Gamma)}, \quad \text{for all } \mathbf{u}_0 \in L^{p'}(\Gamma). \tag{1.7}$$

Remark 1.1. *The constant in Theorem 1.4 depends on the number of the generations of the tree Γ and not on the length of the edges. In the case of a regular tree with all the vertices at the generation k having each one d_{k+1} descendants, $0 \leq k \leq n$, the constants will also depend on the sequence $\{d_k\}_{k=1}^{n+1}$. In the considered case $d_k = 2$ for all $1 \leq k \leq n+1$. At the end of the proof of Theorem 1.4 we will sketch how our argument can be adapted to the case of a regular tree.*

In order to proceed to the proof of Theorem 1.4 we now describe the procedure of indexing the edges and vertices of the tree (see Fig. 1.5). For each index $\overline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \{1,2\}^k$ we denote by $|\overline{\alpha}|$ the number of its components: $|\overline{\alpha}| = k$.

The root of the tree is denoted by O. The remaining vortices and edges will be denoted by $O_{\overline{\alpha}}$, $|\overline{\alpha}| \leq n$ and $e_{\overline{\alpha}}$, $|\overline{\alpha}| \leq n+1$, respectively. Here n and $n+1$ represent the number of generations of vortices, respectively of edges.

The vortices and edges are defined by recurrence in the following way. For each vertex $O_{\overline{\alpha}}$ with the index $|\overline{\alpha}| \leq n$ (possibly empty in the case of the root O) there are two edges that branch out from it $e_{\overline{\alpha\beta}}$ where $\overline{\alpha\beta} = (\alpha_1, \alpha_2, \ldots, \alpha_k, \beta)$ and $\beta \in \{1,2\}$. If $|\overline{\alpha}| \leq n-1$ the other endpoint of $e_{\overline{\alpha\beta}}$ will be denoted by $O_{\overline{\alpha\beta}}$. In the case when $|\overline{\alpha}| = n$ the edges that branch out from these vertices are infinite strips.

With our notations $E = \{e_{\overline{\alpha}} : \overline{\alpha} \in \{1,2\}^k, 1 \leq k \leq n+1\}$. A function $\mathbf{u} : \Gamma \to \mathbb{C}$ is a collection of functions $\{u_{\overline{\alpha}}\}_{e_{\overline{\alpha}} \in E}$ where each component $u_{\overline{\alpha}}$ is a function defined on the corresponding edge $e_{\overline{\alpha}}$, $u_{\overline{\alpha}} : e_{\overline{\alpha}} \to \mathbb{C}$. Each edge $e_{\overline{\alpha}}$ will be identified with the interval $[0, l_{|\overline{\alpha}|}]$ if $|\overline{\alpha}| \leq n$ and with $[0,\infty)$ if $|\overline{\alpha}| = n+1$.

Before starting the proof, since it is quite technical, let us point out its main steps. Equation (1.3) gives us a system of LSE on intervals of the type $(0, l_k)$ or $(0,\infty)$ coupled by Kirchhoff's law. Using a translation in the space variable we transform our problem to a system of coupled linear Schrödinger equations on some intervals $(a_{|\overline{\alpha}|-1}, a_{|\overline{\alpha}|})$ or (a_n, ∞) where the sequence $\{a_k\}_{k=0}^n$ depends on the length of the edges. We then define the functions $Z^{\overline{\alpha}}$ that equal the average of the functions defined on the edges emanating from the vertex $O_{|\alpha|}$ and its descendants. Using an inductive argument on $|\overline{\alpha}|$ we prove dispersive estimates for $Z^{\overline{\alpha}}$ which give us the same ones for $\mathbf{u}$.

Proof of Theorem 1.4. We recall that Theorem 1.1 shows that $S_{\Delta_\Gamma}(t)$ in an $L^2(\Gamma)$-isometry. Thus it is sufficient to prove that for any $t\neq 0$, $S_{\Delta_\Gamma}(t)$ maps continuously $L^1(\Gamma)$ to $L^\infty(\Gamma)$ with a norm less than $C(\Gamma)|t|^{-1/2}$. By density we can consider $\mathbf{u}_0\in D(\Delta_\Gamma)$ and prove the following estimate:

$$\|S_{\Delta_\Gamma}(t)\mathbf{u}_0\|_{L^\infty(\Gamma)} \le C(\Gamma)|t|^{-\frac{1}{2}}\|\mathbf{u}_0\|_{L^1(\Gamma)} \quad \text{for all } t\neq 0.$$

In the following we prove that the above estimate holds with a constant $C(\Gamma)=C(n)$.

Using Theorem 1.1 in the particular case of the tree considered here we have that the solution of system (1.3) satisfies

$$S_{\Delta_\Gamma}(t)\mathbf{u}_0 = (u^{\overline{\alpha}}(t))_{|\overline{\alpha}|\le n+1} \in C(\mathbb{R}, D(\Delta_\Gamma))\cap C^1(\mathbb{R}, L^2(\Gamma)).$$

It means that for all $t\in\mathbb{R}$ the functions $u^{\overline{\alpha}}$, $|\overline{\alpha}|\le n+1$, satisfy

$$u^{\overline{\alpha}}(t) \in \begin{cases} H^2(0, l_{|\alpha|}), & |\overline{\alpha}|\le n, \\ H^2(0,\infty), & |\overline{\alpha}| = n+1. \end{cases}$$

Moreover, the family $\{u^{\overline{\alpha}}\}_{1\le|\overline{\alpha}|\le n+1}$ solves the following system:

$$\begin{cases} iu_t^{\overline{\alpha}}(t,x) + u_{xx}^{\overline{\alpha}}(t,x) = 0, & x\in(0,l_{|\alpha|}), 1\le|\overline{\alpha}|\le n, \\ iu_t^{\overline{\alpha}}(t,x) + u_{xx}^{\overline{\alpha}}(t,x) = 0, & x\in(0,\infty), |\overline{\alpha}| = n+1, \\ \begin{cases} u^{\overline{\alpha}}(t, l_{|\overline{\alpha}|}) = u^{\overline{\alpha\beta}}(t,0), & \beta\in\{1,2\}, 1\le|\overline{\alpha}|\le n, \\ u^1(0,t) = u^2(0,t), \end{cases} \\ \begin{cases} u_x^{\overline{\alpha}}(t, l_{|\overline{\alpha}|}) = \displaystyle\sum_{\beta=1}^{2} u_x^{\overline{\alpha\beta}}(t,0), & 1\le|\overline{\alpha}|\le n, \\ u_x^1(0,t) + u_x^2(0,t) = 0, \end{cases} \\ u^{\overline{\alpha}}(0,x) = u_0^{\overline{\alpha}}(x). \end{cases} \tag{1.8}$$

The first two equations represent the LSE satisfied by each $u^{\overline{\alpha}}$. The second type of properties gives the continuity at the ends of the vertices and the third one is the Kirchhoff type condition on the normal derivatives.

However, the above system is not very useful in order to reduce it to a LSE with discontinuous coefficients as we announced in the introduction. We will rewrite the above system in a convenient manner that will allow us to apply previous results on the dispersive properties of the Schrödinger equation $iu_t + (\sigma u_x)_x = 0$ (see [7]), where σ is a step function taking a finite number of values.

We consider the intervals

$$I_k = \begin{cases} (a_{k-1}, a_k) & \text{if } \quad 1\le k\le n, \\ (a_n, \infty) & \text{if } \quad k = n+1, \end{cases}$$

where $a_0 = 0$ and $a_{k+1} = a_k + l_{k+1}$ for $k = 0,\dots,n-1$.

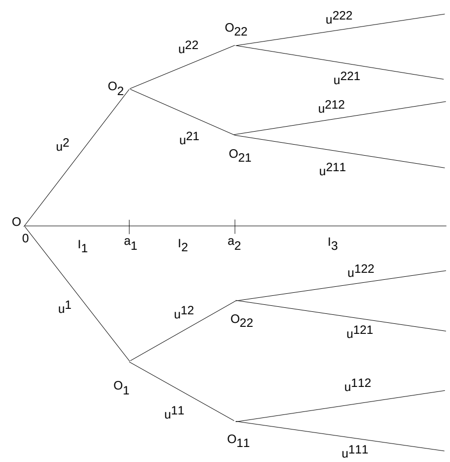

Figure 1.2: The functions situated above each interval are defined on that interval, for example u^1 and u^2 are defined on I_1, etc.

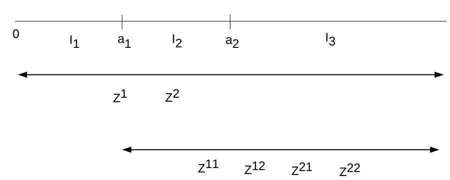

Figure 1.3: The domain where the functions $Z^{\overline{\alpha}}$ are defined

With the new notations, by applying translations in the space variable, system (1.8) can be written in an equivalent form:

$$\begin{cases} iu_t^{\overline{\alpha}}(t,x) + u_{xx}^{\overline{\alpha}}(t,x) = 0, \quad x \in I_{|\overline{\alpha}|}, 1 \leq |\overline{\alpha}| \leq n+1, \\ \begin{cases} u^{\overline{\alpha}}(t, a_{|\overline{\alpha}|}) = u^{\overline{\alpha\beta}}(t, a_{|\overline{\alpha}|}), \quad \beta \in \{1,2\}, 1 \leq |\overline{\alpha}| \leq n, \\ u^1(t,0) = u^2(t,0), \end{cases} \\ \begin{cases} u_x^{\overline{\alpha}}(t, a_{|\overline{\alpha}|}) = \displaystyle\sum_{\beta=1}^{2} u_x^{\overline{\alpha\beta}}(t, a_{|\overline{\alpha}|}), \quad 1 \leq |\overline{\alpha}| \leq n, \\ u_x^1(t,0) + u_x^2(t,0) = 0, \end{cases} \\ u^{\overline{\alpha}}(0,x) = u_0^{\overline{\alpha}}(x), \quad x \in I_{|\alpha|}, 1 \leq |\overline{\alpha}| \leq n. \end{cases} \tag{1.9}$$

In Fig. 1.2 we can visualise where each function $u^{\overline{\alpha}}$ is defined after the translation. We point out that once the dispersive properties are obtained for the second system (1.9) they also hold for the first one (1.8). In the sequel we will concentrate on system (1.9) and prove that for each index $\overline{\alpha}$ with $|\overline{\alpha}| = k$, $1 \leq k \leq n+1$, the following holds for all $t \neq 0$:

$$\max_{|\overline{\alpha}|=k} \|u^{\overline{\alpha}}(t)\|_{L^\infty(I_k)} \leq C(n)|t|^{-1/2} \sum_{1 \leq |\overline{\beta}| \leq n+1} \|u_0^{\overline{\beta}}\|_{L^1(I_{|\overline{\beta}|})}. \tag{1.10}$$

For any $\overline{\alpha}$ with $1 \leq |\overline{\alpha}| \leq n+1$ we define the functions

$$Z^{\overline{\alpha}} : J_{|\overline{\alpha}|} = \bigcup_{k=0}^{n+1-|\overline{\alpha}|} I_{|\overline{\alpha}|+k} \to \mathbb{C}$$

as follows: for each $0 \le k \le n+1-|\alpha|$ and $x \in I_{|\alpha|+k}$ we set

$$Z^{\overline{\alpha}}(t,x) = \frac{\sum_{|\overline{\beta}|=k} u^{\overline{\alpha\beta}}(t,x)}{2^{|\overline{\beta}|}}. \tag{1.11}$$

The domain where each function $Z^{\overline{\alpha}}$ is defined can be viewed in Fig. 1.3.

The definition of the functions $Z^{\overline{\alpha}}$ shows that it coincides with $u^{\overline{\alpha}}$ on the interval $I_{|\overline{\alpha}|}$. Thus it is sufficient to prove that

$$\|Z^{\overline{\alpha}}(t)\|_{L^\infty(J_{\overline{\alpha}})} \le c(n)|t|^{-1/2} \sum_{1\le|\overline{\beta}|\le n+1} \|u_0^{\overline{\beta}}\|_{L^1(I_{\overline{\beta}})}. \tag{1.12}$$

In what follows it will be convenient to write $Z^{\overline{\alpha}}$, $1 \le |\overline{\alpha}| \le n+1$, in a compact form:

$$Z^{\overline{\alpha}}(t,x) = \sum_{k=0}^{n+1-|\overline{\alpha}|} \chi_{\{I_{|\overline{\alpha}|+k}\}}(x)\frac{1}{2^k}\sum_{|\overline{\beta}|=k} u^{\overline{\alpha\beta}}(t,x). \tag{1.13}$$

Moreover, the above identity shows that for each index $\overline{\alpha}$ with $1 \le |\overline{\alpha}| \le n+1$ we have the following inequality:

$$\|Z^{\overline{\alpha}}(0)\|_{L^1(J_{|\overline{\alpha}|})} \le \sum_{|\overline{\alpha}|\le|\overline{\beta}|\le n+1} \|u_0^{\overline{\beta}}\|_{L^1(I_{|\overline{\beta}|})}. \tag{1.14}$$

To prove inequality (1.12) we recall the following result of Banica [7].

Theorem 1.5. *([7]) Consider a partition of the real axis* $-\infty = x_0 < x_1 < \dots < x_{n+1} = \infty$ *and a step function*

$$\sigma(x) = \sigma_i \text{ for } x \in (x_i, x_{i+1}),$$

where σ_i are positive numbers.

The solution u of the Schrödinger equation

$$\begin{cases} iu_t(t,x) + (\sigma(x)u_x)_x(t,x) = 0, & \text{for } x \in \mathbb{R}, t \ne 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

satisfies the dispersion inequality

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \le C|t|^{-1/2}\|u_0\|_{L^1(\mathbb{R})}, \qquad t \ne 0,$$

where the constant C depends on n and on the sequence $\{\sigma_i\}_{i=0}^n$.

As a consequence we obtain the following result.

Lemma 1.1. *Let $n \ge 0$, $-\infty \le a_0 < a_1 < \dots < a_n < a_{n+1} = \infty$ and the function*

$$v \in C(\mathbb{R}, H^2((a_k,a_{k+1}))) \cap C^1(\mathbb{R}, L^2((a_k,a_{k+1}))), \quad k = 0,\dots,n$$

which solves the following system

$$\begin{cases} iv_t(t,x) + v_{xx}(t,x) = 0, & x \in (a_k,a_{k+1}),\ 0 \le k \le n, t \ne 0, \\ v(t,a_k-) = v(t,a_k+), & 1 \le k \le n, \\ v_x(t,a_k-) = c_k v_x(t,a_k+), & t \ne 0, 1 \le k \le n, \\ v(t,a_0) = 0, & \text{if } a_0 > -\infty,\ t \ne 0, \\ v(0,x) = v_0(x), & x \in (a_k,a_{k+1}),\ 0 \le k \le n, \end{cases} \tag{1.15}$$

for some positive constants $\{c_k\}_{k=1}^n$.

If $v_0 \in L^1(a_0,\infty)$ *then there exists a positive constant* $c(n,\{c_k\}_{k=1}^n)$ *such that*

$$\|v(t)\|_{L^\infty((a_0,\infty))} \leq c(n,\{c_k\}_{k=1}^n)|t|^{-1/2}\|v_0\|_{L^1((a_0,\infty))} \quad \text{for all } t \neq 0. \tag{1.16}$$

Remark 1.2. *The constant in the right hand side of* (1.16) *does not depend on the sequence* $\{a_k\}_{k=0}^{n+1}$.

We now prove inequality (1.12) by using an inductive argument following the length of the index $\overline{\alpha}$. We first consider the case when $|\overline{\alpha}| = 1$.

Step 1. The first generation of Z's, $|\overline{\alpha}| = 1$. We consider the functions

$$Z(t,x) = \frac{Z^1(t,x) + Z^2(t,x)}{2}, \quad x \in (0,\infty), t \in \mathbb{R}$$

and

$$\tilde{Z}^1 = Z - Z^1, \quad \tilde{Z}^2 = Z - Z^2.$$

We claim that

$$\|Z(t)\|_{L^\infty((0,\infty))} \leq C(n)|t|^{-1/2}\|Z^1(0) + Z^2(0)\|_{L^1((0,\infty))} \tag{1.17}$$

and

$$\|\tilde{Z}^1(t)\|_{L^\infty((0,\infty))} \leq C(n)|t|^{-1/2}\|Z^1(0) - Z^2(0)\|_{L^1((0,\infty))}. \tag{1.18}$$

A similar estimate will also hold for $\tilde{Z}^2(t)$.

In view of the definition of the functions Z and $\tilde{Z}^1$ estimates (1.17) and (1.18) imply that

$$\|Z^1(t)\|_{L^\infty((0,\infty))} \leq C(n)|t|^{-1/2}(\|Z^1(0)\|_{L^1((0,\infty))} + \|Z^2(0)\|_{L^1((0,\infty))}).$$

Thus, inequality (1.14) gives us that

$$\|Z^1(t)\|_{L^\infty((0,\infty))} \leq C(n)|t|^{-1/2}\sum_{1\leq|\overline{\beta}|\leq n+1}\|u_0^{\overline{\beta}}\|_{L^1(I_{|\overline{\beta}|})},$$

which proves estimate (1.12) in the considered case $|\overline{\alpha}| = 1$.

We now prove estimates (1.17) and (1.18).

For the first one, we observe that Z satisfies the system

$$\begin{cases} iZ_t(t,x) + Z_{xx}(t,x) = 0, & x \in \mathbb{R}\backslash\{a_k, 1 \leq k \leq n\}, \\ Z(t,a_k-) = Z(t,a_k+), & 1 \leq k \leq n, \\ Z_x(t,a_k-) = \frac{1}{2}Z_x(t,a_k+), & 1 \leq k \leq n, \\ Z_x(t,0) = 0, & t \in \mathbb{R}, \\ Z(0,x) = \dfrac{Z^1(0,x) + Z^2(0,x)}{2}, & x \in \mathbb{R}\backslash\{a_k, 1 \leq k \leq n\}. \end{cases} \tag{1.19}$$

Making an even extension of the function Z to the whole real line we enter in the framework of Lemma 1.1 which gives us the following estimates for the function Z:

$$\|Z(t)\|_{L^\infty((0,\infty))} \leq C(n)|t|^{-1/2}\|Z(0)\|_{L^1((0,\infty))} \leq C(n)|t|^{-1/2}\|Z^1(0) + Z^2(0)\|_{L^1((0,\infty))}.$$

This proves (1.17).

We now prove (1.18). The function $\tilde{Z}^1$ satisfies the following system:

$$\begin{cases} i\tilde{Z}^1_t(t,x)+\tilde{Z}^1_{xx}(t,x)=0 & x\in\mathbb{R}\backslash\{a_k,1\le k\le n\},\\ \tilde{Z}^1(t,a_k-)=\tilde{Z}^1(t,a_k+), & 1\le k\le n,\\ \tilde{Z}^1(t,0)=0, & t\in\mathbb{R},\\ \tilde{Z}^1_x(t,a_k-)=\frac{1}{2}\tilde{Z}^1_x(t,a_k+), & 1\le k\le n,\\ \tilde{Z}^1(0,x)=\dfrac{Z^2(0,x)-Z^1(0,x)}{2}, & x\in\mathbb{R}\backslash\{a_k,1\le k\le n\}. \end{cases} \tag{1.20}$$

We now apply Lemma 1.1 and we obtain that $\tilde{Z}^1$ satisfies

$$\|\tilde{Z}^1(t)\|_{L^\infty((0,\infty))}\le C(n)|t|^{-1/2}\|\tilde{Z}^1(0)\|_{L^1((0,\infty))}\le C(n)|t|^{-1/2}\|Z^1(0)-Z^2(0)\|_{L^1((0,\infty))}.$$

The proof of (1.12) in the case $|\overline{\alpha}|=1$ is now finished.

Step II. The next generations of Z's. We assume that we previously proved (1.12) for all indices $\overline{\alpha}$ with its length satisfying $|\overline{\alpha}|=k\le n$. Let us choose an arbitrary index $\overline{\alpha}$ with $|\overline{\alpha}|=k$. We prove that (1.12) also holds for the index $\overline{\alpha\beta}$ with $\beta\in\{1,2\}$.

We consider the function

$$\tilde{Z}^{\overline{\alpha\beta}}(t,x)=Z^{\overline{\alpha\beta}}(t,x)-Z^{\overline{\alpha}}(t,x),\quad x\in J_{k+1}=\bigcup_{m=k+1}^{n+1}I_m.$$

In the case $k\le n-1$ the new function $\tilde{Z}^{\overline{\alpha\beta}}$ satisfies the following system:

$$\begin{cases} i\tilde{Z}^{\overline{\alpha\beta}}_t(t,x)+\tilde{Z}^{\overline{\alpha\beta}}_{xx}(t,x)=0, & t\ne0, x\in\bigcup_{m=k+1}^{n+1}I_m,\\ \tilde{Z}^{\overline{\alpha\beta}}(t,a_k)=0, & t\ne 0,\\ \tilde{Z}^{\overline{\alpha\beta}}(t,a_m-)=\tilde{Z}^{\overline{\alpha\beta}}(t,a_m+), & k+1\le m\le n,\\ \tilde{Z}^{\overline{\alpha\beta}}_x(t,a_m-)=\frac{1}{2}\tilde{Z}^{\overline{\alpha\beta}}_x(t,a_m+), & k+1\le m\le n,\\ \tilde{Z}^{\overline{\alpha\beta}}(0,x)=\tilde{Z}^{\overline{\alpha\beta}}_0(x), & x\in\bigcup_{m=k+1}^{n+1}I_m. \end{cases} \tag{1.21}$$

In the other case, $k=n$, we are dealing with an equation on the last generation of edges of the tree and we get the following system:

$$\begin{cases} i\tilde{Z}^{\overline{\alpha\beta}}_t(t,x)+\tilde{Z}^{\overline{\alpha\beta}}_{xx}(t,x)=0, & t\ne0, x\in I_{n+1},\\ \tilde{Z}^{\overline{\alpha\beta}}(t,a_n)=0, & t\ne0,\\ \tilde{Z}^{\overline{\alpha\beta}}(0,x)=\tilde{Z}^{\overline{\alpha\beta}}_0(x), & x\in I_{n+1}. \end{cases} \tag{1.22}$$

In both systems the second property is a consequence of the fact that $u^{\overline{\alpha}}(t,a_k)=u^{\overline{\alpha\beta}}(t,a_k)$ and thus $Z^{\overline{\alpha}}(t,a_k)=Z^{\overline{\alpha\beta}}(t,a_k)$. The third and fourth properties in system (1.21) are given by the Kirchhoff type conditions imposed in system (1.9).

Systems (1.21) and (1.22) enter in the framework of Lemma 1.1. Thus, we have that

$$\|\tilde{Z}^{\overline{\alpha\beta}}(t)\|_{L^\infty(J_{k+1})}\le C(n)|t|^{-1/2}\|\tilde{Z}^{\overline{\alpha\beta}}_0\|_{L^1(J_{k+1})}.$$

It follows that the same property holds for $Z^{\overline{\alpha\beta}}$ since $Z^{\overline{\alpha\beta}} = \tilde{Z}^{\overline{\alpha\beta}} + Z^{\overline{\alpha}}$. Indeed, for any $t \neq 0$ we have

$$\|Z^{\overline{\alpha\beta}}(t)\|_{L^\infty(J_{k+1})} \leq \|\tilde{Z}^{\overline{\alpha\beta}}(t)\|_{L^\infty(J_{k+1})} + \|Z^{\overline{\alpha}}(t)\|_{L^\infty(J_{k+1})} \leq \|\tilde{Z}^{\overline{\alpha\beta}}(t)\|_{L^\infty(J_{k+1})} + \|Z^{\overline{\alpha}}(t)\|_{L^\infty(J_k)}$$

and using the inductive assumption on $Z^{\overline{\alpha}}$ we obtain that

$$\begin{aligned} \|Z^{\overline{\alpha\beta}}(t)\|_{L^\infty(I_k)} &\leq C(n)|t|^{-1/2}(\|Z_0^{\overline{\alpha\beta}} - Z_0^{\overline{\alpha}}\|_{L^1(J_{k+1})} + \|Z_0^{\overline{\alpha}}\|_{L^1(J_k)}) \\ &\leq C(n)|t|^{-1/2} \sum_{|\overline{\beta}|\leq n+1} \|u_0^{\overline{\beta}}\|_{L^1(I_{\overline{\beta}})}. \end{aligned} \tag{1.23}$$

This implies that (1.12) holds for all indices with length $k+1$.

The proof of (1.12) is now finished. □

Let us now comment about how the above proof can be adapted to obtain similar results in the case of a regular tree. Assume that all the vortices at the generation k have d_{k+1} descendants, $0 \leq k \leq n$, and all the edges of the same generation have the same length.

In this framework we have to modify the functions $Z^{\overline{\alpha}}$ in (1.11) in the following way:

$$Z^{\overline{\alpha}}(t,x) = \begin{cases} u^{\overline{\alpha}} & x \in I_{|\overline{\alpha}|}, \\ \dfrac{\sum_{|\overline{\beta}|=k} u^{\overline{\alpha\beta}}(t,x)}{d_{|\overline{\alpha}|+1} \cdot \dots \cdot d_{|\overline{\alpha}|+k}}, & x \in I_{|\overline{\alpha}|+k}, 1 \leq k \leq n+1-|\overline{\alpha}|. \end{cases}$$

In Step I, we replace Z by

$$Z = \frac{Z^1 + \dots + Z^{d_1}}{d_1}, \tilde{Z}^j = Z - Z^j, j = 1, \dots, d_1,$$

the constant $1/2$ in coupling the derivatives at the points a_k in systems (1.19) and (1.20) by $1/d_{k+1}$ and the initial data in the two systems in agreement to the new definition of the functions $Z^{\overline{\alpha}}$. In Step II we replace in a similar manner the constant $1/2$ in systems (1.21) and (1.22) with $1/d_{m+1}$ and the initial data.

The assumption on the geometry of the tree and the definition of the functions $Z^{\overline{\alpha}}$ as the average of the functions defined on the edges emanating from the vertex $O_{|\alpha|}$ and its descendants allow us to obtain the continuity property at the points $\{a_k\}_{k=1}^n$ in systems (1.19 - 1.22).

With the above changes we can extend the results of Theorem 1.4 and Theorem 1.3 to regular trees.

1.4 The general case

In the previous section we have considered the case of a tree such that each internal vertex has two descendants and the results can be extended to the case of a regular tree as it was defined in [98]. Our proof does not cover the case when the edges of the tree have arbitrary lengths. The fact that all the edges at the same generation have the same length allowed us to consider the averages of the functions $u^{\overline{\alpha}}$'s defined at the same generation of edges by introducing the functions $Z^{\overline{\alpha}}$ in the proof of Theorem 1.4. In the case when the edges of a generation have different lengths we cannot define the functions $Z^{\overline{\alpha}}$ and our argument cannot be applied. This is the case, for example, of the tree in Fig. 1.4 where the functions $u^{\overline{\alpha}}$, $\overline{\alpha} \in \{1,2\}^2$ are not defined on the same interval.

On the other hand the proof of Theorem 1.4 cannot be applied to the case when at some generation we have two vortices with different number of descendants. The fact that at the same generation the number of descendants is the same help us to obtain the continuity property at the points $\{a_k\}_{k=1}^n$ of the functions involved in systems (1.19 - 1.22) and then to use Lemma 1.1.

The objective of this section is to use a different approach. Using an inductive argument we give a description of the solutions of the linear problem that allow us to obtain the dispersive estimates.

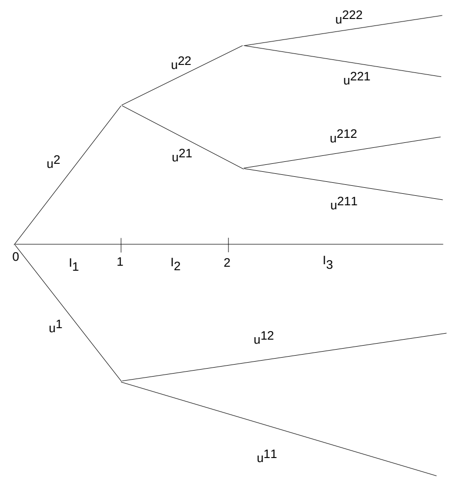

Figure 1.4: A tree where the infinite strips occur at different levels

For $\omega \geq 0$ let R_ω be the resolvent of the Laplacian on a tree

$$R_\omega \mathbf{f} = (-\Delta_\Gamma + \omega^2 I)^{-1}\mathbf{f}.$$

We shall prove in Lemma 1.2 that $\omega R_\omega \mathbf{f}(x)$ can be analytically continued in a region containing the imaginary axis. Therefore we can use a spectral calculus argument to write the solution of the Schrödinger equation with initial data $\mathbf{u}_0$ as

$$e^{it\Delta_\Gamma}\mathbf{u}_0(x) = \int_{-\infty}^{\infty} e^{it\tau^2}\tau R_{i\tau}\mathbf{u}_0(x)\frac{d\tau}{\pi}. \tag{1.24}$$

We shall also obtain in Lemma 1.3 that the following decomposition holds

$$\tau R_{i\tau}\mathbf{u}_0(x) = \sum_{\lambda\in\mathbb{R}} b_\lambda e^{i\tau\psi_\lambda(x)} \int_{I_\lambda} \mathbf{u}_0(y)e^{i\tau\beta_\lambda y}dy, \tag{1.25}$$

with $\psi_\lambda(x), \beta_\lambda \in \mathbb{R}$, $I_\lambda \in \{I_e\}_{e\in E}$ and $\sum_{\lambda\in\mathbb{R}}|b_\lambda| < \infty$. Then decomposition (1.4) is implied by (1.24), (1.25) and the fact that for $t > 0$ and $r \in \mathbb{R}$

$$\int_{-\infty}^{\infty} e^{it\tau^2}e^{i\tau r}d\tau = e^{i\frac{\pi}{4}}\sqrt{\pi}\frac{e^{-\frac{r^2}{4t}}}{\sqrt{t}}.$$

From (1.4) the dispersion estimate (1.5) of Theorem 1.2 follows immediately since $\sum_{\lambda\in\mathbb{R}}|\alpha_\lambda| < \infty$.

Above and in what follows the integration of function $\mathbf{f} = (f^e)_{e\in E}$ on interval I_e means the integral of f^e on the considered interval.

Remark 1.3. *As in [7] we notice that since we can express the solution of the wave equation* $\mathbf{v}_{tt} - \Delta_\Gamma \mathbf{v} = 0$ *with initial data* $(\mathbf{v}_0, 0)$ *as*

$$\mathbf{v}(t,x) = \int_{-\infty}^{\infty} e^{it\tau}R_{i\tau}\mathbf{v}_0(x)i\tau\frac{d\tau}{2\pi},$$

the property

$$\sup_{x\in\Gamma}\int_{-\infty}^{\infty}|\mathbf{v}(t,x)|dt \leq C\|\mathbf{v}_0\|_{L^1(\Gamma)}$$

follows similarly.

We now obtain the expression of the resolvent. The second-order equations

$$(R_\omega \mathbf{f})'' = \omega^2 R_\omega \mathbf{f} - \mathbf{f}$$

must be solved on each edge of the tree together with coupling conditions at each vertex. Then, on each edge parametrized by I_e,

$$R_\omega \mathbf{f}(x) = ce^{\omega x} + \tilde{c}e^{-\omega x} + \frac{1}{2\omega}\int_{I_e} \mathbf{f}(y)\, e^{-\omega|x-y|}dy,\ x \in I_e.$$

Since $R_\omega \mathbf{f}$ belongs to $L^2(\Gamma)$ the coefficients c's are zero on the infinite edges $e \in \mathcal{E}$, parametrized by $[0,\infty)$. If we denote by $\mathcal{I}$ the set of internal edges, we have $2|\mathcal{I}|+|\mathcal{E}|$ coefficients. The Kirchhoff conditions of continuity of $R_\omega \mathbf{f}$ and of transmission of $\partial_x R_\omega \mathbf{f}$ at the vertices of the tree give the system of equations on the coefficients. We have the same number of equations as the number of unknowns. We denote D_Γ the matrix of the system, whose elements are real powers of e^ω.

Therefore the resolvent $R_\omega \mathbf{f}(x)$ is a finite sum of terms :

$$R_\omega \mathbf{f}(x) = \frac{1}{\omega\ \det D_\Gamma(\omega)} \sum_{\lambda=1}^{N(\Gamma)} c_\lambda e^{\pm\omega\Phi_\lambda(x)} \int_{I_\lambda} \mathbf{f}(y)\, e^{\pm\omega y}dy + \frac{1}{2\omega}\int_{I_e} \mathbf{f}(y)\, e^{-\omega|x-y|}dy, \tag{1.26}$$

where $x \in I_e$, $\Phi_\lambda(x) \in \mathbb{R}$, $I_\lambda \in \{I_e\}_{e\in E}$ and $|N(\Gamma)| < \infty$. We shall prove the following proposition that will imply Lemma 1.2 and 1.3 needed for obtaining Theorem 1.2.

Proposition 1.1. *Function* $\det D_\Gamma(\omega)$ *is lower bounded by a positive constant on a strip containing the imaginary axis:*

$$\exists c_\Gamma, \epsilon_\Gamma > 0,\ |\det D_\Gamma(\omega)| > c_\Gamma, \forall \omega \in \mathbb{C}, |\Re\omega| < \epsilon_\Gamma.$$

Lemma 1.2. *Function* $\omega R_\omega \mathbf{f}(x)$ *can be analytically continued in a region containing the imaginary axis.*

Proof. The proof is an immediate consequence of decomposition (1.26) and of Proposition 1.1. □

Lemma 1.3. *The following decomposition holds*

$$\tau R_{i\tau}\mathbf{u}_0(x) = \sum_{\lambda\in\mathbb{R}} b_\lambda e^{i\tau\psi_\lambda(x)} \int_{I_\lambda} \mathbf{u}_0(y) e^{i\tau\beta_\lambda y}dy,$$

with $\psi_\lambda(x), \beta_\lambda \in \mathbb{R}$, $I_\lambda \in \{I_e\}_{e\in E}$ *and* $\sum_{\beta\in\mathbb{R}} |b_\lambda| < \infty$.

Proof. We notice that for $\tau \in \mathbb{R}$, $\det D_\Gamma(i\tau)$ is a finite sum of powers of $e^{i\tau}$. Then, by Proposition 1.1 we are in the framework of a classical theorem in representation theory (S29, Cor.1 of [46]) that asserts that the inverse of $\det D_\Gamma(i\tau)$ is $\sum_{\lambda\in\mathbb{R}} d_\lambda e^{i\tau\lambda}$ with $\sum_{\lambda\in\mathbb{R}} |d_\lambda| < \infty$, and from (1.26) the Lemma follows. □

Let sketch the proof of Proposition 1.1. We shall show by recursion on the number of vertices the following stronger "double" property:

$$P(n): \quad \text{If } \Gamma \text{ has } n \text{ vertices, we have the property } \mathcal{P},$$

$$\mathcal{P}: \exists c_\Gamma, \epsilon_\Gamma > 0, \exists 0 < r_\Gamma < 1,\ |\det D_\Gamma(\omega)| > c_\Gamma,\ \left|\frac{\det \tilde{D}_\Gamma(\omega)}{\det D_\Gamma(\omega)}\right| < r_\Gamma,\ \forall \omega \in \mathbb{C}, |\Re\omega| < \epsilon_\Gamma.$$

We have denoted by $\tilde{D}_\Gamma(\omega)$ the matrix of the system verified by the coefficients, if we impose that on one of the last infinite edges $l \in \mathcal{E}$ we replace in the expression of the resolvent $\tilde{c}e^{-\omega x}$ by $ce^{\omega x}$.

In the case of $P(1)$ we have a star-shaped tree with $m \geq 3$ of edges. All the edges are parametrized by $[0,\infty)$. In particular $D_\Gamma(\omega) = D_\Gamma$. We shall actually prove a stronger property, which implies the property $\mathcal{P}$ for any $\epsilon_\Gamma > 0$:

$$P(1,m): \quad \text{If } \Gamma \text{ has 1 vertex and } m \text{ edges, } \det D_\Gamma(\omega) = m \text{ and } \det \tilde{D}_\Gamma(\omega) = m-2.$$

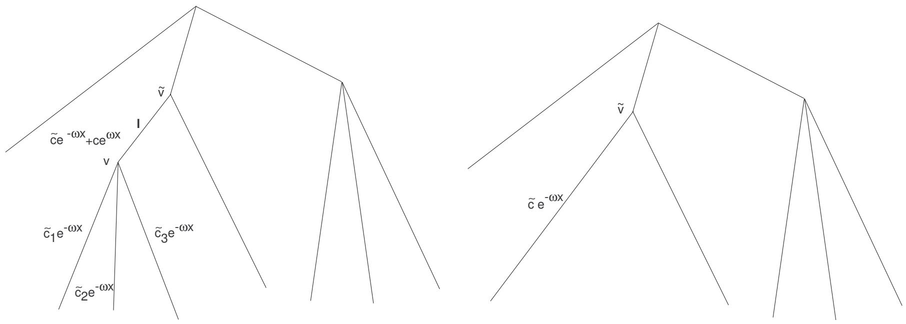

Figure 1.5: With vertex v we obtain tree Γ_4 (left) from Γ_3 (right).

It remains to show that $P(n-1) \to P(n)$ holds. Any tree Γ_n with n vertices, $n \geq 2$, can be seen as a tree Γ_{n-1} with $n-1$ vertices on which we add an extra-vertex. More precisely, let us consider a vertex v from which there start $m \geq 2$ external infinite edges and one internal edge connecting it to the rest of the tree (see Fig. 1.5). Let us notice that such a choice is possible since the graph has no cycles. In particular the edge whose lower extremity is this vertex v is an internal edge l, whose length should be denoted by a, and whose upper vertex we denote by $\tilde{v}$. Now we remove this vertex and transform the internal edge l into an external infinite one. The new graph Γ_{n-1} has $n-1$ vertices.

With respect to the problem on Γ_{n-1}, the resolvent on Γ_n involves a new term $ce^{\omega x}$ aside from $\tilde{c}e^{-\omega x}$ on the interval edge l, and on the external edges emerging from the vertex v it involves terms $\tilde{c}_j e^{-\omega x}$, $1 \leq j \leq m$. We have also the Kirchhoff conditions at the vertex v, which give $m+1$ equations on the coefficients.

We write the square $N \times N$ matrix D_{Γ_n} such that the last $m+2$ column corresponds to the unknowns $\tilde{c}, c, \tilde{c}_1, ..., \tilde{c}_m$. On the last line we write the Kirchhoff derivative condition at the vertex v, and on the $N-j$ lines, $1 \leq j \leq m$ the Kirchhoff continuity conditions at the vertex v. Also, on the $N-m-1$ line we write the derivative condition in the vertex $\tilde{v}$ and on the $N-m-2$ line the continuity condition in $\tilde{v}$ relating $\tilde{c}$, and now also c, to the others coefficients. So D_{Γ_n} is a matrix obtained from the $(N-m-1) \times (N-m-1)$ matrix $D_{\Gamma_{n-1}}$ (whose last column corresponds to the unknown $\tilde{c}$) in the following way

$$D_{\Gamma_n} = \left(\begin{array}{c|ccccccccc}
D_{\Gamma_{n-1}} & & & & & & & & & \\
 & -1 & & & & & & & & \\
 & -1 & & & & & & & & \\
\hline
e^{-\omega a} & e^{\omega a} & -1 & & & & & & & \\
 & & 1 & -1 & & & & & & \\
 & & & 1 & -1 & & & & & \\
 & & & & \cdot & \cdot & & & & \\
 & & \cdot & \cdot & & & & & & \\
 & & & & & & \cdot & \cdot & & \\
 & & & & & & & 1 & -1 & \\
 & & & & & & & & 1 & -1 \\
-e^{-\omega a} & e^{\omega a} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array} \right).$$

We develop $\det D_{\Gamma_n}$ with respect to the last $m+1$ lines, that is as an alternated sum of determinants of $m+1 \times m+1$ minors composed from the last $m+1$ lines of D_{Γ_n} times the determinant of D_{Γ_n} without the lines and columns the minor is made of. The only possibility to obtain a $m+1 \times m+1$ minor composed from the last $m+1$ lines of D_{Γ_n} different from zero is to choose one of the columns $N-m-1$ and $N-m$, together with all last m columns. This follows from the fact that if we eliminate from $\det D_{\Gamma_n}$ both columns $N-m-1$ and $N-m$, together with $m-1$ columns among the last m columns, we obtain a block-diagonal type matrix, with first diagonal block $D_{\Gamma_{n-1}}$ with its last column replaced by zeros, so its determinant vanishes. Therefore

$$\det D_{\Gamma_n} = \det D_{\Gamma_{n-1}} \begin{vmatrix} e^{\omega a} & -1 & & & & & \\ & 1 & -1 & & & & \\ & & 1 & -1 & & & \\ & & & \ddots & \ddots & & \\ & & & & 1 & -1 & \\ & & & & & 1 & -1 \\ e^{\omega a} & 1 & 1 & \cdots & 1 & 1 & 1 \end{vmatrix}$$

$$- \det \tilde{D}_{\Gamma_{n-1}} \begin{vmatrix} e^{-\omega a} & -1 & & & & & \\ & 1 & -1 & & & & \\ & & 1 & -1 & & & \\ & & & \ddots & \ddots & & \\ & & & & 1 & -1 & \\ & & & & & 1 & -1 \\ -e^{-\omega a} & 1 & 1 & \cdots & 1 & 1 & 1 \end{vmatrix}.$$

By developing with respect to the first column the $m+1 \times m+1$ minors,

$$\det D_{\Gamma_n} = \det D_{\Gamma_{n-1}}(e^{\omega a} \det D_{\Gamma^m} + (-1)^{m+2} e^{\omega a}(-1)^m)$$
$$- \det \tilde{D}_{\Gamma_{n-1}}(e^{-\omega a} \det D_{\Gamma^m} - (-1)^{m+2} e^{-\omega a}(-1)^m),$$

so using from the previous subsection that $\det D_{\Gamma^m} = m, \det \tilde{D}_{\Gamma^m}(\omega) = m-2$, we find

$$\det D_{\Gamma_n}(\omega) = (m+1)e^{\omega a} \det D_{\Gamma_{n-1}}(\omega) - (m-1)e^{-\omega a} \det \tilde{D}_{\Gamma_{n-1}}(\omega)$$
$$= (m+1)e^{\omega a} \det D_{\Gamma_{n-1}}(\omega) \left(1 - e^{-2\omega a} \frac{m-1}{m+1} \frac{\det \tilde{D}_{\Gamma_{n-1}}(\omega)}{\det D_{\Gamma_{n-1}}(\omega)}\right).$$

Now, from $P(n-1)$ we have for $|\Re\omega|$ small enough

$$1 - e^{-2\omega a} \frac{m-1}{m+1} \frac{\det \tilde{D}_{\Gamma_{n-1}}(\omega)}{\det D_{\Gamma_{n-1}}(\omega)} > c_0 > 0.$$

Also, $P(n-1)$ gives us the existence of two positive constants $c_{\Gamma_{n-1}}$ and $\epsilon_{\Gamma_{n-1}}$ such that $|\det D_{\Gamma_{n-1}}(\omega)| > c_{\Gamma_{n-1}}, \forall \omega \in \mathbb{C}, |\Re\omega| < \epsilon_{\Gamma_{n-1}}$, so eventually we get

$$\exists c_{\Gamma_n}, \epsilon_{\Gamma_n} > 0, |\det D_{\Gamma_n}(\omega)| > c_{\Gamma_n}, \forall \omega \in \mathbb{C}, |\Re\omega| < \epsilon_{\Gamma_n},$$

and the first part of property $\mathcal{P}$ is proved for $P(n)$.

In a similar way we get

$$\det \tilde{D}_{\Gamma_n}(\omega) = (m-1)e^{\omega a} \det D_{\Gamma_{n-1}}(\omega) - (m-3)e^{-\omega a} \det \tilde{D}_{\Gamma_{n-1}}(\omega),$$

so

$$\frac{\det \tilde{D}_{\Gamma_n}(\omega)}{\det D_{\Gamma_n}(\omega)} = \frac{\frac{m-1}{m+1} - \frac{m-3}{m+1} e^{-2\omega a} \frac{\det \tilde{D}_{\Gamma_{n-1}}(\omega)}{\det D_{\Gamma_{n-1}}(\omega)}}{1 - \frac{m-1}{m+1} e^{-2\omega a} \frac{\det \tilde{D}_{\Gamma_{n-1}}(\omega)}{\det D_{\Gamma_{n-1}}(\omega)}}.$$

Thus we also get the second part of $\mathcal{P}$ for $P(n)$ since

$$\left| \frac{\frac{m-1}{m+1} - \frac{m-3}{m+1} z}{1 - \frac{m-1}{m+1} z} \right| < 1 \iff 0 < (m-2)(|z|^2 - 1) + 2(m-1)(1 - \Re z).$$

1.5 Open Problems

In this paper we have analyzed the dispersive properties for the linear Schrödinger equation on trees. We have assumed that the coupling is given by the classical Kirchhoff's conditions. However there are other coupling conditions (see [77]) which allow to define a "Laplace" operator on a metric graph. To be more precise, let us consider the operator H that acts on functions on the graph Γ as the second derivative $\frac{d^2}{dx^2}$, and its domain consists in all functions f that belong to the Sobolev space $H^2(e)$ on each edge e of Γ and satisfy the following boundary condition at the vertices:

$$A(v)\mathbf{f}(v) + B(v)\mathbf{f}'(v) = 0 \quad \text{for each vertex } v. \tag{1.27}$$

Here $\mathbf{f}(v)$ and $\mathbf{f}'(v)$ are correspondingly the vector of values of f at v attained from directions of different edges converging at v and the vector of derivatives at v in the outgoing directions. For each vertex v of the tree we assume that matrices $A(v)$ and $B(v)$ are of size $d(v)$ and satisfy the following two conditions

1. the joint matrix $(A(v), B(v))$ has maximal rank, i.e. $d(v)$,

2. $A(v)B(v)^T = B(v)A(v)^T$.

Under those assumptions it has been proved in [77] that the considered operator, denoted by $\Delta(A, B)$, is self-adjoint. The case considered in this paper, the Kirchhoff coupling, corresponds to the matrices

$$A(v) = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 1 & -1 \\ 0 & 0 & 0 & \vdots & 0 & 0 \end{pmatrix}, \ B(v) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}.$$

More examples of matrices satisfying the above conditions are given in [77, 78].

The existence of the dispersive properties for the solutions of the Schrödinger on a graph under general coupling conditions on the vertices $iu_t + \Delta_\Gamma(A, B)u = 0$ is mainly an open problem. The resolvent formula obtained in [78] and [80] in terms of the coupling matrices A and B might help to understand the general problem. In the same papers there are also some combinatorial formulations of the resolvent in terms of walks on graphs. Such combinational aspects could clarify if the dispersion is possible only on trees or there are graphs (with some of the edges infinite) with suitable couplings where the dispersion is still true.

It is expected that other results on the Schrödinger equation on $\mathbb{R}$ are still valid on networks. For instance, the smoothing estimate for the linear equation with constant coefficients is still valid. Although its classical proof on $\mathbb{R}$ relies on Fourier analysis, one may easily adapt the proof in [20] which uses only integrations by parts and Besovs spaces that can still be defined on a tree using the heat operator. Strichartz estimates has been used previously to treat controllability issues for the NSE in [92]. The possible applications of the present results in the control context remains to be analyzed. We mention here some previous works on the controllability/stabilization of the wave equation on networks [37], [107].

Another interesting problem consists in the analysis of the same properties on some graphs which combine the periodic structure with the infinite strips. This is the case in Fig. 1.6. We recall that for LSE on the one-dimensional torus Bourgain [16] has analyzed the existence of Strichartz estimates. In the same framework we also mention the work of Burq et. al. [19].

Finally, another problem of interest is the study of the dispersion properties for the magnetic operators analyzed in [81], [79]. The analysis in this case is more difficult since in the presence of

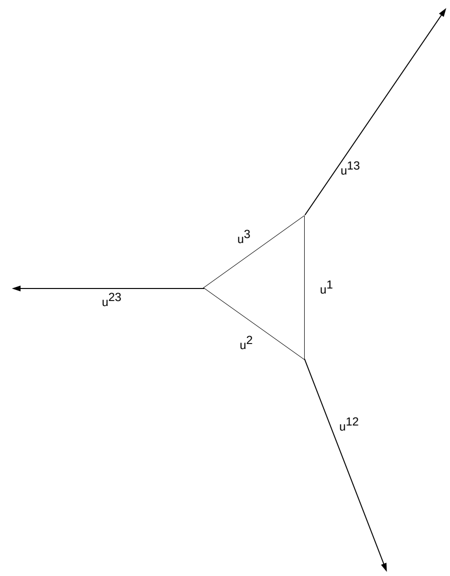

Figure 1.6: A tree where two kinds of structures occur: a periodic one given by the triangle and the infinite strips

an external magnetic field the effect of the topology of the graph becomes more pronounced. In contrast with the analysis done here, in the case of magnetic operators the graphs are viewed as structures in the three dimensional Euclidean space $\mathbb{R}^3$ and the orientation of the edges becomes important.

Chapter 2

Inverse problems on trees

In this chapter we establish global Carleman estimates for the heat and Schrödinger equations on a finite network. The heat equation is considered on a general tree and the Schrödinger equation on a star-shaped tree. The Carleman inequalities are used to prove the Lipschitz stability for an inverse problem consisting in retrieving a stationary potential in the heat (resp. Schrödinger) equation from boundary measurements.

To be more precise we consider the heat equation on a 1-D network Γ given by the edges of a general tree and the Schrödinger equation on a star-shaped tree.

The first system we consider is the following one

$$\begin{cases} \mathbf{u}_t - \Delta_\Gamma \mathbf{u} + \mathbf{p}\mathbf{u} = 0, & \text{in } \Gamma \times (0,T), \\ \mathbf{u} = \mathbf{h}, & \text{on } \partial\Gamma \times (0,T), \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0, & \text{in } \Gamma, \end{cases} \tag{2.1}$$

where Δ_Γ is the Laplace operator on the network Γ. The system is closed with the coupling conditions at the internal nodes of the tree, namely the continuity and the Kirchhoff's law on the flux at all internal vertices of Γ. Here, $\mathbf{u}$ is a collection of functions $u^{\overline{\alpha}}$ each of them satisfying a heat equation on some edge of the network.

Simultaneously with problem (2.1) we consider the following problem

$$\begin{cases} i\mathbf{u}_t + \Delta_\Gamma \mathbf{u} + \mathbf{p}\mathbf{u} = 0, & \text{in } \Gamma \times (0,T), \\ \mathbf{u} = \mathbf{h}, & \text{on } \partial\Gamma \times (0,T), \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0, & \text{in } \Gamma, \end{cases} \tag{2.2}$$

under similar coupling conditions as in the previous model.

In both cases we are interested in determining the potential $\mathbf{p}$, a collection of functions defined on the edges of Γ, from boundary measurements. In the case of the first system, we are able to prove that we can recover $\mathbf{p}$ using only $N-1$ measurements, where N is the total number of exterior nodes of the network Γ. However, in the case of the second system, besides of the fact that we need to deal with a star-shaped network, we only can recover the potential $\mathbf{p}$ from measurements performed at *all* the exterior nodes of Γ.

The use of Carleman estimates to achieve uniqueness and stability results in inverse problems is well known. Some authors use local Carleman inequalities and deduce uniqueness and Hölder estimates. Others make use of global Carleman inequalities and deduce Lipschitz stability results and hence uniqueness results. We shall follow that second approach.

Inverse problems with a finite number of measurements have been widely studied by Bukhgeim and Klibanov (see [18], [74], and [75]) by means of Carleman estimates (see also the book [69] and the references therein). For a wide class of partial differential equations, their method provides the stability in the inverse problem, whenever a suitable Carleman estimate is available. Since [18], there have been many works based upon their methodology.

The theory of global Carleman estimates for parabolic operator has been largely developed since the work by Fursikov-Imanuvilov [45] and it has been applied to many situations (e.g. to prove the controllability along the trajectories or the stability in inverse problems). Since a complete list of references is too long we refer the reader to [113] for a quite complete review of the state of art.

Concerning the Schrödinger equation we refer to [11, 13, 21, 22, 88] where Carleman estimates are proved and used to establish the stability for some inverse problems (see also [68, 85] for some other Carleman estimates for Schrödinger equation).

The same approach has given many results for the wave equation. Since a complete list is too long we quote only some of them, related to the same inverse problem consisting in retrieving a stationary potential in wave equation: [91] and [112] for Dirichlet boundary data and a Neumann measurement and [67] for Neumann boundary data and a Dirichlet measurement. These references are based on the use of local or global Carleman estimates. In the framework of Carleman inequalities on networks we mention the recent paper [10] where the authors establish a global Carleman estimate for the wave equation on a star-shaped tree and used it to derive the Lipschitz stability in an inverse problem. The Carleman estimate in [10] involves some positive definite matrix introduced in [14] to derive a Carleman estimate for the one-dimensional heat equation with discontinuous coefficients.

As far as we know, the determination of a time-independent potential for the heat or Schrödinger equation in a network-like structure has not been addressed in the literature yet. This type of problems has been studied for example for membranes or elastic strings (see for instance [5] and the references therein).

Let us now state the main results of this chaper. For a given initial data $\mathbf{u}_0$ and a given boundary data $\mathbf{h}$, we denote by $\mathbf{u}(\mathbf{p})$ the solution of the above systems associated with the potential $\mathbf{p} \in L^\infty(\Gamma, \mathbb{R})$. We introduce the space

$$H^{2,1}(\Gamma \times (0,T)) := L^2(0,T; H^2(\Gamma)) \cap H^1(0,T; L^2(\Gamma)).$$

We also introduce the ball $B_m(0) := \{\mathbf{q} \in L^\infty(\Gamma, \mathbb{R});\ ||\mathbf{q}||_{L^\infty(\Gamma)} \leq m\}$. Then the following stability results hold.

Theorem 2.1. *Assume that $\mathbf{p} \in L^\infty(\Gamma)$, $\mathbf{u}_0 = \mathbf{u}_0(x)$, $\mathbf{h} = h(x,t)$ and $r > 0$ are such that the solution $\mathbf{u}(\mathbf{p})$ of (2.1) fulfills $\mathbf{u}(\mathbf{p}) \in H^{2,1}(\Gamma \times (0,T))$, $\partial_t \mathbf{u}(\mathbf{p}) \in H^{2,1}(\Gamma \times (0,T))$, and such that for some $t_0 \in (0,T)$ it holds*

$$|\mathbf{u}(\mathbf{p})(\cdot, t_0)| \geq r \ \ a.e. \ \ on\ \Gamma.$$

Then, for any $m > 0$ there exists a constant $C = C(m, ||\partial_t \mathbf{u}(\mathbf{p})||_{L^\infty(\Gamma\times(0,T))}, r)$ such that for any $\mathbf{q} \in B_m(0)$ satisfying

$$\partial_x[\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})](v, .) \in H^1(0,T) \ \ \textit{for all exterior nodes } v,$$

we have

$$\|\mathbf{p}-\mathbf{q}\|_{L^2(\Gamma)} \leq C\Big(\|[\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})](\cdot, t_0)\|_{H^2(\Gamma)} + \sum_{v\in\mathcal{E}} \|\partial_x[\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})](v, \cdot)\|_{H^1(0,T)}\Big),$$

where $\mathcal{E}$ denotes the set of all the exterior vertices of Γ except one.

For the second system, under the assumption that the network is a star-shaped tree, we can prove a similar stability result.

Theorem 2.2. *Assume that $\mathbf{p} \in L^\infty(\Gamma; \mathbb{R})$, $\mathbf{u}_0 = \mathbf{u}_0(x)$, $\mathbf{h} = h(x,t)$ and $r > 0$ are such that the solution of (2.2) satisfies*

- $\mathbf{u}_0(x) \in \mathbb{R}$ *or* $i\mathbf{u}_0(x) \in \mathbb{R}$ *a.e. in* Γ,
- $|\mathbf{u}_0(x)| \geq r > 0$ *a.e. in* Γ, *and*
- $\partial_t \mathbf{u}(\mathbf{p}) \in H^{2,1}(\Gamma \times (0,T))$.

Then, for any $m \geq 0$, *there exists a constant* $C = C(m, ||\partial_t \mathbf{u}(\mathbf{p})||_{H^{2,1}(\Gamma\times(0,T))}, r) > 0$ *such that for any* $\mathbf{q} \in B_m(0)$ *satisfying*

$$\partial_t \mathbf{u}(\mathbf{q}) \in H^{2,1}(\Gamma \times (0,T)),$$

we have

$$||\mathbf{p} - \mathbf{q}||_{L^2(\Gamma)} \leq C \sum_{v\in\partial\Gamma} ||\partial_x[\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})](v,.)||_{H^1(0,T)}.$$

The above theorems extend to networks classical results on inverse problems. To prove those results, we need to establish (new) global Carleman estimates for the heat (resp. the Schrödinger) equation on trees. Note that if we impose Kirchhoff-type conditions to the weight function at the internal vertices, the Carleman estimate cannot be derived. In our Carleman estimates, the weight function has to fulfill some *nonlinear* flux condition at each internal vertex. On the other hand, for the Schrödinger equation posed on a star-shaped tree with N external vertices, we consider a combination of N weight functions in order to cancel some "bad" terms at the internal vertices involving time derivatives. That strategy was used in [12], with two different weight functions, in order to improve the observation region for the wave equation.

The chapter is organized as follows. In Section 2.1 we introduce some notations. Section 2.2 presents the analysis in the case of the heat equation. The Schrödinger equation is considered in Section 2.3. Finally we discuss some open problems in Section 5.

2.1 Notations and Preliminaries

Let $\Gamma = (V, E)$ be a graph where V is the set of vertices and E the set of edges. The edges are assumed to be of finite length and their ends are the vertices of V. For each $v \in V$ we denote $E_v = \{e \in E : v \in e\}$. The *multiplicity* of a vertex of Γ is equal to the number of edges that branch out from it. If the multiplicity is equal to one, the vertex is said to be *exterior*, otherwise it is said to be *interior*. We assume that Γ does not contain vertices with multiplicity two, since they are irrelevant for our models.

From now on, we assume that Γ is a finite tree, that is, Γ is a planar finite connected graph without circuit (closed path). We fix an orientation of Γ and for each oriented edge e, we denote by $I(e)$ its initial vertex and by $T(e)$ its terminal one.

We will use the notations from Chapter 1. The only difference is that here we have all the edges finite. We introduce the space $H_0^1(\Gamma)$ which denotes the set of functions in $H^1(\Gamma)$ that vanish at the exterior vertices. We now introduce the Laplace operator Δ_Γ on the tree Γ. Even if it is a standard procedure, we prefer to recall it following [26], for the sake of completeness. Consider the sesquilinear continuous form φ on $H_0^1(\Gamma)$ defined by

$$\varphi(\mathbf{u}, \mathbf{v}) = (\mathbf{u}_x, \mathbf{v}_x)_{L^2(\Gamma)} = \sum_{e\in E} \int_{I_e} u_x^e(x) \overline{v_x^e}(x) dx.$$

We denote by $D(\Delta_\Gamma)$ the set of all the functions $\mathbf{u} \in H_0^1(\Gamma)$ such that the linear map $\mathbf{v} \in H_0^1(\Gamma) \to \varphi_{\mathbf{u}}(\mathbf{v}) := \varphi(\mathbf{u}, \mathbf{v})$ satisfies

$$|\varphi_{\mathbf{u}}(\mathbf{v})| \leq C\|\mathbf{v}\|_{L^2(\Gamma)} \quad \text{for all } \mathbf{v} \in H_0^1(\Gamma).$$

For $\mathbf{u} \in D(\Delta_\Gamma)$, we can extend $\varphi_{\mathbf{u}}$ to a linear continuous mapping on $L^2(\Gamma)$. There is a unique element in $L^2(\Gamma)$, denoted by $\Delta_\Gamma \mathbf{u}$, such that

$$\varphi(\mathbf{u}, \mathbf{v}) = -(\Delta_\Gamma \mathbf{u}, \mathbf{v})_{L^2(\Gamma)} \quad \text{for all } \mathbf{v} \in H_0^1(\Gamma).$$

We now define the normal exterior derivative of a function $\mathbf{u} = \{u^e\}_{e\in E}$ at the endpoints of the edges. For each $e \in E$ and v an endpoint of e we consider the normal derivative of the restriction of $\mathbf{u}$ to the edge e of E_v evaluated at $i(v,e)$ to be defined by:

$$\frac{\partial u^e}{\partial n_e}(i(v,e)) = \begin{cases} -u^e_x(0+) & \text{if} \quad i(v,e) = 0, \\ u^e_x(l_e-) & \text{if} \quad i(v,e) = l_e. \end{cases}$$

With this notation it is easy to characterize $D(\Delta_\Gamma)$ (see [26]):

$$D(\Delta_\Gamma) = \left\{ \mathbf{u} = \{u^e\}_{e\in E} \in H^2(\Gamma) \cap H^1_0(\Gamma);\ \sum_{e\in E_v} \frac{\partial u^e}{\partial n_e}(i(v,e)) = 0 \quad \text{for any interior vertex } v \right\}$$

and

$$(\Delta_\Gamma \mathbf{u})^e = (u^e)_{xx} \quad \text{for all } e \in E,\ \mathbf{u} \in D(\Delta_\Gamma).$$

In other words $D(\Delta_\Gamma)$ is the space of all the continuous functions $\mathbf{u} = \{u^e\}_{e\in E}$ on Γ, such that for each edge $e \in E$, $u^e \in H^2(I_e)$, and which vanish at each exterior node and fulfill the following Kirchhoff-type condition

$$\sum_{e\in E;\ T(e)=v} u^e_x(l_e-) - \sum_{e\in E;\ I(e)=v} u^e_x(0+) = 0$$

at each interior node v. It is easy to verify that $(\Delta_\Gamma, D(\Delta_\Gamma))$ is a linear, unbounded, self-adjoint, dissipative operator on $L^2(\Gamma)$, i.e. $\Re(\Delta_\Gamma \mathbf{u}, \mathbf{u})_{L^2(\Gamma)} \le 0$ for all $\mathbf{u} \in D(\Delta_\Gamma)$.

We introduce the notations for the elements of the considered tree. We mainly follow the notations of [37]. We first describe the procedure to index the edges and vertices of the tree. We first choose an exterior vertex, called the *root* of the tree and denoted by $\mathcal{R}$. The remaining edges and vertices will be denoted by $e_{\overline{\alpha}}$ and $\mathcal{O}_{\overline{\alpha}}$, respectively, where $\overline{\alpha} = (\alpha_1, \dots, \alpha_k)$ is a multi-index (taking value in $\{1\} \cup \bigcup_{k\ge 2} \mathbb{N}^k$). The multi-indices are defined by induction in the following way. For the edge containing the root $\mathcal{R}$ we choose the index 1. That edge is denoted by $e_{\overline{1}}$ and its second end is denoted by $\mathcal{O}_{\overline{1}}$. Assume now that the interior vertex $\mathcal{O}_{\overline{\alpha}}$, which is the end of the edge $e_{\overline{\alpha}}$, has multiplicity equal to $m_{\overline{\alpha}} + 1$. The $m_{\overline{\alpha}}$ edges, different from $e_{\overline{\alpha}}$, that branch out from $\mathcal{O}_{\overline{\alpha}}$ are denoted by $e_{\overline{\alpha\beta}}$ with $\beta \in \{1, \dots, m_{\overline{\alpha}}\}$. (See Figure 2.1.)

Let now $\mathcal{I}$ be the set of the interior vertices of Γ and $\mathcal{E}$ be the set of the exterior vertices of Γ, $\mathcal{R}$ being excepted. We denote by

$$I_{\mathcal{I}} = \{\overline{\alpha}, \mathcal{O}_{\overline{\alpha}} \in \mathcal{I}\}, \quad I_{\mathcal{E}} = \{\overline{\alpha}, \mathcal{O}_{\overline{\alpha}} \in \mathcal{E}\}.$$

the sets of the indices for the interior and exterior vertices (except the root $\mathcal{R}$). With these notations $I = I_{\mathcal{I}} \cup I_{\mathcal{E}}$ is the set of the indices of all the vertices except the root $\mathcal{R}$.

The length of the edge $e_{\overline{\alpha}}$ will be denoted by $l_{\overline{\alpha}}$. Each $e_{\overline{\alpha}}$ is parameterized by the interval $[0, l_{\overline{\alpha}}]$, so that the end $\mathcal{O}_{\overline{\alpha}}$ of $e_{\overline{\alpha}}$ corresponds to $x = l_{\overline{\alpha}}$ while the origin of $e_{\overline{\alpha}}$ corresponds to $x = 0$.

2.2 The heat equation

In this section we derive a Carleman estimate for the heat equation on a tree and sketch the proof of Theorem 2.1. The following properties for a function $\mathbf{u} = \{u^{\overline{\alpha}}\}_{\overline{\alpha}\in I} : \Gamma \to \mathbb{R}$ will be relevant for our work.

(C1) Continuity condition at the internal vertices: $u^{\overline{\alpha}}(l_{\overline{\alpha}}) = u^{\overline{\alpha\beta}}(0)$ for all $\overline{\alpha} \in I_{\mathcal{I}}$ and $\beta \in [[1, m_{\overline{\alpha}}]]$.

(C2) Flux condition at the internal vertices: $u^{\overline{\alpha}}_x(l_{\overline{\alpha}}) = \sum_{\beta=1}^{m_{\overline{\alpha}}} u^{\overline{\alpha\beta}}_x(0)$ for all $\overline{\alpha} \in I_{\mathcal{I}}$.

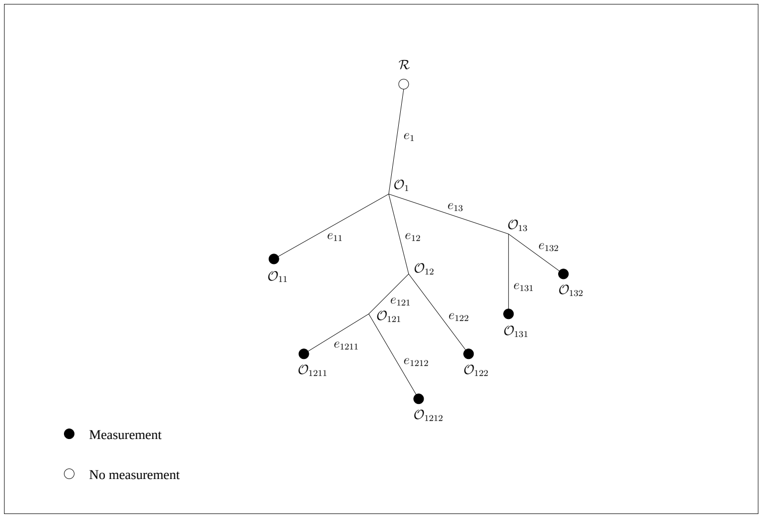

Figure 2.1: A tree with 10 edges.

(C3) Vanishing condition at the root $\mathcal{R}$ and at the external vertices: $\mathbf{u}(v) = 0$ for all $v \in \{\mathcal{R}\} \cup \mathcal{E}$.

We introduce the set

$$\mathcal{Z} = \{\mathbf{u} = \{u^{\overline{\alpha}}\}_{\overline{\alpha}\in I} : \Gamma \times [0,T] \to \mathbb{R};\ u^{\overline{\alpha}} \in C^{2,1}([0, l_{\overline{\alpha}}] \times [0,T]),\ \mathbf{u}(\cdot,t) \text{ satisfies (C1)-(C3)}\}.$$

Note that $\mathbf{u}(\cdot,t) \in D(\Delta_\Gamma)$ for $\mathbf{u} \in \mathcal{Z}$ and $t \in [0,T]$. The aim of this section is to define a continuous weight function $\boldsymbol{\psi} = \{\psi^{\overline{\alpha}}\}_{\overline{\alpha}\in I} : \Gamma \to (0,\infty)$ and a constant $C_{\boldsymbol{\psi}} > 0$ such that if we set

$$\boldsymbol{\theta}(x,t) = \frac{e^{\lambda\boldsymbol{\psi}(x)}}{t(T-t)}, \quad \boldsymbol{\varphi}(x,t) = \frac{e^{\lambda C_{\boldsymbol{\psi}}} - e^{\lambda\boldsymbol{\psi}(x)}}{t(T-t)},\ x \in \Gamma,\ t \in (0,T),$$

we have the following Carleman estimate.

Proposition 2.1. *There exist a continuous function $\boldsymbol{\psi} : \Gamma \to (0,+\infty)$ and some positive constants λ_0, s_0, C such that for all $\lambda \geq \lambda_0$, $s \geq s_0$ and $\mathbf{q} \in \mathcal{Z}$, it holds*

$$\begin{aligned}
&\int_0^T \int_\Gamma \left((s\boldsymbol{\theta})^{-1}(|\mathbf{q}_t|^2 + |\Delta_\Gamma \mathbf{q}|^2) + \lambda^2(s\boldsymbol{\theta})|\mathbf{q}_x|^2 + \lambda^4(s\boldsymbol{\theta})^3|\mathbf{q}|^2\right) e^{-2s\boldsymbol{\varphi}} dxdt \\
&\quad + \int_0^T \lambda(s\boldsymbol{\theta})(|\mathbf{q}_x|^2 e^{-2s\boldsymbol{\varphi}})(\mathcal{R},t)dt \\
&\quad \leq C\Big(\int_0^T \int_\Gamma |\mathbf{q}_t + \Delta_\Gamma \mathbf{q}|^2 e^{-2s\boldsymbol{\varphi}} dxdt + \sum_{v\in\mathcal{E}} \int_0^T \lambda(s\boldsymbol{\theta})(|\mathbf{q}_x|^2 e^{-2s\boldsymbol{\varphi}})(v,t)dt\Big). \qquad (2.3)
\end{aligned}$$

In the above proposition we have used the following notations $|\mathbf{q}|^2 = \{|q^{\overline{\alpha}}|^2\}_{\overline{\alpha}\in I}$, $|\mathbf{q}_t|^2 = \{|q_t^{\overline{\alpha}}|^2\}_{\overline{\alpha}\in I}$, $|\mathbf{q}_x|^2 = \{|q_x^{\overline{\alpha}}|^2\}_{\overline{\alpha}\in I}$, etc. and

$$\int_\Gamma \mathbf{u}\, dx = \sum_{\overline{\alpha}\in I} \int_{I_{\overline{\alpha}}} u^{\overline{\alpha}} dx.$$

Note that the same inequality holds for the operator $\partial_t - \Delta_\Gamma$ instead of $\partial_t + \Delta_\Gamma$ just by changing t into $T - t$. Note also that in the definition of $\mathcal{Z}$ we can replace $C^{2,1}$ by $H^{2,1}$, as well.

Proof. Let us consider the operator $P = \partial_t + \Delta_\Gamma$. Set $\mathbf{u} = e^{-s\boldsymbol{\varphi}}\mathbf{q}$ and $\mathbf{w} = e^{-s\boldsymbol{\varphi}}P(e^{s\boldsymbol{\varphi}}\mathbf{u})$. Following [93] we obtain

$$\mathbf{w} = M\mathbf{u} = \mathbf{u}_t + s\varphi_t\mathbf{u} + (\Delta_\Gamma\mathbf{u} + 2s\varphi_x\mathbf{u}_x + s(\Delta_\Gamma\boldsymbol{\varphi})\mathbf{u} + s^2|\varphi_x|^2\mathbf{u}) = M_1\mathbf{u} + M_2\mathbf{u},$$

where

$$M_1\mathbf{u} = \Delta_\Gamma\mathbf{u} + s\varphi_t\mathbf{u} + s^2|\varphi_x|^2\mathbf{u} \tag{2.4}$$

and

$$M_2\mathbf{u} = \mathbf{u}_t + 2s\varphi_x\mathbf{u}_x + s(\Delta_\Gamma\boldsymbol{\varphi})\mathbf{u} \tag{2.5}$$

are the self-adjoint and skew-adjoint parts of M, respectively. Then

$$\|\mathbf{w}\|^2 = \|M_1\mathbf{u} + M_2\mathbf{u}\|^2 = \|M_1\mathbf{u}\|^2 + \|M_2\mathbf{u}\|^2 + 2(M_1\mathbf{u}, M_2\mathbf{u}),$$

where $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and the inner product of $L^2(\Gamma\times(0,T))$, respectively.

Step 1. *Exact computation of* $(M_1\mathbf{u}, M_2\mathbf{u})$.
Recall that

$$(M_1\mathbf{u}, M_2\mathbf{u}) = \sum_{\overline{\alpha}\in I}\int_0^T\int_0^{l_{\overline{\alpha}}}(M_1\mathbf{u})^{\overline{\alpha}}(M_2\mathbf{u})^{\overline{\alpha}}dxdt.$$

We compute the integral term in the r.h.s. of the above identity only for one (arbitrary) edge $e_{\overline{\alpha}}$, that we denote by e for simplicity. We assume that e is parameterized by $x\in[0,l]$.

For the edge e we have,

$$\int_0^T\int_0^l M_1uM_2u\,dxdt = -2s\int_0^T\int_0^l\varphi_{xx}|u_x|^2 + \int_0^T\int_0^l|u|^2\Big[\frac{s}{2}(\varphi_{4x}-\varphi_{tt}) - s^2(|\varphi_x|^2)_t - s^3\varphi_x(|\varphi_x|^2)_x\Big]$$
$$+\int_0^T\Big[u_xu_t + s\varphi_{xx}uu_x + s|u_x|^2\varphi_x + |u|^2\big(-\frac{s}{2}\varphi_{3x} + s^2\varphi_x\varphi_t + s^3(\varphi_x)^3\big)\Big]\Big|_0^l. \tag{2.6}$$

Summing now the above identity over all the edges $\{e_{\overline{\alpha}}\}_{\overline{\alpha}\in I}$ we obtain the exact expression of the scalar product $(M_1\mathbf{u}, M_2\mathbf{u})$:

$$(M_1\mathbf{u}, M_2\mathbf{u}) = -2s\int_0^T\int_\Gamma(\Delta_\Gamma\boldsymbol{\varphi})|\mathbf{u}_x|^2 + \int_0^T\int_\Gamma|\mathbf{u}|^2\Big[\frac{s}{2}(\boldsymbol{\varphi}_{4x}-\boldsymbol{\varphi}_{tt}) - s^2(|\boldsymbol{\varphi}_x|^2)_t - s^3\boldsymbol{\varphi}_x(|\boldsymbol{\varphi}_x|^2)_x\Big]$$
$$+\sum_{\overline{\alpha}\in I}\int_0^T\Big[u_x^{\overline{\alpha}}u_t^{\overline{\alpha}} + s\varphi_{xx}^{\overline{\alpha}}u^{\overline{\alpha}}u_x^{\overline{\alpha}} + s|u_x^{\overline{\alpha}}|^2\varphi_x^{\overline{\alpha}} + |u^{\overline{\alpha}}|^2\big(-\frac{s}{2}\varphi_{3x}^{\overline{\alpha}} + s^2\varphi_x^{\overline{\alpha}}\varphi_t^{\overline{\alpha}} + s^3(\varphi_x^{\overline{\alpha}})^3\big)\Big]\Big|_0^{l_{\overline{\alpha}}}. \tag{2.7}$$

Step 2. *Terms in the inner product related to the internal nodes.*
Let us pick an internal node $\mathcal{O}_{\overline{\alpha}}$. Using our previous notations, its parent edge is $e_{\overline{\alpha}}$ and its children edges are denoted by $e_{\overline{\alpha\beta}}$ with $\beta\in[[1,m_{\overline{\alpha}}]]$. Let us denote by $X^{\overline{\alpha}}$ the sum of the boundary terms involving this internal node $\mathcal{O}_{\overline{\alpha}}$ in the right hand side of (2.7). Thus

$$X^{\overline{\alpha}} = \int_0^T\Big[u_x^{\overline{\alpha}}u_t^{\overline{\alpha}} + s\varphi_{xx}^{\overline{\alpha}}u^{\overline{\alpha}}u_x^{\overline{\alpha}} + s|u_x^{\overline{\alpha}}|^2\varphi_x^{\overline{\alpha}} + |u^{\overline{\alpha}}|^2\big(-\frac{s}{2}\varphi_{3x}^{\overline{\alpha}} + s^2\varphi_x^{\overline{\alpha}}\varphi_t^{\overline{\alpha}} + s^3(\varphi_x^{\overline{\alpha}})^3\big)\Big](l_{\overline{\alpha}},t)dt$$
$$-\int_0^T\sum_{\beta\in[[1,m_{\overline{\alpha}}]]}\Big[u_x^{\overline{\alpha\beta}}u_t^{\overline{\alpha\beta}} + s\varphi_{xx}^{\overline{\alpha\beta}}u^{\overline{\alpha\beta}}u_x^{\overline{\alpha\beta}} + s|u_x^{\overline{\alpha\beta}}|^2\varphi_x^{\overline{\alpha\beta}}$$
$$+|u^{\overline{\alpha\beta}}|^2\big(-\frac{s}{2}\varphi_{3x}^{\overline{\alpha\beta}} + s^2\varphi_x^{\overline{\alpha\beta}}\varphi_t^{\overline{\alpha\beta}} + s^3(\varphi_x^{\overline{\alpha\beta}})^3\big)\Big](0,t)dt.$$

Moreover, in (2.7) we also have contributions from the exterior nodes in $\mathcal{E}$ and from the root $\mathcal{R}$. These contributions are given by

$$Y = -s\int_0^T|u_x^{\overline{1}}|^2\varphi_x^{\overline{1}}(0,t)dt + s\sum_{\overline{\alpha}\in I_{\mathcal{E}}}\int_0^T|u_x^{\overline{\alpha}}|^2\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)dt. \tag{2.8}$$

Let us now define the weight function $\boldsymbol{\psi} = \{\psi^{\overline{\alpha}}\}_{\overline{\alpha}\in I}$ on the tree as follows. The components $\psi^{\overline{\alpha}} : [0, l_{\overline{\alpha}}] \to \mathbb{R}$ are chosen in such a way that $\psi^{\overline{\alpha}} \in C^\infty([0, l_{\overline{\alpha}}])$ and

(B1) $|\psi_x^{\overline{\alpha}}(x)|^2 + \psi_{xx}^{\overline{\alpha}}(x) \geq 0$ on $[0, l_{\overline{\alpha}}]$,

(B2) $\psi_x^{\overline{\alpha}} > 0$ on $[0, l_{\overline{\alpha}}]$,

(B3) $\frac{3}{4}C_{\boldsymbol{\psi}} \geq \psi^{\overline{\alpha}} > \frac{2}{3}C_{\boldsymbol{\psi}}$ on $[0, l_{\overline{\alpha}}]$, for some positive constant $C_{\boldsymbol{\psi}}$,

(B4) $|\psi_{xx}^{\overline{\alpha}}| \leq K\psi_x^{\overline{\alpha}}$ on $[0, l_{\overline{\alpha}}]$ for some positive constant K,

(B5) $\psi^{\overline{\alpha}}(l_{\overline{\alpha}}) = \psi^{\overline{\alpha\beta}}(0)$ for all $\overline{\alpha} \in I_{\mathcal{I}}$, $\beta \in [[1, m_{\overline{\alpha}}]]$,

(B6) $\psi_x^{\overline{\alpha\beta}}(0) - (m_{\overline{\alpha}} + 1)\psi_x^{\overline{\alpha}}(l_{\overline{\alpha}}) > 0$ for all $\overline{\alpha} \in I_{\mathcal{I}}$, $\beta \in [[1, m_{\overline{\alpha}}]]$,

(B7) $\displaystyle\sum_{\beta\in[[1,m_{\overline{\alpha}}]]} (\psi_x^{\overline{\alpha\beta}}(0))^3 - (\psi_x^{\overline{\alpha}}(l_{\overline{\alpha}}))^3 - (m_{\overline{\alpha}}+1)\psi_x^{\overline{\alpha}}(l_{\overline{\alpha}})\Big|\psi_x^{\overline{\alpha}}(l_{\overline{\alpha}}) - \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} \psi_x^{\overline{\alpha\beta}}(0)\Big|^2 > 0$ for all $\overline{\alpha} \in I_{\mathcal{I}}$.

Finding a set of functions as above is easy. We can even take $\psi^{\overline{\alpha}}$ to be affine, $\psi^{\overline{\alpha}}(x) = a_{\overline{\alpha}}x + b_{\overline{\alpha}}$. The coefficients $a_{\overline{\alpha}}$ and $b_{\overline{\alpha}}$ are positive numbers that satisfy

(P1) $\frac{3}{4}C_\psi \geq a_{\overline{\alpha}}l_{\overline{\alpha}} + b_{\overline{\alpha}} > b_{\overline{\alpha}} > \frac{2}{3}C_\psi$ for all $\overline{\alpha} \in I$,

(P2) $a_{\overline{\alpha}}l_{\overline{\alpha}} + b_{\overline{\alpha}} = b_{\overline{\alpha\beta}}$ for all $\overline{\alpha} \in I_{\mathcal{I}}$ and $\beta \in [[1, m_{\overline{\alpha}}]]$,

(P3) $a_{\overline{\alpha\beta}} - (m_{\overline{\alpha}} + 1)a_{\overline{\alpha}} > 0$ for all $\overline{\alpha} \in I_{\mathcal{I}}$ and $\beta \in [[1, m_{\overline{\alpha}}]]$,

(P4) $\displaystyle\sum_{\beta\in[[1,m_{\overline{\alpha}}]]} (a_{\overline{\alpha\beta}})^3 - (a_{\overline{\alpha}})^3 - (m_{\overline{\alpha}} + 1)a_{\overline{\alpha}}\Big|a_{\overline{\alpha}} - \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} a_{\overline{\alpha\beta}}\Big|^2 > 0$ for all $\overline{\alpha} \in I_{\mathcal{I}}$.

Let us first deal with the conditions (P2)-(P4). We define the constants corresponding to the edge $e_{\overline{1}}$ by $a_{\overline{1}} = 2$ and $b_{\overline{1}} = 1$. Assuming that we have already constructed $a_{\overline{\alpha}}$ and $b_{\overline{\alpha}}$ for some multi-index $\overline{\alpha}$, then $b_{\overline{\alpha\beta}}$ is given by (P2). Next, we have to find $a_{\overline{\alpha\beta}}$ large enough to satisfy (P3)-(P4). Let us choose $a_{\overline{\alpha\beta}} = r_{\overline{\alpha}}a_{\overline{\alpha}}$. Obviously, for large enough $r_{\overline{\alpha}}$, depending on $m_{\overline{\alpha}}$, conditions (P3) and (P4) are satisfied. Finally, assume that all the coefficients $a_{\overline{\alpha}}$ and $b_{\overline{\alpha}}$ have been defined to satisfy (P2)-(P4). Adding $\frac{2}{3}C_{\boldsymbol{\psi}}$ to all the $b_{\overline{\alpha\beta}}$, we see that (P1) is fulfilled for $C_{\boldsymbol{\psi}}$ large enough, while (P2)-(P4) still hold true.

Using the definition of the function $\mathbf{u}$ we have for any index $\overline{\alpha} \in I$ the following identities

$$u_x^{\overline{\alpha}} = e^{-s\varphi^{\overline{\alpha}}}(-s\varphi_x^{\overline{\alpha}}q^{\overline{\alpha}} + q_x^{\overline{\alpha}}), \quad u_t^{\overline{\alpha}} = e^{-s\varphi^{\overline{\alpha}}}(-s\varphi_t^{\overline{\alpha}}q^{\overline{\alpha}} + q_t^{\overline{\alpha}}).$$

Let us set $u(\mathcal{O}_{\overline{\alpha}}, t) = u^{\overline{\alpha}}(l_{\overline{\alpha}}, t) = u^{\overline{\alpha\beta}}(0, t)$ and $\varphi(\mathcal{O}_{\overline{\alpha}}, t) = \varphi^{\overline{\alpha}}(l_{\overline{\alpha}}, t) = \varphi^{\overline{\alpha\beta}}(0, t)$ for any $\overline{\alpha} \in I_{\mathcal{I}}$ and $\beta \in [[1, m_{\overline{\alpha}}]]$. With these notations we have that

$$X^{\overline{\alpha}} \geq Z_1^{\overline{\alpha}} + Z_2^{\overline{\alpha}}$$

where

$$
\begin{aligned}
Z_1^{\overline{\alpha}} = &- \int_0^T \big(- s\varphi_{xt}^{\overline{\alpha}}(l_{\overline{\alpha}},t) + s \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} \varphi_{xt}^{\overline{\alpha\beta}}(0,t)\big) \frac{|u(\mathcal{O}_{\overline{\alpha}},t)|^2}{2} dt \\
&- \frac{s^2}{2} \int_0^T |u(\mathcal{O}_{\overline{\alpha}},t)|^2 \Big[(\frac{T^2}{4}|\varphi_x^{\overline{\alpha}}| + K)^2 |\varphi_x^{\overline{\alpha}}|(l_{\overline{\alpha}},t) + \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} (\frac{T^2}{4}|\varphi_x^{\overline{\alpha\beta}}| + K)^2 |\varphi_x^{\overline{\alpha\beta}}|(0,t) \Big] dt \\
&- (s^3 + \frac{1}{2}s^2)(m_{\overline{\alpha}}+1) \int_0^T |u(\mathcal{O}_{\overline{\alpha}},t)|^2 |\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)| \Big| \varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t) - \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} \varphi_x^{\overline{\alpha\beta}}(0,t) \Big|^2 dt \\
&+ \int_0^T |u(\mathcal{O}_{\overline{\alpha}},t)|^2 \big(- \frac{s}{2}\varphi_{3x}^{\overline{\alpha}} + s^2 \varphi_x^{\overline{\alpha}}\varphi_t^{\overline{\alpha}} + s^3 (\varphi_x^{\overline{\alpha}})^3 \big) \Big] (l_{\overline{\alpha}},t) dt \\
&- \int_0^T |u(\mathcal{O}_{\overline{\alpha}},t)|^2 \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} \big(- \frac{s}{2}\varphi_{3x}^{\overline{\alpha\beta}} + s^2 \varphi_x^{\overline{\alpha\beta}}\varphi_t^{\overline{\alpha\beta}} + s^3 (\varphi_x^{\overline{\alpha\beta}})^3 \big)(0,t) dt
\end{aligned}
$$

and

$$
\begin{aligned}
Z_2^{\overline{\alpha}} = (s - \frac{1}{2}) &\sum_{\beta\in[[1,m_{\overline{\alpha}}]]} \int_0^T \big[|\varphi_x^{\overline{\alpha\beta}}| |u_x^{\overline{\alpha\beta}}|^2 \big](0,t) dt \\
&- (s+\frac{1}{2})(m_{\overline{\alpha}}+1) \int_0^T \Big[|\varphi_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)| \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} |u_x^{\overline{\alpha\beta}}(0,t)|^2 \Big] dt.
\end{aligned}
$$

Looking at the coefficient of s^3 in $Z_1^{\overline{\alpha}}$ and of s in $Z_2^{\overline{\alpha}}$ and using (B6) and (B7) we obtain that for $s \geq s_0$ and $\lambda \geq \lambda_0$ (with s_0, λ_0 large enough)

$$
\begin{aligned}
X^{\overline{\alpha}} &\geq Z_1^{\overline{\alpha}} + Z_2^{\overline{\alpha}} \\
&\geq C \int_0^T s^3\lambda^3\theta^3 |u(\mathcal{O}_{\overline{\alpha}},t)|^2 dt + C \int_0^T s\lambda\theta \Big(|u_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)|^2 + \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} |u_x^{\overline{\alpha\beta}}(0,t)|^2 \Big) dt. \qquad (2.9)
\end{aligned}
$$

In particular, $X^{\overline{\alpha}} > 0$.

Step 3. *Estimation of the integrals along the edges.*
We need the following lemma.

Lemma 2.1. *[93, Claim 1] There exist $\lambda_1 \geq \lambda_0$, $s_1 \geq s_0$ and $A > 0$ such that for all $\lambda \geq \lambda_1$, $s \geq s_1$, it holds*

$$
\int_0^T \int_\Gamma |\mathbf{u}|^2 \Big[\frac{s}{2}(\varphi_{4x} - \varphi_{tt}) - s^2(|\varphi_x|^2)_t - s^3\varphi_x(|\varphi_x|^2)_x \Big] dxdt \geq A\lambda s^3 \int_0^T \int_\Gamma |\mathbf{u}|^2 |\varphi_x|^3 dxdt. \qquad (2.10)
$$

As the proof of [93, Claim 1] does not involve any integration by parts in x, it is still valid in our context.

The following lemma is inspired by [93, Claim 2].

Lemma 2.2. *There exist $s_2 \geq s_1$, $\lambda_2 \geq \lambda_1$, and a positive constant C such that for all $\lambda \geq \lambda_2$ and $s \geq s_2$*

$$
\begin{aligned}
&\lambda s \int_0^T \int_\Gamma |\varphi_x| |\mathbf{u}_x|^2 + \lambda s^{-1} \int_0^T \int_\Gamma |\varphi_x|^{-1} |\Delta_\Gamma \mathbf{u}|^2 \\
&\quad \leq C \Big(s^{-1}\|M_1\mathbf{u}\|^2 + \lambda s^3 \int_0^T \int_\Gamma |\varphi_x|^3 |\mathbf{u}|^2 + \lambda s \int_0^T \sum_{\overline{\alpha}\in I} |\varphi_x^{\overline{\alpha}} u^{\overline{\alpha}} u_x^{\overline{\alpha}}|(0) + |\varphi_x^{\overline{\alpha}} u^{\overline{\alpha}} u_x^{\overline{\alpha}}|(l_{\overline{\alpha}}) \\
&\qquad + \lambda s \int_0^T \sum_{\overline{\alpha}\in I} |(|\varphi_x^{\overline{\alpha}}|)_x| |u^{\overline{\alpha}}|^2(0) + |(|\varphi_x^{\overline{\alpha}}|)_x| |u^{\overline{\alpha}}|^2(l_{\overline{\alpha}}) \Big). \qquad (2.11)
\end{aligned}
$$

Step 4. *Conclusion.*
By (2.7), (2.10) and $(B1)$, we get for $\lambda \geq 1$

$$\begin{aligned}
\|\mathbf{w}\|^2 &= \|M_1\mathbf{u} + M_2\mathbf{u}\|^2 \\
&= \|M_1\mathbf{u}\|^2 + \|M_2\mathbf{u}\|^2 + 2(M_1\mathbf{u}, M_2\mathbf{u}) \\
&= \|M_1\mathbf{u}\|^2 + \|M_2\mathbf{u}\|^2 + 2\Big\{ \sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}} + Y \\
&\quad - 2s\int_0^T \int_\Gamma \boldsymbol{\varphi}_{xx}|\mathbf{u}_x|^2 + \int_0^T\int_\Gamma |\mathbf{u}|^2 \Big[\frac{s}{2}(\boldsymbol{\varphi}_{4x} - \boldsymbol{\varphi}_{tt}) - s^2(|\boldsymbol{\varphi}_x^2|)_t - s^3\boldsymbol{\varphi}_x(|\boldsymbol{\varphi}_x|^2)_x\Big] \Big\} \\
&\geq \|M_1\mathbf{u}\|^2 + \|M_2\mathbf{u}\|^2 + 2\Big\{ \sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}} + Y \\
&\quad + 2s\int_0^T \int_\Gamma (\lambda^2\boldsymbol{\psi}_x^2 + \lambda\boldsymbol{\psi}_{xx})\theta|\mathbf{u}_x|^2 + A\lambda s^3 \int_0^T\int_\Gamma |\mathbf{u}|^2|\boldsymbol{\varphi}_x|^3\Big\} \\
&\geq \|M_1\mathbf{u}\|^2 + \|M_2\mathbf{u}\|^2 + \sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}} + Y + A\lambda s^3\int_0^T\int_\Gamma |\mathbf{u}|^2|\boldsymbol{\varphi}_x|^2. \qquad (2.12)
\end{aligned}$$

Multiplying (2.11) by $A/2C$ and adding it to (2.12) we get

$$\begin{aligned}
&\|M_2\mathbf{u}\|^2 + \|M_1\mathbf{u}\|^2(1-\frac{A}{2s}) + \frac{A\lambda s^3}{2}\iint |\mathbf{u}|^2|\boldsymbol{\varphi}_x|^3 + \frac{A\lambda s}{2C}\iint |\boldsymbol{\varphi}_x||\mathbf{u}_x|^2 + \frac{A\lambda}{2sC}\iint |\boldsymbol{\varphi}_x|^{-1}|\Delta_\Gamma\mathbf{u}|^2 \\
&\quad + \sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}} + Y \leq \|\mathbf{w}\|^2 + \frac{A}{2}B, \qquad (2.13)
\end{aligned}$$

where

$$\begin{aligned}
B &= B_1 + B_2 \\
&= \lambda s\int_0^T \sum_{\overline{\alpha}\in I} |\varphi_x^{\overline{\alpha}} u^{\overline{\alpha}} u_x^{\overline{\alpha}}|(0) + |\varphi_x^{\overline{\alpha}} u^{\overline{\alpha}} u_x^{\overline{\alpha}}|(l_{\overline{\alpha}}) + \lambda s\int_0^T \sum_{\overline{\alpha}\in I} |(|\varphi_x^{\overline{\alpha}}|)_x||u^{\overline{\alpha}}|^2(0) + |(|\varphi_x^{\overline{\alpha}}|)_x||u^{\overline{\alpha}}|^2(l_{\overline{\alpha}})\Big).
\end{aligned}$$

We now prove that for s large enough, the term B is small compared to $\sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}}$, so that B can be absorbed by the left hand side of (2.13). Using (2.9) and the fact that $\mathbf{u}$ vanishes at the vertices of $\mathcal{E}\cup\mathcal{R} = \partial\Gamma$, we see that

$$B_2 \leq C\lambda^3 s\int_0^T \sum_{\overline{\alpha}\in I} \theta|u(\mathcal{O}_{\overline{\alpha}},t)|^2dt = C\lambda^3 s\int_0^T \sum_{\overline{\alpha}\in I_{\mathcal{I}}} \theta|u(\mathcal{O}_{\overline{\alpha}},t)|^2dt \leq \frac{C}{s^2}\sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}}. \qquad (2.14)$$

Using again the fact that $\mathbf{u}$ vanishes at the vertices of $\partial\Gamma$, we obtain with (2.9) that

$$\begin{aligned}
B_1 &\leq C\lambda s\int_0^T \sum_{\overline{\alpha}\in I_{\mathcal{I}}} |\varphi_x^{\overline{\alpha}} u^{\overline{\alpha}}|(\mathcal{O}_{\overline{\alpha}},t)\Big(|u_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)| + \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} |u_x^{\overline{\alpha\beta}}(0,t)|\Big) \\
&\leq C\int_0^T \sum_{\overline{\alpha}\in I_{\mathcal{I}}} \Big((s\lambda)^2|\varphi_x^{\overline{\alpha}}||u^{\overline{\alpha}}|^2(\mathcal{O}_{\overline{\alpha}},t) + |\varphi_x^{\overline{\alpha}}(\mathcal{O}_{\overline{\alpha}},t)|\Big(|u_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)|^2 + \sum_{\beta\in[[1,m_{\overline{\alpha}}]]} |u_x^{\overline{\alpha\beta}}(0,t)|^2\Big)dt \\
&\leq \frac{C}{s}\sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}}. \qquad (2.15)
\end{aligned}$$

Gathering together (2.13), (2.14) and (2.15), we obtain

$$\begin{aligned}
&\|M_2\mathbf{u}\|^2 + \|M_1\mathbf{u}\|^2(1-\frac{A}{2s}) + \frac{A\lambda s^3}{2}\int_0^T\int_\Gamma |\mathbf{u}|^2|\boldsymbol{\varphi}_x|^3 + \frac{A\lambda s}{2C}\iint |\boldsymbol{\varphi}_x||\mathbf{u}_x|^2 + \frac{A\lambda}{2sC}\int_0^T\int_\Gamma |\boldsymbol{\varphi}_x|^{-1}|\Delta_\Gamma\mathbf{u}|^2 \\
&\quad + (1-\frac{C}{s})\sum_{\overline{\alpha}\in I_{\mathcal{I}}} X^{\overline{\alpha}} + Y \leq \|\mathbf{w}\|^2.
\end{aligned}$$

Writing explicitly the term Y and tacking into account the sign of the functions $\psi_x^{\overline{\alpha}}$ occuring in Y, we get for s and λ large enough

$$\|M_1\mathbf{u}\|^2+\|M_2\mathbf{u}\|^2+\lambda s^3\int_0^T\int_\Gamma|\mathbf{u}|^2|\boldsymbol{\varphi}_x|^3+\lambda s\int_0^T\int_\Gamma|\boldsymbol{\varphi}_x||\mathbf{u}_x|^2+\lambda s^{-1}\int_0^T\int_\Gamma|\boldsymbol{\varphi}_x|^{-1}|\Delta_\Gamma\mathbf{u}|^2$$
$$+\sum_{\overline{\alpha}\in I_{\mathcal{I}}}X^{\overline{\alpha}}+\int_0^T\lambda s\theta^{\overline{1}}|u_x|^2(\mathcal{R},t)dt\le C\Big(\|\mathbf{w}\|^2+\int_0^T\sum_{\overline{\alpha}\in I_{\mathcal{E}}}\lambda s\theta^{\overline{\alpha}}|u_x^{\overline{\alpha}}|^2(l_{\overline{\alpha}},t)dt\Big).\quad(2.16)$$

Finally, using the definition of M_2 we get

$$\begin{aligned}\lambda s^{-1}\int_0^T\int_\Gamma|\boldsymbol{\varphi}_x|^{-1}|\mathbf{u}_t|^2dxdt &\le C\lambda s^{-1}\int_0^T\int_\Gamma|\boldsymbol{\varphi}_x|^{-1}\Big(|M_2\mathbf{u}|^2+s^2|\boldsymbol{\varphi}_x|^2|\mathbf{u}_x|^2+s^2|\boldsymbol{\varphi}_{xx}|^2|\mathbf{u}|^2\Big)\\ &\le C\int_0^T\int_\Gamma s^{-1}|M_2\mathbf{u}|^2+\lambda s|\boldsymbol{\varphi}_x||\mathbf{u}_x|^2+\lambda s|\boldsymbol{\varphi}_x|^{-1}|\boldsymbol{\varphi}_{xx}|^2|\mathbf{u}|^2\Big)\quad(2.17)\end{aligned}$$

From (2.16) and (2.17), we infer that for $s\ge s_3$ and $\lambda\ge\lambda_3$ (with s_3,λ_3 large enough) we have that

$$\|M_1\mathbf{u}\|^2+\|M_2\mathbf{u}\|^2+\lambda s^3\int_0^T\int_\Gamma|\mathbf{u}|^2|\boldsymbol{\varphi}_x|^3+\lambda s\int_0^T\int_\Gamma|\boldsymbol{\varphi}_x||\mathbf{u}_x|^2+\lambda s^{-1}\int_0^T\int_\Gamma|\boldsymbol{\varphi}_x|^{-1}(|\Delta_\Gamma\mathbf{u}|^2+|\mathbf{u}_t|^2)$$
$$+\sum_{\overline{\alpha}\in I_{\mathcal{I}}}X^{\overline{\alpha}}+\int_0^T\lambda s\theta^{\overline{1}}|u_x|^2(\mathcal{R},t)dt\le C\Big(\|\mathbf{w}\|^2+\int_0^T\sum_{\overline{\alpha}\in I_{\mathcal{E}}}\lambda s\theta^{\overline{\alpha}}|u_x^{\overline{\alpha}}|^2(l_{\overline{\alpha}},t)dt\Big).$$

Replacing $\mathbf{u}$ by $e^{-s\boldsymbol{\varphi}}\mathbf{q}$ in the last inequality, we readily obtain (2.3). □

Before proving the stability result in Theorem 2.1 we need to analyze the following system:

$$\begin{cases}u_t^{\overline{\alpha}}(x,t)=u_{xx}^{\overline{\alpha}}(x,t)+b^{\overline{\alpha}}(x)u^{\overline{\alpha}}(x,t) & \\ \qquad+R^{\overline{\alpha}}(x,t)f^{\overline{\alpha}}(x), & (x,t)\in(0,l_{\overline{\alpha}})\times(0,T),\ \alpha\in I,\\ u^{\overline{\alpha}}(l_{\overline{\alpha}},t)=0, & t\in(0,T),\ \alpha\in I_{\mathcal{E}},\\ u^{\overline{1}}(0,t)=0, & t\in(0,T),\\ u^{\overline{\alpha}}(l_{\overline{\alpha}},t)=u^{\overline{\alpha\beta}}(0,t), & t\in(0,T),\ \alpha\in I_{\mathcal{I}},\ \beta\in[[1,m_{\overline{\alpha}}]],\\ u_x^{\overline{\alpha}}(l_{\overline{\alpha}},t)=\sum\limits_{\beta=1}^{m_{\overline{\alpha}}}u_x^{\overline{\alpha\beta}}(0,t), & t\in(0,T),\ \overline{\alpha}\in I_{\mathcal{I}},\end{cases}\quad(2.18)$$

where $\mathbf{b}=\{b^{\overline{\alpha}}\}_{\overline{\alpha}\in I}\in L^\infty(\Gamma)$.

We now apply the Carleman estimate (2.3) with $\mathbf{q}=\partial_t\mathbf{u}$ (and some fixed $\lambda>0$).

Proposition 2.2. *Assume that $\mathbf{u}=\{u^{\overline{\alpha}}\}_{\overline{\alpha}\in I}$ is a solution of (2.18) which satisfies $\mathbf{u}_t\in H^{2,1}(\Gamma\times(0,T))$. If $\mathbf{R}=\{R^{\overline{\alpha}}(x,t)\}_{\overline{\alpha}\in I}$ is such that $\mathbf{R}_t\in L^\infty(\Gamma\times(0,T))$ and that*

$$|\mathbf{R}(x,t_0)|\ge r>0,\qquad \textit{for a.e. } x\in\Gamma \textit{ and some } t_0\in(0,T),\quad(2.19)$$

then there exists a positive constant $C=C(||\mathbf{R}_t||_{L^\infty(\Gamma\times(0,T))},||\mathbf{b}||_{L^\infty(\Gamma)},r)$ such that

$$\|\mathbf{f}\|_{L^2(\Gamma)}\le C\left(\|\mathbf{u}(\cdot,t_0)\|_{H^2(\Gamma)}+\sum_{v\in\mathcal{E}}\|\partial_{xt}\mathbf{u}(v,\cdot)\|_{L^2(0,T)}\right)\quad(2.20)$$

for any $\mathbf{f}\in L^2(\Gamma)$.

We are now able to prove the stability result for system (2.1). Let us denote

$$\mathbf{w} = \mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q}).$$

It satisfies the following system

$$\begin{cases} \mathbf{w}_t = \Delta_\Gamma \mathbf{w} - \mathbf{q}\mathbf{w} + \mathbf{R}\mathbf{f} & \text{in } \Gamma \times (0,T), \\ \mathbf{w}(x,t) = 0, & \text{on } \partial\Gamma \times (0,T), \end{cases}$$

where $\mathbf{f} = \mathbf{q} - \mathbf{p}$, $\mathbf{R} = \mathbf{u}(\mathbf{p})$. Note that $\mathbf{R} \in C([0,T]; H^1(\Gamma)) \subset C(\overline{\Gamma} \times [0,T])$, for $u \in L^2(0,T; H^2(\Gamma))$ and $u_t \in L^2(0,T; L^2(\Gamma))$. Using our hypothesis, we see that $|\mathbf{R}(\cdot, t_0)| \geq r > 0$ on Γ, hence we can apply Proposition 2.2 to obtain

$$\|\mathbf{p} - \mathbf{q}\|_{L^2(\Gamma)} \leq C\Big(\|[\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})](\cdot, t_0)\|_{H^2(\Gamma))} + \sum_{v \in \mathcal{E}} \|\partial_x[\mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})](v, \cdot)\|_{H^1(0,T)}\Big),$$

where $C = C(\|\partial_t \mathbf{u}(\mathbf{p})\|_{L^\infty(\Gamma\times(0,T))}, \|\mathbf{q}\|_{L^\infty(\Gamma)}, r)$. The proof is now completed.

2.3 Schrödinger equation on a star-shaped tree

In this section, we consider a network Γ which is a star-shaped tree constituted by N edges e_j (with $N \geq 3$) connected at the internal node $\mathcal{O}$. Here, the parameterization of the edge e_j is chosen so that the origin $\mathcal{O}$ of e_j corresponds to $x = 0$, while the endpoint $\mathcal{O}_j$ of e_j corresponds to $x = l_j$, for all $j \in [[1, N]]$.

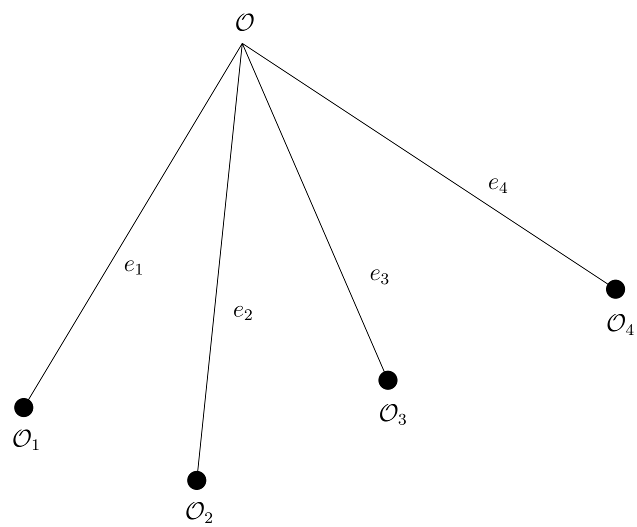

Figure 2.2: A star-shaped tree with 4 edges

We consider the following Cauchy problem

$$\begin{aligned}
iy_{j,t} + y_{j,xx} + p_j(x)y_j &= f_j(x,t), && x \in (0, l_j),\ j \in [[1,N]],\ t \in (0,T), && (2.21)\\
y_j(0,t) &= y_l(0,t), && t \in (0,T),\ j, l \in [[1,N]], && (2.22)\\
\sum_{1\leq j\leq N} y_{j,x}(0,t) &= 0, && t \in (0,T), && (2.23)\\
y(l_j,t) &= 0, && j \in [[1,N]],\ t \in (0,T), && (2.24)\\
y(x,0) &= y_0(x), && x \in \Gamma, && (2.25)
\end{aligned}$$

where $\mathbf{p} = \{p_j\}_{j=1,N} \in L^\infty(\Gamma)$ is some given potential function. Our main aim is to prove the stability for the inverse problem consisting in retrieving the potential $\mathbf{p}$ from the measurement of $y_x(l_j,t)$ for $j \in [[1,N]]$. This is done thanks to some Carleman estimate in following the classical Bukhgeim-Klibanov method.

The first step will be the proof of a Carleman inequality on Γ. The key point is that choosing only one weight function $\boldsymbol{\psi} = \{\psi_j\}_{j=1,N} : \Gamma \to \mathbb{R}$ as in the case of the heat equation is not convenient since we fail to control some boundary terms. Instead, we consider a family of weights $\{\boldsymbol{\psi}^k\}_{k=1,N}$ allowing us to get rid of some *bad* boundary terms.

Assume given a family $(\psi_j^k)_{1\le j,k\le N}$ of functions fulfilling the following properties

$$\psi_j^k : [0,l_j] \to \mathbb{R} \text{ is of class } C^2, \qquad \forall j,k \in [[1,N]], \tag{2.26}$$

$$\psi_{j_1}^{k_1}(0) = \psi_{j_2}^{k_2}(0), \qquad \forall j_1,k_1,j_2,k_2 \in [[1,N]], \tag{2.27}$$

$$|(\psi_j^k)'(x)|^2 + (\psi_j^k)''(x) \ge 0, \qquad \forall x \in [0,l_j],\ \forall j,k \in [[1,N]], \tag{2.28}$$

$$(\psi_j^k)'(x) \neq 0, \qquad \forall x \in [0,l_j],\ \forall j,k \in [[1,N]], \tag{2.29}$$

$$\frac{C}{2} \ge \psi_j^k(x) > \frac{C}{3}, \qquad \forall x \in [0,l_j],\ \forall j,k \in [[1,N]], \tag{2.30}$$

where $C > 0$ is some positive constant. We also assume that the following flux conditions at $x = 0$ are satisfied:

$$\sum_{1\le j\le N} (\psi_j^k)'(0) = 0, \qquad \forall k \in [[1,N]], \tag{2.31}$$

$$\sum_{1\le k\le N} (\psi_j^k)'(0) = 0, \qquad \forall j \in [[1,N]], \tag{2.32}$$

$$\sum_{1\le k\le N} |(\psi_j^k)'(0)|^2 = C_1, \qquad \forall j \in [[1,N]], \tag{2.33}$$

$$\sum_{1\le k\le N} (\psi_j^k)''(0) = C_2, \qquad \forall j \in [[1,N]], \tag{2.34}$$

$$\sum_{1\le k\le N} [(\psi_j^k)'(0)]^3 > 0, \qquad \forall j \in [[1,N]], \tag{2.35}$$

for some constants $C_1 > 0$ and $C_2 \in \mathbb{R}$. Such a family of weights functions $(\psi_j^k)_{1\le j,k\le N}$ exists. It is sufficient to pick (affine) functions of the form $\psi_j^k(x) = a_j^k x + \frac{5}{12}C$ with $C >> 1$ and

$$a_j^k := \begin{cases} N-1 & \text{if } j = k, \\ -1 & \text{if } j \neq k. \end{cases}$$

Let us introduce the families of weights

$$\theta_j^k(x,t) = \frac{e^{\lambda\psi_j^k(x)}}{t(T-t)}, \quad \varphi_j^k(x,t) = \frac{e^{\lambda C} - e^{\lambda\psi_j^k(x)}}{t(T-t)},$$

and the class of functions

$$\mathcal{Z} = \{\mathbf{q} = (q_j)_{j=1,N} \in C(\Gamma\times[0,T]);\ q_j \in C^{2,1}([0,l_j]\times[0,T])\ \forall j \in [[1,N]], \text{ and (2.22)–(2.24) hold}\}.$$

Proposition 2.3. *Assume that the family of weights (ψ_k^j) fulfills (2.26)-(2.35). Then there exist some constants $\lambda_0 \geq 1$, $s_0 \geq 1$ and $C_0 > 0$ such that for all $\lambda \geq \lambda_0$, all $s \geq s_0$, and all $\mathbf{q} \in \mathcal{Z}$, it holds*

$$\sum_{1\leq j,k\leq N} \int_0^T \int_0^{l_j} [\lambda^2 s\theta_j^k |q_{j,x}|^2 + \lambda^4 (s\theta_j^k)^3 |q_j|^2 + |(\widetilde{M}_1^k \mathbf{q})_j|^2 + |(\widetilde{M}_2^k \mathbf{q})_j|^2] e^{-2s\varphi_j^k}\, dxdt$$
$$\leq C_0 \sum_{1\leq j,k\leq N} \left(\int_0^T \int_0^{l_j} |q_{j,t} + iq_{j,xx}|^2 e^{-2s\varphi_j^k}\, dxdt + \int_0^T \lambda s\theta_j^k(l_j) |q_{j,x}(l_j)|^2 e^{-2s\varphi_j^k}\, dt \right), \quad (2.36)$$

where $i = \sqrt{-1}$ and $\widetilde{M}_1^k$ and $\widetilde{M}_2^k$ denote the operators

$$(\widetilde{M}_1^k \mathbf{q})_j := [s(\varphi_{j,t}^k + i\varphi_{j,xx}^k) - 2is^2|\varphi_{j,x}^k|^2] q_j + 2is\varphi_{j,x}^k q_{j,x}, \quad (2.37)$$
$$(\widetilde{M}_2^k \mathbf{q})_j := [-s(\varphi_{j,t}^k + i\varphi_{j,xx}^k) + 2is^2|\varphi_{j,x}^k|^2] q_j + q_{j,t} - 2is\varphi_{j,x}^k q_{j,x} + iq_{j,xx}. \quad (2.38)$$

Remark 2.1. *Note that (2.36) is still valid if, in the definition of $\mathcal{Z}$, one replaces*

$$q_j \in C^{2,1}([0,l_j] \times [0,T]) \quad \forall j \in [[1,N]]$$

by

$$\mathbf{q} \in H^{2,1}(\Gamma \times (0,T)).$$

Proof. In what follows, the letter c will denote a constant (independent of s, λ, $\mathbf{q}$, j, k) which may vary from line to line. Let $\mathbf{q} \in \mathcal{Z}$ be given, and for $j,k \in [[1,N]]$, let

$$u_j^k = e^{-s\varphi_j^k} q_j, \quad w_j^k = e^{-s\varphi_j^k} L(e^{s\varphi_j^k} u_j^k)$$

where L denotes the operator

$$L = \partial_t + i\partial_x^2.$$

Straightforward computations show that $\mathbf{w}^k = M^k \mathbf{u}^k$ with

$$w_j^k = (M^k\mathbf{u}^k)_j := u_{j,t}^k + s\varphi_{j,t}^k u_j^k + i(u_{j,xx}^k + 2s\varphi_{j,x}^k u_{j,x}^k + s\varphi_{j,xx}^k u_j^k + s^2|\varphi_{j,x}^k|^2 u_j^k),$$

the operator M^k acting simply on the components of $\mathbf{u}^k$ along the different edges. Let M_1^k and M_2^k denote respectively the (formal) adjoint and skew-adjoint parts of the operator M^k. We readily obtain that

$$(M_1^k\mathbf{u}^k)_j = i(2s\varphi_{j,x}^k u_{j,x}^k + s\varphi_{j,xx}^k u_j^k) + s\varphi_{j,t}^k u_j^k \quad (2.39)$$
$$(M_2^k\mathbf{u}^k)_j = u_{j,t}^k + i(u_{j,xx}^k + s^2|\varphi_{j,x}^k|^2 u_j^k). \quad (2.40)$$

Letting $(\widetilde{M}_1^k\mathbf{q})_j := e^{s\varphi_j^k}(M_1^k\mathbf{u}^k)_j$ and $(\widetilde{M}_2^k\mathbf{q})_j := e^{s\varphi_j^k}(M_2^k\mathbf{u}^k)_j$, we easily check that (2.37) and (2.38) hold. On the other hand,

$$||\mathbf{w}^k||^2 = ||M_1^k\mathbf{u}^k + M_2^k\mathbf{u}^k||^2 = ||M_1^k\mathbf{u}^k||^2 + ||M_2^k\mathbf{u}^k||^2 + 2\ \mathrm{Re}\ (M_1^k\mathbf{u}^k, M_2^k\mathbf{u}^k)$$

where $(\mathbf{u},\mathbf{v}) := \sum_{1\leq j\leq N} \int_0^T\int_0^{l_j} u_j(x,t)\overline{v_j(x,t)}dxdt$ and $||\mathbf{w}||^2 = (\mathbf{w},\mathbf{w})$. The proof of the Carleman estimate is inspired by those of [88, Proposition 2.1]. In the first step, we compute precisely Re $(M_1^k\mathbf{u}^k, M_2^k\mathbf{u}^k)$. In the second step, we check that the boundary terms related to the internal node $\mathcal{O}$ give positive contributions. The third step is completely similar to the second step in the proof of [88, Proposition 2.1].

Step 1. *Exact computation.* Exact computations of Re $(M_1^k\mathbf{u}^k, M_2^k\mathbf{u}^k)$ give us that

$$\begin{aligned}\sum_{1\le k\le N}||\mathbf{w}^k||^2 &= \sum_{1\le k\le N}\Big[||M_1^k\mathbf{u}^k||^2+||M_2^k\mathbf{u}^k||^2\Big]\\ &\quad+\sum_{1\le j,k\le N}\Big\{-4s\int_0^T\int_0^{l_j}\varphi_{j,xx}^k|u_{j,x}^k|^2-4s\ \mathrm{Im}\int_0^T\int_0^{l_j}\varphi_{j,xt}^k u_j^k\overline{u_{j,x}^k}\\ &\quad+\int_0^T\int_0^{l_j}|u_j^k|^2[s(\varphi_{j,4x}^k-\varphi_{j,tt}^k)-4s^3(\varphi_{j,x}^k)^2\varphi_{j,xx}^k]\\ &\quad+\int_0^T[2s\varphi_{j,x}^k|u_{j,x}^k|^2+(-s\varphi_{j,3x}^k+2s^3(\varphi_{j,x}^k)^3)|u_j^k|^2+2s\varphi_{j,xx}^k\ \mathrm{Re}\ (u_j^k\overline{u_{j,x}^k})\\ &\qquad+2s\varphi_{j,t}^k\ \mathrm{Re}\ (-iu_j^k\overline{u_{j,x}^k})+is\varphi_{j,x}^k(u_j^k\overline{u_{j,t}^k}-u_{j,t}^k\overline{u_j^k})]\Big|_0^l\Big\}.\end{aligned}\tag{2.41}$$

Step 2. *Estimation of the boundary terms at the internal node $\mathcal{O}$.*
We estimate each term in

$$\begin{aligned}&\sum_{j,k}(-2s)\int_0^T\varphi_{j,x}^k(0)|u_{j,x}^k(0)|^2+\sum_{j,k}\int_0^T(s\varphi_{j,3x}^k(0)-2s^3(\varphi_{j,x}^k(0))^3)|\mathbf{u}(0)|^2\\ &+\sum_{j,k}(-2s)\int_0^T\varphi_{j,xx}^k(0)\ \mathrm{Re}\ (\mathbf{u}(0)\overline{u_{j,x}^k(0)})+\sum_{j,k}(-2s)\int_0^T\varphi_{j,t}^k(0)\ \mathrm{Re}\ (-i\mathbf{u}(0)\overline{u_{j,x}^k(0)})\\ &+\sum_{j,k}\int_0^T(-is)\varphi_{j,x}^k(0)(\mathbf{u}(0)\overline{\mathbf{u}_t(0)}-\mathbf{u}_t(0)\overline{\mathbf{u}(0)})=:J_1+J_2+J_3+J_4+J_5,\end{aligned}$$

and conclude that

$$J_1+J_2+J_3+J_4+J_5\ge cs^3\lambda^3\int_0^T\left(\frac{e^{\lambda\psi(0)}}{t(T-t)}\right)^3|\mathbf{u}(0)|^2.\tag{2.42}$$

for $s\ge s_1$, $\lambda\ge\lambda_1$.

Step 3. *Estimation of the integrals along the edges.*
Direct estimations as in [88, Proposition 2.1] yield that for some constant $A>0$

$$\begin{aligned}&\sum_{j,k}\{(-4s)\int_0^T\int_0^{l_j}\varphi_{j,xx}^k|u_{j,x}^k|^2-4s\ \mathrm{Im}\ \int_0^T\int_0^{l_j}\varphi_{j,xt}^k u_j^k\overline{u_{j,x}^k}\\ &\qquad+\int_0^T\int_0^{l_j}|u_j^k|^2[s(\varphi_{j,4x}^k-\varphi_{j,tt}^k)-4s^3(\varphi_{j,x}^k)^2\varphi_{j,xx}^k]\}\\ &\ge A\sum_{j,k}\{\lambda^2 s\int_0^T\int_0^{l_j}\frac{e^{\lambda\psi_j^k}}{t(T-t)}|(\psi_j^k)'u_{j,x}^k|^2+\lambda s^3\int_0^T\int_0^{l_j}|\varphi_{j,x}^k|^3|u_j^k|^2\}\end{aligned}\tag{2.43}$$

provided that $s\ge s_2$, $\lambda\ge\lambda_2$. Combining (2.41), (2.42) and (2.43), we infer that

$$\begin{aligned}&\sum_{j,k}\Big\{\int_0^T\int_0^{l_j}[|(M_1^k\mathbf{u}^k)_j|^2+|(M_2^k\mathbf{u}^k)_j|^2]+\lambda^2 s\int_0^T\int_0^{l_j}\frac{e^{\lambda\psi_j^k}}{t(T-t)}|(\psi_j^k)'u_{j,x}^k|^2\\ &\qquad+\lambda s^3\int_0^T\int_0^{l_j}|\varphi_{j,x}^k|^3|u_j^k|^2+cs^3\lambda^3\int_0^T\Big(\frac{e^{\lambda\psi(0)}}{t(T-t)}\Big)^3|u(0)|^2\Big\}\\ &\qquad\le c\sum_{j,k}\Big(\int_0^T\int_0^{l_j}|w_j^k|^2+s\int_0^T|\varphi_{j,x}^k(l_j)|\,|u_{j,x}^k(l_j)|^2dt\Big).\end{aligned}\tag{2.44}$$

Replacing u_j^k by $e^{-s\varphi_j^k} q_j$ in (2.44) gives (2.36). □

We consider the following boundary initial-value problem

$$\begin{cases} iu_{j,t} + u_{j,xx} + p_j(x)u_j = 0, & x \in (0,l_j),\ j \in [[1,N]],\ t \in (0,T), \\ u_j(0,t) = u_l(0,t), & j,k \in [[1,N]],\ t \in (0,T), \\ \sum_{1\le j\le N} u_{j,x}(0,t) = 0, & t \in (0,T), \\ u_j(l_j,t) = h_j(t), & j \in [[1,N]],\ t \in (0,T), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), & x \in \Gamma. \end{cases} \tag{2.45}$$

In what follows we fix the initial data $\mathbf{u}_0$ and the boundary data $\mathbf{h} = \{h_j\}_{j=1,N}$, and we denote by $\mathbf{u}(\mathbf{p})$ the solution of the system (2.45) associated with the potential $\mathbf{p} \in L^\infty(\Gamma)$.

Pick any $\mathbf{p}, \mathbf{q}$ as in the statement of Theorem 2.2, and introduce the difference $\mathbf{y} := \mathbf{u}(\mathbf{p}) - \mathbf{u}(\mathbf{q})$ of the corresponding solutions of (2.45). Then y fulfills the system

$$\begin{cases} iy_{j,t} + y_{j,xx} + q_j(x)y_j = f_j(x)R_j(x,t), & x \in (0,l_j),\ j \in [[1,N]],\ t \in (0,T), \\ y_j(0,t) = y_l(0,t), & j,k \in [[1,N]],\ t \in (0,T), \\ \sum_{1\le j\le N} y_{j,x}(0,t) = 0, & t \in (0,T), \\ y_j(l_j,t) = 0, & j \in [[1,N]],\ t \in (0,T), \\ y(x,0) = 0, & x \in \Gamma, \end{cases} \tag{2.46}$$

with $f_j = q_j - p_j$ (real valued) and $R_j := (\mathbf{u}(\mathbf{p}))_j$. Theorem 2.2 follows by applying the following result to function $\mathbf{y}$.

Proposition 2.4. *[57, Proposition 4.4] Suppose that $\mathbf{R} = \{R_j\}_{j=1,N}$ satisfies*

- *$\mathbf{R}(x,0) \in \mathbb{R}$ or $i\mathbf{R}(x,0) \in \mathbb{R}$ a.e. in Γ,*
- *$|\mathbf{R}(x,0)| \ge r > 0$ a.e. in Γ,*
- *$\mathbf{R} \in H^1(0,T;L^\infty(\Gamma))$, and*
- *$\partial_t \mathbf{y} \in H^{2,1}(\Gamma \times (0,T))$.*

Then for any $m \ge 0$ there exists a constant $C = C(m, ||\mathbf{R}_t||_{L^2(0,T;L^\infty(\Gamma))}, r)$ such that for any $\mathbf{q} \in L^\infty(\Gamma,\mathbb{R})$ with $||\mathbf{q}||_{L^\infty(\Gamma)} \le m$ and for all $\mathbf{f} \in L^2(\Gamma,\mathbb{R})$, the solution $\mathbf{y}$ of (2.46) satisfies

$$||\mathbf{f}||_{L^2(\Gamma)} \le C \sum_{1\le j\le N} ||y_{j,x}(l_j,.)||_{H^1(0,T)}. \tag{2.47}$$

The complete details of the proof are given in [57].

2.4 Open problems

We now mention a few open problems related to our work. One of them is whether it is possible to reduce the number of measurements at the boundaries. It could be interesting to combine the ideas of the paper with those appearing in [36], [37] where less measurements on the boundary are needed but some rationality assumptions on the lengths of the edges have to be made. For the Schrödinger equation, the question whether a Carleman estimate on a tree with N exterior vertices can be written with only one weight function and $N-1$ boundary observations seems to be challenging.

The extension of the present work to more general graphs with other kind of coupling is also an open problem. We recall here the works of Kostrykin and Schrader [77, 80] where self-adjoint Laplace operators with general coupling conditions are introduced.

Part II

Discrete Equations and Numerical Approximations

Chapter 3

Discrete Schrödinger equations

In this chapter we prove dispersive estimates for the system formed by two coupled discrete Schrödinger equations (DSE). The main goal is to analyze a model which consists in coupling two DSE by Kirchhoff's type condition:

$$\begin{cases} iu_t(t,j)+b_1^{-2}(\Delta_d u)(t,j)=0 & j\leq -1,\ t\neq 0,\\ iv_t(t,j)+b_2^{-2}(\Delta_d v)(t,j)=0 & j\geq 1,\ t\neq 0,\\ u(t,0)=v(t,0), & t\neq 0,\\ b_1^{-2}(u(t,-1)-u(t,0))=b_2^{-2}(v(t,0)-v(t,1)), & t\neq 0,\\ u(0,j)=\varphi(j), & j\leq -1,\\ v(0,j)=\varphi(j), & j\geq 1. \end{cases} \tag{3.1}$$

In the above system $u(t,0)$ and $v(t,0)$ have been artificially introduced to couple the two equations on positive and negative integers. The third condition in the above system requires continuity along the interface $j=0$ and the fourth one can be interpreted as the continuity of the flux along the interface. We will obtain estimates for the resolvent of the discrete operator. The decay of the solutions is obtained by using classical and some new results on oscillatory integrals.

Let us now recall some previous results related with our work. First we consider the following system of difference equations

$$\begin{cases} iu_t+\Delta_d u=0, \quad j\in\mathbb{Z}, t\neq 0,\\ u(0)=\varphi, \end{cases} \tag{3.2}$$

where Δ_d is the discrete laplacian defined by

$$(\Delta_d u)(j)=u_{j+1}-2u_j+u_{j-1}, \quad j\in\mathbb{Z}.$$

Concerning the long time behavior of the solutions of system (3.2) in [99] the authors have proved the following:

$$\|u(t)\|_{l^\infty(\mathbb{Z})}\leq C(|t|+1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z})}, \quad \forall\ t\neq 0. \tag{3.3}$$

The proof of (3.3) consists in writing the solution u of (3.2) as the convolution between a kernel K_t and the initial data φ and then estimate K_t by using Van der Corput's lemma (see Section 3.2). For the linear semigroup $\exp(it\Delta_d)$, Strichartz like estimates similar have been obtained in [99] for pairs (q,r) satinsfying:

$$\frac{1}{q}\leq\frac{1}{3}\Big(\frac{1}{2}-\frac{1}{r}\Big), \quad 2\leq q,r\leq\infty. \tag{3.4}$$

We also mention [63] and [65] where the authors consider a similar equation on $h\mathbb{Z}$ by replacing Δ_d by Δ_d/h^2 and analyze the same properties in the context of numerical approximations of the linear and nonlinear Schrödinger equation.

A more thorough analysis has been done in [76] and [90] where the authors analyze the decay properties of the solutions of equation $iu_t + Au = 0$ where $A = \Delta_d - V$, with V a real-valued potential. In these papers $l^1(\mathbb{Z}) - l^\infty(\mathbb{Z})$ and $l^2_{-\sigma}(\mathbb{Z}) - l^2_\sigma(\mathbb{Z})$ estimates for $\exp(itA)P_{a,c}(A)$ have been obtained where $P_{a,c}(A)$ is the spectral projection to the absolutely continuous spectrum of A and $l^2_{\pm\sigma}(\mathbb{Z})$ are weighted $l^2(\mathbb{Z})$-spaces.

In what concerns the continuous counterpart of the results presented here, i.e. the Schödinger equation with variable coefficients, we mention the results of Banica [7]. Consider a partition of the real axis as follows: $-\infty = x_0 < x_1 < \cdots < x_{n+1} = \infty$ and a step function $\sigma(x) = b_i^{-2}$ for $x \in (x_i, x_{i+1})$, where b_i are positive numbers. The solution u of the Schrödinger equation

$$\begin{cases} iu_t(t,x) + (\sigma(x)u_x)_x(t,x) = 0, & \text{for } x \in \mathbb{R}, t \neq 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

satisfies the dispersion inequality

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq C|t|^{-1/2}\|u_0\|_{L^1(\mathbb{R})}, \quad t \neq 0,$$

where constant C depends on n and on sequence $\{b_i\}_{i=0}^n$. We recall that in [55] the above result was used in the analysis of the long time behavior of the solutions of the linear Schödinger equation on regular trees. In the case of discrete equations the corresponding model is given by

$$\begin{cases} iU_t + AU = 0, \ t \neq 0, \\ U(0) = \varphi, \end{cases} \tag{3.5}$$

where the infinite matrix A is symmetric with a finite number of diagonals nonidentically vanishing. Once a result similar to [7] will be obtained for discrete Schrödinger equations with non-constant coefficients we can apply it to obtain dispersive estimates for discrete Schrödinger equations on trees. But as far as we know the study of the decay properties of solutions of system (3.5) in terms of the properties of A is a difficult task and we try to give here a partial answer to this problem. In the case when A is a diagonal matrix these properties are easily obtained by using the Fourier transform and classical estimates for oscillatory integrals.

The main result of this chapter is given by the following theorem.

Theorem 3.1. *For any $\varphi \in l^2(\mathbb{Z} \setminus \{0\})$ there exists a unique solution $(u,v) \in C(\mathbb{R}, l^2(\mathbb{Z} \setminus \{0\}))$ of system (3.1). Moreover, there exists a positive constant $C(b_1, b_2)$ such that*

$$\|(u,v)(t)\|_{l^\infty(\mathbb{Z}\setminus\{0\})} \leq C(b_1,b_2)(|t|+1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}\setminus\{0\})}, \quad \forall t \in \mathbb{R}, \tag{3.6}$$

holds for all $\varphi \in l^1(\mathbb{Z} \setminus \{0\})$.

Using the well-known results of Keel and Tao [72] we obtain the following Strichartz-like estimates for the solutions of system (3.1).

Theorem 3.2. *For any $\varphi \in l^2(\mathbb{Z} \setminus \{0\})$ the solution (u,v) of system (3.1) satisfies*

$$\|(u,v)\|_{L^q(\mathbb{R},\, l^r(\mathbb{Z}\setminus\{0\}))} \leq C(q,r)\|\varphi\|_{l^2(\mathbb{Z}\setminus\{0\})}$$

for all pairs (q,r) satisfying (3.4).

The chapter is organized as follows: In section 3.1 we present some discrete models, in particular system (3.1) in the case $b_1 = b_2$ and show how it is related with problem (3.2). In addition, a system with a dynamic coupling along the interface is presented. In section 3.2 we present some classical results on oscillatory integrals and make some improvements that we need to obtain the result in Theorem 3.1.

In Section 3.3 we obtain the explicit formula of the resolvent associated with system (3.1) and write a limiting absorption principle. Finally we sketch the proof of Theorem 3.1 and present some open problems.

3.1 Some discrete models

In this section in order to emphasize the main differences and difficulties with respect to the continuous case when we deal with discrete systems we consider two models. In the first case we consider system (3.1) with the two coefficients in the front of the discrete laplacian equal. The second case involves a time dependent ODE coupling at the interface $j = 0$.

In the following we denote $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

Theorem 3.3. *Let us assume that $b_1 = b_2$. For any $\varphi \in l^2(\mathbb{Z}^*)$ there exists a unique solution $u \in C(\mathbb{R}, l^2(\mathbb{Z}^*))$ of system (3.1). Moreover there exists a positive constant $C(b_1)$ such that*

$$\|u(t)\|_{l^\infty(\mathbb{Z}^*)} \leq C(b_1)(|t|+1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}^*)}, \quad \forall\, t \in \mathbb{R}, \tag{3.7}$$

holds for all $\varphi \in l^1(\mathbb{Z}^*)$.

The existence of the solutions of system (3.1) is immediate since we can write it in the form (3.5) where operator A is bounded in $l^2(\mathbb{Z}^*)$:

$$A = \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{2} & -\frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}$$

In the particular case considered here we can reduce the proof of the dispersive estimate (3.7) to the analysis of two problems: one with Dirichlet's boundary condition and another one with a discrete Neumann's boundary condition.

Let us recall that in the case of system (3.2) its solution is given by $u(t) = K_t * \varphi$ where $*$ is the standard convolution on $\mathbb{Z}$ and

$$K_t(j) = \int_{-\pi}^{\pi} e^{-4it\sin^2(\frac{\xi}{2})} e^{ij\xi} d\xi, \quad t \in \mathbb{R},\ j \in \mathbb{Z}.$$

In [99] a simple argument based on Van der Corput's lemma has been used to show that for any real number t the following holds:

$$|K_t(j)| \leq C(|t|+1)^{-1/3}, \quad \forall j \in \mathbb{Z}. \tag{3.8}$$

Let us restrict for simplicity to the case $b_1 = b_2 = 1$. For (u, v) solution of system (3.1) let us set

$$S(j) = \frac{v(j) + u(-j)}{2}, \quad D(j) = \frac{v(j) - u(-j)}{2}, j \geq 0.$$

Observe that u and v can be recovered from S and D as follows

$$(u, v) = ((S - D)(-\cdot), S + D).$$

Writing the equations satisfied by u and v we obtain that D and S solve two discrete Schrödinger equations on $Z^+ = \{j \in \mathbb{Z}, j \geq 1\}$ with Dirichlet, respectively Neumann boundary conditions:

$$\begin{cases} iD_t(t,j) + (\Delta_d D)(t,j) = 0 & j \geq 1,\ t \neq 0, \\ D(t,0) = 0, & t \neq 0, \\ D(0,j) = \frac{\varphi(j) - \varphi(-j)}{2}, & j \geq 1, \end{cases} \tag{3.9}$$

and

$$\begin{cases} iS_t(t,j) + (\Delta_d S)(t,j) = 0 & j \geq 1,\ t \neq 0, \\ S(t,0) = S(t,1), & t \neq 0, \\ S(0,j) = \frac{\varphi(j)+\varphi(-j)}{2}, & j \geq 1. \end{cases} \tag{3.10}$$

Making an odd extension of the function D and using the representation formula for the solutions of (3.2) we obtain that the solution of the Dirichlet problem (3.9) satisfies

$$D(t,j) = \sum_{k\geq 1} (K_t(j-k) - K_t(j+k))D(0,k), \quad t \neq 0,\ j \geq 1. \tag{3.11}$$

A similar even extension of function S permits us to obtain the explicit formula for the solution of the Neumann problem (3.10)

$$S(t,j) = \sum_{k\geq 1} (K_t(k-j) + K_t(k+j-1))S(0,k), \quad t \neq 0,\ j \geq 1. \tag{3.12}$$

Using the decay of the kernel K_t given by (3.8) we obtain that $S(t)$ and $D(t)$ decay as $(|t|+1)^{-1/3}$ and then the same property holds for u and v. This finishes the proof of this particular case.

In our previous analysis has taken into account the particular structure of the equations. When the coefficients b_1 and b_2 are not equal we cannot write an equation verified by functions D or S. We now write system (3.1) in matrix formulation. Using the coupling conditions at $j=0$ system (3.1) can be written in the following equivalent form

$$\begin{cases} iU_t + AU = 0, \\ U(0) = \varphi, \end{cases}$$

where $U = (u,v)^T$, $u = (u(j))_{j\leq -1}$, $v = (v_j)_{j\geq 1}$ and

$$A = \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1^{-2} & -2b_1^{-2} & b_1^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1^{-2} & -b_1^{-2} - \frac{1}{b_1^2+b_2^2} & \frac{1}{b_1^2+b_2^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{b_1^2+b_2^2} & -\frac{1}{b_1^2+b_2^2} - b_2^{-2} & b_2^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_2^{-2} & -2b_2^{-2} & b_2^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}. \tag{3.13}$$

In the particular case $b_1 = b_2 = 1$ the operator A can be decomposed as follows

$$A = \Delta_d + B = \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix} + \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}.$$

However, we do not know how to use the dispersive properties of $\exp(it\Delta_d)$ and the particular structure of B in order to obtain the decay of the new semigroup $\exp(it(\Delta_d + B))$.

Another model of interest is the following one inspired in the numerical approximations of LSE. Set

$$a(x) = \begin{cases} b_1^{-2}, & x < 0, \\ b_2^{-2}, & x > 0. \end{cases}$$

Using the following discrete derivative operator

$$(\partial u)(x) = u(x + \frac{1}{2}) - u(x - \frac{1}{2})$$

we can introduce the second order discrete operator

$$\partial(a\partial u)(j) = a(j+\frac{1}{2})u(j+1) - \big(a(j+\frac{1}{2}) + a(j-\frac{1}{2})\big)u(j) + a(j-\frac{1}{2})u(j-1), j \in \mathbb{Z}.$$

In this case we have to analyze the following system

$$\begin{cases} iu_t(t,j) + b_1^{-2}(\Delta_d u)(t,j) = 0, & j \le -1,\ t \neq 0, \\ iu_t(t,j) + b_2^{-2}(\Delta_d u)(t,j) = 0, & j \ge 1,\ t \neq 0, \\ iu_t(t,0) + b_1^{-2}u(t,-1) - (b_1^{-2} + b_2^{-2})u(t,0) + b_2^{-1}u(t,1) = 0, & t \neq 0, \\ u(0,j) = \varphi(j), & j \in \mathbb{Z}. \end{cases} \tag{3.14}$$

In matrix formulation it reads $iU_t + AU = 0$ where $U = (u(j))_{j\in\mathbb{Z}}$, and the operator A is given by the following one

$$A = \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 \\ 0 & b_1^{-2} & -2b_1^{-2} & b_1^{-2} & 0 & 0 & 0 \\ 0 & 0 & b_1^{-2} & -(b_1^{-2}+b_2^{-2}) & b_2^{-2} & 0 & 0 \\ 0 & 0 & 0 & b_2^{-2} & -2b_2^{-2} & b_2^{-2} & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}. \tag{3.15}$$

Observe that in the case $b_1 = b_2$ the results of [99] give us the decay of the solutions.

Regarding the long time behavior of the solutions of system (3.14) we have the following result.

Theorem 3.4. *For any $\varphi \in l^2(\mathbb{Z})$ there exists a unique solution $u \in C(\mathbb{R}, l^2(\mathbb{Z}))$ of system (3.14). Moreover, there exists a positive constant $C(b_1, b_2)$ such that*

$$\|u(t)\|_{l^\infty(\mathbb{Z})} \le C(b_1,b_2)(|t|+1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z})}, \quad \forall t \in \mathbb{R},$$

holds for all $\varphi \in l^1(\mathbb{Z})$.

The proof of this result is similar to the one of Theorem 3.1.

3.2 Oscillatory integrals

In this section we present some classical tools for oscillatory integrals and we give an improvement of Van der Corput's Lemma that is in some sense similar to the one obtained in [73]. First of all let us recall Van der Corput's lemma(see for example [101], p. 332).

Lemma 3.1. *(Van der Corput) Let $k \ge 1$ be an integer, and $\phi : [a,b] \to \mathbb{R}$ such that $|\phi^{(k)}(x)| \ge 1$ for all $x \in [a,b]$, and ϕ' monotone in the case $k = 1$.*
Then

$$\left|\int_a^b e^{it\phi(x)}\psi(x)dx\right| \le c_k|t|^{-\frac{1}{k}}\big(\|\psi\|_{L^\infty(a,b)} + \int_a^b |\psi'(\xi)|d\xi\big), \quad \forall\ t \neq 0.$$

A first improvement has been obtained in [73] where the authors analyze the smoothing effect of some dispersive equations. We will present here a particular case of the results in [73], that will be sufficient for our purposes. In the sequel Ω will be a bounded interval. We consider class $\mathcal{A}_2$ of real functions $\phi \in C^3(\overline{\Omega})$ satisfying the following conditions:

1) Set $S_\phi = \{\xi \in \Omega : \phi'' = 0\}$ is finite,

2) If $\xi_0 \in S_\phi$ then there exist constants ϵ, c_1, c_2 and $\alpha \ge 2$ such that for all $|\xi - \xi_0| < \epsilon$,

$$c_1|\xi-\xi_0|^{\alpha-2} \le |\phi''(\xi)| \le c_2|\xi-\xi_0|^{\alpha-2},$$

3) ϕ'' has a finite number of changes of monotonicity.

Lemma 3.2. *([73]) Let Ω be a bounded interval, $\phi \in \mathcal{A}_2$ and*

$$I(x,t) = \int_{\Omega} e^{i(t\phi(\xi)-x\xi)} |\phi''(\xi)|^{1/2} d\xi.$$

Then for any $x, t \in \mathbb{R}$

$$|I(x,t)| \leq c_\phi |t|^{-1/2}, \tag{3.16}$$

where c_ϕ depends only on the constants involved in the definition of class $\mathcal{A}_2$.

Remark 3.1. *The results of [73] are more general that the one presented here allowing functions with vertical asymptotics, finite union of intervals or infinite domains.*

In the proof of our main result we will need a result similar to Lemma 3.2 but with $|p'''|^{1/3}$ instead of $|p''|^{1/2}$ in the definition of $I(x,t)$. We define class $\mathcal{A}_3$ of real functions $\phi \in C^4(\overline{\Omega})$ satisfying the following conditions:

1) Set $S_\phi = \{\xi \in \Omega : \phi''' = 0\}$ is finite,

2) If $\xi_0 \in S_\phi$ then there exist constants ϵ, c_1, c_2 and $\alpha \geq 3$ such that for all $|\xi - \xi_0| < \epsilon$,

$$c_1 |\xi - \xi_0|^{\alpha - 3} \leq |\phi'''(\xi)| \leq c_2 |\xi - \xi_0|^{\alpha-3}, \tag{3.17}$$

3) ϕ''' has a finite number of changes of monotonicity.

Lemma 3.3. *([61]) Let Ω be a bounded interval, $\phi \in \mathcal{A}_3$ and*

$$I(x,t) = \int_{\Omega} e^{i(t\phi(\xi)-x\xi)} |\phi'''(\xi)|^{1/3} d\xi.$$

Then for any $x, t \in \mathbb{R}$

$$|I(x,t)| \leq c_\phi |t|^{-1/3}, \tag{3.18}$$

where c_ϕ depends only on the constants involved in the definition of class $\mathcal{A}_3$.

As a consequence of the above results we obtain the following:

Lemma 3.4. *([61]) Let $a \in (0,1]$ and $0 \leq \delta \leq \pi$. There exists $C(a,\delta)$ such that for all real numbers y, z and t*

$$\Big| \int_{\delta}^{\pi} e^{it(2\cos\theta + 2z \arcsin(a \sin\frac{\theta}{2}))} e^{iy\theta} \sin\theta d\theta \Big| \leq C(a,\delta)(|t|+1)^{-1/3} \tag{3.19}$$

and if $\delta > 0$

$$\Big| \int_{0}^{\pi-\delta} e^{it(2\cos\theta + 2z \arcsin(a \sin\frac{\theta}{2}))} e^{iy\theta} d\theta \Big| \leq C(a,\delta)(|t|+1)^{-1/3}. \tag{3.20}$$

We point out here that the proof of Lemma 3.4 is not trivial and needs a careful applications of the three previous lemmas.

3.3 The resolvent

Let us consider the system

$$\begin{cases} iU_t + AU = 0, \\ U(0) = \varphi, \end{cases} \tag{3.21}$$

where $U(t) = (u(t,j))_{j\neq 0}$ and operator A is given by (3.13). We compute explicitly the resolvent $(A - \lambda I)^{-1}$ and we obtain a limiting absorption principle.

We start by localizing the spectrum of operator A and computing the resolvent $R(\lambda) = (A - \lambda I)^{-1}$. We use some classical results on difference equations.

Lemma 3.5. *For any b_1 and b_2 positive the spectrum of operator A satisfies*

$$\sigma(A) = [-4\max\{b_1^{-2}, b_2^{-2}\}, 0]. \tag{3.22}$$

Before computing the resolvent $(A-\lambda I)^{-1}$ we need some results for difference equations.

Lemma 3.6. *For any $\lambda \in \mathbb{C} \setminus [-4, 0]$ and $g \in l^2(\mathbb{Z}^*)$, any solution $f \in l^2(\mathbb{Z}^*)$ of*

$$\Delta_d f(j) - \lambda f(j) = g(j), \quad j \neq 0$$

with $f(0)$ prescribed is given by

$$f(j) = \alpha r^{|j|} + \frac{1}{2r-2-\lambda} \sum_{k\in\mathbb{Z}^*} r^{|j-k|} g(k) \tag{3.23}$$

where α is determined by $f(0)$ and r is the unique solution with $|r| < 1$ of

$$r^2 - 2r + 1 = \lambda r.$$

Moreover

$$f(j) = f(0) r^{|j|} + \frac{1}{r - r^{-1}} \sum_k (r^{|j-k|} - r^{|j|+|k|}) g(k), \quad j \neq 0.$$

As an application of the previous Lemma we have the following result.

Lemma 3.7. *Let $\lambda \in \mathbb{C} \setminus [-4\max\{b_1^{-2}, b_2^{-2}\}, 0]$. For any $g \in l^2(\mathbb{Z}^*)$ there exists a unique solution $f \in l^2(\mathbb{Z}^*)$ of the equation $(A - \lambda I) f = g$. Moreover, it is given by the following formula*

$$\begin{aligned} f(j) = {} & \frac{-r_s^{|j|}}{b_2^{-2}(1-r_2) + b_1^{-2}(1-r_1)} \Big[\sum_{k\in Z_1} r_1^{|k|} g(k) + \sum_{k\in Z_2} r_2^{|k|} g(k) \Big] \\ & + \frac{b_s^2}{r_s - r_s^{-1}} \sum_{k\in Z_s} (r_s^{|j-k|} - r_s^{|j|+|k|}) g(k), \quad j \in Z_s, \end{aligned} \tag{3.24}$$

where for $s \in \{1, 2\}$, $r_s = r_s(\lambda)$ is the unique solution with $|r_s| < 1$ of the equation

$$r_s^2 - 2r_s + 1 = \lambda b_s^2 r_s.$$

We now write a limiting absorption principle. From Lemma 3.7 we know that for any $\lambda \in \mathbb{C} \setminus [-4\max\{b_1^{-2}, b_2^{-2}\}, 0]$ and $\varphi \in l^2(\mathbb{Z}^*)$ there exists $R(\lambda)\varphi = (A-\lambda)^{-1}\varphi \in l^2(\mathbb{Z}^*)$ and it is given by

$$\begin{aligned} (R(\lambda)\varphi)(j) = {} & \frac{-r_s^{|j|}}{b_2^{-2}(1-r_2) + b_1^{-2}(1-r_1)} \Big[\sum_{k\in I_1} r_1^{|k|} \varphi(k) + \sum_{k\in I_2} r_2^{|k|} \varphi(k) \Big] \\ & + \frac{b_s^2}{r_s - r_s^{-1}} \sum_{k\in I_s} (r_s^{|j-k|} - r_s^{|j|+|k|}) \varphi(k), \quad j \in Z_s, \end{aligned} \tag{3.25}$$

where $r_s = r_s(\lambda)$, $s \in \{1, 2\}$, is the unique solution with $|r_s| < 1$ of the equation

$$r_s^2 - 2r_s + 1 = \lambda b_s^2 r_s.$$

Let us now consider $I = [-4\max\{b_1^{-2}, b_2^{-2}\}, 0]$. From Lemma 3.5 we have that $\sigma(A) = I$. For any $\omega \in I$ and $\epsilon \geq 0$ let us denote by $r_{s,\epsilon}^{\pm}$ the unique solution with modulus less than one of

$$r^2 - 2r + 1 = (\omega \pm i\epsilon) b_s^2 r.$$

Denoting $r^+_{s,\epsilon} = \exp(z^+_{s,\epsilon})$ with $z^+_{s,\epsilon} = a^+_{s,\epsilon} + i\tilde{a}^+_{s,\epsilon}$, $a^+_{s,\epsilon} < 0$ and $\tilde{a}^+_{s,\epsilon} \in [-\pi, \pi]$ we obtain by taking the imaginary part in the equation satisfied by $r^+_{s,\epsilon}$ that

$$(\exp(a^+_{s,\epsilon}) - \exp(-a^+_{s,\epsilon}))\sin(\tilde{a}^+_{s,\epsilon}) = \epsilon b_s^2.$$

Thus $\tilde{a}^+_{s,\epsilon} \in [-\pi, 0]$. A similar result holds for $r^-_{s,\epsilon}$, $\tilde{a}^-_{s,\epsilon} \in [0, \pi]$.

Let us set $r^\pm_s = \lim_{\epsilon\downarrow 0} r^\pm_{s,\epsilon}$. Using the sign of the imaginary part of $r^\pm_{s,\epsilon}$ we obtain that $r^\pm_s$ are the solutions with $\Im(r^+_s) \le 0 \le \Im(r^-_s)$ of the equation

$$r^2 - 2r + 1 = \omega b_s^2 r.$$

Also, using that $r^-_{s,\epsilon} = \overline{r^+_{s,\epsilon}}$ we obtain $r^-_s = \overline{r^+_s}$.

For any $\omega \in J = I \setminus \{-4b_1^{-2}, -4b_2^{-2}, 0\}$ and $\varphi \in l^1(\mathbb{Z}^*)$ let us set

$$\begin{aligned}(R^\pm(\omega)\varphi)(j) =& \frac{-(r^\pm_s)^{|j|}}{b_2^{-2}(1-r^\pm_2) + b_1^{-2}(1-r^\pm_1)}\Big[\sum_{k\in I_1}(r^\pm_1)^{|k|}\varphi(k) + \sum_{k\in I_2}(r^\pm_2)^{|k|}\varphi(k)\Big] \\ &+ \frac{b_s^2}{r^\pm_s - (r^\pm_s)^{-1}}\sum_{k\in I_s}((r^\pm_s)^{|j-k|} - (r^\pm_s)^{|j|+|k|})\varphi(k), \quad j \in Z_s.\end{aligned}$$

The new operators $R^\pm(\omega)$ are well defined as bounded operators from $l^1(\mathbb{Z}^*)$ to $l^\infty(\mathbb{Z}^*)$. We point out that we cannot define $R^\pm(\omega)$ for $\omega \in \{-4b_1^{-2}, -4b_2^{-2}, 0\}$ since for $\omega = 0$ we have $r_1 = r_2 = 1$ and for $\omega = 4b_s^{-2}$, $s \in \{1, 2\}$, we have $r_s = -1$. We also emphasize that $R^-(\omega)\varphi = \overline{R^+(\omega)\overline{\varphi}}$. This is a consequence of the fact that for any $\omega \in I$, $r^-_s(\omega) = \overline{r^+_s}(\omega)$. Formally, the above operator equals $R(\omega \pm i\epsilon)$ with $\epsilon = 0$. We point out that as operators on $l^2(\mathbb{Z}^*)$, $R(\omega \pm i\epsilon)$ are defined for any $\omega \in I$ but only if $\epsilon \ne 0$.

Making rigourous the above comments on the the resolvent we obtain that the linear semigroup satisfies the following limiting absorption principle.

Lemma 3.8. *For any $\varphi \in l^1(\mathbb{Z}^*)$ operator $\exp(itA)$ satisfies*

$$e^{itA}\varphi = \frac{1}{2i\pi}\int_I e^{it\omega}[R^+(\omega) - R^-(\omega)]\varphi\, d\omega. \tag{3.26}$$

We now able to sketch the proof of the main result of this Chapter, Theorem 3.1. For any $\varphi \in l^1(\mathbb{Z}^*)$ Lemma 3.8 gives us that

$$(e^{itA}\varphi)(n) = \frac{1}{2\pi i}\int_I e^{it\omega}(R^+(\omega) - R^-(\omega))\varphi(n)ds,\ n \in \mathbb{Z}^*,$$

where $I = [-4\max\{b_1^{-2}, b_2^{-2}\}, 0]$. Using the fact that $R^-(\omega)\varphi = \overline{R^+(\omega)\overline{\varphi}}$ we obtain

$$(e^{itA}\varphi)(n) = \frac{1}{\pi}\int_I e^{it\omega}((\Im R^+)(\omega)\varphi)(n)d\omega,\ n \in \mathbb{Z}^*,$$

where $\Im R^+$ is given by

$$\begin{aligned}(\Im R^+)(\omega)\varphi(j) =& \frac{(R^+(\omega)\varphi)(j) - (R^-(\omega)\varphi)(j)}{2i} \\ =& \sum_{k\in Z_1}\varphi(k)\Im\frac{-(r^+_s)^{|j|}(r^+_1)^{|k|}}{b_2^{-2}(1-r^+_2) + b_1^{-2}(1-r^+_1)} \\ &+ \sum_{k\in Z_2}\varphi(k)\Im\frac{-(r^+_s)^{|j|}(r^+_2)^{|k|}}{b_2^{-2}(1-r^+_2) + b_1^{-2}(1-r^+_1)} \\ &+ \sum_{k\in Z_s}\varphi(k)\Im\frac{b_s^2}{r^+_s - (r^+_s)^{-1}}((r^+_s)^{|j-k|} - (r^+_s)^{|j|+|k|}), \quad j \in Z_s\end{aligned}$$

and for $s \in \{1, 2\}$, r_s^+ is the root of $r^2 - 2r + 1 = \omega b_s^2 r$ with the imaginary part nonpositive.

In order to prove (3.6) it is sufficient to show the existence of a constant $C = C(b_1, b_2)$ such that

$$\sum_{k\in Z_1} |\varphi(k)|\Big|\int_I e^{it\omega}\Im\frac{(r_s^+)^{|j|}(r_1^+)^{|k|}}{b_2^{-2}(1-r_2^+)+b_1^{-2}(1-r_1^+)}d\omega\Big| \leq C(|t|+1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}^*)},\ \forall j\in\mathbb{Z}^*, \quad (3.27)$$

and

$$\sum_{k\in Z_s} |\varphi(k)|\Big|\int_I e^{it\omega}\Im\frac{(r_s^+)^{|j-k|}}{r_s^+-(r_s^+)^{-1}}d\omega\Big| \leq C(|t|+1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}^*)},\ \forall j\in\mathbb{Z}^*. \quad (3.28)$$

The estimates for the other two terms occurring in the representation of $\Im R^+(\omega)$ are similar.

Using the results on oscillatory integrals contained in Section 3.2 we obtain that

$$\sup_{j\in\mathbb{Z}}\Big|\int_I e^{it\omega}\Im\frac{(r_s^+)^{|j|}}{r_s^+-(r_s^+)^{-1}}d\omega\Big| \leq C(b_1,b_2)(|t|+1)^{-1/3}, \quad \forall\ t\in\mathbb{R}. \quad (3.29)$$

and

$$\sup_{j,k\in\mathbb{N}}\Big|\int_I e^{it\omega}\frac{(r_1^+)^{j}(r_2^+)^{k}}{b_2^{-2}(1-r_2^+)+b_1^{-2}(1-r_1^+)}d\omega\Big| \leq C(b_1,b_2)(|t|+1)^{-1/3},\ \forall t\in\mathbb{R},$$

which finishes the proof of the main result of this chapter.

3.4 Open problems

In this papers we have presented the analysis of the dispersive properties of the solutions of a system consisting in coupling two discrete Schrödinger equations. However we did not cover the case when more discrete equations are coupled. The main difficulty is to write in an accurate and clean way the resolvent of the linear operator occurring in the system. Once this case will be understood then we can treat discrete Schödinger equations on trees similar to those considered in [55] in the continuous case.

The analysis presented here mainly concerns the $l^1 - l^\infty$ decay property. In a recent paper [89] the authors use some modifications of the stationary phase method to obtain improved $l^1 - l^p$ decay estimates for the linear Fermi-Pasta-Ulam chain, the Klein-Gordon chain and the discrete nonlinear Schrödinger equation. The optimality of $l^1 - l^p$ estimates for the models presented here remains to be investigated.

There is another question which arises from the presentation given in this chaper. Suppose that we have a system $iU_t + AU = 0$ with an initial datum at $t = 0$, where A is an symmetric operator with a finite number of diagonals not identically vanishing. Under which assumptions on the operator A does solution U decay and how can we characterize the decay property in terms of the properties of A? When A is a diagonal operator we can use Fourier's analysis tools but in the case of a non-diagonal operator this is not useful.

Chapter 4

Convergence rates for dispersive approximation schemes to nonlinear Schrödinger equations

Let us consider the linear (LSE) and the nonlinear (NSE) Schrödinger equations:

$$\begin{cases} iu_t + \partial_x^2 u = 0,\ x \in \mathbb{R},\ t \neq 0, \\ u(0,x) = \varphi(x),\ x \in \mathbb{R} \end{cases} \tag{4.1}$$

and

$$\begin{cases} iu_t + \partial_x^2 u = f(u),\ x \in \mathbb{R},\ t \neq 0, \\ u(0,x) = \varphi(x),\ x \in \mathbb{R}, \end{cases} \tag{4.2}$$

respectively.

The linear equation (4.1) is solved by $u(x,t) = S(t)\varphi$, where $S(t) = e^{it\Delta}$ is the free Schrödinger operator and has two important properties. First, the conservation of the L^2-norm

$$\|u(t)\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}$$

which shows that it is in fact a group of isometries in $L^2(\mathbb{R})$, and a dispersive estimate of the form:

$$|S(t)\varphi(x)| = |u(t,x)| \leq \frac{1}{(4\pi|t|)^{1/2}}\|\varphi\|_{L^1(\mathbb{R})},\ x \in \mathbb{R},\ t \neq 0.$$

The space-time estimate

$$\|S(\cdot)\varphi\|_{L^6(\mathbb{R},\, L^6(\mathbb{R}))} \leq C\|\varphi\|_{L^2(\mathbb{R})}, \tag{4.3}$$

due to Strichartz [102], guarantees that the solutions decay as t becomes large and that they gain some spatial integrability. Inequality (4.3) was generalized by Ginibre and Velo [49]. They proved:

$$\|S(\cdot)\varphi\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}))} \leq C(q)\|\varphi\|_{L^2(\mathbb{R})} \tag{4.4}$$

for the so-called 1/2-admissible pairs (q,r). We recall that the exponent pair (q,r) is α-admissible (cf. [72]) if $2 \leq q, r \leq \infty$, $(q,r,\alpha) \neq (2,\infty,1)$ and

$$\frac{1}{q} = \alpha\left(\frac{1}{2} - \frac{1}{r}\right). \tag{4.5}$$

We see that (4.3) is a particular instance of (4.4) in which $\alpha = 1/2$ and $q = r = 6$.

The extension of these estimates to the inhomogeneous linear Schrödinger equation is due to Yajima [111] and Cazenave and Weissler [30]. These estimates can also be extended to a larger class of equations for which the Laplacian is replaced by any self-adjoint operator such that the L^∞-norm of the fundamental solution behaves like $t^{-1/2}$, [72].

The Strichartz estimates play an important role in the proof of the well-posedness of the nonlinear Schrödinger equation. Typically they are used for nonlinearities for which the energy methods fail to provide well-posedness results. In this way, Tsutsumi [106] proved the existence and uniqueness for $L^2(\mathbb{R})$-initial data for power-like nonlinearities $F(u) = |u|^p u$, in the range of exponents $0 \leq p \leq 4$. More precisely it was proved that the NSE is globally well posed in $L^\infty(\mathbb{R},\, L^2(\mathbb{R})) \cap L^q_{loc}(\mathbb{R},\, L^r(\mathbb{R}))$, where (q, r) is a 1/2-admissible pair depending on the exponent p. This result was complemented by Cazenave and Weissler [31] who proved the local existence in the critical case $p = 4$. The case of H^1-solutions was analyzed by Baillon, Cazenave and Figueira [6], Lin and Strauss [86], Ginibre and Velo [47, 48], Cazenave [27], and, in a more general context, by Kato [70, 71].

This analysis has been extended to semi-discrete numerical schemes for Schrödinger equations by Ignat and Zuazua in [63], [64], [65]. In these articles it was first pointed out that conservative numerical schemes often fail to be dispersive, in the sense that numerical solutions do not fulfill the integrability properties above. This is due to the pathological behavior of high frequency spurious numerical solutions. Then several numerical schemes were developed fulfilling the dispersive properties, uniformly in the mesh-parameter. In the sequel these schemes will be referred to as being *dispersive*. As proved in those articles these schemes may be used in the nonlinear context to prove convergence towards the solutions of the NSE, for the range of exponents p and the functional setting above. The analysis of fully discrete schemes was later developed in [53] where necessary and sufficient conditions were given guaranteeing that the dispersive properties of the continuous model are maintained uniformly with respect to the mesh-size parameters at the discrete level. The present paper is devoted to further analyze the convergence of these numerical schemes, the main goal being the obtention of convergence rates.

Despite of the fact that non-dispersive schemes (in the sense that they do not satisfy the discrete analogue of (4.3)) can not be applied directly in the L^2-setting for nonlinear equations one could still use them by first approximating the L^2-initial data by smooth ones. The paper [66] is devoted to prove that, even if this is done, dispersive schemes are better behaved than the non-dispersive ones in what concerns the order of convergence for rough initial data.

The main results of the chapter are as follows. Theorem 4.4 proves that the error committed when the LSE is approximated by a dispersive numerical scheme in the $L^q(0, T;\, l^r(h\mathbb{Z}))$-norms is of the same order as the one classical consistency+stability analysis yields. Using the ideas of [17], Ch. 6 we can also estimate the error in the $L^q(0, T;\, l^r(h\mathbb{Z}))$-norms, $r > 2$, for non-dispersive schemes; for example for the classical three-point second order approximation of the laplace operator. In this case, in contrast with the good properties of dispersive schemes, for $H^s(\mathbb{R})$-initial data with small s, $1/2 - 1/r \leq s \leq 4 + 1/2 - 1/r$, the error losses a factor of order $h^{3/2(1/2-1/r)}$ with respect to the case $L^\infty(0, T;\, l^2(h\mathbb{Z}))$ which can be handled by classical energy methods (see Example 1 in Section 4.3). Summarizing, we see that dispersive property of numerical schemes is needed to guarantee that the convergence rate of numerical solution is kept in the spaces $L^q(0, T;\, l^r(h\mathbb{Z}))$.

In the the context of the NSE we prove that the dispersive methods introduced in [66] converge to the solutions of NSE with the same order as in the linear problem. To be more precise, in Theorem 4.10 we obtain a polynomial order of convergence, $h^{s/2}$, in the case of a dispersive approximation scheme of order two for the laplace operator for initial data $H^s(\mathbb{R}^d)$ when $0 < s < 4$. In the case of the classical non-dispersive schemes this convergence rate can only be guaranteed for smooth enough initial data, $H^s(\mathbb{R})$, $1/2 < s < 4$ (see Theorem 4.12).

In Section 4.6 we show that non-dispersive numerical schemes with rough data behaves badly. Indeed, when using non-dispersive numerical schemes, combined with a $H^1(\mathbb{R})$-approximation of the initial data $\varphi \in H^s(\mathbb{R}) \backslash H^1(\mathbb{R})$, one gets an order of convergence $|\log h|^{-s/(1-s)}$ which is much weeker than the $h^{s/2}$-one that dispersive schemes ensure.

This Chapter is organized as follows. In Section 4.2 we first obtain a quite general result which allows us to estimate the difference of two families of operators that admit Strichartz estimates. We then particularize it to operators acting on discrete spaces $l^p(h\mathbb{Z})$, obtaining results which will be used in the following sections to get the order of convergence for approximations of the NSE. In Section 4.3 and Section 4.4 we revisit the dispersive schemes for LSE introduced in [62, 63, 64, 65] which are based, respectively, on the use of artificial numerical viscosity and a two-grid preconditioning technique of the initial data.

Section 4.5 is devoted to analyze approximations of the NSE based on the dispersive schemes analyzed in previous sections. Section 4.6 contains classical material on conservative schemes that we include here in order to emphasize the advantages of the dispersive methods.

The analysis in this paper can be extended to fully discrete dispersive schemes introduced and analyzed in [53] and to the multidimensional case. However, several technical aspects need to be dealt with carefully. In particular, one has to take care of the well-posedness of the NSE (see [28]). Furthermore, suitable versions of the technical harmonic analysis results employed in the paper would also be needed (see [52]). This will be the object of future work.

Our methods use Fourier analysis techniques in an essential manner. Adapting this theory to numerical approximation schemes in non-regular meshes is by now a completely open subject.

4.1 Spaces and Notations.

In this section we introduce the spaces we will use along the paper. The computational mesh is $h\mathbb{Z} = \{jh : j \in \mathbb{Z}\}$ for some $h > 0$ and the $l^p(h\mathbb{Z})$ spaces are defined as follows:

$$l^p(h\mathbb{Z}) = \{\varphi : h\mathbb{Z} \to \mathbb{C} : \|\varphi\|_{l^p(h\mathbb{Z})} < \infty\}$$

where

$$\|\varphi\|_{l^p(h\mathbb{Z})} = \begin{cases} \Big(h \sum_{j\in\mathbb{Z}} |u(jh)|^p\Big)^{1/p} & 1 \le p < \infty, \\ \sup_{j\in\mathbb{Z}} |u(jh)| & p = \infty. \end{cases}$$

On the Hilbert space $l^2(h\mathbb{Z})$ we will consider the following scalar product

$$(u, v)_h = \mathrm{Re}\Big(h \sum_{j\in\mathbb{Z}} u(jh)\overline{v(jh)}\Big).$$

When necessary, to simplify the presentation, we will write $(\varphi_j)_{j\in\mathbb{Z}}$ instead of $(\varphi(jh))_{j\in\mathbb{Z}}$.

For a discrete function $\{\varphi(jh)\}_{j\in\mathbb{Z}}$ we denote by $\widehat{\varphi}$ its discrete Fourier transform:

$$\widehat{\varphi}(\xi) = h \sum_{j\in\mathbb{Z}} e^{-ij\xi h}\varphi(jh). \tag{4.6}$$

We will also use the Besov spaces both in the continuous and the discrete framework. It is convenient to consider a function $\eta_0 \in C_c(\mathbb{R})$ such that

$$\eta_0(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \le 1, \\ 0 & \text{if} \quad |\xi| \ge 2, \end{cases}$$

and to define the sequence $(\eta_j)_{j\ge1} \in \mathcal{S}(\mathbb{R})$ by

$$\eta_j = \eta_0\Big(\frac{\xi}{2^j}\Big) - \eta_0\Big(\frac{\xi}{2^{j-1}}\Big)$$

in order to define the Littlewood-Paley decomposition. For any $j \ge 0$ we set the cut-off projectors, $P_j\varphi$, as follows

$$P_j\varphi = (\eta_j\widehat{\varphi})^\vee. \tag{4.7}$$

We point out that these projectors can be defined both for functions of continuous and discrete variables by means of the classical and the semi-discrete Fourier transform.

We introduce the Besov spaces $B^s_{p,2}(\mathbb{R})$ for $1 \leq p \leq \infty$ by $B^s_{p,2} = \{u \in \mathcal{S}'(\mathbb{R}) : \|u\|_{B^s_{p,2}(\mathbb{R})} < \infty\}$ with

$$\|u\|_{B^s_{p,2}(\mathbb{R})} = \|P_0 u\|_{L^p(\mathbb{R})} + \Big(\sum_{j=1}^{\infty} 2^{2sj} \|P_j u\|^2_{L^p(\mathbb{R})}\Big)^{1/2}.$$

Their discrete counterpart $B^s_{p,2}(h\mathbb{Z})$ with $1 < p < \infty$ and $s \in \mathbb{R}$ is given by

$$B^s_{p,2}(h\mathbb{Z}) = \{u : \|u\|_{B^s_{p,2}(h\mathbb{Z})} < \infty\},$$

with

$$\|u\|_{B^s_{p,2}(h\mathbb{Z})} = \|P_0 u\|_{l^p(h\mathbb{Z})} + \Big(\sum_{j=1}^{\infty} 2^{2js} \|P_j u\|^2_{l^p(h\mathbb{Z})}\Big)^{1/2}, \tag{4.8}$$

where $P_j u$ given as in (4.7) are now defined by means of the discrete Fourier transform of the discrete function $u : h\mathbb{Z} \to \mathbb{C}$.

4.2 Estimates on linear semigroups

In this section we will obtain $L^q_t L^r_x$ estimates for the difference of two semigroups $S_A(t)$ and $S_B(t)$ which admit Strichartz estimates. Once this result is obtained in an abstract setting we particularize it to the discrete spaces $l^p(h\mathbb{Z})$.

First we state a well-known result by Keel and Tao [72].

Theorem 4.1. *([72], Theorem 1.2) Let H be a Hilbert space, (X, dx) be a measure space and $U(t) : H \to L^2(X)$ be a one parameter family of mappings with $t \in \mathbb{R}$, which obey the energy estimate*

$$\|U(t)f\|_{L^2(X)} \leq C\|f\|_H \tag{4.9}$$

and the decay estimate

$$\|U(t)U(s)^* g\|_{L^\infty(X)} \leq C|t-s|^{-\alpha}\|g\|_{L^1(X)} \tag{4.10}$$

for some $\alpha > 0$. Then

$$\|U(t)f\|_{L^q(\mathbb{R},\, L^r(X))} \leq C\|f\|_H, \tag{4.11}$$

$$\Big\| \int_{\mathbb{R}} (U(s))^* F(s,\cdot)ds \Big\|_H \leq C\|F\|_{L^{q'}(\mathbb{R},\, L^{r'}(X))}, \tag{4.12}$$

$$\Big\| \int_0^t U(t-s)F(s)ds \Big\|_{L^q(\mathbb{R},\, L^r(X))} \leq C\|F\|_{L^{\tilde{q}'}(\mathbb{R},\, L^{\tilde{r}'}(X))} \tag{4.13}$$

for all (q, r) and $(\tilde{q}, \tilde{r})$, α-admissible pairs.

The following theorem provides the key estimate in obtaining the order of convergence when the LSE is approximated by a dispersive scheme.

Theorem 4.2. *([66]) Let (X, dx) be a measure space, $A : D(A) \to L^2(X)$, $B : D(B) \to L^2(X)$ two linear m-dissipative operators with $D(A) \hookrightarrow D(B)$ continuously and satisfying $AB = BA$. Assume that $(S_A(t))_{t\geq 0}$ and $(S_B(t))_{t\geq 0}$ the semigroups generated by A and B satisfy assumptions (4.9) and (4.10) with $H = L^2(X)$. Then for any two α-admissible pairs (q, r), $(\tilde{q}, \tilde{r})$ the following hold:*
i) There exists a positive constant $C(q)$ such that

$$\|S_A(t)\varphi - S_B(t)\varphi\|_{L^q(I,\, L^r(X))} \leq C(q) \min\Big\{\|\varphi\|_{L^2(X)}, |I|\|(A-B)\varphi\|_{L^2(X)}\Big\} \tag{4.14}$$

for all bounded intervals I and $\varphi \in D(A)$.
ii) There exists a positive constant $C(q,\tilde{q})$ such that

$$\Big\| \int_0^t S_A(t-s)f(s)ds - \int_0^t S_B(t-s)f(s)ds \Big\|_{L^q(I,\, L^r(X))} \tag{4.15}$$

$$\leq C(q,\tilde{q}) \min\Big\{ \|f\|_{L^{\tilde{q}'}(I,\, L^{\tilde{r}'}(X))}, |I| \|(A-B)f\|_{L^{\tilde{q}'}(I,\, L^{\tilde{r}'}(X))} \Big\}$$

for all bounded intervals I and $f \in L^{\tilde{q}'}(I,\, L^{\tilde{r}'}(X))$ such that $(A-B)f \in L^{\tilde{q}'}(I,\, L^{\tilde{r}'}(X))$.

Proof of Theorem 4.2. Using that the operators S_A and S_B verify hypotheses (4.9) and (4.10) of Proposition 4.1 with $H = L^2(X)$, by (4.11) we obtain

$$\|S_A(t)\varphi - S_B(t)\varphi\|_{L^q(I,\, L^r(X))} \leq C(q)\|\varphi\|_{L^2(X)} \tag{4.16}$$

and, by (4.13),

$$\Big\| \int_0^t S_A(t-s)f(s)ds - \int_0^t S_B(t-s)f(s)ds \Big\|_{L^q(\mathbb{R},\, L^r(X))} \leq C(q,\tilde{q})\|f\|_{L^{\tilde{q}'}(\mathbb{R},\, L^{\tilde{r}'}(X))}. \tag{4.17}$$

In view of (4.16) and (4.17) it is then sufficient to prove the following estimates:

$$\|S_A(t)\varphi - S_B(t)\varphi\|_{L^q(I,\, L^r(X))} \leq C(q)|I|\|(A-B)\varphi\|_{L^2(X)} \tag{4.18}$$

and

$$\Big\| \int_0^t S_A(t-s)f(s)ds - \int_0^t S_B(t-s)f(s)ds \Big\|_{L^q(I,\, L^r(X))} \leq C(q,\tilde{q})|I|\|(A-B)f\|_{L^{\tilde{q}'}(I,\, L^{\tilde{r}'}(X))}. \tag{4.19}$$

In the case of (4.18) we write the difference $S_A(\cdot) - S_B(\cdot)$ as follows

$$S_A(t)\varphi - S_B(t)\varphi = \int_0^t S_B(t-s)(A-B)S_A(s)\varphi ds. \tag{4.20}$$

In order to justify this identity let us recall that for any $\varphi \in D(A) \hookrightarrow D(B)$ we have that $u(t) = S_A(t)\varphi \in C([0,\infty), D(A)) \cap C^1([0,\infty), L^2(X))$ and $v(t) = S_B(t)\varphi \in C([0,\infty), D(B)) \cap C^1([0,\infty), L^2(X))$ verify the systems $u_t = Au, u(0) = \varphi$, and $v_t = Bv, v(0) = \varphi$ respectively. Thus $w = u - v \in C([0,\infty), D(B)) \cap C^1([0,\infty), L^2(X))$ satisfy the system $w_t = Bw + (A-B)u, w(0) = 0$. Since $(A-B)u \in C([0,\infty), L^2(X))$ we obtain that w satisfies (4.20).

Going back to (4.20) and using that A and B commute we get the following identity which is the key of our estimates:

$$S_A(t)\varphi - S_B(t)\varphi = \int_0^t S_B(t-s)S_A(s)(A-B)\varphi ds. \tag{4.21}$$

We apply Theorem 4.1 to the semigroup $S_B(\cdot)$ and to function $F(s) = S_A(s)(A-B)\varphi$ in this identity. By (4.13) with $\tilde{r} = 2$ and $\tilde{q} = \infty$, we get

$$\begin{aligned} \|S_A(t)\varphi - S_B(t)\varphi\|_{L^q(I,\, L^r(X))} &\leq C(q)\|S_A(s)(A-B)\varphi\|_{L^1(I,\, L^2(X))} \\ &\leq C(q)|I|\|(A-B)\varphi\|_{L^2(X)}. \end{aligned} \tag{4.22}$$

Thus, (4.18) is proved. As a consequence (4.16) and (4.18) give us (4.14).

We now prove the inhomogenous estimate (4.19). Using again (4.21) we have

$$S_A(t-s)f(s) - S_B(t-s)f(s) = \int_0^{t-s} S_B(t-s-\sigma)S_A(\sigma)(A-B)f(s)d\sigma.$$

We integrate this identity in the s variable. Applying Fubini's theorem on the triangle $\{(s,\sigma):0\le s\le t, 0\le\sigma\le t-s\}$ and using that A and B commute, we get:

$$\Lambda f(t):=\int_0^t S_A(t-s)f(s)ds-\int_0^t S_B(t-s)f(s)=\int_0^t S_A(t-\sigma)\Lambda_1(A-B)f(\sigma)d\sigma$$

where

$$\Lambda_1 g(t)=\int_0^t S_B(t-\tau)g(\tau)d\tau.$$

Applying the inhomogeneous estimate (4.13) to the operator $S_A(\cdot)$ with $(\tilde q',\tilde r')=(1,2)$ we obtain

$$\|\Lambda f\|_{L^q(I,\,L^r(X))}\le C(q)\|\Lambda_1(A-B)f\|_{L^1(I,\,L^2(X))}\le C(q)|I|\|\Lambda_1(A-B)f\|_{L^\infty(I,\,L^2(X))}. \tag{4.23}$$

Using again (4.13) for the semigroup $S_B(\cdot)$, $F=(A-B)f$ and $(q,r)=(\infty,2)$ we get

$$\|\Lambda_1(A-B)f\|_{L^\infty(I,\,L^2(X))}\le C(\tilde q)\|(A-B)f\|_{L^{\tilde q'}(I,\,L^{\tilde r'}(X))}. \tag{4.24}$$

Combining (4.23) and (4.24) we deduce (4.19). Estimates (4.17) and (4.19) finish the proof. □

Remark 4.1. *We point out that, in the the proof of the following estimate*

$$\|S_A(t)\varphi-S_B(t)\varphi\|_{L^q(I,\,l^r(X))}\le C(q)|I|\|(A-B)\varphi\|_{L^2(X)},$$

we do not need that the two operators $S_A(t)$ and $S_B(t)$ admit Strichartz estimates. Indeed, it is sufficient to assume that only one of the involved operators admits Strichartz estimates and the other one to be stable in $L^2(X)$.

In the following we apply the previous results to the particular case $X=h\mathbb{Z}$. We consider operators $A_h$with symbol $a_h:[-\pi/h,\pi/h]\to\mathbb{C}$ such that

$$(A_h\varphi)_j=\int_{-\pi/h}^{\pi/h}e^{ij\xi h}a_h(\xi)\widehat{\varphi}(\xi)d\xi,\ j\in\mathbb{Z}.$$

Also we will consider the operator $|\nabla|^s$ acting on discrete spaces $l^2(h\mathbb{Z})$ whose symbol is given by $|\xi|^s$.

The numerical schemes we shall consider, associated to regular meshes, will enter in this frame by means of the Fourier representation formula of solutions.

Theorem 4.3. *Let $A_h,B_h:l^2(h\mathbb{Z})\to l^2(h\mathbb{Z})$ be two operators whose symbols are a_h and b_h, ib_h being a real function, such that the semigroups they generate, $(S_{A_h}(t))_{t\ge0}$ and $(S_{B_h}(t))_{t\ge0}$, satisfy assumptions (4.9) and (4.10) with some constant C, independent of h. Finally, assume that for some functions $\{\mu(k,h)\}_{k\in F}$, with F a finite set, the following holds for all $\xi\in[-\pi/h,\pi/h]$:*

$$|a_h(\xi)-b_h(\xi)|\le\sum_{k\in F}\mu(k,h)|\xi|^k. \tag{4.25}$$

For any $s>0$, denoting

$$\varepsilon(s,h)=\sum_{k\in F}\mu(k,h)^{\min\{s/k,1\}}, \tag{4.26}$$

the following hold for all (q,r), $(\tilde q,\tilde r)$, α-admissible pairs:
a) There exists a positive constant $C(q)$ such that

$$\|S_{A_h}(t)\varphi-S_{B_h}(t)\varphi\|_{L^q(I,\,l^r(h\mathbb{Z}))}\le C(q)\varepsilon(s,h)\max\{1,|I|\}\|\varphi\|_{B^s_{2,2}(h\mathbb{Z})} \tag{4.27}$$

holds for all $\varphi \in B^s_{2,2}(h\mathbb{Z})$ *uniformly in* $h > 0$.
b) There exists a positive constant $C(s, q, \tilde{q})$ *such that*

$$\Big\| \int_0^t S_{A_h}(t-\sigma)f(\sigma)d\sigma - \int_0^t S_{B_h}(t-\sigma)f(\sigma)d\sigma \Big\|_{L^q(I,\, l^r(h\mathbb{Z}))} \leq C(s,q,\tilde{q})\varepsilon(s,h)\max\{1,|I|\}\|f\|_{L^{\tilde{q}'}(I,\, B^s_{\tilde{r}',2}(h\mathbb{Z}))} \tag{4.28}$$

holds for all $f \in L^{\tilde{q}'}(I,\, B^s_{\tilde{r}',2}(h\mathbb{Z}))$.

Few comments on the above result are needed.

The assumption that the semigroups $(S_{A_h}(t))_{t\geq 0}$ and $(S_{B_h}(t))_{t\geq 0}$, satisfy (4.9) and (4.10) with some constant C, independent of h, means that both of them are $l^2(h\mathbb{Z})$-stable with constants that are independent of h and that the corresponding numerical schemes are dispersive.

Taking into account that both operators, A_h and B_h, commute in view that they are associated to their symbols, the hypotheses of Theorem 4.2 are fulfilled. They also commute with $|\nabla|$ and P_j which are also defined by a Fourier symbol.

The requirement that ib_h is a real function is needed to assure that the semigroup generated by B_h, S_{B_h}, satisfies

$$S_{B_h}(t-\sigma) = S_{B_h}(t)S_{B_h}(-\sigma) = S_{B_h}(t)S_{B_h}(\sigma)^*,$$

identity which will be used in the proof.

In Section 4.3 we will give examples of operators A_h and B_h verifying these hypotheses. In all our estimates we will choose $b_h(\xi) = i\xi^2$, which is the symbol of the continuous Schrödinger semigroup.

4.3 Dispersive schemes for the linear Schrödinger equation

In this section we obtain error estimates for the numerical approximations of the linear Schrödinger equation. We do this not only in the $l^2(h\mathbb{Z})$-norm but also in the auxiliary spaces that are needed in the analysis of the nonlinear Schrödinger equation.

The numerical schemes we shall consider can all be written in the abstract form

$$\begin{cases} iu^h_t(t) + A_h u^h = 0, \quad t > 0, \\ \quad u^h(0) = \mathbf{T}_h\varphi. \end{cases} \tag{4.29}$$

We assume that the operator A_h is an approximation of the $1-d$ Laplacian. On the other hand, $\mathbf{T}_h\varphi$ is an approximation of the initial data φ, $\mathbf{T}_h$ being a map from $L^2(\mathbb{R})$ into $l^2(h\mathbb{Z})$ defined as follows:

$$(\mathbf{T}_h\varphi)(jh) = \int_{-\pi/h}^{\pi/h} e^{ijh\xi}\widehat{\varphi}(\xi)d\xi. \tag{4.30}$$

Observe that this operator acts by truncating the continuous Fourier transform of φ on the interval $(-\pi/h, \pi/h)$ and then considering the discrete inverse Fourier transform on the grid points $h\mathbb{Z}$.

To estimate the error committed in the approximation of the LSE we assume that the operator A_h, approximating the continuous Laplacian, has a symbol a_h which satisfies

$$|a_h(\xi) - \xi^2| \leq \sum_{k\in F} a(k,h)|\xi|^k,\ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right], \tag{4.31}$$

for a finite set of indexes F. As we shall see, different approximation schemes enter in this class for different sets F and orders k.

This condition on the operator A_h suffices to analyze the rate of convergence in the $L^\infty(-T, T;\, l^2(h\mathbb{Z}))$ norm. However, one of our main objectives in this paper is to analyze this error in the auxiliary norms $L^q(-T,T;\, l^r(h\mathbb{Z}))$ which is necessary for addressing the NSE with rough initial data. More precisely, we need to identify classes of approximating operators A_h of the $1-d$ Laplacian so that the semi-discrete semigroup $\exp(itA_h)$ maps uniformly, with respect to parameter h, $l^2(h\mathbb{Z})$ into those spaces.

In the following we consider operators A_h generating dispersive schemes which are $l^2(h\mathbb{Z})$-stable

$$\|\exp(itA_h)\varphi\|_{l^2(h\mathbb{Z})} \leq C\|\varphi\|_{l^2(h\mathbb{Z})},\ \forall\, t \geq 0 \tag{4.32}$$

and satisfy the uniform $l^1(h\mathbb{Z}) - l^\infty(h\mathbb{Z})$ dispersive property:

$$\|\exp(itA_h)\varphi\|_{l^\infty(h\mathbb{Z})} \leq \frac{C}{|t|^{1/2}}\|\varphi\|_{l^1(h\mathbb{Z})},\ \forall\, t \geq 0, \tag{4.33}$$

for all $h > 0$ and for all $\varphi \in l^1(h\mathbb{Z})$, where the above constant C is independent of h. We point out that (4.32) is the standard stability property while the second one, (4.33), holds only for well chosen numerical schemes.

Applying Theorem 4.3 to the operator B_h whose symbol is $-i\xi^2$ and to iA_h, A_h being the approximation of the Laplace operator with the symbol $a_h(\xi)$, we obtain the following result.

Theorem 4.4. *Let $s \geq 0$, A_h satisfying (4.31), (4.32), (4.33), and (q,r) and $(\tilde{q},\tilde{r})$ be two $1/2$-admissible pairs. Denoting*

$$\varepsilon(s,h) = \sum_{k\in F} a(k,h)^{\min\{s/k,1\}}, \tag{4.34}$$

the following hold:

a) There exists a positive constant $C(q)$ such that

$$\|\exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h\exp(it\partial_x^2)\varphi\|_{L^q(0,T;\, l^r(h\mathbb{Z}))} \leq \max\{1,T\}C(q)\varepsilon(s,h)\|\varphi\|_{H^s(\mathbb{R})} \tag{4.35}$$

holds for all $\varphi \in H^s(\mathbb{R})$, $T > 0$ and $h > 0$.

b) There exists a positive constant $C(q,\tilde{q})$ such that

$$\begin{aligned}\Big\|\int_0^t \exp(i(t-\sigma)A_h)\mathbf{T}_h f(\sigma)d\sigma - \int_0^t \mathbf{T}_h\exp(i(t-\sigma)\partial_x^2)f(\sigma)d\sigma\Big\|_{L^q(0,T;\, l^r(h\mathbb{Z}))} \\ \leq C(q,\tilde{q})\max\{1,T\}\varepsilon(s,h)\|f\|_{L^{\tilde{q}'}(0,T;\, B^s_{\tilde{r}',2}(\mathbb{R}))},\end{aligned} \tag{4.36}$$

holds for all $T > 0$, $f \in L^{\tilde{q}'}(0,T;\, B^s_{\tilde{r}',2}(\mathbb{R}))$ and $h > 0$.

In the particular case when $(q,r) = (\infty,2)$ and the set F of indices k entering in the definition (4.34) of $\varepsilon(s,h)$ is reduced to a simple element, the statements in this Theorem are proved in [103] (Theorem 10.1.2, p. 201):

$$\|\exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h\exp(it\partial_x^2)\varphi\|_{L^\infty(0,T;\, l^2(h\mathbb{Z}))} \leq C(q)T\varepsilon(s,h)\|\varphi\|_{H^s(\mathbb{R})}. \tag{4.37}$$

We observe that for $s \geq s_0 = \max\{k :\ k \in F\}$ the function $s \to \varepsilon(s,h)$ is independent of the s-variable:

$$\varepsilon(s,k) = \varepsilon(s_0,k) = \sum_{k\in F} a(k,h).$$

This means that imposing more than $H^{s_0}(\mathbb{R})$ regularity on the initial data does not improve the order of convergence in (4.35) and (4.36).

In the case $0 \leq s \leq s_0$, with s_0 as above, the estimate $H^{s_0}(\mathbb{R}) \to L^\infty(0,T;\, l^2(h\mathbb{Z}))$ in (4.35) and the one given by the stability of the scheme $L^2(\mathbb{R}) \to L^\infty(0,T;\, l^2(h\mathbb{Z}))$, allow to obtain, using an interpolation argument, a weaker estimate:

$$\|\exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h\exp(it\partial_x^2)\varphi\|_{L^\infty(0,T;\, l^2(h\mathbb{Z}))} \leq C(T)\varepsilon(s_0,h)^{s/s_0}\|\varphi\|_{H^s(\mathbb{R})}.$$

If the set F has an unique element then this estimate is equivalent to (4.35). However, the improved estimates (4.35) and (4.36) cannot be proved without using Paley-Littlewood's decomposition, as in the proof of Theorem 4.3.

In the following we analyze various operators A_h which approximate the $1-d$ Laplace operator ∂_x^2.

Example 1. The 3-point conservative approximation. The simplest example of approximation scheme for the Laplace operator ∂_x^2 is given by the classical finite difference approximation Δ_h

$$(\Delta_h u)_j = \frac{u_{j+1} + u_{j-1} - 2u_j}{h^2}. \tag{4.38}$$

It satisfies hypothesis (4.31) with $F = \{4\}$ and $a(4, h) = h^2$. Thus, we are dealing with an approximation scheme of order two. Indeed, we have:

$$\Big|\frac{4}{h^2}\sin^2\Big(\frac{\xi h}{2}\Big) - \xi^2\Big| \lesssim h^2|\xi|^4, \,\forall\, \xi \in \Big[-\frac{\pi}{h}, \frac{\pi}{h}\Big].$$

However, this operator does not satisfy (4.33) with a constant C independent of the mesh size h, (see [63], Theorem 1.1) and Theorem 4.4 cannot be applied. This means that we cannot obtain the same estimate as for second order dispersive schemes:

$$\|\exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h\exp(it\partial_x^2)\varphi\|_{L^q(0,T;\, l^r(h\mathbb{Z}))} \le C(q,T)\|\varphi\|_{H^s(\mathbb{R})}\begin{cases} h^{s/2}, & s\in(0,4),\\ h^2, & s>4.\end{cases} \tag{4.39}$$

However, using the ideas of Brenner on the order of convergence in the $l^r(h\mathbb{Z})$-norm, $r > 2$, ([17], Ch. 6, Theorem 3.2, Theorem 3.3 and Ch.3, Corollary 5.1) we can get the following estimates:

$$\begin{aligned}
&\|\exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h\exp(it\partial_x^2)\varphi\|_{L^q(0,T;\, l^r(h\mathbb{Z}))}\\
&\qquad\le C(q,T)\|\varphi\|_{B^s_{r,\infty}(\mathbb{R})}\begin{cases} h^{\frac{1}{2}(s-1+\frac{2}{r})}, & s\in(0,4+1-\frac{2}{r}),\\ h^2, & s\ge 4+1-\frac{2}{r},\end{cases}\\
&\qquad\le C(q,T)\|\varphi\|_{H^{s+\frac12-\frac1r}(\mathbb{R})}\begin{cases} h^{\frac{1}{2}(s-1+\frac{2}{r})}, & s\in(0,4+1-\frac{2}{r}),\\ h^2, & s\ge 4+1-\frac{2}{r},\end{cases}
\end{aligned}$$

where we have used that $H^{s_0}(\mathbb{R}) = B^{s_0}_{2,2}(\mathbb{R}) \hookrightarrow B^s_{r,\infty}(\mathbb{R})$ when $s_0 - 1/2 = s - 1/r$.

Observe that in the case $s \in (0, 4)$ the above estimate guarantees that

$$\begin{aligned}
&\|\exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h\exp(it\partial_x^2)\varphi\|_{L^q(0,T;\, l^r(h\mathbb{Z}))}\\
&\qquad\le C(q,T)\|\varphi\|_{H^{s+\frac12-\frac1r}(\mathbb{R})} h^{\frac12(s+\frac12-\frac1r)} h^{-\frac32(\frac12-\frac1r)}.
\end{aligned} \tag{4.40}$$

Moreover for any $\sigma \in (1/2 - 1/r, 4 + 1/2 - 1/r)$ we can find $s \in (0, 4)$ with $\sigma = s + 1/2 - 1/r$ and using (4.40) we obtain

$$\|\exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h\exp(it\partial_x^2)\varphi\|_{L^q(0,T;\, l^r(h\mathbb{Z}))} \le C(q,T)\|\varphi\|_{H^\sigma(\mathbb{R})} h^{\frac{\sigma}{2}} h^{-\frac32(\frac12-\frac1r)}. \tag{4.41}$$

In the case of an approximation of order two one could expect the error in the above estimate to be of order $h^{\sigma/2}$ as in the $L^\infty(0,T;\, l^2(h\mathbb{Z}))$ case. But, here we get an extra factor of order $h^{-3/2(1/2-1/r)}$ which diverges unless $r = 2$, which corresponds to the classical energy estimate in $L^\infty(0,T;\, L^2(\mathbb{R}))$. This does not happen in the case of a second order dispersive approximation of the Schrödinger operator, where Theorem 4.4 give us an order of error as in (4.39).

Note that, according to Theorem 4.4, this loss in the rate of convergence is due to the lack of dispersive properties of the scheme.

Also we point out that to obtain an error of order h^2 in (4.40) we need to consider initial data in $H^{4+1-2/r}(\mathbb{R})$. So we need to impose an extra regularity condition of $1-2/r$ derivatives on the initial data φ to assure the same order of convergence as the one in (4.39) for dispersive schemes.

Example 2. Fourier filtering of the 3-point conservative approximation. Another example is given by the spectral filtering $\Delta_{h,\gamma}$ defined by:

$$\Delta_{h,\gamma}\varphi = \Delta_h(\mathbf{1}_{(-\frac{\gamma\pi}{h},\frac{\gamma\pi}{h})}\widehat{\varphi})^\vee,\ \gamma < \frac{1}{2}. \tag{4.42}$$

In other words, $\Delta_{h,\gamma}$ is a discrete operator whose action is as follows:

$$(\Delta_{h,\gamma}\varphi)_j = \int_{-\gamma\pi/h}^{\gamma\pi/h} \frac{4}{h^2}\sin^2\big(\frac{\xi h}{2}\big)e^{ijh\xi}\widehat{\varphi}(\xi)d\xi,\ j\in\mathbb{Z},$$

i.e. it has the symbol

$$a_{h,\gamma}(\xi) = \frac{4}{h^2}\sin^2\big(\frac{\xi h}{2}\big)\mathbf{1}_{(-\gamma\pi/h,\gamma\pi/h)}.$$

In this case

$$|a_{h,\gamma}(\xi)-\xi^2| \leq c(\gamma)\begin{cases} h^2\xi^4, & |\xi|\leq \pi\gamma/h,\\ \xi^2, & |\xi|\geq \pi\gamma/h\end{cases} \leq c(\gamma)h^2\xi^4 \quad \text{for all} \quad \xi\in\Big[-\frac{\pi}{h},\frac{\pi}{h}\Big].$$

Thus $\Delta_{h,\gamma}$ constitutes an approximation of the Laplace operator Δ of order two and the semigroup generated by $i\Delta_{h,\gamma}$ has uniform dispersive properties (see [64]). Theorem 4.4, which exploits the dispersive character of the numerical scheme, gives us

$$\|\exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h\exp(it\Delta)\varphi\|_{L^q(0,T;\,l^r(h\mathbb{Z}))} \leq C(q,T)\|\varphi\|_{H^s(\mathbb{R})}\begin{cases} h^{s/2}, & s\in(0,4),\\ h^2, & s>4.\end{cases}$$

We note that using the same arguments based on $l^r(h\mathbb{Z})$-error estimates (given in [17]), as in the Example 1, we can obtain the same result only if $r=2$ or assuming more regularity of the initial data φ.

This scheme, however, has a serious drawback to be implemented in nonlinear problems since it requires the Fourier filtering to be applied on the initial data and also on the nonlinearity, which is computationally expensive.

Example 3. Viscous approximation. To overcome the lack of uniform $L^q(I,l^r(h\mathbb{Z}))$ estimates, in [64] and [54] numerical schemes based in adding extra numerical viscosity have been introduced. The first possibility is to take $A_h = \Delta_h + ia(h)\Delta_h$ with $a(h) = h^{2-1/\alpha(h)}$ and $\alpha(h)\to 1/2$ such that $a(h)\to 0$. In this case (4.31) is satisfied as follows:

$$\Big|\frac{4}{h^2}\sin^2\big(\frac{\xi h}{2}\big) + ia(h)\frac{4}{h^2}\sin^2\big(\frac{\xi h}{2}\big) - \xi^2\Big| \leq h^2\xi^4 + a(h)\xi^2. \tag{4.43}$$

This numerical approximation of the Schrödinger semigroup has been used in [64] and [65] to construct convergent numerical schemes for the NSE. However, the special choice of the function $a(h)$ that is required, shows that the error in the right hand side of (4.43) goes to zero slower that any polynomial function of h and thus, at least theoretically, the convergence towards LSE, and, consequently to the NSE, will be very slow. Thus, we will not further analyze this scheme.

Example 4. A higher order viscous approximation. A possibility to overcome the drawbacks of the previous scheme, associated to the different behavior of the $l^1(h\mathbb{Z})-l^\infty(h\mathbb{Z})$ decay rate of the solutions, is to choose higher order dissipative schemes as introduced in [54]:

$$A_h = \Delta_h - ih^{2(m-1)}(-\Delta_h)^m,\ m\geq 2. \tag{4.44}$$

In this case, hypothesis (4.31) reads:

$$\Big|\frac{4}{h^2}\sin^2\big(\frac{\xi h}{2}\big)+ih^{2(m-1)}\Big(\frac{4}{h^2}\sin^2\big(\frac{\xi h}{2}\big)\Big)^m-\xi^2\Big|\le h^2\xi^4+h^{2(m-1)}\xi^{2m}. \tag{4.45}$$

Theorem 4.4 then guarantees that for any $0\le s\le 4$ the following estimate holds:

$$\begin{aligned}\|\exp(itA_h)\mathbf{T}_h\varphi-\mathbf{T}_h\exp(it\Delta)\varphi\|_{L^q(0,T;\,l^r(h\mathbb{Z}))}&\le\max\{1,T\}(h^{s/2}+h^{(m-1)s/m})\|\varphi\|_{H^s(\mathbb{R})}\\&\le\max\{1,T\}h^{s/2}\|\varphi\|_{H^s(\mathbb{R})}.\end{aligned}$$

Thus we obtain the same order of error as for the discrete Laplacian $A_h=\Delta_h$ but this time not only in the $L^\infty(I;\,l^2(h\mathbb{Z}))$-norm but in all the auxiliary $L^q(I,\,l^r(h\mathbb{Z}))$-norms. We thus get the same optimal results as for the other dispersive scheme in Example 2 based on Fourier filtering.

4.4 A two-grid algorithm

In this section we analyze one further strategy introduced in [62] and [64] to recover the uniformity of the dispersive properties. It is based on the two-grid algorithm that we now describe. We consider the standard conservative 3-point approximation of the laplacian: $A_h=\Delta_h$. But, this time, in order to avoid the lack of dispersive properties associated with the high frequency components, the scheme will be restricted to the class of slowly oscillatory data obtained by a two-grid algorithm. The main advantage of this filtering method with respect to the Fourier one is that the filtering can be realized in the physical space.

The method, inspired by [50], is roughly as follows. We consider two meshes: the coarse one of size $4h$, $h>0$, $4h\mathbb{Z}$, and the finer one, the computational one, $h\mathbb{Z}$, of size $h>0$. The method relies basically on solving the finite-difference semi-discretization on the fine mesh $h\mathbb{Z}$, but only for slowly oscillating data, interpolated from the coarse grid $4h\mathbb{Z}$. The $1/4$ ratio between the two meshes is important to guarantee the dispersive properties of the method. This particular structure of the data cancels the pathology of the discrete symbol at the points $\pm\pi/2h$.

To be more precise we introduce the extension operator $\mathbf{\Pi}_h^{4h}$ which associates to any function $\psi:4h\mathbb{Z}\to\mathbb{C}$ a new function $\mathbf{\Pi}_h^{4h}\psi:h\mathbb{Z}\to\mathbb{C}$ obtained by an interpolation process:

$$(\mathbf{\Pi}_h^{4h}\psi)_j=(\mathbf{P}^1_{4h}\psi)(jh),\ j\in\mathbb{Z},$$

where $\mathbf{P}^1_{4h}\psi$ is the piecewise linear interpolator of ψ.

The semi-discrete method we propose is the following:

$$\begin{cases} iu^h_t(t)+\Delta_h u^h=0, & t>0,\\[2mm] u^h(0)=\mathbf{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi. \end{cases} \tag{4.46}$$

The Fourier transform of the two-grid initial datum can be characterized as follows (see Lemma 5.2, [64]):

$$(\mathbf{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi)^\wedge(\xi)=m(h\xi)\widetilde{\widehat{\mathbf{T}_{4h}\varphi}}(\xi),\ \xi\in\Big[-\frac{\pi}{h},\frac{\pi}{h}\Big], \tag{4.47}$$

where $\widetilde{\widehat{\mathbf{T}_{4h}\varphi}}(\xi)$ is the extension by periodicity of the function $\widehat{\mathbf{T}_{4h}\varphi}$, initially defined on $[-\pi/4h,\pi/4h]$, to the interval $[-\pi/h,\pi/h]$, and

$$m(\xi)=\left(\frac{e^{4i\xi}-1}{4(e^{i\xi}-1)}\right)^2,p\ge 2. \tag{4.48}$$

The following result, proved in [62], guarantees that system (4.46) is dispersive in the sense that the discrete version of the Strichartz inequalities hold, uniformly on $h>0$.

Theorem 4.5. *Let (q,r), $(\tilde q,\tilde r)$ be two $1/2$-admissible pairs. The following properties hold*
i) There exists a positive constant $C(q)$ such that

$$\|e^{it\Delta_h}\mathbf{\Pi}_h^{4h}\varphi\|_{L^q(\mathbb{R},\,l^r(h\mathbb{Z}))}\leq C(q)\|\mathbf{\Pi}_h^{4h}\varphi\|_{l^2(h\mathbb{Z})} \tag{4.49}$$

uniformly on $h>0$.
ii) There exists a positive constant $C(d,r,\tilde r)$ such that

$$\Big\|\int_{s<t}e^{i(t-s)\Delta_h}\mathbf{\Pi}_h^{4h}f(s)ds\Big\|_{L^q(\mathbb{R},\,l^r(h\mathbb{Z}))}\leq C(q,\tilde q)\|\mathbf{\Pi}_h^{4h}f\|_{L^{\tilde q'}(\mathbb{R},\,l^{\tilde r'}(h\mathbb{Z}))} \tag{4.50}$$

for all $f\in L^{\tilde q'}(\mathbb{R},\,l^{\tilde r'}(4h\mathbb{Z}))$, uniformly in $h>0$.

In the following lemma we estimate the error introduced by the two-grid algorithm.

Theorem 4.6. *Let $s\geq 0$ and (q,r), $(\tilde q,\tilde r)$ be two admissible pairs.*
a) There exists a positive constant $C(q,s)$ such that

$$\begin{aligned}\|\exp(it\Delta_h)\mathbf{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi-&\mathbf{T}_h\exp(it\partial_x^2)\varphi\|_{L^q(I;\,l^r(h\mathbb{Z}))} \\ &\leq C(q,s)\max\{1,|I|\}\big(h^{\min\{s/2,2\}}+h^{\min\{s,1\}}\big)\|\varphi\|_{H^s(\mathbb{R})},\end{aligned} \tag{4.51}$$

holds for all $\varphi\in H^s(\mathbb{R})$ and $h>0$.
b) There exists a positive constant $C(q,\tilde q,s)$ such that

$$\begin{aligned}\Big\|\int_{s<t}\exp(i(t-s)\Delta_h)\mathbf{\Pi}_h^{4h}\mathbf{T}_{4h}f(s)ds-\int_{s<t}\mathbf{T}_h\exp(i(t-s)\partial_x^2)f(s)ds\Big\|_{L^q(I;\,l^r(h\mathbb{Z}))} \\ \leq C(q,\tilde q,s)\max\{1,|I|\}\big(h^{\min\{s/2,2\}}+h^{\min\{s,1\}}\big)\|f\|_{L^{\tilde q'}(I;\,B^s_{\tilde r',2}(\mathbb{R}))}.\end{aligned} \tag{4.52}$$

There are two error terms in the above estimates: $h^{\min\{s/2,2\}}$ and $h^{\min\{s,1\}}$. The first one comes from a second order numerical scheme generated by the approximation of the laplacian ∂_x^2 with Δ_h and the second one from the use of a two-grid interpolator. Observe that for initial data $\varphi\in H^s(\mathbb{R})$, $s\in(0,2)$ the results are the same as in the case of the second order schemes. Also, imposing more than $H^2(\mathbb{R})$ regularity on the initial data does not improve the order of convergence. This is a consequence of the fact that the two-grid interpolator appears. The multiplier $m(\xi)$ defined in (4.48) satisfies $m(\xi)-1\simeq\xi$ as $\xi\sim 0$ and then the following estimate, which occurs in the proof of Theorem 4.6,

$$\int_{-\pi/4h}^{\pi/4h}|m(h\xi)-1|^2|\widehat{\varphi}(\xi)|^2d\xi\lesssim(h\|\varphi\|_{H^1(\mathbb{R})})^2,$$

cannot be improved by imposing more regularity on the function φ.

4.5 Convergence of the dispersive method for the NSE

In this section we introduce numerical schemes for the NSE based on dispersive approximations of the LSE. We first present some classical results on well-posedness and regularity of solutions of the NSE. Secondly we obtain the order of convergence for the approximations of the NSE described above.

We consider the NSE with nonlinearity $f(u)=|u|^pu$ and $\varphi\in H^s(\mathbb{R})$. We are interested in the case of $H^s(\mathbb{R})$ initial data with $s\leq 1$. The following well-posedness result is known.

Theorem 4.7. *Let $f(u)=|u|^pu$ with $p\in(0,4)$. Then*
i) (Global existence and uniqueness, [28], Th. 4.6.1, Ch. 4, p. 109)
For any $\varphi\in L^2(\mathbb{R})$, there exists a unique global solution u of (4.2) in the class

$$u\in C(\mathbb{R},L^2(\mathbb{R}))\cap L^q_{loc}(\mathbb{R},L^r(\mathbb{R}))$$

for all $1/2$*-admissible pairs* (q,r) *such that*

$$\|u(t)\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}, \forall\, t \in \mathbb{R}.$$

ii) (Stability, [28], Th. 4.6.1, Ch. 4, p. 109) Let φ *and* ψ *be two* $L^2(\mathbb{R})$ *functions, and* u *and* v *the corresponding solutions of the NSE. Then for any* $T>0$ *there exists a positive constant* $C(T, \|\varphi\|_{L^2(\mathbb{R})}, \|\psi\|_{L^2(\mathbb{R})})$ *such that the following holds*

$$\|u-v\|_{L^\infty(0,T;\,L^2(\mathbb{R}))} \leq C(T, \|\varphi\|_{L^2(\mathbb{R})}, \|\psi\|_{L^2(\mathbb{R})})\|\varphi-\psi\|_{L^2(\mathbb{R})} \tag{4.53}$$

iii) (Regularity) Moreover if $\varphi \in H^s(\mathbb{R})$, $s \in (0,1/2)$ *then ([28], Theorem 5.1.1, Ch. 5, p. 147)*

$$u \in C(\mathbb{R}, H^s(\mathbb{R})) \cap L^q_{loc}(\mathbb{R}, B^s_{r,2}(\mathbb{R}))$$

for every admissible pairs (q,r).

Also if $\varphi \in H^1(\mathbb{R})$ *then* $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ *([28], Theorem 5.2.1, Ch. 5, p. 149).*

Remark 4.2. *The embedding* $B^s_{r,2}(\mathbb{R}) \hookrightarrow W^{s,r}(\mathbb{R}), r \geq 2$, *(see [28], Remark 1.4.3, p. 14) guarantees that, in particular,* $u \in L^q_{loc}(\mathbb{R}, W^{s,r}(\mathbb{R}))$. *Moreover,* $f(u) \in L^{q'}_{loc}(\mathbb{R}, B^s_{r',2}(\mathbb{R}))$ *and for any* $0 < s \leq 1$ *(see [28], formula (4.9.20), p. 128)*

$$\|f(u)\|_{L^{q'}(I, B^s_{r',2}(\mathbb{R}))} \lesssim |I|^{\frac{4-p(1-2s)}{4}} \|u\|^{p+1}_{L^q(I, B^s_{r,2}(\mathbb{R}))}. \tag{4.54}$$

The fixed point argument used to prove the existence and uniqueness result in Theorem 4.7 gives us also quantitative information of the solutions of NSE in terms of the $L^2(\mathbb{R})$-norm of the initial data. The following holds:

Lemma 4.1. *Let* $\varphi \in L^2(\mathbb{R})$ *and* u *be the solution of the NSE with initial data* φ *and nonlinearity* $f(u) = |u|^p u$, $p \in (0,4)$, *as in Theorem 4.7. There exists* $c(p) > 0$ *and* $T_0 = c(p)\|\varphi\|^{-4p/(4-p)}_{L^2(\mathbb{R})}$ *such that for any* $1/2$*-admissible pairs* (q,r), *there exists a positive constant* $C(p,q)$ *such that*

$$\|u\|_{L^q(I;\,L^r(\mathbb{R}))} \leq C(p,q)\|\varphi\|_{L^2(\mathbb{R})} \tag{4.55}$$

holds for all intervals I *with* $|I| \leq T_0$.

Proof of Lemma 4.1. Let us fix an admissible pair (q,r). The fixed point argument used in the proof of Theorem 4.7 (see ([27], Th. 5.5.1, p. 15) gives us the existence of a time T_0,

$$T_0 = c(p)\|\varphi\|^{-\frac{4p}{4-p}}_{L^2(\mathbb{R})},$$

such that

$$\|u\|_{L^q(0,T_0;\,L^r(\mathbb{R}))} \leq C(p,q)\|\varphi\|_{L^2(\mathbb{R})}.$$

The same argument applied to the interval $[(k-1)T_0, kT_0]$, $k \geq 1$, and the conservation of the $L^2(\mathbb{R})$-norm of the solution u of the NSE gives us that

$$\|u\|_{L^q((k-1)T_0,\,kT_0;\,L^r(\mathbb{R}))} \leq C(p,q)\|u((k-1)T_0)\|_{L^2(\mathbb{R})} = C(p,q)\|\varphi\|_{L^2(\mathbb{R})}.$$

This proves (4.55) and finishes the proof of Lemma 4.1. □

We now consider a numerical scheme for the NSE based on approximations of the LSE that has uniform dispersive properties of Strichartz type. Examples of such schemes have been given in Section 4.3 and Section 4.4.

To be more precise, we deal with the following numerical schemes:

- Consider

$$\begin{cases} iu_t^h + A_h u^h = f(u^h), \quad t>0, \\ u^h(0) = \varphi^h, \end{cases} \tag{4.56}$$

where A_h is an approximation of Δ such that $\exp(itA_h)$ has uniform dispersive properties of Strichartz type. We also assume that A_h satisfies $\mathrm{Re}(iA_h\varphi,\varphi)_h \le 0$, $\Re$ being the real part, and has a symbol $a_h(\xi)$ which verifies

$$|a_h(\xi) - \xi^2| \le \sum_{k\in F} a(k,h)|\xi|^k,\, \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]. \tag{4.57}$$

- The two-grid scheme. The two-grid scheme can be adapted to the nonlinear frame as follows. Consider the equation

$$\begin{cases} iu_t^{0,h} + \Delta_h u^{0,h} = \mathbf{\Pi}_h^{4h} f((\mathbf{\Pi}_h^{4h})^* u^{0,h}), \quad t>0, \\ u^{0,h}(0) = \mathbf{\Pi}_h^{4h}\varphi^h, \end{cases} \tag{4.58}$$

where $(\mathbf{\Pi}_h^{4h})^* : l^2(h\mathbb{Z}) \to l^2(4h\mathbb{Z})$ is the adjoint of $\mathbf{\Pi}_h^{4h} : l^2(4h\mathbb{Z}) \to l^2(h\mathbb{Z})$ and φ^h is an approximation of φ.

By [62], Theorem 4.1, *for any $p\in(0,4)$ there exists of a positive time $T_0 = T_0(\|\varphi\|_{L^2(\mathbb{R})})$ and a unique solution $u^{h,0} \in C(0,T_0;\, l^2(h\mathbb{Z}^d)) \cap L^q(0,T_0;\, l^{p+2}(h\mathbb{Z}^d))$, $q = 4(p+2)/p$, of the system (4.58). Moreover, $u^{h,0}$ satisfies*

$$\|u^h\|_{L^\infty(\mathbb{R},\, l^2(h\mathbb{Z}^d))} \le \|\mathbf{\Pi}_h^{4h}\varphi^h\|_{l^2(h\mathbb{Z}^d)} \tag{4.59}$$

and

$$\|u^h\|_{L^q(0,T_0;\, l^{p+2}(h\mathbb{Z}^d))} \le c(T_0)\|\mathbf{\Pi}_h^{4h}\varphi^h\|_{l^2(h\mathbb{Z}^d)}, \tag{4.60}$$

where the above constant is independent of h.

With T_0 obtained above, for any $k\ge 1$ we consider $u^{k,h} : [kT_0,(k+1)T_0] \to \mathbb{C}$ the solution of the following system

$$\begin{cases} iu_t^{k,h} + \Delta_h u^{k,h} = \mathbf{\Pi}_h^{4h} f((\mathbf{\Pi}_h^{4h})^* u^{k,h}), \quad t\in[kT_0,(k+1)T_0], \\ u^{k,h}(kT_0) = \mathbf{\Pi}_h^{4h} u^{k-1,h}(kT_0). \end{cases} \tag{4.61}$$

Once, $u^{k,h}$ are computed the approximation u^h of NSE is defined as

$$u^h(t) = u^{k,h}(t),\, t\in[kT_0,(k+1)T_0). \tag{4.62}$$

We point out that systems (4.58) and (4.61) have always a global solution in the class $C(\mathbb{R},\, l^2(h\mathbb{Z}))$ (use the embedding $l^2(h\mathbb{Z}) \subset l^\infty(h\mathbb{Z})$, a classical fix point argument and the conservation of the $l^2(h\mathbb{Z})$-norm). However, estimates in the $L^q(0,T;\, l^r(h\mathbb{Z}))$-norm, uniformly with respect to the mesh-size parameter $h>0$, cannot be proved without using Strichartz estimates given by Theorem 4.5. Thus we need to take initial data obtained through a two-grid process. Since the two-grid class of functions is not invariant under the flow of system (4.58) we need to update the solution at some time-step T_0 which depends only on $L^2(\mathbb{R})$-norm of the initial data φ.

The following theorems give us the existence and uniqueness of solutions for the above systems as well as quantitative dispersive estimates of solutions u^h, similar to those obtained in Lemma 4.1 for the continuous NSE, uniformly on the mesh-size parameter $h>0$.

Theorem 4.8. *Let $p \in (0,4)$, $f(u) = |u|^p u$ and A_h be such that $\mathrm{Re}(iA_h\varphi,\varphi)_h \leq 0$ and (4.33) holds. Then for every $\varphi^h \in l^2(h\mathbb{Z})$, there exists a unique global solution $u^h \in C(\mathbb{R}, l^2(h\mathbb{Z}))$ of (4.56) which satisfies*

$$\|u^h\|_{L^\infty(\mathbb{R},\, l^2(h\mathbb{Z}))} \leq \|\varphi^h\|_{l^2(h\mathbb{Z})}. \tag{4.63}$$

Moreover, there exist $c(p) > 0$ and $C(p,q) > 0$ such that for any finite interval I with $|I| \leq T_0 = c(p)\|\varphi^h\|_{l^2(h\mathbb{Z})}^{-4p/(4-p)}$

$$\|u^h\|_{L^q(I,\, l^r(h\mathbb{Z}))} \leq C(p,q)\|\varphi^h\|_{l^2(h\mathbb{Z})}, \tag{4.64}$$

where (q,r) is a $1/2$-admissible pair and the above constant is independent of h.

Proof. Condition $\mathrm{Re}(iA_h\varphi,\varphi)_h \leq 0$ implies the $l^2(h\mathbb{Z})$ stability property (4.32). Then local existence is obtained by using Strichartz estimates given by Proposition 4.1 applied to the operator $\exp(itA_h)$ and a classical fix point argument in a suitable Banach space (see [64] and [65] for more details). The global existence of solutions and estimate (4.63) are guaranteed by the property $\mathrm{Re}(iA_h\varphi,\varphi)_h \leq 0$, and that $\mathrm{Re}(if(u_h),u_h)_h = 0$ and the energy identity:

$$\frac{d}{dt}\|u^h(t)\|^2_{l^2(h\mathbb{Z})} = 2\mathrm{Re}(iA_h u^h, u^h)_h + 2\mathrm{Re}(if(u^h),u^h)_h \leq 0. \tag{4.65}$$

Once the global existence is proved, estimate (4.64) is obtained in a similar manner as Lemma 4.1 and we will omit its proof. ◻

Theorem 4.9. *Let $p \in (0,4)$ and $q = 4(p+2)/p$. Then for all $h > 0$ and for every $\varphi^h \in l^2(4h\mathbb{Z})$, there exists a unique global solution $u^h \in C(\mathbb{R}, l^2(h\mathbb{Z})) \cap L^q_{loc}(\mathbb{R}, l^{p+2}(h\mathbb{Z}^d))$ of (4.58)-(4.62) which satisfies*

$$\|u^h\|_{L^\infty(\mathbb{R},\, l^2(h\mathbb{Z}))} \leq \|\mathbf{\Pi}_h^{4h}\varphi^h\|_{l^2(h\mathbb{Z})}. \tag{4.66}$$

Moreover, there exist $c(p) > 0$ and $C(p,q) > 0$ such that for any finite interval I with $|I| \leq T_0 = c(p)\|\varphi^h\|_{l^2(h\mathbb{Z})}^{-4p/(4-p)}$

$$\|u^h\|_{L^q(I,\, l^{p+2}(h\mathbb{Z}))} \leq C(p,q)\|\mathbf{\Pi}_h^{4h}\varphi^h\|_{l^2(h\mathbb{Z})}, \tag{4.67}$$

where (q,r) is a $1/2$-admissible pair and the above constant is independent of h.

Proof. The existence in the interval $(0,T_0)$, $T_0 = T_0(\|\varphi^h\|_{l^2(h\mathbb{Z})})$ for system (4.56) is obtained by using the Strichartz estimates given by Theorem 4.5 and a classical fix point argument in a suitable Banach space (see [64] and [65] for more details).

For any $k \geq 1$ the same arguments guarantee the local existence for systems (4.61). To prove that each system has solutions on an interval of length T_0 we have to prove a priori that the $l^2(h\mathbb{Z})$-norm of u^h does not increase. The particular approximation we have introduced of the nonlinear term in (4.58)-(4.61) gives us (after multiplying these equations by $u^{k,h}$ and taking the $l^2(h\mathbb{Z})$-norm) that for any $t \in [kT_0,(k+1)T_0]$

$$\|u^{k,h}(t)\|_{l^2(h\mathbb{Z})} = \|u^{k,h}(kT_0)\|_{l^2(h\mathbb{Z})} \leq \|u^{k-1,h}(kT_0)\|_{l^2(h\mathbb{Z})}$$

and then

$$\|u^{k,h}(t)\|_{l^2(h\mathbb{Z})} \leq \|u^{0,h}(0)\|_{l^2(h\mathbb{Z})} = \|\mathbf{\Pi}_h^{4h}\varphi^h\|_{l^2(h\mathbb{Z})}.$$

This proves (4.66) and the fact that for any $k \geq 1$ system (4.61) has a solution on the whole interval $[kT_0,(k+1)T_0]$. Estimate (4.67) is obtained locally on each interval $[kT_0,(k+1)T_0]$ together with the local existence result. ◻

Let us consider u^h the solution of the semidiscrete problem (4.56) and u of the continuous one (4.2). In the following theorem we evaluate the difference between u^h and $\mathbf{T}_h u$.

Theorem 4.10. *Let $p \in (0,4)$, $s \in (0,1/2)$, $f(u) = |u|^p u$ and A_h be as in Theorem 4.8 satisfying (4.57). For any $\varphi \in H^s(\mathbb{R})$, we consider u^h and $u \in L^\infty(\mathbb{R},\, H^s(\mathbb{R})) \cap L^{q_0}_{loc}(\mathbb{R}, B^s_{p+2,2}(\mathbb{R}))$, $q_0 = 4(p+2)/p$ solutions of problems (4.56) and (4.2), respectively. Then for any $T > 0$ there exists a positive constant $C(T, \|\varphi\|_{L^2(\mathbb{R})})$ such that*

$$\begin{aligned}
&\|u^h - \mathbf{T}_h u\|_{L^{q_0}(0,T;\, l^{p+2}(h\mathbb{Z}))} + \|u^h - \mathbf{T}_h u\|_{L^\infty(0,T;\, l^2(h\mathbb{Z}))} \\
&\quad \le C(T, \|\varphi\|_{L^2(\mathbb{R})}, p)\Big[\varepsilon(s,h)\|u\|_{L^\infty(0,T;\, H^s(\mathbb{R}))} + \Big(h^s + \varepsilon(s,h)\Big)\|u\|^{p+1}_{L^{q_0}(0,T;\, B^s_{p+2,2}(\mathbb{R}))}\Big]
\end{aligned} \tag{4.68}$$

holds for all $h > 0$.

In the case of the two-grid method, the solution u^h of system (4.58) approximates the solution u of the NSE (4.2) and the error committed is given by the following theorem.

Theorem 4.11. *Let $p \in (0,4)$, $s \in (0,1/2)$, $f(u) = |u|^p u$. For any $\varphi \in H^s(\mathbb{R})$, we consider u^h and $u \in L^\infty(\mathbb{R},\, H^s(\mathbb{R})) \cap L^{q_0}_{loc}(\mathbb{R}, B^s_{p+2,2}(\mathbb{R}))$, $q_0 = 4(p+2)/p$, solutions of problems (4.58)-(4.62) and (4.2), respectively. Then for any $T > 0$ there exists a positive constant $C(T, \|\varphi\|_{L^2(\mathbb{R})})$ such that*

$$\begin{aligned}
&\|u^h - \mathbf{T}_h u\|_{L^{q_0}(0,T;\, l^{p+2}(h\mathbb{Z}))} + \|u^h - \mathbf{T}_h u\|_{L^\infty(0,T;\, l^2(h\mathbb{Z}))} \\
&\quad \le C(T, \|\varphi\|_{L^2(\mathbb{R})}, p)\Big[h^{s/2}\|u\|_{L^\infty(0,T; H^s(\mathbb{R}))} + \Big(h^s + h^{s/2}\Big)\|u\|^{p+1}_{L^{q_0}(0,T;\, B^s_{p+2,2}(\mathbb{R}))}\Big]
\end{aligned} \tag{4.69}$$

holds for all $h > 0$.

Remark 4.3. *Using classical results on the solutions of the NSE (see for example [27], Theorem 5.1.1, Ch. 5, p. 147) we can state the above result in a more compact way: For any $T > 0$ there exists a positive constant $C(T, \|\varphi\|_{H^s(\mathbb{R})})$ such that*

$$\|u^h - \mathbf{T}_h u\|_{L^{q_0}(0,T;\, l^{p+2}(h\mathbb{Z}))} + \|u^h - \mathbf{T}_h u\|_{L^\infty(0,T;\, l^2(h\mathbb{Z}))} \le C(T, \|\varphi\|_{H^s(\mathbb{R})}) h^{s/2} \tag{4.70}$$

holds for all $h > 0$.

Theorem 4.10 shows that if $h^s \le \varepsilon(s,h)$ then the error committed to approximate the nonlinear problem is the same as for the linear problem with the same initial data. As we proved in Section 4.3, for the higher order dissipative scheme $A_h = \Delta_h - ih^{2(m-1)}(-\Delta_h)^m$, $m \ge 2$, and for the two-grid method, $\varepsilon(s,h) = h^{s/2} \ge h^s$. So these schemes enter in this framework. It is also remarkable that the use of dispersive schemes allows to prove the convergence for the NSE and to obtain the convergence rate for $H^s(\mathbb{R})$ initial data with $0 < s < 1/2$. We point out that the energy method does not provide any error estimate in this case, the minimal smoothing required for the energy method being $H^s(\mathbb{R})$, with $s > 1/2$ (see Section 4.6 for all the details).

The idea of the proof of Theorem 4.10 is that there exists a time T_1 depending on the $L^2(\mathbb{R})$-norm of the initial data:

$$T_1 \simeq \min\{1, \|\varphi\|^{-4p/(4-p)}_{L^2(\mathbb{R})}\},$$

such that the error in the approximation of the nonlinear problem

$$\mathrm{err}_h(t) = u^h(t) - \mathbf{T}_h u(t),$$

when considered in the $L^{q_0}(0,T_1;\, l^{p+2}(h\mathbb{Z})) \cap L^\infty(0,T_1;\, l^2(h\mathbb{Z}))$-norm is controlled by the error produced in the linear part

$$\mathrm{err}^{lin}_h(t) = \exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h \exp(it\partial_x^2)\varphi.$$

The proof of Theorem 4.11 is similar to that of Theorem 4.10 since the estimates in any interval $(0,T)$ are obtained reiterating the argument in each interval $(kT_0, (k+1)T_0)$, $k \ge 0$, for some $T_0 = T_0(\|\varphi\|_{L^2(\mathbb{R})})$ in view of the structure of the scheme. All the details are given in [66].

4.6 Nondispersive methods

In this section we will consider a numerical scheme for which the operator A_h has no uniform (with respect to the mesh size h) dispersive properties of Strichartz type. Accordingly we may not use $L^q_t L^r_x$ estimates for the linear semigroup $\exp(itA_h)$ and all the possible convergence estimates need to be based on the fact that the solution u of the continuous problem is uniformly bounded in space and time: $u \in L^\infty((0,T); L^\infty(\mathbb{R}))$. Thus, the only estimates we can use are those that the L^2-theory may yield. When working with $H^s(\mathbb{R})$-data with $s > 1/2$, using $L^\infty(\mathbb{R}; H^s(\mathbb{R}))$ estimates on solutions and Sobolev's embedding we can get L^2-estimates.

There is a classical argument that works whenever the nonlinearity f satisfies

$$|f(u) - f(v)| \le C(|u|^p + |v|^p)|u - v|. \tag{4.71}$$

Standard error estimates (see Theorem 4.4 with the particular case $(q, r) = (\infty, 2)$ or [103], Theorem 10.1.2, p. 201) and Gronwall's inequality yield when $0 \le t \le T$:

$$\begin{aligned} &\|u^h(t) - \mathbf{T}_h u(t)\|_{l^2(h\mathbb{Z})} \\ &\qquad \le h^{1/2} C(T)\big(\|\varphi\|_{H^1(\mathbb{R})} + \|u\|^{p+1}_{L^\infty(0,T;\,H^1(\mathbb{R}))}\big) \exp(T\|u\|^p_{L^\infty(0,T;\,H^1(\mathbb{R}))}), \end{aligned} \tag{4.72}$$

for the conservative semi-discrete finite-difference scheme. For the sake of completness we will prove this estimate in Section 4.6.1.

We emphasize that in order to obtain estimate (4.72) we need to use that the solution u, which we want to approximate, belongs to the space $L^\infty(\mathbb{R})$, condition which is guaranteed by assuming that the initial data is smooth enough. However, obviously, in general, solutions of the NSE do not belong to $L^\infty(\mathbb{R})$ and therefore these estimates can not be applied. One can overcome this drawback assuming that the initial data belong to $H^1(\mathbb{R})$ or even to $H^s(\mathbb{R})$ with $s > 1/2$ since in this case $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Using H^1-energy estimates and Sobolev's embedding we can deduce L^∞-bounds on solutions allowing to apply (4.72). We emphasize that this standard approach fails to provide any error estimate for initial data in $H^s(\mathbb{R})$ with $s < 1/2$.

However, this type of error estimate can also be used for $H^s(\mathbb{R})$-initial data with $s < 1/2$ (or even for $L^2(\mathbb{R})$-initial data), by a density argument. Indeed, given $\varphi \in H^s(\mathbb{R})$ with $0 \le s < 1/2$, for any $\delta > 0$ we may choose $\varphi_\delta \in H^1(\mathbb{R})$ such that

$$\|\varphi - \varphi_\delta\|_{H^s(\mathbb{R})} \le \delta.$$

Let u_δ be the solution of NSE corresponding to φ_δ. Obviously, φ_δ being $H^1(\mathbb{R})$-smooth, we can apply standard results as (4.72) to u_δ. On the other hand, stability results for NSE allow us to prove the proximity of u and u_δ in $H^s(\mathbb{R})$. This allows showing the convergence of numerical approximations of u_δ, that we may denote by $u_{\delta,h}$, towards the solution u associated to φ as both $\delta \to 0$ and $h \to 0$. But for this to be true h needs to be exponentially small of the order of $\exp(-1/\delta)$ which is much smaller than the typical mesh-size needed to apply the results of the previous sections on dispersive schemes that required h to be of the order of $\delta^{2/s}$.

4.6.1 A classical argument for smooth initial data

In this section we present the technical details of the error estimates in the case of $H^1(\mathbb{R})$-initial data. In this case we do not require the numerical scheme to be dispersive, the only ingredient being the Sobolev's embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.

Theorem 4.12. *Let $f(u) = |u|^p u$ with $p \in (0, 4)$ and $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ be solution of (4.2) with initial data $\varphi \in H^1(\mathbb{R})$. Also assume that A_h is an approximation of order two of the laplace operator ∂_x^2 and u^h is the solution of the following system*

$$\begin{cases} iu^h_t + A_h u^h = f(u^h), \quad t > 0, \\ \quad u^h(0) = \mathbf{T}_h \varphi, \end{cases} \tag{4.73}$$

satisfying $\|u^h\|_{L^\infty((0,T)\times h\mathbb{Z})} \le C(T, \|\varphi\|_{H^1(\mathbb{R})})$.

Then for all $T>0$ *and* $h>0$

$$\|u^h(t) - \mathbf{T}_h u(t)\|_{l^2(h\mathbb{Z})} \le h^{1/2} \max\{T, T^2\}\big(\|\varphi\|_{H^1(\mathbb{R})} + \|u\|^{p+1}_{L^\infty(0,T:\,H^1(\mathbb{R}))}\big) \exp(T\|u\|^p_{L^\infty(0,T;\,H^1(\mathbb{R}))}).$$

We now give an example where the hypotheses of the above theorem are verified. We consider the following NSE:

$$\begin{cases} iu_t + \partial_x^2 u = |u|^p u, \ x\in\mathbb{R}, \ t>0, \\ u(0,x) = \varphi(x), \ x \in \mathbb{R}, \end{cases} \tag{4.74}$$

and its numerical approximation

$$\begin{cases} iu^h_t + \Delta_h u^h = |u^h|^p u^h, \quad t>0, \\ \qquad u^h(0) = \varphi^h. \end{cases} \tag{4.75}$$

In the case of the continuous problem we have the following conservation laws (see [28], Corollary 4.3.4, p. 93):

$$\|u(t)\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}$$

and

$$\frac{d}{dt}\Big(\frac12 \int_{\mathbb{R}} |u_x(t,x)|^2 dx + \frac{1}{p+2}\int_{\mathbb{R}} |u(t,x)|^{p+2} dx\Big) = 0.$$

The same identities apply in the semi-discrete case (it suffices to multiply the equation (4.75) by $\overline{u}^h$, respectively $\overline{u}^h_t$, to sum over the integers and to take the real part of the resulting identity):

$$\|u^h(t)\|_{l^2(h\mathbb{Z})} = \|\varphi^h\|_{l^2(h\mathbb{Z})}$$

and

$$\frac{d}{dt}\Big(\frac{h}{2}\sum_{j\in\mathbb{Z}} \Big|\frac{u^h_{j+1}(t) - u^h_j(t)}{h}\Big|^2 + \frac{h}{p+2}\sum_{j\in\mathbb{Z}} |u^h_j(t)|^{p+2}\Big) = 0.$$

In view of the above identities, the hypotheses of Theorem 4.12 are verified.

Proof of Theorem 4.12. Using the variations of constants formula we get

$$\mathbf{T}_h u(t) = \mathbf{T}_h \exp(it\partial_x^2)\varphi + \int_0^t \mathbf{T}_h \exp(i(t-\sigma)\partial_x^2) f(u(\sigma)) d\sigma$$

and

$$u^h(t) = \exp(itA_h)\mathbf{T}_h\varphi + \int_0^t \exp(i(t-\sigma)A_h) f(u^h(\sigma)) d\sigma.$$

Then

$$\begin{aligned} \mathrm{err}_h(t) &:= \|u^h(t) - \mathbf{T}_h u(t)\|_{l^2(h\mathbb{Z})} \\ &\le \|\exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h \exp(it\partial_x^2)\varphi\|_{l^2(h\mathbb{Z})} \\ &\quad + \int_0^t \|\exp(i(t-\sigma)A_h)\Big(f(u^h(\sigma)) - \mathbf{T}_h f(u(\sigma))\Big) d\sigma\|_{l^2(h\mathbb{Z})} d\sigma \\ &\quad + \int_0^t \|\exp(i(t-\sigma)A_h)\mathbf{T}_h f(u(\sigma)) - \mathbf{T}_h \exp((t-\sigma)\partial_x^2) f(u(\sigma))\|_{l^2(h\mathbb{Z})} d\sigma. \end{aligned} \tag{4.76}$$

Now, applying the error estimates for the linear terms as in (4.37) with $\varepsilon(1,h) = h^{1/2}$, we get

$$\|\exp(itA_h))\mathbf{T}_h\varphi - \mathbf{T}_h \exp(it\partial_x^2)\varphi\|_{l^2(h\mathbb{Z})} \le Th^{1/2}\|\varphi\|_{H^1(\mathbb{R})}. \tag{4.77}$$

Also, using that $f(u)=|u|^p u$ we have that $\|f(u)\|_{H^1(\mathbb{R})}\leq C\|u\|^p_{H^1(\mathbb{R})}$ and then by (4.37) we get

$$\begin{aligned}\int_0^t \|\exp(i(t-\sigma)A_h)\mathbf{T}_h f(u(\sigma)) - \mathbf{T}_h\exp(i(t-\sigma)\partial_x^2)f(u(\sigma))\|_{l^2(h\mathbb{Z})}d\sigma \\ \leq CTh^{1/2}\|f(u)\|_{L^1(0,T;\,H^1(\mathbb{R}))}\leq CT^2h^{1/2}\|u\|^{p+1}_{L^\infty(0,T;\,H^1(\mathbb{R}))}.\end{aligned} \tag{4.78}$$

Using the $l^2(h\mathbb{Z})$-stability of $\exp(itA_h)$, (4.76), (4.77) and (4.78) we obtain

$$\mathrm{err}_h(t)\leq Th^{1/2}\|\varphi\|_{H^1(\mathbb{R})}+CT^2h^{1/2}\|u\|^{p+1}_{L^\infty(0,T;\,H^1(\mathbb{R}))}+\int_0^t\|f(u^h(\sigma))-\mathbf{T}_h f(u(\sigma)\|_{l^2(h\mathbb{Z}))}.$$

Now we write $f(u^h(s))-\mathbf{T}_h f(u(s)) = I_1^h(s)+I_2^h(s)$ where

$$I_1^h(s)=f(u^h(s))-f(\mathbf{T}_h u(s)),\; I_2^h(s)=f(\mathbf{T}_h u(s))-\mathbf{T}_h f(u(s)).$$

In the case of I_1^h we use that f satisfies (4.71) to get

$$\begin{aligned}\|I_1^h(s)\|_{l^2(h\mathbb{Z})}&\leq C\Big(\|u^h(s)\|^p_{l^\infty(h\mathbb{Z})}+\|\mathbf{T}_h u(s)\|^p_{l^\infty(h\mathbb{Z})}\Big)\|u^h(s)-\mathbf{T}_h u(s)\|_{l^2(h\mathbb{Z})}\\ &\leq C(\|u^h\|^p_{L^\infty((0,T)\times h\mathbb{Z})}+\|u\|^p_{L^\infty((0,T)\times\mathbb{R})})\|u^h(s)-\mathbf{T}_h u(s)\|_{l^2(h\mathbb{Z})}\\ &\leq C\|u\|^p_{L^\infty(0,T;\,H^1(\mathbb{R}))}\mathrm{err}_h(s).\end{aligned}$$

Also we obtain that

$$\|I_2^h(s)\|_{l^2(h\mathbb{Z})}\leq h\|u(s)\|^{p+1}_{H^1(\mathbb{R})}.$$

Putting together all the above estimates, for any $0\leq t\leq T$ we obtain:

$$\begin{aligned}\mathrm{err}_h(t)&\leq h^{1/2}T\|\varphi\|_{H^1(\mathbb{R})}+\|u\|^p_{L^\infty(0,T;\,H^1(\mathbb{R}))}\int_0^t\mathrm{err}_h(\sigma)d\sigma\\ &\quad + hT\|u\|^{p+1}_{L^\infty(0,T;H^1(\mathbb{R}))}+T^2h^{1/2}\|u\|^{p+1}_{L^\infty(0,T:\,H^1(\mathbb{R}))}\\ &\leq h^{1/2}\max\{T,T^2\}\big(\|\varphi\|_{H^1(\mathbb{R})}+\|u\|^{p+1}_{L^\infty(0,T:\,H^1(\mathbb{R}))}\big)+\|u\|^p_{L^\infty(0,T;\,H^1(\mathbb{R}))}\int_0^t\mathrm{err}_h(s)ds.\end{aligned}$$

Applying Gronwall's Lemma we obtain

$$\mathrm{err}_h(t)\lesssim h^{1/2}\max\{T,T^2\}\big(\|\varphi\|_{H^1(\mathbb{R})}+\|u\|^{p+1}_{L^\infty(0,T:\,H^1(\mathbb{R}))}\big)\exp(T\|u\|^p_{L^\infty(0,T;\,H^1(\mathbb{R}))}). \tag{4.79}$$

The proof is now finished. □

4.6.2 Approximating $H^s(\mathbb{R})$, $s<1/2$, solutions by smooth ones.

Given $\varphi\in H^s(\mathbb{R})$ we choose an approximation $\tilde\varphi\in H^1(\mathbb{R})$ such that $\|\varphi-\tilde\varphi\|_{H^s(\mathbb{R})}$ is small (a similar analysis can be done by considering $\varphi_\delta\in H^{s_1}$ with $s_1>1/2$). For $\tilde\varphi$ we consider the following approximation of $\tilde u$ solution of the NSE (4.2) with initial data $\tilde\varphi$:

$$\begin{cases} i\partial_t\tilde u_h(t)+A_h\tilde u_h=f(\tilde u_h), & t>0,\\ \tilde u_h(0)=\mathbf{T}_h\tilde\varphi, \end{cases} \tag{4.80}$$

where the operator A_h is a second order approximation of the Laplace operator. We do not require the linear scheme associated to the operator A_h to satisfy uniform dispersive estimates.

Solving (4.80) we obtain an approximation $\tilde u_h$ of the solutions $\tilde u$ of NSE with initial datum $\tilde\varphi$, which itself is an approximation of the solution u of the NSE with initial datum φ.

In the following Theorem we give an explicit estimate of the distance between $\tilde u_h$ and u.

Theorem 4.13. *Let $0 \le s < 1/2$, $\varphi \in H^s(\mathbb{R})$, and $u \in C(\mathbb{R}; H^s(\mathbb{R}))$ be the solution of NSE with initial datum φ given by Theorem 4.7. For any $T > 0$ there exists a positive constant $C(T, \|\varphi\|_{L^2(\mathbb{R})})$ such that the following holds*

$$\|\mathbf{T}_h u - \tilde{u}_h\|_{L^\infty(0,T;\, l^2(h\mathbb{Z}))} \le C(T, p, \|\varphi\|_{L^2(\mathbb{R})})\|\varphi - \tilde{\varphi}\|_{L^2(\mathbb{R})} + h^{1/2}\exp\left(T\|\tilde{u}\|^p_{L^\infty(0,T;\, H^1(\mathbb{R}))}\right) \quad (4.81)$$

for all $h > 0$ and $\delta > 0$.

In the following we show that the above method of regularizing the initial data $\varphi \in H^s(\mathbb{R})$ and then applying the $H^1(\mathbb{R})$ theory for that approximation does not give the same rate of convergence $h^{s/2}$ obtained in the case of a dispersive method of order two (see (4.70)). This occurs since for $\|\varphi - \tilde{\varphi}\|_{L^2(\mathbb{R})}$ to be small, $\|\tilde{\varphi}\|_{H^1(\mathbb{R})}$ needs to be large and $\|\tilde{u}\|_{L^\infty(0,T;\, H^1(\mathbb{R}))}$ too.

To simplify the presentation we will consider the case $p = 2$.

Theorem 4.14. *Let $p = 2$, $0 < s < 1/2$, $\varphi \in H^s(\mathbb{R})$ and $u \in C(\mathbb{R}, H^s(\mathbb{R}))$ be solution of NSE with initial data φ given by Theorem 4.7 and u_h^* be the best approximation with $H^1(\mathbb{R})$-initial data as given by (4.80) with the conservative approximation $A_h = \Delta_h$. Then for any time T, there exists a constant $C(\|\varphi\|_{H^s(\mathbb{R})}, T, s)$ such that*

$$\|\mathbf{T}_h u - u_h^*\|_{L^\infty(0,T;\, l^2(h\mathbb{Z}))} \le C(\|\varphi\|_{H^s(\mathbb{R})}, T, s)|\log h|^{-\frac{s}{1-s}}. \quad (4.82)$$

To prove this result we will use in an essential manner the following Lemma.

Lemma 4.2. *Let $0 < s < 1$ and $h \in (0,1)$. Then for any $\varphi \in H^s(\mathbb{R})$ the functional $J_{h,\varphi}$ defined by*

$$J_{h,\varphi}(g) = \frac{1}{2}\|\varphi - g\|^2_{L^2(\mathbb{R})} + \frac{h}{2}\exp(\|g\|^2_{H^1(\mathbb{R})}) \quad (4.83)$$

satisfies:

$$\min_{g \in H^1(\mathbb{R})} J_{h,\varphi}(g) \le C(\|\varphi\|_{H^s(\mathbb{R})}, s)|\log h|^{-s/(1-s)}. \quad (4.84)$$

Moreover, the above estimate is optimal in the sense that the power of the $|\log h|$ term cannot be improved: for any $0 < \epsilon < 1 - s$ there exists $\varphi_\epsilon \in H^s(\mathbb{R})$ such that

$$\liminf_{h \to 0} \frac{\min_{g \in H^1(\mathbb{R})} J_{h,\varphi_\varepsilon}(g)}{|\log h|^{-(s+\varepsilon)/(1-s-\varepsilon)}} > 0.$$

Remark 4.4. *We point out that, to obtain (4.84) and (4.2), we will use in an essential manner that $s < 1$. In fact in the case $s = 1$ the minimum of J_h over $H^1(\mathbb{R})$ is of order h. This can be seen by choosing $g = \varphi$ and observing that $J_h(\varphi) = h\exp(\|\varphi\|_{H^1(\mathbb{R})})$. This choice cannot be done if $\varphi \in H^s(\mathbb{R})\backslash H^1(\mathbb{R})$.*

Proof of Theorem 4.14. Let us choose $\tilde{\varphi} \in H^1(\mathbb{R})$ which approximates φ in $H^s(\mathbb{R})$. Then by Theorem 4.13 we get

$$\begin{aligned}\|\mathbf{T}_h u - \tilde{u}_h\|^2_{L^\infty(0,T;\, l^2(h\mathbb{Z}))} &\le C(T, \|\varphi\|_{L^2(\mathbb{R})})\|\varphi - \tilde{\varphi}\|^2_{L^2(\mathbb{R})} + h\exp\left(2T\|\tilde{\varphi}\|^2_{H^1(\mathbb{R})}\right)\\ &\le C(T, \|\varphi\|_{L^2(\mathbb{R})})J_{h,\sqrt{2T}\varphi}(\sqrt{2T}\tilde{\varphi}),\end{aligned}$$

where $\tilde{u}_h$ is the solution of (4.80) with initial data $\mathbf{T}_h\tilde{\varphi}$.

For each h fixed, in order to obtain the best approximation u_h^* of $\mathbf{T}_h u$, we have to choose in the right hand side of the above inequality the function φ^* which minimizes the functional $J_{h,\sqrt{2T}\varphi}(\cdot)$ defined by (4.83) over $H^1(\mathbb{R})$. Using estimate (4.84) from Lemma 4.2 we obtain the desired result:

$$\begin{aligned}\|\mathbf{T}_h u - u_h^*\|_{L^\infty(0,T;\, l^2(h\mathbb{Z}))} &\le C(\|\varphi\|_{H^s(\mathbb{R})}, T, s)\min_{\tilde{\varphi} \in H^1(\mathbb{R})} J_{h,\sqrt{2T}\varphi}(\sqrt{2T}\tilde{\varphi})\\ &\le C(\|\varphi\|_{H^s(\mathbb{R})}, T, s)|\log h|^{-\frac{s}{1-s}}\end{aligned}$$

where u_h^* is the solution of (4.80) with initial data $\mathbf{T}_h\varphi^*$ $\square$

Proof of Theorem 4.13. Using the stability result (4.53) for the NSE we obtain

$$\begin{aligned}\|u-\tilde u\|_{L^\infty(0,T;\,L^2(\mathbb{R}))} &\le C(T,p,\|\varphi\|_{L^2(\mathbb{R})},\|\tilde\varphi\|_{L^2(\mathbb{R})})\|\varphi-\tilde\varphi\|_{L^2(\mathbb{R})}\\ &\le C(T,p,\|\varphi\|_{L^2(\mathbb{R})})\|\varphi-\tilde\varphi\|_{L^2(\mathbb{R})}.\end{aligned}$$

Now using the classical results for smooth initial data presented in Section 4.6.1, by (4.79) we get

$$\|\mathbf{T}_h\tilde u-\tilde u_h^h\|_{L^\infty(0,T;\,l^2(h\mathbb{Z}))}\le Ch^{1/2}\exp(T\|\tilde u\|^p_{L^\infty(0,T;\,H^1(\mathbb{R}))}).$$

Thus

$$\begin{aligned}\|\mathbf{T}_h u-\tilde u_h\|_{L^\infty(0,T;\,l^2(h\mathbb{Z}))} &\le \|\mathbf{T}_h u-\mathbf{T}_h\tilde u\|_{L^\infty(0,T;\,l^2(h\mathbb{Z}))}+\|\mathbf{T}_h\tilde u-\tilde u_h\|_{L^\infty(0,T;\,l^2(h\mathbb{Z}))}\\ &\le \|u-\tilde u\|_{L^\infty(0,T;\,L^2(\mathbb{R}))}+\|\mathbf{T}_h\tilde u-\tilde u_h\|_{L^\infty(0,T;\,l^2(h\mathbb{Z}))}\\ &\le C(T,p,\|\varphi\|_{L^2(\mathbb{R})})\|\varphi-\tilde\varphi\|_{L^2(\mathbb{R})}+h^{1/2}\exp(T\|\tilde u\|^p_{L^\infty(0,T;\,H^1(\mathbb{R}))}).\end{aligned}$$

This yields (4.81). $\square$

Chapter 5

A splitting method for the Nonlinear Schrödinger equation

Let us consider the nonlinear Schrödinger equation (NSE):

$$\begin{cases} \dfrac{du}{dt} = i\Delta u + i\lambda |u|^p u, & x \in \mathbb{R}^d, t \neq 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^d. \end{cases} \tag{5.1}$$

For any $0 \leq p < 4/d$, $\lambda \in \mathbb{R}$ and $\varphi \in L^2(\mathbb{R}^d)$, equation (5.1) has a unique global solution $u \in C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^q_{loc}(\mathbb{R}, L^r(\mathbb{R}^d))$ for some suitable pairs (q,r). This has been proved by Tsutsumi in [106] by using a fix point argument and the so-called Strichartz estimates [102]. These estimates show that the semigroup generated by the linear Schrödinger equation (LSE), $S(t) = \exp(it\Delta)$, satisfies

$$\|S(\cdot)\varphi\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} \leq C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)} \quad \text{for all} \quad \varphi \in L^2(\mathbb{R}^d), \tag{5.2}$$

for the so-called admissible pairs (q,r) (cf. [72]): $2 \leq q, r \leq \infty$, $(q,r,d) \neq (2,\infty,2)$ and

$$\frac{1}{q} = \frac{d}{2}\left(\frac{1}{2} - \frac{1}{r}\right).$$

In addition, in [106] the stability of solutions under perturbation of the initial data has been proved. In fact there exists a time T, depending on the $L^2(\mathbb{R}^d)$-norm of the initial data, such that on the interval $(0,T)$ the difference between two solutions of equation (5.1) is controlled by the error made in the linear part $S(t)(\varphi_1 - \varphi_2)$ in a certain $L^q(0,T,L^r(\mathbb{R}^d))$-norm. Thus, Strichartz's estimate (5.2) shows that, locally, the error between two solutions u_1 and u_2 can be estimated in terms of the $L^2(\mathbb{R}^d)$-norm of the difference of the initial data $\varphi_1 - \varphi_2$. Using the global well-posedness of system (5.1) the same procedure can be extended to any bounded time interval. We will adapt this idea to the numerical context in order to estimate the error committed when approximating the solutions of (5.1) by a splitting method.

A splitting method consists in decomposing the flow (5.1) in two flows, which in principle should be computed easily. To be more precise, we define the flow $N(t)$ for the differential equation:

$$\begin{cases} \dfrac{du}{dt} = i\lambda |u|^p u, & x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{5.3}$$

i.e.

$$N(t)\varphi = \exp(it\lambda|\varphi|^p)\varphi. \tag{5.4}$$

The idea of splitting methods is to approximate the solutions of (5.1) by combining the two flows $S(t)$ and $N(t)$. For a fixed time interval $[0,T]$ we can choose a small positive time step τ and consider either the Lie approximation:

$$Z(n\tau) = (S(\tau)N(\tau))^n\varphi,\ 0 \le n\tau \le T, \tag{5.5}$$

or Strang approximation

$$Z(n\tau) = (S(\tau/2)N(\tau)S(\tau/2))^n\varphi,\ 0 \le n\tau \le T. \tag{5.6}$$

In the two-dimensional case, Besse et al. [15] have analyzed the convergence of the above methods for globally Lipschitz-continuous nonlinearities. Also Lubich [87] analyzed the Strang method for the Schrödinger-Poisson equation and the cubic NSE in the case of $H^4(\mathbb{R}^3)$-initial data. There, the $H^4(\mathbb{R}^3)$-regularity was imposed to guarantee that the approximate solution Z remains bounded in the $H^2(\mathbb{R}^3)$-norm.

In this chapter we introduce a splitting method for the NSE with $1 \le p < 4/d$ and prove the convergence in the $L^2(\mathbb{R}^d)$-norm for $H^2(\mathbb{R}^d)$-initial data. The scheme we analyse is based on an approximation $S_\tau(t)$ of the linear semigroup $S(t)$ which admits Strichartz-like estimates in some time discrete spaces. We make use of these new estimates to establish uniform bounds on the numerical solution in the auxiliary spaces $l^q_{loc}(\tau\mathbb{Z}, L^r(\mathbb{R}^d))$ without assuming more than $L^2(\mathbb{R}^d)$-regularity on the initial data. Once these bounds are obtained we will need the $H^2(\mathbb{R}^d)$ regularity in order to obtain the order of error.

The idea behind the numerical schemes for the LSE which admit uniform (with respect to discretization parameters) estimates of Strichartz type is that when they are applied in the context of NSE, the error committed is controlled by the error committed in approximating the LSE. The application of these numerical schemes for NSE has been previously used in the case of semidiscrete space approximations [62, 63, 65] and in the fully discrete case in [53].

In this chapter we will concentrate on Lie's approximation method. We remark that Z defined by (5.5) satisfies

$$Z(n\tau) = S(n\tau)\varphi + \tau\sum_{k=0}^{n-1} S(n\tau - k\tau)\frac{N(\tau)-\mathcal{I}}{\tau}Z(k\tau), \quad n \ge 1. \tag{5.7}$$

Since Z is defined on a discrete set of points we need to evaluate Z in some discrete time norms $l^q(\tau\mathbb{Z}, L^r(\mathbb{R}^d))$. We emphasize that for $(q,r) \ne (\infty, 2)$ even the linear part $S(n\tau)\varphi$ does not satisfy Strichartz-like estimates:

$$\|S(n\tau)\varphi\|_{l^q(\tau\mathbb{Z},L^r(\mathbb{R}^d))} \le C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)} \quad \text{for all} \quad \varphi \in L^2(\mathbb{R}^d),$$

where

$$\|u\|_{l^q(\tau\mathbb{Z},L^r(\mathbb{R}^d))} = \Big(\tau\sum_{n\in\mathbb{Z}}\|u(k\tau)\|^q_{L^r(\mathbb{R}^d)}\Big)^{1/q}.$$

Indeed, in contrast with the classical estimate (5.2), the above inequality implies that

$$\tau^{1/q}\|S(\tau)\varphi\|_{L^r(\mathbb{R}^d)} \le C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)},$$

inequality which does not hold for all $\varphi \in L^2(\mathbb{R}^d)$ (choose $\varphi = S(-\tau)\psi$ with $\psi \in L^2(\mathbb{R}^d)\backslash L^r(\mathbb{R}^d)$ for $r \ne 2$). This implies that we have to choose an approximation $S_\tau(t)$ of the linear semigroup $S(t)$ such that $S_\tau(t)$ admits Strichartz-like estimates which are discrete in time and moreover, these estimates are uniform with respect to the time parameter τ:

$$\|S_\tau(n\tau)\varphi\|_{l^q(\tau\mathbb{Z},L^r(\mathbb{R}^d))} \le C\|\varphi\|_{L^2(\mathbb{R}^d)}, \quad \forall\,\varphi \in L^2(\mathbb{R}^d).$$

One of the possible choices is the filtered operator

$$S_\tau(t)\varphi = S(t)\Pi_\tau\varphi$$

where Π_τ filters the high frequencies as follows

$$\widehat{\Pi_\tau \varphi}(\xi) = \widehat{\varphi}(\xi)\mathbf{1}_{\{|\xi|\leq \tau^{-1/2}\}}(\xi), \quad \xi \in \mathbb{R}^d. \tag{5.8}$$

For other possible choices of the operator S_τ we refer to the previous work on dispersive methods for LSE [62, 63, 65]. Also as initial data we have to choose a filtration of φ, $\Pi_\tau\varphi$, since otherwise $Z_\tau(0)\varphi = \varphi$ does not belong to $L^r(\mathbb{R}^d)$ and we cannot evaluate the $l^q(0 \leq n\tau \leq T, L^r(\mathbb{R}^d))$-norm of the approximation Z_τ.

The splitting scheme we propose is the following one:

$$Z_\tau(n\tau) = (S_\tau(\tau)N(\tau))^n \Pi_\tau \varphi, \quad n \geq 0. \tag{5.9}$$

Observe that in this scheme only the linear equation is filtered while the nonlinear one is solved exactly.

In the following, for any interval I with $|I| \geq \tau$, the space $l^q(n\tau \in I, L^r(\mathbb{R}^d))$ contains all functions defined on $\tau\mathbb{Z} \cap I$ with values in $L^r(\mathbb{R}^d)$ and the norm on this space is defined by

$$\|u\|_{l^q(n\tau\in I,\, L^r(\mathbb{R}^d))} = \Big(\tau \sum_{n\in\mathbb{Z}} \|u(k\tau)\|^q_{L^r(\mathbb{R}^d)}\Big)^{1/q}.$$

In the sequel we always assume that τ is a small parameter, in the sense that there exists $\tau_0 = \tau_0(\|\varphi\|_{L^2(\mathbb{R}^d)})$ such that all the results holds for $\tau \leq \tau_0$.

The main results of this chapter are the following.

Theorem 5.1. *(Stability) Let $0 < p < 4/d$. For any $\varphi \in L^2(\mathbb{R}^d)$ the approximation Z_τ introduced in (5.9) satisfies:*
i) a uniform $L^2(\mathbb{R}^d)$-bound

$$\max_{n\geq 0} \|Z_\tau(n\tau)\|_{L^2(\mathbb{R}^d)} \leq \|\varphi\|_{L^2(\mathbb{R}^d)}, \tag{5.10}$$

ii) there exists $T_0 \simeq \|\varphi\|^{-\frac{4p}{4-dp}}$ such that for any interval I with $|I| \leq T_0$ and for any admissible pair (q, r) the following

$$\|Z_\tau(n\tau)\|_{l^q(n\tau\in I,\, L^r(\mathbb{R}^d))} \leq C(d, p, q)\|\varphi\|_{L^2(\mathbb{R}^d)} \tag{5.11}$$

holds for some constant $C(d, p, q)$ independent of the time step τ,

iii) for any $T > 0$ and (q, r) admissible-pair the following

$$\|Z_\tau(n\tau)\|_{l^q(0\leq n\tau\leq T;\, L^r(\mathbb{R}^d))} \leq C(T, d, p, q)\|\varphi\|_{L^2(\mathbb{R}^d)} \tag{5.12}$$

holds for some constant $C(T, d, p, q)$ independent of the time step τ.

Theorem 5.2. *(Convergence) Let $d \leq 3$, $p \in [1, 4/d)$ and $\varphi \in H^2(\mathbb{R}^d)$. The numerical solution Z_τ has a first-order error bound in $L^2(\mathbb{R}^d)$:*

$$\max_{0\leq n\tau\leq T} \|Z_\tau(n\tau) - u(n\tau)\|_{L^2(\mathbb{R}^d)} \leq \tau C(T, d, p, \|\varphi\|_{H^2(\mathbb{R}^d)}).$$

We point out that Theorem 5.2 works in the case $d \leq 3$ which is quite restrictive. The restriction $p \geq 1$ comes from the fact that in our proof we need to guarantee that u solution of (5.1) belongs to $C(0, T, H^2(\mathbb{R}^d))$ (see [28], Ch. 5.3).

We now comment on the possible analysis of the order of error in the case of less regularity or other nonlinearities. It is convenient to write u in the semigroup formulation:

$$u(t) = S(t)\varphi + i\lambda \int_0^t S(t-s)|u|^p u(s)ds, \quad t \geq 0. \tag{5.13}$$

Looking at (5.7), we observe that Z (or Z_τ) defined by (5.5) (or (5.9)), differs from u in two important facts: the integral in (5.13) is replaced by a sum in (5.7) and the nonlinear term $f(u) = \lambda|u|^p u$ is replaced by $\tau^{-1}(N(\tau) - \mathcal{I})Z$. In view of this, it seems to be reasonable that Z better approximates the solution of the following NSE:

$$\begin{cases} \dfrac{dv}{dt} = i\Delta v + \dfrac{\exp(i\lambda\tau|v|^p) - 1}{\tau} v, & x \in \mathbb{R}^d, t > 0, \\ v(x,0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{5.14}$$

whose solution satisfies

$$v(t) = S(t)\varphi + \int_0^t S(t-s)\frac{N(\tau) - \mathcal{I}}{\tau} v(s)ds, \quad t \geq 0. \tag{5.15}$$

When $0 \leq p < 4/d$ and $\varphi \in H^1(\mathbb{R}^d)$, equation (5.14) has a global $H^1(\mathbb{R}^d)$-solution (see [28], Theorem 5.2.1). We conjecture that in this case similar results to those obtained in this chapter could be obtained.

In what concerns the range $4/d < p < 4/(d-2)$, $d \geq 3$, ($4/d < p < \infty$ if $d \in \{1,2\}$) equation (5.1) entries in the subcritical H^1-case and there are instances where the solution is global (see [28], Ch. 6 for a precise statement) since we have the following conservation of energy:

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 - \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |u|^{p+1}.$$

However, in this range of p's we cannot guarantee that system (5.14) has a global H^1-solution since it is not obvious what is the *energy* which is preserved. This suggests that the $H^1(\mathbb{R}^d)$-stability for large time intervals for the splitting methods (5.5)-(5.6) will be very difficult to prove, or even impossible, even though the solutions of (5.1) are global and belong to $H^1(\mathbb{R}^d)$ at any positive time. It has been proved in [87] that the $H^1(\mathbb{R}^3)$-stability of the numerical scheme can be established assuming more regularity on the initial data, for example $H^3(\mathbb{R}^3)$ in the case $p = 2$.

Since in the case $4/d < p < 4/(d-2)$, $d \geq 3$, ($4/d < p < \infty$ if $d \in \{1,2\}$) the global existence of an H^1-solution for (5.14) is not an easy task we can only guarantee the existence of a local solution v in some time interval $[0, T_0]$ with $T_0 = T_0(\|\varphi\|_{H^1(\mathbb{R}^d)})$. In what concerns the splitting method we conjecture that there exists a positive time $T_1 \simeq T_0$ such that the solution $\{Z(n\tau)\}_{0 \leq n\tau \leq T_1}$ is uniformly bounded with respect to the time parameter τ in the $H^1(\mathbb{R}^d)$-norm. This smallness on the time interval has been also previously imposed by Fröhlich in [44] where the order of error has been obtained in the case of the Schrödinger-Poisson equation. The error analysis for small intervals of time remains to be analysed in a future work.

The analysis presented here can be extended to splitting methods in fully discrete framework by using the schemes introduced and analyzed in [53].

We now give discrete in time Strichartz-like estimates for the operator S_τ introduced in previous section. Similar estimates for space semidiscretizations and fully discrete schemes have been obtained in [62, 63] and [53]. Once the Strichartz estimates are obtained we can apply them to obtain uniform bounds on the discrete solution Z_τ.

Theorem 5.3. *The semigroup $\{S_\tau(t)\}_{t\in\mathbb{R}}$ satisfies*

$$\|S_\tau(t)\varphi\|_{L^2(\mathbb{R}^d)} \leq \|\varphi\|_{L^2(\mathbb{R}^d)}, \quad \forall\, t \in \mathbb{R}, \tag{5.16}$$

and

$$\|S_\tau(t)\varphi\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C(d)}{\tau^{d/2} + |t|^{d/2}}\|\varphi\|_{L^1(\mathbb{R}^d)}, \quad \forall\, t \in \mathbb{R}. \tag{5.17}$$

Moreover, for any admissible pairs (q,r) and $(\tilde{q},\tilde{r})$ the following hold

i) Continuous in time estimates:

$$\|S_\tau(\cdot)\varphi\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} \leq C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)}, \tag{5.18}$$

$$\Big\| \int_{\mathbb{R}} S_\tau(s)^* f(s) ds \Big\|_{L^2(\mathbb{R}^d)} \leq C(d,\tilde{q}) \|f\|_{L^{\tilde{q}'}(\mathbb{R}, L^{\tilde{r}'}(\mathbb{R}^d))}, \tag{5.19}$$

and

$$\Big\| \int_{s<t} S_\tau(t-s) f(s) ds \Big\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} \leq C(d,q,\tilde{q}) \|f\|_{L^{\tilde{q}'}(\mathbb{R}, L^{\tilde{r}'}(\mathbb{R}^d))}, \tag{5.20}$$

ii) Discrete in time estimates:

$$\|S_\tau(\cdot)\varphi\|_{l^q(\tau\mathbb{Z}, L^r(\mathbb{R}^d))} \leq C(d,q) \|\varphi\|_{L^2(\mathbb{R}^d)}, \tag{5.21}$$

$$\Big\| \tau \sum_{n\in\mathbb{Z}} S_\tau(n\tau)^* f(n\tau) \Big\|_{L^2(\mathbb{R}^d)} \leq C(d,\tilde{q}) \|f\|_{l^{\tilde{q}'}(\tau\mathbb{Z}, L^{\tilde{r}'}(\mathbb{R}^d))}, \tag{5.22}$$

and

$$\Big\| \tau \sum_{k=-\infty}^{n-1} S_\tau((n-k)\tau) f(k\tau) \Big\|_{l^q(\tau\mathbb{Z}, L^r(\mathbb{R}^d))} \leq C(d,q,\tilde{q}) \|f\|_{l^{\tilde{q}'}(\tau\mathbb{Z}, L^{\tilde{r}'}(\mathbb{R}^d))}. \tag{5.23}$$

Remark 5.1. *Inequalities* (5.16) *and* (5.17) *give us estimates for S_τ in norms which are discrete in time. When considering continuous in time norms $L^q(\mathbb{R}, L^r(\mathbb{R}^d))$ we obtain similar results since* (5.17) *implies that*

$$\|S_\tau(t)S_\tau(s)^*\varphi\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{|t-s|^{d/2}} \|\varphi\|_{L^1(\mathbb{R}^d)}, \quad \forall\, t \neq s,$$

and we apply the results of Keel and Tao, [72], Theorem 1.2.

With the above estimates we are able to prove the stability result in Theorem 5.1 and the error in Theorem 5.2. The complete proof is contained in [56].

Part III

Nonlocal Diffusion

Chapter 6

A nonlocal convection-diffusion equation

In this chapter we analyze a nonlocal equation that takes into account convective and diffusive effects. We deal with the nonlocal evolution equation

$$\begin{cases} u_t(t,x) = (J * u - u)(t,x) + (G * (f(u)) - f(u))(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{6.1}$$

Let us state first our basic assumptions. The functions J and G are nonnegatives and verify $\int_{\mathbb{R}^d} J(x)dx = \int_{\mathbb{R}^d} G(x)dx = 1$. Moreover, we consider J smooth, $J \in \mathcal{S}(\mathbb{R}^d)$, the space of rapidly decreasing functions, and radially symmetric and G smooth, $G \in \mathcal{S}(\mathbb{R}^d)$, but not necessarily symmetric. To obtain a diffusion operator similar to the Laplacian we impose in addition that J verifies

$$\frac{1}{2}\partial^2_{\xi_i\xi_i}\widehat{J}(0) = \frac{1}{2}\int_{supp(J)} J(z)z_i^2\,dz = 1.$$

This implies that

$$\widehat{J}(\xi) - 1 + \xi^2 \sim |\xi|^3, \qquad \text{for } \xi \text{ close to } 0.$$

Here $\widehat{J}$ is the Fourier transform of J and the notation $A \sim B$ means that there exist constants C_1 and C_2 such that $C_1 A \le B \le C_2 A$. We can consider more general kernels J with expansions in Fourier variables of the form $\widehat{J}(\xi) - 1 + A\,\xi^2 \sim |\xi|^3$. Since the results (and the proofs) are almost the same, we do not include the details for this more general case, but we comment on how the results are modified by the appearance of A.

The nonlinearity f will de assumed nondecreasing with $f(0) = 0$ and locally Lipschitz continuous (a typical example that we will consider below is $f(u) = |u|^{q-1}u$ with $q > 1$).

Equations like $w_t = J * w - w$ and variations of it, have been recently widely used to model diffusion processes, for example, in biology, dislocations dynamics, etc. See, for example, [9], [24], [33], [34], [35], [41], [42], [110] and [114]. As stated in [41], if $w(t,x)$ is thought of as the density of a single population at the point x at time t, and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $(J * w)(t,x) = \int_{\mathbb{R}^N} J(y-x)w(t,y)\,dy$ is the rate at which individuals are arriving to position x from all other places and $-w(t,x) = -\int_{\mathbb{R}^N} J(y-x)w(t,x)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density w satisfies the equation $w_t = J * w - w$.

In our case, see the equation in (6.1), we have a diffusion operator $J * u - u$ and a nonlinear convective part given by $G * (f(u)) - f(u)$. Concerning this last term, if G is not symmetric then individuals have greater probability of jumping in one direction than in others, provoking a convective effect.

We will call equation (6.1), a *nonlocal convection-diffusion equation.* It is nonlocal since the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$ and its derivatives at that point (t,x), but on all the values of u in a fixed spatial neighborhood of x through the convolution terms $J*u$ and $G*(f(u))$ (this neighborhood depends on the supports of J and G).

First, we prove existence, uniqueness and well-possedness of a solution with an initial condition $u(0,x)=u_0(x)\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$.

Theorem 6.1. *For any $u_0\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ there exists a unique global solution*

$$u\in C([0,\infty);L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)).$$

If u and v are solutions of (6.1) corresponding to initial data $u_0,v_0\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ respectively, then the following contraction property

$$\|u(t)-v(t)\|_{L^1(\mathbb{R}^d)}\le\|u_0-v_0\|_{L^1(\mathbb{R}^d)}$$

holds for any $t\ge 0$. In addition,

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)}\le\|u_0\|_{L^\infty(\mathbb{R}^d)}.$$

We have to emphasize that a lack of regularizing effect occurs. This has been already observed in [32] for the linear problem $w_t=J*w-w$. In [39], the authors prove that the solutions to the local convection-diffusion problem, $u_t=\Delta u+b\cdot\nabla f(u)$, satisfy an estimate of the form $\|u(t)\|_{L^\infty(\mathbb{R}^d)}\le C(\|u_0\|_{L^1(\mathbb{R}^d)})\,t^{-d/2}$ for any initial data $u_0\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. In our nonlocal model, we cannot prove such type of inequality independently of the $L^\infty(\mathbb{R}^d)$-norm of the initial data. Moreover, in the one-dimensional case with a suitable bound on the nonlinearity that appears in the convective part, f, we can prove that such an inequality does not hold in general, see Section 6.2. In addition, the $L^1(\mathbb{R}^d)-L^\infty(\mathbb{R}^d)$ regularizing effect is not available for the linear equation, $w_t=J*w-w$, see Section 6.1.

Concerning (6.1) we can obtain a solution to a standard convection-diffusion equation

$$v_t(t,x)=\Delta v(t,x)+b\cdot\nabla f(v)(t,x),\qquad t>0,\ x\in\mathbb{R}^d, \tag{6.2}$$

as the limit of solutions to (6.1) when a scaling parameter goes to zero. In fact, let us consider

$$J_\varepsilon(s)=\frac{1}{\varepsilon^d}J\left(\frac{s}{\varepsilon}\right),\qquad G_\varepsilon(s)=\frac{1}{\varepsilon^d}G\left(\frac{s}{\varepsilon}\right),$$

and the solution $u_\varepsilon(t,x)$ to our convection-diffusion problem rescaled adequately,

$$\begin{cases} (u_\varepsilon)_t(t,x) &= \dfrac{1}{\varepsilon^2}\displaystyle\int_{\mathbb{R}^d} J_\varepsilon(x-y)(u_\varepsilon(t,y)-u_\varepsilon(t,x))\,dy \\ & \qquad +\dfrac{1}{\varepsilon}\displaystyle\int_{\mathbb{R}^d} G_\varepsilon(x-y)(f(u_\varepsilon(t,y))-f(u_\varepsilon(t,x)))\,dy, \\ u_\varepsilon(x,0) &= u_0(x). \end{cases} \tag{6.3}$$

Remark that the scaling is different for the diffusive part of the equation $J*u-u$ and for the convective part $G*f(u)-f(u)$. The same different scaling properties can be observed for the local terms Δu and $b\cdot\nabla f(u)$.

Theorem 6.2. *With the above notations, for any $T>0$, we have*

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\|u_\varepsilon-v\|_{L^2(\mathbb{R}^d)}=0,$$

where $v(t,x)$ is the unique solution to the local convection-diffusion problem (6.2) with initial condition $v(x,0)=u_0(x)\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ and $b=(b_1,...,b_d)$ given by

$$b_j=\int_{\mathbb{R}^d}x_j\,G(x)\,dx,\qquad j=1,...,d.$$

This result justifies the use of the name “nonlocal convection-diffusion problem” when we refer to (6.1).

From our hypotheses on J and G it follows that they verify $|\widehat{G}(\xi) - 1 - ib\cdot\xi| \le C|\xi|^2$ and $|\widehat{J}(\xi) - 1 + \xi^2| \le C|\xi|^3$ for every $\xi \in \mathbb{R}^d$. These bounds are exactly what we are using in the proof of this convergence result.

Remark that when G is symmetric then $b = 0$ and we obtain the heat equation in the limit. Of course the most interesting case is when $b \ne 0$ (this happens when G is not symmetric). Also we note that the conclusion of the theorem holds for other $L^p(\mathbb{R}^d)$-norms besides $L^2(\mathbb{R}^d)$, however the proof is more involved.

We can consider kernels J such that

$$A = \frac{1}{2}\int_{supp(J)} J(z) z_i^2\, dz \ne 1.$$

This gives the expansion $\widehat{J}(\xi) - 1 + A\xi^2 \sim |\xi|^3$, for ξ close to 0. In this case we will arrive to a convection-diffusion equation with a multiple of the Laplacian as the diffusion operator, $v_t = A\Delta v + b\cdot\nabla f(v)$.

Next, we want to study the asymptotic behaviour as $t \to \infty$ of solutions to (6.1). To this end we first analyze the decay of solutions taking into account only the diffusive part (the linear part) of the equation. These solutions have a similar decay rate as the one that holds for the heat equation, see [32] and [59] where the Fourier transform play a key role. For the linear problem we have the following result concerning the asymptotic behaviour.

Theorem 6.3. *Let $p \in [1,\infty]$. For any $u_0 \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ the solution $w(t,x)$ of the linear problem*

$$\begin{cases} w_t(t,x) = (J * w - w)(t,x), & t>0,\ x\in\mathbb{R}^d, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^d, \end{cases} \tag{6.4}$$

satisfies the decay estimate

$$\|w(t)\|_{L^p(\mathbb{R}^d)} \le C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})\, \langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})}.$$

Throughout this paper we will use the notation $A \le \langle t\rangle^{-\alpha}$ to denote $A \le (1+t)^{-\alpha}$.

Now we are ready to face the study of the asymptotic behaviour of the complete problem (6.1). To this end we have to impose some grow condition on f. Therefore, in the sequel we restrict ourselves to nonlinearities f that are pure powers

$$f(u) = |u|^{q-1}u,\ \ q>1. \tag{6.5}$$

The analysis is more involved than the one performed for the linear part and we cannot use here the Fourier transform directly (of course, by the presence of the nonlinear term). Our strategy is to write a variation of constants formula for the solution and then prove estimates that say that the nonlinear part decay faster than the linear one. For the local convection diffusion equation this analysis was performed by Escobedo and Zuazua in [39]. However, in the previously mentioned reference energy estimates were used together with Sobolev inequalities to obtain decay bounds. These Sobolev inequalities are not available for the nonlocal model, since the linear part does not have any regularizing effect, see the remark at the end of Section 6.4. Therefore, we have to avoid the use of energy estimates and tackle the problem using a variant of the Fourier splitting method proposed by Schonbek to deal with local problems, see [94], [95] and [96].

We state our result concerning the asymptotic behaviour (decay rate) of the complete nonlocal model as follows:

Theorem 6.4. *Let f satisfies (6.5) with $q > 1$ and $u_0 \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Then, for every $p\in[1,\infty)$ the solution u of equation (6.1) verifies*

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \le C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})\, \langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})}. \tag{6.6}$$

Finally, we look at the first order term in the asymptotic expansion of the solution. For $q > (d+1)/d$, we find that this leading order term is the same as the one that appears in the linear local heat equation. This is due to the fact that the nonlinear convection is of higher order and that the radially symmetric diffusion leads to gaussian kernels in the asymptotic regime, see [32] and [59].

Theorem 6.5. *Let f satisfies (6.5) with $q > (d+1)/d$ and let the initial condition u_0 belongs to $L^1(\mathbb{R}^d, 1+|x|) \cap L^\infty(\mathbb{R}^d)$. For any $p \in [2,\infty)$ the following holds*

$$t^{-\frac{d}{2}(1-\frac{1}{p})} \|u(t) - MH(t)\|_{L^p(\mathbb{R}^d)} \leq C(J,G,p,d)\, \alpha_q(t),$$

where

$$M = \int_{\mathbb{R}^d} u_0(x)\, dx,$$

$H(t)$ is the Gaussian,

$$H(t) = \frac{e^{-\frac{x^2}{4t}}}{(2\pi t)^{\frac{d}{2}}},$$

and

$$\alpha_q(t) = \begin{cases} \langle t \rangle^{-\frac{1}{2}} & \text{if} \quad q \geq (d+2)/d, \\ \langle t \rangle^{\frac{1-d(q-1)}{2}} & \text{if} \quad (d+1)/d < q < (d+2)/d. \end{cases}$$

Remark that we prove a weak nonlinear behaviour, in fact the decay rate and the first order term in the expansion are the same that appear in the linear model $w_t = J * w - w$, see [59].

As before, recall that our hypotheses on J imply that $\widehat{J}(\xi) - (1 - |\xi|^2) \sim B|\xi|^3$, for ξ close to 0. This is the key property of J used in the proof of Theorem 6.5. We note that when we have an expansion of the form $\widehat{J}(\xi) - (1 - A|\xi|^2) \sim B|\xi|^3$, for $\xi \sim 0$, we get as first order term a Gaussian profile of the form $H_A(t) = H(At)$.

Also note that $q = (d+1)/d$ is a critical exponent for the local convection-diffusion problem, $v_t = \Delta v + b \cdot \nabla(v^q)$, see [39]. When q is supercritical, $q > (d+1)/d$, for the local equation it also holds an asymptotic simplification to the heat semigroup as $t \to \infty$.

The first order term in the asymptotic behaviour for critical or subcritical exponents $1 < q \leq (d+1)/d$ is left open. One of the main difficulties that one has to face here is the absence of a self-similar profile due to the inhomogeneous behaviour of the convolution kernels.

The rest of the paper is organized as follows: in Section 6.1 we deal with the estimates for the linear semigroup that will be used to prove existence and uniqueness of solutions as well as for the proof of the asymptotic behaviour. In Section 6.2 we prove existence and uniqueness of solutions, Theorem 6.1. In Section 6.3 we show the convergence to the local convection-diffusion equation, Theorem 6.2 and finally in Sections 6.4 and 6.5 we deal with the asymptotic behaviour, we find the decay rate and the first order term in the asymptotic expansion, Theorems 6.4 and 6.5.

6.1 The linear semigroup

In this section we analyze the asymptotic behavior of the solutions of the equation

$$\begin{cases} w_t(t,x) = (J * w - w)(t,x), & t > 0,\, x \in \mathbb{R}^d, \\ w(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{6.7}$$

When J is nonnegative and compactly supported, this equation shares many properties with the classical heat equation, $w_t = \Delta w$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and perturbations propagate with infinite speed, see [41]. However, there is no regularizing effect in general. In fact, the singularity of the source solution, that is a solution to (6.7) with initial condition a delta measure, $u_0 = \delta_0$, remains with an exponential decay. In fact, this fundamental solution can be decomposed as $S(t,x) = e^{-t}\delta_0 + K_t(x)$ where $K_t(x)$ is smooth, see Lemma 6.1. In this way we see that there is no regularizing effect since the solution w of (6.7) can be written as $w(t) = S(t) * u_0 = e^{-t}u_0 + K_t * u_0$ with K_t smooth, which means that $w(\cdot,t)$ is as regular as u_0 is. This fact makes the analysis of (6.7) more involved.

Lemma 6.1. *The fundamental solution of (6.7), that is the solution of (6.7) with initial condition $u_0 = \delta_0$, can be decomposed as*

$$S(t,x) = e^{-t}\delta_0(x) + K_t(x), \tag{6.8}$$

with $K_t(x) = K(t,x)$ smooth. Moreover, if u is the solution of (6.7) it can be written as

$$w(t,x) = (S * u_0)(t,x) = \int_{\mathbb{R}} S(t,x-y)u_0(y)\,dy.$$

In the following we will give estimates on the regular part of the fundamental solution K_t defined by:

$$K_t(x) = \int_{\mathbb{R}^d} (e^{t(\widehat{J}(\xi)-1)} - e^{-t})\, e^{ix\cdot\xi}\,d\xi, \tag{6.9}$$

that is, in the Fourier space,

$$\widehat{K}_t(\xi) = e^{t(\widehat{J}(\xi)-1)} - e^{-t}.$$

The behavior of $L^p(\mathbb{R}^d)$-norms of K_t will be obtained by analyzing the cases $p=\infty$ and $p=1$. The case $p=\infty$ follows by Hausdorff-Young's inequality. The case $p=1$ follows by using the fact that the $L^1(\mathbb{R}^d)$-norm of the solutions to (6.7) does not increase.

The following lemma gives us the decay rate of the $L^p(\mathbb{R}^d)$-norms of the kernel K_t.

Lemma 6.2. *For any $p\geq 1$ there exists a positive constant $c(p,J)$ such that K_t, defined in (6.9), satisfies:*

$$\|K_t\|_{L^p(\mathbb{R}^d)} \leq c(p,J)\,\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})} \tag{6.10}$$

for any $t>0$.

Remark 6.1. *In fact, when $p=\infty$, a stronger inequality can be proven,*

$$\|K_t\|_{L^\infty(\mathbb{R}^d)} \leq Cte^{-\delta t}\|\widehat{J}\|_{L^1(\mathbb{R}^d)} + C\,\langle t\rangle^{-d/2},$$

for some positive $\delta = \delta(J)$. Moreover, for $p=1$ we have,

$$\|K_t\|_{L^1(\mathbb{R}^d)} \leq 2$$

and for any $p\in[1,\infty]$

$$\|S(t)\|_{L^p(\mathbb{R}^d)-L^p(\mathbb{R}^d)} \leq 3.$$

Proof of Lemma 6.2. We analyze the cases $p=\infty$ and $p=1$, the others can be easily obtained applying Hölder's inequality.

Case $p=\infty$. Using Hausdorff-Young's inequality we obtain that

$$\|K_t\|_{L^\infty(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} |e^{t(\widehat{J}(\xi)-1)} - e^{-t}|d\xi.$$

Let us choose $R>0$ such that

$$|\widehat{J}(\xi)| \leq 1 - \frac{|\xi|^2}{2} \text{ for all } |\xi|\leq R. \tag{6.11}$$

Once R is fixed, there exists $\delta=\delta(J)$, $0<\delta<1$, with

$$|\widehat{J}(\xi)| \leq 1-\delta \text{ for all } |\xi|\geq R. \tag{6.12}$$

For any $|\xi| \geq R$,

$$|e^{t(\widehat{J}(\xi)-1)} - e^{-t}| \leq t|\widehat{J}(\xi)| \max\{e^{-t}, e^{t(\widehat{J}(\xi)-1)}\} \leq te^{-\delta t}|\widehat{J}(\xi)|. \tag{6.13}$$

Then the following integral decays exponentially,

$$\int_{|\xi|\geq R} |e^{t(\widehat{J}(\xi)-1)} - e^{-t}| d\xi \leq e^{-\delta t} t \int_{|\xi|\geq R} |\widehat{J}(\xi)| d\xi.$$

Using that this term is exponentially small, it remains to prove that

$$I(t) = \int_{|\xi|\leq R} |e^{t(\widehat{J}(\xi)-1)} - e^{-t}| d\xi \leq C\langle t\rangle^{-d/2}. \tag{6.14}$$

To handle this case we use the following estimates:

$$|I(t)| \leq \int_{|\xi|\leq R} e^{t(\widehat{J}(\xi)-1)} d\xi + e^{-t}C(R) \leq \int_{|\xi|\leq R} d\xi + e^{-t}C(R) \leq C(R)$$

and

$$\begin{aligned} |I(t)| &\leq \int_{|\xi|\leq R} e^{t(\widehat{J}(\xi)-1)} d\xi + e^{-t}C(R) \leq \int_{|\xi|\leq R} e^{-\frac{t|\xi|^2}{2}} + e^{-t}C(R) \\ &= t^{-d/2} \int_{|\eta|\leq Rt^{1/2}} e^{-\frac{|\eta|^2}{2}} + e^{-t}C(R) \leq Ct^{-d/2}. \end{aligned}$$

The last two estimates prove (6.14) and this finishes the analysis of this case.

Case $p = 1$. First we prove that the $L^1(\mathbb{R}^d)$-norm of the solutions to equation (6.4) does not increase. Multiplying equation (6.4) by $\operatorname{sgn}(w(t,x))$ and integrating in space variable we obtain,

$$\frac{d}{dt} \int_{\mathbb{R}^d} |w(t,x)|\, dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x-y)|w(t,y)|\, dx\, dy - \int_{\mathbb{R}^d} |w(t,x)|\, dx \leq 0,$$

which shows that the $L^1(\mathbb{R}^d)$-norm does not increase. Hence, for any $u_0 \in L^1(\mathbb{R}^d)$, the following holds:

$$\int_{\mathbb{R}^d} |e^{-t}u_0(x) + (K_t * u_0)(x)|\, dx \leq \int_{\mathbb{R}^d} |u_0(x)|\, dx,$$

and as a consequence,

$$\int_{\mathbb{R}^d} |(K_t * u_0)(x)|\, dx \leq 2 \int_{\mathbb{R}^d} |u_0(x)|\, dx.$$

Choosing $(u_0)_n \in L^1(\mathbb{R}^d)$ such that $(u_0)_n \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$ we obtain in the limit that

$$\int_{\mathbb{R}^d} |K_t(x)| dx \leq 2.$$

This ends the proof of the L^1-case and finishes the proof. □

The following lemma will play a key role when analyzing the decay of the complete problem (6.1). In the sequel we will denote by $L^1(\mathbb{R}^d, a(x))$ the following space:

$$L^1(\mathbb{R}^d, a(x)) = \left\{ \varphi : \int_{\mathbb{R}^d} a(x)|\varphi(x)| dx < \infty \right\}.$$

Lemma 6.3. *Let $p \geq 1$ and $J \in \mathcal{S}(\mathbb{R}^d)$. There exists a positive constant $c(p, J)$ such that*

$$\|K_t * \varphi - K_t\|_{L^p(\mathbb{R}^d)} \leq c(p)\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}} \|\varphi\|_{L^1(\mathbb{R}^d, |x|)}$$

holds for all $\varphi \in L^1(\mathbb{R}^d, 1 + |x|)$.

We now prove a decay estimate that takes into account the linear semigroup applied to the convolution with a kernel G.

Lemma 6.4. *Let $1 \le p \le r \le \infty$, $J \in \mathcal{S}(\mathbb{R}^d)$ and $G \in L^1(\mathbb{R}^d, |x|)$. There exists a positive constant $C = C(p, J, G)$ such that the following estimate*

$$\|S(t) * G * \varphi - S(t) * \varphi\|_{L^r(\mathbb{R}^d)} \le C \langle t\rangle^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}}(\|\varphi\|_{L^p(\mathbb{R}^d)} + \|\varphi\|_{L^r(\mathbb{R}^d)}). \tag{6.15}$$

holds for all $\varphi \in L^p(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$.

Remark 6.2. *In fact the following stronger inequality holds:*

$$\|S(t) * G * \varphi - S(t) * \varphi\|_{L^r(\mathbb{R}^d)} \le C\, \langle t\rangle^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}} \|\varphi\|_{L^p(\mathbb{R}^d)} + C\, e^{-t}\|\varphi\|_{L^r(\mathbb{R}^d)}.$$

Proof. We write $S(t)$ as $S(t) = e^{-t}\delta_0 + K_t$ and we get

$$S(t) * G * \varphi - S(t) * \varphi = e^{-t}(G * \varphi - \varphi) + K_t * G * \varphi - K_t * \varphi.$$

The first term in the above right hand side verifies:

$$e^{-t}\|G * \varphi - \varphi\|_{L^r(\mathbb{R}^d)} \le e^{-t}(\|G\|_{L^1(\mathbb{R}^d)}\|\varphi\|_{L^r(\mathbb{R}^d)} + \|\varphi\|_{L^r(\mathbb{R}^d)}) \le 2e^{-t}\|\varphi\|_{L^r(\mathbb{R}^d)}.$$

For the second one, by Lemma 6.3 we get that K_t satisfies

$$\|K_t * G - K_t\|_{L^a(\mathbb{R}^d)} \le C(r, J)\|G\|_{L^1(\mathbb{R}^d, |x|)}\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{a})-\frac{1}{2}}$$

for all $t \ge 0$ where a is such that $1/r = 1/a + 1/p - 1$. Then, using Young's inequality we end the proof. □

6.2 Existence and uniqueness

In this section we use the previous results and estimates on the linear semigroup to prove the existence and uniqueness of the solution to our nonlinear problem (6.1). The proof is based on the variation of constants formula and uses the previous properties of the linear diffusion semigroup.

Proof of Theorem 6.1. Recall that we want prove the global existence of solutions for initial conditions $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

Let us consider the following integral equation associated with (6.1):

$$u(t) = S(t) * u_0 + \int_0^t S(t-s) * (G * (f(u)) - f(u))(s)\, ds, \tag{6.16}$$

the functional

$$\Phi[u](t) = S(t) * u_0 + \int_0^t S(t-s) * (G * (f(u)) - f(u))(s)\, ds$$

and the space

$$X(T) = C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty([0,T];\, \mathbb{R}^d)$$

endowed with the norm

$$\|u\|_{X(T)} = \sup_{t\in[0,T]} \left(\|u(t)\|_{L^1(\mathbb{R}^d)} + \|u(t)\|_{L^\infty(\mathbb{R}^d)}\right).$$

It follows that Φ is a contraction in the ball of radius R, B_R, of X_T, if T is small enough. This proves the local existence of the solutions.

To prove the global well posedness of the solutions we have to guarantee that both $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$-norms of the solutions do not blow up in finite time. We will apply the following lemma to control the $L^\infty(\mathbb{R}^d)$-norm of the solutions.

Lemma 6.5. *Let $\theta \in L^1(\mathbb{R}^d)$ and K be a nonnegative function with mass one. Then for any $\mu \geq 0$ the following hold:*

$$\int_{\theta(x)>\mu} \int_{\mathbb{R}^d} K(x-y)\theta(y)\,dy\,dx \leq \int_{\theta(x)>\mu} \theta(x)\,dx \tag{6.17}$$

and

$$\int_{\theta(x)<-\mu} \int_{\mathbb{R}^d} K(x-y)\theta(y)\,dy\,dx \geq \int_{\theta(x)<-\mu} \theta(x)\,dx. \tag{6.18}$$

Control of the L^1-norm. As in the previous section, we multiply equation (6.1) by $\operatorname{sgn}(u(t,x))$ and integrate in $\mathbb{R}^d$ to obtain the following estimate

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} |u(t,x)|\,dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)u(t,y)\operatorname{sgn}(u(t,x))\,dy\,dx - \int_{\mathbb{R}^d} |u(t,x)|\,dx \\ &+ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)f(u(t,y))\operatorname{sgn}(u(t,x))\,dy\,dx - \int_{\mathbb{R}^d} f(u(t,x))\operatorname{sgn}(u(t,x))\,dx \\ &\leq 0,\end{aligned}$$

which shows that the L^1-norm does not increase.

Control of the L^∞-norm. Let us denote $m = \|u_0\|_{L^\infty(\mathbb{R}^d)}$. Multiplying the equation in (6.1) by $\operatorname{sgn}(u-m)^+$ and integrating in the x variable we get,

$$\frac{d}{dt}\int_{\mathbb{R}^d} (u(t,x)-m)^+ dx = I_1(t) + I_2(t)$$

where

$$I_1(t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)u(t,y)\operatorname{sgn}(u(t,x)-m)^+\,dy\,dx - \int_{\mathbb{R}^d} u(t,x)\operatorname{sgn}(u(t,x)-m)^+\,dx$$

and

$$\begin{aligned}I_2(t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} &G(x-y)f(u)(t,y)\operatorname{sgn}(u(t,x)-m)^+\,dy\,dx \\ &- \int_{\mathbb{R}^d} f(u)(t,x)\operatorname{sgn}(u(t,x)-m)^+\,dx.\end{aligned}$$

We claim that both I_1 and I_2 are negative. Thus $(u(t,x)-m)^+ = 0$ a.e. $x \in \mathbb{R}^d$ and then $u(t,x) \leq m$ for all $t > 0$ and a.e. $x \in \mathbb{R}^d$.

In the case of I_1, applying Lemma 6.5 with $K = J$, $\theta = u(t)$ and $\mu = m$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)u(t,y)\operatorname{sgn}(u(t,x)-m)^+\,dy\,dx &= \int_{u(x)>m}\int_{\mathbb{R}^d} J(x-y)u(t,y)\,dy\,dx \\ &\leq \int_{u(x)>m} u(t,x)\,dx.\end{aligned}$$

To handle the second one, I_2, we proceed in a similar manner. Applying Lemma 6.5 with

$$\theta(x) = f(u)(t,x) \quad \text{and} \quad \mu = f(m)$$

we obtain

$$\int_{f(u(t,x))>f(m)} \int_{\mathbb{R}^d} G(x-y)f(u)(t,y)\,dy\,dx \leq \int_{f(u(t,x))>f(m)} f(u)(t,x)\,dx.$$

Using that f is a nondecreasing function, we rewrite this inequality in an equivalent form te obtain the desired inequality:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)f(u)(t,y)\,\mathrm{sgn}(u(t,x)-m)^+\,dy\,dx = \int_{u(t,x)\geq m}\int_{\mathbb{R}^d} G(x-y)f(u)(t,y)\,dy\,dx$$
$$= \int_{f(u)(t,x)\geq f(m)}\int_{\mathbb{R}^d} G(x-y)f(u)(t,y)\,dy\,dx \leq \int_{u(t,x)\geq m} f(u)(t,x)\,dx.$$

In a similar way, by using inequality (6.18) we get

$$\frac{d}{dt}\int_{\mathbb{R}^d}(u(t,x)+m)^-\,dx \leq 0,$$

which implies that $u(t,x)\geq -m$ for all $t>0$ and a.e. $x\in\mathbb{R}^d$.

We conclude that $\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}$.

Step III. Uniqueness and contraction property. Let us consider u and v two solutions corresponding to initial data u_0 and v_0 respectively. We will prove that for any $t>0$ the following holds:

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)-v(t,x)|\,dx \leq 0.$$

To this end, we multiply by $\mathrm{sgn}(u(t,x)-v(t,x))$ the equation satisfied by $u-v$ and using the symmetry of J, the positivity of J and G and that their mass equals one we obtain,

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)-v(t,x)|\,dx \leq\ 0.$$

Thus we get the uniqueness of the solutions and the contraction property

$$\|u(t)-v(t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0-v_0\|_{L^1(\mathbb{R}^d)}.$$

This ends the proof of Theorem 6.1. □

Now we prove that, due to the lack of regularizing effect, the $L^\infty(\mathbb{R})$-norm does not get bounded for positive times when we consider initial conditions in $L^1(\mathbb{R})$. This is in contrast to what happens for the local convection-diffusion problem, see [39].

Proposition 6.1. *Let $d=1$ and $|f(u)|\leq C|u|^q$ with $1\leq q<2$. Then*

$$\sup_{u_0\in L^1(\mathbb{R})}\sup_{t\in[0,1]}\frac{t^{\frac12}\|u(t)\|_{L^\infty(\mathbb{R})}}{\|u_0\|_{L^1(\mathbb{R})}} = \infty.$$

Proof. Assume by contradiction that

$$\sup_{u_0\in L^1(\mathbb{R})}\sup_{t\in[0,1]}\frac{t^{\frac12}\|u(t)\|_{L^\infty(\mathbb{R})}}{\|u_0\|_{L^1(\mathbb{R})}} = M<\infty. \tag{6.19}$$

Using the representation formula (6.16) we get:

$$\|u(1)\|_{L^\infty(\mathbb{R})} \geq \|S(1)*u_0\|_{L^\infty(\mathbb{R})} - \left\|\int_0^1 S(1-s)*(G*(f(u))-f(u))(s)\,ds\right\|_{L^\infty(\mathbb{R})}$$

Using Lemma 6.4 the last term can be bounded as follows:

$$\left\|\int_0^1 S(1-s)*(G*(f(u))-f(u))(s)\,ds\right\|_{L^\infty(\mathbb{R})} \leq \int_0^1 \langle 1-s\rangle^{-\frac12}\|f(u(s))\|_{L^\infty(\mathbb{R})}\,ds$$
$$\leq C\int_0^1\|u(s)\|^q_{L^\infty(\mathbb{R})}ds \leq CM^q\|u_0\|^q_{L^1(\mathbb{R})}\int_0^1 s^{-\frac{q}{2}}ds$$
$$\leq CM^q\|u_0\|^q_{L^1(\mathbb{R})},$$

provided that $q < 2$.

This implies that the $L^\infty(\mathbb{R})$-norm of the solution at time $t = 1$ satisfies

$$\begin{aligned}\|u(1)\|_{L^\infty(\mathbb{R})} &\geq \|S(1) * u_0\|_{L^\infty(\mathbb{R})} - CM^q \|u_0\|^q_{L^1(\mathbb{R})}\\ &\geq e^{-1}\|u_0\|_{L^\infty(\mathbb{R})} - \|K_1\|_{L^\infty(\mathbb{R})}\|u_0\|_{L^1(\mathbb{R})} - CM^q\|u_0\|^q_{L^1(\mathbb{R})}\\ &\geq e^{-1}\|u_0\|_{L^\infty(\mathbb{R})} - C\|u_0\|_{L^1(\mathbb{R})} - CM^q\|u_0\|^q_{L^1(\mathbb{R})}.\end{aligned}$$

Choosing now a sequence $u_{0,\varepsilon}$ with $\|u_{0,\varepsilon}\|_{L^1(\mathbb{R})} = 1$ and $\|u_{0,\varepsilon}\|_{L^\infty(\mathbb{R})} \to \infty$ we obtain that

$$\|u_{0,\varepsilon}(1)\|_{L^\infty(\mathbb{R})} \to \infty,$$

a contradiction with our assumption (6.19). The proof of the result is now completed. □

6.3 Convergence to the local problem

In this section we prove the convergence of solutions of the nonlocal problem to solutions of the local convection-diffusion equation when we rescale the kernels and let the scaling parameter go to zero.

As we did in the previous sections we begin with the analysis of the linear part.

Lemma 6.6. *Assume that $u_0 \in L^2(\mathbb{R}^d)$. Let w_ε be the solution to*

$$\begin{cases} (w_\varepsilon)_t(t,x) = \dfrac{1}{\varepsilon^2}\displaystyle\int_{\mathbb{R}^d} J_\varepsilon(x-y)(w_\varepsilon(t,y) - w_\varepsilon(t,x))\,dy,\\ w_\varepsilon(0,x) = u_0(x),\end{cases} \tag{6.20}$$

and w the solution to

$$\begin{cases} w_t(t,x) = \Delta w(t,x),\\ w(0,x) = u_0(x).\end{cases} \tag{6.21}$$

Then, for any positive T,

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]} \|w_\varepsilon - w\|_{L^2(\mathbb{R}^d)} = 0.$$

Next we use a lemma that provides us with a uniform (independent of ε) decay for the nonlocal convective part.

Lemma 6.7. *There exists a positive constant $C = C(J, G)$ such that*

$$\left\|\Big(\frac{S_\varepsilon(t) * G_\varepsilon - S_\varepsilon(t)}{\varepsilon}\Big) * \varphi\right\|_{L^2(\mathbb{R}^d)} \leq C\, t^{-\frac{1}{2}}\, \|\varphi\|_{L^2(\mathbb{R}^d)}$$

holds for all $t > 0$ and $\varphi \in L^2(\mathbb{R}^d)$, uniformly on $\varepsilon > 0$. Here $S_\varepsilon(t)$ is the linear semigroup associated to (6.20).

Also, the following result will be useful in the proof of Theorem 6.2.

Lemma 6.8. *Let be $T > 0$ and $M > 0$. Then the following*

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\int_0^t \left\|\Big(\frac{S_\varepsilon(s) * G_\varepsilon - S_\varepsilon(s)}{\varepsilon} - b\cdot\nabla H(s)\Big) * \varphi(s)\right\|_{L^2(\mathbb{R}^d)} ds = 0,$$

holds uniformly for all $\|\varphi\|_{L^\infty([0,T];L^2(\mathbb{R}^d))} \leq M$. Here H is the linear heat semigroup given by the Gaussian

$$H(t) = \frac{e^{-\frac{x^2}{4t}}}{(2\pi t)^{\frac{d}{2}}}$$

and $b = (b_1, ..., b_d)$ *is given by*

$$b_j = \int_{\mathbb{R}^d} x_j \, G(x)\, dx, \qquad j = 1, ..., d.$$

Now we are ready to prove Theorem 6.2.

Proof of Theorem 6.2. First we write the two problems in the semigroup formulation,

$$u_\varepsilon(t) = S_\varepsilon(t) * u_0 + \int_0^t \frac{S_\varepsilon(t-s) * G_\varepsilon - S_\varepsilon(t-s)}{\varepsilon} * f(u_\varepsilon(s))\, ds$$

and

$$v(t) = H(t) * u_0 + \int_0^t b \cdot \nabla H(t-s) * f(v(s))\, ds.$$

Then

$$\sup_{t\in[0,T]} \|u_\varepsilon(t) - v(t)\|_{L^2(\mathbb{R}^d)} \le \sup_{t\in[0,T]} I_{1,\varepsilon}(t) + \sup_{t\in[0,T]} I_{2,\varepsilon}(t) \tag{6.22}$$

where

$$I_{1,\varepsilon}(t) = \|S_\varepsilon(t) * u_0 - H(t) * u_0\|_{L^2(\mathbb{R}^d)}$$

and

$$I_{2,\varepsilon}(t) = \left\| \int_0^t \frac{S_\varepsilon(t-s) * G_\varepsilon - S_\varepsilon(t-s)}{\varepsilon} * f(u_\varepsilon(s)) - \int_0^t b \cdot \nabla H(t-s) * f(v(s)) \right\|_{L^2(\mathbb{R}^d)}.$$

In view of Lemma 6.6 we have

$$\sup_{t\in[0,T]} I_{1,\varepsilon}(t) \to 0 \qquad \text{as } \varepsilon \to 0.$$

So it remains to analyze the second term $I_{2,\varepsilon}$. To this end, we split it again

$$I_{2,\varepsilon}(t) \le I_{3,\varepsilon}(t) + I_{4,\varepsilon}(t)$$

where

$$I_{3,\varepsilon}(t) = \int_0^t \left\| \frac{S_\varepsilon(t-s) * G_\varepsilon - S_\varepsilon(t-s)}{\varepsilon} * \big(f(u_\varepsilon(s)) - f(v(s))\big) \right\|_{L^2(\mathbb{R}^d)} ds$$

and

$$I_{4,\varepsilon}(t) = \int_0^t \left\| \left(\frac{S_\varepsilon(t-s) * G_\varepsilon - S_\varepsilon(t-s)}{\varepsilon} - b \cdot \nabla H(t-s) \right) * f(v(s)) \right\|_{L^2(\mathbb{R}^d)} ds.$$

Using Young's inequality and that from our hypotheses we have an uniform bound for u_ε and u in terms of $\|u_0\|_{L^1(\mathbb{R}^d)}$, $\|u_0\|_{L^\infty(\mathbb{R}^d)}$ we obtain

$$\begin{aligned} I_{3,\varepsilon}(t) &\le \int_0^t \frac{\|f(u_\varepsilon(s)) - f(v(s))\|_{L^2(\mathbb{R}^d)}}{|t-s|^{\frac12}} ds \\ &\le \|f(u_\varepsilon) - f(v)\|_{L^\infty((0,T);\, L^2(\mathbb{R}^d))} \int_0^t \frac{ds}{|t-s|^{\frac12}} \\ &\le 2T^{1/2} \|u_\varepsilon - v\|_{L^\infty((0,T);\, L^2(\mathbb{R}^d))} C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}). \end{aligned} \tag{6.23}$$

By Lemma 6.8 (see [58] for the full details) we obtain that

$$\sup_{t\in[0,T]} I_{4,\varepsilon} \le C\varepsilon^{\frac13} \|f(v)\|_{L^\infty((0,T);\, L^2(\mathbb{R}^d))} \le C\varepsilon^{\frac13} C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}). \tag{6.24}$$

Using (6.22), (6.23) and (6.24) we get:

$$\begin{aligned}\|u_\varepsilon - v\|_{L^\infty((0,T);\, L^2(\mathbb{R}^d))} \leq \quad & \|I_{1,\varepsilon}\|_{L^\infty((0,T);\, L^2(\mathbb{R}^d))} \\ & + T^{\frac{1}{2}} C(\|u_0\|_{L^1(\mathbb{R})}, \|u_0\|_{L^\infty(\mathbb{R})}) \|u_\varepsilon - v\|_{L^\infty((0,T);\, L^2(\mathbb{R}^d))}.\end{aligned}$$

Choosing $T = T_0$ sufficiently small, depending on $\|u_0\|_{L^1(\mathbb{R})}$ and $\|u_0\|_{L^\infty(\mathbb{R})}$ we get

$$\|u_\varepsilon - v\|_{L^\infty((0,T);\, L^2(\mathbb{R}^d))} \leq \|I_{1,\varepsilon}\|_{L^\infty((0,T);\, L^2(\mathbb{R}^d))} \to 0,$$

as $\varepsilon \to 0$.

Using the same argument in any interval $[\tau, \tau + T_0]$, the stability of the solutions of the equation (6.3) in $L^2(\mathbb{R}^d)$-norm and that for any time $\tau > 0$ it holds that

$$\|u_\varepsilon(\tau)\|_{L^1(\mathbb{R}^d)} + \|u_\varepsilon(\tau)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)} + \|u_0\|_{L^\infty(\mathbb{R}^d)},$$

we obtain

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_\varepsilon - v\|_{L^2(\mathbb{R}^d)} = 0,$$

as we wanted to prove. □

6.4 Long time behaviour of the solutions

The aim of this section is to obtain the first term in the asymptotic expansion of the solution u to (6.1). The main ingredient for our proofs is the following lemma inspired in the Fourier splitting method introduced by Schonbek, see [94], [95] and [96].

Lemma 6.9. *Let R and δ be such that the function $\widehat{J}$ satisfies:*

$$\widehat{J}(\xi) \leq 1 - \frac{|\xi|^2}{2}, \qquad |\xi| \leq R \tag{6.25}$$

and

$$\widehat{J}(\xi) \leq 1 - \delta, \qquad |\xi| \geq R. \tag{6.26}$$

Let us assume that the function $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following differential inequality:

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u(t,x)|^2 \, dx \leq c \int_{\mathbb{R}^d} (J * u - u)(t,x) u(t,x) \, dx, \tag{6.27}$$

for any $t > 0$. Then for any $1 \leq r < \infty$ there exists a constant $a = rd/c\delta$ such that

$$\int_{\mathbb{R}^d} |u(at,x)|^2 dx \leq \frac{\|u(0)\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}} + \frac{rd\omega_0 (2\delta)^{\frac{d}{2}}}{(t+1)^{rd}} \int_0^t (s+1)^{rd - \frac{d}{2} - 1} \|u(as)\|^2_{L^1(\mathbb{R}^d)} ds \tag{6.28}$$

holds for all positive time t where ω_0 is the volume of the unit ball in $\mathbb{R}^d$. In particular

$$\|u(at)\|_{L^2(\mathbb{R}^d)} \leq \frac{\|u(0)\|_{L^2(\mathbb{R}^d)}}{(t+1)^{\frac{rd}{2}}} + \frac{(2\omega_0)^{\frac{1}{2}} (2\delta)^{\frac{d}{4}}}{(t+1)^{\frac{d}{4}}} \|u\|_{L^\infty([0,\infty);\, L^1(\mathbb{R}^d))}. \tag{6.29}$$

Remark 6.3. *The differential inequality (6.27) can be written in the following form:*

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u(t,x)|^2 \, dx \leq -\frac{c}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x-y)(u(t,x) - u(t,y))^2 \, dx \, dy.$$

This is the nonlocal version of the energy method used in [39]. However, in our case, exactly the same inequalities used in [39] could not be applied.

We also need the nonlocal version of the well known identity

$$\int_{\mathbb{R}^d} \Delta u\, |u|^{p-1} \operatorname{sgn}(u)\, dx = -\frac{4(p-1)}{p^2} \int_{\mathbb{R}^d} |\nabla(|u|^{p/2})|^2\, dx.$$

Lemma 6.10. *([58]) Let $2 \le p < \infty$. For any function $u : \mathbb{R}^d \mapsto \mathbb{R}^d$, $I(u)$ defined by*

$$I(u) = \int_{\mathbb{R}^d} (J * u - u)(x)|u(x)|^{p-1} \operatorname{sgn}(u(x))\, dx$$

satisfies

$$\begin{aligned} I(u) &\le \frac{4(p-1)}{p^2} \int_{\mathbb{R}^d} (J * |u|^{p/2} - |u|^{p/2})(x)|u(x)|^{p/2}\, dx \\ &= -\frac{2(p-1)}{p^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x-y)(|u(y)|^{p/2} - |u(x)|^{p/2})^2\, dx\, dy. \end{aligned}$$

Now we are ready to proceed with the proof of Theorem 6.4.

Proof of Theorem 6.4. Let u be the solution to the nonlocal convection-diffusion problem. Then, by the same arguments that we used to control the $L^1(\mathbb{R}^d)$-norm, we obtain the following:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |u(t,x)|^p dx &= p \int_{\mathbb{R}^d} (J * u - u)(t,x)|u(t,x)|^{p-1} \operatorname{sgn}(u(t,x))\, dx \\ &\quad + \int_{\mathbb{R}^d} (G * f(u) - f(u))(t,x)|u(t,x)|^{p-1} \operatorname{sgn}(u(t,x))\, dx \\ &\le p \int_{\mathbb{R}^d} (J * u - u)(t,x)|u(t,x)|^{p-1} \operatorname{sgn}(u(t,x))\, dx. \end{aligned}$$

Using Lemma 6.10 we get that the $L^p(\mathbb{R}^d)$-norm of the solution u satisfies the following differential inequality:

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u(t,x)|^p\, dx \le \frac{4(p-1)}{p} \int_{\mathbb{R}^d} (J * |u|^{p/2} - |u|^{p/2})(x)|u(x)|^{p/2}\, dx. \tag{6.30}$$

First, let us consider $p = 2$. Then

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u(t,x)|^2\, dx \le 2 \int_{\mathbb{R}^d} (J * |u| - |u|)(t,x)|u(t,x)|\, dx.$$

Applying Lemma 6.9 with $|u|$, $c = 2$, $r = 1$ and using that $\|u\|_{L^\infty([0,\infty);\, L^1(\mathbb{R}^d))} \le \|u_0\|_{L^1(\mathbb{R}^d)}$ we obtain

$$\begin{aligned} \|u(td/2\delta)\|_{L^2(\mathbb{R})} &\le \frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{(t+1)^{\frac{d}{2}}} + \frac{(2\omega_0)^{\frac{1}{2}}(2\delta)^{\frac{d}{4}}}{(t+1)^{\frac{d}{4}}} \|u\|_{L^\infty([0,\infty);\, L^1(\mathbb{R}^d))} \\ &\le \frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{(t+1)^{\frac{d}{2}}} + \frac{(2\omega_0)^{\frac{1}{2}}(2\delta)^{\frac{d}{4}}}{(t+1)^{\frac{d}{4}}} \|u_0\|_{L^1(\mathbb{R}^d))} \le \frac{C(J, \|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})}{(t+1)^{\frac{d}{4}}}, \end{aligned}$$

which proves (6.6) in the case $p = 2$. Using that the $L^1(\mathbb{R}^d)$-norm of the solutions to (6.1), does not increase, $\|u(t)\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)}$, by Hölder's inequality we obtain the desired decay rate (6.6) in any $L^p(\mathbb{R}^d)$-norm with $p \in [1,2]$.

In the following, using an inductive argument, we will prove the result for any $r = 2^m$, with $m \ge 1$ an integer. By Hölder's inequality this will give us the $L^p(\mathbb{R}^d)$-norm decay for any $2 < p < \infty$.

Let us choose $r = 2^m$ with $m \ge 1$ and assume that the following

$$\|u(t)\|_{L^r(\mathbb{R}^d)} \le C\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{r})}$$

holds for some positive constant $C = C(J, \|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})$ and for every positive time t. We want to show an analogous estimate for $p = 2r = 2^{m+1}$.

We use (6.30) with $p = 2r$ to obtain the following differential inequality:

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u(t,x)|^{2r} dx \le \frac{4(2r-1)}{2r}\int_{\mathbb{R}^d} (J * |u|^r - |u|^r)(t,x)|u(t,x)|^r dx.$$

Applying Lemma (6.9) with $|u|^r$, $c(r) = 2(2r-1)/r$ and $a = rd/c(r)\delta$ we get:

$$\begin{aligned}
\int_{\mathbb{R}^d} |u(at)|^{2r} &\le \frac{\|u_0^r\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}} + \frac{d\omega_0(2\delta)^{\frac{d}{2}}}{(t+1)^{rd}}\int_0^t (s+1)^{rd-\frac{d}{2}-1}\|u^r(as)\|^2_{L^1(\mathbb{R}^d)} ds \\
&\le \frac{\|u_0\|^{2r}_{L^{2r}(\mathbb{R}^d)}}{(t+1)^{rd}} + \frac{C(J)}{(t+1)^{rd}}\int_0^t (s+1)^{rd-\frac{d}{2}-1}\|u(as)\|^{2r}_{L^r(\mathbb{R}^d)} ds \\
&\le \frac{C(J, \|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})}{(t+1)^d}\left(1 + \int_0^t (s+1)^{rd-\frac{d}{2}-1}(s+1)^{-dr(1-\frac{1}{r})} ds\right) \\
&\le \frac{C}{(t+1)^{dr}}(1 + (t+1)^{\frac{d}{2}}) \le C(t+1)^{\frac{d}{2}-dr}
\end{aligned}$$

and then

$$\|u(at)\|_{L^{2r}(\mathbb{R}^d)} \le C(J, \|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})(t+1)^{-\frac{d}{2}(1-\frac{1}{2r})},$$

which finishes the proof. $\square$

Let us close this section with a remark concerning the applicability of energy methods to study nonlocal problems. If we want to use energy estimates to get decay rates (for example in $L^2(\mathbb{R}^d)$), we arrive easily to

$$\frac{d}{dt}\int_{\mathbb{R}^d} |w(t,x)|^2 \, dx = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(w(t,x)-w(t,y))^2 \, dx \, dy$$

when we deal with a solution of the linear equation $w_t = J * w - w$ and to

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u(t,x)|^2 \, dx \le -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(t,x)-u(t,y))^2 \, dx \, dy$$

when we consider the complete convection-diffusion problem. However, we can not go further since an inequality of the form

$$\left(\int_{\mathbb{R}^d} |u(x)|^p \, dx\right)^{\frac{2}{p}} \le C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(x)-u(y))^2 \, dx \, dy \tag{6.31}$$

is not available for $p > 2$. However, in the next chapter we will develop a method which avoid the use of inequality (6.31).

6.5 Weakly nonlinear behaviour

In this section we find the leading order term in the asymptotic expansion of the solution to (6.1). We use ideas from [39] showing that the nonlinear term decays faster than the linear part.

We recall a previous result of [59] that extends to nonlocal diffusion problems the result of [38] in the case of the heat equation.

Lemma 6.11. *Let $J \in \mathcal{S}(\mathbb{R}^d)$ such that*

$$\widehat{J}(\xi) - (1 - |\xi|^2) \sim B|\xi|^3, \qquad \xi \sim 0,$$

for some constant B. For every $p \in [2,\infty)$, there exists some positive constant $C = C(p,J)$ such that

$$\|S(t)*\varphi - MH(t)\|_{L^p(\mathbb{R}^d)} \le Ce^{-t}\|\varphi\|_{L^p(\mathbb{R}^d)} + C\|\varphi\|_{L^1(\mathbb{R}^d,|x|)}\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}},\ t>0, \qquad (6.32)$$

for every $\varphi \in L^1(\mathbb{R}^d, 1+|x|)$ with $M = \int_{\mathbb{R}} \varphi(x)\,dx$, where

$$H(t) = \frac{e^{-\frac{x^2}{4t}}}{(2\pi t)^{\frac{d}{2}}},$$

is the gaussian.

Remark 6.4. *We can consider a condition like $\widehat{J}(\xi) - (1 - A|\xi|^2) \sim B|\xi|^3$ for $\xi \sim 0$ and obtain as profile a modified Gaussian $H_A(t) = H(At)$, but we omit the tedious details.*

Remark 6.5. *The case $p \in [1,2)$ is more subtle. The analysis performed in the previous sections to handle the case $p = 1$ can be also extended to cover this case when the dimension d verifies $1 \le d \le 3$. Indeed in this case, if J satisfies $\widehat{J}(\xi) \sim 1 - A|\xi|^s$, $\xi \sim 0$, then s has to be grater than $[d/2]+1$ and $s = 2$ to obtain the Gaussian profile.*

Proof. We write $S(t) = e^{-t}\delta_0 + K_t$. Then it is sufficient to prove that

$$\|K_t*\varphi - MK_t\|_{L^p(\mathbb{R}^d)} \le C\|\varphi\|_{L^1(\mathbb{R}^d,|x|)}\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}$$

and

$$t^{\frac{d}{2}(1-\frac{1}{p})}\|K_t - H(t)\|_{L^p(\mathbb{R}^d)} \le C\langle t\rangle^{-\frac{1}{2}}.$$

The first estimate follows by Lemma 6.3. The second one uses the hypotheses on $\widehat{J}$. A detailed proof can be found in [59]. □

Now we are ready to prove that the same expansion holds for solutions to the complete problem (6.1) when $q > (d+1)/d$.

Proof of Theorem 6.5. In view of (6.32) it is sufficient to prove that

$$t^{-\frac{d}{2}(1-\frac{1}{p})}\|u(t) - S(t)*u_0\|_{L^p(\mathbb{R}^d)} \le C\langle t\rangle^{-\frac{d}{2}(q-1)+\frac{1}{2}}.$$

Using the representation (6.16) we get that

$$\|u(t) - S(t)*u_0\|_{L^p(\mathbb{R}^d)} \le \int_0^t \|[S(t-s)*G - S(t-s)]*|u(s)|^{q-1}u(s)\|_{L^p(\mathbb{R}^d)}\,ds.$$

We now estimate the right hand side term as follows: we will split it in two parts, one in which we integrate on $(0,t/2)$ and another one where we integrate on $(t/2,t)$. Concerning the second term, by Lemma 6.4, Theorem 6.4 we have,

$$\begin{aligned}
&\int_{t/2}^t \|[S(t-s)*G - S(t-s)]*|u(s)|^{q-1}u(s)\|_{L^p(\mathbb{R}^d)}ds \\
&\qquad \le C(J,G)\int_{t/2}^t \langle t-s\rangle^{-\frac{1}{2}}\|u(s)\|^q_{L^{pq}(\mathbb{R}^d)}ds \\
&\qquad \le C(J,G,\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R})})\int_{t/2}^t \langle t-s\rangle^{-\frac{1}{2}}\langle s\rangle^{-\frac{d}{2}(q-\frac{1}{p})}ds \\
&\qquad \le C\langle t\rangle^{-\frac{d}{2}(q-\frac{1}{p})+\frac{1}{2}} \le Ct^{-\frac{d}{2}(1-\frac{1}{p})}\langle t\rangle^{-\frac{d}{2}(q-1)+\frac{1}{2}}.
\end{aligned}$$

To bound the first term we proceed as follows,

$$\begin{aligned}
&\int_0^{t/2} \|[S(t-s)*G - S(t-s)] * |u(s)|^{q-1}u(s)\|_{L^p(\mathbb{R}^d)}\, ds \\
&\quad \le C(p,J,G)\int_0^{t/2} \langle t-s\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}(\||u(s)|^q\|_{L^1(\mathbb{R}^d)} + \||u(s)|^q\|_{L^p(\mathbb{R}^d)})\, ds \\
&\quad \le C\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}\Big(\int_0^{t/2}\|u(s)\|^q_{L^q(\mathbb{R}^d)}ds + \int_0^{t/2}\|u(s)\|^q_{L^{pq}(\mathbb{R}^d)}ds\Big) \\
&\quad = C\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}(I_1(t)+I_2(t)).
\end{aligned}$$

By Theorem 6.4, for the first integral, $I_1(t)$, we have the following estimate:

$$I_1(t) \le \int_0^{t/2}\|u(s)\|^q_{L^q(\mathbb{R}^d)}ds \le C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})\int_0^{t/2}\langle s\rangle^{-\frac{d}{2}(q-1)}ds,$$

and an explicit computation of the last integral shows that

$$\langle t\rangle^{-\frac{1}{2}}\int_0^{t/2}\langle s\rangle^{-\frac{d}{2}(q-1)}ds \le C\langle t\rangle^{-\frac{d}{2}(q-1)+\frac{1}{2}}.$$

Arguing in the same manner for I_2 we get

$$\begin{aligned}
\langle t\rangle^{-\frac{1}{2}}I_2(t) &\le C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})\langle t\rangle^{-\frac{1}{2}}\int_0^{t/2}\langle s\rangle^{-\frac{dq}{2}(1-\frac{1}{pq})}ds \\
&\le C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})\langle t\rangle^{-\frac{d}{2}(q-\frac{1}{p})+\frac{1}{2}} \\
&\le C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})\langle t\rangle^{-\frac{d}{2}(q-1)+\frac{1}{2}}.
\end{aligned}$$

This ends the proof. $\square$

Chapter 7

Decay estimates for nonlocal problems via energy methods

In this chapter our main aim is to apply energy methods to obtain decay estimates for solutions to nonlocal evolution equations.

First, let us introduce the prototype of nonlocal equation that we have in mind. Let $G : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative, compactly supported, radial, continuous function with $\int_{\mathbb{R}^d} G(z)\,dz = 1$. Nonlocal evolution equations of the form

$$u_t(x,t) = (G * u - u)(x,t) = \int_{\mathbb{R}^d} G(x-y)u(y,t)\,dy - u(x,t), \tag{7.1}$$

and variations of it, have been recently widely used to model diffusion processes.

The asymptotic behavior as $t \to \infty$ in for the nonlocal model (7.1) was studied in [32], see also [58] and [59], where the authors prove that every solution to (7.1) with an initial condition u_0 such that $u_0, \widehat{u_0} \in L^1(\mathbb{R}^d)$ has an asymptotic behavior given by $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le Ct^{-d/2}$.

The proof of this fact is based on a explicit representation formula for the solution in Fourier variables. In fact, from equation (7.1) we obtain $\widehat{u}_t(\xi,t) = (\widehat{G}(\xi) - 1)\widehat{u}(\xi,t)$, and hence the solution is given by, $\widehat{u}(\xi,t) = e^{(\widehat{G}(\xi)-1)t}\widehat{u_0}(\xi)$. From this explicit formula it can be obtained the decay in $L^\infty(\mathbb{R}^d)$ of the solutions, see [32] and [58]. This decay, together with the conservation of mass, gives the decay of the $L^q(\mathbb{R}^d)$-norms by interpolation. It holds, $\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \le C\,t^{-d/2(1-1/q)}$. Note that the asymptotic behavior is the same as the one for solutions of the heat equation and, as happens for the heat equation, the asymptotic profile is a gaussian, [32].

As we have mentioned, our main task here is to develop an energy method to obtain decay estimates. Our motivation to introduce energy methods to deal with nonlocal problems is twofold, first we want to see how energy methods can be applied to equations possibly without any regularization effect and moreover we want to deal with nonlinear problems for which there are no explicit representation formula for the solution (in general, Fourier methods are not applicable to nonlinear problems).

To begin our analysis, we first deal with a linear nonlocal diffusion operator with a nonlinear source, that is, we consider the following evolution problem

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)(u(y,t) - u(x,t))\,dy + f(u)(x,t) \tag{7.2}$$

with f a locally Liptshitz function satisfying the sign condition $f(s)s \le 0$ and $J(x,y)$ a symmetric nonnegative kernel.

We generalize the previous results in two ways, we allow a nonlinear term $f(u)$ imposing only a dissipativity condition, $f(s)s \le 0$, and, what is even more relevant, we can consider equations in which the nonlocal part is not given by a convolution but for a general operator of the form $\int_{\mathbb{R}^d} J(x,y)(u(y) - u(x))\,dy$.

Our first result reads as follows: *under adequate hypothesis on J (see Theorem 7.1 in Section 7.1) and f a locally Liptshitz function satisfying the sign condition $f(s)s \le 0$, consider an initial condition $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with $d \ge 3$. Then, for any $1 \le q < \infty$ the solution to (7.2) verifies the following decay bound,*

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \le C\, t^{-\frac{d}{2}\left(1-\frac{1}{q}\right)}.$$

Our main hypotheses on J can be summarized as follows: $J(x,y)$ is strictly positive ($\ge c_1 > 0$) for $|y - a(x)| \le c_2$, where a is a function with bounded derivatives.

We remark that this decay bound need not be optimal, in the final section we present examples of functions J that give exponential decay in $L^2(\mathbb{R})$. To obtain a complete classification of all possible decay rates seems a very difficult but challenging problem.

Our energy approach not only simplifies the proof of the asymptotic decay in the linear case but also can de applied to handle nonlinear operators, like a nonlocal analogous to the $p-$Laplacian. Let $p > 2$ and consider

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)|u(y,t) - u(x,t)|^{p-2}(u(y,t) - u(x,t))\, dy. \tag{7.3}$$

This problem, with a convolution kernel, $J(x,y) = G(x-y)$ was considered in [3] and [4] where the authors found existence, uniqueness and the convergence of the solutions to solutions of the local $p-$Laplacian evolution problem, $v_t = \mathrm{div}(|\nabla v|^{p-2}\nabla v)$ when a rescaling parameter (that measures the size of the support of the convolution kernel G) goes to zero.

In this case the asymptotic decay is described as follows: *given $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a unique solution to (7.3). Moreover, under adequate hypothesis on J (see Theorem 7.1 in Section 7.1) and $2 \le p < d$, its asymptotic decay is bounded by*

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \le C\, t^{-\left(\frac{d}{d(p-2)+p}\right)\left(1-\frac{1}{q}\right)},$$

for $1 \le q < \infty$.

This asymptotic decay is the same one that holds for solutions to the local $p-$Laplacian, $v_t = \mathrm{div}(|\nabla v|^{p-2}\nabla v)$, see Chapter 11 in [108].

The assumption on the initial data, $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, is imposed since, in general, nonlocal evolution equations have no regularizing $L^1(\mathbb{R}^d) - L^q(\mathbb{R}^d)$ effect. In the particular case of a convolution kernel $J(x,y) = G(x-y)$, i.e. equation (7.1), in [32] it is proved that solutions u can be written as $u(t) = e^{-t}u_0 + K_t * u_0$, where K_t is a smooth function. As a consequence at any time $t > 0$, the solution u is as regular as the initial datum u_0 is. Thus, it is hopeless to guarantee that at any time $t > 0$, the solution $u(t)$ belongs to $L^q(\mathbb{R}^d)$ without assuming that $u_0 \in L^q(\mathbb{R}^d)$.

We also have to mention that we are assuming the following hypothesis on the kernel $J(x;\cdot) \in L^1(\mathbb{R}^d)$. This excludes the analysis of the possibility of a faster decay for u if for example J has fat tails, as happens for equations involving generators of Levy processes.

The rest of the chapter is organized as follows: In Section 7.1 we collect some preliminaries and prove a decomposition theorem that will be used to apply energy methods; in Section 7.2 we deal with the decay of solutions with linear nonlocal diffusion and a nonlinear dissipative source and in Section 7.3 we prove the decay for the nonlocal $p-$Laplacian. Finally in Section 7.4 we present examples of J for which we can prove exponential decay bounds for the linear problem.

7.1 Preliminaries

In this section we collect some preliminaries and state and prove a crucial decomposition theorem. In what follows we denote by

$$p^* = \frac{pd}{(d-p)}$$

the usual Sobolev exponent, while

$$p' = \frac{p}{p-1}$$

denotes the usual conjugate exponent.

First, let us describe briefly how the energy method can be applied to obtain decay estimates for local problems. Let us begin with the simpler case of the estimate for solutions to the heat equation in $L^2(\mathbb{R}^d)$-norm,

$$u_t = \Delta u.$$

If we multiply by u and integrate in $\mathbb{R}^d$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)dx = -\int_{\mathbb{R}^d} |\nabla u(x,t)|^2 dx.$$

Now we use Sobolev's inequality

$$\int_{\mathbb{R}^d} |\nabla u|^2(x,t)\,dx \geq C\left(\int_{\mathbb{R}^d} |u|^{2^*}(x,t)\,dx\right)^{2/2^*}$$

to obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)\,dx \leq -C\left(\int_{\mathbb{R}^d} |u|^{2^*}(x,t)\,dx\right)^{2/2^*}.$$

If we use interpolation and conservation of mass, that implies $\|u(t)\|_{L^1(\mathbb{R}^d)} \leq C$ for any $t>0$, we have

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \leq \|u(t)\|^{\alpha}_{L^1(\mathbb{R}^d)}\|u(t)\|^{1-\alpha}_{L^{2^*}(\mathbb{R}^d)} \leq C\|u(t)\|^{1-\alpha}_{L^{2^*}(\mathbb{R}^d)}$$

with α determined by

$$\frac{1}{2} = \alpha + \frac{1-\alpha}{2^*}, \qquad \text{that is,} \qquad \alpha = \frac{2^*-2}{2(2^*-1)}.$$

Hence we get

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)\,dx \leq -C\left(\int_{\mathbb{R}^d} u^2(x,t)\,dx\right)^{\frac{1}{1-\alpha}}$$

from where the decay estimate

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \leq C\,t^{-\frac{d}{2}\left(1-\frac{1}{2}\right)}, \qquad t>0,$$

follows.

In the case of the $p-$Laplacian in the whole space,

$$u_t = \Delta_p u,$$

the argument is similar, we multiply by u, integrate in $\mathbb{R}^d$ and use Sobolev inequality, that in this case reads,

$$\int_{\mathbb{R}^d} |\nabla u|^p(x,t)\,dx \geq C\left(\int_{\mathbb{R}^d} |u|^{p^*}(x,t)\,dx\right)^{p/p^*}$$

and interpolation to get a similar inequality for the L^2-norm of a solution

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)\,dx \leq -C\left(\int_{\mathbb{R}^d} u^2(x,t)\,dx\right)^{\theta}$$

for an explicit $\theta<1$ that depends on p and d. As before this inequality implies a decay bound for the L^2-norm.

We want to mimic the steps for the nonlocal evolution problem

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)(u(y,t) - u(x,t))\, dy.$$

Hence, we multiply by u and integrate in $\mathbb{R}^d$ to obtain,

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)\, dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)(u(y,t) - u(x,t))\, dy\, u(x,t)\, dx. \tag{7.4}$$

Now, we need to "integrate by parts". Therefore, let us begin by a simple algebraic identity (whose proof is immediate) that plays the role of an integration by parts formula for nonlocal operators.

Lemma 7.1. *If J is symmetric, $J(x,y) = J(y,x)$ then it holds*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)(\varphi(y) - \varphi(x))\psi(x)dydx = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)(\varphi(y) - \varphi(x))(\psi(y) - \psi(x))dydx.$$

If we apply this lemma to (7.4) we get

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)dx = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)(u(y,t) - u(x,t))^2\, dy\, dx,$$

but now we run into troubles since there is no analogous to Sobolev inequality. In fact, an inequality of the form

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)(u(y,t) - u(x,t))^2\, dy\, dx \geq C\left(\int_{\mathbb{R}^d} u^q(x,t)\, dx\right)^{2/q}$$

can not hold for any $q > 2$.

Now the idea is to split the function u as the sum of two functions $u = v + w$, where on the function v (the "smooth"part of the solution) the nonlocal operator acts as a gradient and on the function w (the "rough"part) it does not increase its norm significatively.

Therefore, we need to obtain estimates for the $L^p(\mathbb{R}^d)$-norm of the nonlocal operators. The main result of this section is the following.

Theorem 7.1. *Let $p \in [1,\infty)$ and $J(\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$ be a symmetric nonnegative function satisfying*

HJ1) *There exists a positive constant $C < \infty$ such that*

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\, dx \leq C.$$

HJ2) *There exist positive constants c_1, c_2 and a function $a \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ satisfying*

$$\sup_{x\in\mathbb{R}^d} |\nabla a(x)| < \infty \tag{7.5}$$

such that the set

$$B_x = \{y \in \mathbb{R}^d : |y - a(x)| \leq c_2\} \tag{7.6}$$

verifies

$$B_x \subset \{y \in \mathbb{R}^d : J(x,y) > c_1\}.$$

Then, for any function $u \in L^p(\mathbb{R}^d)$ there exist two functions v and w such that $u = v + w$ and

$$\|\nabla v\|^p_{L^p(\mathbb{R}^d)} + \|w\|^p_{L^p(\mathbb{R}^d)} \leq C(J,p)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)|u(x) - u(y)|^p\, dx\, dy. \tag{7.7}$$

Moreover, if $u \in L^q(\mathbb{R}^d)$ *with* $q \in [1,\infty]$ *then the functions* v *and* w *satisfy*

$$\|v\|_{L^q(\mathbb{R}^d)} \le C(J,q)\|u\|_{L^q(\mathbb{R}^d)} \tag{7.8}$$

and

$$\|w\|_{L^q(\mathbb{R}^d)} \le C(J,q)\|u\|_{L^q(\mathbb{R}^d)}. \tag{7.9}$$

Before the proof we collect some remarks and a prove a corollary.

Remark 7.1. *The above result says that there exists a decomposition of* u *in a smooth part,* v*, and a rough part,* w*, such that the action of the nonlocal operator is like a gradient on the smooth part and as the identity on the rough part.*

Remark 7.2. *The constant* $C(J;q)$ *in the theorem depends only on the constants of* HJ1) *and* HJ2) *and not on any other characteristic of the kernel* J*.*

Remark 7.3. *We note that in the case* $1 \le p < d$ *using the classical Sobolev's inequality* $\|v\|_{L^{p^*}(\mathbb{R}^d)} \le \|\nabla v\|_{L^p(\mathbb{R}^d)}$ *we get that* (7.7) *implies*

$$\|v\|^p_{L^{p^*}(\mathbb{R}^d)} + \|w\|^p_{L^p(\mathbb{R}^d)} \le C(J,p)\int_{\mathbb{R}^r}\int_{\mathbb{R}^d} J(x,y)|u(x)-u(y)|^p\,dx\,dy.$$

Remark 7.4. *In particular, we can consider* $a(x) = x$*, that is, the case of a convolution kernel,* $J(x,y) = G(x-y)$*, with* $G(0) > 0$*. In fact, it is reasonable to assume that* $J(x,x) > 0$ *since in biological models this means that the probability that some individuals that are in* x *at time* t *remain at the same position is positive.*

To simplify the notation let us note by $\langle A_p u, u\rangle$ the following quantity,

$$\langle A_p u, u\rangle := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)|u(x)-u(y)|^p\,dx\,dy.$$

Observe that, in order that the above quantity to be finite, we have to assume a priori that u belongs to $L^p(\mathbb{R}^d)$.

Note that our main result of this section, Theorem 7.1 gives estimates from below for $\langle A_p u, u\rangle$. A corollary of this result is the following.

Corollary 7.1. *Let* $J(\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *be a symmetric nonnegative function satisfying hypotheses* HJ1) *and* HJ2) *in Theorem 7.1 and* $p \in [1,d)$*. There exist two positive constants* $C_1 = C_1(J,p)$ *and* $C_2 = C_2(J,p)$ *such that for any* $u \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ *the following holds:*

$$\|u\|^p_{L^p(\mathbb{R}^d)} \le C_1\|u\|^{p(1-\alpha(p))}_{L^1(\mathbb{R}^d)}\langle A_p u, u\rangle^{\alpha(p)} + C_2\langle A_p u, u\rangle, \tag{7.10}$$

where $\alpha(p)$ *satisfies:*

$$\frac{1}{p} = \frac{\alpha(p)}{p^*} + 1 - \alpha(p).$$

Remark 7.5. *The explicit value of* $\alpha(p)$ *is given by*

$$\alpha(p) = \frac{p^*}{p'(p^*-1)} = \frac{d(p-1)}{d(p-1)+p}. \tag{7.11}$$

Remark 7.6. *In the case of the local operator* $B_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$*, using Sobolev's inequality and interpolation inequalities we have the following estimate*

$$\|u\|^p_{L^p(\mathbb{R}^d)} \le C_1\|u\|^{p(1-\alpha(p))}_{L^1(\mathbb{R}^d)}\langle B_p u, u\rangle^{\alpha(p)}.$$

In the nonlocal case an extra term involving $\langle A_p u, u\rangle$ *occurs, see* (7.10)*.*

Proof of Corollary 7.1. We use the decomposition $u = v + w$ given by Theorem 7.1 to obtain

$$\|u\|^p_{L^p(\mathbb{R}^d)} \leq \|v\|^p_{L^p(\mathbb{R}^d)} + \|w\|^p_{L^p(\mathbb{R}^d)}.$$

Also, by (7.7), we have

$$\|\nabla v\|^p_{L^p(\mathbb{R}^d)} \leq C(J,p)\langle A_p u, u\rangle$$

and

$$\|w\|^p_{L^p(\mathbb{R}^d)} \leq C(J,p)\langle A_p u, u\rangle.$$

Then, from the interpolation inequality

$$\|v\|_{L^p(\mathbb{R}^d)} \leq \|v\|^{\alpha(p)}_{L^{p^*}(\mathbb{R}^d)}\|v\|^{1-\alpha(p)}_{L^1(\mathbb{R}^d)},$$

we obtain that the $L^p(\mathbb{R}^d)$-norm of u satisfies

$$\begin{aligned}\|u\|^p_{L^p(\mathbb{R}^d)} &\leq \|v\|^{\alpha(p)p}_{L^{p^*}(\mathbb{R}^d)}\|v\|^{(1-\alpha(p))p}_{L^1(\mathbb{R}^d)} + \|w\|^p_{L^p(\mathbb{R}^d)} \leq \|\nabla v\|^{\alpha(p)p}_{L^p(\mathbb{R}^d)}\|u\|^{(1-\alpha(p))p}_{L^1(\mathbb{R}^d)} + C(J,p)\langle A_p u, u\rangle \\ &\leq C_1\|u\|^{(1-\alpha(p))p}_{L^1(\mathbb{R}^d)}\langle A_p u, u\rangle^{\alpha(p)} + C_2\langle A_p u, u\rangle,\end{aligned}$$

as we wanted to prove. □

Now we proceed with the proof of the decomposition theorem.

Proof of Theorem 7.1. We divide the proof in two steps. First of all, we prove under the assumptions HJ1)-HJ2) the existence of a function $\rho(\cdot,\cdot)$ satisfying

H1) $\rho(x,\cdot) \in C_c^\infty(\mathbb{R}^d)$ for a.e. $x \in \mathbb{R}^d$,

H2) $\int_{\mathbb{R}^d} \rho(x,y)\,dy = 1$ for a.e. $x \in \mathbb{R}^d$,

H3) $\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x,y)\,dx \leq M < \infty$,

H4) $\operatorname{supp}\rho(x,\cdot) \subset B_x$ for a.e. $x \in \mathbb{R}^d$,

H5) $\sup_{x\in\mathbb{R}^d} \|\rho(x,\cdot)\|_{L^{p'}(\mathbb{R}^d)} \leq M < \infty$,

H6) $\sum_{k=1}^d \sup_{x\in\mathbb{R}^d} \|\partial_{x_k}\rho(x,\cdot)\|_{L^{p'}(\mathbb{R}^d)} \leq M < \infty$.

Next, we define

$$v(x) = \int_{\mathbb{R}^d} \rho(x,y)u(y)\,dy, \qquad \text{and} \qquad w = u - v,$$

and prove (7.7), (7.8) and (7.9).

Step I. Construction of ρ. With c_2 given by HJ2) we consider a smooth function $\psi \in C_c^\infty(\mathbb{R}^d)$ supported in the ball $B_{c_2}(0)$, $0 \leq \psi \leq C$ and having mass one:

$$\int_{B_{c_2}(0)} \psi(x)\,dx = 1.$$

For any $x \in \mathbb{R}^d$ we consider the function $a(x)$ and the set B_x as in (7.6), see HJ2). We then define $\rho(x,y)$ by

$$\rho(x,y) = \psi(y - a(x)). \tag{7.12}$$

We will prove properties H3) and H6) since the others easily follow with a constant $M(J)$. We point out that the assumption on the existence of a ball B_x centered at $a(x)$ with radius c_2 is necessary in proving H5). Otherwise, $\inf_{x\in\mathbb{R}^d}|B_x| = 0$ and by Hölder inequality, we get

$$\|\rho(x,\cdot)\|_{L^{p'}(\mathbb{R}^d)} \geq \frac{\int_{\mathbb{R}^d}\rho(x,y)dy}{|B_x|^{1/p}} = \frac{1}{|B_x|^{1/p}}$$

and then

$$\sup_{x\in\mathbb{R}^d}\|\rho(x,\cdot)\|_{L^{p'}(\mathbb{R}^d)} \geq \frac{1}{\inf_{x\in\mathbb{R}^d}|B_x|^{1/p}} = \infty.$$

Therefore, we cannot obtain property H5).

We now prove property H3). Observe that, by definition (7.12) of the function $\rho(\cdot,\cdot)$ and the fact that $\psi \leq C$ we have

$$\begin{aligned}\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\rho(x,y)\,dx = \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(y-a(x))\,dx = \sup_{y\in\mathbb{R}^d}\int_{|y-a(x)|\leq c_2}\psi(y-a(x))\,dx\\ \leq C\sup_{y\in\mathbb{R}^d}\big|\{x : |y-a(x)|\leq c_2\}\big|.\end{aligned}$$

It remains to show that the last term in the right hand side is finite. Indeed, given y, we have

$$|\{x : |y-a(x)|\leq c_2\}| \leq \int_{\{x:|y_n-a(x)|\leq c_2\}}\frac{J(x,y)}{c_1}\,dx \leq \frac{1}{c_1}\int_{\mathbb{R}^d}J(x,y)\,dx \leq C.$$

We now prove HJ6). By definition (7.12) for any $x\in\mathbb{R}^d$ we have

$$\|\partial_{x_k}\rho(x,\cdot)\|_{L^{p'}(\mathbb{R}^d)} = \|\nabla\psi(\cdot-a(x))\cdot\partial_{x_k}a(x)\|_{L^{p'}(\mathbb{R}^d)} \leq |\partial_{x_k}a(x)|\|\nabla\psi\|_{L^{p'}(\mathbb{R}^d)}.$$

Using (7.5) and the construction of ψ we obtain HJ6).

Step II. Proof of the estimates on u, v and w. We have proved that there exists a function ρ satisfying hypotheses H1)-H6). Let us take

$$v(x) = \int_{\mathbb{R}^d}\rho(x,y)u(y)\,dy, \qquad \text{and} \qquad w = u - v.$$

First we prove (7.8) and (7.9). Hölder's inequality applied to the function v and H2) guarantee that

$$|v(x)|^q \leq \int_{\mathbb{R}^d}\rho(x,y)|u(y)|^q\,dy\left(\int_{\mathbb{R}^d}\rho(x,y)\,dy\right)^{\frac{q}{q'}} = \int_{\mathbb{R}^d}\rho(x,y)|u(y)|^q\,dy,$$

Then, property H3) gives us

$$\begin{aligned}\int_{\mathbb{R}^d}|v(x)|^q\,dx \leq \int_{\mathbb{R}^d}|u(y)|^q\int_{\mathbb{R}^d}\rho(x,y)\,dx\,dy \leq \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\rho(x,y)dx\int_{\mathbb{R}^d}|u(y)|^q\,dy\\ \leq M\int_{\mathbb{R}^d}|u(y)|^q\,dy\end{aligned}$$

which proves (7.8).

Also, we obviously have

$$\|w\|_{L^q(\mathbb{R}^d)} \leq \|u\|_{L^q(\mathbb{R}^d)} + \|v\|_{L^q(\mathbb{R}^d)} \leq (1+M^{1/q})\|u\|_{L^q(\mathbb{R}^d)}.$$

We now proceed to prove (7.7). To do that we prove the following inequalities:

$$\|w\|^p_{L^p(\mathbb{R}^d)} \leq c_1^{-1}\sup_{x\in\mathbb{R}^d}\|\rho(x,\cdot)\|^p_{L^{p'}(\mathbb{R}^d)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}J(x,y)|u(x)-u(y)|^p\,dx\,dy \tag{7.13}$$

and

$$\|\nabla v\|^p_{L^p(\mathbb{R}^d)} \leq \sum_{k=1}^d c_1^{-1}\sup_{x\in\mathbb{R}^d}\|\partial_{x_k}\rho(x,\cdot)\|^p_{L^{p'}(\mathbb{R}^d)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}J(x,y)|u(x)-u(y)|^p\,dx\,dy. \tag{7.14}$$

The fact that for any $x \in \mathbb{R}^d$, $\rho(x,\cdot)$ is supported in the set B_x and has mass one gives the following

$$
\begin{aligned}
w(x) &= u(x) - \int_{\mathbb{R}^d} \rho(x,y)u(y)\,dy = \int_{\mathbb{R}^d} \rho(x,y)(u(x)-u(y))\,dy \\
&= \int_{B_x} \rho(x,y)(u(x)-u(y))\,dy.
\end{aligned}
$$

Then by Hölder's inequality we get:

$$
\begin{aligned}
\|w\|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \left| \int_{B_x} \rho(x,y)(u(x)-u(y))dy \right|^p dx \\
&\leq \int_{\mathbb{R}^d} \int_{B_x} |u(x)-u(y)|^p dy \left(\int_{B_x} \rho(x,y)^{p'} dy \right)^{\frac{p}{p'}} dx \\
&\leq \sup_{x\in\mathbb{R}^d} \left(\int_{B_x} \rho(x,y)^{p'} dy \right)^{\frac{p}{p'}} \int_{\mathbb{R}^d} \int_{B_x} |u(x)-u(y)|^p \,dy\,dx \\
&\leq \sup_{x\in\mathbb{R}^d} \|\rho(x,\cdot)\|^p_{L^{p'}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{B_x} |u(x)-u(y)|^p \,dy\,dx.
\end{aligned}
$$

Using now that for any $x \in \mathbb{R}^d$ and $y \in B_x$ we have $J(x,y) > c_1$ we obtain

$$
\begin{aligned}
\|w\|^p_{L^p(\mathbb{R}^d)} &\leq c_1^{-1} \sup_{x\in\mathbb{R}^d} \|\rho(x,\cdot)\|^p_{L^{p'}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{B_x} J(x,y)|u(x)-u(y)|^p \,dy\,dx \\
&\leq c_1^{-1} \sup_{x\in\mathbb{R}^d} \|\rho(x,\cdot)\|^p_{L^{p'}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x,y)|u(x)-u(y)|^p \,dy\,dx
\end{aligned}
$$

which proves (7.13).

In the case of v we proceed in a similar manner, by tacking into account that for any $x \in \mathbb{R}$ the mass of $\partial_{x_k}\rho(x,y)$, $k = 1,\dots,d$ vanishes:

$$
\int_{\mathbb{R}^d} \partial_{x_k}\rho(x,y)\,dy = \partial_{x_k}\Big(\int_{\mathbb{R}^d} \rho(x,y)dy \Big) = 0.
$$

The definition of v and this mass property gives,

$$
\partial_{x_k} v(x) = \int_{\mathbb{R}^d} \partial_{x_k}\rho(x,y)(u(y)-u(x))\,dy = \int_{B_x} \partial_{x_k}\rho(x,y)(u(y)-u(x))\,dy.
$$

Thus, by Hölder inequality and the fact that $J(x,y) > c_1$ for all $y \in B_x$ we obtain,

$$
\begin{aligned}
\|\partial_{x_k} v\|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \left| \int_{B_x} \partial_{x_k}\rho(x,y)(u(y)-u(x))dy \right|^p dx \\
&\leq \int_{\mathbb{R}^d} \int_{B_x} |u(y)-u(x)|^p \,dy \left(\int_{B_x} |\partial_{x_k}\rho(x,y)|^{p'} dy \right)^{\frac{p}{p'}} dx \\
&= \sup_{x\in\mathbb{R}^d} \|\partial_{x_k}\rho(x,\cdot)\|^p_{L^{p'}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{B_x} |u(y)-u(x)|^p \,dx\,dy \\
&\leq c^{-1} \sup_{x\in\mathbb{R}^d} \|\partial_{x_k}\rho(x,\cdot)\|^p_{L^{p'}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{B_x} J(x,y)|u(y)-u(x)|^p \,dx\,dy \\
&\leq c^{-1} \sup_{x\in\mathbb{R}^d} \|\partial_{x_k}\rho(x,\cdot)\|^p_{L^{p'}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x,y)|u(y)-u(x)|^p \,dx\,dy.
\end{aligned}
$$

Summing the above inequalities for all $k = 1,\dots,d$ we get (7.14).

The proof is now finished since (7.13) and (7.14) imply (7.7). □

Now we present a similar result to Corollary 7.1 which can be used to obtain less accurate bounds (hence we prefer to use the more general result presented above) in the particular case of the nonlocal laplacian, i.e. $p = 2$, and $J(x, y) = G(x - y)$. The result is no so general as Corollary 7.1, but it is obtained using Fourier analysis tools and has the advantage that the previous decomposition $u = v + w$ can be better understood. We include it here just for this purpose. In fact this decomposition can be viewed as a Fourier splitting of the function u in two parts, the first one, v, corresponding to the low frequencies (the smooth part) of u, and the second one, w, corresponds to the high frequencies component (the rough part) of u.

We will use that in the particular case $p = 2$ and $J(x, y) = G(x - y)$, G with mass one, the operator $\langle A_2 u, u\rangle$ can be represented by means of the Fourier transform of G as follows

$$\langle A_2 u, u\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)|u(x)-u(y)|^2\,dx\,dy = \int_{\mathbb{R}^d}(1-\widehat{G}(\xi))|\widehat{u}(\xi)|^2\,d\xi.$$

Lemma 7.2. *Let $d \geq 3$ and G be such that its Fourier transform $\widehat{G}(\xi)$ satisfies*

$$\begin{cases} \widehat{G}(\xi) \leq 1 - \frac{|\xi|^2}{2}, & |\xi| \leq R, \\ \widehat{G}(\xi) \leq 1-\delta, & |\xi| \geq R, \end{cases} \tag{7.15}$$

for some positive numbers R and δ. Then, for any $\varepsilon \in (0,1)$ there exists a constant $C = C(\varepsilon, \delta, R, d)$ such that the following

$$\|u\|^2_{L^2(\mathbb{R})} \leq C\|u\|^{2(1-\beta(\varepsilon))}_{L^{1+\varepsilon}(\mathbb{R}^d)}\langle A_2 u, u\rangle^{\beta(\varepsilon)} + \langle A_2 u, u\rangle \tag{7.16}$$

holds for all $u \in L^{1+\varepsilon}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ where

$$\beta(\varepsilon) = \frac{(1-\varepsilon)d}{d+2-\varepsilon(d-2)}.$$

Remark 7.7. *The limit case $\varepsilon = 0$ cannot be obtained since an estimate of the type*

$$\|(\mathbf{1}_{\{|\xi|\leq R\}}\widehat{u})^{\vee}\|_{L^1(\mathbb{R}^d)} \leq \|u\|_{L^1(\mathbb{R}^d)}$$

does not hold for all functions $u \in L^1(\mathbb{R}^d)$. In dimension one this can be seen by choosing a sequence u_ε with $\|u_\varepsilon\|_{L^1(\mathbb{R}^d)} = 1$ such that $u_\varepsilon \to \delta_0$, the Dirac delta. Then

$$(\mathbf{1}_{\{|\xi|\leq R\}}\widehat{u}_\varepsilon)^{\vee} = u_\varepsilon * \frac{\sin(Rx)}{Rx} \to \frac{\sin Rx}{Rx}$$

and the last function does not belong to $L^1(\mathbb{R}^d)$. Thus $\|(\mathbf{1}_{\{|\xi|\leq R\}}\widehat{u}_\varepsilon)^{\vee}\|_{L^1(\mathbb{R}^d)} \to \infty$ but $\|u_\varepsilon\|_{L^1(\mathbb{R}^d)} = 1$.

Remark 7.8. *The same arguments can be used to obtain estimates for any function G which satisfies*

$$\begin{cases} \widehat{G}(\xi) \leq 1 - \frac{|\xi|^{2s}}{2}, & |\xi| \leq R, \\ \widehat{G}(\xi) \leq 1-\delta, & |\xi| \geq R, \end{cases}$$

for some positive numbers R, δ and s.

Proof of Lemma 7.2. For any function $u \in L^2(\mathbb{R}^d)$ we define its projections on the low and high frequencies respectively,

$$v := (\mathbf{1}_{\{|\xi|\leq R\}}\widehat{u})^{\vee}, \qquad w := (\mathbf{1}_{\{|\xi|\geq R\}}\widehat{u})^{\vee}. \tag{7.17}$$

Using that the function $\widehat{G}$ satisfies (7.15) we obtain the following estimate for the operator A_2:

$$
\begin{aligned}
\langle A_2 u, u\rangle &= \int_{\mathbb{R}^d} (1-\widehat{G}(\xi))|\widehat{u}(\xi)|^2\, d\xi \geq \int_{|\xi|\leq R} \frac{|\xi|^2}{2}|\widehat{u}(\xi)|^2\, d\xi + \delta \int_{|\xi|\geq R} |\widehat{u}(\xi)|^2\, d\xi \qquad (7.18)\\
&= \frac{1}{2}\int_{\mathbb{R}^d} |\xi|^2 |\widehat{v}(\xi)|^2 d\xi + \delta \int_{\mathbb{R}^d} |\widehat{w}(\xi)|^2 d\xi\\
&\geq c(\delta)\Big(\|\nabla v\|^2_{L^2(\mathbb{R}^d)} + \|w\|^2_{L^2(\mathbb{R}^d)}\Big)\\
&\geq c(\delta)(\|v\|^2_{L^{2^*}(\mathbb{R}^d)} + \|w\|^2_{L^2(\mathbb{R}^d)}\Big).
\end{aligned}
$$

In order to estimate from above the $L^2(\mathbb{R}^d)$-norm of u as in (7.16), using the orthogonality of v and w it is sufficient to estimate each projection v and w since

$$\|u\|^2_{L^2(\mathbb{R}^d)} = \|v\|^2_{L^2(\mathbb{R}^d)} + \|w\|^2_{L^2(\mathbb{R}^d)}.$$

In the case of w, using (7.17) and (7.18) we have the rough estimate:

$$\|w\|^2_{L^2(\mathbb{R}^d)} \leq \frac{1}{c(\delta)}\langle A_2 u, u\rangle. \qquad (7.19)$$

Next we estimate the $L^2(\mathbb{R}^d)$-norm of v. We recall that classical results on Fourier multipliers (see Chapter 4 in [100]) give us that for any $p\in(1,\infty)$ the $L^p(\mathbb{R}^d)$-norm of v, defined by (7.17), can be bounded from above by the $L^p(\mathbb{R}^d)$-norm of u as follows:

$$\|v\|_{L^p(\mathbb{R}^d)} \leq C(p,d)\|u\|_{L^p(\mathbb{R}^d)}. \qquad (7.20)$$

Using this estimate and interpolation inequalities we obtain that v, the low frequency projection of u, satisfies

$$
\begin{aligned}
\|v\|^2_{L^2(\mathbb{R}^d)} &\leq \Big(\|v\|^{1-\beta(\varepsilon)}_{L^{1+\varepsilon}(\mathbb{R}^d)}\|v\|^{\beta(\varepsilon)}_{L^{2^*}(\mathbb{R}^d)}\Big)^2 \leq \Big(c(\varepsilon,d)\|u\|^{1-\beta(\varepsilon)}_{L^{1+\varepsilon}(\mathbb{R}^d)}\|v\|^{\beta(\varepsilon)}_{L^{2^*}(\mathbb{R}^d)}\Big)^2 \qquad (7.21)\\
&\leq c^2(\varepsilon,d)c(\delta)^{-\beta(\varepsilon)}\|u\|^{2(1-\beta(\varepsilon))}_{L^{1+\varepsilon}(\mathbb{R}^d)}\langle A_2 u,u\rangle^{\beta(\varepsilon)},
\end{aligned}
$$

where $c(\varepsilon,d)$ is given by applying (7.20) with $p=1+\varepsilon$ and $\beta(\varepsilon)$ by

$$\frac{1}{2} = \frac{1-\beta(\varepsilon)}{1+\varepsilon} + \frac{\beta(\varepsilon)}{2^*},$$

that is,

$$\beta(\varepsilon) = \frac{(1-\varepsilon)d}{d+2-\varepsilon(d-2)}.$$

Combining (7.18), (7.19) and (7.21) we obtain

$$\|u\|^2_{L^2(\mathbb{R}^d)} \leq c(\varepsilon,\delta,d)\|u\|^{2(1-\beta(\varepsilon))}_{L^{1+\beta(\varepsilon)}(\mathbb{R}^d)}\langle A_2 u,u\rangle^{\alpha(\varepsilon)} + \langle A_2 u, u\rangle. \qquad (7.22)$$

The proof is now finished. □

7.2 Decay estimates for a nonlinear problem.

In this section we will obtain the long time behavior of the solutions u to the following equation

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)(u(y,t)-u(x,t))\, dy + f(u)(x,t) \qquad (7.23)$$

under suitable assumptions on the kernel J and the nonlinearity f. Our goal is to obtain here a proof of the decay rate of the solution u to (7.23) by using energy methods.

The main result of this section is the following theorem.

Theorem 7.2. *Let $J(x,y)$ be a symmetric nonnegative kernel satisfying* HJ1) *as in Theorem 7.1 and f be a locally Lipshitz function with $f(s)s \le 0$. For any $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a unique solution to equation (7.23) which satisfies*

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)} \quad \text{and} \quad \|u(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)} \tag{7.24}$$

for every $t > 0$.

Moreover, if $d \ge 3$ and J also satisfies HJ2) *then the following holds:*

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \le C(q,d)\|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{q})} \tag{7.25}$$

for all $q \in [1,\infty)$ and for all t sufficiently large.

Remark 7.9. *The proof uses the results of Theorem 7.1 and Corollary 7.1 obtained in Section 7.1 in the particular case $p = 2$. In order to apply Corollary 7.1 we need to assume $d > 2$, i.e. $d \ge 3$.*

The same arguments we use here also work for the convection diffusion equation:

$$\begin{cases} u_t(t,x) = (G_1 * u - u)(t,x) + \big(G_2 * (|u|^{r-1}u) - |u|^{r-1}u\big)(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{7.26}$$

where $r > 1$ and G_1 and G_2 are positive functions with mass one. We have to mention that this time the dissipativity condition on the nonlinear part have to be understood in the following sense

$$\int_{\mathbb{R}^d} \big(G * (|u|^{r-1}u) - |u|^{r-1}u\big)\,|u|^{q-2}u \le 0$$

for any $q \ge 1$.

In the case of equation (7.26), the same decay as in (7.25) has been obtained in [58] by means of the so-called *Fourier Splitting method* introduced by Schonbek in [94], [95] and [96] in the context of the local convection-diffusion equation. Our method also works if the convolution terms in (7.26) are replaced by integral operators as in (7.23).

The following lemma will be used in the proof of Theorem 7.2.

Lemma 7.3. *Let $d > 2$ and u such that $u(t) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for all $t \ge 0$ satisfying:*

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)dx + \langle A_2 u(t), u(t)\rangle \le 0, \qquad \text{for all } t > 0,$$

with J as in Theorem 7.1. Assuming that

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \le \|u(0)\|_{L^1(\mathbb{R}^d)}, \qquad \text{for all } t > 0, \tag{7.27}$$

there exists a constant $c(d,J)$ such that

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \le c(d,J)\|u(0)\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})}$$

holds for all t large enough.

Remark 7.10. *Under the same hypotheses we can replace the initial time $t = 0$ with any positive time t_0, the result being the same for large time t,*

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \le c(d)\|u(t_0)\|_{L^1(\mathbb{R}^d)} (t - t_0)^{-\frac{d}{2}(1-\frac{1}{2})}.$$

Proof of Lemma 7.3. By Corollary 7.1 and property (7.27) we obtain

$$\begin{aligned}\|u(t)\|^2_{L^2(\mathbb{R}^d)} &\le C_1(J)\|u(t)\|^{2(1-\alpha(2))}_{L^1(\mathbb{R}^d)}\langle A_2u(t),u(t)\rangle^{\alpha(2)} + C_2(J)\langle A_2u(t),u(t)\rangle\\ &\le C_1(J)\|u(0)\|^{2(1-\alpha(2))}_{L^1(\mathbb{R}^d)}\langle A_2u(t),u(t)\rangle^{\alpha(2)} + C_2(J)\langle A_2u(t),u(t)\rangle\end{aligned}$$

where $\alpha(2)=d/(d+2)$ is given by (7.11). To simplify the presentation we will assume without loss of generality that $C_1(J)=C_2(J)=1$ (otherwise one can track the constants that appear in each step of the proof). Then for any $t>0$, $\langle A_2u(t),u(t)\rangle$ satisfies

$$H^{-1}(\|u(t)\|^2_{L^2(\mathbb{R}^d)}) \le \langle A_2u(t),u(t)\rangle$$

where

$$H(x)=\|u(0)\|^{2(1-\alpha(2))}_{L^1(\mathbb{R}^d)}x^{\alpha(2)}+x.$$

Analyzing the function $H_{a,\beta}(x)=ax^\beta+x$, $a>0$, $\beta\in(0,1)$, we find that $\langle A_2u(t),u(t)\rangle$ verifies:

$$\langle A_2u(t),u(t)\rangle \ge \begin{cases} \frac12\|u(t)\|^2_{L^2(\mathbb{R}^d)}, & \|u(t)\|^2_{L^2(\mathbb{R}^d)} > 2\,\|u(0)\|^2_{L^1(\mathbb{R}^d)},\\ \left(\frac{\|u(t)\|^2_{L^2(\mathbb{R}^d)}}{2\|u(0)\|^{2(1-\alpha(2))}_{L^1(\mathbb{R}^d)}}\right)^{\frac{1}{\alpha(2)}}, & \|u(t)\|^2_{L^2(\mathbb{R}^d)} < 2\,\|u(0)\|^2_{L^1(\mathbb{R}^d)}.\end{cases}$$

Then, $\phi(t)=\|u(t)\|^2_{L^2(\mathbb{R})}$ satisfies the following differential inequality for all $t\ge 0$:

$$\phi_t(t)+\frac{\phi(t)}{2}\chi_{\left\{\phi(t)>2\|u(0)\|^2_{L^1(\mathbb{R}^d)}\right\}}+\left(\frac{\phi(t)}{2\|u(0)\|^{2(1-\alpha(2))}_{L^1(\mathbb{R}^d)}}\right)^{\frac{1}{\alpha(2)}}\chi_{\left\{\phi(t)<2\|u(0)\|^2_{L^1(\mathbb{R}^d)}\right\}}\le 0.$$

Thus, there exists t_0 such that for all $t\ge t_0$, $\phi(t)$ satisfies the following differential inequality for all $t\ge t_0$:

$$\phi_t(t)+\left(\frac{\phi(t)}{2\|u(0)\|^{2(1-\alpha(2))}_{L^1(\mathbb{R}^d)}}\right)^{\frac{1}{\alpha(2)}}\le 0.$$

Integrating it on (t_0,t) we get that ϕ satisfies

$$\phi(t)\le C\|u(0)\|^2_{L^1(\mathbb{R}^d)}(t-t_0)^{-d(1-\frac12)},\ t>t_0,$$

in other words

$$\|u(t)\|_{L^2(\mathbb{R}^d)}\le C\|u(0)\|_{L^1(\mathbb{R}^d)}t^{-\frac{d}{2}(1-\frac12)}$$

holds for all time t large enough. □

Proof of Theorem 7.2. **Step I. Global existence and uniqueness.** First, let us prove the existence and uniqueness of a local solution. To this end we use a fixed point argument.

Let us consider the space

$$X=C^0\big([0,T];\,L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)\big)$$

with the norm

$$\|u\|_X=\max_{t\in[0,T]}\Big\{\|u(t)\|_{L^1(\mathbb{R}^d)}+\|u(t)\|_{L^\infty(\mathbb{R}^d)}\Big\}.$$

Since the map $u\to f(u)$ is Lipschitz continuous on bounded subsets of X (as a consequence of the properties of f) classical results on semilinear evolution problems (see for example [29], Proposition 4.3.3) guarantees the existence of a unique local solution u.

We now prove (7.24) which guarantee the global existence of solutions to equation (7.23). We multiply equation (7.23) with $\operatorname{sgn}(u)$ and integrate on $\mathbb{R}^d$. Using Lemma 7.1 and the fact that $f(s)s \leq 0, s \in \mathbb{R}$, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}|u(x,t)|dx &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)(u(y,t)-u(x,t))\operatorname{sgn}(u(x,t))dydx\\ &= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)(u(y,t)-u(x,t))(\operatorname{sgn}(u(y,t))-\operatorname{sgn}(u(x,t)))dydx\\ &\leq 0.\end{aligned}$$

From here it follows that

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}.$$

Now, multiplying the equation by $(u(x,t)-M)_+$, where $M = \|u_0\|_{L^\infty(\mathbb{R}^d)}$, and integrating on $\mathbb{R}^d$ we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\frac{(u(x,t)-M)_+^2}{2}dx &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)(u(y,t)-u(x,t))(u(x,t)-M)_+dydx\\ &= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)(u(y,t)-u(x,t))\big((u(y,t)-M)_+-(u(x,t)-M)_+\big)dx\,dy\\ &\leq -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\big|(u(y,t)-M)_+-(u(x,t)-M)_+\big|^2dx\,dy.\end{aligned}$$

Therefore, we obtain that $u(x,t) \leq M$ for all $t \geq 0$ and a.e. $x \in \mathbb{R}^d$. In a similar way we get $u(x,t) \geq -M$ for all $t \geq 0$ and a.e. $x \in \mathbb{R}^d$.

We conclude that $\|u\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}$ and that the solution u is global.

Step II. Proof of the long time behaviour. We divide the proof in several steps.

Step II a). The case $p = 2$. Multiplying equation (7.23) by $\operatorname{sgn}(u)$ and u we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)|\,dx \leq 0 \tag{7.28}$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^d}u^2(t)dx + \langle A_2u(t),u(t)\rangle \leq 0. \tag{7.29}$$

Inequality (7.28) implies that (7.27) holds.

Inequalities (7.28) and (7.29) allow us to apply Lemma 7.3. Thus we obtain that

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}t^{-\frac{d}{2}(1-\frac{1}{2})}.$$

holds for large enough t. This gives us, by interpolation, the long time behaviour of the solution u in any $L^q(\mathbb{R}^d)$-norm when $1 \leq q \leq 2$.

Step II b). The case $p = 2^{n+1}$. We use an iterative argument to prove that once the result is assumed for $p = 2^n$ we get the result for $p = 2^{n+1}$.

Assume that it holds for $p = 2^n$. Then

$$\|u(t)\|_{L^{2^n}(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}t^{-\frac{d}{2}(1-\frac{1}{2^n})}$$

holds for all t large enough.

Let us fix $r = 2^{n+1}$. We multiply equation (7.23) with u^{r-1} to obtain

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_{\mathbb{R}^d}u^r(x,t)dx &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}J(x,y)(u(x,t)-u(y,t))u^{r-1}(x,t)\,dx\,dy\\ &= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}J(x,y)(u(x,t)-u(y,t))(u^{r-1}(x,t)-u^{r-1}(y,t))\,dx\,dy\\ &\leq -c(r)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}J(x,y)(u^{r/2}(x,t)-u^{r/2}(y,t))^2\,dx\,dy.\end{aligned}$$

Then $v = u^{r/2}$ verifies:

$$\frac{d}{dt}\int_{\mathbb{R}^d} v^2(x,t)dx + c(r)\langle A_2 v(t), v(t)\rangle \leq 0, \quad t > 0.$$

By Lemma 7.3 and Remark 7.10 we obtain that for large time t the following holds:

$$\|v(t)\|_{L^2(\mathbb{R}^d)} \leq \|v(t/2)\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})}.$$

Then

$$\|u^{r/2}(t)\|_{L^2(\mathbb{R}^d)} \leq \|u^{r/2}(t/2)\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})}$$

and using that $r = 2^{n+1}$:

$$\|u(t)\|^{2^n}_{L^{2^{n+1}}(\mathbb{R}^d)} \leq C(d,n)\|u(t/2)\|^{2^n}_{L^{2^n}(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})}.$$

Using the hypothesis on the $L^{2^n}(\mathbb{R}^d)$-norm of u we get

$$\begin{aligned}\|u(t)\|_{L^{2^{n+1}}(\mathbb{R}^d)} &\leq C(d,n)\|u(t/2)\|_{L^{2^n}(\mathbb{R}^d)} t^{-\frac{d}{2}(\frac{1}{2^n}-\frac{1}{2^{n+1}})}\\ &\leq C(d,n)\|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2^n})} t^{-\frac{d}{2}(\frac{1}{2^n}-\frac{1}{2^{n+1}})}\\ &\leq C(d,n)\|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2^{n+1}})}.\end{aligned}$$

The proof is now finished since we can interpolate between the cases $r = 2^n$ and $r = 2^{n+1}$, $n \geq 0$ an integer. ∎

7.3 Decay estimates for the nonlocal $p-$Laplacian

In this section we deal with the following nonlocal analogous to the $p-$laplacian evolution,

$$u_t(x,t) = \int_{\mathbb{R}^d} J(x,y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy. \tag{7.30}$$

Existence and uniqueness of a solution follows from the results in [4] (see also [3] for the Neumann problem). Again for this case we have to note that in those references a convolution kernel was considered $J(x,y) = G(x-y)$ but it can be checked that the same proof gives existence and uniqueness for a general $J(x,y)$.

Theorem 7.3. *([4], Proposition 2.4) Let $1 < p < \infty$. For any initial condition $u_0 \in L^p(\mathbb{R}^d)$ there exists a unique global solution $u \in C([0,\infty): L^p(\mathbb{R}^d)) \cap W^{1,1}((0,\infty): L^p(\mathbb{R}^d))$ of equation (7.30).*

Concerning the long time behaviour of the solutions of equation (7.30) we have the following result.

Theorem 7.4. *Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $2 \leq p < d$. For any $1 \leq q < \infty$ the solution to (7.30) verifies*

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \leq C t^{-\left(\frac{d}{d(p-2)+p}\right)\left(1-\frac{1}{q}\right)} \tag{7.31}$$

for all t sufficiently large.

Remark 7.11. *The condition $p \geq 2$ is used in the inductive step in our proof. Also $p < d$ is necessary in order to use Corollary 7.1.*

Proof. We multiply equation (7.30) by $|u|^{r-2}u(x)$, $1 \leq r < \infty$, and integrate to obtain, using Lemma 7.1,

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u|^r(x,t)\,dx \leq -C(p,r)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x,y)\left||u|^{\frac{p+r-2}{p}}(y,t) - |u|^{\frac{p+r-2}{p}}(x,t)\right|^p dy\,dx.$$

The above inequality gives us that for any $1 \leq r < \infty$, u, the solution to (7.30), satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u|^r(t,x)dx + C(p,r)\langle A_p|u(t)|^{\frac{p+r-2}{p}}, |u(t)|^{\frac{p+r-2}{p}}\rangle \leq 0 \tag{7.32}$$

This inequality is crucial to obtain the long time behaviour (7.31) of a solution u to (7.30).

Next, we will prove by induction that the sequence $\{p_n\}_{n\geq 0}$ defined by

$$p_0 = 1, \qquad p_{n+1} = pp_n - p + 2, \qquad n \geq 0,$$

satisfies

$$\|u(t)\|_{L^{p_n}(\mathbb{R}^d)} \leq Ct^{-d_n} \tag{7.33}$$

where

$$d_n = \frac{d}{d(p-2)+p}\left(1 - \frac{1}{p_n}\right).$$

As the sequence p_n verifies $p_n \to \infty$ as $n \to \infty$ the desired inequality (7.31) follows by interpolation.

□

7.4 Examples of exponential decay

In this section we present a simple example of $J(x,y)$ for which we obtain exponential decay of the solutions to the linear problem

$$u_t(x,t) = \int_{\mathbb{R}} J(x,y)(u(y,t) - u(x,t))\, dy. \tag{7.34}$$

Note that, to simplify, we restrict ourselves to one space dimension.

Lemma 7.4. *Let $a : \mathbb{R} \to \mathbb{R}$ be a diffeomorfism. Assume that*

$$J(x,y) \geq \frac{1}{2} \quad on \quad |y - a(x)| \leq 1,$$

where the function a satisfies

$$\sup_{\mathbb{R}} |(a^{-1})_x| < 1 \quad or \quad \inf_{\mathbb{R}} |(a^{-1})_x| > 1$$

then there exists a positive constant C such that

$$\langle A_2 u, u\rangle \geq C\|u\|^2_{L^2(\mathbb{R})}.$$

Proof. Using the symmetry of the function J we get

$$J(x,y) \geq \frac{1}{4}\chi_{\{|x-a(y)|<1\}} + \frac{1}{4}\chi_{\{|y-a(x)|<1\}}. \tag{7.35}$$

Let us consider $\psi : \mathbb{R} \to \mathbb{R}$ a smooth positive function, supported on $(-1,1)$. Then

$$2\|\psi\|_{L^\infty(\mathbb{R})} J(x,y) \geq \rho(x,y) := \psi(x - a(y)) + \psi(y - a(x))$$

and

$$2\|\psi\|_{L^\infty(\mathbb{R})}\langle A_2 u, u\rangle \geq \iint_{\mathbb{R}^2} \rho(x,y)(u(x) - u(y))^2 dxdy. \tag{7.36}$$

Let be θ a positive constant which will be fixed latter. We have

$$\iint_{\mathbb{R}^2} \rho(x,y)(u(x)-u(y))^2\,dx\,dy$$
$$\geq \begin{cases} \dfrac{1-\theta}{\theta}\displaystyle\int_{\mathbb{R}}\psi(y)dy\int_{\mathbb{R}}u^2(x)\Big(\theta-\frac{\sup_{x\in\mathbb{R}}\psi*|(a^{-1})_x|}{\int_{\mathbb{R}}\psi(y)dy}\Big)dx, & \theta<1,\\ \dfrac{1-\theta}{\theta}\displaystyle\int_{\mathbb{R}}\psi(y)\,dy\int_{\mathbb{R}}u^2(x)\Big(\theta-\frac{\inf\psi*(a^{-1})_x}{\int_{\mathbb{R}}\psi(y)\,dy}\Big)\,dx, & \theta>1.\end{cases}$$

If $\sup_{x\in\mathbb{R}}|(a^{-1})_x(x)|<1$ we choose θ satisfying

$$\frac{\sup_{\mathbb{R}}\psi*|(a^{-1})_x|}{\int_{\mathbb{R}}\psi(y)dy}<\theta<1$$

and thus by (7.36)

$$2\|\psi\|_{L^\infty(\mathbb{R})}\langle A_2u,u\rangle\geq C(\theta,\psi,a)\|u\|^2_{L^2(\mathbb{R})}.$$

The other case

$$\inf_{x\in\mathbb{R}}|(a^{-1})_x(x)|>1$$

can be treated in a similar way. □

Theorem 7.5. *Let $u_0\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$. Then the solution to (7.34) verifies*

$$\|u(\cdot,t)\|_{L^2(\mathbb{R})}\leq Ce^{-Ct}$$

for all $t>t_0$.

Proof. Multiplying equation (7.34) by u we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d}u^2(t)\,dx+\langle A_2u(t),u(t)\rangle\leq 0,$$

and using our previous estimate (Lemma 7.4) we get

$$\frac{d}{dt}\int_{\mathbb{R}^d}u^2(t)\,dx+C\int_{\mathbb{R}^d}u^2(t)dx\leq 0,$$

from where the result follows. □

Chapter 8

Final comments and further directions of research

The activity of L. Ignat in the last five years has been mainly carried out within the Institute of Mathematics "Simion Stoilow" of the Romanian Academy. The teaching activity has been concretized in three master courses on Partial Differential Equations at "Şcoala Normală Superioară Bucureşti". For short periods of times L. Ignat was visiting researcher at Basque Center for Applied Mathematics (Spain) and Henri Poincare Institute (France).

The author of this thesis has also advised one master thesis at SNSB Bucharest: Diana Stan in 2010. Diana is now a PhD student at ICMAT Madrid under the advice of Juan-Luis Vasquez. Another master thesis is under my supervision: that of Cristian Gavrus which will be defended in 2012. This shows the capacity of the author to advise PhD theses in the future.

We want to point out that in September 2010, L. Ignat was member of the committee of Aurora Marica's PhD thesis at Universidad Autonoma de Madrid even this would have been impossible in Romania under the Romanian legislation at the time. Aurora Marica visited IMAR with a Bitdefender fellowship working under the supervision of L. Ignat.

The author of this thesis has obtained three grants in the last five years:

1. Analysis, Control and Numerical Approximations of Partial Differential Equation, CNCS-UEFICDI, 05/10/2011-04/10/2014, 1.5000.00 RON=350.000 EUR

2. Qualitative properties of partial differential equations and their numerical approximations, CNCSIS, PN II, TE-4/2010, 28/07/2010 - 27/07/2013, 750.000RON=175.000 EUR.

3. Qualitative properties of diffusion and dispersion in the study of the nonlinear problems and their numerical approximations, CNCSIS, RP-3,10/2007-09/2009, 500.000RON=125000EUR.

He has also been a member in Spanish research projects as well as in the EU grant NUMERI-WAVES whose IP was/is Enrique Zuazua.

In the last four years, L. Ignat have taught four courses, three at SNSB and another one at Basque Center for Applied Mathematics, Bilbao, Spain.

1. Numerical Methods for Partial Differential Equations, SNSB, 2010-2011,

2. Numerical schemes for dispersive equations, February 08, 2010 - February 12, 2010, BCAM, Bilbao, Spain,

3. Evolution equations, SNSB, 2009-2010,

4. Evolution equations: dissipation and dispersion, SNSB, 2008-2009.

The activity of L. Ignat in the last five years therefore shows his capacity to advise PhD students, a thing that was impossible until now under the Romanian legislation. All the author's future plans will also depend on the capacity of the Romanian institutions to attract persons that have obtained their habilitation.

The future plan of L. Ignats evolution will include advising students and continuing the research lines that have been productive in the last five years. To transform all these plans into reality the author will apply to open positions in the Romanian academic environment. L. Ignat will also apply to grant calls. This was productive in the past since the author has obtained grants in all the competitions he has applied to in the last five years, even in the case of those destined to senior researchers.

Some Open Problems

Finally we want to present only a few of the open problems that are under consideration to be analyzed in the future. We mention that these problems have also been presented in the chapters of this thesis.

I. Schrödinger equation on networks. In Chapter 1 we have analyzed the dispersive properties for the linear Schrödinger equation on trees. We have assumed that the coupling is given by Kirchhoff's classical conditions. However there are other coupling conditions (see [77]) which allow us to define a "Laplace" operator on a metric graph. To be more precise, let us consider the operator H that acts on functions on the graph Γ as the second derivative $\frac{d^2}{dx^2}$, and its domain consists in all functions f that belong to the Sobolev space $H^2(e)$ on each edge e of Γ and satisfy the following boundary condition at the vertices:

$$A(v)\mathbf{f}(v)+B(v)\mathbf{f}'(v)=0 \quad \text{for each vertex } v. \tag{8.1}$$

Here $\mathbf{f}(v)$ and $\mathbf{f}'(v)$ are correspondingly the vector of values of f at v attained from directions of different edges converging at v and the vector of derivatives at v in the outgoing directions. For each vertex v of the tree we assume that matrices $A(v)$ and $B(v)$ are of size $d(v)$ and satisfy the following two conditions

1. the joint matrix $(A(v), B(v))$ has maximal rank, i.e. $d(v)$,
2. $A(v)B(v)^T = B(v)A(v)^T$.

Under these assumptions it has been proved in [77] that the considered operator, denoted by $\Delta(A, B)$, is self-adjoint. The case considered in this paper, the Kirchhoff coupling, corresponds to the matrices

$$A(v)=\begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0\\ 0 & 1 & -1 & \dots & 0 & 0\\ 0 & 0 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \vdots & 1 & -1\\ 0 & 0 & 0 & \vdots & 0 & 0 \end{pmatrix},\quad B(v)=\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 0 & 0\\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}.$$

More examples of matrices satisfying the above conditions are given in [77, 78].

The existence of the dispersive properties for the solutions of the Schrödinger equation on a graph under general coupling conditions on the vertices $iu_t + \Delta_\Gamma(A, B)u = 0$ is mainly an open problem. The resolvent formula obtained in [78] and [80] in terms of the coupling matrices A and B might help one to understand the general problem. In the same papers there are also

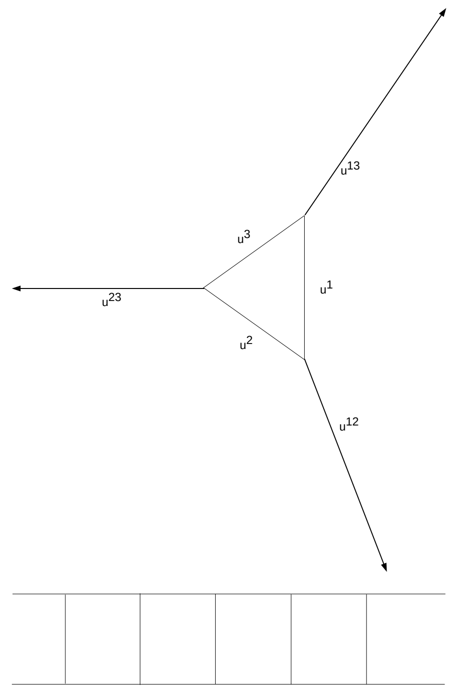

Figure 8.1: Networks where two kinds of structures occur: a periodic one coupled with an infinite one

some combinatorial formulations of the resolvent in terms of walks on graphs. Such combinational aspects could clarify if the dispersion is possible only on trees or if there are graphs (with some of the edges infinite) with suitable couplings where the dispersion is still true.

It is expected that other results on the Schrödinger equation on $\mathbb{R}$ are still valid on networks. For instance, the smoothing estimate for the linear equation with constant coefficients is still valid. Although its classical proof on $\mathbb{R}$ relies on Fourier analysis, one may easily adapt the proof in [20] which uses only integrations by parts and Besovs spaces that can still be defined on a tree using the heat operator. Strichartz estimates were used previously to treat controllability issues for the NSE in [92]. The possible applications of the present results in the control context are still to be analyzed. We mention here some previous works on the controllability/stabilization of the wave equation on networks [37], [107].

Another interesting problem consists in the analysis of the same properties on some graphs which combine the periodic structure with the infinite strips. This is the case in Fig. 8.1 and Fig. 8.2. We recall that for LSE on the one-dimensional torus, Bourgain [16] has analyzed the existence of Strichartz estimates. In the same framework we also mention works [19] and [104].

Finally, another problem of interest is the study of the dispersion properties for the magnetic operators analyzed in [81], [79]. The analysis in this case is more difficult since in the presence of an external magnetic field the effect of the topology of the graph becomes more pronounced. In contrast with the analysis done here, in the case of magnetic operators the graphs are viewed as structures in the three dimensional Euclidean space $\mathbb{R}^3$ and the orientation of the edges becomes important.

II. Inverse problems on networks. We now mention a few open problems related to our work in Chapter 2. One of them is whether it is possible to reduce the number of measurements at the boundaries. It could be interesting to combine the ideas of the paper with those appearing in [36], [37] where less measurements on the boundary are needed but some rationality assumptions on the lengths of the edges have to be made. For the Schrödinger equation, the question whether a

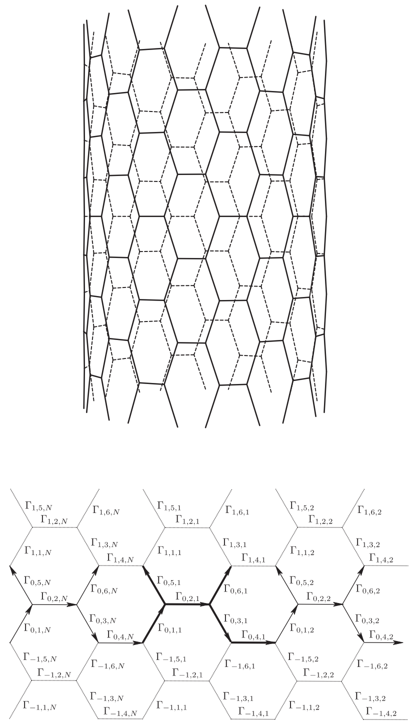

Figure 8.2: Exotic structures

Carleman estimate on a tree with N exterior vertices can be written with only one weight function and $N-1$ boundary observations seems to be challenging.

The extension of the present work to more general graphs with other kind of coupling is also an open problem. We recall here the works of Kostrykin and Schrader [77, 80] where self-adjoint Laplace operators with general coupling conditions are introduced.

III. Discrete equations. In Chapter 3 we have analyzed the dispersive properties of the solutions of a system consisting in coupling two discrete Schrödinger equations. However we do not cover the case when more discrete equations are coupled. The main difficulty is to write in an accurate and clean way the resolvent of the linear operator occurring in the system. Once this case is understood then we can treat discrete Schödinger equations on trees similarly to those considered in [55] in the continuous case.

The analysis presented in this paper mainly concerns the l^1-l^∞ decay property. In a recent paper [89] the authors used some modifications of the stationary phase method to obtain improved l^1-l^p decay estimates for the linear Fermi-Pasta-Ulam chain, the Klein-Gordon chain and the discrete nonlinear Schrödinger equation. The optimality of l^1-l^p estimates for the models presented here remains to be investigated.

There is another question which arises from this paper. Suppose that we have a system $iU_t + AU = 0$ with an initial datum at $t=0$, where A is a symmetric operator with a finite number of diagonals not identically vanishing. Under which assumptions on the operator A does solution U decay and how can we characterize the decay property in terms of the properties of A? When A is a diagonal operator we can use Fourier's analysis tools but in the case of a non-diagonal operator this is not useful.

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