TOEPLITZ ALGEBRAS ARISING FROM ACTIONS OF $\mathbb{N}^r$

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We define a semidirect product groupoid of a system of partially defined local homeomorphims $T = (T_1, \ldots, T_r)$. We prove that this construction gives rise to amenable groupoids. The algebra associated is a Cuntz-like algebra. We use this construction for higher rank graph algebras in order to give a topological interpretation for the duality in E-theory between $C^*(\Lambda)$ and $C^*(\Lambda^{op})$.

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INTRODUCTION

Toeplitz algebras have been used to define extensions of C^*-algebras. The beginning of this paper is in [20] where a Toeplitz algebra was the main tool in constructing a K-homology class for higher rank graph algebras. Let (Λ, σ) be a higher rank graph with shape (see [17]), Λ^* the set of morphism of nonzero shape and $\overline{\Lambda} = \Omega \cup \Lambda^*$ where Ω is a symbol (the vacuum morphism) which does not belong to Λ^*. We define left and right creations on the Fock space $\mathsf{F}_\Lambda = \mathsf{F} = l^2(\overline{\Lambda})$:

$$L_\lambda \delta_\mu = \begin{cases} \delta_{\lambda\mu} & \text{if } s(\lambda) = t(\mu) \\ 0 & \text{otherwise,} \end{cases}$$

$$R_\lambda \delta_\mu = \begin{cases} \delta_{\mu\lambda} & \text{if } s(\mu) = t(\lambda) \\ 0 & \text{otherwise,} \end{cases} \qquad L_\lambda \Omega = R_\lambda \Omega = \delta_\lambda.$$

The left sided Toeplitz algebra is $\mathcal{E}_l = C^*(L_\lambda; \lambda \in \Lambda^*)$ the right sided Toeplitz algebra is $\mathcal{E}_r = C^*(R_\lambda; \lambda \in \Lambda^*)$ and the two sided Toeplitz algebra is $\mathcal{E} = C^*(L_\lambda, R_\lambda; \lambda \in \Lambda^*)$. These algebras are well known in the rank one case since they give rise to short exact sequences. For example, one can use $\mathcal{E}$ to define

$$0 \to \mathcal{K} \to \mathcal{E} \to O_E \otimes O_{E^{op}} \to 0,$$

where E is an ordinary oriented graph with certain conditions which give the uniqueness of the algebras O_E and $O_{E^{op}}$, E^{op} is the graph obtain by reversing the arrows. In the higher rank case the algebra $\mathcal{E}$ has a more complicated ideal structure. It is this fact which gives a longer exact sequence. It would give a KK-class if the sequence was semisplit [20, 24]. This would follow immediately from the nuclearity of $\mathcal{E}$. We tried to give a more conceptual construction of this algebra as a groupoid algebra of an amenable groupoid. We have not been able to do it except in the rank one case. However, the construction appearing in [22] can be generalized to give a groupoid description of one-sided Toeplitz algebras of a higher rank graph. We believe that this construction can be generalized to actions of other semigroups. For example actions of cones in $\mathbb{R}^n$ should give algebras generated by Wiener-Hopf operators (see [18, 19]).

The organization of this paper is as follows. In the first section we give the main construction of this paper, the semidirect product of an action of $\mathbb{N}^r$ by partial local homeomorphism which we also call a multiply generated dynamical system. Such an action is obtained by choosing r commuting partially defined local homeomorphisms. The main examples come from shifts on finite, semifinite and infinite paths of a higher rank graph. There are two groupoids associated to such an action, the groupoid of germs of the pseudogroup generated by these local homeomorphism and the semidirect product groupoid. We think that these groupoids have been around in the study of Toeplitz algebras of higher rank graphs as well as in the general theory of crossed products by partial actions (see for example [11, 12]). The semidirect product can be defined only if we have a condition on the domains of the partial maps. Otherwise it may not even give rise to an algebraic groupoid. This condition is fullfiled for one sided Toeplitz algebras associated to higher rank graphs. Following the lines in [22], we prove that these two groupoids are isomorphic if and only if the dynamical system is essentially free. Next we prove the amenability of the semidirect product. This is done by decomposing the action in subaction and then applying a result on the amenability of extensions.

In the second section we give a groupoid approach for the two-sided Toeplitz extension of a rank-one graph. The description is ad-hoc since we do not know an elegant construction like a semidirect product. The definition of the unit space is indicated by the diagonal algebra $\mathcal{E}\cap l^\infty(\overline{\Lambda})$ which is generated by the projections $L_\lambda L_\lambda^*$ and $R_\lambda R_\lambda^*$. These groupoids are not Hausdorff and this fact is best illustrated by the graph with one vertex and one edge (whose C^*-algebra is $C(\mathbb{T})$).

In the third section we remind several facts about duality in a bivariant theory. We give the definition of Spanier-Whitehead duality and a condition which is often taken as definition in literature. We remind then a way to construct exact sequences starting from an algebra and a tuple of ideals. We prove that irrational rotation algebras are dual. Here we give an operator approach and a long exact sequence approach. We start with simple observations of the duality of circles along the lines of [15]. Ideas appearing in this example are inspiring for more general cases, even for higher rank graph algebras. The example of rotation algebras is included in the last section using a twist of the Cuntz-Krieger relation for a higher rank graph.

In the last section we improve the results in [20] and give a groupoid approach to the duality of higher rank graph algebras. We make some simple observations inspired by the example of rotation algebras to improve the result to any locally finite higher rank graph with finite set of objects, regardless the uniqueness of the generating relation of our graph algebras as for examples the graph $\mathbb{N}^r$. We obtain a duality for the universal C^*-algebras $C^*(\Lambda)$ and $C^*(\Lambda^{op})$. The K-theory fundamental class is given by r partial unitaries which can be best described in the groupoid picture as two-sided shifts. Even if we do not have a conceptual groupoid approach to the two-sided Toeplitz algebra, we can define it as a groupoid of germs. The main problem in understanding better this groupoid is the unit space X which is given by the spectrum of the diagonal algebra $\mathcal{E} \cap l^\infty(\overline{\Lambda})$. The isometries $L^*_\lambda R_\mu$ with $\sigma(\lambda) = \sigma(\mu) = e_j$ are not given by something like shift equivalence. However, the partial isometries can be viewed as partial homeomorphisms using Gelfand duality . The space X has three properties inherited from the graph Λ: the shape σ extends to a map from X to $\overline{\mathbb{N}}^r$, the multiplication $\lambda\mu\nu$ extends to a multiplication $\lambda x \mu$, and the unique factorization $\alpha = \lambda\alpha'\mu$ where $\sigma(\alpha) \geq \sigma(\lambda) + \sigma(\mu)$ extends to a unique factorization $x = \lambda x' \mu$ where $\sigma(x) \geq \sigma(\lambda) + \sigma(\mu)$. These properties are enough to define a MGDS given by shifts on the space $X \times \Lambda^\infty$ together with an equivariant map to the MGDS $\overline{\mathbb{N}}^r \times \Lambda^\infty$ given again by shifts. This map induces a map of the semidirect product groupoids which in turn induces a morphism between the algebras $\mathcal{T}^{\otimes r} \otimes C^*(\Lambda)$ and $\mathcal{E} \otimes C^*(\Lambda)$. This morphism is the crucial step in the proof of the duality.

Finally, we want to draw the attention to the results in [9, 10]. Our groupoid approach may give a hint to what a higher rank hyperbolic group would be.

1. PRELIMINARIES

In this paper we shall use locally compact r-discrete groupoids (possibly non-Hausdorff) ([23]). The canonical Haar system λ^u is the discrete measure. A 2-cocycle α in $Z^2(G, \mathbb{T})$ is a map α defined on the set G^2 of composable pairs to $\mathbb{T}$ such that $\alpha(x, yz)\alpha(y, z) = \alpha(xy, z)\alpha(x, y)$ for any $(x, y), (y, z) \in G^2$ ([21]). The convolution algebra $C_c(G, \sigma)$ is given by the operation

$$f * g(x) = \int f(xy)g(y^{-1})\alpha(xy, y^{-1})\mathrm{d}\lambda^{d(x)}(y), \quad f^*(x) = f(x^{-1})\alpha(x, x^{-1}).$$

We shall use mainly the reduced algebra $C_r(G,\alpha)$ which is a certain completion of $C_c(G,\alpha)$. Any open invariant set of G^0 gives rise to an ideal of $C_r(G,\alpha)$. We use the cocycle σ in order to include in our study rotation algebras and some twisted higher rank graph algebras.

Two partial maps on a set X, $L:\mathrm{dom}(L)\to\mathrm{ran}(L)$ and $R:\mathrm{dom}(R)\to\mathrm{ran}(R)$ can be composed if one defines

$$\mathrm{dom}(LR)=\{x\in\mathrm{dom}(R);\ R(x)\in\mathrm{dom}(L)\}$$

and $LR(x)=L(R(x)$ for any $x\in\mathrm{dom}(LR)$. We say that two partial maps L and R on a set X commute if $\mathrm{dom}(LR)=\mathrm{dom}(RL)$ and $LR=RL$ on $\mathrm{dom}(LR)$. We set $L^0=\mathrm{id}$ for any partial map on X. If L is injective, we denote by L^{-1} the partial map $L^{-1}:\mathrm{ran}(L)\to\mathrm{dom}(L)$. A local homeomorphism is a map $\phi:X\to Y$ with the property that each point x has a neighborhood U such that $\phi|_U:U\to\phi(U)$ is a homeomorphism.

A partial homeomorphism on X is a local homeomorphism S from an open set $\mathrm{dom}(S)$ of X onto an open set $\mathrm{ran}(S)$ of X. A set $\mathcal{G}$ of partial homeomorphisms of X which is closed under composition, inversion and containing the identity is called a pseudogroup ([2]). For any set S of partial homeomorphisms on X, there exists the smallest pseudogroup $[S]$ generated by S. The semi-direct product groupoid $X\rtimes\mathcal{G}$ of a pseudogroup $\mathcal{G}$ is the set of triples (x,S,y) where $S\in\mathcal{G},y\in\mathrm{dom}(S),x=S(y)$ with the obvious operations

$$(x,S,y)(y,T,z)=(x,ST,y),\quad (x,S,y)^{-1}=(y,S^{-1},x).$$

The topology is given by the product topology of X and $\mathcal{G}$ where $\mathcal{G}$ has the discrete topology. The groupoid of germs is a quotion of the semidirect product groupoid by the equivalence relation $(x_1,S_1,y_1)\sim(x_2,S_2,y_2)$ if and only if $y_1=y_2$ and $S_1=S_2$ on a neighborhood of y_1. The topology is the quotient topology. These two groupoids are r-discrete, that is the range and source maps are local homeomorphisms.

In the monoid $\mathbb{N}^r$ we write $n=(n_1,\ldots,n_r)$ with $n_j\in\mathbb{N}$ and e_k the coordinates $(0,\ldots,1,\ldots,0)$.

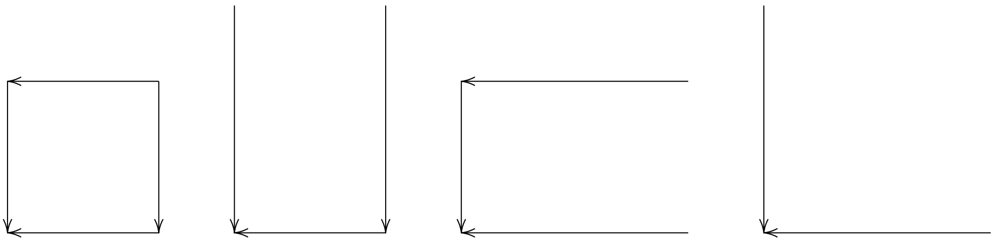
2. CUNTZ-LIKE ALGEBRAS ASSOCIATED WITH MULTIPLY GENERATED DYNAMICAL SYSTEMS

Definition 2.1 (Conform [22]). A multiply generated dynamical system (MGDS) is a pair (X,T) where X is a topological space and $T=(T_1,\ldots,T_r)$ a system of r commuting partial homeomorphisms on X.

Examples 2.2. (i) ([12], Definition 5.1) For $m\in\overline{\mathbb{N}}^r$ let $\mathbb{N}^r_m$ be the higher rank graph $\{(n,n')\in\mathbb{N}^r\times\mathbb{N}^r : n\le n'\le m\}$ where $s(n,k)=k$, $t(n,k)=n$, $\sigma(n,k)=k-n$ and the composition is given by $(n,k)(k,p)=(n,p)$. Let Λ be a finitely aligned rank r graph and

$$X_\Lambda = \{x : \mathbb{N}^r_m \to \Lambda;\ m \in \overline{\mathbb{N}}^r_m\}$$

the space of finite, semifinite and infinite paths. The shape σ can be extended to X_Λ, $\sigma(x) = m$ where x is defined on $\mathbb{N}^r_m$. For $r = 2$ an element in X_Λ can be seen graphically as one of the following:



A basis of a topology on X_Λ is given by the sets $\{x : x(0,k) = \lambda, k \le \sigma(x) \le m\}$ where $k \in \mathbb{N}^r_m$, $\lambda \in \Lambda$ and $m \in \overline{\mathbb{N}}^r_m$. Then the partial homeomorphisms T_k with $\mathrm{dom}(T_k) = \{x \in X_\Lambda : \sigma(x) \ge k\}$, $T_k(x) : \mathbb{N}^r_{\sigma(x)-k} \to \Lambda$, $T_k(n, n') = x(n + k, n' + k)$ give a MGDS.

(ii) In the example above the restriction to boundary paths ∂X gives a subsystem. If Λ has no sources then $\partial X = \Lambda^\infty = \{x : \sigma(x) = (\infty \dots \infty)\}$ (see [12], Definition 5.10 and [17]).

(iii) Let X be the free monoid on r letters $a_1, \dots, a_r$, that is the set of words $a_{i_1} \dots a_{i_k}$. Put on X the discrete topology and define $T = (L, R)$ the translations on the left and on the right $\mathrm{dom}(L) = \mathrm{dom}(R) = \{a_{i_1} \dots a_{i_k} \colon k \ge 1\}$ (the set of nonvoid words), $L(a_{i_1} a_{i_2} \dots a_{i_k}) = a_{i_2} \dots a_{i_k}$, $R(a_{i_1} \dots a_{i_{k-1}} a_{i_k}) = a_{i_1} \dots a_{i_{k-1}}$. For instance, if $r = 1$ then $X = \mathbb{N}$ and $\mathrm{dom}(L) = \mathrm{dom}(R) = \mathbb{N} \setminus \{0\}$, $L(k) = R(k) = k - 1$. It is clear that L and R commute.

We denote by $\mathcal{G}(X, T)$ the full pseudogroup generated by the restrictions of $T_j|_U$ where U is an open subset of X on which T_j is injective. Because of the commutation conditions on T, we can define $T^n = T_1^{n_1} T_2^{n_2} \cdots T_k^{n_r}$ for $n \in \mathbb{N}^k$. One can see a MGDS as an action of the semigroup $\mathbb{N}^r$ on X by partial local homeomorphisms. We write sometimes T_n instead of T^n when we want to view T as a semigroup. We need a technical condition on the domains of T^n (domain condition)

$$\text{(DC)} \qquad \mathrm{dom}(T^n) \cap \mathrm{dom}(T^m) \subset \mathrm{dom}(T^{n \vee m}),$$

where $n \vee m$ is the componentwise maximum. The following lemma is basically Lemma 2.4 from [22] for MGDS with (DC).

Lemma 2.3. *Let (X, T) be a MGDS with (DC).*

(i) *A partial homeomorphism S belongs to $\mathcal{G}(X,T)$ if and only if it is locally of the form $(T^m_{|U})^{-1}T^n_{|V}$ where $m,n \in \mathbb{N}^r$, U is an open set on which T^m is injective and V is an open set on which T^n is injective.*

(ii) *Let $a \in X$. Suppose that $(T^m_{|U})^{-1}T^n_{|V}$ and $(T^p_{|W})^{-1}T^q_{|Y}$ are two partial homeomorphisms as in* (i) *having a in their domains and $(T^m_{|U})^{-1}T^n_{|V}a = (T^p_{|W})^{-1}T^q_{|Y}a$. If $m-n=p-q$, then $(T^m_{|U})^{-1}T^n_{|V}$ and $(T^p_{|W})^{-1}T^q_{|Y}$ have the same germ at a.*

Proof. (i) It is clear that $(T^m_{|U})^{-1}T^n_{|V} \in \mathcal{G}(X,T)$ and, since $\mathcal{G}(X,T)$ is full, it is still true for a partial homeomorphism locally of this form. The inverse of $(T^m_{|U})^{-1}T^n_{|V}$ is $(T^n_{|V})^{-1}T^m_{|U}$ which belongs to $\mathcal{G}(X,T)$. It remains to show that the product of $(T^m_{|U})^{-1}T^n_{|V}$ and $(T^p_{|W})^{-1}T^q_{|Y}$ is locally of the same form. When $r=1$ this is the alternative $n \geq p$ or $n \leq p$ given in the proof of Lemma 2.4 of [22]. In our setting this alternative does not work since $\mathbb{N}^r$ is not totally ordered. For this reason we need the condition (DC). Let $x \in Y$ such that $(T^p_{|W})^{-1}T^q_{|Y}x = y \in V$. We can suppose that this happens in a neighborhood of x and so we assume that it is true on Y. Then with $z = (T^m_{|U})^{-1}T^n_{|V}y$ we have $T^q x = T^p y$ and $T^n y = T^m z$. From condition (DC) we have $y \in \mathrm{dom}(T^{p\vee n})$ and $T^{q+p\vee n-p}x = T^{p\vee n}y = T^{m+p\vee n-n}z$ for any $x \in Y$ so that $(T^m_{|U})^{-1}T^n_{|V}(T^p_{|W})^{-1}T^q_{|Y}$ is locally of the form $(T^{m+p\vee n-n}_{|Z})^{-1}T^{q+p\vee n-p}_{|Z'}$.

(ii) Taking neighborhoods of a and $(T^m_{|U})^{-1}T^n_{|V}a$, we may assume that $W=U$ and $Y=V$, $T^p(U)=T^q(V)$ and $T^{m\vee p}$ is injective on U. For $x \in V$, $(T^m_{|U})^{-1}T^n_{|V}x = y \in U$, $(T^p_{|U})^{-1}T^q_{|V}x = y' \in U$ we have $T^m y = T^n x$ and $T^p y' = T^q x$. As $U \subset \mathrm{dom}(T^m) \cap \mathrm{dom}(T^p) \subset \mathrm{dom}(T^{m\vee p})$ we have $T^{m\vee p}y = T^{n+m\vee p-m}x = T^{q+m\vee p-p}x = T^{m\vee p}y'$ and therefore $y=y'$ so $(T^m_{|U})^{-1}T^n_{|V} = (T^p_{|U})^{-1}T^q_{|V}$. □

Having defined a pseudogroup, we denote by $\mathrm{Germ}(X,T)$ the groupoid of germs of $\mathcal{G}(X,T)$.

Following [22], Definition 2.5 we consider another groupoid, the semidirect product groupoid (simply replacing $\mathbb{Z}$ with $\mathbb{Z}^r$):

Definition 2.4. Let (X,T) be a MGDS with (DC). Its semidirect groupoid is

$$G(X,T) = \{(x, m-n, y);\ m,n \in \mathbb{N}^r,\ x \in \mathrm{dom}(T^m),\ y \in \mathrm{dom}(T^n),\ T^m x = T^n y\}$$

with the groupoid structure induced by the product structure of the trivial groupoid $X \times X$ and of the group $\mathbb{Z}^r$. The topology is defined by the basic open sets

$$\mathcal{U}(U;m,n;V) = \{(x,m-n,y) : (x,y) \in U \times V,\ T^m x = T^n y\},$$

where U (respectively V) is an open subset of the domain of T^m (respectively T^n) on which T^m (respectively T^n) is injective.

The family of given subsets is indeed a basis for a topology since

$$\mathcal{U}(U; m, n; V) \cap \mathcal{U}(U'; m', n'; V') \supset \mathcal{U}(U \cap U'; m \vee m', n \vee n'; V \cap V').$$

Thus, $\gamma = (x, z, y)$ and $\eta = (x', z', y')$ in $G(X, T)$ are composable if and only if $y = x'$ and then $\gamma\eta = (x, z + z', y')$. The range and domain are $r(x, z, y) = x$ and $d(x, z, y) = y$. This is a groupoid indeed since $y \in \mathrm{dom}(T^{n\vee m'})$ so $T^{m+n\vee m'-n}x = T^{n'+n\vee m'-m'}y'$. In the absence of the condition (DC), $G(X, T)$ may not be a groupoid. Consider, for example, (X, L, R) as in Example 2.2(iii). The condition (DC) is not satisfied since $\mathrm{dom}(L) \cap \mathrm{dom}(R)$ is the set of nonvoid words, while $\mathrm{dom}(LR)$ is the set of words with length greater or equal to 2. Let $|\cdot|$ be the word length on X and Ω the empty word. For $x, y \in X$ we have $\gamma = (x, (|x|, -|y|), y) \in G(X, T)$ and $\eta = (y, (|y|, 0), \Omega) \in G(X, T)$ since $T^{(|x|,0)}x = L^{|x|}x = \Omega = R^{|y|}y = T^{(0,|y|)}$ and $T^{(|y|,0)}y = L^{|y|}y = \Omega = T^0\Omega$ but $\gamma\eta = (x, (|x| + |y|, -|y|), \Omega) \notin G(X, T)$ since $x \notin \mathrm{dom}(T^n)$ for any $n \in \mathbb{N}^2$ with $n_1 > |x|$.

We assume again that (X, T) have (DC). According to (ii) of the previous lemma, there is a map π from $G(X, T)$ onto $\mathrm{Germ}(X, T)$ which sends $(x, m - n, y)$ into the germ $[x, (T^m_{|U})^{-1}T^n_{|V}, y]$ where U is an open neighborhood of x on which T^m is injective and V is an open neighborhood of y on which T^n is injective. This map is continuous and is a groupoid homomorphism. It is an isomorphism when (X, T) is essentially free.

Definition 2.5 (Conform [22], Definition 2.6). We shall say that a MGDS (X, T) is essentially free if for every pair of distinct $m, n \in \mathbb{N}^r$, there is no open set on which T^n and T^m agree.

LEMMA 2.6 (Conform [22], Lemma 2.7). *Let (X, T) be an essentially free MGDS with (DC). Then*

(i) *If $(T^m_{|U})^{-1}T^n_{|V}$ and $(T^p_{|W})^{-1}T^q_{|Y}$, where $m, n, p, q \in \mathbb{N}$ and U, V, W, Y are open sets such that $T^m_{|U}, T^n_{|V}, T^p_{|W}, T^q_{|Y}$ are injective and have the same germ at a, then $m - n = p - q$.*

(ii) *The map $c : \mathrm{Germ}(X, T) \to \mathbb{Z}^r$ such that $c[(T^m_{|U})^{-1}T^n_{|V}x, (T^m_{|U})^{-1}T^n_{|V}, x] = m - n$ is a continuous homomorphism.*

Proof. (i) By assumption, we have $T^m y = T^n x$ and $T^p y = T^q x$ for x and y in neighborhoods of a respectively $b = (T^m_{|U})^{-1}T^n_{|V}a$. Then on these neighborhoods we have $T^{n+m\vee p-m}x = T^{m\vee p}y = T^{q+m\vee p-p}x$. The essential freeness implies that $n + m \vee p - m = q + m \vee p - p$ so $n - m = q - p$.

(ii) We have seen in the proof of the previous lemma that the product of $(T^m_{|U})^{-1}T^n_{|V}$ and $(T^p_{|W})^{-1}T^q_{|Y}$ is locally of the form $(T^{m+p\vee n-n}_{|Z})^{-1}T^{q+p\vee n-p}_{|Z'}$

and that the inverse of $(T^m_{|U})^{-1}T^n_{|V}$ is $(T^n_{|V})^{-1}T^m_{|U}$. This shows that c is a homomorphism and by construction it is locally constant. □

PROPOSITION 2.7. *Let (X,T) be a MGDS with (DC). Then (X,T) is essentially free if and only if the above surjection $\pi : G(X,T) \to \mathrm{Germ}(X,T)$ is an isomorphism.*

Proof. Word with word as in [22], Proposition 2.8. □

This homomorphism induces a morphism of algebras even if (X,T) is not essentially free. We construct it in

LEMMA 2.8. *Let $\pi : G_1 \to G_2$ a surjective morphism between two r-discrete groupoids with the property that if $s(\pi(x)) = r(\pi(y))$ then $s(x) = r(y)$ Then the correspondence $C_c(G_1) \ni f \to \tilde{\pi}(f) \in C_c(G_2)$, $\tilde{\pi}(f)(x) = \sum_{\pi(x')=x} f(x')$, is a *-homomorphism between the topological algebras $C_c(G_1)$ and $C_c(G_2)$. Composing it with the regular representation of $C_c(G_2)$ we obtain a bounded representation of $C_c(G_1)$ therefore a morphism $\tilde{\pi} : C^*(G_1) \to C^*_r(G_2)$.*

Proof. The condition in the statement is $(\pi \times \pi)^{-1}(G_2^2) = G_1^2$. This means that composable pairs lifts only to composable pairs. In particular, the restriction π^0 of π to the unit space must be a bijection. Moreover, π satisfies this condition if π^0 is a bijection which identifies the equivalence relations associated to G_1 and G_2. In particular if $\pi(u) = r(x)$ then $\{x';\, \pi(x') = x\} \subset G_1^u$. Indeed, if $\pi(x') = \pi(y') = x$ then $\pi(y'^{-1})$ and $\pi(x')$ are composable, so y'^{-1} and x' are composable, hence $r(x') = r(y')$. This says that $\pi(G_1^u) = G_2^{\pi^0(u)}$. If $f \in C_c(G_1)$ then $\{x' : \pi(x') = x\} \subset \mathrm{supp}(f) \cap G_1^u$ which is finite. Therefore, the sum which defines $\tilde{\pi}(f)$ is finite. We have $\mathrm{supp}(\tilde{\pi}(f)) \subset \pi(\mathrm{supp}(f)$ so $\tilde{\pi}(f) \in C_c(G_2)$.

We compute

$$\tilde{\pi}(f \star g)(x) = \sum_{\pi(x')=x} f \star g(x') = \sum_{\pi(y'z')=x} f(y')g(z'),$$

$$\tilde{\pi}(f) \star \tilde{\pi}(g)(x) = \sum_{yz=x} \tilde{\pi}(f)(y)\tilde{\pi}(g)(z).$$

Since π is surjective we have

$$\tilde{\pi}(f) \star \tilde{\pi}(g)(x) = \sum_{\pi(y')\pi(z')=x} f(y')g(z').$$

The first sum runs over $\{(y', z') : \pi(y')\pi(z') = x\}$ and the second over sum runs over $\{(y', z') : \pi(y'z') = x\}$. These two sets are equal by the assumption on π so $\tilde{\pi}$ is an algebraic morphism.

If $f_n \in C_c(G_1)$, $\text{supp}(f_n) \subset K$, $f_n \to^u f$ on K, then $\tilde{\pi}(f_n) \to^u \tilde{\pi}(f)$ since the cardinal of the set $\{x' : \pi(x') = x\}$ is bounded when x runs over a compact set.

Finally, one has

$$\|\tilde{\pi}(f)\|_I = \sup_{u \in G_2^0} \sum_{r(x)=u} |\tilde{\pi}(f)(x)| =$$

$$= \sup_{u \in G_2^0} \sum_{r(x)=u} \left| \sum_{\pi(x')=x} f(x') \right| \leq \sup_{u \in G_2^0} \sum_{r(x)=u,\, \pi(x')=x} |f(x')|.$$

Since $\pi(G_1^u) = G_2^{\pi^0(u)}$ and π^0 is a bijection, the last sum is

$$\sup_{u \in G_1^0} \sum_{r(x)=u} |f(x)| = \|f\|_I \geq \|\tilde{\pi}(f)\|_I.$$

Therefore, $\|f\|_I \geq \|\tilde{\pi}(f)\|_{B(L^2(G_2))}$ since the regular representation is bounded. $\square$

Proposition 2.9. *Lemma* 2.8 *holds for* $G_1 = G(X,T)$, $G_2 = \text{Germ}(X,T)$ *and* π *the morphism of Lemma* 2.3.

Proof. π is continuous and surjective. $\pi(x,z,y)$ and $\pi(x',z',y')$ are composable if and only if $y = x'$ which is the condition for (x,z,y) and (x',z',y') to be composable. $\square$

We assume from now on that X is Hausdorff, second countable and locally compact. Then $G(X,T)$ becomes a Hausdorff, locally compact étale groupoid. In the next theorem we prove the amenability of $G(X,T)$. For the proof we need some notations. The motivation comes from Example 2.2. For $1 \leq j \leq r$ let $X_j = \bigcap_{n \in \mathbb{N}} \text{dom}(T_j^n)$ and for a subset $J \subset \{1,\ldots,r\}$ (J may be void) we denote by $X_J = \bigcap_{j \in J} X_j \cap \bigcap_{j \notin J}(X \backslash X_j)$. Clearly, $X_J \cap X_{J'} = \Phi$ for $J \neq J'$ and for $x \in X$ we have $x \in X_{J_x}$ where $J_x = \{j : x \in X_j\}$ so the subsets X_J provide a partition of X. When $r = 1$ we get the partition in the end of the proof of Proposition 2.9 (i) in [22]. For $x \in X$ let

$$\sigma(x) = \sup\{n \in \mathbb{N}^r : x \in \text{dom}(T^n)\} \in \overline{\mathbb{N}}^r.$$

For the rank one case $\sigma(x)$ may be considered as the exit time of T, the first time when x escapes the domain of T. In general $\sigma(x)_j$ can be thought of as the exit time of T_j so we call $\sigma(x)$ the *exit time of* T. The crucial assumption (DC) ensures the existence of this supremum. Then $X_J = \{x \in X : \sigma(x)_j = \infty \text{ for } j \in J \text{ and } \sigma(x)_j \text{ finite for } j \notin J\}$. For $n \in \overline{\mathbb{N}}^r$ and $J = \{j : n_j = \infty\}$ we have $\sigma^{-1}(n) = \bigcap_{j \in J} X_j \cap \bigcap_{j \notin J} \text{dom}(T^{n_j}) \backslash \text{dom}(T^{n_j + e_j}) \subset X_J$ which is Borel and analytic so that a preimage by σ is analytic. Note that from the condition (DC) we have

$$\bigcap_{j\notin J} \mathrm{dom}(T^n)\backslash\mathrm{dom}(T^{n+e_j}) = \bigcap_{j\notin J} \mathrm{dom}(T_j^{n_j})\backslash\mathrm{dom}(T_j^{n_j+1}).$$

Denote by $\mathbb{Z}^J$ the subgroup of $\mathbb{Z}^r$ given by the inclusion $\mathbb{Z}^{|J|} \ni z \mapsto \sum_{j\in J} z_j e_j \in \mathbb{Z}^r$ and, for $z \in \mathbb{Z}^r$, let $z_J = \sum_{j\in J} z_j e_j \in \mathbb{Z}^J$. We use a similar notation for $\mathbb{N}^J$ and n_J.

Lemma 2.10. (i) *For $x, y \in X$ and $n \in \mathbb{N}^r$ we have $\sigma(T^n x) = \sigma(x) - n$.*
(ii) *For $(x, z, y) \in G(X,T)|_{X_J}$ we have $z_{J^c} = \sigma(x)_{J^c} - \sigma(y)_{J^c}$, where $J^c = \{1, \ldots, r\}\backslash J$.*

Proof. (i) By definition of σ we have $\sigma(T^n x) = \sup\{m \in \mathbb{N}^r : x \in \mathrm{dom}(T^{m+n})\} = \sup\{m - n \in \mathbb{N}^r : m \geq n, x \in \mathrm{dom}(T^m)\} = \sup\{m \in \mathbb{N}^r : m \geq n, x \in \mathrm{dom}(T^m)\} - n\sigma(x) - n$.
(ii) By definition of $G(X,T)$ there exist $n, m \in \mathbb{N}^r$ such that $T^n x = T^m y$ so by (i) we have $\sigma(T^n x) = \sigma(x) - n = \sigma(T^m y) = \sigma(y) - m$. We can substract the finite J^c coordinates to get $z_{J^c} = n_{J^c} - m_{J^c} = \sigma(x)_{J^c} - \sigma(y)_{J^c}$. □

The following lemma is very likely folklore

Lemma 2.11. *Let G be a Borel groupoid, $(X_j)_{j\in J}$, J a finite set, a partition of G^0 by invariant Borel sets. Then G is measurewise amenable if and only if $G|_{X_j}$ is measurewise amenable.*

Proof. This follows directly from the definition of amenability ([1], Definition 3.2.8). Precisely, we denote by $\sqcup$ the disjoint union of Borel set with the obvious Borel structure. We know that $G^0 = \sqcup X_j$, $G = \sqcup G|_{X_j}$ so $L^\infty(G) = \oplus L^\infty(G|_{X_j})$ and $L^\infty(G^0) = \oplus L^\infty(X_j)$. Therefore a mean $m : L^\infty(G) \to L^\infty(G^0)$ is invariant if and only if the restrictions $m_j : L^\infty(G|_{X_j}) \to L^\infty(X_j)$ are invariant. Conversely, we can piece m_j together to define $m = \oplus m_j$ an invariant mean $L^\infty(G) \to L^\infty(G^0)$. □

Theorem 2.12 (Conform [22], Proposition 2.9). *Let (X,T) be a MGDS with (DC). Then*
i) *$G(X,T)$ is amenable;*
ii) *the full and reduced C^*-algebras coincide;*
iii) *the C^*-algebra $C^*(X,T) = C^*(G(X,T))$ is nuclear.*

Proof. (i) We will check measurewise amenability (according to [1], 3.3.7, it is equivalent to topological amenability for étale groupoids). Each X_J is an invariant Borel set for $G(X,T)$. By Lemma 2.11 it is enough to prove that the reduction of $G(X,T)$ on X_J is amenable. By the definition of X_J, the homomorphism $c_J : G(X,T)|_{X_J} \to \mathbb{Z}^J$ defined by $c_J(x,z,y) = z_J$ is strongly surjective in the sense given in [1], Definition 5.3.7. We shall show that $R_J = c_J^{-1}(0)$ is an amenable equivalence relation. Once this proven, we can now apply a result on the amenability of an extension ([1], 5.3.14) to conclude that $G(X,T)|_{X_J}$ is amenable, hence $G(X,T)$ amenable.

We have

$$R_J = \{(x, m-n, y) : x, y \in X_J, \exists n, m \in \mathbb{N}^r, \text{ with } m_J = n_J,\\ x \in \mathrm{dom}(T^n),\ y \in \mathrm{dom}(T^m),\ T^m(x) = T^n(y)\}.$$

If $(x,z,y) \in R_J$ we have $z_J = 0$ and $z_{J^c} = \sigma(x)_{J^c} - \sigma(y)_{J^c}$ (conform Lemma 2.10(ii)) so z depends only on x, y and $R_J \subset X_J \times X_J$. We leave z out when we have such an element. We show that R_J is an equivalence relation. If $(x,y), (y,z) \in R_J$ then we can find $n, n', m, m' \in \mathbb{N}^r$ such that $n_J = n'_J$, $T^n x = T^{n'} y$, $m_J = m'_J$, $T^m y = T^{m'} z$. As we have seen before, we can assume $n_{J^c} = \sigma(x)_{J^c}$, $n'_{J^c} = \sigma(y)_{J^c}$, $m_{J^c} = \sigma(y)_{J^c}$, $m'_{J^c} = \sigma(z)_{J^c}$. (DC) gives again $T^{n_J \vee m_J + \sigma(x)_{J^c}} x = T^{n_J \vee m_J + \sigma(y)_{J^c}} y = T^{n_J \vee m_J + \sigma(z)_{J^c}} z$ and then $(x,z) \in R_J$ which proves that R_J is an equivalence relation.

We shall show that R_J is an inductive limit of amenable equivalence relations. For $N \in \mathbb{N}^J$ let

$$R_J^N = \{(x,y) \in R_J : \exists n, m \in \mathbb{N}^r \text{ such that } n_J = m_J \le N,\ T^n(x) = T^m(y)\}.$$

By Lemma 2.10(ii), we can choose n, m such that $n_{J^c} = \sigma(x)_{J^c}$ and $m_{J^c} = \sigma(y)_{J^c}$ so R_J^N can be described as

$$R_J^N = \{(x,y) \in R_J : \exists\, n \in \mathbb{N}^J,\ n \le N,\ \text{such that } T^{n+\sigma(x)_{J^c}}(x) = T^{n+\sigma(y)_{J^c}}(y)\}.$$

R_J^N is an equivalence relation on X_J since we have seen before that

$$T^{n_J \vee p_J + \sigma(x)_{J^c}} x = T^{n_J \vee p_J + \sigma(z)_{J^c}} z$$

for $(x,y,m,n), (y,z,p,q)$ defining two elements in R_J^N as above. $R_J = \bigcup_{N \in \mathbb{N}^J} \cdot$ $\cdot R_J^N$, $(R_J^N)^0 = X_J = (R_J)^0$ and $R_J^N = R_J^{N+1}$. We shall show that R_J^N is a proper equivalence relation. This ensures that R_J is the inductive limit of R_J^N in the sense of [1], Chapter 5.3.f. Then Proposition 5.3.37 of [1] gives the amenability of R_J.

Since R_J^N is a discrete equivalence relation we have to show that the space X_J/R_J^N is analytic (conform [1], Example 2.1.4(2)). According to [3], Corollary 4.12, this follows from the countable separability of X_J/R_J^N. Since σ is a Borel map, we can find a sequence Y_i of Borel subsets of X_J such that $Y_i \subset \{x \in X_J : \sigma(x)_{J^c} = k\}$ for some $k \in \mathbb{N}^{J^c}$ and the restriction of T^n to Y_i is injective. We can suppose furthermore that $(Y_i)_i$ is separating for X_J and is closed under finite intersections. The saturation of a Borel set $A \subset X_J$ is $[A] = \bigcup_{n \in \mathbb{N}^J, n \le N} (T^{n+\sigma(\cdot)_{J^c}})^{-1} T^{n+\sigma(\cdot)_{J^c}}(A)$ which is a Borel set. We use here the notation $T^{n+\sigma_{J^c}(\cdot)}$ for the Borel map $x \mapsto T^{n+\sigma_{J^c}(x)}(x)$. Let $(x,y) \notin R_J^N$, $x \in Y_i$. We prove that we can choose Y_j such that $y \notin [Y_j]$ so $\pi(Y_j)$ separates $\pi(x)$ and $\pi(y)$ where π is the projection from X_J to X_J/R_J^N. If $y \notin Y_i$ we are done. If $y \in [Y_i]$, there exists $y' \in Y_i$ such that $(y,y') \in R_J^N$. Since the family $(Y_i)_i$ separates X_J and is closed under finite intersections, we can find another set $Y_j \subset Y_i$ which separates x and y', that is $x \in Y_j$ and $y' \notin Y_j$. If $y \notin [Y_j]$ we are back to the case $y \notin [Y_i]$. If $y \in [Y_j]$ then $(y, y'') \in R_J^N$ for some $y'' \in Y_j \subset Y_i$ so $(y', y'') \in R_J^N$. This means that there exists $n \in \mathbb{N}^J$, $n \le N$ such that $T^{n+\sigma(y')_{J^c}}(y') = T^{n+\sigma(y'')_{J^c}}(y'')$. Since $y', y'' \in Y_i$ we have $\sigma(y')_{J^c} = \sigma(y'')_{J^c}$. $T^{n+\sigma(\cdot)_{J^c}}$ is injective on Y_i so $y' = y''$. This contradicts the choice of Y_j ($y' \notin Y_j$). Therefore, $y \notin [Y_j]$. □

Corollary 2.13. *There is a morphism*

$$\tilde{\pi} : C_r(G(X,T)) \to C_r(\mathrm{Germ}(X,T))$$

induced by the canonical morphism $\pi : G(X,T) \to \mathrm{Germ}(X,T)$.

Proof. According to Proposition 2.9 there is a morphism

$$C_r^*(G(X,T)) = C^*(G(X,T)) \to C_r^*(\mathrm{Germ}(X,T)). \quad \square$$

3. GRUPOIDS OF TWO-SIDED TOEPLITZ ALGEBRAS OF A DIRECTED GRAPH

Let us start with the classical Toeplitz extension. The two-sided Toeplitz extension is given by pulling back the Toeplitz extension of $C(\mathbb{T})$ with the diagonal morphism. If S is a proper isometry this Toeplitz algebra is $\mathcal{T} = C^*((S,z_1),(S,z_2)) \subset C^*(S) \oplus C(\mathbb{T}) \otimes C(\mathbb{T})$. If we think of $C(\mathbb{T})$ as $C^*(\mathbb{Z})$ the diagonal morphism is given by the morphism $\mathbb{Z}\times\mathbb{Z} \ni (x,y) \mapsto x+y \in \mathbb{Z}$. $C^*(S)$ is a Cuntz-like algebra of the groupoid given by $(\overline{\mathbb{N}}, T)$ where $T : \overline{\mathbb{N}} \setminus \{0\} \to \overline{\mathbb{N}}, T(n) = n-1$. We replace now the points (∞, z, ∞) by $(\infty, (z_1,z_2), \infty)$ where $z_1, z_2 \in \mathbb{Z}$ to get another groupoid, G. Two pairs (x,z,y) and (x',z',y') are composable if $y = x'$ and then $(x,z,y)(y,z',y') = (x, z+z', y')$ where $z+z'$ is the addition in $\mathbb{Z}$ or $\mathbb{Z}^2$ accordingly. As an algebraic object, G is the reduction of $G(\overline{\mathbb{N}},T) \times G(\overline{\mathbb{N}},T)$ to the diagonal of $\overline{\mathbb{N}} \times \overline{\mathbb{N}}$. Its reduction to $\mathbb{N}$ (an invariant subset) is $\mathbb{N}\times\mathbb{N}$ and to $\{\infty\}$ is $\mathbb{Z}\times\mathbb{Z}$. The basis of the topology is given by $\{(n, n-m, m)\}, n,m \in \mathbb{N}$ and

$$B_{z_1,z_2} = \{(n+z_1+z_2, z_1+z_2, n) : n \in \mathbb{N}\} \cup \{(\infty, (z_1,z_2), \infty)\}.$$

This is not a Hausdorff topology since if $z_1 + z_2 = z_1' + z_2'$ then the sequence $(n, z_1+z_2, n+z_1+z_2)$ converges both to $(\infty, (z_1,z_2), \infty)$ and $(\infty, (z_1', z_2'), \infty)$. However, the sets in the above base are Hausdorff and compact, $G^u, u \in \overline{\mathbb{N}}$

is a discrete set in the relative topology from G and $\overline{\mathbb{N}}$, the unit space, is Hausdorff. G is therefore a non-Hausdorff r-discrete groupoid. The algebra $C_c(G)$ is generated by $1_{\{(m,n)\}}$ with $m,n\in\mathbb{N}$ and $1_{B(z_1,z_2)}$. It is easy to check that the map which sends $1_{B(1,0)}$ to(S,z_1) and $1_{B(0,1)}$ to (S,z_2) can be extended to an isomorphism from $C^*(G)$ to $\mathcal{T}$.

We define next a groupoid which gives rise to the algebra $C^*(L_\lambda\oplus s_\lambda, R_\lambda\oplus t_\lambda)\subset\mathcal{E}\oplus C^*(\Lambda)\otimes C^*(\Lambda^{op})$ where Λ is a rank one graph. Let $X=\overline{\Lambda}\cup(\Lambda^\infty\times(\Lambda^{op})^\infty)$ be the space of finite words and pairs (x,y) of infinite words given by the graphs Λ respectively Λ^{op}. The sets $\{\lambda\},\lambda\in\overline{\Lambda}$ and $\{\lambda x\mu : x\in\overline{\Lambda}\}\cup\{(\lambda x,\mu y)(x,y)\in\Lambda^\infty\times(\Lambda^{op})^\infty\}$ provide a basis for a Hausdorff locally compact topology on X. X is a compactification of $\overline{\Lambda}$ provided that Λ is connected (there is at least one finite paths between any two objects). We assume this hereafter. A sequence in $\lambda_n\in\Lambda$ converges in X if and only if it is eventually constant or converges in Λ^∞ and $(\Lambda^{op})^\infty\}$. Equivalently, it means that (λ_n,λ_n) converges in $X_\Lambda\times X_{\Lambda^{op}}$. This shows again that the unit space is a kind of diagonal. Note that for $x\in\Lambda$, $\lambda x\mu$ is $\mu(\lambda x)$ if we think of λx as an element in Λ^{op}. We define a groupoid G

$$\{(\lambda,\sigma(\lambda)-\sigma(\mu),\mu):\lambda,\mu\in\overline{\Lambda}\}\cup\{((\lambda x,\mu y),(\sigma(\lambda)-\sigma(\lambda'),$$
$$\sigma(\mu)-\sigma(\mu')),(\lambda' x,\mu' y)) : (\lambda,\mu),(\lambda',\mu')\in\overline{\Lambda}\times\overline{\Lambda^{op}},\ (x,y)\in\Lambda^\infty\times(\Lambda^{op})^\infty\},$$

where two pairs (x,z,y) and (x',z',y') are composable if $y=x'$ and then $(x,z,y)(y,z',y')=(x,z+z',y')$ where $z+z'$ is the addition in $\mathbb{Z}$ or $\mathbb{Z}^2$ accordingly. We see that the piece of information $\sigma(\lambda)-\sigma(\mu)$ in $(\lambda,\sigma(\lambda)-\sigma(\mu),\mu)$ is redundant, that is G reduced to $\overline{\Lambda}$ is the principal groupoid $\overline{\Lambda}\times\overline{\Lambda}$. We freely avoid it whenever convenient. As an algebraic object G can be thought of as the reduction of $G(X_\Lambda,T_\Lambda)\times G(X_{\Lambda^{op}},T_{\Lambda^{op}})$ to X. But G does not have the induced topology which is rather odd. A basis for the topology is given by the sets $\{(\lambda,\mu)\}$, $\lambda,\mu\in\overline{\Lambda}$ and

$$B(\lambda,\mu,\lambda',\mu')=\{(\lambda x\mu,\lambda' x\mu'):x\in\overline{\Lambda}\}\cup\{((\lambda x,\mu y),(\sigma(\lambda)-\sigma(\lambda'),$$
$$\sigma(\mu)-\sigma(\mu')),(\lambda' x,\mu' y));(\lambda,\mu),(\lambda',\mu')\in\overline{\Lambda}\times\overline{\Lambda^{op}},\ (x,y)\in\Lambda^\infty\times(\Lambda^{op})^\infty\}.$$

This groupoid may not be Hausdorff, the non-Hausdorffness appearing from periodic infinite paths. To see it, we construct a sequence which converges to two distinct points. Start with $\lambda\in\Lambda^*$, $o(\lambda)=t(\lambda)$. Then the sequence $(\lambda^n,\lambda^n)_n$ converges to $((\lambda^\infty,\lambda^\infty),(\sigma(\lambda^k),-\sigma(\lambda^k),(\lambda^\infty,\lambda^\infty))$ for any $k\in\mathbb{N}$. Nevertheless, the topology induced on the sets $B(\lambda,\mu,\lambda',\mu')$ is Hausdorff. Indeed, it is clear that the points (λ,μ) are separated being open sets. Let $(x,z,y)\neq(x',z,y')\in B(\lambda,\mu,\lambda',\mu')$ such that $x\in\Lambda^\infty\times(\Lambda^{op})^\infty$. Here $z=(\sigma(\lambda)-\sigma(\lambda'),\sigma(\mu)-\sigma(\mu'))$. Let us suppose that $x=(x_1,x_2)\neq x'=(x_1',x_2')$, the case $y\neq y'$ being similar. Assume $x_1\neq x_1'$ the case $x_2\neq x_2'$ being similar. Then there exist $\gamma\neq\gamma'\in\overline{\Lambda}$, $a,b\in\Lambda^\infty$ such that $\gamma\geq\lambda$, $\gamma'\geq\lambda'$, $\sigma(\gamma)=\sigma(\gamma')$

and $x_1 = \lambda a$, $x_1' = \gamma' b$. The sets $B(\gamma, \delta, \mu, \mu')$ and $B(\gamma', \delta', \mu, \mu')$ are disjoint for any $\delta, \delta' \in \overline{\Lambda}$, $\delta, \delta' \geq \lambda'$, $\sigma(\delta) = \sigma(\delta') = \sigma(\lambda') + \sigma(\gamma) - \sigma(\lambda)$. Therefore, G is a non-Hausdorff locally compact groupoid. The correspondence $1_{B(\lambda,\Omega,\Omega,\Omega)} \to L_\lambda \oplus s_\lambda$ extends to an isomorphism from $C^*(G)$ to $C^*(L_\lambda \oplus s_\lambda, R_\lambda \oplus t_\lambda)$.

4. DUALITY IN BIVARIANT THEORIES

K-theory and K-homology are particular cases of the Kasparov groups. The product in KK-theory provides the analogue of the cup and cap products in algebraic topology. Therefore KK-theory and other bivariant theories are the framework for notions of duality. KK-theory is suitable when geometric classes are available Ext-theory was used in [15, 9, 10]. The novelty in [20] is to use Ext^r but a technical condition of semisplitness leads to E-theory. We give first a general notion of duality analogous with the Spanier-Whitehead duality in algebraic topology. Then we give a condition which implies this duality. It appeared for the first time in [16], Chapter 4, Theorem 6, but was highlighted by Connes in [7], Chapter VI.4.β. After that we recall the construction of long exact sequences starting from an algebra and a tuple of ideals.

Let F be any of the bivariant theories KK-theory, E-theory, Ext-theory or KExt-theory. For $x \in F(A_1 \otimes \ldots \otimes A_n, B_1 \otimes \ldots \otimes B_m)$ we define the flip maps $\sigma_{ij}(x)$ and $\sigma^{ij}(x)$ the induced maps on F-groups by the flip homomorphisms of C^*-algebras $A_1 \otimes \ldots \otimes A_i \otimes \ldots \otimes A_j \otimes A_n \to A_1 \otimes \ldots \otimes A_j \otimes \ldots \otimes A_i \otimes \ldots \otimes A_n$ respectively $B_1 \otimes \ldots \otimes B_i \otimes \ldots \otimes B_j \otimes \ldots \otimes B_n \to B_1 \otimes \ldots \otimes B_j \otimes \ldots \otimes B_i \otimes \ldots \otimes B_n$.

Definition 4.1 (Spanier-Whitehead duality, [15], Definition 2.2). Let A, B be separable C^*-algebras, $\Delta \in F^r(A \otimes B, \mathbb{C})$, $\delta \in F^r(\mathbb{C}, A \otimes B)$. We say that A and B are Spanier-Whitehead r-dual in F-theory if the maps $\Delta_i : K_i(A) \to K^{i+r}(B)$, $\Delta_i(x) = x \otimes_A \Delta$ and $\sigma^{12}(\Delta)_i : K_i(B) \to K^{i+r}(A)$, $\sigma^{12}(\Delta)_i(x) = x \otimes_B \Delta$ are isomorphisms with inverses $\delta_i : K^{i+r}(B) \to K_i(A)$, $\delta_i(y) = \delta \otimes_B y$ respectively $\sigma_{12}(\delta)_i : K^{i+r}(A) \to K_i(B)$, $\sigma_{12}(\delta)_i(y) = \sigma_{12}(\delta) \otimes_A y$.

The duality classes, Δ and δ, are the K-homology and the K-theory classes.

THEOREM 4.2 ([10], Theorem 11). *Let A, B, Δ, δ be C^*-algebras and classes as above such that*

$$\delta \otimes_B \sigma^{12}(\Delta) = [1_A], \quad \sigma_{12}(\delta) \otimes_A \Delta = [(-1)^i 1_B].$$

Then the maps Δ_i and δ_i defined above are isomorphisms.

The proof can be found in [10]. We shall call such algebras simply r-dual. The sign in the second condition comes from the change of sign under flip maps σ. The sign is not given in [16] and [7]. We do not know of any example of two dual C^*-algebras such that the assumptions of Theorem 4.2 are not fulfilled.

If B is the same as A we say that A is a Poincaré duality algebra. The dual is not unique since if A has a dual B and A is KK-equivalent to A' then A' is also dual with B.

Bivariant theories work well when the algebra in the first argument is separable. For KK-theory this is forced by the Kasparov technical theorem. Since in the definition of duality both algebras appear in the first argument we are forced to work with separable algebras. For a separable algebra the K-theory group is countable. For this reason not every algebra has a dual. For example M_{2^∞}, the CAR algebra, does not have a dual. If there was a dual B for M_{2^∞} we would have $K_i(B) = K^1(M_{2^\infty})$ where i is 0 or 1. But $K^1(M_{2^\infty})$ is $\mathbb{Z}_{(2)}/\mathbb{Z}$ ([14]), where $\mathbb{Z}_{(2)}$ is the group of 2-adic numbers, which is uncountable. Therefore, B cannot be separable.

In [20] the K-homology class was given as an E-theory class associated to a long exact sequences. We recall the construction of long exact sequences starting from an algebra and a tuple of ideals.

Let $\mathcal{E}$ be an arbitrary C^*-algebra and $J_1, \ldots, J_r$ be r ideals in it. We shall describe next a procedure of defining an r-fold exact sequence starting with $B = \bigcap_{i=1}^r J_i$ and ending with $A = \mathcal{E} \big/ J_1 + \ldots + J_r$. The motivation comes from the Zekri's Yonneda product ([24]) $\epsilon_1 \otimes_{\mathbb{C}} \epsilon_2$ of two 1-fold extensions $\epsilon_1 \in \mathrm{Ext}(A_1, B_1)$ and $\epsilon_2 \in \mathrm{Ext}(A_2, B_2)$. Let

$$\begin{aligned} 0 \to B_1 \to E_1 \to A_1 \to 0 \\ 0 \to B_2 \to E_2 \to A_2 \to 0 \end{aligned}$$

be the extensions ϵ_1 and ϵ_2. Forgetting for the moment the troubles caused by tensor products, the product $\epsilon_1 \otimes_{\mathbb{C}} \epsilon_2$ is computed by tensoring ϵ_1 with B_2 on the right, ϵ_2 with A_1 on the left and then splicing:

$$0 \to B_1 \otimes B_2 \to E_1 \otimes B_2 \to A_1 \otimes E_2 \to A_1 \otimes A_2 \to 0.$$

Define $\mathcal{E} = E_1 \otimes E_2$, $J_1 = B_1 \otimes E_2$, $J_2 = E_1 \otimes B_2$. We have

$$\begin{aligned} B_1 \otimes B_2 = J_1 \cap J_2 = J_1 J_2, \quad E_1 \otimes B_2 = J_2, \\ A_1 \otimes E_2 = \mathcal{E}/J_1, \quad A_1 \otimes A_2 = \mathcal{E}/J_1 + J_2 \end{aligned}$$

so the exact sequence is

$$0 \to J_1 \cap J_2 \to J_2 \to \mathcal{E}/J_1 \to \mathcal{E}/J_1 + J_2 \to 0.$$

For a nonempty subset $S \subseteq \{1, \ldots, r\}$ let us define

$$J^S = \bigcap_{j \in S} J_j \quad \text{and} \quad J_S = \sum_{j \in S} J_j.$$

We shall use the abbreviation $\{k, k+1, \ldots, l\} = \overline{k,l}$. Define

$$\mathcal{E}_0 = J_1 \cap \ldots \cap J_r = J_{\overline{1,r}},\ \mathcal{E}_1 = J_2 \cap \ldots \cap J_r = J_{\overline{2,r}},$$

$$\mathcal{E}_k = {J_{k+1} \cap \ldots \cap J_r} \Big/ {(J_1 + \ldots + J_{k-1}) \cap J_{k+1} \cap \ldots \cap J_r} =$$

$$= J^{\overline{k+1,r}} \Big/ J_{\overline{1,k-1}} \cap J^{\overline{k+1,r}},$$

for any $k \in \{2, \ldots r-1\}$ and

$$\mathcal{E}_r = \mathcal{E} \Big/ J_1 + \ldots + J_{r-1} = \mathcal{E} \Big/ J^{\overline{1,r-1}},$$

$$\mathcal{E}_{r+1} = \mathcal{E} \Big/ J_1 + \ldots + J_r = \mathcal{E} \Big/ J^{\overline{1,r}}.$$

$\mathcal{E}_0 \subseteq \mathcal{E}_1$ so we can define $i_0 : \mathcal{E}_0 \hookrightarrow \mathcal{E}_1$. Since $\mathcal{E}_1 \subseteq J_3 \cap \ldots \cap J_r$, there is a map $i_1 : \mathcal{E}_1 \to \mathcal{E}_2$ given by the inclusion composed with the quotient map. For $k \in \{2, \ldots, r-2\}$ we have

$$J^{\overline{k+1,r}} = J_{k+1} \cap \ldots \cap J_r \subseteq J_{k+2} \cap \ldots \cap J_r = J^{\overline{k+2,r}}$$

and

$$J_{\overline{1,k-1}} \cap J^{\overline{k+1,r}} \subseteq J_{\overline{1,k}} \cap J^{\overline{k+2,r}}$$

so that we can again define a map $i_k : \mathcal{E}_k \to \mathcal{E}_{k+1}$ as the bottom line of the diagram

$$\begin{array}{ccc} J^{\overline{k+1,r}} & \hookrightarrow & J^{\overline{k+2,r}} \\ \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi_{k+1}} \\ \mathcal{E}_k & \xrightarrow{i_k} & \mathcal{E}_{k+1} \end{array}$$

where the vertical arrows are projections. Using the isomorphism $J \Big/ J \cap I \simeq J + I \Big/ I$ we write

$$\mathcal{E}_{r-1} J_r \Big/ J_{\overline{1,r-2}} \cap J_r \quad J_1 + \cdots + J_{r-2} + J_r \Big/ J_1 + \cdots + J_{r-2}.$$

Therefore, there is also a natural homomorphism $i_{r-1} : \mathcal{E}_{r-1} \to \mathcal{E}_r$ since $J_1 + \cdots + J_{r-2} + J_r \subseteq \mathcal{E}$ and $J_1 + \cdots + J_{r-2} \subseteq J_1 + \cdots + J_{r-1}$. Finally, $i_r : \mathcal{E}_r \to \mathcal{E}_{r+1}$ is defined since $J_1 + \cdots + J_{r-1} \subseteq J_1 + \cdots + J_r$.

Theorem 4.3 ([20], Proposition 3.1). *The r-fold sequence*

$$0 \longrightarrow B \xrightarrow{i_0} \mathcal{E}_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{r-1}} \mathcal{E}_r \xrightarrow{i_r} A \longrightarrow 0$$

is exact.

Such sequences define Ext^r classes provided they are semisplit. In [20] this technical condition was avoided using E-theory, that is thinking of such a sequence as an E^r-theory class. However, as a consequence of the fact that $\mathrm{Ext}^1(A,B) = \overline{\mathrm{Ext}}^1(A,B)$ is a group and the Yonneda product is bilinear, we can prove that $\mathrm{Ext}^r(A,B)$ is a group. Indeed, we can decompose any $\epsilon \in \overline{\mathrm{Ext}^r}(A,B)$ as a Yonneda product $\gamma(\epsilon_1, \epsilon_2)$ with $\epsilon_1 \in \overline{\mathrm{Ext}}^1(A, A_1)$. Since $\mathrm{Ext}^1(A, A_1)$ is a group, we can find an extension, $\epsilon_1' \in \mathrm{Ext}^1(A, A_1)$ such that $\epsilon_1 \oplus \epsilon_1'$ is a null (i.e., split) extension in $\mathrm{ext}^1(A, A_1)$. The inverse of ϵ is now $\gamma(\epsilon_1', \epsilon_2)$. As a consequence, if A is nuclear then ϵ_1 is always semisplit so $\gamma(\epsilon_1', \epsilon_2)$ provides an inverse for ϵ in $\overline{\mathrm{Ext}}^r(A,B)$. Therefore, $\overline{\mathrm{Ext}}^r(A,B)$ is a group which contains $\mathrm{Ext}^r(A,B)$. We conjecture that they are equal if A is nuclear. More generally, we can define a group if in the semisplitness condition we only require that ϵ_r, the rightmost 1-fold exact sequence, is semisplit. We believe that this is equal to $\mathrm{Ext}^r(A,B)$. In any case it factors through KK as a consequence of abstract characterizations ([8]).

For example if G is a groupoid and $X_1, \ldots, X_r$ are r open invariant subsets of G^0, we get a tuple of ideals of $C_r^*(G)$ $(C_r^*(G|_{X_1}), C_r^*(G|_{X_2}), \ldots, C_r^*(G|_{X_r}))$. The algebras $\mathcal{E}_k$ corresponds to $C_r^*(G|_{Y_k})$, where

$$Y_0 = X_1 \cap \ldots \cap X_r, \quad Y_1 = X_2 \cap \ldots \cap X_r,$$

$$Y_k = (X_{k+1} \cap \ldots \cap X_r) \setminus (X_1 \cup \ldots \cup X_{k-1}),$$

$$Y_r = X \setminus (X_1 \cup \ldots \cup X_{r-1}), \quad Y_{r+1} = X \setminus (X_1 \cup \ldots \cup X_r).$$

In particular, if (X,T) is a MGDS with (DC) condition and X_j are the set used in the proof of the amenability of $G(X,T)$ we have

$$Y_k = \{x \in X : \sigma(x)_j < \infty \text{ for } j \geq k+1 \text{ and } \sigma(x)_j = \infty \text{ for } j \leq k-1\}.$$

5. DUALITY FOR ROTATION ALGEBRAS

We begin with the simple example of the circle. We give here two approaches, using extensions and operators. Ideas in the computation below appear again and again for more complicated algebras so it is worth having in mind this simple example. The K-homology class $\Delta \in KK^1(C(\mathbb{T}) \otimes C(\mathbb{T}), \mathbb{C})$ is the pull-back of the Toeplitz extension by the diagonal morphism, that is the top row of the diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K} & \longrightarrow & \mathcal{E} & \longrightarrow & C(\mathbb{T}) \otimes C(\mathbb{T}) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{K} & \longrightarrow & \mathcal{T} & \longrightarrow & C(\mathbb{T}) & \longrightarrow & 0
\end{array}$$

where $\mathcal{E} = C^*((S, z_1), (S, z_2)) \subset \mathcal{T} \oplus C(\mathbb{T}) \otimes C(\mathbb{T})$.

The K-theory class $\delta \in KK^1(\mathbb{C}, C(\mathbb{T})\otimes C(\mathbb{T}))$ is given by the unitary $\overline{z}\otimes z$ but we would like to view it as a morphism $\delta : \mathcal{S} \to C(\mathbb{T}) \otimes C(\mathbb{T})$ given by the restriction to $\mathcal{S}$ of the morphism $C(\mathbb{T}) \ni z \mapsto \overline{z}\otimes z \in C(\mathbb{T})\otimes C(\mathbb{T})$. We have to compute the product $\delta\otimes_{C(\mathbb{T})}\sigma^{12}\Delta$ in $KK^1(\mathbb{C}, C(\mathbb{T})\otimes C(\mathbb{T}))\otimes_{C(\mathbb{T})} KK^1(C(\mathbb{T})\otimes C(\mathbb{T}), \mathbb{C})$, that is in $KK(\mathcal{S} \otimes \underline{C(\mathbb{T})}, C(\mathbb{T}) \otimes C(\mathbb{T}) \otimes \underline{C(\mathbb{T})}) \otimes KK^1(\underline{C(\mathbb{T})} \otimes C(\mathbb{T}) \otimes C(\mathbb{T}), \underline{C(\mathbb{T})})$. We use the underline in order to highlight the fact that $\underline{C(\mathbb{T})} \to \underline{C(\mathbb{T})}$ is the identity homomorphism. Since δ is a morphism, this product is given by a pull-back, the top line of the following diagram pulled-back further by the inclusion $\mathcal{S} \otimes \underline{C(\mathbb{T})} \subset C(\mathbb{T}) \otimes \underline{C(\mathbb{T})}$:

$$\begin{CD}
\Delta' : 0 @>>> \underline{C(\mathbb{T})} \otimes \mathbb{K} @>>> \mathcal{E}' @>>> C(\mathbb{T}) \otimes \underline{C(\mathbb{T})} @>>> 0 \\
@. @| @VVV @VV{\delta\otimes 1_{C(\mathbb{T})}}V @. \\
0 @>>> \underline{C(\mathbb{T})} \otimes \mathbb{K} @>>> \underline{C(\mathbb{T})} \otimes \mathcal{E} @>>> \underline{C(\mathbb{T})} \otimes C(\mathbb{T}) \otimes C(\mathbb{T}) @>>> 0
\end{CD}$$

Here $\mathcal{E}' = C^*(\overline{z}\otimes(S, z_1), 1\otimes(S, z_2))$ with $\overline{z}\otimes(S, z_1) \mapsto z\otimes 1$ and $1\otimes(S, z_2) \mapsto 1\otimes z$. Informally, this shows that $\underline{C(\mathbb{T})}$ of $\underline{C(\mathbb{T})}\otimes\mathbb{K}$ acts on $C(\mathbb{T})$ of $C(\mathbb{T})\otimes\underline{C(\mathbb{T})}$ in the top row of the diagram above. We want to have it act on $\underline{C(\mathbb{T})}$. We do this formally by applying a pull-back with a rotation morphism on the right. This is called an untwist in [15]. Namely, let $\Theta : C(\mathbb{T})\otimes C(\mathbb{T}) \to C(\mathbb{T})\otimes C(\mathbb{T})$ defined by $z \otimes 1 \mapsto z \otimes 1$ and $1 \otimes z \mapsto \overline{z} \otimes z$. If we view $C(\mathbb{T})\otimes C(\mathbb{T})$ as $C(\mathbb{T} \to C(\mathbb{T}))$, Θ can be expressed in terms of the gauge action γ as $\Theta(f)(z) = \gamma_z(f(z))$. This is important since Θ restricted to $\mathcal{S} \otimes C(\mathbb{T})$ is given by $\Theta(f)(t) = \gamma_{e^{2\pi i t}}(f(t))$ so it is homotopic to the identity by $\Theta_s(f)(t) = \gamma_{e^{2\pi i s t}}(f(t))$.

Now we pull back Δ' with Θ but it is enough to notice a subsequence:

$$\begin{CD}
0 @>>> \mathbb{K} \otimes C(\mathbb{T}) @>>> \mathcal{T} \otimes C(\mathbb{T}) @>>> C(\mathbb{T}) \otimes C(\mathbb{T}) @>>> 0 \\
@. @VVV @VVV @VV{\Theta}V @. \\
\Delta' : 0 @>>> \underline{C(\mathbb{T})} \otimes \mathbb{K} @>>> \mathcal{E}' @>>> C(\mathbb{T}) \otimes \underline{C(\mathbb{T})} @>>> 0
\end{CD}$$

Indeed, by the pull-back construction $\Theta(z \otimes 1) = z \otimes 1 \leftarrow \overline{z} \otimes (S, z_1) = W$ and $\Theta(1 \otimes z) = \overline{z} \otimes z \leftarrow z \otimes (1, \overline{z}_1 z_2) = W^*(1 \otimes (S, z_2))$. Then $C^*(W, W^*(1 \otimes (S, z_2))) \simeq \mathcal{T} \otimes C(\mathbb{T})$. The inclusion $\mathbb{K} \otimes C(\mathbb{T}) \subset \underline{C(\mathbb{T})} \otimes \mathbb{K}$ is given by a full corner since $C(\mathbb{T}) \otimes P_{\xi_0}$ is a full corner in both. Therefore, $\delta \otimes_{C(\mathbb{T})} \sigma^{12}(\Delta) = \tau_{C(\mathbb{T})}(\mathcal{T})$ so by Bott periodicity this is the same with $[1_{C(\mathbb{T})}]$.

It is shown in [7], Chapter VI.4.β that A_θ, the irrational rotation algebra, is a Poincaré duality algebra. This was one of the first non-commutative examples. In fact one can notice that A_θ is in the bootstrap category so its KK-theory is given by the K-theory. The 2-torus $\mathbb{T}^2$ has the same K-theory. By a consequence of the universal coefficient theorem ([4], Corollary 23.10.2) they are KK-equivalent and the dual of A_θ can be taken A_θ. But the interest is to give an explicit description of Δ and δ.

The dual of A_θ constructed in [7] is A_θ^{op}, the opposite algebra, which is isomorphic to $A_{-\theta}$ and A_θ. Indeed, let $\lambda = \mathrm{e}^{2\pi\mathrm{i}\theta}$ and u, v, $\overline{u}$, $\overline{v}$ be the canonical generators of A_θ respectively $A_{-\theta}$, that is $uv = \lambda vu$ and $\overline{uv} = \overline{\lambda vu}$. An isomorphism between A_θ and $A_{-\theta}$ is given by $u \mapsto \overline{v}$, $v \mapsto \overline{u}$. The arguments given below work for θ arbitrary if we consider A_θ as the universal C^*-algebra generated by the commutation relation $uv = \lambda vu$. The K-homology class is similar to the K-homology class of the torus $\mathbb{T}^2$. It is most easily defined by an unbounded Fredholm operator. It is known that an orthonormal basis in $L^2(\tau)$, τ the canonical trace on A_θ, is given by $u^n v^m$ which we shall also denote by $\xi_{n,m}$. The algebra A_θ acts on $L^2(\tau)$ by left multiplication $u\xi_{m,n} = \xi_{m+1,n}$, $v\xi_{m,n} = \lambda^{-m}\xi_{m,n+1}$ and the algebra $A_{-\theta} = A_\theta^{op}$ acts by right multiplication $\overline{u}\xi_{m,n} = \lambda^{-n}\xi_{m+1,n}$, $\overline{v}\xi_{m,n} = \xi_{m,n+1}$. The action of A_θ commutes with the action of $A_{-\theta}$ so we have an action of $A_\theta \otimes A_{-\theta}$ on H. On $L^2(\tau)$ there exist two canonical unbounded operators $D_1\xi_{n,m} = n\xi_{n,m}$ and $D_2\xi_{n,m} = m\xi_{n,m}$. These come from the canonical derivations of the gauge action, $\delta_1 x = \frac{\partial}{\partial z_1}\gamma_{z_1}(x)$ and $\delta_2 x = \frac{\partial}{\partial z_2}\gamma_{z_2}(x)$. Let $H = L^2(\tau) \oplus L^2(\tau)$ with the natural grading and

$$D = \begin{pmatrix} 0 & D_1 + \mathrm{i}D_2 \\ D_1 - \mathrm{i}D_2 & 0 \end{pmatrix} \subset \mathbb{L}(H).$$

Then D is an essentially selfadjoint unbounded operator with $D^2 = D_1^2 + D_2^2$. Therefore, $(1 + D^2)^{-1}\xi_{m,n} = \frac{1}{m^2+n^2}$ so $(1 + D^2)^{-1}$ is compact. We could pass to a bounded operator by a standard procedure: $F = \frac{D}{\sqrt{D^2+1}}$ but we prefer to use the unbounded Kasparov triples ([4], Definition 17.11.1). What is crucial for the computation below is that D is the Dirac operator of $\mathbb{T}^2$ if we identify $L^2(\tau)$ with $l^2(\mathbb{Z}^2)$. It is easy to check that $[D, u]$, $[D, v]$, $[D, \overline{u}]$, $[D, \overline{v}]$ are bounded operators since this is true if we replace D by D_1 or D_2. Therefore, the pair (H, D) gives an element Δ in $KK(A_\theta \otimes A_{-\theta}, \mathbb{C})$. The K-theory class is defined by a homomorphism $\delta : \mathcal{S}^{\otimes 2} \to A_\theta \otimes A_{-\theta}$. First define $\delta : C(\mathbb{T}^2) \to A_\theta \otimes A_{-\theta}$ by $\delta(z_1) = u \otimes \overline{u}$, $\delta(z_2) = v \otimes \overline{v}$ and then restrict to $\mathcal{S}^{\otimes 2} = C_0((0, 1) \times (0, 1)) \subset C(\mathbb{T}^2)$. The definition makes sense since $u \otimes \overline{u}$ commutes with $v \otimes \overline{v}$.

To compute the product we use an untwist morphism, an idea from [15] which is a rotation argument. Let $\Theta : C(\mathbb{T}^2) \otimes A_\theta \to C(\mathbb{T}^2) \otimes A_\theta$ defined on generators by $z_j \otimes 1 \mapsto z_j \otimes 1$, $1 \otimes u \mapsto \overline{z_1} \otimes u$, $1 \otimes v \mapsto \overline{z_2} \otimes v$. We denote also by Θ the restriction to $\mathcal{S}^{\otimes 2}$. If we use the identification $C(\mathbb{T}^2) \otimes A_\theta \simeq C(\mathbb{T}^2 \to A_\theta)$ then Θ is given by $\Theta(f)(z) = \gamma_z(f(z))$. The restriction to $\mathcal{S}^{\otimes 2} = C_0((0, 1) \times (0, 1))$ is $\Theta(f)(t_1, t_2) = \gamma_{(\mathrm{e}^{2\pi\mathrm{i}t_1}, \mathrm{e}^{2\pi\mathrm{i}t_2})}(f(t_1, t_2))$. There is an obvious homotopy between Θ and the identity, $\Theta_s(f)(t_1, t_2) = \gamma_{(\mathrm{e}^{2\pi\mathrm{i}st_1}, \mathrm{e}^{2\pi\mathrm{i}st_2})}(f(t_1, t_2))$, $s \in [0, 1]$.

Therefore, the product $[\Theta]\otimes\tau_{A_\theta}(\delta)\otimes\tau^{A_\theta}(\sigma^{12}(\Delta))$ is equal to $\delta\otimes\sigma^{12}(\Delta)$. This is given by the Kasparov module $(A_\theta\otimes H,\psi,1_{A_\theta}\otimes D)$. The homomorphism $\psi:\mathcal{S}^{\otimes 2}\otimes A_\theta\to\mathbb{L}(A_\theta\otimes H)$ is given by the restriction to $\mathcal{S}^{\otimes 2}$ of the homomorphism $C(\mathbb{T}^2)\otimes A_\theta\to\mathbb{L}(A_\theta\otimes H)$, which we shall denote again by ψ,

$$z_1\mapsto u^*\otimes\overline{u},\quad z_2\mapsto v^*\otimes\overline{v},$$

$$u\mapsto u\otimes\overline{u}^*\otimes u,\quad v\mapsto v\otimes\overline{v}^*\otimes v,$$

where the first component acts as an operator on A_θ while the second and the third component act on H. We shall define a unitary operator in $U\in\mathbb{L}(A_\theta)\otimes H$ such that $(A_\theta\otimes H,U\psi U^*,1\otimes D)$ is the class $\tau^{A_\theta}(\mathcal{T}^{\otimes 2})\in KK(A_\theta\otimes C(\mathbb{T})^2,A_\theta)$. By Bott periodicity this is the same as 1_{A_θ}. Let $U(a\otimes\xi_{m,n})=\lambda^{mn}v^n\ u^m a\otimes\xi_{m,n}$ when restricted to $A_\theta\otimes\mathcal{S}^{\otimes 2}$. Then $U^*(a\otimes\xi_{m,n})=\lambda^{-mn}u^{-m}v^{-n}a\otimes\xi_{m,n}$. We have $U\psi(z_1\otimes 1)U^*(a\otimes\xi_{m,n})=a\otimes\xi_{m+1,n}$, $U\psi(z_2\otimes 1)U^*(a\otimes\xi_{m,n})=a\otimes\xi_{m,n+1}$, $U\psi(1\otimes u)U^*(a\otimes\xi_{m,n})=ua\otimes\xi_{m,n}$ and $U\psi(1\otimes v)U^*(a\otimes\xi_{m,n})=va\otimes\xi_{m,n}$. Then $U\psi(1\otimes u)U^*=u\otimes 1$, $U\psi(1\otimes v)U^*=v\otimes 1$ and $U(1\otimes D)U^*=1\otimes D$. Therefore, the cycle $(A_\theta\otimes H,\psi,1\otimes D)$ is unitary equivalent to $\tau^{A_\theta}(\mathcal{T}^{\otimes 2})$.

We give next another approach using exact sequences. More exactly, we express the Dirac class as a four term exact sequence. Let $s_1,s_2,\overline{s}_1,\overline{s}_2:l^2(\mathbb{N}^2)\to l^2(\mathbb{N}^2)$ be the operators $s_1\xi_{m,n}=\xi_{m+1,n}$, $s_2\xi_{m,n}=\lambda^{-m}\xi_{m,n+1}$, $\overline{s}_1\xi_{m,n}=\lambda^{-n}\xi_{m+1,n}$, $\overline{s}_2\xi_{m,n}=\xi_{m,n+1}$. To define a Toeplitz algebra $\mathcal{E}$, a kind of pull-back, let $S_1=s_1\oplus u\otimes 1$, $S_2=s_2\oplus v\otimes 1$, $\overline{S}_1=\overline{s}_1\oplus 1\otimes\overline{u}$, $\overline{S}_2=\overline{s}_2\oplus 1\otimes\overline{v}$ in $\mathbb{L}(l^2(\mathbb{N}^2))\oplus A_\theta\otimes A_{-\theta}$. Let

$$\mathcal{E}=C^*(S_1,S_2,\overline{S}_1,\overline{S}_2),$$

$$P_1=S_1^*S_1-S_1S_1^*=\overline{S}_1^*\overline{S}_1-\overline{S}_1\overline{S}_1^*=P_{[\xi_{0,n};\,n\in\mathbb{N}]},$$

$$P_2=S_2^*S_2-S_2S_2^*=\overline{S}_2^*\overline{S}_2-\overline{S}_2\overline{S}_2^*=P_{[\xi_{m,0};\,m\in\mathbb{N}]},$$

and J_1,J_2 the ideals generated in $\mathcal{E}$ by P_1 respectively P_2. Then $J_1=(\mathbb{K}(l^2(\mathbb{N}))\otimes\mathbb{L}(l^2(\mathbb{N})))\cap\mathcal{E}$ and $J_2=(\mathbb{L}(l^2(\mathbb{N}))\otimes\mathbb{K}(l^2(\mathbb{N})))\cap\mathcal{E}$. It is easy to see that J_1 and J_2 are nuclear. Indeed, $P_1J_1P_1=C^*(P_1s_2P_1)$ is a Toeplitz algebra generated by an isometry and at the same time a full corner in J_1.

Proposition 5.1. *We have* $J_1J_2\simeq\mathbb{K}(l^2(\mathbb{N}^2))$ *and* $\mathcal{E}/(J_1+J_2)\simeq A_\theta\otimes A_{-\theta}$.

Proof. For the first isomorphism we show that P_1xP_2y is a finite rank operator for any x,y finite products of $S_j,\overline{S}_j,S_j^*,\overline{S}_j^*$. Indeed, if $P_2y\xi_{m,n}\neq 0$ then n is bounded by a constant depending only on y and if $P_1x\xi_{m,n}\neq 0$ then m is bounded by a constant depending only on x. Therefore if $P_1xP_2y\xi_{m,n}\neq 0$ then m and n are bounded by a constant depending only on x and y so $J_1J_2\subset\mathbb{K}$. The opposite inclusion follows from $P_1P_2=P_{\xi_{0,0}}$. For the second isomorphism we have the following commutation relations (the hat means modulo J_1+J_2) $\widehat{S_1}\widehat{S_2}=\lambda\widehat{S_2}\widehat{S_1}$, $\widehat{S_2}\widehat{S_2}^*=\widehat{S_2}^*\widehat{S_2}=1$, $\widehat{S_1}\widehat{S_2}=\lambda\widehat{S_2}\widehat{S_1}$, $\widehat{\overline{S}_1}\widehat{\overline{S}_2}=$

$\overline{\lambda}\widehat{\overline{S}_2}\widehat{\overline{S}_1}$, $S_i\overline{S}_j = \overline{S}_j S_i$, $S_i^*\overline{S}_j = \overline{S}_j S_i^*$. By the universality of A_θ we can define a morphism $A_\theta \otimes A_{-\theta} \to \mathcal{E}/J_1 + J_2$ by $u \mapsto \widehat{S_1}$, $v \mapsto \widehat{S_2}$, $\overline{u} \mapsto \widehat{\overline{S}_1}$, $\overline{v} \mapsto \widehat{\overline{S}_2}$. This is injective since the diagram

$$\begin{array}{ccc} A_\theta \otimes A_{-\theta} & \longrightarrow & \mathcal{E}/J_1 + J_2 \\ & \overset{\text{id}}{\searrow} & \downarrow \\ & & A_\theta \otimes A_{-\theta} \end{array}$$

is commutative, where the vertical arrow is the restriction to $\mathcal{E}$ of the projection $\mathbb{L}(l^2(\mathbb{N}^2)) \oplus A_\theta \otimes A_{-\theta} \to A_\theta \otimes A_{-\theta}$. Surjectivity is obvious. $\square$

The above proof shows that if θ is irrational we do not have to consider that kind of pull-back since the algebra $A_\theta \otimes A_{-\theta}$ is simple. By Theorem 4.3 we have a 4-term exact sequence

$$0 \to \mathbb{K}(l^2(\mathbb{N}^2)) \to J_2 \to \mathcal{E}/J_1 \to A_\theta \otimes A_{-\theta} \to 0.$$

The nuclearity of $\mathcal{E}$ follows from the nuclearity of $J_1 + J_2$ Therefore, we have a class $\Delta \in \mathrm{Ext}^2(A_\theta \otimes A_{-\theta}, \mathbb{C})$. This is our K-homology class. The K-theory class and the morphism Θ is defined as in the Fredholm operator approach above. The product $[\Theta] \otimes \tau_{A_\theta}(\delta) \otimes \tau^{A_\theta}(\sigma^{12}(\Delta))$, which is equal to $\delta \otimes \sigma^{12}(\Delta)$, is given by a pull-back:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A_\theta \otimes \mathbb{K} & \longrightarrow & \mathcal{E}_1' & \longrightarrow & \mathcal{E}_2' & \longrightarrow & C(\mathbb{T}^2) \otimes A_\theta & \longrightarrow & 0 \\ & & \| & & & & & & \downarrow \Theta & & \\ & & \| & & & & & & C(\mathbb{T}^r) \otimes A_\theta & & \\ & & \| & & & & & & \downarrow \delta\otimes\text{id} & & \\ 0 & \longrightarrow & A_\theta \otimes \mathbb{K} & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & A_\theta \otimes A_{-\theta} \otimes A_\theta & \longrightarrow & 0 \end{array}$$

Thinking of the 2-fold exact sequence as 2 splices of 1-fold exact sequences this pull-back is constructed inductively. In fact, only $\mathcal{E}_r'$ is changed.

Let $W_1 = u^* \otimes \overline{S}_1$, $W_2 = v^* \otimes \overline{S}_2$, $u' = W_1^*(1 \otimes S_1)$, $v' = W_2^*(1 \otimes S_2)$.

Proposition 5.2. *One has*
(i) $W_1 W_2 = W_2 W_1$;
(ii) $W_1^* W_2 = W_2 W_1^*$;
(iii) $W_j(1 \otimes S_i) = (1 \otimes S_i)W_j$ *for any* $i, j \in \{1, 2\}$;
(iv) $W_j^*(1 \otimes S_i) = (1 \otimes S_i)W_j^*$ *for any* $i, j \in \{1, 2\}$;
(v) $W_j^* W_j - W_j W_j^* = 1 \otimes P_j$;
(vi) $W_j u' = u' W_j$, $W_j v' = v' W_j$ *for* $j \in \{1, 2\}$;
(vii) $u'v' = \lambda v'u'$.

Proof. (i) and (ii) follow from the relations $u^*v^* = \lambda v^*u^*$ and $\overline{S}_1\overline{S}_2 = \overline{\lambda S_2}\overline{S}_1$.

(iii) and (iv) follow from the commutation between $C^*(S_i)$ and $C^*(\overline{S}_j)$.

(iv) follows from $P_j = \overline{S}_j^*\overline{S}_j - \overline{S}_j\overline{S}_j^*$.

(v) is an immediate consequence of (i)–(iv).

(vi) and (vii) follow from $S_1S_2 = \lambda S_2S_1$. □

A consequence of Proposition 5.2 is that there is a morphism $\mathcal{T}^{\otimes 2}\otimes A_\theta \to C^*(W_1, W_2, u', v')$ with $S\otimes 1 \mapsto W_1$, $1\otimes S \mapsto W_2$, $u \mapsto u'$, $v \mapsto v'$. Indeed, $\mathcal{T}^{\otimes 2} \to C^*(W_1, W_2)$ is a morphism by the universality of Toeplitz algebra, $u' = u \otimes \overline{S}_1^*S_1$, $v' = v \otimes \overline{S}_2^*S_2$, $\overline{S}_1^*S_1$, $\overline{S}_2^*S_2$ are unitaries with $u'v' = \lambda v'u'$ so $C^*(u', v')$ is an image of A_θ.

Now, we have the diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{K}\otimes A_\theta & \longrightarrow & \mathcal{T}\otimes\mathbb{K}\otimes A_\theta & \longrightarrow & C(\mathbb{T})\otimes\mathcal{T}\otimes A_\theta & \longrightarrow & C(\mathbb{T}^2)\otimes A_\theta & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A_\theta\otimes\mathbb{K}(l^2(\mathbb{N}^2)) & \longrightarrow & \mathcal{E}_1' & \longrightarrow & \mathcal{E}_2' & \longrightarrow & C(\mathbb{T}^2)\otimes A_\theta & \longrightarrow & 0
\end{array}$$

By Proposition 5.2(v) we have $A_\theta \otimes P_{[\xi_{0,0}]} \subset C^*(W_j, u', v')$ so the leftmost vertical arrow in the diagram above sends a full corner onto a full corner. Therefore, the two sequences give rise to the same Ext-theory class.

6. A GRUPOID PICTURE OF THE DUALITY FOR HRG ALGEBRA

One can improve the result of [20] for any higher rank graph Λ with the following finitness condition

(F) the set $\{\lambda : \sigma(\lambda) = n\}$ is nonvoid and finite for any $n \in \mathbb{N}^r$

such that $0 < \#\Lambda_n(v) < \infty$ and $0 < \#\Lambda^n(v) < \infty$ where $n \in \mathbb{N}^r$, $v \in \Lambda^0$. One direction is to give up the aperiodicity condition. Inspired by the example of rotation algebras, we can replace the Toeplitz algebra $\mathcal{E}$ with the subalgebra $\tilde{\mathcal{E}}$ of $\mathcal{E} \oplus C^*(\Lambda) \otimes C^*(\Lambda^{op})$ generated by $\tilde{L}_\lambda = L_\lambda \oplus (s_\lambda \otimes 1)$ and $\tilde{R}_\lambda = R_\lambda \oplus (1 \otimes t_\lambda)$. We refer to $\mathcal{E}$ as the first summand of $\tilde{\mathcal{E}}$ and to $C^*(\Lambda) \otimes C^*(\Lambda^{op})$ as to the second summand. This trick will give rise to a kind of pull-back for long exact sequences. The constructions are the same as in [20] with

O_Λ replaced by $C^*(\Lambda)$ and $O_{\Lambda^{op}}$ replaced by $C^*(\Lambda^{op})$. In this way we avoid aperiodicity conditions.

Another way to improve the result is to change one Cuntz-Krieger relations which define the graph algebra. We want it in order to include rotation algebras. We want a map $c : \Lambda \times \Lambda \to \mathbb{T}$ and isometries s_λ such that

$$s_\lambda s_\mu = c(\lambda, \mu) s_{\lambda\mu}.$$

For associativity we must have

$$s_\alpha(s_\beta s_\gamma) = s_\alpha c(\beta,\gamma) s_{\beta\gamma} = c(\beta,\gamma)c(\alpha,\beta\gamma)s_{\alpha\beta\gamma},$$
$$(s_\alpha s_\beta)s_\gamma = c(\alpha,\beta)s_{\alpha\beta}s_\gamma = c(\alpha,\beta)c(\alpha\beta,\gamma)s_{\alpha\beta\gamma},$$

so c must satisfy the condition

$$c(\alpha,\beta)c(\alpha\beta,\gamma) = c(\alpha,\beta\gamma)c(\beta,\gamma),$$

where $s(\alpha) = t(\beta), s(\beta) = t(\gamma)$. An example is

$$c(\lambda,\mu) = \prod_{i<j} c_{ij}^{\sigma(\lambda)_i \sigma(\mu)_j},$$

where $c_{ij} \in \mathbb{T}$. When Λ is the monoid $\mathbb{N}^r$, we get the universal rotation algebras generated by n unitaries $u_1, \ldots, u_n$ subject to the relations $u_i u_j = c_{ij} u_j u_i$, $i < j$.

The K-theory class is the same as in [20]. It is given by n partial unitaries

$$w_k = \sum_{\sigma(\lambda)=e_k} s_\lambda^* \otimes t_\lambda.$$

More exactly, $\delta \in E^r(\mathbb{C}, C^*(\Lambda) \otimes C^*(\Lambda^{op}))KK(\mathcal{S}^{\otimes r}, C^*(\Lambda) \otimes C^*(\Lambda^{op}))$ is defined by restricting the morphism $C(\mathbb{T}^r) \to C^*(\Lambda) \otimes C^*(\Lambda^{op})$, $z_k \mapsto w_k$ to $\mathcal{S}^{\otimes r}$. We note that the condition (F) we imposed on Λ at the beginning of this section is crucial for these operators to be well defined (the sum must be finite) and partial unitaries.

To define the K-homology class we twist the creations

$$L_\lambda \delta_\mu = \begin{cases} c(\lambda,\mu)\delta_{\lambda\mu} & \text{if } s(\lambda) = t(\mu) \\ 0 & \text{otherwise,} \end{cases}$$

$$R_\lambda \delta_\mu = \begin{cases} c(\mu,\lambda)\delta_{\mu\lambda} & \text{if } s(\mu) = t(\lambda) \\ 0 & \text{otherwise.} \end{cases}$$

$$L_\lambda \Omega = R_\lambda \Omega = \delta_\lambda \quad \text{for any } \lambda \in \Lambda^*.$$

One can check that $L_\lambda L_\mu \delta_\alpha = c(\mu,\alpha)c(\lambda,\mu\alpha)\delta_{\lambda\mu\alpha}$ and $L_{\lambda\mu}\delta_\alpha = c(\lambda\mu,\alpha)\delta_{\lambda\mu\alpha}$. Using the relation $c(\lambda,\mu)c(\lambda\mu,\alpha) = c(\mu,\alpha)c(\lambda,\mu\alpha)$ we get $L_\lambda L_\mu = c(\lambda,\mu)L_{\lambda\mu}$. Similarly, one has $R_\lambda R_\mu = c(\mu,\lambda)R_{\mu\lambda}$.

The closed linear span of a set S in a normed linear space is denoted by $[S]$ and the projection onto a closed subspace $\mathcal{L}$ of a Hilbert space $\mathcal{H}$ by $P_{\mathcal{L}}$. For any $j \in \{1,\dots,r\}$ and $a \in \Lambda^o$ we define

$$P_a = P_{[\Omega,\delta_\lambda | o(\lambda)=a]}, \quad P_a^j = P_{[\Omega,\delta_\lambda | o(\lambda)=a \text{ and } \sigma(\lambda)_j=0]},$$
$$Q_a = P_{[\Omega,\delta_\lambda | t(\lambda)=a]}, \quad Q_a^j = P_{[\Omega,\delta_\lambda | t(\lambda)=a \text{ and } \sigma(\lambda)_j=0]},$$
$$P^j = P_{[\Omega,\delta_\lambda | \sigma(\lambda)_j=0]}.$$

Define $L_\Omega = 1 = R_\Omega$ and for $a \in \Lambda^o$ let L_a be the projection P_a and R_a the projection Q_a. It is easy to check that for any $\mu \in \Lambda^*$, $j \in \{1,\dots,r\}$ and $a \in \Lambda^o$ we have

$$L_\mu^* L_\mu = P_{s(\mu)}, \quad P_a = L_a \sum_{t(\lambda)=a;\, \sigma(\lambda)=e_j} L_\lambda L_\lambda^* + P_a^j,$$

$$R_\mu^* R_\mu = Q_{t(\mu)}, \quad Q_a = R_a \sum_{t(\lambda)=a;\, \sigma(\lambda)=e_j} R_\lambda R_\lambda^* + Q_a^j.$$

Note also that

$$1 - \sum_{\sigma(\lambda)=e_j} L_\lambda L_\lambda^* = P^j = 1 - \sum_{\sigma(\lambda)=e_j} R_\lambda R_\lambda^*$$

and for all $k \in \mathbb{N}^r$ we have

$$\sum_{\sigma(\lambda)=k} L_\lambda L_\lambda^* = \sum_{\sigma(\lambda)=k} R_\lambda R_\lambda^* = P_{[\delta_\mu | \sigma(\mu)\geq k]}.$$

We have similar relations in the algebra $\tilde{\mathcal{E}}$ if we replace L with $\tilde{L}$ and R with $\tilde{R}$

$$\tilde{L}_a = \sum_{t(\lambda)=a;\, \sigma(\lambda)=e_j} \tilde{L}_\lambda \tilde{L}_\lambda^* + P_a^j, \quad \tilde{R}_a = \sum_{t(\lambda)=a;\, \sigma(\lambda)=e_j} \tilde{R}_\lambda \tilde{R}_\lambda^* + Q_a^j.$$

$$1 - \sum_{\sigma(\lambda)=e_j} \tilde{L}_\lambda \tilde{L}_\lambda^* = P^j = 1 - \sum_{\sigma(\lambda)=e_j} \tilde{R}_\lambda \tilde{R}_\lambda^*$$

As in [20], the Toeplitz algebra $\mathcal{E}$ contains a tuple of r ideals $(J_1,\dots,J_r)$, $J_j = \langle P^j \rangle$ the closed two-sided ideals generated by P^j in $\tilde{\mathcal{E}}$. Note that the ideals J_k are in the first summand.

As in [20], Lemma 2.1 we have $\bigcap_{j=1}^r J_j = \mathbb{K}(\mathsf{F})$ and as in [20], Theorem 2.2, there is a morphism $C^*(\Lambda) \otimes C^*(\Lambda^{op}) \simeq \tilde{\mathcal{E}}/J_{\{1,\dots,r\}}$ given by $s_\lambda \otimes 1 \mapsto \widehat{\tilde{L}_\lambda}$ and $1 \otimes t_\lambda \mapsto \widehat{\tilde{R}_\lambda}$. The pull back construction, more exactly the second summand, is used to prove the injectivity of the above isomorphism which used the uniqueness of the Cuntz-Krieger relations associated to the graph. Indeed, since $J_{\{1,\ldots,r\}} \subset \pi_1(\mathcal{E})$ we have the diagram

$$\begin{array}{ccc} C^*(\Lambda)\otimes C^*(\Lambda^{op}) & \longrightarrow & \tilde{\mathcal{E}}/J_{\{1,\ldots,r\}} \\ & \searrow^{\mathrm{id}} & \downarrow \\ & & C^*(\Lambda)\otimes C^*(\Lambda^{op}) \end{array}$$

where the vertical arrow is the projection onto the second summand $\mathcal{E}/J_{\{1,\ldots,r\}} \ni \hat{x}\oplus y \mapsto y \in C^*(\Lambda)\otimes C^*(\Lambda^{op})$. From the construction in Section 4 we get a long exact sequence

$$0 \to \mathbb{K}(\mathsf{F}) \to \mathcal{E}_1 \to \ldots \to \mathcal{E}_r \to C^*(\Lambda)\otimes C^*(\Lambda^{op}) \to 0,$$

therefore, an E-theory class $\Delta \in E^r(C^*(\Lambda)\otimes C^*(\Lambda^{op}), \mathbb{C}) = K^r(C^*(\Lambda)\otimes C^*(\Lambda^{op}))$. To compute the product $\delta \otimes_{C^*(\Lambda^{op})} \sigma_{12}(\Delta) = \tau_{C^*(\Lambda)}(\delta)\otimes\tau^{C^*(\Lambda)}\cdot(\sigma_{12}(\Delta))$, which is a class in $E^0(\mathcal{S}^{\otimes r}\otimes C^*(\Lambda), \mathcal{S}^{\otimes r}\otimes C^*(\Lambda))$, we find $[\Theta]\otimes\tau_{C^*(\Lambda)}(\delta)\otimes\tau^{C^*(\Lambda)}(\sigma_{12}(\Delta)) = [\tau_{C^*(\Lambda)}(\delta)\circ\Theta]\otimes\tau^{C^*(\Lambda)}(\sigma_{12}(\Delta))$, where $\Theta : \mathcal{S}^{\otimes r}\otimes C^*(\Lambda) \to \mathcal{S}^{\otimes r}\otimes C^*(\Lambda)$. This is defined by restricting the morphism $\Theta : C(\mathbb{T}^r)\otimes C^*(\Lambda) \to C(\mathbb{T}^r)\otimes C^*(\Lambda)$, $\Theta(f)(z) = \gamma_z(f(z))$, γ the gauge action, to $\mathcal{S}^{\otimes r}\otimes C^*(\Lambda)$. This restriction is in turn homotopic to the identity. Our product is a pull-back ([13], Proposition 5.8), that is the top row of the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C^*(\Lambda)\otimes\mathcal{E}_0 & \longrightarrow & \mathcal{E}_1' & \longrightarrow & \cdots & \longrightarrow & \mathcal{E}_r' & \longrightarrow & \\ & & \Vert & & & & & & & & \\ 0 & \longrightarrow & C^*(\Lambda)\otimes\mathcal{E}_0 & \longrightarrow & C^*(\Lambda)\otimes\mathcal{E}_1 & \longrightarrow & \cdots & \longrightarrow & C^*(\Lambda)\otimes\mathcal{E}_r & \longrightarrow & \end{array}$$

$$\begin{array}{ccccc} \longrightarrow & C(\mathbb{T}^r)\otimes C^*(\Lambda) & \longrightarrow & 0 \\ & \downarrow{\scriptstyle\Theta} & & \\ & C(\mathbb{T}^r)\otimes C^*(\Lambda) & & \\ & \downarrow{\scriptstyle\delta\otimes\mathrm{id}} & & \\ \longrightarrow & C^*(\Lambda)\otimes C^*(\Lambda^{op})\otimes C^*(\Lambda) & \longrightarrow & 0 \end{array}$$

To show that it gives the same element as $\tau_{C^*(\Lambda)}(\mathcal{T}^{\otimes r}) \in E^r(\mathcal{S}^{\otimes r}\otimes C^*(\Lambda), C^*(\Lambda))$ (which is $1_{C^*(\Lambda)} \in E(C^*(\Lambda), C^*(\Lambda))$ by Bott periodicity), we shall identify a subsequence which gives $\tau_{C^*(\Lambda)}(\mathcal{T}^{\otimes r})$. This is done by identifying in $C^*(\Lambda)\otimes\ \tilde{\mathcal{E}}$ an image $\mathcal{E}'$ of the algebra $\mathcal{T}^{\otimes r}\otimes C^*(\Lambda)$ which preserves the canonical tuple of ideals of $\mathcal{T}^{\otimes r} \otimes C^*(\Lambda)$. Therefore, it gives a commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & C^*(\Lambda) \otimes \mathcal{E}_0 & \longrightarrow & \mathcal{E}'_1 & \longrightarrow \cdots \longrightarrow & \mathcal{E}'_r & \longrightarrow & & \\
 & \uparrow & & \uparrow & & \uparrow & & & \\
0 \longrightarrow & \mathcal{T}_0 \otimes C^*(\Lambda) & \longrightarrow & \mathcal{T}_1 \otimes C^*(\Lambda) & \longrightarrow \cdots \longrightarrow & \mathcal{T}_r \otimes C^*(\Lambda) & \longrightarrow & &
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow & C(\mathbb{T}^r) \otimes C^*(\Lambda) & \longrightarrow 0 \\
 & \| & \\
\longrightarrow & C(\mathbb{T}^r) \otimes C^*(\Lambda) & \longrightarrow 0
\end{array}$$

In addition, the leftmost vertical arrow gives an E-theoretic equivalence since the algebras $C^*(\Lambda) \otimes P_\Omega \subset C^*(\Lambda) \otimes \mathcal{E}_0$ and $P_0 \otimes C^*(\Lambda) \subset \mathcal{T}_0 \otimes C^*(\Lambda)$ are full corners. To construct the algebra $\mathcal{E}'$ we define

$$W_j = \sum_{\sigma(\lambda)=e_j} s_\lambda^* \otimes \tilde{R}_\lambda, \quad V_\lambda = \big[W^{\sigma(\lambda)}\big]^*(1 \otimes \tilde{L}_\lambda),$$

where for a tuple of r commuting operators $x = (x_1, \ldots, x_r)$ and $k \in \mathbb{N}^r$ we define $x^k = x_1^{k_1} \ldots x_r^{k_r}$. One can prove as in [20], Proposition 6.4 that $C^*(\Lambda) \simeq C^*(V_\lambda \mid \lambda \in \Lambda^*)$, the isomorphism being given by $s_\lambda \mapsto V_\lambda$. The pull-back construction replaces the uniqueness of O_Λ invoked in [20], Proposition 6.4 and gives the injectivity. The isometries $(W_k)_k$ commute with each other and with V_λ's. So we have a morphism $\mathcal{T}^{\otimes r} \otimes C^*(\Lambda) \to C^*(W_k, V_\lambda)$ which is the identity on the full corner $P_\Omega \otimes C^*(\Lambda)$. All these statements follow exactly as in [20].

We want to give this duality a groupoid interpretation. First we make some comments about our twisted algebras which include rotation algebras. We do not know a 2-cocycle α on $G(X,T)$ as in Section 2 which gives the twisted algebra given by a cocycle c. However, for the particular cocycles $c(\lambda,\mu) = \prod_{i<j} c_{ij}^{\sigma(\lambda)_i \sigma(\mu)_j}$, which include the rotation algebras, we can define

$$\alpha((x,z,y),(x',z',y')) = \prod_{i<j} c_{ij}^{z_i z'_j}.$$

Let us start with the K-theory class. More precisely, from the groupoid picture of the algebra $C^*(\Lambda) \otimes C^*(\Lambda)$ the partial unitaries w_k are two-sided shifts on the double infinite paths space. Indeed, conform [20], Proposition 4.1

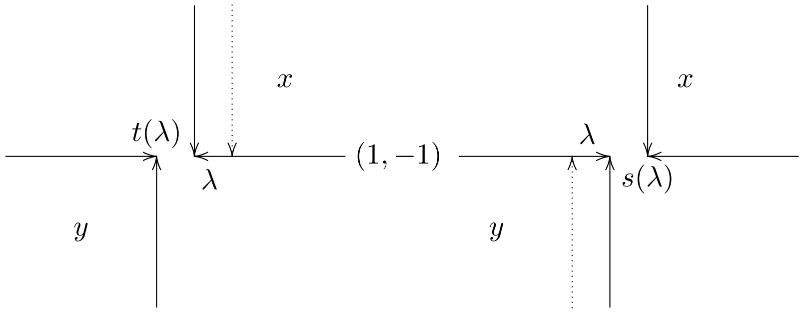
$$w_k w_k^* = w_k^* w_k = \sum_{a \in \Lambda^o} s_a \otimes t_a.$$

In the groupoid $G(\Lambda^\infty, T) \times G((\Lambda^{op})^\infty, T^{op})$, this projection corresponds to the set 1_Z where

$$Z = \{(x, y) \in \Lambda^\infty \times (\Lambda^{op})^\infty;\ s(x) = s(y)\}.$$

The isometry w_k is given by the bisection

$$\sum_{\sigma(\lambda)=e_k} 1_{\{((\lambda x, y), (1,-1), (x, \lambda y))\}}.$$



Identifying the set Z with the set of two-sided infinite words, we can say that w_k is the two sided shift in the direction k. The transformation group of the n shifts on Z, $Z \rtimes \mathbb{Z}^n$ is an open subgroupoid of $G(\Lambda^\infty, T) \times G((\Lambda^{op})^\infty, T^{op})$. Note that here one can see clearly the necessity of the condition (F) on Λ. Without it the space Z may not be locally compact or the two-sided shifts on the double infinite path space may not be well defined on Z (when the graph has sinks or sources). The morphism $C(\mathbb{T}^r) \to C^*(\Lambda) \otimes C^*(\Lambda^{op})$, $z_k \mapsto w_k$ is given by the inclusion of $C^*(\mathbb{Z}^r) \to C(Z) \ltimes \mathbb{Z}^r$. The twist morphism Θ is given by the topological isomorphism of groupoids

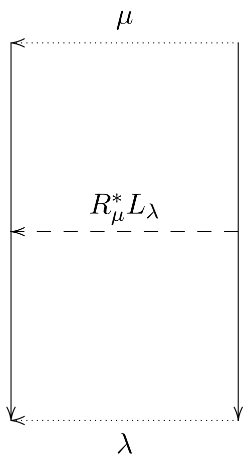
$$\mathbb{Z}^r \times G(\Lambda^\infty, T) \ni (z, x) \mapsto (z + b(x), x) \in \mathbb{Z}^r \times G(\Lambda^\infty, T),$$

where b is the cocycle which gives the gauge action $b(x, z, y) = z$. The restriction morphism is $\mathbb{R}^r \times G(\Lambda^\infty, T) \ni (t, x) \mapsto (t + b(x), x) \in \mathbb{R}^r \times G(\Lambda^\infty, T)$. The homotopy of this morphism and the identity is obtained by multiplying $b(x)$ with a parameter $s \in [0, 1]$.

To give a groupoid description of the K-homology class it would be ideal to have a conceptual groupoid description (like semidirect product) of the two-sided Toeplitz algebra $\mathcal{E}$. We have not been able to do it but we can still use a groupoid of germs. Let X_l, X_r, X be the Gelfand spectrum of the commutative unital algebras $\mathcal{E}_l \cap l^\infty(\overline{\Lambda})$, $\mathcal{E}_r \cap l^\infty(\overline{\Lambda})$ respectively $\mathcal{E} \cap l^\infty(\overline{\Lambda})$ where $l^\infty(\Lambda) \subset \mathbb{L}(l^2(\overline{\Lambda}))$ is the algebra of diagonal operators. There is an isomorphism from $L_\lambda^* L_\lambda C(X) L_\lambda^* L_\lambda$ to $L_\lambda L_\lambda^* C(X) L_\lambda L_\lambda^*$ given by $f \mapsto L_\lambda f L_\lambda^*$. Therefore, the isometries $\{L_\lambda : \lambda \in \Lambda\}$ give rise by Gelfand duality to partial homeomorphisms of X_l, so they generate a pseudogroup $\mathcal{G}_l$. Similarly,

$\{R_\lambda : \lambda \in \Lambda\}$ give rise to partial homeomorphisms of X_r and they generate a pseudogroup $\mathcal{G}_r$. The set of partial isometries $\{L_\lambda, R_\lambda : \lambda \in \Lambda\}$ gives also partial homeomorphisms $\{l_\lambda, r_\lambda : \lambda \in \Lambda\}$ of X which generate a pseudogroup $\mathcal{G}$. We denote by G_l, G_r, G the corresponding groupoids of germs. It is important to have in mind that the maps l_λ and r_λ extend the maps $s(\lambda)\Lambda \ni x \mapsto \lambda x \in \lambda\Lambda$ respectively $\Lambda t(\lambda) \ni x \mapsto x\lambda \in \Lambda\lambda$ so we shall freely use the notation $\lambda x = l_\lambda x$ and $y\lambda = r_\lambda y$ whenever $x \in \mathrm{dom}(l_\lambda)$, $y \in \mathrm{dom}(r_\lambda)$. Here our convention is that $\Omega x = x$ for any $x \in X$.

X_l is easily described since the algebra $\mathcal{E}_l \cap l^\infty(\overline{\Lambda})$ is generated by the projections $L_\lambda L_\lambda^*$. It is the space appearing in Example 2.2(i) of [12]. X_r is the same space when we replace Λ with Λ^{op}. However, the space X seems to be much more complicated. The difficulty appears from the projections $P_j L_\lambda^* R_\mu R_\mu^* L_\mu P_j$ with $\sigma(\lambda) = \sigma(\mu) = e_j$ which are not in the algebra generated by the projections $L_\lambda L_\lambda^*$ and $R_\lambda R_\lambda^*$. This projection is the domain of the partial isometry $R_\mu^* L_\mu P_j$ which is not given by shift equivalence.



This does not happen in the rank one case since left and right creations commute up to compact operators.

Lemma 6.1. (i) *The set $\overline{\Lambda}$ is an open dense discrete subset of X (X is a compactification of $\overline{\Lambda}$).*

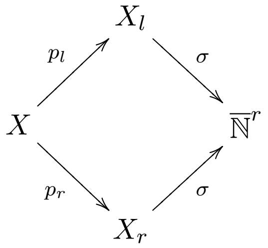
(ii) *The algebra $\mathcal{E}$ is the image of the canonical representation π of $C_r^*(G)$ on $l^2(\overline{\Lambda}) = \mathsf{F}$.*

Proof. (i) $\mathcal{E}$ contains the ideal of compact operators $\mathbb{K}(\mathsf{F})$ so $\mathcal{E} \cap l^\infty(\overline{\Lambda})$ contains the diagonal algebra $\mathbb{K}(\mathsf{F}) \cap l^\infty(\overline{\Lambda}) = C_0(\overline{\Lambda})$ as an essential ideal. Therefore, X contains the discrete set $\overline{\Lambda}$ as a dense open subset. X is compact since it is the spectrum of an unital algebra.

(ii) The set $\overline{\Lambda}$ is invariant with respect to the partial homeomorphisms l_λ, r_λ so it is invariant for the groupoid G. It is also open so it gives rise to an ideal of $C_r^*(G)$. Since the reduction $G|_{\overline{\Lambda}}$ is the transitive groupoid, this ideal is the ideal of compact operators. We have a representation π of $C_r^*(G)$ on $l^2(\overline{\Lambda})$ induced by the canonical representation of the ideal $\mathbb{K}(l^2(\overline{\Lambda}))$: $\pi(T)\delta_\lambda = \pi(TP_{[\delta_\lambda]})\delta_\lambda$, $T \in C_r^*(G)$. The image $\pi(C_r^*(G))$ in $\mathbb{L}(l^2(\overline{\Lambda}))$ is exactly the algebra $\mathcal{E}$. Indeed, a basis for the topology of the groupoid G is given by (Y, w) where 1_Y is a projection in $\mathcal{E} \cap l^\infty(\overline{\Lambda})$ and v is a finite word in $l_\lambda, l_\lambda^{-1}, r_\lambda, r_\lambda^{-1}$ such that v^{-1} is a homeomorphism on Y. Therefore, $\pi(1_{(Y,w)}) = 1_Y w \in \mathcal{E}$, where w is the word in $L_\lambda, L_\lambda^*, R_\lambda, R_\lambda^*$ obtained by replacing l and r with L and R. Therefore, the image of the representation π is contained in $\mathcal{E}$ and it is clear that L_λ, R_λ are in this image so $\pi(C_r^*(G)) = \mathcal{E}$. $\square$

We want now to identify the morphism $\mathcal{T}^{\otimes r} \otimes C^*(\Lambda) \to \mathcal{E} \otimes C^*(\Lambda)$, $S_i \to W_i$ and $s_\lambda \to V_\lambda$. In the following proposition we give the main properties of X. The continuous map σ defined on X_l and X_r is the extended shape given in Example 2.2(i).

PROPOSITION 6.2. (i) *There are canonical continuous surjective maps p_l, p_r and σ such that the diagram*



is commutative where $p_l(\lambda) = p_r(\lambda) = \lambda$ and σ extends the shape on $\overline{\Lambda}$.

(ii) *The source and terminal maps s and t on Λ extends to continuous maps on X.*

(iii) *X has the factorization property: if $x \in X$, $\sigma(x) \geq k + p$, $k, p \in \mathbb{N}^r$ then there exists unique $\lambda, \mu \in \Lambda$ with $\sigma(\lambda) = k$, $\sigma(\mu) = p$ and $y \in X$ with $\sigma(y) = \sigma(x) - k - p$ such that $x = \lambda y \mu$.*

(iv) *$J_k = \pi(C_r^*(G|_{X_k}))$ where $X_k = \{x \in X : \sigma(x)_k < \infty\}$.*

(v) *Let $Z = \{(x, y) \in X \times \Lambda^\infty : s(x) = t(y)\}$. Then the map*

$$\phi = (\sigma, \psi) : (\Lambda \times \Lambda^\infty) \cap Z \ni (x, y) \mapsto (\sigma(x), xy) \in \overline{\mathbb{N}}^n \times \Lambda^\infty$$

extends to a continuous surjective map $\phi : Z \to \overline{\mathbb{N}}^n \times \Lambda^\infty$.

Proof. (i) We have the inclusion of algebras $\mathcal{E}_l \cap l^\infty(\overline{\Lambda}) \subset \mathcal{E} \cap l^\infty(\overline{\Lambda})$, that is $C(X_l) \subset C(X)$, with identity on $C_0(\Lambda)$ hence there is a continuous surjection $X \to X_l$ which is the identity on the set $\overline{\Lambda}$. Similarly, there is a continuous surjection $X \to X_r$ which is the identity on the set $\overline{\Lambda}$. The diagram is commutative because $\sigma p_l = \sigma p_r$ on $\overline{\Lambda}$ which is a dense set. We denote by σ the shape extended to X_l, X_r or X.

(ii) X is a disjoint union of $\mathrm{dom}(l_a)$ so we can define $s(x) = a$ if $x \in \mathrm{dom}(l_a)$ and $t(x) = a$ if $x \in \mathrm{dom}(r_a)$. s and t are continuous since $s^{-1}(\{a\}) = \mathrm{dom}(l_a)$ and $t^{-1}(\{a\}) = \mathrm{dom}(r_a)$.

(iii) We have a disjoint union

$$X = \bigcup_{\sigma(\lambda)=k} \mathrm{dom}(l_\lambda) = \bigcup_{\sigma(\lambda)=k} \mathrm{dom}(r_\lambda).$$

Therefore, there is one and only one λ with $\sigma(\lambda) = k$ such that $x \in \mathrm{dom}(l_\lambda)$. We can take $x' = l_\lambda^{-1}(x)$ and then $x = \lambda x'$. Moreover, $\sigma(x') = \sigma(x) - k$. This can be checked on the set $\{\lambda\mu : \mu \in \overline{\Lambda}\}$ which is a dense set in $\mathrm{dom}(l_\lambda^*) = \mathrm{ran}(l_\lambda)$. We repeat now this reason for right factorization of x'.

(iv) The projection P_k corresponds to the open set $\{x \in X : \sigma(x)_k = 0\}$ which generates the open G-invariant set P_k is $\{x \in X : \sigma(x)_k < \infty\}$.

(v) We have to show that the maps $(\Lambda \times \Lambda^\infty) \cap Z \ni (x, y) \mapsto \sigma(x) \in \overline{\mathbb{N}}^r$ and $(\Lambda \times \Lambda^\infty) \cap Z \ni (x, y) \mapsto xy \in \Lambda^\infty$ extend to continuous maps. The first claim is proven in (i) above. To prove the second let (λ_n, y_n) be a convergent sequence in Z. We have to prove that $(\lambda_n y_n)(0, m)$ is eventually constant. Since y_n is convergent in Λ^∞ we know that $y_n(0, m)$ is eventually constant so we can suppose it is μ. We know that the sequence $(\lambda_n\mu)_n$ converges in X since R_μ is continuous so it converges also in X_l. This implies that $(\lambda_n\mu)(0, m) = (\lambda_n y_n)(0, m)$ is eventually constant. For the surjectivity of ϕ we use the density of the set $\mathbb{N}^r \times \Lambda^\infty$ in $\overline{\mathbb{N}}^r \times \Lambda^\infty$. This dense set is in the image of ϕ which is closed since Z is compact. □

From the factorization in (iii) of the proposition above, we can define commuting left and right semigroups of shifts (L, R) on X with $\mathrm{dom}(L_k) = \mathrm{dom}(R_k) = \{x \in X : \sigma(x) \geq k\}$

$$L_k x = x', \ R_k = x'' \text{ where } x = \lambda x' = x''\mu \text{ with } \sigma(\lambda) = \sigma(\mu) = k.$$

These shifts are continuous since their restrictions to $\mathrm{dom}(l_\lambda)$ respectively $\mathrm{dom}(r_\lambda)$ are l_λ^{-1} and r_λ^{-1}. We believe that our desired conceptual description of the groupoid G comes from these two shifts but we have not been able to find it.

Of particular interest is the set $X_\infty = X \setminus \bigcup_{j=1}^r X_j$.

Lemma 6.3. *We have $X_\infty \subset \Lambda^\infty \times (\Lambda^{op})^\infty$ and $G_{|X_\infty} \subset \mathrm{Germ}(\Lambda^\infty \times \Lambda^{op}, T_1 \times T_2)$ where T_1 and T_2 are the MGDS on Λ^∞ and $(\Lambda^{op})^\infty$ given in Example* 2.2 (i).

Proof. From the previous proposition (iv), we have

$$C(X_\infty) = \mathcal{E} \cap l^\infty(\overline{\Lambda}) \big/ ((J_1 + \ldots + J_r) \cap l^\infty(\overline{\Lambda})).$$

From [20], Theorem 2.2 the left and right creations and their adjoints commute modulo $J_1 + \ldots + J_r$. Therefore the algebra $C(X_\infty)$ is generated by $L_\lambda L_\lambda^* R_\mu R_\mu^* \big/ (J_1 + \ldots + J_r)$. There is a surjection $C^*(s_\lambda s_\lambda^* \otimes 1, 1 \otimes t_\lambda t_\lambda^*) \to C(X_\infty)$, which sends

$$s_\lambda s_\lambda^* \otimes 1 \text{ to } L_\lambda L_\lambda^* \big/ J_1 + \ldots + J_r \quad \text{and} \quad 1 \otimes t_\lambda t_\lambda^* \text{ to } R_\mu R_\mu^* \big/ J_1 + \ldots + J_r.$$

By Gelfand duality this map gives rise to an injective map from X_∞ to $\Lambda^\infty \times (\Lambda^{op})^\infty$. Moreover, the restrictions to X_∞ of l_λ and r_λ is in the restriction to X_∞ of the pseudogroup generated by T_1 and T_2 on $\Lambda^\infty \times (\Lambda^{op})^\infty$. □

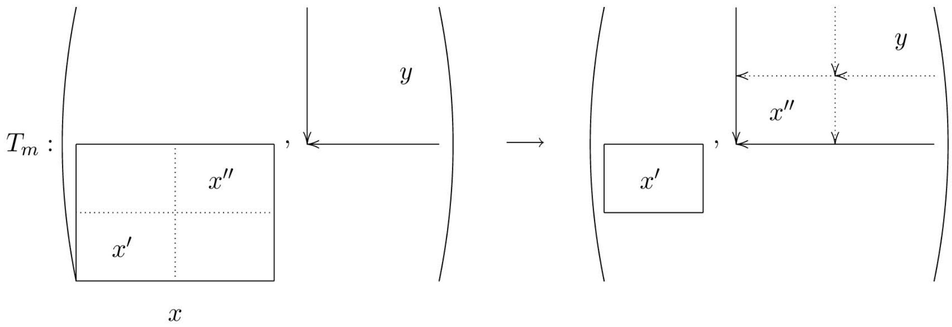
Proposition 6.4. (i) *The maps*

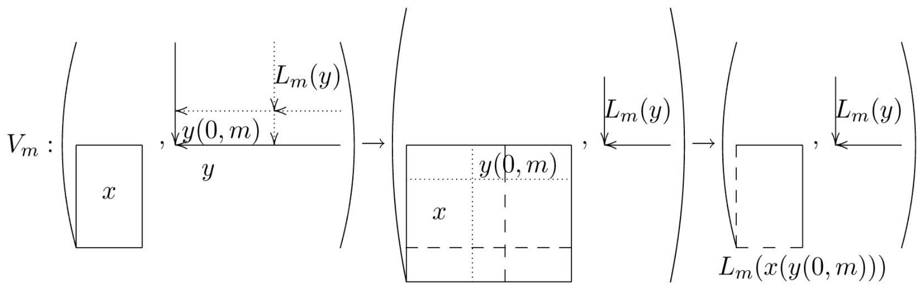
$$T_m : \{(x, y) \in Z : x = x'x'', \sigma(x'') = m\} \ni (x, y) \mapsto (x', x''y'') \in Z$$

$$V_m : Z \to Z, \quad V_m(x, y) = (L_m(x(y(0, m))), L_m(y))$$

define a MGDS (T, V) on Z which satisfies the condition (DC).

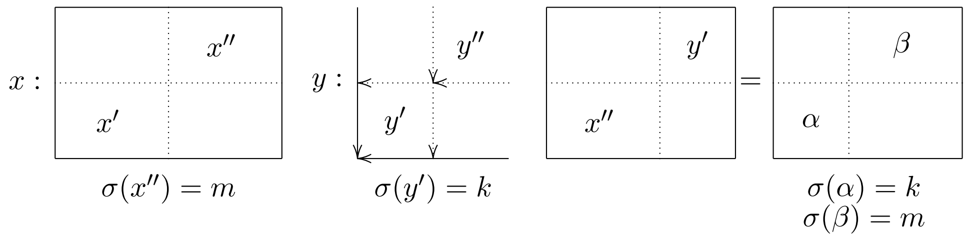
(ii) *The exit time map σ defined in Section 2 is the same as the map σ in Proposition 6.2.*





Proof. (i) We note that $\mathrm{dom}(T_m) = \{(x, y) \in Z : \sigma(x) \geq m\}$ and $\mathrm{dom}(V_m) = Z$ so $\mathrm{dom}(T_m V_k) = \mathrm{dom}(V_k T_m) = \mathrm{dom}(T_m)$. T_m is a local homeomorphism from $Z \cap (R_\lambda R_\lambda^* \times \Lambda^\infty)$ to $Z \cap (X \times s_\lambda s_\lambda^*)$ and V_m is a homeomorphism from $\{(x, y) \in Z : xy(0, m) \in L_\lambda L_\lambda^*\}$ to $\{(x, y) \in Z : t(x) = s(\lambda)\}$

with $\sigma(\lambda) = m$. Using the map ϕ from Proposition 6.2, we can see the map (T_m, V_k) as a homeomorphism from the set $\phi^{-1}(\{(p,x) : p \geq m, x \in \mathrm{dom}(l_\lambda)\})$ to $\phi^{-1}(\{(p,x) : p \geq 0, x \in \mathrm{dom}(l_{s(\lambda)})\})$. The semigroup condition of T and V is easily verified. We prove now that $T_m V_k = V_k T_m$. Let $(x,y) \in Z$ with $\sigma(x) \geq m$. Write $x = x'x''$, $y = y'y''$, $x''y' = \alpha\beta$ with $\sigma(x'') = \sigma(\beta) = m$, $\sigma(y') = \sigma(\alpha) = k$.



One has

$$\begin{aligned} T_m V_k(x,y) &= T_m(L_k(x'x''y'), y'') = T_m(L_k(x'\alpha\beta), y'') \\ &= T_m(L_k(x'\alpha)\beta, y'')) = (L_k(x'\alpha), \beta y''), \\ V_k T_m(x,y) &= V_k T_m(x'x'', y'y'') = V_k(x', x''y'y'') \\ &= V_k(x', \alpha\beta y'') = (L_k(x'\alpha), \beta y''). \end{aligned}$$

The (DC) condition is satisfied since $\mathrm{dom}(T_m V_k) \cap \mathrm{dom}(T_{m'} V_{k'}) = \mathrm{dom}(T_m) \cap \mathrm{dom}(T_{m'}) = \{(x,y) \in Z : \sigma(x) \geq m \vee m'\} = \mathrm{dom}(T_{m \vee m'})$.

(ii) Recall that the map σ in Section 2 was $\sigma(x) = \sup\{m : x \in \mathrm{dom}(T_m)\}$. In our case this supremum is $\sup\{m : m \leq \sigma(x)\} = \sigma(x)$. □

There is a natural equivariant map between MGDS $(Z, (T,V))$ and the MGDS which gives the algebra $\mathcal{T}^{\otimes r} \otimes C^*(\Lambda)$. Here, an equivariant map between two MGDS (X,T) and (Y,S) means a map $\phi : X \to Y$ such that $\phi \circ T_m = S_m \circ \phi$.

Proposition 6.5. *Let* $(\overline{\mathbb{N}}^r, S), (\Lambda^\infty, W)$ *be the MGDS of Example* 2.2 (i), (ii). *The map* ϕ *of Proposition* 6.2 *is an equivariant map between* $(Z,(T,V))$ *and* $(\overline{\mathbb{N}}^n \times \Lambda^\infty, S \times W)$

Proof. Regarding domains, it is enough to check that $\mathrm{dom}(S_k\phi) = \mathrm{dom}(\phi T_k)$ since W_k and V_k are everywhere defined. We have

$$\phi(\mathrm{dom}(T_k) = \phi(\{(x,y) \in Z;\, \sigma(x) \geq k\}) = \{(n,x);\, n \geq k\} = \mathrm{dom}(S_k)\phi.$$

Now, it remains to to check that $S_k\phi(x,y) = \phi T_k(x,y)$ and $W_k\phi(x,y) = \phi V_k(x,y)$ for $(x,y) \in Z$ with $\sigma(x) \in \mathbb{N}^n$ since $\overline{\Lambda}$ is a dense set in X. If

$x = x'x''$ with $\sigma(x'') = k$ one has

$$S_k(\phi(x,y)) = S_k(\sigma(x), xy) = (\sigma(x) - k, xy)$$
$$= (\sigma(x'), x'x''y) = \phi(x', x''y) = \phi(T^k(x,y)).$$

If $y = y'y''$ and $xy' = \alpha\beta$ with $\sigma(y') = \sigma(\alpha) = k$ (so $\sigma(x) = \sigma(\beta) = k$) one has

$$W_k(\phi(x,y)) = W_k(\sigma(x), xy'y'') = W_k(\sigma(x), \alpha\beta y'')$$
$$= (\sigma(x), \beta y'') = (\sigma(\beta), \beta y'') = \phi(\beta, y'') = \phi(V_k(x,y)). \quad \square$$

The next theorem gives a condition for a morphism of r-discrete groupoids to induce a morphism of the corresponding reduced algebras.

LEMMA 6.6. *Let $\phi : G_1 \to G_2$ be a proper morphism of two r-discrete groupoids such that $\phi : G_1^x \to G_1^{\phi(x)}$ is a bijection for any $x \in G_1^0$. Then the map $C_c(G_1) \ni f \mapsto f \circ \phi \in C_c(G_2)$ extends to a morphism $\tilde{\phi} : C_r^*(G_2) \to C_r^*(G_1)$.*

Proof. We have to show that the map $C_c(G_2) \ni f \mapsto L(f \circ \phi) \in \mathbb{L}(L^2(G_1))$ is a bounded representation, where $L^2(G_1)$ is the Hilbert module given by the field of Hilbert spaces G_1^0, $l^2(G_1^x)$. Indeed,

$$\widetilde{f \star g}(t) = f \star g(\phi(t)) = \sum_{s \in G_2^{\phi(s(t))}} f(\phi(t)s)g(s^{-1}).$$

We use now the assumption that $\phi : G_1^x \to G_1^{\phi(x)}$ is a bijection and we get

$$\widetilde{f \star g}(t) = \sum_{v \in G_1^{s(t)}} f(\phi(t)\phi(v))g(\phi(v)^{-1}) = \tilde{f} \star \tilde{g}(t). \quad \square$$

PROPOSITION 6.7. *The previews lemma holds if $G_1 = G(Z, (T,V))$ and $G_2 = G(\overline{\mathbb{N}}^n \times \Lambda^\infty, S \times W)$.*

Proof. First let us prove that the lemma holds for the map σ and the pairs of groupoids $G_1 = G(Z,T)$ and $G_2 = G(\overline{\mathbb{N}}^n, S)$. To prove the surjectivity let $(\lambda, x) \in Z$, $(\sigma(\lambda), n-m, k) \in G(\overline{\mathbb{N}}^r, S)$ such that $n \le \sigma(\lambda)$, $m \le k$ and $\sigma(\lambda) - n = k - m$. We construct $(\lambda', x') \in Z$ such that $\sigma(\lambda') = k$ and $T_n(\lambda, x) = T_m(\lambda', x')$. Since $n \le \sigma(\lambda)$ one can decompose $\lambda = \alpha\beta$ with $\sigma(\beta) = n$. Then $\lambda' = \alpha((\beta x)(0,m)))$ and $x' = (\beta x)(m, \infty)$ have the properties required To prove the injectivity let $(\lambda, x) \in Z$, $(\lambda', x') \in Z$, $(\lambda'', x'') \in Z$, $n, m, p, q \in \mathbb{N}^r$ such that $n - m = p - q$, $\sigma(\lambda') = \sigma(\lambda'')$ $T_n(\lambda, x) = T_m(\lambda', x')$ and $T_p(\lambda, x) = T_q(\lambda'', x'')$. Because of the condition (DC) of T we have $T_{n\vee p}(\lambda, x) = T_{m+n\vee p-n}(\lambda', x')$ and $T_{n\vee p}(\lambda, x) = T_{q+n\vee p-p}(\lambda'', x'')$ so $T_k(\lambda', x') = T_k(\lambda'', x'')$ where $k = m + n \vee p - n = q + n \vee p - p$, hence $\lambda' = \alpha\beta'$, $\lambda'' = \alpha\beta''$ with $\sigma(\beta') = \sigma(\beta) = k$ and $\beta'x' = \beta''x''$. The uniqueness of the factorization of a word in Λ^∞ implies that $\lambda' = \lambda''$ and $x' = x''$.

We prove now that the above lemma is true for ψ, $G_1 = G(Z,V)$ and $G_2 = G(\Lambda^\infty, W)$. To prove the surjectivity, let $(\lambda x, n-m, y) \in G(\Lambda^\infty, W)$ with $\sigma(\alpha) = n$, $\sigma(\beta) = m$, $\lambda x = \alpha z$, $y = \beta z$. One has $\lambda x(0,n) = \alpha\gamma$ so $T_n(\lambda, x) = (\gamma, x(n,\infty))$. Let λ' be given by the factorization $\beta\gamma = \lambda'\beta'$ with $\sigma(\beta)' = m$ and $x' = \beta' x(n,\infty)$. Then $T_m(\lambda', x') = (\gamma, x(n,\infty)) = T_n(\lambda, x)$ so $((\lambda, x), n-m, (\lambda', x')) \in G(Z,V)$ and $\lambda' x' = \lambda'\beta' x(n,\infty) = \beta\gamma x(n,\infty) = \beta z = y$. To prove the injectivity let $((\lambda,x), n-m, (\lambda', x')), ((\lambda,x), p-q, (\lambda'', x'')) \in G(Z,V)$ such that $n-m = p-q$, $\lambda' x' = \lambda'' x''$. As for $G(Z,T)$ we have $V_k(\lambda', x') = V_k(\lambda'', x'')$ where $k = m + n \vee p - n = q + n \vee p - p$. From the definition of V one has the factorizations $\lambda'\alpha = \alpha'\gamma$, $\lambda''\beta = \beta'\gamma$, $x' = \alpha y$, $x'' = \beta y$ where $\sigma(\alpha) = \sigma(\alpha') = \sigma(\beta) = \sigma(\beta') = k$. Then we have $\alpha'\gamma y = \beta'\gamma y$, hence $\alpha' = \beta'$ and $x' = x''$. We also have $\lambda'\alpha = \lambda''\beta = \alpha'\gamma$ so $\lambda' = \lambda''$.

We show now that lemma holds for ϕ, $G_1 = G(Z,(T,V))$ and $G_2 = G(\overline{\mathbb{N}}^n \times \Lambda^\infty, S \times W)$. First we show the injectivity. Let $(\lambda, x), (\lambda', x'), (\lambda'', x'') \in Z$, $n, m, n', m', p'q, p', q' \in \mathbb{N}^r$ such that $V_n(T_m(\lambda, x)) = V_{n'}(T_{m'}(\lambda', x'))$, $V_p(T_q(\lambda, x)) = V_{p'}(T_{q'}(\lambda'', x''))$, $n - n' = p - p'$, $m - m' = q - q'$, $\sigma(\lambda') = \sigma(\lambda'')$, $\lambda' x' = \lambda'' x''$. As before we have $V_n(T_{m\vee q}(\lambda, x)) = V_{n'}(T_{m'+m\vee q-m}(\lambda', x'))$ and $V_p(T_{m\vee q}(\lambda, x)) = V_{p'}(T_{q'+m\vee q-q}(\lambda', x'))$ so $V_n(T_{m\vee q}(\lambda, x)) = V_{n'}(T_k(\lambda', x'))$ and $V_p(T_{m\vee q}(\lambda, x)) = V_{p'}(T_k(\lambda'', x''))$ with $k = m' + m \vee q - m = q' + m \vee q - q$. As $\psi(T_k(\lambda, x)) = \psi(\lambda, x)$ the injectivity result proven above for ψ, $G(Z,V)$ and $G(\Lambda^\infty, \times W)$ shows that $T_k(\lambda', x') = T_k(\lambda'', x'')$. Now we apply the injectivity result for σ, $G(Z,T)$ and $G(\overline{\mathbb{N}}^n, S)$ to get $(\lambda', x') = (\lambda'', x'')$.

To prove the surjectivity let $(\lambda, x) \in Z$ and $((\sigma(\lambda), \lambda x), (n - n', m - m'), (k, y)) \in G(\overline{\mathbb{N}}^n \times \Lambda^\infty, S \times W)$. We apply the surjectivity proven above for ψ, $G(Z,V)$ and $G(\Lambda^\infty, \times W)$ in the point $T_m(\lambda, x)$. There is $(\alpha, \beta) \in Z$ such that $\alpha\beta = y$ and $V_n(T_m(\lambda, x)) = V_{n'}(\alpha, \beta)$. It follows that $\sigma(\alpha) = \sigma(\lambda) - m = k - m'$. We define now $\lambda' = \alpha\beta(0, m')$ and $x' = \beta(m', \infty)$. One has $\sigma(\lambda') = k$, $\lambda' x' = y$ and $T_{m'}(\lambda', x') = (\alpha, \beta)$ so $V_n T_m(\lambda, x) = V_{n'} T_{m'}(\lambda', x')$. $\square$

From Lemma 6.6 we have an induced homomorphism $\tilde{\phi} : \mathcal{T}^{\otimes n} \otimes C^*(\Lambda) \to C^*(Z, (T,V))$. Proposition 2.9 gives a homomorphism $\tilde{\pi} : \mathcal{T}^{\otimes n} \otimes C^*(\Lambda)$ to $\mathrm{Germ}(Z,(T,V))$. But $\mathrm{Germ}(Z,(T,V))$ is an open subgroupoid of $G \times \mathrm{Germ}(\Lambda^\infty, W)$. The MGDS (Λ^∞, W) is essentially free so by Proposition 2.7 we have a map from $C^*(\mathrm{Germ}(Z,(T,V)))$ to $\mathcal{E} \otimes C^*(\Lambda)$. Composing these homomorphisms we get the map from $\mathcal{T}^{\otimes n} \otimes C^*(\Lambda)$ to $\mathcal{E} \otimes C^*(\Lambda)$. Schematically, we view this in terms of groupoids by the diagram

$$\begin{array}{ccc} G(Z,(T,V)) \to^{\phi} & & G(\overline{\mathbb{N}}^r \times \Lambda^\infty, S \times W) \\ \big\downarrow \pi & & \\ \mathrm{Germ}(Z,(T,V)) \subset_{\text{open}} & & G \times \mathrm{Germ}(\Lambda^\infty), W \end{array}$$

ϕ restricted to the set $\Omega \times \Lambda^\infty$ gives an identification between $G(Z,(T,V))|_{\{\Omega\}\times\Lambda^\infty}$ and $G(\overline{\mathbb{N}}^r \times \Lambda^\infty, S \times W)|_{\{0\}\times\Lambda^\infty}$, so an isomorphism between full corners.

7. CONCLUSIONS

As a conclusion we want to draw the attention to the results in [9, 10]. Our groupoid approach to the duality between higher rank graph algebras can lead to a notion of higher rank hyperbolic groups. The higher rank version of a tree is a Euclidean building so the $\tilde{A}_n$-groups in [5] and [6] have to fall into this class. One can then think of the one-sided and two-sided Toeplitz algebras as subalgebras of $\mathbb{L}(l^2(\Gamma))$, where Γ is such a group. For example the left-sided Toeplitz algebra associated to the free group $\mathbb{F}_n$ is $C(\overline{\mathbb{F}_n}) \rtimes \mathbb{F}_n$ where $\overline{\mathbb{F}_n}$ is a compactification of $\mathbb{F}_n$ using the left-sided distance $d_l(\alpha,\beta) = l(\alpha^{-1}\beta)$, l the word length. Then the right-sided Toeplitz algebras is given by the same algebra $C(\partial\overline{\mathbb{F}_n}) \rtimes \mathbb{F}_n$ but the compactification is given by the right-sided distance $d_r(\alpha,\beta) = l(\alpha\beta^{-1})$. The two-sided Toeplitz algebra is the algebra generated by these two in their canonical representation on $\mathbb{L}(l^2(\mathbb{F}_n))$.

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