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Morse inequalities, a function space integral approach

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Abstract

In this paper we prove the Morse inequalities in the non-degenerate and degenerate cases. Like the approach of J.-M. Bismut, ours is based on the idea suggested by Witten. In fact, if anything, our approach is closer to Witten's original idea than Bismut's.

Keywords: Heat kernel; Integration on path spaces; Stochastic analysis on manifolds; Morse theory; Supersolution of partial differential equations; Vector bundles

0. Introduction

Let M be a d-dimensional compact manifold and $h : M \to \mathbb{R}$ a Morse function. Set ind_c, the index of h at the critical point c to be the number of negative eigenvalues of $\operatorname{hess}_c h$. If m_k is the number of critical points of index k and b_k is the kth Betti number, i.e., the

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dimension of the kth cohomology group with real coefficients, the statement of the Morse inequalities in the non-degenerate case is

$$m_k - m_{k-1} + \cdots + (-1)^k m_0 \geqslant b_k - b_{k-1} + \cdots + (-1)^k b_0, \quad 0 \leqslant k \leqslant d, \tag{0.1}$$

with equality for $k = d$.

We describe now our analytic approach for proving this result. By the Morse lemma one can find a local coordinate chart around each critical point so that h is quadratic in each of these. We choose a metric on M which is flat in these coordinate charts. Following Witten [11], we define for $\alpha > 0$ the operators

$$d^{\alpha h} = e^{-\alpha h} d e^{\alpha h}, \qquad \delta^{\alpha h} = e^{\alpha h} \delta e^{-\alpha h}, \tag{0.2}$$

$$\Box^{\alpha} = d^{\alpha h}\delta^{\alpha h} + \delta^{\alpha h} d^{\alpha h}, \tag{0.3}$$

where d is the usual exterior differential operator and δ its dual.

We consider $\mathbf{p}_k^{\alpha}(t, x, y)$ the kernel of the operator $e^{-t\Box^{\alpha}/2}$ acting on k-forms and take

$$Q_k^{\alpha}(t) = \int_M \operatorname{Tr} \mathbf{p}_k^{\alpha}(t, x, x)\, dx, \tag{0.4}$$

where Tr stands for the trace on $\bigwedge^k(M)$ and dx is the volume measure on M.

The starting point in the proving Morse inequalities is the following inequality due to Bismut [1, Theorem 1.3]

$$Q_k^{\alpha}(t) - Q_{k-1}^{\alpha}(t) + \cdots + (-1)^k Q_0^{\alpha}(t) \geqslant b_k - b_{k-1} + \cdots + (-1)^k b_0 \tag{0.5}$$

for $t > 0, \alpha > 0$, with equality for $k = d$.

Our main result is the following:

$$\lim_{\alpha \to \infty} Q_k^{\alpha}(t) = m_k \quad \text{for } 0 \leqslant k \leqslant d, \text{ and all } t > 0. \tag{0.6}$$

This and (0.5) imply (0.1).

Before discussing the general case, let us show how one can prove this in the simple situation when the manifold M is $\mathbb{R}^d$ and $h(x) = -\frac{1}{2}(x_1^2 + \cdots + x_s^2) + \frac{1}{2}(x_{s+1}^2 + \cdots + x_d^2)$. The only critical point is 0 and its index is s. Using Mehler formula, and performing simple computations, one gets

$$\int_{\mathbb{R}^d} \operatorname{Tr} p_k^{\alpha}(x, x)\, dx = \sum_{\substack{k_1 + k_2 = k \\ 0 \leqslant k_1 \leqslant d-s \\ 0 \leqslant k_2 \leqslant s}} \exp\bigl(-\alpha t(k_1 + s - k_2)\bigr).$$

Therefore

$$\lim_{\alpha\to\infty}\int_{\mathbb{R}^d}\operatorname{Tr} p_k^\alpha(x,x)\,dx=\begin{cases}1 & \text{if } k=s,\\ 0 & \text{otherwise,}\end{cases}$$

which is the proof of (0.6) for this basic model.

In the general case, we try to localize the proof of (0.6) to ones similar to the above for each critical point.

Our approach is based on a function space representation of $\mathbf{p}_k^\alpha(t,x,x)$. We write $\mathbf{p}_k^\alpha(t,x,x)$ via the Malliavin calculus and we analyze separately the cases for x away from the critical points and for x close to the critical points.

- When x stays away from the critical set, $\mathbf{p}_k^\alpha(t,x,x)$ is exponentially decaying to 0 as $\alpha\to\infty$. Using elementary analysis, the Markov property and estimates on exit times from balls of M, we reduce the analysis to one involving an integral over paths in an Euclidean ball. Finally, the estimate of this last integral comes down to estimates on the solution to a linear PDE problem (cf. (2.14) and (2.16)) in balls of $\mathbb{R}^d$, which is done in Section 2.2.
- When x is close to the critical set, $\mathbf{p}_k^\alpha(t,x,x)$ is, up to an exponentially decaying term in α, an integral over paths staying up to time t inside a small neighborhood of the critical set. Finally, replacing the manifold M by $\mathbb{R}^d$ and reversing the localization procedure described above, the computation of $\lim_{\alpha\to\infty} Q_k^\alpha(t)$ is reduced to one described above for the case of $\mathbb{R}^d$.

The degenerate case is more involved because the geometry near the critical set is in general non-trivial.

Let h be a Bott–Morse function on the compact manifold M with critical connected sub-manifolds $N_1,N_2,\dots,N_l$. The degenerate Morse lemma says that there are disjoint tubular neighborhoods $B_1,B_2,\dots,B_l$ of $N_1,N_2,\dots,N_l$ in M, open sets $\mathcal{V}_1,\mathcal{V}_2,\dots,\mathcal{V}_l$, $N_i\subset\mathcal{V}_i\subset B_i$, $i=1,\dots,l$, and Euclidean bundles $B_i^\pm$ of dimensions $\nu_i^\pm$, such that $B_i=B_i^+\oplus B_i^-$ with the property that h restricted to $\mathcal{V}_i$ is given by

$$h(z)=h\restriction N_i+\frac{1}{2}\bigl(|y^+|^2-|y^-|^2\bigr),\tag{0.7}$$

where $y^\pm$ are the $E_i^\pm$ components of z seen as a vector in $(E_i)_{\rho_i(z)}$, and $\rho_i:B_i\to N_i$ the canonical projection.

The statement of the degenerate Morse inequalities is

$$m_k-m_{k-1}+\cdots+(-1)^k m_0\geqslant b_k-b_{k-1}+\cdots(-1)^k b_0,\quad 0\leqslant k\leqslant d,\tag{0.8}$$

with equality for $k=d$, where

$$m_k=\sum_{i=1}^{l}\dim H^{k-\nu_i^-}\bigl(N_i;o\bigl(B_i^-\bigr)\bigr),\tag{0.9}$$

and $H^k(N_i; o(B_i^-))$ is the kth cohomology group of N_i twisted by the orientation bundle of B_i^-.

In order to do analysis we need a metric on M. Using standard procedures, in Section 3.1 we construct a metric and a compatible connection, called the Bismut connection, on each $T(B_i)$. Once this is done, we take any metric on M whose restriction to $\mathcal{V}_i$ is the metric on B_i.

We point out that in the non-degenerate case, the only connection we work with is the Levi-Civita connection. In the degenerate framework, the Bismut connection on B_i fits better for computational purposes than the Levi-Civita connection. For example, the parallel transport along paths in B_i with respect to the Bismut connection sends fibers into fibers, while the Levi-Civita connection does not. This is one feature that makes the Bismut connection more appropriate for fiberwise computations. On the other hand the Bismut and the Levi-Civita share a number of important features. For instance with respect to either connection, the Laplacians on functions coincide and the Hessian of the function h is the same.

The Brownian motion on the bundle B_i is given as the parallel transport of the Brownian paths in the fiber along the Brownian paths on the basis. This appears in Bismut's paper [1], and we discuss it in Section 3.3.

With the same definitions and notations as in the non-degenerate case (cf. (0.2), (0.3) and (0.4)), the inequality (0.5) holds. Our main result in this context is

$$\lim_{t\to\infty}\lim_{\alpha\to\infty} Q_k^\alpha(t) = m_k \quad \text{for } 0 \leqslant k \leqslant d, \tag{0.10}$$

with m_k defined in (0.9). Then, (0.8) follows from this and (0.5).

The program of proving (0.10) is the same as the one in the non-degenerate case. That is, we estimate $\mathbf{p}_k^\alpha(t,x,x)$ for x away or near the critical set.

- For x away from the critical set, one can prove that $\mathbf{p}_k^\alpha(t,x,x)$ is decaying exponentially fast to 0 as $\alpha \to \infty$. The idea is basically the same as in the degenerate case outlined above. The only technical issue is due to the fact that one needs to use the representation of the Brownian motion on B_i we mentioned above. With this one can reduce the analysis to a PDE estimate on a ball in the fibers of B_i.
- Following the same route as in the non-degenerate case for x near the critical set, say near N_i, leads one to a comparison between $\mathbf{p}_k^\alpha(t,x,x)$ and the heat kernel of the operator $\square^\alpha$ on B_i. We would like to reduce the computations to computations for harmonic oscillator in fibers of B_i, but the operator $\square^\alpha$ does not preserve horizontal and vertical forms on B_i. Nevertheless, comparing the heat kernel $\mathbf{p}_k^\alpha(t,x,x)$ on M with the heat kernel of various other operators (based on the Bismut connection on B_i, which does preserve the horizontal and vertical forms), we show how one can reduce the computations to one for harmonic oscillators in fibers. This comparison analysis is done in two parts of Section 3.4. The first part consists in proving that the heat kernel of a more general class of operators is bounded by a quantity which is based on the harmonic oscillator in fibers. The second part is carried out in Theorem (3.42) and uses the first part to compare the heat kernels of various operators.

Given this analysis, the proof of (0.10) is given in Section 3.5 and is straightforward.

We mention here one technical subtlety which appears in the degenerate case. Integration by parts on the path space over a compact manifold or in Euclidean space is well known, but in our case the vector bundles B_i are non-compact manifolds. Nonetheless, the Ricci curvature is bounded from below and from above in terms of the square of the distance function, and this turns out to be enough to prove the integration by parts on the path space.

The paper is organized as follows. In Section 1, we describe various facts from geometry, probability, and estimates on the exit time from balls on Riemannian manifolds.

Section 2 is the analysis of the non-degenerate case. The main theme of the paper is in Section 2.1, which shows how one can estimate the heat kernel $\mathbf{p}_k^\alpha(t,x,x)$ for x away from the critical case. The heart of the matter is Section 2.2 which is dedicated to proving Theorem 2.14, achieved by constructing a super-solution to the PDE (2.16). In Section 2.3 we analyze $\mathbf{p}_k^\alpha(t,x,x)$ when x is close to the critical case, and in Section 2.4 we give the proof of the Morse inequalities.

Section 3 deals with the degenerate case. Section 3.1 discusses the metric and the Bismut connection on a vector bundle over a compact manifold. Section 3.2 contains the representation of the Brownian motion on a vector bundle. Section 3.4 is the backbone of the degenerate case. First we analyze the boundedness of the heat kernel of a general operator described in (3.28). In Theorem 3.42 we show that the traces of the heat kernels of the two operators we are primarily interested in are close to each other. The degenerate Morse inequalities are proved in Section 3.5.

The first part of the appendix is about geometric computations needed in the degenerate case. The second part is a general discussion on the existence, representation and basic estimates of heat kernels of operators acting on forms. The third part is the justification of integration by parts on the path space of a Riemannian manifold with the Ricci curvature satisfying bounds given by (B.1), and the applications we use in the degenerate case.

1. General facts

Differential geometry

Given a d-dimensional compact Riemannian manifold M and a function $h:M\to\mathbb{R}$, we denote by *Crit* the set of critical points and set

$$d^{\alpha h}=e^{-\alpha h}de^{\alpha h},\qquad \delta^{\alpha h}=e^{\alpha h}\delta e^{-\alpha h},\qquad \Box^\alpha=d^{\alpha h}\delta^{\alpha h}+\delta^{\alpha h}d^{\alpha h},\tag{1.1}$$

where d is the usual exterior differential operator on smooth forms and δ its adjoint.

We consider the kernel $\mathbf{p}_k^\alpha(t,x,y):\bigwedge_y^k(M)\to\bigwedge_x^k(M)$ of the operator $e^{-t\Box^\alpha/2}$ acting on k-forms, and take

$$Q_k^\alpha(t)=\int_M \operatorname{Tr}\mathbf{p}_k^\alpha(t,x,x)\,dx,\tag{1.2}$$

where Tr stands for the trace on $\bigwedge^k(M)$ and dx for the volume measure on M.

Bismut, [1, Theorem 1.3] proved the following basic fact.

Theorem 1.3 *(The basic inequality). For any $\alpha > 0$, $t > 0$ we have*

$$Q_k^\alpha(t) - Q_{k-1}^\alpha(t) + \cdots + (-1)^k Q_0^\alpha(t) \geqslant b_k - b_{k-1} + \cdots + (-1)^k b_0, \quad 0 \leqslant k \leqslant d,$$

with equality for $k = d$.

Definition 1.4. (1) If ∇ is a connection on M and $X \in T(M)$, define ∇_X to be the derivation on $\bigwedge(M)$ so that

(a) if f is a function, $\nabla_X f = Xf$;
(b) if ω is a 1-form, $\nabla_X \omega$ is the 1-form given by

$$(\nabla_X \omega)(Y) = X\big(\omega(Y)\big) - \omega(\nabla_X Y) \quad \text{for any } Y \in T(M).$$

(2) Given a connection ∇ on M, we define Δ^∇, the corresponding Laplacian on forms by

$$\big(\Delta^\nabla \omega\big)(x) = \sum_{j=1}^{d} (\nabla_{(E_j)_x} \nabla_{E_j} \omega - \nabla_{\nabla_{(E_j)_x} E_j} \omega),$$

where ω, $(E_j)_{j=1}^d$ are, respectively, a smooth form and a smooth local orthonormal basis defined in a neighborhood of x.

(3) If S is a $2k$-tensor, we define D^*S, its action on $\bigwedge(M)$, by

$$D^* S_x = \sum_{j_1,\ldots,j_{2k}=1}^{d} S_x(E_{j_1}, E_{j_2}, \ldots, E_{j_{2k}})\big(E_{j_1}^* \wedge i_{E_{j_2}}\big) \circ \cdots \circ \big(E_{j_{2k-1}}^* \wedge i_{E_{j_{2k}}}\big),$$

where $(E_j)_{j=1}^d$ is any orthonormal basis in $T_x(M)$, and for $X \in T_x(M)$, i_X is the usual contraction operator determined by X. We call a tensor $\bar S$ even if $\bar S = S_1 + \cdots + S_r$, where S_i is a $2k_i$ tensor for $i = 1, \ldots, r$. We denote by $D^*\bar S$ its extension to forms by linearity.

(4) If S is a $(2k+1)$-tensor, we define $(D^*S)_x : T_x(M) \to \mathrm{End}(\bigwedge_x(M))$ by

$$\big(D^* S\big)_x(X_x) = \sum_{j_1,\ldots,j_{2k}=1}^{d} S_x(X_x, E_{j_1}, E_{j_2}, \ldots, E_{j_{2k}})\big(E_{j_1}^* \wedge i_{E_{j_2}}\big) \circ \cdots \circ \big(E_{j_{2k-1}}^* \wedge i_{E_{j_{2k}}}\big)$$

where $(E_j)_{j=1}^d$ is any orthonormal basis in $T_x(M)$. We call a tensor $\bar S$ odd if $\bar S = S_1 + \cdots + S_r$, where S_i is a $2k_i + 1$ tensor for $i = 1, \ldots, r$. $D^*\bar S$ denotes its extension to forms by linearity.

For the remainder of this section we work only with the Levi-Civita connection and Δ is its associated Laplacian.

The Hessian of h is the 2-form given by

$$\operatorname{hess}_x h(X_x, Y_x) = \langle \nabla_{X_x} \operatorname{grad} h, Y_x \rangle,$$

for any $X_x, Y_x \in T_x(M)$. $D^* \operatorname{hess} h$ and $D^* R$ are the extensions to $\bigwedge(M)$ of the Hessian $\operatorname{hess} h$ and the curvature tensor R.

With these notations, we have the following decomposition (cf. (A.5))

$$\square^\alpha = -\Delta + \alpha^2 |\operatorname{grad} h|^2 - \alpha \Delta h + 2\alpha D^* \operatorname{hess} h - D^* R. \tag{1.5}$$

Probabilistic preliminaries

For a detailed discussion of things presented here, we refer the reader to [8], especially to Chapter 8 in there. However for the reader's convenience we present here the main ideas.

In order to describe our representation of $\mathbf{p}_k^\alpha(t, x, y)$, we first introduce the Wiener measure $\mathcal{W}_d$ on $\mathcal{P}(\mathbb{R}^d)$ and describe the map $\mathbf{w} \in \mathcal{P}(\mathbb{R}^d) \to p(\cdot, x, \mathbf{w}) \in \mathcal{P}(M)$, whose distribution is the Wiener measure on M based at x. Here and elsewhere, for a manifold N, we set $\mathcal{P}(N) := C([0, \infty); N)$.

The measure $\mathcal{W}_d$ is the measure on $\mathcal{P}(\mathbb{R}^d)$ with the following properties:

- $\mathcal{W}_d(\mathbf{w}(0) = \mathbf{0}) = 1$;
- if $\Pi_t(\mathbf{w}) = \mathbf{w}(t)$, then for $0 \leqslant s < t$, the $\mathcal{W}_d$-distribution of $\Pi_t - \Pi_s$ is a centered normal (i.e., Gaussian) with covariance $(t - s)I_d$;
- if $0 \leqslant t_1 < t_2 < \cdots < t_n$, the random variables $\Pi_{t_1}, \Pi_{t_2} - \Pi_{t_1}, \ldots, \Pi_{t_n} - \Pi_{t_{n-1}}$ are $\mathcal{W}_d$-independent.

Consider the orthonormal frame bundle over M given by

$$\mathcal{O}(M) = \bigl\{\mathfrak{f} = \bigl(x, (E_j)_{j=1}^d\bigr):\ (E_j)_{j=1}^d \text{ orthonormal basis in } T_x(M)\bigr\}$$

with the canonical projection $\pi : \mathcal{O}(M) \to M$, $\pi(\mathfrak{f}) = x$. Fixing an orthonormal basis $(e_j)_{j=1}^d$ in $\mathbb{R}^d$, one can naturally interpret $\mathfrak{f}$ as an isometry from $\mathbb{R}^d$ to $T_x(M)$ which sends e_j to E_j. Using parallel transport one can define the horizontal lift of a vector $X_x \in T_x(M)$ to a vector $\mathfrak{X}_\mathfrak{f} \in T_\mathfrak{f}(\mathcal{O}(M))$ with $\pi(\mathfrak{f}) = x$. For $\xi \in \mathbb{R}^d$, we define the canonical vector field $\mathfrak{E}(\xi)_\mathfrak{f}$ to be the lift of the vector $\mathfrak{f}\xi \in T_{\pi(\mathfrak{f})}(M)$.

We explain now the construction of $\mathbf{w} \in \mathcal{P}(\mathbb{R}^d) \to p(\cdot, x, \mathbf{w}) \in \mathcal{P}(M)$. First, for $\mathfrak{f} \in \mathcal{O}(M)$ and a piecewise smooth $\mathbf{w} \in \mathcal{P}(\mathbb{R}^d)$, we construct the path $\mathfrak{p}(\cdot, \mathfrak{f}, \mathbf{w})$ by the following prescription

$$\dot{\mathfrak{p}}(t, \mathfrak{f}, \mathbf{w}) = \mathfrak{E}\bigl(\dot{\mathbf{w}}(t)\bigr)_{\mathfrak{p}(t, \mathfrak{f}, \mathbf{w})} \quad \text{with } \mathfrak{p}(0, \mathfrak{f}, \mathbf{w}) = \mathfrak{f}. \tag{1.6}$$

To extend this map to any $\mathcal{W}_d$-almost every path $\mathbf{w}$, consider $\mathbf{w}_n$, the piecewise linear path on each interval $[k/2^n, (k+1)/2^n]$ with $\mathbf{w}_n(k/2^n) = \mathbf{w}(k/2^n)$, $k \geqslant 0$ integer. Then, for any $\mathcal{W}_d$-almost every path $\mathbf{w}$,

$$\mathfrak{p}(\cdot, \mathfrak{f}, \mathbf{w}_n) \to \mathfrak{p}(\cdot, \mathfrak{f}, \mathbf{w})$$

uniformly on any interval $[0, T]$. Set

$$p(t, x, \mathbf{w}) = \pi \mathfrak{p}(t, \mathfrak{f}, \mathbf{w}) \quad \text{for } \mathfrak{f} \in \pi^{-1} x, \tag{1.7}$$

and μ_x^M, the Wiener measure on M based at x, the $\mathcal{W}_d$-distribution of $\mathbf{w} \in \mathcal{P}(\mathbb{R}^d) \to p(\cdot, x, \mathbf{w}) \in \mathcal{P}(M)$. There is a natural notion of parallel transport along $p(\cdot, x, \mathbf{w})$ given by $\tau_{p(\cdot,x,\mathbf{w})\restriction[0,t]} = \mathfrak{p}(t, \mathfrak{f}, \mathbf{w})\mathfrak{f}^{-1}$. Therefore, the parallel transport $\tau_{p\restriction[0,t]}$ is well defined for μ_x^M-almost any path $p \in \mathcal{P}(M)$.

Theorem C.3 gives the representation of the heat kernel of $\Box^\alpha$ acting on k forms by

$$\begin{aligned} \mathbf{p}_k^\alpha(t, x, y) = \mathbb{E}^{\mathcal{W}_d}\Bigg[&\exp\left(-\frac{\alpha^2}{2}\int_0^t \left|\operatorname{grad} h\big(p(v, x, \mathbf{w})\big)\right|^2 dv + \frac{\alpha}{2}\int_0^t \Delta h\big(p(v, x, \mathbf{w})\big)\, dv\right) \\ &\times V_k^\alpha\big(t, p(\cdot, x, \mathbf{w})\big)\tau_{p(\cdot,x,\mathbf{w})\restriction[t,0]}\delta_y\big(p(t, x, \mathbf{w})\big)\Bigg], \end{aligned} \tag{1.8}$$

where the integration is interpreted via a repeated integration by parts on path space, and for μ_x^M-almost any path $p \in \mathcal{P}(M)$, V_k^α is the solution to the ODE on $\operatorname{End}(\bigwedge_{p(0)}^k(M))$

$$\begin{cases} \dot{V}_k^\alpha(t, p) = V_k^\alpha(t, p)(\tau_{p\restriction[t,0]}(-\alpha D^* \operatorname{hess}_{p(t)} h + \frac{1}{2} D^* R_{p(t)})\tau_{p\restriction[0,t]}), \\ V_k^\alpha(0, p) = \operatorname{Id}_{\bigwedge_{p(0)}^k(M)}. \end{cases} \tag{1.9}$$

About exit times

We record here some estimates on the exit times from balls useful in various situations. The following result follows easily from [8, Theorem 8.62].

Theorem 1.10. *Let M be a complete Riemannian manifold, and assume that for a fixed point o and some $\gamma \geqslant 0$,*

$$\langle \operatorname{Ric}_x X_x, X_x\rangle \geqslant -\gamma\big(1 + \operatorname{dist}(x, o)^2\big)|X_x|^2 \quad \textit{for } x \in M,\ X_x \in T_x(M).$$

Then, for each compact set $K \subset M$, there exists a constant $C = C(K, \gamma) > 0$ such that

$$\sup_{x\in K} \mu_x^M(\zeta_r \leqslant t) \leqslant C \exp\left(-\frac{r^2}{4te^{Ct}}\right) \quad \textit{for any } r, t > 0 \tag{1.11}$$

where ζ_r is the first exit time from the ball $B(x, r)$.

If M is a compact manifold or $\mathbb{R}^d$, there is a constant $C > 0$ so that

$$\mu_x^M(\zeta_r \leqslant t) \leqslant C \exp\left(-\frac{r^2}{4t}\right) \quad \text{for any } r, t > 0,\ x \in M. \tag{1.12}$$

What we will need is the following.

Corollary 1.13. *Let M be a complete Riemannian manifold so that for a fixed point o and $\gamma \geqslant 0$,*

$$\langle \mathrm{Ric}_x X_x, X_x\rangle \geqslant -\gamma\big(1 + \mathrm{dist}(x, o)^2\big)|X_x|^2 \quad \text{for } x \in M,\ X_x \in T_x(M).$$

Then, for each compact set $K \subset M$, there exists a constant $C = C(K)$ such that

$$\sup_{x\in K} \mathbb{E}^{\mu_x^M}\big[e^{-\beta(t\wedge\zeta_r)}\big] \leqslant \exp(-t\beta) + C\exp\big(-re^{-Ct}\sqrt{\beta}/2\big) \quad \text{for all } r, t, \beta > 0. \tag{1.14}$$

If M is a compact manifold or $\mathbb{R}^d$, there is a constant $C > 0$ so that

$$\mathbb{E}^{\mu_x^M}\big[e^{-\beta(t\wedge\zeta_r)}\big] \leqslant \exp(-t\beta) + C\exp\big(-r\sqrt{\beta}/2\big) \quad \text{for all } r, t, \beta > 0,\ x \in M. \tag{1.15}$$

Proof. First,

$$\begin{aligned}
\mathbb{E}^{\mu_x^M}\big[e^{-\beta(t\wedge\zeta_r)}\big] &= e^{-\beta t}\mu_x^M(\zeta_r > t) + \mathbb{E}^{\mu_x^M}\big[e^{-\beta\zeta_r}, \zeta_r \leqslant t\big] \\
&= e^{-\beta t} + \beta\int_0^t e^{-\beta\sigma}\mu_x^M(\zeta_r \leqslant \sigma)\,d\sigma.
\end{aligned}$$

By (1.11),

$$\begin{aligned}
\int_0^t e^{-\beta\sigma}\mu_x^M(\zeta_r \leqslant \sigma)\,d\sigma &\leqslant C\int_0^t \exp\left(-\beta\sigma - \frac{r^2}{4\sigma e^{C\sigma}}\right)d\sigma \\
&\leqslant C\int_0^\infty \exp\left(-\beta\sigma - \frac{r^2 e^{-Ct}}{4\sigma}\right)d\sigma.
\end{aligned}$$

For given $a, b > 0$, use the change of variable $\xi = a\sigma^{1/2} - b\sigma^{-1/2}$ to justify that

$$\begin{aligned}\int_0^\infty e^{-a^2\sigma - \frac{b^2}{\sigma}}\,d\sigma &= \frac{e^{-2ab}}{2a^2}\int_{-\infty}^{\infty}\left(\sqrt{\xi^2+4ab} + \frac{\xi^2}{\sqrt{\xi^2+4ab}}\right)e^{-\xi^2}\,d\xi \\ &\leqslant \frac{e^{-2ab}}{a^2}\int_{-\infty}^{\infty}\left(\sqrt{\xi^2+4ab}\right)e^{-\xi^2}\,d\xi \\ &\leqslant \frac{e^{-2ab}}{a^2}\int_{-\infty}^{\infty}\left(|\xi| + 2\sqrt{ab}\right)e^{-\xi^2}\,d\xi \\ &\leqslant 2\left(1+\sqrt{ab}\right)\frac{e^{-2ab}}{a^2}. \end{aligned} \tag{$*$}$$

Taking $a = \sqrt{\beta}$ and $b = \frac{1}{2}re^{-Ct/2}$, for another constant C, we get (1.14). For the second part, by (1.12),

$$\int_0^t e^{-\beta\sigma}\mu_x^M(\zeta_r \leqslant \sigma)\,d\sigma \leqslant C\int_0^\infty \exp\left(-\beta\sigma - \frac{r^2}{4\sigma}\right)d\sigma.$$

Taking, $a = \sqrt{\beta}$ and $b = r/2$ in $(*)$, we get (1.15). □

2. Non-degenerate Morse inequalities

In this section M is a d-dimensional compact manifold and $h : M \to \mathbb{R}$ a Morse function with the critical set $Crit = \{c_1, \ldots, c_l\}$. For each $c \in Crit$, by the Morse lemma, one can find coordinate charts (U_c, φ_c), with $c \in U_c$ such that $\varphi_c(c) = 0$ and

$$h\left(\varphi_c^{-1}(x_1, \ldots, x_d)\right) = h(c) - \frac{1}{2}\sum_{j=1}^{\operatorname{ind}(c)} x_j^2 + \frac{1}{2}\sum_{k=\operatorname{ind}(c)+1}^{d} x_k^2,$$

where $\operatorname{ind}(c)$ is the index of c. Using these coordinate charts we choose a metric on M which is flat in each U_c.

For $r > 0$, set $Crit_r = \{x \in M,\ \operatorname{dist}(x, Crit) < r\}$ and $\Lambda_r = \{x \in M,\ \operatorname{dist}(x, Crit) > r\}$. We fix $r > 0$, small enough so that $B(c, 5r) \subset U_c$, the balls $\{B(c, 5r);\ c \in Crit\}$ are disjoint and the metric on each of them is flat. All the constants appearing in this section may depend on this fixed r.

Although for the purpose of this section it suffices to analyze the heat kernel for $t = 1$, for further study we will analyze this for arbitrary $t > 0$.

2.1. Away from the critical set

Our goal here is to prove the following.

Theorem 2.1 *(The away case). There exist constants $C_1, C_2, C_3 > 0$ so that for $t > 0$ and $\alpha \geqslant C_1 e^{C_1 t}$,*

$$\left\|\mathbf{p}_k^\alpha(t,x,y)\right\| \leqslant t^{-d/2}\alpha^{C_3}\left(\exp\left(-C_2 t\alpha^2\right) + \exp(-C_2\alpha)\right) \quad \text{for } (x,y) \in \Lambda_r \times M,$$

where the norm $\|\cdot\|$ is the Hilbert–Schmidt norm on the space $\mathrm{Hom}(\bigwedge_y(M), \bigwedge_x(M))$ of linear maps from $\bigwedge_y(M)$ to $\bigwedge_x(M)$.

The first step is to estimate the contribution of V_k^α (cf. (1.9)). Choose a smooth function f_k so that,

$$\begin{aligned} &f_k(x) = \mathrm{ind}(c) \quad \text{for } c \in \mathit{Crit} \text{ and } x \in B(c, 4r), \\ &-D^* \,\mathrm{hess}_x h \leqslant f_k(x)\,\mathrm{Id}_{\bigwedge_x^k(M)} \quad \text{for all } x \in M. \end{aligned} \tag{2.2}$$

By Lemma B.24 and Proposition C.5 we know there is a constant $C > 0$, depending on the size of the curvature, so that,

$$\left\|\mathbf{p}_k^\alpha(t,x,y)\right\| \leqslant e^{tC} E^{\mathcal{W}_d}\left[\exp\left(\int_0^t H_k^\alpha\big(p(v,x,\mathbf{w})\big)\,dv\right)\delta_y\big(p(t,x,\mathbf{w})\big)\right], \tag{2.3}$$

where

$$H_k^\alpha(y) = \left(-\frac{\alpha^2}{2}\left|\mathrm{grad}\, h(y)\right|^2 + \frac{\alpha}{2}\Delta h(y) + \alpha f_k(y)\right). \tag{2.4}$$

Now we need to estimate the expectation in (2.3).

Proposition 2.5. *Given $0 < \eta \leqslant 1$, set*

$$H_k^{\alpha,\eta}(y) = (1+\eta)\left(-\frac{\alpha^2}{2}\left|\mathrm{grad}\, h(y)\right|^2 + \frac{\alpha}{2}\Delta h(y) + \alpha f_k(y)\right). \tag{2.6}$$

Then, there is a polynomial $P(\alpha, t)$ so that for $t, \alpha > 0$

$$\left\|\mathbf{p}_k^\alpha(t,x,y)\right\| \leqslant \frac{e^{Ct}P(\alpha,t)}{\eta^d t^{d/2}}\left\{\mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^t H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right)\right]\right\}^{1/(1+\eta)}, \tag{2.7}$$

uniformly on M in x, y.

Proof. By repeated application of integration by parts (see Corollary B.26 and Theorem B.19 for details) we have

$$\mathbb{E}^{\mathcal{W}_d}\left[\exp\left(\int_0^t H_k^\alpha\big(p(v,x,\mathbf{w})\big)\,dv\right)\delta_y\big(p(t,x,\mathbf{w})\big)\right]$$
$$\leqslant \sum_{i=0}^d t^{-d}\alpha^i\mathbb{E}^{\mathcal{W}_d}\left[\big\|A_i(t,x,y,\mathbf{w})\big\|\exp\left(\int_0^t H_k^\alpha\big(p(v,x,\mathbf{w})\big)\,dv\right)\right],$$

each $A_i(t,x,y,\mathbf{w})$ being a linear combination of products of $\mathrm{End}(\bigwedge_x(M))$-valued iterated Itô or Riemann integrals. Using Hölder's inequality (see for instance [9, Exercise 7.2.15] for precise constants), (2.7) follows. □

Set

$$q_k^{\alpha,\eta}(t,x)=\mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^t H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right)\right].$$

As (2.7) makes clear, estimates on $\mathbf{p}_k^\alpha(t,x,y)$ will follow from estimates on $q_k^{\alpha,\eta}(t,x)$. If the path is staying in $\Lambda_{r/2}$, then the quadratic term in α in the expression of $H_k^{\alpha,\eta}(v,p)$ dominates. Thus the contribution of these paths to the integral should be exponentially small. On the other hand, the contribution of the paths that are getting close to the critical points requires a more careful analysis.

To carry out this heuristic argument, we first define the following sequences of stopping times. Set $\sigma_1=\sigma$ to be the first exit time from $\Lambda_{r/2}$, $\zeta=\zeta_1$ the first exit time from $Crit_r$ and

$$\begin{aligned}\zeta_n(p)&=\inf\big\{t\geqslant\sigma_n(p)\mid p(t)\in\Lambda_r\big\},\\ \sigma_{n+1}(p)&=\inf\big\{t\geqslant\zeta_n(p)\mid p(t)\in Crit_{r/2}\big\}.\end{aligned}\tag{2.8}$$

Using the continuity of paths, we have $\sigma_n\nearrow\infty$, and therefore,

$$q_k^{\alpha,\eta}(t,x)=\lim_{n\to\infty}\mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{t\wedge\sigma_n} H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right)\right].$$

The following theorem shows that the expectation in the above expression is non-increasing as n increases.

Theorem 2.9. *There exists $C > 0$ such that, for $t > 0$, $e^{-Ct} \geqslant \eta \geqslant 0$, $\alpha > Ce^{Ct}$ and $n \geqslant 1$, if $x \in \Lambda_r$,*

$$\mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{t\wedge\sigma_{n+1}} H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right)\right] \leqslant \mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{t\wedge\sigma_n} H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right)\right], \tag{2.10}$$

and if $x \in Crit_{r/2}$,

$$\mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{t\wedge\zeta_{n+1}} H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right), \Gamma\right] \leqslant \mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{t\wedge\zeta_n} H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right), \Gamma\right], \tag{2.11}$$

for $\Gamma \in \mathcal{F}_\zeta = \{\Gamma;\ \Gamma \cap \{\zeta < t\} \in \mathcal{F}_t$ for any $t > 0\}$ where $\mathcal{F}_t$ is the sigma-algebra generated by $\{p \upharpoonright [0,t],\ p \in \mathcal{P}(M)\}$.

Before proving these, we will show how Theorem 2.1 follows from them.

Proof of Theorem 2.1. There is a constant $c_1 > 0$ so that $|\operatorname{grad}_x h| \geqslant 4c_1 \operatorname{dist}(x, Crit)$ for all $x \in M$. On the other hand, if $p(0) \in \Lambda_{r/2}$, then

$$\begin{gathered} H_k^{\alpha,\eta}\big(p(v)\big) \leqslant -2c_1^2 r^2 \alpha^2 + c_2\alpha \leqslant -c_1^2 r^2 \alpha^2 \\ \text{for } 0 \leqslant v \leqslant \sigma(p) \text{ and } \alpha \geqslant \frac{c_2}{r^2 c_1^2},\ 0 < \eta \leqslant 1, \end{gathered} \tag{2.12}$$

with c_2 a constant depending only on the bounds of the Hessian and the Laplacian of h on M. Hence, by (2.8) and (2.10),

$$q_k^{\alpha,\eta}(t,x) \leqslant \mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{t\wedge\sigma(p)} H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right)\right] \leqslant \mathbb{E}^{\mu_x^M}\left[\exp\big(-r^2c_1^2\alpha^2(t\wedge\sigma)\big)\right].$$

Theorem 2.1 follows from this, (1.15) and (2.7). □

Now we return to

Proof of Theorem 2.9. We prove only (2.10), since the proof of (2.11) is similar.

Step 1. Here we show that

$$\mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{t\wedge\sigma_{n+1}(p)} H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right)\right] \leqslant \mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{t\wedge\zeta_n(p)} H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right)\right]. \tag{\#}$$

To see this, first apply the Markov property to justify that

$$\mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{t\wedge\sigma_{n+1}(p)} H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right)\right]$$

$$=\int\left(\int\exp\left(\int_0^{\sigma(\psi)\wedge(t-t\wedge\zeta_n(p))} H_k^{\alpha,\eta}\big(\psi(v)\big)\,dv\right)\mu^M_{p(\zeta_n)}(d\psi)\right)$$

$$\times\exp\left(\int_0^{t\wedge\zeta_n(p)} H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right)\mu_x^M(dp).$$

Now we remark that the path ψ starts at $p(\zeta_n)$ and runs till $\sigma(\psi)\wedge(t-t\wedge\zeta_n(p))$, thus it stays in $\Lambda_{r/2}$. By (2.12) we get that $H_k^{\alpha,\eta}(\psi(v))\leqslant 0$ for $0\leqslant v\leqslant\sigma(\psi)\wedge(t-t\wedge\zeta_n(p))$ and $\alpha\geqslant\frac{c_2}{r^2c_1^2}$, $0<\eta\leqslant 1$. This shows that the inside integral in the double integral is less than 1 and so (#) follows.

Step 2. Now we show that

$$\mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{t\wedge\zeta_n(p)} H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right)\right]\leqslant\mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{t\wedge\sigma_n(p)} H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right)\right].\qquad(\#\#)$$

To this end, use the Markov property to justify that

$$\mathbb{E}^{\mu_x^M}\left[\exp\left(\int_0^{t\wedge\zeta_n} H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right)\right]$$

$$=\int\left(\int\exp\left(\int_0^{\zeta(\psi)\wedge(t-t\wedge\sigma_n(p))} H_k^{\alpha,\eta}\big(\psi(v)\big)\,dv\right)\mu^M_{p(\sigma_n)}(d\psi)\right)$$

$$\times\exp\left(\int_0^{t\wedge\sigma_n(p)} H_k^{\alpha,\eta}\big(p(v)\big)\,dv\right)\mu_x^M(dp).\qquad(*)$$

Notice now that the point $p(\sigma_n)$ is on one of the spheres $\{S(c,r/2);\ c\in Crit\}$. Also notice that each of the balls $B(c,r)$ is just an Euclidean ball with the corresponding Euclidean metric on it and $H_k^{\alpha,\eta}(\psi(v))=((1+\eta)\alpha d-(1+\eta)\alpha^2|\psi(v)|^2)/2$ if $\psi(v)\in B(c,r)$. Let's fix a critical point $c\in Crit$. Because the inside integral in $(*)$ runs up to the exit time from the ball, the translation invariance of Brownian motion in the Euclidean space allows us to take $c = 0$. Now define the function $u_\eta^\alpha : [0,\infty) \times B(0,r) \to \mathbb{R}$

$$u_\eta^\alpha(s,y) = \int \exp\left(\frac{1}{2}\int_0^{\zeta(\psi)\wedge s} (1+\eta)\alpha d - (1+\eta)\alpha^2 \left|y+\psi(v)\right|^2 dv\right) \mathcal{W}_d(d\psi). \tag{2.13}$$

At this stage, what we need is the following estimate, which will be proved in the next section.

Theorem 2.14. *There exists $C > 0$ such that for any $t > 0$,*

$$\sup_{s\in[0,t],\, r/2 \leqslant |y| \leqslant r} u_\eta^\alpha(s,y) \leqslant 1 \quad \textit{for } \alpha \geqslant Ce^{Ct},\ e^{-Ct} \geqslant \eta \geqslant 0.$$

Clearly Theorem 2.14 is all that we need to complete the proof of (##) and consequently of (2.11). $\square$

2.2. The proof of Theorem 2.14

The following is a standard result in stochastic analysis.

Proposition 2.15. *u_η^α is the solution $u = u_\eta^\alpha$ to the initial-boundary problem on $[0,\infty) \times B(0,r)$*

$$\begin{cases} \partial_s u(s,x) = \frac{1}{2}\Delta u(s,x) + \frac{1}{2}((1+\eta)\alpha d - (1+\eta)\alpha^2|x|^2)u(s,x), \\ u(0,x) = 1 \quad \textit{if } x \in B(0,r), \\ u(s,y) = 1 \quad \textit{if } s > 0,\ y \in \partial B(0,r). \end{cases} \tag{2.16}$$

Next we want to get estimates on the solution to Eq. (2.16). We will do this by constructing a supersolution. The bound of the solution in terms of a supersolution is contained in the following result, which is an extension of the classical comparison for the heat equation.

Proposition 2.17. *Let $V \in C(B(0,r))$ be a continuous function which is bounded above, $A \subset B(0,r)$ a closed set, and two functions*

$$u \in C^{1,2}\big((0,t] \times B(0,r)\big) \cap C\big(\big([0,t] \times B(0,r)\big) \cup \big((0,t] \times \partial B(0,r)\big)\big),$$
$$v \in C^{1,2}\big((0,t] \times \big(B(0,r) \setminus A\big)\big) \cap C\big(\big([0,t] \times B(0,r)\big) \cup \big((0,t] \times \partial B(0,r)\big)\big),$$

be given. Assume that

(1) *for any $a \in A$ there exists a unit vector w_a such that for any $s \in (0,t]$, the function $\sigma \to \kappa(\sigma) = v(s, a + \sigma w_a)$ defined around 0 has left and right derivatives and*

$$\kappa_l'(0) > \kappa_r'(0); \tag{$*$}$$

(2) *on $(0,t] \times B(0,r) \setminus A$,*

$$\partial_s u = \frac{1}{2}\Delta u + Vu, \qquad \partial_s v \geqslant \frac{1}{2}\Delta v + Vv; \tag{$**$}$$

(3) *for any $x \in B(0,r)$ and $(s,y) \in (0,t] \times \partial B(0,r)$*

$$u(0,x) \leqslant v(0,x), \qquad u(s,y) \leqslant v(s,y). \tag{$***$}$$

Then, $u \leqslant v$ on $[0,t] \times B(0,r)$.

Proof. By taking the difference $v - u$, we may assume $u = 0$. After taking $\bar{v}(s,x) = e^{Cs} v(s,x)$ with $C > \sup_{x \in B(0,r)} V(x)$, we may also assume, without loss of generality, that $\sup_{x \in B(0,r)} V(x) < 0$. Now we prove that $v \geqslant 0$ on $[0,t] \times B(0, r-\epsilon)$ for small ϵ. The idea is the same as the proof of classical minimum principle given in [3, Theorem 9, Chapter 7].

Replacing $v(s,x)$ by $v(s,x) + \delta s$, $\delta > 0$, we may assume that

$$\partial_s v(s,x) > \frac{1}{2}\Delta v(s,x) + V(x) v(s,x) \quad \text{for all } (s,x) \in [0,t] \times B(0,r) \setminus A. \tag{#}$$

For $\epsilon > 0$, choose a point (s_ϵ, x_ϵ) to be a minimum point of v on the set $[0,t] \times \overline{B(0, r-\epsilon)}$. We claim that

$$v(s_\epsilon, x_\epsilon) \geqslant \min_{([0,t] \times \partial B(0,r-\epsilon)) \cup (0 \times B(0,r-\epsilon))} (v \wedge 0). \tag{##}$$

If this were not the case, then from (##), $v(s_\epsilon, x_\epsilon) < 0$ and we would be in one of the following three cases:

(1) $(s_\epsilon, x_\epsilon) \in (0,t) \times B(0, r-\epsilon) \setminus A$. In this case, $\partial_s v(s_\epsilon, x_\epsilon) = 0$ and $\Delta v(s_\epsilon, x_\epsilon) \geqslant 0$ which contradicts (#).
(2) $(s_\epsilon, x_\epsilon) \in \{t\} \times B(0, r-\epsilon) \setminus A$. We have that $\partial_s v(s_\epsilon, x_\epsilon) \leqslant 0$ and $\Delta v(s_\epsilon, x_\epsilon) \geqslant 0$, again contradicts (#).
(3) If $x_\epsilon \in A \cap B(0, r-\epsilon)$ then certainly the point (s_ϵ, x_ϵ) cannot be a local minimum because, if it were, then the function κ associated with (s_ϵ, x_ϵ) would have a local minimum at 0 and then

$$\kappa_r' \geqslant 0 \geqslant \kappa_l'$$

in contradiction with the assumption made in ($***$).

To end the proof, we let ϵ tend to 0 in (##) to get that $v \geqslant 0$ in $[0,t] \times B(0,r)$. $\square$

We are now prepared to begin the proof of Theorem 2.14 which will be done in two separate lemmas. In order to simplify the notation, remark that, by rescaling, it suffices to deal with the case $r = 1$. Also, for the following two lemmas we will replace α by $\alpha\sqrt{1+\eta}$ and will take $\epsilon = \sqrt{1+\eta} - 1$.

Lemma 2.18. *Let $\bar{u} = \bar{u}_\epsilon^\alpha$ be the solution in $(0,\infty) \times B(0,1)$ to*

$$\begin{cases} \partial_s \bar{u}(s,x) = \frac{1}{2}\Delta \bar{u}(s,x) + \frac{1}{2}(d(1+\epsilon)\alpha - \alpha^2|x|^2)\bar{u}(s,x), \\ \bar{u}(0,x) = 1 - \exp\big(\frac{(\alpha-73d)(|x|^2-1)}{2}\big) \quad \textit{for } |x| < 1, \\ \bar{u}(s,y) = 0 \quad \textit{for } s > 0,\ |y| = 1. \end{cases} \tag{2.19}$$

Then, for any $t > 0$, $1 \wedge \frac{1}{4dt} \geqslant \epsilon \geqslant 0$, and $\alpha \geqslant 156d$,

$$\sup_{s\in[0,t]} \bar{u}_\epsilon^\alpha(s,x) \leqslant 1 - \exp\left(\frac{(\alpha-73d)(|x|^2-1)}{2}\right) \quad \textit{for } 1/2 \leqslant |x| \leqslant 1. \tag{2.20}$$

Proof. For simplicity of calculations we set $\beta = \alpha - 73d$. Consider the following functions:

$$k(x) = 1 - \exp\left(\frac{\beta(|x|^2-1)}{2}\right) \quad \text{and} \quad w(s,x) = \left(\frac{2e^{\epsilon\alpha s}}{1+e^{-2\alpha s}}\right)^{d/2} \exp\left(-\frac{\alpha \tanh \alpha s}{2}|x|^2\right).$$

A simple computation shows that w is the solution on $[0,\infty) \times \mathbb{R}^d$ to the equation

$$\partial_s w(s,x) = \frac{1}{2}\Delta w(s,x) + \frac{1}{2}\big(d(1+\epsilon)\alpha - \alpha^2|x|^2\big)w(s,x), \tag{$*$}$$

with the initial condition $w(0,x) = 1$.

Let t_α be the unique solution of $\tanh \alpha t = \frac{d+\beta}{2\alpha}$. The idea is to show that for the time interval $[0,t_\alpha]$, $\bar{u}$ is bounded above by kw and for the time interval $[t_\alpha,t]$, $\bar{u}$ is bounded above by w itself.

Claim 1. *For $s \in [0,t_\alpha]$ and $x \in B(0,1)$*

$$\bar{u}(s,x) \leqslant k(x)w(s,x). \tag{i}$$

Proof. We check that the right-hand side of (i) is a supersolution on $[0,t_\alpha] \times B(0,1)$. This comes down to checking

$$\partial_s(kw)(s,x) \geqslant \frac{1}{2}\Delta(kw)(s,x) + \frac{1}{2}\big(d(1+\epsilon)\alpha - \alpha^2|x|^2\big)k(x)w(s,x)$$

which, because of $(*)$, becomes

$$(\Delta k)(x)w(s,x) + 2\langle \nabla k(x), \nabla w(s,x)\rangle \leqslant 0 \quad \text{or} \quad |x|^2\big(2\alpha \tanh(\alpha s) - \beta\big) \leqslant d,$$

which is true for $|x| \leqslant 1$ and $0 \leqslant s \leqslant t_\alpha$. Since both $\bar{u}$ and kw have the same initial-boundary values, the claim follows by a simple application of the classical maximum principle. □

Claim 2. *For $s \in [t_\alpha, t]$ and $x \in B(0,1)$ we have*

$$\bar{u}(s,x) \leqslant w(s,x). \tag{ii}$$

Proof. Because w satisfies $(*)$ and on the parabolic boundary it dominates $\bar{u}$, this is again a simple application of the classical maximum principle. □

With these two claims at hand, we now check (2.20).

Case *($s \in [0, t_\alpha]$).* By Claim 1, we need to verify that for $1/2 \leqslant |x| \leqslant 1$, $k(x)w(s,x) \leqslant k(x)$ or equivalently, $w(s,x) \leqslant 1$. In view of the expression for w, what we need to check is

$$\left(\frac{2e^{\epsilon\alpha s}}{1+e^{-2\alpha s}}\right)^{d/2} \exp\left(-\frac{\alpha \tanh(\alpha s)}{8}\right) \leqslant 1 \quad \text{for } s \in [0, t_\alpha]. \tag{$**$}$$

Because $s \in [0, t_\alpha]$ and $\tanh(\alpha t_\alpha) = \frac{d+\beta}{2\alpha}$, we know $\tanh(\alpha s) \leqslant \frac{1}{2}$. We claim that for $1 \geqslant \epsilon \geqslant 0$ and $\alpha \geqslant 10d$, $(**)$ is true. To see this, take $\sigma = \tanh(\alpha s)$. Then $e^{-2\alpha s} = \frac{1-\sigma}{1+\sigma}$ and the expression on the left-hand side in $(**)$ is

$$\varrho(\sigma) = \frac{(1+\sigma)^{d(\epsilon+2)/4}}{(1-\sigma)^{d\epsilon/4}} e^{-\alpha\sigma/8}.$$

Now,

$$\varrho'(\sigma) = \left(-\frac{\alpha}{8} + \frac{d(\epsilon+2)}{4(1+\sigma)} + \frac{d\epsilon}{4(1-\sigma)}\right)\varrho(\sigma)$$

and for $\sigma \in [0, \frac{1}{2}]$ this is negative for $1 \geqslant \epsilon \geqslant 0$ and $\alpha \geqslant 10d$. Therefore $\varrho(\sigma) \leqslant \varrho(0) = 1$ which proves $(**)$.

Case *($s \in [t_\alpha, t]$).* Claim 2 says $\bar{u} \leqslant w$. At the same time, for $|x| \in [1/2, 1]$, $\frac{1}{4dt} \geqslant \epsilon \geqslant 0$, and $\alpha \geqslant 156d$

$$\begin{aligned} w(s,x) &\leqslant \left(\frac{2e^{\epsilon\alpha s}}{1+e^{-2\alpha s}}\right)^{d/2} \exp\left(-\frac{\alpha \tanh \alpha s}{8}\right) \\ &\leqslant 2^{d/2} \exp\left(\frac{d\alpha\epsilon t}{2} - \frac{d+\beta}{4}\right) \leqslant \exp\left(-\frac{\alpha - 148d}{8}\right) \leqslant e^{-1}, \end{aligned}$$

and further,

$$e^{-1} < 1 - e^{-1} \leqslant 1 - e^{-3\beta/8} \leqslant k(x),$$

which implies $\bar{u}(s,x) \leqslant k(x)$ for $s \in [t_\alpha, t]$. $\square$

Lemma 2.21. *Let* $\underline{u} = \underline{u}_\epsilon^\alpha$ *be the solution in* $(0,\infty) \times B(0,1)$ *to*

$$\begin{cases} \partial_s \underline{u}(s,x) = \frac{1}{2}\Delta \underline{u}(s,x) + \frac{1}{2}(d(1+\epsilon)\alpha - \alpha^2|x|^2)\underline{u}(s,x), \\ \underline{u}(0,x) = \exp\left(\frac{(\alpha-73d)(|x|^2-1)}{2}\right) \quad \text{for } |x|<1, \\ \underline{u}(s,y) = 1 \quad \text{for } s>0,\ |y|=1. \end{cases} \tag{2.22}$$

Then, for any $t > 0$, $e^{-72dt} \geqslant \epsilon \geqslant 0$ *and* $\alpha \geqslant 190de^{72td}$,

$$\sup_{s\in[0,t]} \underline{u}_\epsilon^\alpha(s,x) \leqslant \exp\left(\frac{(\alpha-73d)(|x|^2-1)}{2}\right), \quad \text{for } 1/2 \leqslant |x| \leqslant 1, \tag{2.23}$$

and

$$\sup_{s\in[0,t]} \underline{u}_\epsilon^\alpha(s,x) \leqslant \exp\left(-\frac{\alpha|x|^2}{2} - \frac{\alpha}{2}\right), \quad \text{for } 0 \leqslant |x| \leqslant 1/2. \tag{2.24}$$

Proof. The strategy for proving this lemma is to make use of Corollary 2.17 and to construct the supersolution on subregions of the ball $B(0,1)$.

To simplify the writing, we set $\beta = \alpha - 73d$. Take $\delta > 0$ a small number and the set A in Corollary 2.17 to be $\{x \in B(0,1),\ |x| = 1/2\} \cup \{x \in B(0,1),\ |x| = 1/6\}$.

Region 1 *(*$\{x\colon 1/2 \leqslant |x| \leqslant 1\}$*).* In this region we take

$$v_1(s,x) = \exp\left(\frac{(\beta-2\delta)(|x|^2-1)}{2}\right).$$

(1) Checking $\partial_s v_1(s,x) \geqslant \frac{1}{2}\Delta v_1(s,x) + \frac{1}{2}(d(1+\epsilon)\alpha - \alpha^2|x|^2)v_1(s,x)$ is equivalent to checking

$$(\alpha - \beta + 2\delta)(\alpha + \beta - 2\delta)|x|^2 \geqslant d\big((1+\epsilon)\alpha + \beta - 2\delta\big) \quad \text{for } 1/2 \leqslant |x| \leqslant 1.$$

This is true for any $1/2 \leqslant |x| \leqslant 1$ iff

$$\frac{1}{4} \geqslant \frac{d(2+\epsilon)\alpha - d(73d+2\delta)}{2(73d+2\delta)\alpha - (73d+2\delta)^2},$$

which is certainly the case for $1 > \delta > 0$, $1 \geqslant \epsilon > 0$, and $\alpha \geqslant 40d$.

(2) The initial-boundary condition is satisfied for $s = 0$, $1/2 \leqslant |x| \leqslant 1$ and also for any $s > 0$ and $|x| = 1$.

Region 2 *($\{x\colon\ 1/6 \leqslant |x| \leqslant 1/2\}$).* In this region we take

$$v_2(s,x) = \exp\left((\beta-\delta)|x|\left(|x|-\frac{1}{2}\right)\right)\exp\left(-\frac{3(\beta-2\delta)}{8}\right).$$

(1) Checking $\partial_s v_2(s,x) \geqslant \frac{1}{2}\Delta v_2(s,x) + \frac{1}{2}(d(1+\epsilon)\alpha - \alpha^2|x|^2)v_2(s,x)$ is equivalent to

$$0 \geqslant (\beta-\delta)\left(2d - \frac{(d-1)}{2|x|}\right) + (\beta-\delta)^2\left(2|x|-\frac{1}{2}\right)^2 + d\alpha(1+\epsilon) - \alpha^2|x|^2$$

which will be fulfilled if

$$0 \geqslant 8d(\beta-\delta) + (\beta-\delta)^2\big(4|x|-1\big)^2 + 4d\alpha(1+\epsilon) - 4\alpha^2|x|^2 \quad \text{for } 1/6 \leqslant |x| \leqslant 1/2.$$

Observe that the right-hand side of this is quadratic in $|x|$ with dominant coefficient $16(\beta-\delta)^2 - 4\alpha^2$. Hence, for $\alpha \geqslant 148d$, this coefficient is positive and so it suffices to check that the expression is less than 0 for $|x| = 1/2$ and $|x| = 1/6$. This comes down to verifying the following inequalities:

- For $|x| = 1/2$: $0 \geqslant 8d(\beta-\delta) + (\beta-\delta)^2 + 4d\alpha(1+\epsilon) - \alpha^2$ which is equivalent to

$$\frac{1}{4} \geqslant \frac{d(3+\epsilon)\alpha - d(146d+2\delta)}{2(73d+\delta)\alpha - (73d+\delta)^2}.$$

  This is true for $1 \geqslant \delta > 0$, $1 \geqslant \epsilon \geqslant 0$ and $\alpha \geqslant 148d$.
- For $|x| = 1/6$: $0 \geqslant 72d(\beta-\delta) + (\beta-\delta)^2 + 36d\alpha(1+\epsilon) - \alpha^2$ is equivalent to

$$\frac{1}{36} \geqslant \frac{d(3+\epsilon)\alpha - d(146d+2\delta)}{2(73d+\delta)\alpha - (73d+\delta)^2}$$

  which is again true for $1 \geqslant \delta > 0$, $1 \geqslant \epsilon \geqslant 0$ and $\alpha \geqslant 148d$.

(2) The boundary condition reduces here to checking that

$$v_2(0,x) \geqslant \exp\left(\frac{\beta(|x|^2-1)}{2}\right) \quad \text{for } 1/6 \leqslant |x| \leqslant 1/2, \tag{$*$}$$

which comes down to

$$4(\beta-\delta)|x|\big(2|x|-1\big) - 3(\beta-2\delta) \geqslant 4\beta\big(|x|^2-1\big), \quad \text{or}$$
$$\beta\big(2|x|-1\big)^2 \geqslant 4\delta\big(|x|\big(2|x|-1\big) - 1/2\big),$$

and is certainly satisfied for any $\alpha \geqslant 73d$, $\delta \geqslant 0$, $0 \leqslant |x| \leqslant 1/2$.

(3) We need to check the conditions required in Proposition 2.17. We start by pointing out that $v_2(s,x)=v_1(s,x)$ for any $s\geqslant 0$ and x with $|x|=1/2$. For a with $|a|=1/2$, we choose $w_a=\frac{a}{|a|}$ and

$$\kappa(\sigma)=\begin{cases} v_1(s,a+\sigma w_a) & \text{if } \sigma\geqslant 0,\\ v_2(s,a+\sigma w_a) & \text{if } \sigma<0.\end{cases}$$

Then

$$\kappa_l'(0)=\big\langle \nabla v_2(s,a), w_a\big\rangle=(1/2)(\beta-\delta)v_2(s,a),$$
$$\kappa_r'(0)=\big\langle \nabla v_1(s,a), w_a\big\rangle=(1/2)(\beta-2\delta)v_1(s,a)$$

which shows that $\kappa_l'(0)>\kappa_r'(0)$.

Region 3 *($\{x\colon\ 0\leqslant |x|\leqslant 1/6\}$).* In this region we take the function

$$v_3(s,x)=\exp\left(\frac{\alpha}{2}\big(1-e^{-72ds}\big)\left(\frac{1}{36}-|x|^2\right)\right)v_2(s,1/6).$$

(1) Checking $\partial_s v_3(s,x)\geqslant \frac{1}{2}\Delta v_3(s,x)+\frac{1}{2}(d(1+\epsilon)\alpha-\alpha^2|x|^2)v_3(s,x)$ is equivalent to

$$de^{-72ds}-72d|x|^2e^{-72ds}\geqslant d\epsilon-2\alpha|x|^2e^{-72ds}+\alpha|x|^2e^{-144ds}.$$

If $e^{-72dt}\geqslant\epsilon$, the above inequality is satisfied if the following is true:

$$\alpha\big(2-e^{-72ds}\big)-72d\geqslant 0\quad \text{for all } 0\leqslant s\leqslant t,$$

which is indeed the case if $\alpha\geqslant 73d$.

(2) The initial-boundary comparison in this case is

$$v_3(0,x)\geqslant \exp\left(\frac{\beta(|x|^2-1)}{2}\right)\quad \text{for } 0\leqslant |x|\leqslant 1/6,$$

and is fulfilled because (cf. $(*)$)

$$v_3(0,x)=v_2(0,1/6)\geqslant \exp\left(\frac{\beta(\frac{1}{36}-1)}{2}\right)\geqslant \exp\left(\frac{\beta(|x|^2-1)}{2}\right).$$

(3) We now verify the conditions of Proposition 2.17. Observe that $v_3(s,x)=v_2(s,x)$ for $s\in[0,t]$ and x with $|x|=1/6$. Now for a given a with $|a|=1/6$ we choose $w_a=\frac{a}{|a|}$ and the function

$$\kappa(\sigma)=\begin{cases} v_2(s,a+\sigma w_a) & \text{if } \sigma\geqslant 0,\\ v_3(s,a+\sigma w_a) & \text{if } \sigma<0.\end{cases}$$

Then

$$\kappa_l'(0) = \langle \nabla v_3(s,a), w_a \rangle = -(1/6)\alpha\big(1-e^{-72ds}\big)v_2(s,a),$$
$$\kappa_r'(0) = \langle \nabla v_2(s,a), w_a \rangle = -(1/6)(\beta-\delta)v_2(s,a).$$

Thus $\kappa_l'(0) > \kappa_r'(0)$ iff $\alpha(1-e^{-72ds}) < \beta-\delta$ for all $s \in [0,t]$. This last one is true if $\alpha > (1+73d)e^{72ds}$ for $s \in [0,t]$, which is obviously the case if $\alpha \geqslant 74de^{72dt}$.

Now define

$$v(s,x) = \begin{cases} v_1(s,x) & \text{if } 1/2 \leqslant |x| \leqslant 1, \\ v_2(s,x) & \text{if } 1/6 \leqslant |x| \leqslant 1/2, \\ v_3(s,x) & \text{if } |x| \leqslant 1/6. \end{cases}$$

Then v satisfies all the requirements in Proposition 2.17, and this gives an upper bound on $\underline{u}$. Because this is true for all small $\delta > 0$, we finally get for $0 \leqslant s \leqslant t$

$$\underline{u}(s,x) \leqslant \begin{cases} \exp\big(\frac{\beta(|x|^2-1)}{2}\big) & \text{if } 1/2 \leqslant |x| \leqslant 1, \\ \exp\big(\frac{\beta}{8}(8|x|^2-8|x|-3)\big) & \text{if } 1/6 \leqslant |x| \leqslant 1/2, \\ \exp\big(\frac{\alpha}{72}(1-e^{-72ds})(1-36|x|^2)-\frac{37\beta}{72}\big) & \text{if } 0 \leqslant |x| \leqslant 1/6. \end{cases}$$

From this, (2.23) and (2.24) follow easily. □

2.3. Near the critical set

In this section we analyze the heat kernel $\mathbf{p}_k^\alpha(t,x,x)$ for x close to the critical set. We show here that the heat kernel is exponentially close to the heat kernel of an harmonic oscillator type operator on $\mathbb{R}^d$ described in the introduction. Since the analysis is based on the same ideas as in the away case, we will only point out the basic steps, leaving the details to the reader.

Using integration by parts on the path space (see Corollary B.26 and Theorem B.19 for details), we can write

$$\mathbf{p}_k^\alpha(t,x,y) = \mathbb{E}^{\mathcal{W}_d}\big[\Phi^\alpha(t,x,y,\mathbf{w})\big], \tag{2.25}$$

where $\Phi^\alpha(t,x,y,\cdot)$ is $\mathrm{Hom}(\bigwedge_y(M),\bigwedge_x(M))$-valued Wiener functional with the property that there exist a measurable map $(t,\mathbf{w}) \in [0,\infty) \times \mathcal{P}(\mathbb{R}^d) \to R_{x,y}^\alpha(t,\mathbf{w}) \in \mathbb{R}$ and a polynomial $P(t,\alpha)$ so that with H_k^α given by (2.4),

$$\big\|\Phi^\alpha(t,x,y,\mathbf{w})\big\|_{H.S} \leqslant R_{x,y}^\alpha(t,\mathbf{w}) \exp\left(\int_0^t H_k^\alpha\big(p(v,x,\mathbf{w})\big)\,dv\right) \quad \text{and}$$

$$\big\|R_{x,y}^\alpha(t,\mathbf{w})\big\|_{L^p(\mathcal{W}_d)} \leqslant \frac{p^d P(t,\alpha)}{t^{d/2}} \quad \text{for all } p \geqslant 1,\ t > 0,\ x,y \in M. \tag{2.26}$$

Moreover, if $\zeta_\Gamma(\mathbf{w}) = \inf\{t > 0\colon\ p(t, x, \mathbf{w}) \in \Gamma\}$, $x \in \Gamma$ and $\Gamma \subset M$ an open set, then the map $\Phi^\alpha(t, x, y, \cdot)$ on $\{\zeta_\Gamma \geqslant t\}$ depends only on the curvature, its derivatives and h inside Γ.

We now fix $c \in Crit$, take ζ to be the first exit time from $B(c, 2r)$, and set

$$I^{\mathrm{ext}}_{x,y}(t,\alpha) = \mathbb{E}^{\mathcal{W}_d}\big[\Phi^\alpha(t,x,y,\mathbf{w}), \zeta\big(p(\cdot,x,\mathbf{w})\big) < t\big],$$
$$I^{\mathrm{int}}_{x,y}(t,\alpha) = \mathbb{E}^{\mathcal{W}_d}\big[\Phi^\alpha(t,x,y,\mathbf{w}), \zeta\big(p(\cdot,x,\mathbf{w})\big) \geqslant t\big].$$

The next result describes the behavior of $I^{\mathrm{ext}}_{x,y}(t,\alpha)$.

Theorem 2.27. *There exist constants $C_1, C_2, N > 0$ so that for $t > 0$ and $\alpha \geqslant C_1 e^{C_1 t}$,*

$$I^{\mathrm{ext}}_{x,y}(t,\alpha) \leqslant t^{-d/2}\alpha^N \exp\left(-C_2\alpha - \frac{\alpha|x|^2}{2}\right) \quad \textit{for all } x \in B(c,r),\ y \in M. \tag{2.28}$$

Proof. We follow the same route as we did in proving Theorem 2.1. First, from (2.26), a simple application of Hölder's inequality shows that for another polynomial $P(t,\alpha)$ and $0 < \eta < 1$,

$$I^{\mathrm{ext}}_{x,y}(t,\alpha) \leqslant \frac{P(t,\alpha)}{\eta^d t^{d/2}} \left\{\mathbb{E}^{\mu^M_x}\left[\exp\left(\int_0^t H^{\alpha,\eta}_k\big(p(v)\big)\,dv\right), \zeta(p) < t\right]\right\}^{1/(1+\eta)},$$

where $H^{\alpha,\eta}_k$ is defined by (2.6). By (2.11),

$$\mathbb{E}^{\mu^M_x}\left[\exp\left(\int_0^t H^{\alpha,\eta}_k\big(p(v)\big)\,dv\right), \zeta(p) < t\right] \leqslant u(t,x) \quad \text{for any } x \in B(c,r),$$

where

$$u(s,x) = \mathbb{E}^{\mu^M_x}\left[\exp\left(\int_0^{\zeta(p)} H^{\alpha,\eta}_k\big(p(v)\big)\,dv\right), \zeta(p) < s\right].$$

Using a coordinate chart to identify $B(c,2r)$ with the Euclidean ball $B(0,2r)$, we have

$$H^{\alpha,\eta}_k\big(p(v)\big) = (1+\eta)\alpha d - (1+\eta)\alpha^2\big|p(v)\big|^2 \quad \text{for } v \leqslant \zeta(p),$$

and u is the solution to the initial-boundary problem in $(0,\infty) \times B(0,2r)$:

$$\begin{cases} \partial_s u(s,x) = \frac{1}{2}\Delta u(s,x) + \frac{1}{2}(1+\eta)(\alpha d - \alpha^2|x|^2)u(s,x), \\ u(0,x) = 0 \quad \text{for } x \in B(0,2r), \\ u(s,y) = 1 \quad \text{for } s > 0,\ y \in \partial B(0,2r). \end{cases}$$

Rescaling and using (2.24), one gets (2.28). □

We now turn to the integral $I^{\mathrm{int}}_{x,y}(t,\alpha)$, with $x \in B(c,r)$. Using a coordinate chart near c, we can identify $B(c,2r)$ with the ball $B(0,2r)$ in $\mathbb{R}^d$ and the metric with the Euclidean metric on $\mathbb{R}^d$. Define $h_c : \mathbb{R}^d \to \mathbb{R}$ by

$$h_c(x) = -\frac{1}{2}\sum_{j=1}^{\mathrm{ind}(c)} x_j^2 + \frac{1}{2}\sum_{k=\mathrm{ind}(c)+1}^{d} x_k^2.$$

Notice that the function h on M coincides with h_c on $B(c,2r)$. The main point here is the fact that the expression for $I^{\mathrm{int}}_{x,y}(t,\alpha)$ depends only on the curvature, its derivatives and the function h inside the ball $B(c,2r)$. Now, we take the starting manifold to be $\mathbb{R}^d$, the Morse function h_c, the operator $\square^\alpha_c$ on $\bigwedge(\mathbb{R}^d)$ given by

$$\square^\alpha_c = -\Delta + \alpha^2|x|^2 - \alpha\big(d - 2\,\mathrm{ind}(c)\big) + 2D^*\,\mathrm{hess}\,h_c,$$

and denote its heat kernel by $\bar{\mathbf{p}}^\alpha_{c,k}(t,x,y)$. In estimating the heat kernel on the compact manifold M we needed two important ingredients. One was the boundedness of the curvature to justify (2.3), and the other was the exit time estimates (1.15) used in the proof of Theorem 2.1. Both these key ingredients are available if we replace the manifold M by $\mathbb{R}^d$. Indeed, the curvature is 0 and the estimates on the exit time in (1.15) are valid. Running the same argument for the heat kernel $\bar{\mathbf{p}}^\alpha_{c,k}(t,x,y)$ we see that the results in Theorems 2.1 and 2.27 hold true. Moreover, if $x,y \in B(0,r)$, the integral $\bar{I}^{\mathrm{int}}_{x,y}(t,\alpha)$ corresponding to $\bar{\mathbf{p}}^\alpha_{c,k}(t,x,y)$ is exactly $I^{\mathrm{int}}_{x,y}(t,\alpha)$. Consequently, we have

Theorem 2.29. *There exist C_1, $C_2 > 0$ such that for $t > 0$, $\alpha > C_2 e^{C_2 t}$,*

$$\big\|\mathbf{p}^\alpha_k(t,x,y) - \bar{\mathbf{p}}^\alpha_{c,k}(t,x,y)\big\| \leqslant t^{-d/2}\exp\left(-C_1\alpha - \frac{\alpha|x|^2}{2}\right) \quad \text{for } x,y \in B(c,r). \tag{2.30}$$

Therefore, (*cf.* (1.2) *and* (2.1)) *there are K_1, $K_2 > 0$, so that for $t > 0$, $\alpha \geqslant K_2 e^{K_2 t}$,*

$$\left|Q^\alpha_k(t) - \sum_{c\in Crit}\int_{B(c,r)} \mathrm{Tr}\,\bar{\mathbf{p}}^\alpha_{c,k}(t,x,x)\,dx\right| \leqslant K_2 t^{-d/2}\big(\exp\big(-K_1 t\alpha^2\big) + \exp(-K_1\alpha)\big).$$

2.4. The proof of the non-degenerate Morse inequalities

We are now ready to give the proof of the Morse inequalities.

Because

$$\bigwedge\nolimits^k(\mathbb{R}^d) = \bigoplus_{\substack{k_1+k_2=k\\ 0\leqslant k_1\leqslant d-\mathrm{ind}(c),\, 0\leqslant k_2\leqslant \mathrm{ind}(c)}} \bigwedge\nolimits^{k_1}\big(\mathbb{R}^{d-\mathrm{ind}(c)}\big) \wedge \bigwedge\nolimits^{k_2}\big(\mathbb{R}^{\mathrm{ind}(c)}\big),$$

and the Hessians of h_c is diagonal, the representation in (1.8) yields

$$\begin{aligned}&\operatorname{Tr}\bar{\mathbf{p}}_{c,k}^{\alpha}(t,x,x)\\&\quad=\sum_{\substack{k_1+k_2=k\\0\leqslant k_1\leqslant d-\operatorname{ind}(c)\\0\leqslant k_2\leqslant\operatorname{ind}(c)}} e^{t\alpha(-k_1+k_2-\operatorname{ind}(c))}\mathbb{E}^{\mathcal{W}_d}\left[\exp\left(-\frac{\alpha^2}{2}\int_0^t\left|x+\varphi(\sigma)\right|^2d\sigma\right)\delta\big(\varphi(t)\big)\right],\end{aligned}$$

where δ is the delta function at 0. The integral above can be identified with the heat kernel of the operator $\frac{1}{2}\Delta-\frac{\alpha^2}{2}|x|^2$ on $L^2(\mathbb{R}^d)$, which, by Mehler's formula, makes the expression above equal to

$$\sum_{\substack{k_1+k_2=k\\0\leqslant k_1\leqslant d-\operatorname{ind}(c)\\0\leqslant k_2\leqslant\operatorname{ind}(c)}}\left(\frac{\alpha}{\pi(1-e^{-2t\alpha})}\right)^{d/2}e^{-\alpha\tanh(t\alpha/2)|x|^2+t\alpha(-k_1+k_2-\operatorname{ind}(c))}.$$

Taking the integral over $B(0,r)$, and changing $x\to\frac{1}{\sqrt{\alpha\tanh(t\alpha/2)}}x$, we get that

$$\int_{B(0,r)}\operatorname{Tr}\bar{\mathbf{p}}_{c,k}^{\alpha}(t,x,x)\,dx=\sum_{\substack{k_1+k_2=k\\0\leqslant k_1\leqslant d-\operatorname{ind}(c)\\0\leqslant k_2\leqslant\operatorname{ind}(c)}}\exp\big(-\alpha tk_1-\alpha t\big(\operatorname{ind}(c)-k_2\big)\big)A(\alpha)$$

where

$$A(\alpha)=\left(\frac{1}{\pi(1-e^{-t\alpha})^2}\right)^{d/2}\int_{B(0,r\sqrt{\alpha\tanh(t\alpha/2)})}e^{-|x|^2}\,dx.$$

Because $\lim_{\alpha\to\infty}A(\alpha)=1$, the integral above tends either to 0 or 1. It tends to 1 only in the case $k_1=0$ and $k_2=\operatorname{ind}(c)$ which is equivalent to $k=\operatorname{ind}(c)$. Taking the sum over all critical points we arrive at the following theorem.

Theorem 2.31. *For $t>0$, $\lim_{\alpha\to\infty}Q_k^{\alpha}(t)=m_k$.*

From this and Theorem 1.3 we get

Theorem 2.32 *(Non-degenerate Morse inequalities). For $0\leqslant k\leqslant d$,*

$$m_k-m_{k-1}+\cdots+(-1)^k m_0\geqslant b_k-b_{k-1}+\cdots+(-1)^k b_0,\tag{2.33}$$

with equality for $k=d$.

3. Degenerate Morse inequalities

Let h be a Bott–Morse function on the d-dimensional compact manifold M with critical connected submanifolds $\{N_i\colon\ 1 \leqslant i \leqslant l\}$ such that $Crit = \bigcup_{i=1}^{l} N_i$. By the degenerate Morse lemma, for each $i = 1, \dots, l$, there are ν_i-dimensional embedded vector bundles B_i in M endowed with a metric on fibers, $\nu_i^{\pm}$-dimensional vector sub-bundles $B_i^{\pm} \subset B_i$ so that B_i^{+} and B_i^{-} are orthogonal to each other, and open sets $\mathcal{V}_i \subset M$, such that $N_i \subset \mathcal{V}_i \subset B_i$ and

$$h(z_i) = h \restriction N_i + \frac{1}{2}\left(\left|y_i^+\right|^2 - \left|y_i^-\right|^2\right) \quad \text{for } z_i \in \mathcal{V}_i.$$

For each $i = 1, \dots, l$, and $z \in B_i$, $T_z(B_i)$ has a natural vertical subspace $T_z^{\mathrm{V}}(B_i)$, which is identified with the fiber $(B_i)_x$, where x is the projection of z onto N_i. Naturally, one can move the metric from $(B_i)_x$ to $T_z^{\mathrm{V}}(B_i)$. We will define also a horizontal subspace $T_z^{\mathrm{H}}(B_i)$ which is constructed by a standard procedure using a vertical connection (i.e., a connection on sections of B_i which is compatible with the metric). Once this is done, we choose a metric on N_i and lift it to the horizontal subspaces $T^{\mathrm{H}}(B_i)$, therefore, by declaring the vertical and the horizontal spaces orthogonal, we get a well-defined metric on $T(B_i)$.

Given the metric on $T(B_i)$, the Levi-Civita connection is a natural connection to work with. Unfortunately the parallel transport with respect to this connection along curves in B_i does not preserve the horizontal and vertical subspaces. For this reason, we will define the Bismut connection, whose parallel transport does preserve the horizontal and vertical components and is more useful for fiberwise computations.

We choose a metric on $T(M)$, so that on each $T(\mathcal{V}_i)$ it is the metric on $T(B_i)$. For $r > 0$, set $Crit_r = \{x \in M,\ \operatorname{dist}(x, Crit) < r\}$ and $\Lambda_r = \{x \in M,\ \operatorname{dist}(x, Crit) > r\}$.

In Section 3.2 we show that the Brownian motion on B_i has a representation as the parallel transport of the vertical Brownian motion along the Brownian paths on N_i. In Section 3.4 we discuss the heat kernel estimates, and give a comparison between the heat kernel of the operators $\square^{\alpha}$ and another operator computed in terms of the Bismut connection. Section 3.5 is dedicated to proving the Morse inequalities in the degenerate case.

3.1. Bismut's connection on a vector bundle

In this section we will construct and analyze Bismut's connection, as defined in [1], on a vector bundle over a Riemannian manifold.

Let B be a ν-dimensional vector bundle over a Riemannian manifold N of dimension m with the projection map $\rho : B \to N$.

Convention. *We will use z to denote a generic point on B, x a generic point on N, and y a generic point in a fiber of B. Thus, the point z will be identified with its corresponding coordinates (x, y), where $x = \rho(z)$, and y its identification as a vector in the fiber B_x.*

Assume that the bundle B has a smooth metric on fibers and a compatible connection $\nabla^{\mathrm{V}} : T(N) \times \Gamma(N, B) \to \Gamma(N, B)$, where $\Gamma(N, B)$ is the set of sections in B. ∇^{V} defines the notion of parallel transport τ^{V} of fibers along curves in N. Given a smooth curve $c:[0,a]\to N$ and a point $y\in B_{c(0)}$ we define its *lift*, $\mathbf{c}:[0,a]\to B$ starting at y by the prescription

$$\mathbf{c}(t)=\bigl(c(t),\tau^{\mathrm{V}}_{c\restriction[0,t]}y\bigr).$$

Using this lift we define the lift of vectors in $T(N)$ to vectors in $T(B)$. For $X_x\in T_x(N)$ take a curve such that $c(0)=x$, $\dot c(0)=X_x$ and its lift $\mathbf{c}$ starting at $y\in B_x$. Then the lift of X_x is defined to be

$$X^{\mathrm{H}}_z=\dot{\mathbf{c}}(0),$$

and is called the *horizontal* lift of X_x. In fact if we choose a local coordinate system $\{x_i\colon\ i=1,\dots,m\}$ on $U\subset N$ and $\{y_j\colon\ j=1,\dots,\nu\}$ a trivialization of B over U and determine $\Gamma^q_{i,j}$ by

$$\nabla^{\mathrm{V}}_{(\frac{\partial}{\partial x_i})_x}\frac{\partial}{\partial y_j}=\Gamma^q_{i,j}(x)\frac{\partial}{\partial y_q},\tag{3.1}$$

where we used Einstein's summation convention, then

$$\left(\frac{\partial}{\partial x_i}\right)^{\mathrm{H}}_z=\left(\frac{\partial}{\partial x_i}\right)_x-y^j\Gamma^p_{i,j}(x)\frac{\partial}{\partial y_p}.\tag{3.2}$$

Denote by $T^{\mathrm{H}}_z(B)$ the space of all horizontal lifts from $T_x(N)$ to $T_z(B)$ and by $T^{\mathrm{V}}_z(B)$ the space of vertical vectors at z in $T_z(B)$, i.e., the set of vectors in $T_z(B)$ tangent to the fiber B_x. Then

$$T_z(B)=T^{\mathrm{H}}_z(B)\oplus T^{\mathrm{V}}_z(B).\tag{3.3}$$

For a vector $X\in T_z(B)$, we use X^{H} and X^{V} to denote its horizontal and vertical components and we will identify these vectors with vectors in $T_x(N)$, respectively B_x. For a vector X which is already in either $(B_i)_x$ or $T_x(N_i)$ we will drop the superscript V or H for their identifications with vertical or horizontal vectors.

Using the decomposition (3.3), we construct the metric on $T_z(B)$ by lifting the metric from $T_x(N)$ to a metric on $T^{\mathrm{H}}_z(B)$, the metric on the fiber B_x to one on $T^{\mathrm{V}}_z(B)$, and declaring the spaces $T^{\mathrm{H}}_z(B)$ and $T^{\mathrm{V}}_z(B)$ orthogonal to each other.

We describe Bismut's connection by describing the associated parallel transport. For a curve $\mathbf{c}:[0,a]\to B$ and $X_0\in T_{\mathbf{c}(0)}(B)$, take $c(t)=\rho(\mathbf{c}(t))$ and X^{H}_0, X^{V}_0 the horizontal and vertical components of X_0. The *parallel transport* X_t of X_0 along $\mathbf{c}$ is the vector whose components X^{H}_t, X^{V}_t are obtained by parallel transport in N along c, respectively by vertical parallel transport along c. We set $\tau^{\mathrm{B}}_{\mathbf{c}\restriction[0,t]}X_0=X_t$. We will use the notation $\tau^{\mathrm{B}}_{\mathbf{c}\restriction[t,0]}$ for the parallel transport along the curve $\mathbf{c}$ from $\mathbf{c}(t)$ to $\mathbf{c}(0)$.

Definition 3.4 *(Bismut's connection on $T(B)$).* For $X_z \in T_z(B)$, a curve $\mathbf{c}$ such that $\mathbf{c}(0) = z$, $\dot{\mathbf{c}}(0) = X_z$, and Y a vector field along $\mathbf{c}$, we define

$$\nabla^{\mathrm{B}}_{X_z} Y = \frac{d}{dt} \tau^{\mathrm{B}}_{\mathbf{c}\upharpoonright[t,0]} Y_{\mathbf{c}(t)}.$$

Proposition 3.5. *If Y is a smooth vector field defined in a neighborhood of z, $\{(F_j)_x\colon j = 1, \dots, \nu\}$ is any basis in B_x, $Y^{\mathrm{V}}_x = \sum_{j=1}^{\nu} f_j (F_j)_x$ is the representation of the vector Y^{V} restricted to B_x, then,*

$$\nabla^{\mathrm{B}}_{X_z} Y = \nabla^{\mathrm{H}}_{X^{\mathrm{H}}_x} Y^{\mathrm{H}} + \nabla^{\mathrm{V}}_{X^{\mathrm{H}}_x} Y^{\mathrm{V}} + X^{\mathrm{V}}_z \bigl(Y^{\mathrm{V}}\bigr), \tag{3.6}$$

where $X^{\mathrm{V}}_z (Y^{\mathrm{V}}) = \sum_{j=1}^{\nu} X^{\mathrm{V}}_z (f_j) F_j$. Moreover, ∇^{B} is a covariant derivative which is compatible with the metric on $T(B)$.

Proof. Using the definition of the parallel transport we have

$$\tau^{\mathrm{B}}_{\mathbf{c}\upharpoonright[t,0]} Y_{\mathbf{c}(t)} = \tau^{\mathrm{H}}_{c\upharpoonright[t,0]} Y^{\mathrm{H}}_{\mathbf{c}(t)} + \tau^{\mathrm{V}}_{c\upharpoonright[t,0]} Y^{\mathrm{V}}_{\mathbf{c}(t)}, \tag{$*$}$$

with $c = \rho \circ \mathbf{c}$. Fix a geodesic ball $B(x, r)$ in N and an orthonormal basis $(E_i)_x$, $i = 1, \dots, m$ in $T_x(N)$. We identify the fibers of B over $B(r, x)$ with the fiber B_x by using parallel transport along geodesics radiating from x. In particular the basis $(F_j)_x$ is naturally extended to a section F_j in the bundle B defined on $B(x, r)$. Now we extend each $(E_i)_x$ to a vector field E_i defined on $B(r, x)$ by parallel transport with respect to the Levi-Civita connection along geodesics starting from x. Notice that we can write $Y = \sum_{i=1}^{m} g_i E_i + \sum_{j=1}^{\nu} f_j F_j$, for some functions $g_i : B(x, r) \to \mathbb{R}$ and $f_j : B(x, r) \times B_x \to \mathbb{R}$. Thus we can write $\mathbf{c}(t) = (c(t), \gamma(t))$, where γ is the curve in B_x with $\dot{\gamma}(0) = [\dot{\mathbf{c}}(0)]^{\mathrm{V}}$ and

$$\tau^{\mathrm{B}}_{\mathbf{c}\upharpoonright[t,0]} Y_{\mathbf{c}(t)} = \sum_{i=1}^{m} g_i\bigl(c(t)\bigr)(E_i)_{c(t)} + \sum_{j=1}^{\nu} f_j\bigl(c(t), \gamma(t)\bigr)(F_j)_{c(t)}.$$

From this, taking derivative with respect to t at 0, one gets (3.6). The compatibility with the metric on B follows from $(*)$ and the fact that both horizontal and vertical parallel transports preserve the length of vectors transported. □

Define the vertical curvature by $R^{\mathrm{V}}(X, Y)Z = \nabla^{\mathrm{V}}_X \nabla^{\mathrm{V}}_Y Z - \nabla^{\mathrm{V}}_X \nabla^{\mathrm{V}}_Y Z - \nabla^{\mathrm{V}}_{[X,Y]} Z$ for $X, Y \in T(N)$ and $Z \in \Gamma(N, B)$. Set R^{H} for the curvature of the Levi-Civita connection ∇^{H} on N.

Theorem 3.7.

(1) *The torsion of the connection ∇^{B} at $z = (x, y)$ is given by*

$$T^{\mathrm{B}}(X, Y) = R^{\mathrm{V}}\bigl(X^{\mathrm{H}}, Y^{\mathrm{H}}\bigr) y \quad \textit{for } X, Y \in T_z(B), \tag{3.8}$$

where y is interpreted as a vector in $T^{\mathrm{V}}_z(B)$.

(2) *The curvature of* ∇^{B} *is given by*

$$R^{\mathrm{B}}(X,Y)Z = R^{\mathrm{H}}\big(X^{\mathrm{H}},Y^{\mathrm{H}}\big)Z^{\mathrm{H}} + R^{\mathrm{V}}\big(X^{\mathrm{H}},Y^{\mathrm{H}}\big)Z^{\mathrm{V}} \quad \textit{for } X,Y,Z \in T_z(B). \tag{3.9}$$

(3) *If* N *is compact and* R^{LC} *is the curvature of the Levi-Civita connection* ∇^{LC} *on* B, *then, there is a constant* $C \geqslant 0$ *such that*

$$\big|R_z^{\mathrm{LC}}(X,Y)Z\big|_z \leqslant C\big(1+|y|^2\big)|X|_z|Y|_z|Z|_z \quad \textit{for } X,Y,Z \in T_z(B). \tag{3.10}$$

Moreover, for each $n \geqslant 1$, *there is a constant* $C_n \geqslant 0$ *so that for any* $X_1,\dots,X_n,X,Y,Z \in T_z(B)$,

$$\begin{aligned}&\big|\big(\nabla^{\mathrm{LC}}_{X_1}\dots\big(\nabla^{\mathrm{LC}}_{X_{n-1}}\big(\nabla^{\mathrm{LC}}_{X_n}R^{\mathrm{LC}}\big)(X,Y)Z\big)\dots\big)\big|_z\\ &\quad\leqslant C_n\big(1+|y|\big)^{2+n}|X_1|_z\dots|X_{n-1}|_z|X_n|_z|X|_z|Y|_z|Z|_z.\end{aligned} \tag{3.11}$$

(4) *The Laplacians* (*cf. Definition* 1.4) *on functions with respect to* ∇^{B} *and* ∇^{LC} *on* B *are the same.*

(5) *If* $h: B \to \mathbb{R}$ *is given by* $h(z) = |y|^2$, *then its Hessian* (*cf. Definition* A.2) *with respect to either connection* ∇^{B} *or* ∇^{LC} *is*

$$(\mathrm{hess}_z\, h)X_z = 2X_z^{\mathrm{V}} \quad \textit{for } X_z \in T_z(B). \tag{3.12}$$

On forms (*cf. Definition* 1.4)

$$\big(D^*\,\mathrm{hess}_z\, h\big)\omega_z^{\mathrm{H}} \wedge \omega_z^{\mathrm{V}} = 2\deg\big(\omega_z^{\mathrm{V}}\big)\omega_z^{\mathrm{H}} \wedge \omega_z^{\mathrm{V}} \tag{(3.12)$'$}$$

for any horizontal form ω_z^{H} *and any vertical form* ω_z^{V}, *where* deg *stands for the degree of the form.*

Proof. (1) and (2). It suffices to prove these for vertical and horizontal vector fields. For this purpose, take local coordinates in $U \subset N$, a trivialization of the vector bundle over U and the vectors $\frac{\partial}{\partial y_j}$, $j = 1,\dots,\nu$, and $(\frac{\partial}{\partial x_i})^{\mathrm{H}}$, $i = 1,\dots,m$, given by (3.2). The rest is a straightforward computation and is left to the reader.

(3) First we show that for any coordinate system on N and a corresponding trivialization of B there is a constant C_U such that

$$\big|R_z^{\mathrm{LC}}(X,Y)Z\big| \leqslant C_U\big(1+|y|^2\big)|X|_z|Y|_z|Z|_z \quad \text{for } z \in \rho^{-1}(U),\ X,Y,Z \in T_z(B).$$

Given a coordinate system on N, we take the vectors $(\frac{\partial}{\partial x_i})_z^{\mathrm{H}}$, $i = 1,\dots,m$, $\frac{\partial}{\partial y_j}$, $j = 1,\dots,\nu$, as the basis vectors for the space $T_z(B)$. To simplify the writing we will index them as Z_α, $\alpha = 1,\dots,m+\nu$. Consider $\bar{\Gamma}^{\gamma}_{\alpha,\beta}$, the Christoffel coefficients of the Levi-Civita connection given by

$$\nabla^{\mathrm{LC}}_{Z_\alpha} Z_\beta = \bar{\Gamma}^{\gamma}_{\alpha\beta} Z_\gamma.$$

We study how these coefficients depend on y. For this purpose remember that

$$\bigl\langle \nabla^{\mathrm{LC}}_{Z_\alpha} Z_\beta, Z_\gamma\bigr\rangle = \frac{1}{2}\bigl\{Z_\alpha\langle Z_\beta, Z_\gamma\rangle + Z_\beta\langle Z_\gamma, Z_\alpha\rangle - Z_\gamma\langle Z_\alpha, Z_\beta\rangle \\ + \bigl\langle [Z_\alpha, Z_\beta], Z_\gamma\bigr\rangle - \bigl\langle [Z_\beta, Z_\gamma], Z_\alpha\bigr\rangle + \bigl\langle [Z_\gamma, Z_\alpha], Z_\beta\bigr\rangle\bigr\},$$

and so

$$\bar{\Gamma}^\delta_{\alpha\beta} = \frac{1}{2}\bigl\{Z_\alpha g_{\beta,\gamma} + Z_\beta g_{\gamma,\alpha} - Z_\gamma g_{\alpha,\beta} \\ + \bigl\langle [Z_\alpha, Z_\beta], Z_\gamma\bigr\rangle - \bigl\langle [Z_\beta, Z_\gamma], Z_\alpha\bigr\rangle + \bigl\langle [Z_\gamma, Z_\alpha], Z_\beta\bigr\rangle\bigr\} g^{\gamma\delta},$$

where $g_{\alpha\beta} = \langle Z_\alpha, Z_\beta\rangle$ and $(g^{\alpha\beta})^{m+\nu}_{\alpha,\beta=1}$ is the inverse of the matrix $(g_{\alpha\beta})^{m+\nu}_{\alpha,\beta=1}$. We look at the dependence on y of each term in the above sum. In the first place, one has to notice that by construction, $g_{\alpha\beta}$ does not depend on y, and is a smooth function of x, thus the same is true for $g^{\alpha\beta}$. Secondly, by (3.2), the vector field Z_α is at most a first order polynomial in y. Hence, if we write $[Z_\alpha, Z_\beta] = A^l_{\alpha\beta} Z_l$, then, by, each coefficient $A^l_{\alpha\beta}$ depends linearly on y. Hence, all $\bar{\Gamma}^\delta_{\alpha\beta}$ are first degree polynomials in y. Now,

$$\begin{aligned} &R^{\mathrm{LC}}(Z_\alpha, Z_\beta)Z_\gamma \\ &\quad = \nabla^{\mathrm{LC}}_{Z_\alpha}\nabla^{\mathrm{LC}}_{Z_\beta} Z_\gamma - \nabla^{\mathrm{LC}}_{Z_\beta}\nabla^{\mathrm{LC}}_{Z_\alpha} Z_\gamma - \nabla^{\mathrm{LC}}_{[Z_\alpha, Z_\beta]} Z_\gamma \\ &\quad = \nabla^{\mathrm{LC}}_{Z_\alpha}\bar{\Gamma}^m_{\beta\gamma} Z_m - \nabla^{\mathrm{LC}}_{Z_\beta}\bar{\Gamma}^m_{\alpha\gamma} Z_m - A^l_{\alpha\beta}\nabla^{\mathrm{LC}}_{Z_l} Z_\gamma \\ &\quad = Z_\alpha\bigl(\bar{\Gamma}^m_{\beta\gamma}\bigr)Z_m + \bar{\Gamma}^m_{\beta\gamma}\nabla^{\mathrm{LC}}_{Z_\alpha} Z_m - Z_\beta\bigl(\bar{\Gamma}^m_{\alpha\gamma}\bigr)Z_m - \bar{\Gamma}^m_{\alpha\gamma}\nabla^{\mathrm{LC}}_{Z_\beta} Z_m - A^l_{\alpha\beta}\bar{\Gamma}^\delta_{l\gamma} Z_\delta \\ &\quad = Z_\alpha\bigl(\bar{\Gamma}^m_{\beta\gamma}\bigr)Z_m + \bar{\Gamma}^m_{\beta\gamma}\bar{\Gamma}^\delta_{\alpha m} Z_\delta - Z_\beta\bigl(\bar{\Gamma}^m_{\alpha\gamma}\bigr)Z_m - \bar{\Gamma}^m_{\alpha\gamma}\bar{\Gamma}^\delta_{\beta m} Z_\delta - A^l_{\alpha\beta}\bar{\Gamma}^\delta_{l\gamma} Z_\delta, \end{aligned} \tag{$*$}$$

which shows that $\langle R^{\mathrm{LC}}(Z_\alpha, Z_\beta)Z_\gamma, Z_\delta\rangle$ is a second order polynomial in y with smooth coefficients in x. Hence, there exists a constant $C_U \geqslant 0$ such that,

$$\bigl|\bigl\langle R^{\mathrm{LC}}_z(Z_\alpha, Z_\beta)Z_\gamma, Z_\delta\bigr\rangle\bigr| \leqslant C_U\bigl(1 + |y|^2\bigr).$$

Since N is compact, this implies (3.10). To deal with the estimates on the derivatives, first observe that again, by compactness, it suffices to do this locally. We do this when the X_i's are Z_α. To carry this out, we apply induction and to pass from step n to $n+1$ we take $n+1$ derivatives in $(*)$ and use the estimates from step n.

(4) On functions, we have $\Delta^{\nabla^{\mathrm{B}}} f = \sum_{j=1}^d E_j E_j f - (\nabla^{\mathrm{B}}_{E_j} E_j) f$, for any local orthonormal basis $\{E_j\colon\ j = 1, \ldots, d\}$ in $T(B)$. Therefore, in order to prove that the Laplacians are the same on functions, it suffices to check it for a particular choice of E_j. Take a basis consisting of either vertical or horizontal vectors. For such a choice we have that $\nabla^{\mathrm{B}}_{E_j} E_j = \nabla^{\mathrm{LC}}_{E_j} E_j$, $j = 1, \ldots, m+\nu$. Indeed, because of

$$\bigl\langle\bigl(\nabla^{\mathrm{B}}_X - \nabla^{\mathrm{LC}}_X\bigr)Y, Z\bigr\rangle = \frac{1}{2}\bigl\{\bigl\langle T^{\mathrm{B}}(X, Y), Z\bigr\rangle - \bigl\langle T^{\mathrm{B}}(Y, Z), X\bigr\rangle + \bigl\langle T^{\mathrm{B}}(Z, X), Y\bigr\rangle\bigr\}, \tag{3.13}$$

and (3.8), we get

$$\big\langle\big(\nabla^{\mathrm{B}}_{E_j}-\nabla^{\mathrm{LC}}_{E_j}\big)E_j,E_k\big\rangle=\big\langle T^{\mathrm{B}}(E_k,E_j),E_j\big\rangle=0\quad\text{for } k=1,\dots,m+\nu.$$

(5) We start by computing the gradient of the function $h(z)=|y|^2$ as

$$\operatorname{grad}_z h=2y. \tag{3.14}$$

If X is a horizontal vector field at z, then $\langle\operatorname{grad}_z h,X\rangle=0$. Indeed, if c is a curve on N starting at x such that $\dot c(0)=\rho_*X$, and $\mathbf{c}$ its lift starting at y, then

$$\langle\operatorname{grad}_z h,X\rangle=Xh=\frac{d}{dt}h\big(\mathbf{c}(t)\big)\Big|_{t=0}=\frac{d}{dt}\big|\mathbf{c}(t)\big|^2\Big|_{t=0}=0$$

since $\mathbf{c}(t)$ is obtained by parallel transport and is of constant length in fibers. If X is a vertical vector at z, then take its identification as a $v\in B_x$ and consider the curve $\mathbf{c}(t)=y+tv$. Then,

$$\langle\operatorname{grad}_z h,X\rangle=Xh=\frac{d}{dt}h\big(\mathbf{c}(t)\big)\Big|_{t=0}=\frac{d}{dt}|y+tv|^2\Big|_{t=0}=2\langle y,v\rangle=2\langle y,X\rangle,$$

hence (3.14). Returning to the Hessian, we have by definition,

$$\big\langle\big(\operatorname{hess}^{\nabla^{\mathrm{B}}}h\big)X,Y\big\rangle=\big\langle\nabla^{\mathrm{B}}_X\operatorname{grad}h,Y\big\rangle\quad\text{and}\quad\big\langle\big(\operatorname{hess}^{\nabla^{\mathrm{LC}}}h\big)X,Y\big\rangle=\big\langle\nabla^{\mathrm{LC}}_X\operatorname{grad}h,Y\big\rangle$$

and by (3.13), the difference between these is

$$\frac12\big\langle T^{\mathrm{B}}(X,Y),\operatorname{grad}h\big\rangle-\frac12\big\langle T^{\mathrm{B}}(Y,\operatorname{grad}h),X\big\rangle+\frac12\big\langle T^{\mathrm{B}}(\operatorname{grad}h,X),Y\big\rangle.$$

Since $\operatorname{grad}h$ is vertical, the last two expressions are zero. Now $\langle T^{\mathrm{B}}(X,Y),\operatorname{grad}h\rangle=\langle R^{\mathrm{V}}(X^{\mathrm{H}},Y^{\mathrm{H}})y,y\rangle$ which is 0 because of the skew-symmetry of R^{V}. Moreover, by $\nabla^{\mathrm{V}}_{X_z^{\mathrm{H}}}y=0$ and (3.6) we get

$$\big\langle\big(\operatorname{hess}^{\nabla^{\mathrm{B}}}h\big)_zX_z,Y_z\big\rangle=\big\langle\nabla^{\mathrm{B}}_{X_z}\operatorname{grad}h,Y_z\big\rangle=2\big\langle\nabla^{\mathrm{B}}_{X_z}y,Y_z\big\rangle=2\big\langle X_z^{\mathrm{V}}(y),Y_z\big\rangle=2\big\langle X_z^{\mathrm{V}},Y_z\big\rangle.$$

The rest is straightforward. □

3.2. Frame bundle and Brownian motion

3.2.1. Orthonormal frame bundle

In this section we first discuss the orthonormal frame bundle over a complete Riemannian manifold endowed with a compatible connection. We then specialize to the vector bundle case and show how the Bismut connection is manifested there, in particular we prove Proposition 3.21, which is the key ingredient in the representation of the Brownian motion given in 3.24.

Assume M is a d-dimensional complete Riemannian manifold endowed with a compatible connection ∇. The orthonormal frame bundle is given by

$$\mathcal{O}(M) = \{\mathfrak{f} = (x, e(x)),\ e(x) = (e_1, \dots, e_d) \text{ orthonormal basis of } T_x(M)\}.$$

$\pi : \mathcal{O}(M) \to M$ stands for the canonical projection given by $\pi((x, e(x))) = x$. Fixing an orthonormal basis in $\mathbb{R}^d$, we can naturally identify $\mathfrak{f}$ with the isometry from $\mathbb{R}^d$ onto $T_{\pi(f)}(M)$ sending the basis from $\mathbb{R}^d$ into the basis $\{e_i\colon\ i = 1, \dots, d\}$ in $T_{\pi(\mathfrak{f})}(M)$. For $O \in O(d)$, we define $R_O : \mathcal{O}(M) \to \mathcal{O}(M)$ by

$$(R_O\mathfrak{f})(\xi) = \mathfrak{f}(O\xi), \quad \mathfrak{f} \in \mathcal{O}(M),\ \xi \in \mathbb{R}^d.$$

For a in the Lie algebra $o(d)$ of $O(d)$, we define $\lambda(a) \in T_\mathfrak{f}\mathcal{O}(M)$ by

$$\lambda(a)_\mathfrak{f} = \frac{d}{dt} R_{\exp(ta)}\mathfrak{f}\bigg|_{t=0}.$$

Set $\mathcal{V}_\mathfrak{f}\mathcal{O}(M) = \{\lambda(a)_\mathfrak{f};\ a \in o(d)\}$, the vertical subspace of $T\mathcal{O}(M)$. This subspace is canonically defined and is independent of any connection.

Given a compatible connection ∇, we construct a complement of the vertical subspace in $T\mathcal{O}(M)$. For the case of the Levi-Civita connection this is described in [8, Chapter 8].

Using the parallel transport with respect to ∇, one can define the horizontal lift to $\mathfrak{f} \in \pi^{-1}(c(a))$ of the smooth curve $t \in [a, b] \to c(t) \in M$ to the smooth curve $t \in [a, b] \to \mathfrak{c}(t) = (c(t), e(c(t))) \in \mathcal{O}(M)$ starting at $\mathfrak{f}$ such that the frame $e(c(t))$ is obtained by parallel transport of $e(c(a))$ along $c \restriction [0, t]$. For a vector $X_x \in T_x(M)$, take a curve c such that $c(0) = x$, $\dot{c}(0) = X_x$, $\mathfrak{c}$ its lift to $\mathfrak{f} \in \pi^{-1}(x)$, and set $\mathfrak{H}_\mathfrak{f} X_x = \dot{\mathfrak{c}}(0)$. This is a well-defined notion, in the sense that the lift of X_x depends only on X_x and not on the curve c. Now define $\mathcal{H}^\nabla_\mathfrak{f}\mathcal{O}(M)$, the subspace of horizontal lifts at $\mathfrak{f}$. The following decomposition holds

$$T_\mathfrak{f}\mathcal{O}(M) = \mathcal{H}^\nabla_\mathfrak{f}\mathcal{O}(M) \oplus \mathcal{V}_\mathfrak{f}\mathcal{O}(M).$$

We denote by $\mathfrak{X}^{\mathrm{H}}$ and $\mathfrak{X}^{\mathrm{V}}$ the horizontal and vertical components of the vector $\mathfrak{X}$ given by this decomposition.

The *canonical vector field* $\mathfrak{E}^\nabla(\xi)_\mathfrak{f}$ is the horizontal lift to $\mathfrak{f}$ of $\mathfrak{f}\xi$. The ∇-*Bochner Laplacian* is

$$\Delta^\nabla_\mathrm{b} = \sum_{j=1}^d \mathfrak{E}^\nabla(e_j)^2,$$

for any orthonormal basis $\{e_j\colon\ j = 1, \dots, d\}$ in $\mathbb{R}^d$. For $f \in C^2(M)$ we have

$$\Delta^\nabla f = \Delta^\nabla_\mathrm{b}(f \circ \pi). \tag{3.15}$$

Orthonormal frame sub-bundle. An interesting situation arises when we take the manifold to be the bundle B with the Bismut connection ∇^{B} constructed in Section 3.1. In this case we define the sub-bundle $\mathcal{O}^{\mathrm{h,v}}(B)$ of $\mathcal{O}(B)$ by

$$\mathcal{O}^{\mathrm{h,v}}(B)=\{\mathfrak{f}=(z,e'(z),e''(z))\},$$

with $e'(z)=(e'_1,\dots,e'_m)$, an orthonormal basis of $T_z^{\mathrm{H}}(B)$ and $e''(z)=(e''_1,\dots,e''_\nu)$ orthonormal basis of $T_z^{\mathrm{V}}(B)$. Recall that m is the dimension of the base space N and ν is the dimension of the fibers of B. Taking orthonormal bases in $\mathbb{R}^m$ and $\mathbb{R}^\nu$ and the corresponding natural orthonormal basis in $\mathbb{R}^d=\mathbb{R}^m\times\mathbb{R}^\nu$, one can interpret $\mathfrak{f}\in\mathcal{O}^{\mathrm{h,v}}(B)$ as an isometry from $\mathbb{R}^d$ into $T_z(B)$ which sends the orthonormal basis of $\mathbb{R}^m$ into an orthonormal basis of $T_z^{\mathrm{H}}(B)$ and the orthonormal basis of $\mathbb{R}^\nu$ into an orthonormal basis of $T_z^{\mathrm{V}}(B)$.

In terms of principal bundles, the structure subgroup of $\mathcal{O}^{\mathrm{h,v}}(B)$ is the subgroup $o(m)\times o(\nu)$ of $o(d)$, and so the vertical subspace $\mathcal{V}_{\mathfrak{f}}\mathcal{O}^{\mathrm{h,v}}(B)$ is the subspace $\{\lambda(a)_{\mathfrak{f}};\ a\in o(m)\times o(\nu)\}$. Due to the fact that the parallel transport with respect to Bismut's connection preserves the horizontal and vertical subspaces $T^{\mathrm{H}}(B)$ and $T^{\mathrm{V}}(B)$, the horizontal subspace $\mathcal{H}_{\mathfrak{f}}^{\nabla^{\mathrm{B}}}\mathcal{O}^{\mathrm{h,v}}(B)$ is the same as the horizontal subspace $\mathcal{H}_{\mathfrak{f}}^{\nabla^{\mathrm{B}}}\mathcal{O}(B)$ at any $\mathfrak{f}\in\mathcal{O}^{\mathrm{h,v}}(B)$. Thus,

$$T_{\mathfrak{f}}\mathcal{O}^{\mathrm{h,v}}(B)=\mathcal{H}_{\mathfrak{f}}^{\nabla^{\mathrm{B}}}\mathcal{O}(B)\oplus\mathcal{V}_{\mathfrak{f}}\mathcal{O}^{\mathrm{h,v}}(B).$$

We denote $\mathfrak{X}^{\mathrm{H}}$, $\mathfrak{X}^{\mathrm{V}}$ the horizontal and vertical components of the vector $\mathfrak{X}$ given by this decomposition.

The canonical vector field $\mathfrak{E}^{\nabla}(\xi)$ is given at $\mathfrak{f}$ by the horizontal lift to $\mathfrak{f}\in\mathcal{O}^{\mathrm{h,v}}(B)$ of $\mathfrak{f}\xi$. The $o(m)\times o(\nu)$*-valued canonical* 1*-form* $\omega: T\mathcal{O}^{\mathrm{h,v}}(B)\to o(m)\times o(\nu)$ is given by the prescription

$$\omega(\mathfrak{X}_{\mathfrak{f}})=a,$$

where $a\in o(m)\times o(\nu)$ with $\lambda(a)_{\mathfrak{f}}=\mathfrak{X}_{\mathfrak{f}}^{\mathrm{V}}$. Otherwise stated, this is completely characterized by the relations

$$\omega(\lambda(a)_{\mathfrak{f}})=a\quad\text{and}\quad\omega(\mathfrak{E}^{\nabla}(\xi)_{\mathfrak{f}})=0\quad\text{for } a\in o(m)\times o(\nu),\ \xi\in\mathbb{R}^m.$$

The $\mathbb{R}^d$*-valued canonical* 1*-form* $\theta: T\mathcal{O}^{\mathrm{h,v}}(B)\to\mathbb{R}^d$ is given by

$$\theta(\mathfrak{X}_{\mathfrak{f}})=\mathfrak{f}^{-1}(\pi_*\mathfrak{X}_{\mathfrak{f}}).$$

We have for any vector $\mathfrak{X}\in T_{\mathfrak{f}}\mathcal{O}^{\mathrm{h,v}}(B)$

$$\mathfrak{X}=\mathfrak{E}^{\nabla}(\theta(\mathfrak{X}))_{\mathfrak{f}}+\lambda(\omega(\mathfrak{X}))_{\mathfrak{f}}. \tag{3.16}$$

For $\mathfrak{X},\mathfrak{Y}\in T\mathcal{O}^{\mathrm{h,v}}(B)$ the *torsion form* is defined by

$$\mathcal{T}(\mathfrak{X},\mathfrak{Y})=\mathfrak{X}^{\mathrm{H}}\theta(\mathfrak{Y}^{\mathrm{H}})-\mathfrak{Y}^{\mathrm{H}}\theta(\mathfrak{X}^{\mathrm{H}})-\theta([\mathfrak{X}^{\mathrm{H}},\mathfrak{Y}^{\mathrm{H}}]) \tag{3.17}$$

and the *curvature form* by

$$\mathcal{R}(\mathfrak{X},\mathfrak{Y}) = -\omega\big(\big[\mathfrak{X}^{\mathrm{H}},\mathfrak{Y}^{\mathrm{H}}\big]\big). \tag{3.18}$$

Note that these definitions depend only on the values of $\mathfrak{X},\mathfrak{Y}$ at $\mathfrak{f}$.

The relationship with the usual torsion and curvature tensors on B is the following. If $\mathfrak{X},\mathfrak{Y} \in T_{\mathfrak{f}}\mathcal{O}^{\mathrm{h,v}}(B)$, and $Z \in T_{\pi(\mathfrak{f})}(B)$, then

$$T^{\mathrm{B}}(\pi_*\mathfrak{X},\pi_*\mathfrak{Y}) = \mathfrak{f}\big(\mathcal{T}(\mathfrak{X},\mathfrak{Y})\big) \tag{3.19}$$

where T^{B} is the torsion of the connection ∇^{B},

$$R^{\mathrm{B}}(\pi_*\mathfrak{X},\pi_*\mathfrak{Y})Z = \mathfrak{f}\big(\mathcal{R}(\mathfrak{X},\mathfrak{Y})\mathfrak{f}^{-1}Z\big) \tag{3.20}$$

where R^{B} is the curvature of the connection ∇^{B}. For a proof of these, see [4, Chapter 3, Theorem 5.1].

Proposition 3.21. *If $\xi \in \mathbb{R}^m$, $\eta \in \mathbb{R}^\nu$, then* $[\mathfrak{E}^{\nabla^{\mathrm{B}}}(\xi),\mathfrak{E}^{\nabla^{\mathrm{B}}}(\eta)] = 0$.

Proof. Using (3.19) (3.20), (3.8) and (3.9), one gets that $\mathcal{T}(\mathfrak{E}^{\nabla^{\mathrm{B}}}(\xi),\mathfrak{E}^{\nabla^{\mathrm{B}}}(\eta)) = 0$ and $\mathcal{R}(\mathfrak{E}^{\nabla^{\mathrm{B}}}(\xi),\mathfrak{E}^{\nabla^{\mathrm{B}}}(\eta)) = 0$. By (3.17), (3.18) and the fact that $\theta(\mathfrak{E}^{\mathrm{B}}(\xi)) = \xi$, $\theta(\mathfrak{E}^{\mathrm{B}}(\eta)) = \eta$, one obtains

$$\theta\big(\big[\mathfrak{E}^{\nabla^{\mathrm{B}}}(\xi),\mathfrak{E}^{\nabla^{\mathrm{B}}}(\eta)\big]\big) = -\mathcal{T}\big(\mathfrak{E}^{\nabla^{\mathrm{B}}}(\xi),\mathfrak{E}^{\nabla^{\mathrm{B}}}(\eta)\big) = 0.$$

In the same way,

$$\omega\big(\big[\mathfrak{E}^{\nabla^{\mathrm{B}}}(\xi),\mathfrak{E}^{\nabla^{\mathrm{B}}}(\eta)\big]\big) = -\mathcal{R}\big(\mathfrak{E}^{\nabla^{\mathrm{B}}}(\xi),\mathfrak{E}^{\nabla^{\mathrm{B}}}(\eta)\big) = 0,$$

and the conclusion follows from (3.16). □

3.3. Parallel transport and Brownian motion

In this section we discuss parallel transport notions along Brownian paths. The first case is for a complete Riemannian manifold with a compatible connection which has the same Laplacian as the Levi-Civita connection. We specialize this to the case of a vector bundle endowed with the Bismut connection and prove the representation of the Brownian motion given in Proposition 3.24, as the parallel transport of the vertical motion along the horizontal motion.

The general setup is the following. The manifold M is a complete Riemannian manifold with the growth condition on the Ricci curvature of the Levi-Civita connection

$$\big\langle \mathrm{Ric}_x^{\mathrm{LC}} X_x, X_x\big\rangle \geqslant -C\big(1+\mathrm{dist}(x,o)^2\big)|X_x|^2 \quad \text{for } X_x \in T_x(M), \tag{3.22}$$

where o is a fixed reference point and $C > 0$ is a constant. We choose a connection ∇ which is compatible with the metric on M. Our main assumption about ∇ is that on functions, the Laplacian Δ^{∇} is the same as the Laplacian Δ^{LC} for the Levi-Civita connection.

When the connection ∇ is the Levi-Civita connection, in Section 1 we defined $\mathbf{w} \in \mathcal{P}(\mathbb{R}^d) \to \mathfrak{p}(\cdot, \mathfrak{f}, \mathbf{w}) \in \mathcal{P}(\mathcal{O}(M))$, and its projection $p(t, x, \mathbf{w}) = \pi \mathfrak{p}(t, \mathfrak{f}, \mathbf{w})$, where $\mathfrak{f} \in \pi^{-1}x$. The distribution of $\mathbf{w} \to p(\cdot, x, \mathbf{w})$ is the Wiener measure on M and the parallel transport along Brownian paths is given by

$$\tau_{p(\cdot,x,\mathbf{w})\restriction[0,t]} = \mathfrak{p}(t, \mathfrak{f}, \mathbf{w})\mathfrak{f}^{-1}.$$

Given the connection ∇ on M, we can define a map $\mathbf{w} \in \mathcal{P}(\mathbb{R}^d) \to \mathfrak{p}^{\nabla}(\cdot, \mathfrak{f}, \mathbf{w}) \in \mathcal{P}(\mathcal{O}(M))$ so that for any piecewise smooth path $\mathbf{w} \in \mathcal{P}(\mathbb{R}^d)$,

$$\dot{\mathfrak{p}}^{\nabla}(t, \mathfrak{f}, \mathbf{w}) = \mathfrak{E}^{\nabla}\big(\dot{\mathbf{w}}(t)\big)_{\mathfrak{p}^{\nabla}(t,\mathfrak{f},\mathbf{w})} \quad \text{with } \mathfrak{p}^{\nabla}(0, \mathfrak{f}, \mathbf{w}) = \mathfrak{f}. \tag{3.23}$$

As in the Levi-Civita case, one can extend $\mathfrak{p}^{\nabla}(\cdot, \mathfrak{f}, \mathbf{w})$ to almost any $\mathcal{W}_d$ path $\mathbf{w} \in \mathcal{P}(\mathbb{R}^d)$. The only concern is the explosion.

We show that almost sure there no explosion. To this end, first define $p^{\nabla}(t, x, \mathbf{w}) = \pi \mathfrak{p}^{\nabla}(t, \mathfrak{f}, \mathbf{w})$ if $x = \pi(\mathfrak{f})$. The distribution of the map $\mathbf{w} \to \mathfrak{p}^{\nabla}(\cdot, \mathfrak{f}, \mathbf{w})$ is the solution to the martingale problem of $\Delta_{\mathrm{b}}^{\nabla}$ starting at $\mathfrak{f}$. From (3.15), one deduces that the distribution of the map $\mathbf{w} \to p^{\nabla}(\cdot, x, \mathbf{w})$ is the solution to the martingale problem starting at x associated to the operator Δ^{∇}. Since $\Delta^{\nabla} = \Delta^{\mathrm{LC}}$, it follows, from the uniqueness of the solution to the martingale problem and the non-explosion of $p(\cdot, x, \mathbf{w})$, that the distribution of $\mathbf{w} \to p^{\nabla}(\cdot, x, \mathbf{w})$ is μ_x^M, hence, $p^{\nabla}(\cdot, x, \mathbf{w})$ does not explode.

Therefore, $\mathfrak{p}^{\nabla}(t, \mathfrak{f}, \mathbf{w})$ is well defined for all $t \geqslant 0$. We define the parallel transport along the path $p^{\nabla}(\cdot, x, \mathbf{w})$ by

$$\tau^{\nabla}_{p^{\nabla}(\cdot,x,\mathbf{w})\restriction[0,t]} = \mathfrak{p}^{\nabla}(t, \mathfrak{f}, \mathbf{w})\mathfrak{f}^{-1}.$$

Since the distribution of $\mathbf{w} \to p^{\nabla}(\cdot, x, \mathbf{w})$ is μ_x^M, this shows that the parallel transport is well defined along μ_x^M almost any path $p \in \mathcal{P}(M)$.

For the case of a vector bundle B endowed with the Bismut connection ∇^{B}, we point out that for any $\mathfrak{f} \in \mathcal{O}^{\mathrm{h,v}}(B)$, and any piecewise smooth $\mathbf{w} \in \mathcal{P}(\mathbb{R}^d)$, the uniqueness of ordinary differential equations guarantees that $\mathfrak{p}^{\nabla^{\mathrm{B}}}(t, \mathfrak{f}, \mathbf{w})$ lives in the sub-bundle $\mathcal{O}^{\mathrm{h,v}}(B)$. Hence, $\mathfrak{p}^{\nabla^{\mathrm{B}}}(t, \mathfrak{f}, \mathbf{w})$ lives in $\mathcal{O}^{\mathrm{h,v}}(B)$ for any $\mathfrak{f} \in \mathcal{O}^{\mathrm{h,v}}(B)$, $t \geqslant 0$ and $\mathcal{W}_d$-almost any path $\mathbf{w} \in \mathcal{P}(\mathbb{R}^d)$.

For a given $\mathfrak{f} \in \mathcal{O}^{\mathrm{h,v}}(B)$ and a piecewise smooth path $\mathbf{w} = (\mathbf{w}', \mathbf{w}'') \in \mathcal{P}(\mathbb{R}^m) \times \mathcal{P}(\mathbb{R}^{\nu})$, where m is the dimension of the base manifold N and ν is the dimension of the fiber of B, we set

$$\mathfrak{p}^{\mathrm{h}}(\cdot, \mathfrak{f}, \mathbf{w}') = \mathfrak{p}^{\nabla^{\mathrm{B}}}\big(\cdot, \mathfrak{f}, (\mathbf{w}', 0)\big) \quad \text{and} \quad \mathfrak{p}^{\mathrm{v}}(\cdot, \mathfrak{f}, \mathbf{w}'') = \mathfrak{p}^{\nabla^{\mathrm{B}}}\big(\cdot, \mathfrak{f}, (0, \mathbf{w}'')\big).$$

Notice that $\mathfrak{p}^{\mathrm{h}}(\cdot, \mathfrak{f}, \mathbf{w}')$ is the solution to a differential equation involving only the canonical vector fields $\mathfrak{E}^{\nabla^{\mathrm{B}}}(\xi)$, $\xi \in \mathbb{R}^m$ and $\mathfrak{p}^{\mathrm{v}}(\cdot, \mathfrak{f}, \mathbf{w}'')$ is a solution to a differential equation involving only the canonical vector fields $\mathfrak{E}^{\nabla^{\mathrm{B}}}(\eta)$, $\eta \in \mathbb{R}^{\nu}$.

Now, invoking Proposition 3.21, and the non-explosion of $\mathfrak{p}^{\nabla^{\mathrm{B}}}(\cdot, \mathfrak{f}, \mathbf{w})$, one can prove that ([8, Section 3.4.2] for a more detailed discussion)

$$\mathfrak{p}^{\mathrm{h}}(\cdot, \mathfrak{f}, \mathbf{w}_n') \to \mathfrak{p}^{\mathrm{h}}(\cdot, \mathfrak{f}, \mathbf{w}') \quad \text{and} \quad \mathfrak{p}^{\mathrm{v}}(\cdot, \mathfrak{f}, \mathbf{w}_n'') \to \mathfrak{p}^{\mathrm{h}}(\cdot, \mathfrak{f}, \mathbf{w}'')$$

and

$$\mathfrak{p}^{\nabla^{\mathrm{B}}}\big(t, \mathfrak{f}, (\mathbf{w}', \mathbf{w}'')\big) = \mathfrak{p}^{\mathrm{h}}\big(t, \mathfrak{p}^{\mathrm{v}}(t, \mathfrak{f}, \mathbf{w}''), \mathbf{w}'\big) = \mathfrak{p}^{\mathrm{v}}\big(t, \mathfrak{p}^{\mathrm{h}}(t, \mathfrak{f}, \mathbf{w}'), \mathbf{w}''\big).$$

If $\mathfrak{f} \in \pi^{-1}(z) \subset \mathcal{O}^{\mathrm{h,v}}(B)$, we set

$$p^{\nabla^{\mathrm{B}}}(\cdot, z, \mathbf{w}) = \pi \mathfrak{p}^{\nabla^{\mathrm{B}}}(\cdot, \mathfrak{f}, \mathbf{w}), \qquad p^{\mathrm{h}}(\cdot, z, \mathbf{w}') = \pi \mathfrak{p}^{\mathrm{h}}(\cdot, \mathfrak{f}, \mathbf{w}'),$$
$$p^{\mathrm{v}}(\cdot, z, \mathbf{w}'') = \pi \mathfrak{p}^{\mathrm{v}}(\cdot, \mathfrak{f}, \mathbf{w}'').$$

We say that $p^{\nabla^{\mathrm{B}}}(t, z, \mathbf{w})$ is the Wiener process or the Brownian motion on B starting at z, while $p^{\mathrm{h}}(t, z, \mathbf{w}')$ and $p^{\mathrm{v}}(t, z, \mathbf{w}'')$ are the horizontal and the vertical Brownian motions on B starting at z.

Next we want to identify the horizontal and vertical motions. Given the horizontal and vertical motions, we can define the parallel transport along them. For $\mathfrak{f} \in \pi^{-1}(z) \subset \mathcal{O}^{\mathrm{h,v}}(B)$, the natural choice is the following

$$\tau^{\mathrm{B}}_{p^{\mathrm{h}}(\cdot,z,\mathbf{w}')\restriction[0,t]} = \mathfrak{p}^{\mathrm{h}}(t, \mathfrak{f}, \mathbf{w}')\mathfrak{f}^{-1} \quad \text{and} \quad \tau^{\mathrm{B}}_{p^{\mathrm{v}}(\cdot,z,\mathbf{w}'')\restriction[0,t]} = \mathfrak{p}^{\mathrm{v}}(t, \mathfrak{f}, \mathbf{w}'')\mathfrak{f}^{-1}.$$

As the name suggests, the horizontal Brownian motion $p^{\mathrm{h}}(\cdot, z, \mathbf{w}')$ should be the Brownian motion on the base manifold N if $z \in N$. And this is indeed the case since for a piecewise smooth $\mathbf{w}' \in \mathcal{P}(\mathbb{R}^m)$,

$$\dot{p}^{\mathrm{h}}(t, x, \mathbf{w}') = \tau^{\mathrm{B}}_{p^{\mathrm{h}}(\cdot,x,\mathbf{w}')\restriction[0,t]}\mathfrak{f}\dot{\mathbf{w}}'(t) \quad \text{with } p^{\mathrm{h}}(0, x, \mathbf{w}') = x,$$
$$\dot{p}_N(t, x, \mathbf{w}') = \tau_{p_N(\cdot,x,\mathbf{w}')\restriction[0,t]}\mathfrak{f}\dot{\mathbf{w}}'(t) \quad \text{with } p_N(0, x, \mathbf{w}') = x,$$

where $p_N(\cdot, x, \mathbf{w}')$ is the corresponding map constructed for the manifold N and τ stands for the parallel transport with respect to the Levi-Civita connection there. Because of the definition of the parallel transport with respect to the Bismut connection and uniqueness of the solution to ODE's, it follows that $p^{\mathrm{h}}(t, x, \mathbf{w}_n') = p_N(t, x, \mathbf{w}_n')$ and $p^{\mathrm{h}}(t, x, \mathbf{w}') = p_N(t, x, \mathbf{w}')$ for $\mathcal{W}_m$-almost any path $\mathbf{w}'$.

A similar argument works for the vertical direction. Namely, given a point $z = (x, y)$, identify the fiber B_x with $\mathbb{R}^\nu$ using $\mathfrak{f} \in \pi^{-1}(z) \subset \mathcal{O}^{\mathrm{h,v}}(B)$. Then, one can prove that $p^{\mathrm{v}}(t, z, \mathbf{w}'') = y + \mathbf{w}''$ for $\mathcal{W}_\nu$-almost any path $\mathbf{w}''$.

The next step is to elucidate what happens with the horizontal motion which does not start at a point on the base manifold N. For this purpose, for a given vector $X^{\mathrm{V}} \in B_x$ we define the *vertical parallel transport* of X^{V} along the Brownian motion on the base by

$$\tau^{\mathrm{V}}_{p_N(\cdot,x,\mathbf{w}')\restriction[0,t]}X^{\mathrm{V}} = \tau^{\mathrm{B}}_{p^{\mathrm{h}}(\cdot,x,\mathbf{w}')\restriction[0,t]}X^{\mathrm{V}}.$$

Notice that this is consistent with the definition for the case $\mathbf{w}'$ is a piecewise smooth path. At this point one can check that

$$p^{\mathrm{h}}(t, z, \mathbf{w}') = \tau^{\mathrm{V}}_{p_N(\cdot, x, \mathbf{w}')\restriction[0,t]} y \quad \text{for } z = (x, y),$$

where y is interpreted here as a vertical vector. This can be seen by first checking it for piecewise smooth paths and, by approximation, for almost any path. This representation of $p^{\mathrm{h}}(t, z, \mathbf{w}')$ says that *the horizontal Brownian paths starting at z are obtained by vertical parallel transport of y along the Brownian paths on N.*

We summarize in the following proposition.

Proposition 3.24. *The distribution of $\mathbf{w} \to p^{\nabla^{\mathrm{B}}}(\cdot, z, \mathbf{w})$ is μ^B_z. Moreover, using $\mathfrak{f} \in \pi^{-1}(z) \subset \mathcal{O}^{\mathrm{h,v}}(B)$ to identify B_x with $\mathbb{R}^\nu$, then we have*

$$p^{\nabla^{\mathrm{B}}}(t, z, \mathbf{w}) = \big(p_N(\cdot, x, \mathbf{w}'), \tau^{\mathrm{V}}_{p_N(\cdot, x, \mathbf{w}')\restriction[0,t]}\big(y + \mathbf{w}''(t)\big)\big).$$

3.4. Heat kernel estimates

In this section we turn to the analysis of the heat kernel of operator $\square^\alpha$. In the non-degenerate case, the main idea was to show that the heat kernel of $\square^\alpha$ is close to the heat kernel of an harmonic oscillator operator. The situation here is complicated by the fact that the geometry near the critical submanifolds is not flat as it was in the non-degenerate case. Thus, the flat Euclidean space, where the harmonic oscillator was defined, must be replaced here by the non-compact bundles B_i near each N_i.

Recall that

$$\square^\alpha = -\Delta + \alpha^2 |\operatorname{grad} h|^2 - \alpha \Delta h + 2\alpha D^* \operatorname{hess} h - D^* R, \tag{3.25}$$

where R is the curvature of the Levi-Civita connection on M. The first step is to show that the heat kernel $\mathbf{p}^\alpha_k(t, z_1, z_2)$ decays exponentially as $\alpha \to \infty$ when z_1 or z_2 is away from the critical set.

Now on each $\bigwedge(B_i)$ we define the operator

$$\square^\alpha_i = -\Delta^{\mathrm{B}} + \alpha^2 |y|^2 - \alpha\big(\nu_i^+ - \nu_i^-\big) + 2\alpha D^* \operatorname{hess} \bar{h} - D^* R^{\mathrm{H}} \tag{3.26}$$

where Δ^{B} is the Laplacian on forms on B_i with respect to the Bismut connection ∇^{B}, R^{H} is (see (3.9)) the curvature of the Levi-Civita connection on N_i extended to $T^{\mathrm{H}}(B_i)$ and

$$\bar{h}(z) = \frac{1}{2}\big(|y^+|^2 - |y^-|^2\big), \tag{3.27}$$

$\mathbf{p}^{i,\alpha}_k(t, z_1, z_2)$ will be the heat kernel of $\frac{1}{2}\square^\alpha_i$ on k-forms. For z_1, z_2 near N_i, our main goal is to compare $\mathbf{p}^\alpha_k(t, z_1, z_2)$ with $\mathbf{p}^{i,\alpha}_k(t, z_1, z_2)$.

Set $B(N_i, r) = \{z = (x, y) \in B_i\colon |y| < r\} \subset B_i$, the ball bundle of radius r around the critical submanifold N_i.

In this section, the manifold we are working on is either the original manifold M, or one of the bundles B_i, and the function h considered is either the original function on M or the function $\bar{h}$ defined by (3.27). To deal with both situations at the same time, we introduce and study operators of the form

$$\mathcal{L}^\alpha = -\Delta^\nabla + \alpha^2 |\operatorname{grad} h|^2 - \alpha \Delta h + 2\alpha D^* \operatorname{hess} h + \sum_{j=1}^d A_1(E_j)\nabla_{E_j} + A_2, \tag{3.28}$$

where the obvious components satisfy:

(1) The connection ∇:
  (a) compatibility with the metric of the manifold;
  (b) ∇-Laplacian on functions is the same as the Levi-Civita Laplacian;
  (c) the Hessian of the function h with respect to the connection ∇ (cf. Definition A.2) is the same as the Hessian with respect to the Levi-Civita connection.
(2) In the notation introduced in (3) of Definition 1.4

$$A_1 = D^* S \quad \text{and} \quad A_2 = D^* T$$

where S is an odd tensor and T is an even tensor with the condition that $(A_1)_z(X_z)$ is skew-symmetric for $z \in M$, $X_z \in T_z(M)$.

Denote by $\mathbf{p}_k^{\mathcal{L}^\alpha}(t, z_1, z_2) : \bigwedge_{z_2}^k(M) \to \bigwedge_{z_1}^k(M)$ the heat kernel of the operator $\frac{1}{2}\mathcal{L}^\alpha$ acting on k-forms. Using Proposition C.5 and Lemma B.24, one knows that there is a constant $C > 0$ depending on A_1 and A_2 so that for any $t > 0$, $\alpha > 0$,

$$\left\| \mathbf{p}_k^{\mathcal{L}^\alpha}(t, z_1, z_2) \right\| \leqslant e^{Ct} \mathbb{E}^{\mathcal{W}_d}\left[\exp\left(\int_0^t H^\alpha\big(p(v, z_1, \mathbf{w})\big)\, dv \right) \delta_{z_2}\big(p(t, z_1, \mathbf{w})\big) \right], \tag{3.29}$$

where

$$H^\alpha(y) = -\frac{\alpha^2}{2} \left|\operatorname{grad} h(y)\right|^2 + \frac{\alpha}{2} \Delta h(y) + \alpha f(y)$$

and f is a smooth function satisfying

$$\begin{aligned} f(z) &= \nu_i^- \quad \text{for } z \text{ close to the critical submanifold } N_i, \\ -\operatorname{hess}_z h|_{\bigwedge_z^k(M)} &\leqslant f(z) \operatorname{Id}_{\bigwedge_z^k(M)} \quad \text{for any } z \in M,\ k = 1, \dots, d. \end{aligned} \tag{3.30}$$

If the manifold is B_i, we can choose this function to be $f(z) = \nu_i^-$.

Set

$$\mathcal{P}^{\alpha}(t, z_1, z_2) = \mathbb{E}^{\mathcal{W}_d}\left[\exp\left(\int_0^t H^{\alpha}\big(p(v, z_1, \mathbf{w})\big)\right)\delta_{z_2}\big(p(t, z_1, \mathbf{w})\big)\right]. \tag{3.31}$$

The first result of this section is the following.

Theorem 3.32. *For small enough $r > 0$, there exist $C_1 = C_1(r) > 0$, $C_2 = C_2(r) > 0$ and $C_3 > 0$, such that for $t > 0$ and $\alpha \geqslant C_1 e^{C_1 t}$,*

$$\mathcal{P}^{\alpha}(t, z_1, z_2) \leqslant t^{-d/2}\alpha^{C_3}\big(\exp\big(-C_2 t\alpha^2\big) + \exp(-C_2\alpha)\big), \quad \textit{for } (z_1, z_2) \in \Lambda_r \times M.$$

Proof. Because the proof is just a repetition of the argument we gave in the non-degenerate case, we will only point out the main steps. Using integration by parts on pathspace and Hölder's inequality, one can show that there is a polynomial in $P(\alpha, t)$, such that for $0 < \eta \leqslant 1$, for $(z_1, z_2) \in \Lambda_r \times M$

$$\mathcal{P}^{\alpha}(t, z_1, z_2) \leqslant \frac{P(\alpha, t)}{\eta^d t^{d/2}}\left\{\mathbb{E}^{\mu_{z_1}^M}\left[\exp\left(\int_0^t (1+\eta)H^{\alpha}\big(p(v)\big)\,dv\right)\right]\right\}^{1/(1+\eta)}.$$

Next, we need to estimate

$$q^{\alpha,\eta}(t, z) = \mathbb{E}^{\mu_z^M}\left[\exp\left(\int_0^t (1+\eta)H^{\alpha}\big(p(v)\big)\,dv\right)\right]. \tag{3.33}$$

As in the non-degenerate case (cf. Theorem 2.9 and its proof), we use the Markov property to segregate the contribution from the paths which stay away from the critical set from that of paths which stay close to the critical set. The only difference from the non-degenerate case comes from the contribution of the paths staying inside a neighborhood of a critical submanifold N_i. We replace here the balls around the critical points used in the proof of Theorem 2.9 with the ball bundles $B(N_i, r)$, and the corresponding function u there with $u : [0, \infty) \times B(N_i, r) \to \mathbb{R}$ given by

$$u_{\eta}^{\alpha}(s, z) = \int \exp\left(\int_0^{\zeta(p)\wedge s} (1+\eta)H^{\alpha}\big(p(v)\big)\,dv\right)\mu_z^M(dp),$$

with ζ the exit time from the ball bundle $B(N_i, r)$. Since when r is small enough, for any $z \in B(N_i, r)$, $h(z) = \frac{1}{2}(|y^+|^2 - |y^-|^2)$, we get (cf. (3.12))

$$H^{\alpha}\big(p(v)\big) = \left(-\frac{\alpha^2}{2}\big|p^{\mathrm{V}}(v)\big|^2 + \frac{\alpha\nu_i}{2}\right) \quad \text{for } v \leqslant \zeta(p).$$

At this point, we identify the fiber $(B_i)_{x_1}$ with $\mathbb{R}^{\nu_i}$, and then use the representation of the Brownian motion on the bundle B_i given by Proposition 3.24, which, because parallel transport is an isometry in fibers, yields

$$u_\eta^\alpha(s,z) = u_\eta^\alpha(s,y) = \mathbb{E}^{\mathcal{W}_{\nu_i}}\left[\exp\left(\int_0^{\zeta(\mathbf{w}'')\wedge s}(1+\eta)\left(-\frac{\alpha^2}{2}\left|y+\mathbf{w}''(v)\right|^2+\frac{\alpha\nu_i}{2}\right)dv\right)\right],$$

where ζ becomes the exit time from the ball of radius r in $\mathbb{R}^{\nu_i}$. From here, the argument runs the way it did in the non-degenerate case. □

This result gives the exponential decay of $\mathcal{P}$ in α when one of the points z_1 or z_2 is away from the critical set.

When z_1 and z_2 are both close to one of the critical submanifold N_i, set

$$\bar{\mathcal{P}}_i^\alpha(t,z_1,z_2) = \mathbb{E}^{\mathcal{W}_d}\left[\exp\left(-\frac{\alpha^2}{2}\int_0^t\left|y_1+\mathbf{w}''(s)\right|^2ds+\frac{\alpha t\nu_i}{2}\right)\delta_{z_2}\big(p(t,z_1,\mathbf{w})\big)\right] \tag{3.34}$$

where $\mathbf{w}''$ stands for the Brownian motion starting at 0 on the fiber $(B_i)_{x_1}$ identified with $\mathbb{R}^{\nu_i}$. Then we have the following result.

Theorem 3.35. *For small enough $r>0$, there exist $C_1=C_1(r)>0$, $C_2=C_2(r)>0$, $C_3>0$, such that for $t>0$, $\alpha\geqslant C_1e^{C_1t}$, and $z_1,z_2\in B(N_i,r)$ (cf. (3.31)),*

$$\left|\mathcal{P}^\alpha(t,z_1,z_2)-\bar{\mathcal{P}}_i^\alpha(t,z_1,z_2)\right|\leqslant t^{-d/2}\alpha^{C_3}\exp\left(-\alpha C_2-\alpha\frac{|y_1|^2}{2}\right). \tag{3.36}$$

Proof. The proof is based on the same idea outlined in Section 2.3. Using integration by parts on the path space (cf. (B.26)), write

$$\mathcal{P}^\alpha(t,z_1,z_2) = \mathbb{E}^{\mathcal{W}_d}\big[\Phi^\alpha(t,z_1,z_2,\mathbf{w})\big],$$

where $\Phi^\alpha(t,z_1,z_2,\cdot)$ is $\mathrm{Hom}(\bigwedge_{z_2}(M),\bigwedge_{z_1}(M))$-valued Wiener functional with the property that there exist a measurable map $(t,\mathbf{w})\in[0,\infty)\times\mathcal{P}(\mathbb{R}^d)\to R_{z_1,z_2}^\alpha(t,\mathbf{w})\in\mathbb{R}$ and a polynomial $P(t,\alpha)$ so that

$$\left\|\Phi^\alpha(t,z_1,z_2,\mathbf{w})\right\|_{H.S}\leqslant R_{z_1,z_2}^\alpha(t,\mathbf{w})\exp\left(\int_0^t H^\alpha\big(p(v,x,\mathbf{w})\big)\,dv\right)\quad\text{and}$$

$$\left\|R_{z_1,z_2}^\alpha(t,\mathbf{w})\right\|_{L^s(\mathcal{W}_d)}\leqslant\frac{s^dP_s(t,\alpha)}{t^{d/2}}\quad\text{for all } s>1,\ t>0,\ z_1,z_2\in M.$$

Take ζ the first exit time from $B(N_i,2r)$ and set

$$I^{\mathrm{ext}}_{z_1,z_2}(t,\alpha)=\mathbb{E}^{\mathcal{W}_d}\big[\Phi^\alpha(t,z_1,z_2,\mathbf{w}),\zeta\big(p(\cdot,z_1,\mathbf{w})\big)<t\big],$$
$$I^{\mathrm{int}}_{z_1,z_2}(t,\alpha)=\mathbb{E}^{\mathcal{W}_d}\big[\Phi^\alpha(t,z_1,z_2,\mathbf{w}),\zeta\big(p(\cdot,z_1,\mathbf{w})\big)\geqslant t\big].$$

As in Theorem 2.27, there are constants $C_1=C_1(r), C_2=C_2(r)>0$ so that for any $t>0, \alpha>C_2e^{C_2t}$,

$$I^{\mathrm{ext}}_{z_1,z_2}(t,\alpha)\leqslant t^{-d/2}\exp\left(-C_1\alpha-\frac{\alpha|y_1|^2}{2}\right). \tag{$*$}$$

The proof of this is basically the same as the one for Theorem 2.27, the only difference is that we replace the ball $B(c,2r)$ there with $B(N_i,2r)$ here and the corresponding function u there has to be replaced here with

$$u^\alpha_\eta(s,z)=\mathbb{E}^{\mu^M_z}\left[\exp\left(\int_0^{\zeta(p)}(1+\eta)H^\alpha\big(p(v)\big)\,dv\right),\zeta(p)<s\right].$$

Identifying the fiber $(B_i)_x$ with $\mathbb{R}^{\nu_i}$ and using the representation of the Brownian motion on the bundle B_i given by Proposition 3.24, and the fact that the parallel transport is an isometry on fibers, we get that $u^\alpha_\eta(s,z)=u^\alpha_\eta(s,y)$, with

$$u^\alpha_\eta(s,y)=\mathbb{E}^{\mathcal{W}_{\nu_i}}\left[\exp\left(\int_0^{\zeta(\mathbf{w}'')}(1+\eta)\left(-\frac{\alpha^2}{2}\big|y+\mathbf{w}''(v)\big|^2+\frac{\alpha\nu_i}{2}\right)dv\right),\zeta(\mathbf{w}'')<s\right],$$

where now ζ is the exit time from the ball of radius $2r$ in $\mathbb{R}^{\nu_i}$. From here, the rest goes as in the proof of Theorem 2.27.

Now, we turn to the integral $I^{\mathrm{int}}_z(\alpha)$. The main point is that this integral can be computed also as the integral coming from the quantity $\bar{\mathcal{P}}^\alpha_i$ (cf. (3.34)). One can see this by replacing the manifold M with the manifold B_i, the function h with the function $\bar{h}$, and write $\bar{\mathcal{P}}^\alpha$ as the sum of $\bar{I}^{\mathrm{int}}_{z_1,z_2}(t,\alpha)$ and $\bar{I}^{\mathrm{ext}}_{z_1,z_2}(t,\alpha)$. The same exponential decay holds for $\bar{I}^{\mathrm{ext}}_{z_1,z_2}(t,\alpha)$ as in $(*)$, while the factors $\Phi^\alpha(t,z_1,z_2,\mathbf{w})$'s in the expressions of $I^{\mathrm{int}}_{z_1,z_2}(t,\alpha)$ and $\bar{I}^{\mathrm{int}}_{z_1,z_2}(t,\alpha)$, are equal (cf. (B.19)). The rest follows. □

Next we want to estimate $\bar{\mathcal{P}}^\alpha_i(t,z_1,z_2)$.

Proposition 3.37. *If $\mathbf{p}_{N_i}(t,x_1,x_2)$ is the heat kernel of the Laplacian acting on functions on the submanifold N_i, then*

$$\bar{\mathcal{P}}^\alpha_i(t,z_1,z_2)\leqslant\mathbf{p}_{N_i}(t,x_1,x_2)\left(\frac{\alpha}{\pi(1-e^{-2t\alpha})}\right)^{\nu_i/2}$$
$$\times\exp\left(-\frac{\alpha\coth(t\alpha)}{2}\left(|y_1|^2-\frac{|y_1||y_2|}{\cosh(t\alpha)}+|y_2|^2\right)\right). \tag{3.38}$$

In particular, there is a constant $C(t, N_i)$ depending only on t and the submanifold N_i so that for all $\alpha > 0$,

$$\int_{B(N_i,r)} \bar{\mathcal{P}}_i^\alpha(t,z,z)\,dz \leqslant C(t,N_i). \tag{3.39}$$

Proof. Recalling the representation of the Brownian motion given by Proposition 3.24, one can prove that for a compactly supported function $f : B_i \to \mathbb{R}$,

$$\begin{aligned}
&\int_{B_i} \bar{\mathcal{P}}_i^\alpha(t,z_1,u) f(u)\,du \\
&\quad = \mathbb{E}^{\mathcal{W}_d}\Big[e^{-\frac{\alpha^2}{2}\int_0^t |y_1+\mathbf{w}''(s)|^2\,ds+\frac{\alpha t \nu_i}{2}} f\big(p_{N_i}(t,x_1,\mathbf{w}'), \tau^{\mathrm{V}}_{p_{N_i}(\cdot,x_1,\mathbf{w}')\restriction[0,t]}\big(y_1+\mathbf{w}''(t)\big)\big)\Big] \\
&\quad = \mathbb{E}^{\mathcal{W}_{d-\nu_i}}\Bigg[\int_{(B_i)_{x_1}} \mathcal{Q}_i^\alpha(t,y_1,\xi) f\big(p_{N_i}(t,x_1,\mathbf{w}'), \tau^{\mathrm{V}}_{p_{N_i}(\cdot,x_1,\mathbf{w}')\restriction[0,t]}(\xi)\big)\,d\xi\Bigg]
\end{aligned}$$

where $\mathcal{Q}_i^\alpha(t,\zeta,\xi)$ is the heat kernel for the Hermite like operator $\frac{1}{2}\Delta - \frac{\alpha^2}{2}|y|^2 + \frac{\alpha\nu_i}{2}$ on $\mathbb{R}^{\nu_i}$. To get the estimate for $\bar{\mathcal{P}}_i^\alpha(t,z_1,z_2)$, we will estimate $\mathcal{Q}_i^\alpha(t,\zeta,\xi)$ when ξ is near y_2 and then we will replace f with an approximation of δ_{z_2}.

We can use the formula (see for example [10, p. 390])

$$\mathcal{Q}_i^\alpha(t,\zeta,\xi) = \left(\frac{\alpha}{\pi(1-e^{-2t\alpha})}\right)^{\nu_i/2} \exp\left(-\frac{\alpha\coth(t\alpha)}{2}\left(|\zeta|^2 - \frac{2\langle\zeta,\xi\rangle}{\cosh(t\alpha)} + |\xi|^2\right)\right), \tag{3.40}$$

to show that, for any $\xi \in B(y_2,\epsilon)$,

$$\begin{aligned}
\mathcal{Q}_i^\alpha(t,y_1,\xi) &\leqslant \left(\frac{\alpha}{\pi(1-e^{-2t\alpha})}\right)^{\nu_i/2} \\
&\quad \times \exp\left(-\frac{\alpha\coth(t\alpha)}{2}\left(|y_1|^2 - \frac{2|y_1|(|y_2|+\epsilon)}{\cosh(t\alpha)} + |y_2|^2 - \epsilon|y_2| - \epsilon^2\right)\right).
\end{aligned}$$

In particular, if $f \geqslant 0$ has support in $B(z_2,\epsilon)$, then, by the fact that the parallel transport is an isometry in fibers, we have the inequality

$$\begin{aligned}
&\int_{B_i} \bar{\mathcal{P}}_i^\alpha(t,z_1,u) f(u)\,du \\
&\quad \leqslant \mathbb{E}^{\mathcal{W}_{d-\nu_i}}\Bigg[\int_{(B_i)_{x_1}} f\big(p_{N_i}(t,x_1,\mathbf{w}'), \tau^{\mathrm{V}}_{p_{N_i}(\cdot,x_1,\mathbf{w}')\restriction[0,t]}(\xi)\big)\,d\xi\Bigg]\left(\frac{\alpha}{\pi(1-e^{-2t\alpha})}\right)^{\nu_i/2} \\
&\qquad \times \exp\left(-\frac{\alpha\coth(t\alpha)}{2}\left(|y_1|^2 - \frac{2|y_1|(|y_2|+\epsilon)}{\cosh(t\alpha)} + |y_2|^2 - \epsilon|y_2| - \epsilon^2\right)\right).
\end{aligned}$$

Now, we choose a geodesic ball centered at x_2 in N_i and identify the fibers of B_i with $(B_i)_{x_2}$ by parallel transport along geodesics in N_i radiating from x_2. Then, we can choose a smooth compactly supported approximation f_n^{v} of δ_{y_2} in the fiber $(B_i)_{x_2}$ and we extend this in an obvious way to nearby fibers. Next we choose an approximation f_n^{h} with support in a small neighborhood of x_2 for δ_{x_2} on N_i. Finally, if one takes $f_n(x,y) = f_n^{\mathrm{h}}(x) f_n^{\mathrm{v}}(y)$ in the above inequality, after letting n tend to infinity and then ϵ to 0, one gets (3.38). To prove (3.39), one has to integrate (3.38) and make a change of variable. $\square$

Now we will carry out the program outlined at the beginning of this section, namely the comparison of the heat kernels of $\square^\alpha$ and $\square_i^\alpha$. Before stating the next result, we introduce the following notation. We define

$$\bigwedge\nolimits_z^{q,q^+,q^-}(B_i) = \bigwedge\nolimits^q \big(T(B_i)_z^{\mathrm{H}}\big) \wedge \bigwedge\nolimits_x^{q^+}\big(B_i^+\big) \wedge \bigwedge\nolimits_x^{q^-}\big(B_i^-\big),$$

and observe that there exists the following natural decomposition

$$\bigwedge\nolimits_z^k(B_i) = \bigoplus_{\substack{q+q^++q^-=k\\ 0\leqslant q\leqslant \dim(N_i)\\ 0\leqslant q^+\leqslant \nu_i^+,\,0\leqslant q^-\leqslant \nu_i^-}} \bigwedge\nolimits_z^{q,q^+,q^-}(B_i). \tag{3.41}$$

Since $B_i = B_i^+ \oplus B_i^-$, the Bismut connection on B_i is the direct sum of the Bismut connections of B_i^+ and B_i^-. Moreover, because the Bismut connection of B_i preserves the horizontal and vertical subspaces of $T(B_i)$, the operator $\square_i^\alpha$, and consequently its heat kernel, preserve the spaces $\bigwedge_z^{q,q^+,q^-}(B_i)$. If $k = q + q^+ + q^-$, we will denote by $\mathbf{p}^{i,\alpha}_{(q,q^+,q^-)}(t,z_1,z_2)$ the restriction of the heat kernel $\mathbf{p}_k^{i,\alpha}(t,z_1,z_2)$ to $\bigwedge_z^{q,q^+,q^-}(B_i)$.

The main result is contained in the following theorem.

Theorem 3.42. *For every $t>0$, there exist constants $C_1 = C_1(r) > 0$ and $C_2(t) > 0$, such that for $\alpha \geqslant C_1 e^{C_1 t}$,*

$$\left| \int_M \operatorname{Tr} \mathbf{p}_k^\alpha(t,z,z)\,dz - \sum_{i=1}^l \int_{B(N_i,r)} \operatorname{Tr} \mathbf{p}^{i,\alpha}_{(k-\nu_i^-,0,\nu_i^-)}(t,z,z)\,dz \right| \leqslant r C_2(t). \tag{3.43}$$

Proof. The proof of this result in given in several steps, and each step will involve a comparison of heat kernels of various operators.

Assume r_0 is a number so that $B(N_i, 4r_0) \subset \mathcal{V}_i$. Choose a cut-off function $\varphi : M \to [0,1]$ on M which is 0 outside $\bigcup_{i=1}^l B(N_i, r_0)$, $i = 1,\dots,l$, and 1 on $\bigcup_{i=1}^l B(N_i, r_0/2)$. For small enough $r>0$, define

$$\varphi_r(z_i) = \varphi\left(x_i, \frac{y_i}{r}\right) \quad \text{for } z_i \text{ close to } N_i, \tag{3.44}$$

and the connection ∇^r on M by

$$\nabla^r = \varphi_r \nabla_i^{\mathrm{B}} + (1 - \varphi_r)\nabla^{\mathrm{LC}}. \tag{3.45}$$

Notice that (cf. Theorem 3.7(4)), the Laplacian on functions of this connection is the same as the Laplacian on functions of the Levi-Civita connection and (cf. (3.12)), the Hessian of the function h is the same as the Hessian computed with respect to the Levi-Civita connection. Now set

$$\diamond^{\alpha,r} = d^{\nabla^r,\alpha h}\delta^{\nabla^r,\alpha h} + \delta^{\nabla^r,\alpha h}d^{\nabla^r,\alpha h},$$

where the operators $d^{\nabla^r,\alpha h}$ and $\delta^{\nabla^r,\alpha h}$ are defined in (A.0). From (A.5), we have

$$\begin{aligned}\diamond^{\alpha,r} = {} & -\Delta^{\nabla^r} + \alpha^2|\operatorname{grad} h|^2 - \alpha\Delta h + 2\alpha \operatorname{hess} h + D^* R^{\nabla^r} \\ & + \varphi_r \sum_{j,k,l}^{d} \langle T^{\mathrm{B}}(E_j, E_k), E_l\rangle E_j^* \wedge i_{E_k}\nabla^r_{E_l},\end{aligned}$$

where R^{∇^r} is the curvature of the connection ∇^r and T^{B} is the torsion of the Bismut connection.

The next observation is that, by combining (A.1), (3.13) and (3.8), one can prove that there is a constant $C > 0$ such that for any form ω and $\alpha > 0$,

$$\left|\left(d^{\nabla^r,\alpha h} - d^{\alpha h}\right)\omega\right| \leqslant rC\varphi_r|\omega| \quad \text{and} \quad \left|\left(\delta^{\nabla^r,\alpha h} - \delta^{\alpha h}\right)\omega\right| \leqslant rC\varphi_r|\omega|. \tag{3.46}$$

Now we define $\diamond_i^\alpha$ on B_i by

$$\begin{aligned}\diamond_i^\alpha = {} & -\Delta^{\mathrm{B}} + \alpha^2|\operatorname{grad} h|^2 - \alpha\Delta h + 2\alpha \operatorname{hess} h \\ & + D^* R^{\mathrm{B}} + \varphi \sum_{j,k,l}^{d} \langle T^{\mathrm{B}}(E_j, E_k), E_l\rangle E_j^* \wedge i_{E_k}\nabla^{\mathrm{B}}_{E_l},\end{aligned} \tag{3.47}$$

where Δ^{B}, T^{B} and R^{B} are the Laplacian on forms, the torsion, respectively the curvature of the Bismut connection on B_i. Notice that both of $\diamond^{\alpha,r}$ and $\diamond_i^\alpha$ are in the form (3.28), therefore inequality (3.29) holds (eventually with a different constant C) and one can get upper bounds on their heat kernels on forms using Theorems 3.32, 3.35, and 3.37. Because the operator $\diamond_i^\alpha$ leaves the spaces $\bigwedge_z^{q,q^+,q^-}(B_i)$ invariant, so does its heat kernel. We will denote by $\mathbf{p}_{(q,q^+,q^-)}^{\diamond_i^\alpha}(t, z_1, z_2)$, the restriction of $\mathbf{p}_k^{\diamond_i^\alpha}(t, z_1, z_2)$ to $\bigwedge_z^{q,q^+,q^-}(B_i)$ if $k = q + q^+ + q^-$.

The proof of the theorem will be given in three steps. First, we compare the integral of the traces of the heat kernels for $\square^\alpha$ and $\diamond^{\alpha,r}$. Second, near the submanifold N_i, we compare the heat kernels of $\diamond^{\alpha,r}$ with the one of $\diamond_i^\alpha$. At last, we compare the heat kernel of $\diamond_i^\alpha$ and $\square_i$ on B_i.

Claim 1. *For every $t > 0$, there are constants $C_1 = C_1(r) > 0$ and $C_2(t) > 0$ so that for $\alpha \geqslant C_1 e^{C_1 t}$,*

$$\left| \int_M \operatorname{Tr} \mathbf{p}_k^{\alpha}(t,z,z)\,dz - \int_M \operatorname{Tr} \mathbf{p}_k^{\diamond^{\alpha,r}}(t,z,z)\,dz \right| \leqslant rC(t). \tag{3.48}$$

Proof of Claim 1. Using standard arguments, one can show that

$$\int_M \operatorname{Tr} \mathbf{p}_k^{\alpha}(t,z,z)\,dz = \operatorname{Tr} e^{-t\square_k^{\alpha}/2} = \sum_{j=0}^{\infty} e^{-t\lambda_j(\alpha)/2},$$

$$\int_M \operatorname{Tr} \mathbf{p}_k^{\diamond^{\alpha,r}}(t,z,z)\,dz = \operatorname{Tr} e^{-t(\diamond^{\alpha,r})_k/2} = \sum_{j=0}^{\infty} e^{-t\lambda_j^r(\alpha)/2},$$

where $\lambda_j(\alpha)$ and $\lambda_j^r(\alpha)$ are the eigenvalues, arranged in non-decreasing order, of the operators $\diamond^{\alpha,r}$ and $\square^{\alpha}$ acting on H_k^2, the completion of $\bigwedge^k(M)$ in the norm $\|u\|_{2,k} = \sqrt{\|du\|_{L^2}^2 + \|u\|_{L^2}^2}$.

For any operator which is bounded bellow, the min–max formula gives

$$\lambda_j(L) = \inf_{\substack{V \subset H_k^2 \\ \dim(V)=j}} \max\left\{ \int_M \langle (Lu)(z), u(z)\rangle\,dz \;\middle|\; u \in V,\ \|u\| = 1 \right\}, \tag{$*$}$$

where $\|\cdot\|$ is the L^2 norm of sections in $L^2(\bigwedge^k(M))$. From

$$\langle \square^{\alpha} u, u\rangle = \left|d^{\alpha h} u\right|^2 + \left|\delta^{\alpha h} u\right|^2 \quad \text{and} \quad \langle \diamond^{\alpha,r} u, u\rangle = \left|d^{\nabla^r,\alpha h} u\right|^2 + \left|\delta^{\nabla^r,\alpha h} u\right|^2,$$

and the elementary inequality,

$$\left|\left(a^2+b^2\right)^{1/2} - \left(c^2+d^2\right)^{1/2}\right| \leqslant |a-c| + |b-d| \quad \text{for } a,b,c,d \text{ real numbers},$$

we get

$$\left|\langle \square_r^{\alpha} u, u\rangle^{1/2} - \langle \diamond^{\alpha} u, u\rangle^{1/2}\right| \leqslant \left|d^{\nabla^r,\alpha h} u - d^{\alpha h} u\right| + \left|\delta^{\nabla^r,\alpha h} u - \delta^{\alpha h} u\right|,$$

which combined with (3.46) give

$$\left|\langle \diamond^{\alpha,r} u, u\rangle^{1/2} - \langle \square^{\alpha} u, u\rangle^{1/2}\right| \leqslant rC|u|.$$

Hence, $(*)$ now implies that for any $\alpha > 0$,

$$\left|\sqrt{\lambda_j^r(\alpha)} - \sqrt{\lambda_j(\alpha)}\right| \leqslant rC.$$

Now, for $f(a)=e^{-a^2/2}$, one has $|f'(a)|\leqslant 1$ and then $|e^{-a^2/2}-e^{-b^2/2}|\leqslant |a-b|$, which implies that for $x,y\geqslant 0$, $|e^{-x}-e^{-y}|\leqslant |\sqrt{x}-\sqrt{y}|(e^{-x/2}+e^{-y/2})$. From this, one gets

$$\begin{aligned}\left|\sum_{j=0}^{\infty} e^{-t\lambda_j(\alpha)/2}-\sum_{j=0}^{\infty} e^{-t\lambda_j^r(\alpha)/2}\right| &\leqslant rC\sum_{j=0}^{\infty}\bigl(e^{-t\lambda_j(\alpha)/4}+e^{-t\lambda_j^r(\alpha)/4}\bigr)\\ &= rC\biggl(\int_M \operatorname{Tr}\mathbf{p}_k^{\alpha}(t/2,z,z)\,dz+\int_M \operatorname{Tr}\mathbf{p}_k^{\Diamond^{\alpha,r}}(t/2,z,z)\,dz\biggr),\end{aligned}$$

which combined with (3.29), (3.32), (3.35) and (3.39) ends the proof of the claim. □

Our next step is to compare the heat kernel for $\Diamond^{\alpha,r}$ with the heat kernel of the operator $\Diamond_i$ on B_i.

Claim 2. *There exist $C_1=C_1(r)>0$, $C_2=C_2(r)>0$, $C_3=C_3(r)>0$ and $C_4>0$, such that for $t>0$ and $\alpha\geqslant C_1e^{C_1t}$,*

$$\begin{aligned}&\bigl\|\mathbf{p}_k^{\Diamond^{\alpha,r}}(t,z_1,z_2)\bigr\|\leqslant t^{-d/2}\alpha^{C_4}\bigl(\exp\bigl(-C_2t\alpha^2\bigr)+\exp(-C_2\alpha)\bigr)\quad \text{for } (z_1,z_2)\in\Lambda_r\times M,\\ &\bigl\|\mathbf{p}_k^{\Diamond^{\alpha,r}}(t,z_1,z_2)-\mathbf{p}_k^{\Diamond_i^{\alpha}}(t,z_1,z_2)\bigr\|\\ &\qquad\leqslant t^{-d/2}\alpha^{C_4}\exp\biggl(-\alpha C_2-\alpha\frac{|y_1|^2}{2}\biggr)\quad \text{for } z_1,z_2\in B(N_i,r).\end{aligned}$$

Proof of Claim 2. The first part follows from (3.29) and (3.32). The proof of the second part is based on the same ideas as the ones in the proofs of Theorems 3.32 and 3.35. Using integration by parts, one can write the heat kernels $\mathbf{p}_k^{\Diamond_r^{\alpha}}(t,z_1,z_2)$ and $\mathbf{p}_k^{\Diamond_i^{\alpha}}(t,z_1,z_2)$ as regular integrals (i.e., without the delta functions). Then, split the resulting integrals into the ones on paths leaving $B(N_i,2r)$ before time t, and the others on paths staying inside $B(N_i,2r)$ till time t. The integrals over paths which leave decay exponentially. On the other hand, since near B_i the connections ∇^r and ∇^{B} are the same, the other two integrals are equal, and from this the second part follows. □

Next we compare the heat kernel for the operators $\Diamond_i^{\alpha}$ and $\Box_i^{\alpha}$ on B_i.

Claim 3. *There exists a constant $C>0$, independent of r and t, so that for any $t>0$ and $\alpha>0$ (cf. (3.34)),*

$$\begin{aligned}&\text{if}\quad 0<q^+\quad\text{or}\quad q^-<\nu_i^-,\quad \bigl\|\mathbf{p}_{(q,q^+,q^-)}^{\Diamond_i^{\alpha}}(t,z_1,z_2)\bigr\|\leqslant e^{-t(q^++\nu_i^--q^-)\alpha+Ct}\bar{\mathcal{P}}_i^{\alpha}(t,z_1,z_2),\\ &\text{if}\quad q^+=0\quad\text{and}\quad q^-=\nu_i^-,\quad \text{on } \textstyle\bigwedge^{q,0,\nu_i^-}(B_i)\text{: } \mathbf{p}_{(q,0,\nu_i^-)}^{\Diamond_i^{\alpha}}(t,z_1,z_2)=\mathbf{p}_{(q,0,\nu_i^-)}^{i,\alpha}(t,z_1,z_2).\end{aligned}\tag{3.49}$$

Proof of Claim 3. The first part follows from the fact that the Laplacian of $\bar{h}$ is $\nu_i^+ - \nu_i^-$, and the Hessian on $\bigwedge_z^{q,q^+,q^-}(B_i)$ is the multiplication by $(q^+ - q^-)$. Therefore the first part is a consequence of Proposition C.5 and the definition of $\bar{\mathcal{P}}_i^\alpha(t, z_1, z_2)$ in (3.34).

For the second part, we show that the operators $\diamondsuit_i^\alpha$ and $\Box_i^\alpha$ are the same on $\bigwedge_z^{q,0,\nu_i^-}(B_i)$. This amounts to verifying two claims. The first one is that the action of $\sum_{j,k,l}^d \langle T^{\mathrm{B}}(E_j, E_k), E_l\rangle E_j^* \wedge i_{E_k} \nabla_{E_l}^{\mathrm{B}}$ in (3.47) on $\bigwedge_z^{q,0,\nu_i^-}(B_i)$ is 0, and the other is that the actions of $D^* R^{\mathrm{B}}$ and $D^* R^{\mathrm{H}}$ are the same on $\bigwedge_z^{q,0,\nu_i^-}(B_i)$. To see these, one has to use (3.8) and a choice of the basis $\{E_j\colon\ j = 1, \ldots, d\}$ containing only vertical or horizontal vectors. Fix a point $z \in B_i$, a horizontal basis $\{F_k\colon\ k = 1, \ldots, \dim(N_i)\}$, in a neighborhood of x, and a vertical basis $\{F_j^\pm\colon\ j = 1, \ldots, \nu_i^\pm\}$ in a neighborhood of y. Then every form in $\bigwedge_z^{q,0,\nu_i^-}(B_i)$ can be written as a linear combination of forms $\omega = f F_{k_1}^* \wedge \cdots \wedge F_{k_q}^* \wedge (F_1^-)^* \wedge \cdots \wedge (F_{\nu_i^-}^-)^*$, where $f : B_i \to \mathbb{R}$ is a smooth function. Now, the term $\langle T^{\mathrm{B}}(E_j, E_k), E_l\rangle$ is non-zero only if E_j and E_k are horizontal and E_l is vertical, in which case, (3.6) and the fact that $F_j^* \wedge i_{F_k} \nabla_{F_l^-}^B \omega = 0$ imply the first claim. For the second claim one has to look at the formula that gives the curvature of the Bismut connection in (3.9) and observe that

$$
\begin{aligned}
D^* R^{\mathrm{B}} \omega = {} & \sum_{j,k,l,m=1}^{\dim(N_i)} \bigl\langle R^{\mathrm{H}}(F_j, F_k) F_l, F_m \bigr\rangle \bigl(F_j^* \wedge i_{F_k}\bigr) \circ \bigl(F_l^* \wedge i_{F_m}\bigr) \omega \\
& + \sum_{j,k=1}^{\dim(N_i)} \sum_{l,m=1}^{\nu_i^+} \bigl\langle R^{\mathrm{V}}(F_j, F_k) F_l^+, F_m^+ \bigr\rangle \bigl(F_j^* \wedge i_{F_k}\bigr) \circ \bigl(\bigl(F_l^+\bigr)^* \wedge i_{F_m^+}\bigr) \omega \\
& + \sum_{j,k=1}^{\dim(N_i)} \sum_{l,m=1}^{\nu_i^-} \bigl\langle R^{\mathrm{V}}(F_j, F_k) F_l^-, F_m^- \bigr\rangle \bigl(F_j^* \wedge i_{F_k}\bigr) \circ \bigl(\bigl(F_l^-\bigr)^* \wedge i_{F_m^-}\bigr) \omega \\
= {} & D^* R^{\mathrm{H}} \omega,
\end{aligned}
$$

where the second line above is 0 (there is no part involving F^+-vectors in ω) and the third line, $((F_l^-)^* \wedge i_{F_m^-})\omega$ is non-zero only if $l = m$, in which case $\langle R^{\mathrm{V}}(F_j, F_k) F_l^-, F_m^-\rangle = 0$ by the skew-symmetry of the curvature R^{V}. $\Box$

After combining these claims and estimates, (3.32), (3.35) and (3.37), one can easily complete the proof of Theorem 3.42. $\Box$

3.5. The proof of the degenerate Morse inequalities

In this section we prove the Morse inequalities in the degenerate case. Theorem 3.42 provides the estimates which allow us to reduce everything to studying the asymptotic behavior of the heat kernel $\mathbf{p}_{(k-\nu_i^-,0,\nu_i^-)}^{i,\alpha}(t, z_1, z_2)$ on $\bigwedge^{k-\nu_i^-,0,\nu_i^-}(B_i)$.

Set $T_x^-(N_i) = T_x(N_i) \otimes \bigwedge^{\nu_i^-}((B_i)_x)$, $\nabla^{i,-} = \nabla^{\mathrm{LC}} \otimes \nabla^{\mathrm{V}}$ the connection on $T^-(N_i)$, τ^- and R_i^-, the parallel transport, respectively, the curvature associated to $\nabla^{i,-}$. Also, set $\mathcal{F}_x^q = \bigwedge_x^q(N_i) \otimes \bigwedge^{\nu_i^-}((B_i)_x)$. Using Definition 1.4, we extend the Levi-Civita connection ∇^{LC} to forms on N_i and extend $\nabla^{i,-}$ naturally to $\mathcal{F}_x^q$. The parallel transport also has a natural extension from $T^-(N_i)$ to $\mathcal{F}_x^q$.

Topologically, the bundle $\mathcal{F}^q$ is the same as $\bigwedge^q(T(N_i) \otimes o(B_i^-))$, with $o(B_i^-)$ the orientation bundle of B_i^-, therefore (cf. [5, Chapter I, Section 7]) the differential d_i^- is well defined and the cohomology groups of the sequence

$$0 \to \mathcal{F}^0 \xrightarrow[d_i^-]{} \mathcal{F}^1 \xrightarrow[d_i^-]{} \cdots \xrightarrow[d_i^-]{} \mathcal{F}^{\dim(N_i)} \xrightarrow[d_i^-]{} 0$$

are $H(N_i; o(B_i^-))$. Analytically, one can write

$$\left(d_i^-\right)_x = \sum_{j=1}^{\dim(N_i)} F_j^* \wedge \nabla_{F_j}^{i,-},$$

for any orthonormal basis $\{F_j\colon\ j = 1, \ldots, \dim(N_i)\}$ at $T_x(N_i)$. The bundles $\mathcal{F}^q$ can be endowed with a natural metric inherited from $\bigwedge_x^q(M)$ and $\bigwedge^{\nu_i^-}((B_i^-)_x)$. This allows one to define δ_i^-, the adjoint of d_i^-. Next we set

$$\square^{i,-} = d_i^- \delta_i^- + \delta_i^- d_i^-, \tag{3.50}$$

with the convention that $\square_q^{i,-}$ is its action on $\mathcal{F}^q$. The Weitzenböck formula in this framework is

$$\square^{i,-} = -\Delta^{i,-} - D^* R^-,$$

where $\Delta^{i,-}$ is the Laplacian associated to the connection $\nabla^{i,-}$ by the recipe given in Definition 1.4. We will denote the heat kernel of $\square_q^{i,-}$ by $\mathbf{p}_q^{i,-}(t,x,y)$.

We recall that $p_{N_i}(\cdot, x, \mathbf{w}) \in \mathcal{P}(N_i)$ stands for the map associated to the Riemannian manifold N_i with the property that the distribution of $\mathbf{w} \in \mathcal{P}(\mathbb{R}^{\dim(N_i)}) \to p_{N_i}(\cdot, x, \mathbf{w}) \in \mathcal{P}(N_i)$ is the Wiener measure on N_i. For μ_{N_i}-almost any path $\psi \in \mathcal{P}(N_i)$ the parallel transport $\tau^-_{\psi\restriction[s,0]}$ is well defined, consequently, for μ_{N_i}-almost any $\psi \in \mathcal{P}(N_i)$, one can define $W_q^-(t,\psi)$ to be the solution to the ODE on $\mathcal{F}^q_{\psi(0)}$,

$$\begin{cases} \dot{W}_q^-(s,\psi) = W_q^-(s,\psi)\left(\tau^-_{\psi\restriction[s,0]} \frac{1}{2} D^* R^-_{\psi(s)} \tau^-_{\psi\restriction[0,s]}\right), \\ W_q^-(0,\psi) = \mathrm{Id}_{\bigwedge^{q,-}_{\psi(0)}(N_i)}. \end{cases} \tag{3.51}$$

Using this map (cf. Theorem C.3), the heat kernel of $\square_q^{i,-}$ has the representation

$$\mathbf{p}_q^{i,-}(t,x_1,x_2) = \mathbb{E}^{\mathcal{W}_{\dim(N_i)}}\left[W_q^-\left(t, p_{N_i}(\cdot,x,\mathbf{w})\right)\tau^-_{p_{N_i}(\cdot,x_1,\mathbf{w}))\restriction[t,0]}\delta_{x_2}\left(p_{N_i}(t,x_1,\mathbf{w})\right)\right].$$

For any $z \in B_i$, one can naturally identify $\bigwedge_z^{q,0,\nu_i^-}(B_i)$ with $\mathcal{F}_q = \bigwedge_x^q(N_i) \otimes \bigwedge^{\nu_i^-}((B_i)_x)$, therefore, we can write

$$\mathbf{p}^{i,\alpha}_{(q,0,\nu_i^-)}(t, z_1, z_2) = \mathbb{E}^{\mathcal{W}_d}\Bigg[\exp\Bigg(-\frac{\alpha^2}{2}\int_0^t \big|p(v, z_1, \mathbf{w})\big|^2\, dv + \frac{\alpha t \nu_i}{2}\Bigg)$$
$$\times \big(W_q^-\big(t, \pi\big(p(\cdot, z_1, \mathbf{w})\big)\big)\tau^-_{\pi(p(\cdot,z_1,\mathbf{w}))\restriction[t,0]}\big)\delta_{z_2}\big(p(t, z_1, \mathbf{w})\big)\Bigg],$$

where $\pi : B_i \to N_i$ is the projection from B_i onto N_i. To estimate $\mathbf{p}^{i,\alpha}_{(q,0,\nu_i^-)}(t, z_1, z_2)$, we follow the same line of reasoning as the one in proving Proposition 3.37. First, for a compactly supported function $f : B_i \to \mathbb{R}$, we write (cf. Proposition 3.24),

$$\int_{B_i} \mathbf{p}^{i,\alpha}_{(q,0,\nu_i^-)}(t, z_1, u) f(u)\, du$$
$$= \mathbb{E}^{\mathcal{W}_d}\big[e^{-\frac{\alpha^2}{2}\int_0^t |p(v,z_1,\mathbf{w})|^2\, dv + \frac{\alpha t \nu_i}{2}} W_q^-\big(t, \pi\big(p(\cdot, z_1, \mathbf{w})\big)\big)\tau^-_{\pi(p(\cdot,x_1,\mathbf{w}))\restriction[t,0]} f\big(p(t, z_1, \mathbf{w})\big)\big]$$
$$= \mathbb{E}^{\mathcal{W}_{\dim(N_i)}}\Bigg[W_q^-\big(t, p_{N_i}(\cdot, x_1, \mathbf{w}')\big)\tau^-_{p_{N_i}(\cdot,x_1,\mathbf{w}')\restriction[t,0]}$$
$$\times \int_{(B_i)_{x_1}} \mathcal{Q}_i^\alpha(t, y_1, \xi) f\big(p_{N_i}(t, x_1, \mathbf{w}'), \tau^{\mathrm{V}}_{p_{N_i}(\cdot,x_1,\mathbf{w}')\restriction[0,t]}(\xi)\big)\, d\xi\Bigg], \tag{$*$}$$

where $\mathcal{Q}_i^\alpha$ is defined in (3.40), and $\mathbf{w}'$ stands for paths in $\mathcal{P}(\mathbb{R}^{\dim(N_i)})$. Then we get the estimates for $\mathbf{p}^{i,\alpha}_{(q,0,\nu_i^-)}(t, z_1, z_2)$ by estimating $\mathcal{Q}_i^\alpha(t, y_1, \xi)$ and then replacing f with an approximation of the delta-function δ_{z_2}. Indeed, elementary estimates show that for $|y_1| \leqslant r$ and $|\xi| = |\eta| \leqslant r$,

$$\big|\mathcal{Q}_i^\alpha(t, y_1, \xi) - \mathcal{Q}_i^\alpha(t, y_2, \eta)\big| \leqslant 2\sinh\bigg(\frac{\alpha r^2}{2\sinh(t\alpha)}\bigg)\bigg(\frac{\alpha}{\pi(1 - e^{-2t\alpha})}\bigg)^{\nu_i/2}$$
$$\times \exp\bigg(-\frac{\alpha \coth(t\alpha)(|y_1|^2 + |\eta|^2)}{2}\bigg). \tag{$**$}$$

Then, choosing an approximation f_n to δ_{z_2} as in the proof of Proposition 3.37, and recalling that vertical parallel translation along Brownian paths on N_i is an isometry on fibers, one can deduce from $(*)$ and $(**)$ that for a constant $C > 0$, independent of t and r, and any $z_1, z_2 \in B(N_i, r)$,

$$
\begin{aligned}
&\left\|\mathbf{p}^{i,\alpha}_{(q,0,\nu_i^-)}(t,z_1,z_2)-\mathcal{Q}^\alpha_i(t,y_1,y_2)\mathbf{p}^{i,-}_q(t,x_1,x_2)\right\| \\
&\quad\leqslant 2e^{Ct}\mathbf{p}_{N_i}(t,x_1,x_2)\sinh\left(\frac{\alpha r^2}{2\sinh(t\alpha)}\right)\left(\frac{\alpha}{\pi(1-e^{-2t\alpha})}\right)^{\nu_i/2} \\
&\qquad\times\exp\left(-\frac{\alpha\coth(t\alpha)(|y_1|^2+|y_2|^2)}{2}\right),
\end{aligned}
$$

where $\mathbf{p}_{N_i}(t,x_1,x_2)$ is the heat kernel of the Laplacian on functions on N_i. After integrating this last inequality on $B(N_i,r)$ and performing elementary computations, one gets

$$
\left|\int\limits_{B(N_i,r)}\operatorname{Tr}\mathbf{p}^{i,\alpha}_{(q,0,\nu_i^-)}(t,z,z)\,dz-A(r,\alpha)\int\limits_{N_i}\operatorname{Tr}\mathbf{p}^{i,-}_q(t,x,x)\,dx\right|\leqslant K(t)\sinh\left(\frac{\alpha r^2}{2\sinh(t\alpha)}\right)
$$

for a constant $K(t)>0$, depending only on t, and with

$$
A(r,\alpha)=\left(\frac{1}{\pi(1-e^{-t\alpha})^2}\right)^{\nu_i/2}\int\limits_{B(0,r\sqrt{\alpha\tanh(t\alpha/2)})}e^{-|y|^2}\,dy.
$$

This combined with the result from Theorem 3.42 and the fact that

$$
\int\limits_{N_i}\operatorname{Tr}\mathbf{p}^{i,-}_q(t,x,x)\,dx=\operatorname{Tr}e^{-t\square^{i,-}_q},
$$

proves that for some constants, $K(t)>0$ and $C=C(r)>0$, any $\alpha>Ce^{Ct}$, and any integer $k\in\{0,1,\dots,d\}$,

$$
\left|\int\limits_M\operatorname{Tr}\mathbf{p}^\alpha_k(t,z,z)\,dz-A(r,\alpha)\sum_{i=1}^l\operatorname{Tr}e^{-t\square^{i,-}_{k-\nu_i^-}}\right|\leqslant K(t)r+K(t)\sinh\left(\frac{\alpha r^2}{2\sinh(t\alpha)}\right).
$$

Because $\lim_{\alpha\to\infty}A(r,\alpha)=1$, one can argue that

$$
\begin{aligned}
-K(t)r+\sum_{i=1}^l\operatorname{Tr}e^{-t\square^{i,-}_{k-\nu_i^-}}&\leqslant\liminf_{\alpha\to\infty}\int\limits_M\operatorname{Tr}\mathbf{p}^\alpha_k(t,z,z)\,dz \\
&\leqslant\limsup_{\alpha\to\infty}\int\limits_M\operatorname{Tr}\mathbf{p}^\alpha_k(t,z,z)\,dz\leqslant K(t)r+\sum_{i=1}^l\operatorname{Tr}e^{-t\square^{i,-}_{k-\nu_i^-}}.
\end{aligned}
$$

Finally, this is true for any $r>0$, and therefore, it yields

$$
\lim_{\alpha\to\infty}\int\limits_M\operatorname{Tr}\mathbf{p}^\alpha_k(t,z,z)\,dz=\sum_{i=1}^l\operatorname{Tr}e^{-t\square^{i,-}_{k-\nu_i^-}}.
$$

From this, when t tends to infinity, using standard arguments in Hodge theory one can prove the following.

Theorem 3.52 *(Degenerate Morse inequalities).*

$$m_k - m_{k-1} + \cdots + (-1)^k m_0 \geqslant b_k - b_{k-1} + \cdots + (-1)^k b_0 \tag{3.53}$$

where

$$m_k = \sum_{i=1}^{l} \dim H^{k-\nu_i^-}\big(M_i; o\big(B_i^-\big)\big),$$

with $H^k(M_i; o(B_i^-))$ the cohomology group of N_i twisted by the orientation bundle of E_i^-. This inequality becomes equality for $k = d$.

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Appendix A. Geometric computations

In this section, M is a d-dimensional Riemannian manifold, ∇ a compatible connection, i.e., $\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = X\langle Y, Z\rangle$ for any vector fields X, Y, $Z \in T(M)$, and $h : M \to \mathbb{R}$ a smooth function.

Define the operators on forms (cf. Definition 1.4)

$$\begin{aligned} &\bullet\ d_z^\nabla = \textstyle\sum_{j=1}^d (E_j^*)_z \wedge \nabla_{(E_j)_z} \text{ and } \delta^\nabla \text{ its adjoint,}\\ &\bullet\ d^{\nabla,h} = e^{-h} d^\nabla e^h \text{ and } \delta^{\nabla,h} = e^h \delta^\nabla e^{-h},\\ &\bullet\ \Box^\nabla = d^\nabla \delta^\nabla + \delta^\nabla d^\nabla \text{ and } \Box^{\nabla,h} = d^{\nabla,h}\delta^{\nabla,h} + \delta^{\nabla,h} d^{\nabla,h}. \end{aligned} \tag{A.0}$$

Notice that the definition above does not depend on the orthonormal basis $\{(E_j)_x\colon j = 1, \ldots, d\}$ and all operators send smooth forms into smooth forms.

We collect a number of basic facts in the following proposition.

Proposition A.1. *Let T be the torsion of the connection ∇. Then,*

(1) $\nabla_X i_Y - i_Y \nabla_X = i_{\nabla_X Y}$, *for any vector fields X, Y.*
(2) $\langle \nabla_X \omega, \eta\rangle + \langle \omega, \nabla_X \eta\rangle = X\langle \omega, \eta\rangle$, *for any vector field X and any forms ω, η.*

(3) $\delta_x^{\nabla} = -\sum_{j=1}^{d} i_{(E_j)_x}\nabla_{(E_j)_x} + \sum_{j,k=1}^{d}\langle T((E_k)_x,(E_j)_x),(E_k)_x\rangle i_{(E_j)_x}$, *for any orthonormal basis* $\{(E_j)_x\colon\ j=1,\dots,d\}$ *of* $T_x(M)$.
(4) $d^{\nabla,h}\omega = d^{\nabla}\omega + dh\wedge\omega$ *and* $\delta^{\nabla,h}\omega = \delta^{\nabla}\omega + i_{\operatorname{grad} h}$ *for any form* ω.

Proof. We choose an orthonormal basis $\{E_i\colon\ i=1,\dots,d\}$ in a neighborhood of $x\in M$.

(1) Because both sides are anti-derivations on $\bigwedge(M)$, one needs only check the equality on functions and 1-forms. For functions both are 0. For 1-forms, it suffices to do this for $\omega = fE_1^*$ and this is straightforward.

(2) It suffices to prove it for $\omega = fE_1^*\wedge E_2^*\wedge\cdots\wedge E_k^*$, $\eta = gE_{j_1}^*\wedge E_{j_2}^*\wedge\cdots\wedge E_{j_k}^*$. The rest consists in direct computations and is left to the reader.

(3) Since ∇ is a derivation, $\nabla_{E_j}(E_j^*\wedge\omega) = (\nabla_{E_j}E_j^*)\wedge\omega + E_j^*\wedge(\nabla_{E_j}\omega)$ for any form ω, hence $(E_j^*\wedge\nabla_{E_j})^* = i_{E_j}\nabla_{E_j}^* + i_{\sum_{k=1}^{d}\langle E_j,\nabla_{E_j}E_k\rangle E_k}$. Now using (2), one can justify that $\nabla_X^* = -\nabla_X - \operatorname{div}(X)$ for any vector field X. The rest follows from the fact that $\operatorname{div}(E_j) = \sum_{k=1}^{d}\langle E_k,[E_k,E_j]\rangle$, $\langle E_k,\nabla_{E_k}E_j\rangle = \langle E_k,\nabla_{E_j}E_k + [E_k,E_j] + T(E_k,E_j)\rangle$ and $\langle E_k,\nabla_{E_j}E_k\rangle = 0$.

(4) This is straightforward. □

We now want a decomposition of the operator $\square^{\nabla,h}$. For this purpose, set

$$L_X^{\nabla} = d^{\nabla}i_X + i_X d^{\nabla}.$$

A simple computation gives

$$\square^{\nabla,h} = \square^{\nabla} + |\operatorname{grad} h|^2 + L_{\operatorname{grad} h}^{\nabla} + \bigl(L_{\operatorname{grad} h}^{\nabla}\bigr)^*.$$

Definition A.2. The Hessian $\operatorname{hess}^{\nabla}$, respectively the symmetric Hessian $\operatorname{Shess}^{\nabla}$, with respect to the connection ∇ are prescribed by

$$\bigl\langle\bigl(\operatorname{hess}_x^{\nabla} h\bigr)X_x, Y_x\bigr\rangle = \langle\nabla_{X_x}\operatorname{grad} h, Y_x\rangle,$$
$$\bigl\langle\bigl(\operatorname{Shess}_x^{\nabla} h\bigr)X_x, Y_x\bigr\rangle = \frac{1}{2}\bigl(\langle\nabla_{X_x}\operatorname{grad} h, Y_x\rangle + \langle X_x,\nabla_{Y_x}\operatorname{grad} h\rangle\bigr).$$

Proposition A.3. *We have*

$$L_X^{\nabla} + \bigl(L_X^{\nabla}\bigr)^* = -\operatorname{div}(X) + \sum_{j,k=1}^{d}\bigl(\langle\nabla_{E_j}X,E_k\rangle + \langle\nabla_{E_k}X,E_j\rangle\bigr)E_j^*\wedge i_{E_k}.$$

Therefore,

$$\square^{\nabla,h} = \square^{\nabla} + |\operatorname{grad} h|^2 - \Delta h + 2D^*\operatorname{Shess}^{\nabla} h,$$

where $D^*\operatorname{Shess} h$ *is the extension of the symmetric Hessian given by Definition* 1.4.

Proof. Choose a local orthonormal basis $\{E_i\colon\ i=1,\dots,d\}$ and use Proposition A.1 to get,

$$\begin{aligned}
L_X^\nabla &= \sum_{j=1}^d \bigl(E_j^* \wedge \nabla_{E_j} i_X + i_X E_j^* \wedge \nabla_{E_j}\bigr)\\
&= \sum_{j=1}^d \bigl(E_j^* \wedge \nabla_{E_j} i_X + \langle X, E_j\rangle \nabla_{E_j} - E_j^* i_X \wedge \nabla_{E_j}\bigr)\\
&= \sum_{j=1}^d \bigl(E_j^* \wedge i_{\nabla_{E_j} X} + \langle X, E_j\rangle \nabla_{E_j}\bigr) = \sum_{j,k=1}^d \langle \nabla_{E_j} X, E_k\rangle E_j^* \wedge i_{E_k} + \nabla_X.
\end{aligned}$$

Since $\nabla_X^* = -\operatorname{div}(X) - \nabla_X$,

$$\bigl(L_X^\nabla\bigr)^* = \nabla_X^* + \sum_{j,k=1}^d \langle \nabla_{E_j} X, E_k\rangle E_k^* \wedge i_{E_j} = -\operatorname{div}(X) - \nabla_X + \sum_{j,k=1}^d \langle \nabla_{E_j} X, E_k\rangle E_k^* \wedge i_{E_j},$$

which ends the proof. $\square$

Theorem A.4. *If T and R are, respectively, the torsion and the curvature of the connection ∇, then for any local orthonormal basis $\{E_i\colon\ i=1,\dots,d\}$ in a neighborhood of a point x, we have*

$$\begin{aligned}
\square^\nabla = {}& -\Delta^\nabla - D^* R + \sum_{j,k,l}^d \bigl\langle T(E_j, E_k), E_l\bigr\rangle E_j^* \wedge i_{E_k} \nabla_{E_l}\\
&+ \sum_{j,k=1}^d \bigl\langle T(E_k, E_j), E_k\bigr\rangle \nabla_{E_j} + \sum_{j,k,l=1}^d E_j^* \wedge i_{\nabla_{E_j}(\langle T(E_k, E_l), E_k\rangle E_l)}
\end{aligned}$$

where D^*R is defined by Definition 1.4. *In particular, if for any $X \in T_x(M)$, the torsion T satisfies $\sum_{j=1}^d \langle T(E_j, X), E_j\rangle = 0$, then*

$$\begin{aligned}
\square^{\nabla,h} = {}& -\Delta^\nabla + |\operatorname{grad} h|^2 - \Delta h + 2D^* \operatorname{Shess}^\nabla h\\
&+ \sum_{j,k,l}^d \bigl\langle T(E_j, E_k), E_l\bigr\rangle E_j^* \wedge i_{E_k} \nabla_{E_l} - D^* R.
\end{aligned} \tag{A.5}$$

Proof. For simplicity we are going to use the summation convention in which any time an index appears twice it is summed from 1 to d. Now,

$$
\begin{aligned}
d^\nabla\delta^\nabla+\delta^\nabla d^\nabla &= -E_j^*\wedge\nabla_{E_j}(i_{E_k}\nabla_{E_k})+E_j^*\wedge\nabla_{E_j}i_{\langle T(E_k,E_l),E_k\rangle E_l}\\
&\quad -i_{E_k}\nabla_{E_k}\bigl(E_j^*\wedge\nabla_{E_j}\bigr)+i_{\langle T(E_k,E_l),E_k\rangle E_l}E_j^*\wedge\nabla_{E_j}\\
&\overset{(\mathrm{A}.1)(1)}{=} -E_j^*\wedge i_{E_k}\nabla_{E_j}\nabla_{E_k}-E_j^*\wedge i_{\nabla_{E_j}E_k}\nabla_{E_k}\\
&\quad -i_{E_k}\bigl(\nabla_{E_k}E_j^*\bigr)\wedge\nabla_{E_j}-i_{E_k}\bigl(E_j^*\wedge\nabla_{E_k}\nabla_{E_j}\bigr)\\
&\quad +E_j^*\wedge i_{\nabla_{E_j}(\langle T(E_k,E_l),E_k\rangle E_l)}+E_j^*\wedge i_{\langle T(E_k,E_l),E_k\rangle E_l}\nabla_{E_j}\\
&\quad +\bigl\langle T(E_k,E_j),E_k\bigr\rangle\nabla_{E_j}-E_j^*\wedge i_{\langle T(E_k,E_l),E_k\rangle E_l}\nabla_{E_j}\\
&\qquad \bigl(\text{notice that } E_j^*\wedge i_{E_k}+i_{E_k}E_j^*\wedge=\delta_{jk}\bigr)\\
&= -\nabla_{E_j}\nabla_{E_j}-E_j^*\wedge i_{E_k}(\nabla_{E_j}\nabla_{E_k}-\nabla_{E_k}\nabla_{E_j})\\
&\quad -\langle\nabla_{E_j}E_k,E_l\rangle E_j^*\wedge i_{E_l}\nabla_{E_k}\\
&\quad -\bigl\langle\nabla_{E_k}E_j^*,E_k^*\bigr\rangle\nabla_{E_j}+\bigl\langle\nabla_{E_k}E_j^*,E_l^*\bigr\rangle E_l^*\wedge i_{E_k}\nabla_{E_j}\\
&\quad +E_j^*\wedge i_{\nabla_{E_j}(\langle T(E_k,E_l),E_k\rangle E_l)}+\bigl\langle T(E_k,E_j),E_k\bigr\rangle\nabla_{E_j}.
\end{aligned}
$$

The last line in this computation is the last line in the formula we want to prove. Since $\langle\nabla_{E_j}E_l,E_k\rangle=-\langle\nabla_{E_j}E_k,E_l\rangle$, one gets

$$
\begin{aligned}
&-\nabla_{E_j}\nabla_{E_j}-E_j^*\wedge i_{E_k}(\nabla_{E_j}\nabla_{E_k}-\nabla_{E_k}\nabla_{E_j})-\langle\nabla_{E_j}E_k,E_l\rangle E_j^*\wedge i_{E_l}\nabla_{E_k}\\
&\quad =-\Delta^\nabla-\nabla_{\nabla_{E_j}E_j}-E_j^*\wedge i_{E_k}R(E_j,E_k)-E_j^*\wedge i_{E_k}\nabla_{[E_j,E_k]}\\
&\qquad -\langle\nabla_{E_j}E_k,E_l\rangle E_j^*\wedge i_{E_l}\nabla_{E_k}+\langle\nabla_{E_j}E_j,E_k\rangle\nabla_{E_k}-\langle\nabla_{E_l}E_k,E_j\rangle E_l^*\wedge i_{E_k}\nabla_{E_j}\\
&\quad =-\Delta^\nabla-D^*R-\bigl\langle[E_j,E_k],E_l\bigr\rangle E_j^*\wedge i_{E_k}\nabla_{E_l}\\
&\qquad +\langle\nabla_{E_j}E_k,E_l\rangle E_j^*\wedge i_{E_k}\nabla_{E_l}-\langle\nabla_{E_k}E_j,E_l\rangle E_j^*\wedge i_{E_k}\nabla_{E_l}\\
&\quad =-\Delta^\nabla-D^*R+\sum_{j,k,l}^{d}\bigl\langle T(E_j,E_k),E_l\bigr\rangle E_j^*\wedge i_{E_k}\nabla_{E_l},
\end{aligned}
$$

which ends the proof. □

Appendix B. Integration by parts on the path space of a Riemannian manifold

B.1. Integration by parts

Integration by parts on the path space of a compact Riemann manifold is well understood (see for example [2] for a detailed exposition). In this paper we need an integration by parts on a vector bundle endowed with a particular metric. The bundle is not a compact manifold but the metric has enough properties to guarantee the integration by parts on its path space. For example, the condition given by (3.10) turns out to be sufficient for what we need.

We will assume in this section that M is a d-dimensional Riemannian manifold so that for a fixed reference point o on M, and C and m positive constants, the Ricci curvature satisfies

$$-C\bigl(1+\mathrm{dist}(x,o)^2\bigr)|X|_x^2 \leqslant \mathrm{Ric}_x(X_x,X_x) \leqslant C\bigl(1+\mathrm{dist}(x,o)^m\bigr)|X|_x^2, \qquad \text{(B.1)}$$

for any $x\in M$, $X_x\in T_x(M)$.

The idea outlined in [7] for the proof of the integration by parts on the path space of a compact manifold is based on a perturbation scheme. Recall the map $\mathbf{w}\to\mathfrak{p}(\cdot,\mathfrak{f},\mathbf{w})$, defined in Section 1. Since $\mathfrak{f}$ will be fixed throughout this section, we will simply drop it from our notations. Our perturbation is driven by $\mathbf{h}:[0,\infty]\to\mathbb{R}^d$, with the property that there is a function $\dot{\mathbf{h}}\in L^2([0,\infty];\mathbb{R}^d)$ such that $\mathbf{h}(\sigma)=\int_0^\sigma\dot{\mathbf{h}}(v)\,dv$ for $\sigma\geqslant 0$.

The perturbation $\mathbf{w}\to\mathfrak{p}_s(\cdot,\mathbf{w})$, $s\in[0,1]$, for a piecewise smooth path $\mathbf{w}$ is prescribed by

$$\theta\bigl(\dot{\mathfrak{p}}_0(t,\mathbf{w})\bigr)=\dot{\mathbf{w}}(t)\quad\text{and}\quad\theta\bigl(\mathfrak{p}_s'(t,\mathbf{w})\bigr)=h(t) \qquad \text{(B.2)}$$

for any $t\geqslant 0$ and any $s\in[0,1]$, where here the superscript dot and the prime denote the differentiation with respect to t and s, respectively. Here θ is the connection 1-form (see for example [8, 8.16]).

Assume for the moment that M is a compact manifold. We will characterize $\mathbf{w}\to\mathfrak{p}_s(t,\mathbf{w})$ as the integral curve of a time dependent vector field. In order to do this, we first introduce some spaces. For a Hilbert space V, we denote by $L^2([0,1];V)$ the space of L^2-integrable functions with values in V. $H([0,1];V)$ will be the space of absolutely continuous functions $s\in[0,1]\to\psi_s\in V$ so that $\psi_s'\in L^2([0,1];V)$. We set $L^2([0,1];\mathcal{O}(M))$ to be the set of $s\in[0,1]\to\mathfrak{p}_s\in\mathcal{O}(M)$ so that $\theta(\mathfrak{p}_s)\in L^2([0,1];\mathbb{R}^d)$ and $\omega(\mathfrak{p}_s)\in L^2([0,1];o(d))$, where ω is the $o(d)$-valued 1-form (see for example [8, 8.18]). $H([0,1];\mathcal{O}(M))$ is the set of absolutely continuous functions $s\in[0,1]\to\mathfrak{p}_s\in\mathcal{O}(M)$ with $\theta(\mathfrak{p}_s)\in H([0,1];\mathbb{R}^d)$ and $\omega(\mathfrak{p}_s)\in H([0,1];o(d))$.

Given $\xi\in\mathbb{R}^d$, we define $\mathbf{X}(t,\xi):H([0,1];\mathcal{O}(M))\to L^2([0,1];\mathcal{O}(M))$ by

$$\bigl[\mathbf{X}(t,\xi)_{\mathfrak{p}.}\bigr](s)=\mathfrak{E}\left(O_s(\mathfrak{p}.)\left(\xi+\int_0^s O_v(\mathfrak{p}.)^{\mathsf{T}}\dot{\mathbf{h}}(t)\,dv\right)\right)_{\mathfrak{p}_t}$$

where $s\in[0,1]\to O_s(\mathfrak{p}.)\in O(d)$ is the solution to the ODE

$$O_s'(\mathfrak{p}.)+\theta(\mathfrak{p}_t')O_s(\mathfrak{p}.)=0\quad\text{with }O_0(\mathfrak{p}.)=I. \qquad \text{(B.3)}$$

For a given piecewise smooth path $\mathbf{w}$, the perturbation $\mathbf{w}\to\mathfrak{p}_s(t,\mathbf{w})$, $s\in[0,1]$, is the integral curve of the time dependent vector field $\mathbf{X}(t,\dot{\mathbf{w}}(t))$,

$$\dot{\mathfrak{p}}.(t,\mathbf{w})=\bigl[\mathbf{X}\bigl(t,\dot{\mathbf{w}}(t)\bigr)_{\mathfrak{p}.(t,\mathbf{w})}\bigr]\quad\text{with }\mathfrak{p}_0(t,\mathbf{w})=\mathfrak{p}(t,\mathbf{w}).$$

One can prove that $\mathfrak{p}_s(t,\mathbf{w})$ can be extended from piecewise smooth paths $\mathbf{w}$ to generic Brownian paths and that (cf. [7, 4.14]), for continuous functions $F:\mathcal{P}(\mathcal{O}(M))\to\mathbb{R}$ of $\mathfrak{p}\upharpoonright[0,t]$,

$$\mathbb{E}^{\mathcal{W}_d}\big[F\big(\mathfrak{p}_s(\cdot,\mathbf{w})\big)E_s(t,\mathbf{w})\big]=\mathbb{E}\big[F\big(\mathfrak{p}(\cdot,\mathbf{w})\big)\big] \tag{B.4}$$

where

$$E_s(t,\mathbf{w})=\exp\left[-\int_0^t\big(\mathbf{b}_s(\sigma,\mathbf{w}),d\mathbf{w}(\sigma)\big)-\frac{1}{2}\int_0^t\big|\mathbf{b}_s(\sigma,\mathbf{w})\big|^2\,d\sigma\right], \tag{B.5}$$

and, with $O_s(t,\mathbf{w})=O_s(\mathfrak{p}_\cdot(t,\mathbf{w}))$,

$$\mathbf{b}_s(t,\mathbf{w})=\int_0^s O_v(t,\mathbf{w})^{\top}\left(\dot{\mathbf{h}}(t)+\frac{1}{2}\mathfrak{R}_{\mathfrak{p}_v(t,\mathbf{w})}\mathbf{h}(t)\right)dv \tag{B.6}$$

where the superscript $\top$ is standing for the transpose of a matrix.

Now, for a bounded smooth $F:\mathcal{P}(\mathcal{O}(M))\to\mathbb{R}$, which depends only on the path $\mathfrak{p}\upharpoonright[0,t]$, we define

$$\big[X(\mathbf{h})F\big](\mathbf{w})=\frac{d}{ds}F\big(\mathfrak{p}_s(\cdot,\mathbf{w})\big)\bigg|_{s=0}.$$

Considering $F=F_1F_2$, with F_1, F_2 bounded smooth functions depending only on the path $\mathfrak{p}\upharpoonright[0,t]$, and taking derivatives with respect to s at 0 in (B.4), we get the integration by parts formula

$$\begin{aligned}&\mathbb{E}^{\mathcal{W}_d}\big[\big[X(\mathbf{h})F_1\big](\mathbf{w})F_2\big(\mathfrak{p}(\cdot,\mathbf{w})\big)\big]\\&\quad=-\mathbb{E}^{\mathcal{W}_d}\big[F_1\big(\mathfrak{p}(\cdot,\mathbf{w})\big)\big[X(\mathbf{h})F_2\big](\mathbf{w})\big]\\&\qquad+\mathbb{E}^{\mathcal{W}_d}\left[F_1F_2\big(\mathfrak{p}(\cdot,\mathbf{w})\big)\int_0^t\left(\dot{\mathbf{h}}(\sigma)+\frac{1}{2}\,\mathrm{Ric}_{\mathfrak{p}(\sigma,\mathbf{w})}\,\mathbf{h}(\sigma)\right)d\sigma\right].\end{aligned}$$

If the manifold is not compact, the main idea is to prove a localized version of this integration by parts formula and then pass to the full version by using the integrability of the Ricci curvature. Here, there are two key facts. The first is that (cf. (B.2)) the perturbed path is staying in finite distance from the original path. To be more precise, let $\zeta_R(\mathbf{w})$ be the exit time of the path $\mathfrak{p}(\cdot,\mathbf{w})$ from $\pi^{-1}B(o,R)$ (recall $\pi:\mathcal{O}(M)\to M$ is the projection). Then, for large enough R, and any $(s,\sigma)\in[0,1]\times[0,\zeta_R(\mathbf{w})]$, $\mathfrak{p}_s(\sigma,\mathbf{w})\in\pi^{-1}B(o,2R)$. The second fact is a version of the integration by parts for the stopped path $\mathfrak{p}(\cdot\wedge\zeta_R(\mathbf{w}),\mathbf{w})$.

Now we assume M is a complete Riemannian manifold whose Ricci tensor satisfies the condition (B.1). The perturbation scheme for the path $\mathfrak{p}$ is the one given in (B.2). However, for the application we have in mind, we will have to consider the joint distribution of $\mathfrak{p}$, $\mathbf{w}$ and another map which we will describe now.

Assume we are given $\mathfrak{S} \in C^\infty(\mathcal{O}(M); \mathrm{Hom}(\mathbb{R}^d; so(d)))$, where $so(d)$ is the Lie algebra of the group $o(d)$, i.e., the set of skew-symmetric $d \times d$-matrices. This will be used to construct a map which will be considered part of our perturbation.

First, for a smooth path $\mathbf{w}$, define $\mathbf{W}_s(t, \mathbf{w})$, its perturbation, given by $\dot{\mathbf{W}}_s(t, \mathbf{w}) = \theta(\dot{\mathfrak{p}}_s(t, \mathbf{w}))$. Now define $\mathfrak{o}(t, \mathbf{w})$ to be the solution to

$$\dot{\mathfrak{o}}(t, \mathbf{w}) = \mathfrak{o}(t, \mathbf{w})\mathfrak{S}_{\mathfrak{p}(t,\mathbf{w})}\big(\dot{\mathbf{w}}(t)\big) \quad \text{with } \mathfrak{o}(0, \mathbf{w}) = I,$$

and its perturbation $\mathfrak{O}_s(t, \mathbf{w}) \in O(d)$ to be given by the solution to the equation

$$\dot{\mathfrak{O}}_s(t, \mathbf{w}) = \mathfrak{O}_s(t, \mathbf{w})\mathfrak{S}_{\mathfrak{p}_s(t,\mathbf{w})}\big(\dot{\mathbf{W}}_s(t, \mathbf{w})\big) \quad \text{with } \mathfrak{O}_s(0, \mathbf{w}) = I. \tag{B.7}$$

Notice that, since $\mathfrak{S}_{\mathfrak{f}}(\xi)$ is skew-symmetric, both $\mathfrak{o}(t, \mathbf{w})$ and $\mathfrak{O}_s(t, \mathbf{w})$ are in $o(d)$. The map $\mathbf{w} \to \mathfrak{z}_s(\cdot, \mathbf{w}) := (\mathbf{W}_s(\cdot, \mathbf{w}), \mathfrak{p}_s(\cdot, \mathbf{w}), \mathfrak{O}_s(\cdot, \mathbf{w}))$ is the perturbation we will consider here.

Let O_s be given by the prescription in (B.3), and construct the following time dependent vector fields sending smooth functions $s \in [0, 1] \to \mathfrak{p}_s \in \mathcal{O}(M)$ into smooth functions by the following recipe,

$$\begin{aligned} \big[A_0(t)_{\mathfrak{p}.}\big](s) &= O_s(\mathfrak{p}.) \int_0^s O_v(\mathfrak{p}.)^{\mathsf{T}} \dot{\mathbf{h}}(t)\, dv, \\ \big[A_k(t)_{\mathfrak{p}.}\big](s) &= O_s(\mathfrak{p}.)e_k \quad \text{for } 1 \leqslant k \leqslant d, \end{aligned} \tag{B.8}$$

where $(e_k)_{k=1..d}$ is an orthonormal basis of $\mathbb{R}^d$ which will be fixed for the rest of this section. For a smooth map $s \to (\mathbf{v}_s, \mathfrak{p}_s, \mathfrak{O}_s) \in \mathbb{R}^d \times \mathcal{O}(M) \times O(d)$, we set

$$\big[\mathfrak{X}_0(t)_{(\mathbf{v}.,\mathfrak{p}.,\mathfrak{O}.)}\big](s) = (\partial_{[A_0(t)_{\mathfrak{p}.}](s)})_{\mathbf{v}_s} + \big(\mathfrak{E}\big(\big[A_0(t)_{\mathfrak{p}.}\big](s)\big)\big)_{\mathfrak{p}_s} + (\partial_{\mathfrak{O}_s \mathfrak{S}([A_0(t)_{\mathfrak{p}.}](s))})_{\mathfrak{O}_s},$$
$$\big[\mathfrak{X}_k(t)_{(\mathbf{v}.,\mathfrak{p}.,\mathfrak{O}.)}\big](s) = (\partial_{[A_k(t)_{\mathfrak{p}.}](s)})_{\mathbf{v}_s} + \big(\mathfrak{E}\big(\big[A_k(t)_{\mathfrak{p}.}\big](s)\big)\big)_{\mathfrak{p}_s} + (\partial_{\mathfrak{O}_s \mathfrak{S}([A_k(t)_{\mathfrak{p}.}](s))})_{\mathfrak{O}_s}.$$

Then, for a piecewise smooth $\mathbf{w}$, the map $t \to \mathfrak{z}.(t, \mathbf{w})$ is the unique solution to the following differential equation:

$$d\mathfrak{z}.(t, \mathbf{w}) = \mathfrak{X}_0(t)_{\mathfrak{z}.(t,\mathbf{w})} + \sum_{k=1}^d \mathfrak{X}_k(t)_{\mathfrak{z}.(t,\mathbf{w})}\big\langle \dot{\mathbf{w}}(t), e_k\big\rangle, \quad \text{with } \mathfrak{z}.(0, \mathbf{w}) = (\mathbf{0}, \mathfrak{f}, I).$$

Using an approximation scheme with piecewise smooth paths, one can prove that $\mathfrak{z}.(t, \mathbf{w})$ is well defined up to a set of $\mathcal{W}_d$-measure zero and satisfies the Stratonovich stochastic differential equation,

$$d\mathfrak{z}.(t, \mathbf{w}) = \mathfrak{X}_0(t)_{\mathfrak{z}.(t,\mathbf{w})}\, dt + \sum_{k=1}^d \mathfrak{X}_k(t)_{\mathfrak{z}.(t,\mathbf{w})} \circ d\big\langle \mathbf{w}(t), e_k\big\rangle, \quad \mathfrak{z}.(0, \mathbf{w}) = (\mathbf{0}, \mathfrak{f}, I). \tag{B.9}$$

For details on proving this, we refer to [6, Corollary 10.28]. However, we should mention two key ingredients needed for the argument. One is the non-explosion of Brownian motion and the associated estimates on the rate at which Brownian motion leaves balls, which are consequences of the quadratic lower bound on the Ricci curvature. The other is the fact, alluded to earlier, that the perturbed paths stay close to the unperturbed (i.e., Brownian) paths.

We now define the vector field given by

$$\mathfrak{Y}(\xi)_{(\mathfrak{f},\mathbf{v},\mathfrak{o})} = \mathfrak{E}(\xi)_{\mathfrak{f}} + (\partial_\xi)_{\mathbf{v}} + (\partial_{\mathfrak{o}\mathfrak{S}_{\mathfrak{f}}(\xi)})$$

for $\mathfrak{f} \in \mathcal{O}(M)$, $\mathbf{v} \in \mathbb{R}^d$, $\mathfrak{o} \in o(d)$. For any $\varphi \in C_c^2(\mathbb{R}^d \times \mathcal{O}(M) \times o(d))$, one can prove (using Itô's formula and the idea in the proof of [6, Corollary 10.28]) that for each $s \in [0,1]$,

$$\varphi\big(\mathfrak{z}_s(t,\mathbf{w})\big) - \int_0^t \Bigg(\mathfrak{Y}\big(\mathbf{b}_s(\sigma,\mathbf{w})\big) + \frac{1}{2}\sum_{k=1}^d \mathfrak{Y}(e_k)^2\Bigg)\varphi\big(\mathfrak{z}_s(\sigma,\mathbf{w})\big)\,d\sigma, \tag{B.10}$$

is a $\mathcal{W}_d$-martingale with $\mathbf{b}_s(t,\mathbf{w})$ defined in (B.6).

Now we can state and prove the main result of this section.

Theorem B.11 *(Integration by parts on path space). Under the growth condition on the Ricci curvature given by* (B.1)*, for any bounded smooth function $F : \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathcal{O}(M)) \times \mathcal{P}(o(d)) \to \mathbb{R}$, depending only on the path $\mathfrak{z} \restriction [0,t]$, define*

$$\big[X(\mathbf{h})F\big](\mathbf{w}) = \frac{d}{ds}F\big(\mathfrak{z}_s(\cdot,\mathbf{w})\big)\bigg|_{s=0}. \tag{B.12}$$

If $\mathfrak{z}(t,\mathbf{w}) = \mathfrak{z}_0(t,\mathbf{w})$, then, for bounded smooth functions $F_1, F_2 : \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathcal{O}(M)) \times \mathcal{P}(o(d)) \to \mathbb{R}$ depending only on the path $\mathfrak{z} \restriction [0,t]$, we have

$$\begin{aligned} &\mathbb{E}^{\mathcal{W}_d}\big[\big[X(\mathbf{h})F_1\big](\mathbf{w})F_2\big(\mathfrak{z}(\cdot,\mathbf{w})\big)\big] \\ &\quad = -\mathbb{E}^{\mathcal{W}_d}\big[F_1\big(\mathfrak{z}(\cdot,\mathbf{w})\big)\big[X(\mathbf{h})F_2\big](\mathbf{w})\big] \\ &\qquad + \mathbb{E}^{\mathcal{W}_d}\Bigg[F_1F_2\big(\mathfrak{z}(\cdot,\mathbf{w})\big)\int_0^t \bigg(\dot{\mathbf{h}}(\sigma) + \frac{1}{2}\mathrm{Ric}_{\mathfrak{z}(\sigma,\mathbf{w})}\,\mathbf{h}(\sigma)\bigg)\,d\sigma\Bigg], \end{aligned} \tag{B.13}$$

provided that $\mathbf{w} \to [X(\mathbf{h})F_1](\mathbf{w})F_2(\mathfrak{z}(\cdot,\mathbf{w}))$ and $\mathbf{w} \to F_1(\mathfrak{z}(\cdot,\mathbf{w}))[X(\mathbf{h})F_2](\mathbf{w})$ are $\mathcal{W}_d$-integrable.

Proof. The difficulty here, which is not present in the compact case, is caused by the unboundedness of the Ricci curvature, which leads to the possibility that the $E_s(t,\mathbf{w})$ in (B.5) is not integrable and so (B.4) no longer holds. To get around this difficulty, we use a localization procedure based on stopping times.

Take a large $R > 0$, and consider a compactly supported function $\psi_R : \mathcal{O}(M) \to [0,1]$, which is 1 on $U_{2R} = \pi^{-1}B(o, 2R)$. Then set

$$E_s^R(t, \mathbf{w}) = \exp\left(-\int_0^t \langle \mathbf{b}_s^R(\sigma, \mathbf{w}), d\mathbf{w}(\sigma)\rangle - \frac{1}{2}\int_0^t \left|\mathbf{b}_s^R(\sigma, \mathbf{w})\right|^2 d\sigma\right), \tag{B.14}$$

where

$$\mathbf{b}_s^R(t, \mathbf{w}) = \int_0^s \psi_R\big(\mathfrak{p}_v(t, \mathbf{w})\big) O_v(t, \mathbf{w})^\intercal \left(\dot{\mathbf{h}}(t) + \frac{1}{2}\mathfrak{R}_{\mathfrak{p}_v(t,\mathbf{w})}\mathbf{h}(t)\right) dv.$$

Now, $E_s^R(t, \mathbf{w})$ is integrable since $\mathbf{b}_s^R$ is bounded. Let $\mathcal{F}_t$ be the σ-algebra generated by $\{p \restriction [0,t],\ p \in \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathcal{O}(M)) \times \mathcal{P}(\mathrm{End}(\mathbb{R}^d))\}$ and take $\zeta_R : \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathcal{O}(M)) \times \mathcal{P}(o(d)) \to [0, \infty)$, so that $\zeta_R(\mathbf{w}, \mathfrak{p}, \mathfrak{O})$ is the first time the path $\mathfrak{p}(\cdot, \mathbf{w})$ exits the set $U_R = \pi^{-1}B(o, R)$. Note here the crucial fact that (cf. (B.2)), for large R, $\mathfrak{p}_s(t, \mathbf{w}) \in U_{2R}$ for all $(s, t) \in [0,1] \times [0, \zeta_R(\mathbf{w}, \mathfrak{p}, \mathfrak{O})]$.

Define the probabilities $\mathbb{P}$ and $\mathbb{Q}_s^R$ on $\mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathcal{O}(M)) \times \mathcal{P}(\mathrm{End}(\mathbb{R}^d))$ by

$$\mathbb{P}(C) = \mathbb{E}^{\mathcal{W}_d}\big[\mathfrak{z}(t, \mathbf{w}) \in C\big] \quad \text{and} \quad \mathbb{Q}_s^R(C) = \mathbb{E}^{\mathcal{W}_d}\big[E_s^R(t, \mathbf{w}), \mathfrak{z}_s(t, \mathbf{w}) \in C\big].$$

Now, for any $\varphi \in C_c^2(\mathbb{R}^d \times \mathcal{O}(M) \times \mathcal{P}(\mathrm{End}(\mathbb{R}^d))$

$$\varphi\big(\mathbf{w}(t \wedge \zeta_R), \mathfrak{p}(t \wedge \zeta_R), \mathfrak{o}(t \wedge \zeta_R)\big) - \frac{1}{2}\int_0^{t \wedge \zeta_R} \sum_{k=1}^d \big(\mathfrak{Y}(e_k)\big)^2 \varphi\big(\mathbf{w}(\sigma), \mathfrak{p}(\sigma), \mathfrak{o}(\sigma)\big)\, d\sigma$$

is a $\mathbb{Q}_s^R$-martingale. On the other hand, one can see that

$$\varphi\big(\mathbf{w}(t), \mathfrak{p}(t), \mathfrak{o}(t)\big) - \frac{1}{2}\int_0^t \sum_{k=1}^d \big(\mathfrak{Y}(e_k)\big)^2 \varphi\big(\mathbf{w}(\sigma), \mathfrak{p}(\sigma), \mathfrak{o}(\sigma)\big)\, d\sigma$$

is a $\mathbb{P}$-martingale. Therefore, invoking [8, Theorem 3.10], we conclude that

$$\mathbb{Q}_s^R \restriction \mathcal{F}_{\zeta_R \wedge t} = \mathbb{P} \restriction \mathcal{F}_{\zeta_R \wedge t},$$

which implies that for large enough R, $s \in [0,1]$, and any bounded $\mathcal{F}_t$-measurable function $F : \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathcal{O}(M)) \times \mathcal{P}(\mathrm{End}(\mathbb{R}^d)) \to \mathbb{R}$,

$$\mathbb{E}^{\mathcal{W}_d}\big[E_s^{\mathbf{h},R}(\cdot \wedge t \wedge \zeta_R, \mathbf{w}) F\big(\mathfrak{z}_s(\cdot \wedge t \wedge \zeta_R, \mathbf{w})\big)\big] = \mathbb{E}^{\mathcal{W}_d}\big[F\big(\mathfrak{z}(\cdot \wedge t \wedge \zeta_R, \mathbf{w})\big)\big].$$

From this, taking derivatives with respect to s and using the bounds on the Ricci curvature and the estimates in [8, Theorem 8.62], one gets (B.13). $\square$

B.2. Application

The main motivation for this application is given at the end of this section for conditional expectations coming from heat kernels. In the next section we will apply these things to real heat kernels. Throughout this section, unless explicitly specified, we will drop the dependence on $\mathfrak{f}$ from $\mathfrak{p}(t, \mathfrak{f}, \mathbf{w})$. We begin with a definition.

Definition B.15. If N is a smooth Riemannian manifold and $(V, \|\cdot\|)$ is a normed vector space, we say that a smooth map $\mathcal{A}: \mathcal{O}(M) \times N \to V$, has at most polynomial growth in all its derivatives if for any positive integer n, there exist $N_n, C_n \geqslant 0$ so that for any vector fields $\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_n \in \{(X, Y) \in T\mathcal{O}(M) \times TN \mid |\pi_* X| \leqslant 1, |Y| \leqslant 1\}$ and $(\mathfrak{f}, y) \in \mathcal{O}(M) \times N$,

$$\left\|(\mathfrak{X}_1)_{(\mathfrak{f},y)}\mathfrak{X}_2 \ldots \mathfrak{X}_n \mathcal{A}\right\| \leqslant C_n\bigl(1 + \operatorname{dist}(\pi\mathfrak{f}, o)\bigr)^{N_n}. \tag{B.16}$$

For a given normed vector space $(V, \|\cdot\|)$, consider a map $\mathfrak{A}: \mathcal{O}(M) \times O(d) \to \operatorname{End}(V)$, and for $(\mathfrak{p}, \mathfrak{o}) \in \mathcal{P}(\mathcal{O}(M)) \times \mathcal{P}(O(d))$, define $\mathfrak{U}_t(\mathfrak{p}, \mathfrak{o})$ to be the solution to the ODE

$$\begin{cases} \dot{\mathfrak{U}}_t(\mathfrak{p}, \mathfrak{o}) = \mathfrak{U}_t(\mathfrak{p}, \mathfrak{o})\mathfrak{A}(\mathfrak{p}(t), \mathfrak{o}(t)), \\ \mathfrak{U}_0(\mathfrak{p}, \mathfrak{o}) = I_V. \end{cases} \tag{B.17}$$

Also, recall the map $\mathfrak{S} \in C^\infty(\mathcal{O}(M); \operatorname{Hom}(\mathbb{R}^d; o(d)))$ used in defining Eq. (B.7). Here Ω is the $o(d)$-valued 2-form defined by $[\mathfrak{E}(\xi), \mathfrak{E}(\eta)] = -\lambda(\Omega(\xi, \eta))$ (see [8, 8.44]).

Assumption. We assume that $\mathfrak{S}$, Ω and $\mathfrak{A}$ are of at most polynomial growth in all their derivatives and there is a smooth function $\phi: \mathcal{O}(M) \to \mathbb{R}$, and a constant $C > 0$ such that,

$$\bigl\langle \mathfrak{A}(\mathfrak{f}, \mathfrak{o})\xi, \xi \bigr\rangle \leqslant \phi(\mathfrak{f})|\xi|^2 \leqslant C \operatorname{dist}(\pi\mathfrak{f}, o)|\xi|^2 \quad \text{for } \mathfrak{f} \in \mathcal{O}(M),\ \xi \in V. \tag{B.18}$$

The main result is the following.

Theorem B.19. *For $f \in C_c^\infty(M; \mathbb{R})$, $G \in C^\infty(O(d); \operatorname{End}(V))$, and any horizontal vector fields $\mathfrak{X}_1, \ldots, \mathfrak{X}_m$, there exist $\mathfrak{T}_1(t), \mathfrak{T}_2(t), \ldots, \mathfrak{T}_l(t)$ products of multiple* $\operatorname{End}(V)$*-valued classic or stochastic integrals with the integrands bounded polynomially in terms of the distance function and $\Psi(t, \cdot) \in \bigcap_{p \geqslant 1} L^p(\mathcal{W}_d; \operatorname{End}(V))$, so that*

$$\bigl\|\Psi(t, \mathbf{w})\bigr\| \leqslant \bigl(\bigl\|\mathfrak{T}_1(t, \mathbf{w})\bigr\| + \cdots + \bigl\|\mathfrak{T}_l(t, \mathbf{w})\bigr\|\bigr) \exp\left(\int_0^t \phi\bigl(\mathfrak{p}(\sigma, \mathbf{w})\bigr)\, d\sigma\right), \tag{B.20}$$

and

$$\begin{aligned} &\mathbb{E}^{\mathcal{W}_d}\bigl[\bigl(\mathfrak{X}_1 \ldots \mathfrak{X}_n(f \circ \pi)\bigr)\bigl(\mathfrak{p}(t, \mathbf{w})\bigr)\mathfrak{U}_t\bigl(\mathfrak{p}(\cdot, \mathbf{w}), \mathfrak{o}(\cdot, \mathbf{w})\bigr)G\bigl(\mathfrak{o}(t, \mathbf{w})\bigr)\bigr] \\ &\quad = \mathbb{E}^{\mathcal{W}_d}\bigl[f\bigl(\pi\bigl(\mathfrak{p}(t, \mathbf{w})\bigr)\bigr)\Psi(t, \mathbf{w})\bigr]. \end{aligned} \tag{B.21}$$

Moreover, if $\Gamma \subset \mathcal{O}(M)$ is an open set, and $\zeta_\Gamma(\mathbf{w}) = \inf\{t > 0, \mathfrak{p}(\cdot, \mathbf{w}) \in \Gamma^c\}$ the first exit time of $\mathfrak{p}(\cdot, \mathbf{w})$ from Γ, then $\Psi(t)$ on the set $\{\zeta_\Gamma \geqslant t\}$ depends only on the support of f, the vectors $\mathfrak{X}_1, \ldots, \mathfrak{X}_m$, and the restriction to Γ and $\Gamma \times O(d)$ of f, $\mathfrak{S}$, Ω, respectively $\mathfrak{A}$.

Proof. The idea is to use Theorem B.11 to move the derivatives off of $f \circ \pi$ to the other side.

Because the function $f \circ \pi$ has compact support, we may assume that the vector fields $\mathfrak{X}_i$, $i = 1, \ldots, m$, also have compact support. Next, notice that a compactly supported horizontal vector field $\mathfrak{X}$, can be written as $\mathfrak{X} = \sum_{k=1}^d a_k \mathfrak{E}(e_k)$ where a_k, $k = 1, \ldots, d$, are compactly supported functions. Therefore, one can reduce the problem to one in which all vectors fields $\mathfrak{X}_i$, $i = 1, \ldots, m-1$, are of the form $\mathfrak{E}(e_{j_i})$, and $\mathfrak{X}_m = a\mathfrak{E}(e_{j_m})$ with a a compactly supported function. If $a : \mathcal{O}(M) \to \mathbb{R}$, $\mathbf{h}_k(s) = \frac{s}{t} e_k$ and $\Phi_a(\mathfrak{p}) = a(\mathfrak{p}(t))$, then

$$\big[X(\mathbf{h}_k)\Phi_a\big]\big(\mathfrak{p}(\cdot, \mathbf{w})\big) = \mathfrak{E}(e_k)_{\mathfrak{p}(t,\mathbf{w})} a + \sum_{l=1}^d \lambda\left(\int_0^t \Omega_{\mathfrak{p}(\sigma,\mathbf{w})}\big(e_l, \mathbf{h}_k(\sigma)\big) \circ d\mathbf{w}_l(\sigma)\right) a.$$

Applying this to $f \circ \pi$, one gets

$$\big(\mathfrak{E}(e_k)(f \circ \pi)\big)\big(\mathfrak{p}(t, \mathbf{w})\big) = \big[X(\mathbf{h}_k)\Phi_{f\circ\pi}\big]\big(\mathfrak{p}(\cdot, \mathbf{w})\big),$$

and consequently, one can use Theorem B.11 to move all the derivatives of $f \circ \pi$. After doing this, all that one has to do is to estimate derivatives of solutions to stochastic integral equations.

For a purely stochastic or classical integral, the estimates are obtained by simply computing the derivatives. For example, in the case of an integral involving the Ricci curvature, we point out that

$$\begin{aligned} &X(\mathbf{h}_{l_r}) \ldots X(\mathbf{h}_{l_1}) \int_0^t \left\langle \dot{\mathbf{h}}_k(\sigma) + \frac{1}{2}\mathfrak{R}_{\mathfrak{p}(\sigma,\mathbf{w})}\mathbf{h}_k(\sigma), d\mathbf{w}(\sigma)\right\rangle \\ &\quad = \frac{1}{2}\int_0^t \big\langle \big(\mathfrak{E}(e_{l_r})_{\mathfrak{p}(\sigma,\mathbf{w})} \ldots \mathfrak{E}(e_{l_1})\mathfrak{R}\big)\mathbf{h}_k(\sigma), d\mathbf{w}(\sigma)\big\rangle. \end{aligned} \tag{B.22}$$

Similar equalities can be obtained for other integrals involving the full curvature.

For the estimates of $X(\mathbf{h}_{l_r}) \ldots X(\mathbf{h}_{l_1})G$ or $X(\mathbf{h}_{l_r}) \ldots X(\mathbf{h}_{l_1})\mathfrak{U}_t$, we use induction.

First observe that the estimates of $X(\mathbf{h}_{l_1}) \ldots X(\mathbf{h}_{l_m})G$ are reduced to estimates of $X(\mathbf{h}_{l_1}) \ldots X(\mathbf{h}_{l_m})\mathfrak{o}$. For this purpose, notice that, since $\mathfrak{S}$ is skew-symmetric, $\mathfrak{o}(t, \mathbf{w})$ is an orthogonal matrix. Set

$$\mathfrak{o}^m = X(\mathbf{h}_{l_m}) \ldots X(\mathbf{h}_{l_1})\mathfrak{o}.$$

From (B.7), $\mathfrak{O}_s(t,\mathbf{w})$ satisfies

$$d\mathfrak{O}_s(t,\mathbf{w}) = \mathfrak{O}_s(t,\mathbf{w})\mathfrak{S}_{\mathfrak{P}_s(t,\mathbf{w})}\big(\circ d\mathbf{W}_s(t,\mathbf{w})\big),$$

with $\mathfrak{O}_s(0,\mathbf{w}) = I$, thus the equation for $\mathfrak{o}^1$ is

$$d\mathfrak{o}^1(t,\mathbf{w}) = \mathfrak{o}^1(t,\mathbf{w})\mathfrak{S}_{\mathfrak{p}(t,\mathbf{w})}(\circ d\mathbf{w}) + \mathcal{Z}(t,\mathbf{w}) \tag{B.23}$$

with the initial condition $\mathfrak{o}^1(0,\mathbf{w}) = 0$, and $\mathcal{Z}(t,\mathbf{w})$, a multiple classic and stochastic integrals involving the curvature, $\mathfrak{o}$, canonical vector fields and $\mathfrak{S}$. In general, the solution to the Stratonovich stochastic differential equation

$$dY_t = Y_t\mathfrak{S}_{\mathfrak{p}(t,\mathbf{w})}(\circ d\mathbf{w}) + \circ dZ_t$$

with the initial condition $Y_0 = 0$, is given by

$$Y_t = \int_0^t (\circ dZ_\sigma)\mathfrak{o}(\sigma,\mathbf{w})^*\mathfrak{o}(t,\mathbf{w}).$$

In the same way we can estimate higher derivatives. The equation satisfied by $\mathfrak{o}^k$ is written in terms of the action of $X(\mathbf{h}_{l_s})\dots X(\mathbf{h}_{l_1})$, $0 \leqslant s \leqslant k-1$, on $\mathfrak{o}$, $\mathfrak{S}$ and the curvature, we already know how to control.

We point out the general structure of these derivatives, namely they are iterated Stratonovich integrals where the integrands are polynomially bounded in terms of the distance function on M.

For the case of $\mathfrak{U}_t$, set

$$\mathfrak{U}_t^m = X(\mathbf{h}_{l_m})\dots X(\mathbf{h}_{l_1})\mathfrak{U}_t.$$

The following lemma is needed here and has a standard proof, therefore will be omitted.

Lemma B.24. *Let $(H,\langle\cdot,\cdot\rangle)$ be a finite-dimensional vector space with an inner product. Let $t \in [0,\infty) \to X_t, Y_t \in \operatorname{End}(H)$ be two continuous maps, and $t:[0,\infty) \to \psi(t) \in \mathbb{R}$ a locally integrable function with the property that $\langle X_t\xi,\xi\rangle \leqslant \psi(t)|\xi|^2$ for any $t \geqslant 0$. If U_t is the solution to*

$$\dot{U}_t = U_tX_t + Y_t$$

then,

$$\|U_t\| \leqslant e^{\int_0^t \psi(s)\,ds}\left(\|U_0\| + \int_0^t \|Y_s\|e^{-\int_0^s \psi(\sigma)\,d\sigma}\,ds\right) \quad \text{for } t \geqslant 0,$$

where $\|\cdot\|$ is the operator norm.

From this lemma, with $Y = 0$ and $W_0 = I$, one gets

$$\mathfrak{U}_t\big(\mathfrak{p}(\cdot,\mathbf{w}),\mathfrak{o}(\cdot,\mathbf{w})\big) \leqslant C\exp\left(\int_0^t \phi\big(\mathfrak{p}(\sigma,\mathbf{w})\big)\,d\sigma\right).$$

Next, write the equation for $\mathfrak{U}^1$:

$$\begin{cases} d\mathfrak{U}_t^1 = \mathfrak{U}_t^1\mathfrak{A}(\mathfrak{p}(t,\mathbf{w}),\mathfrak{o}(t,\mathbf{w})) + \mathfrak{U}_t\mathfrak{A}^1(\mathfrak{p}(t,\mathbf{w}),\mathfrak{o}(t,\mathbf{w})), \\ \mathfrak{U}_0^1 = 0, \end{cases} \tag{$*$}$$

where, $\mathfrak{A}^1$ is given in terms of the action of the vector fields $X(\mathbf{h}_i)$, $i = 1,\dots,m$ on $\mathfrak{A}$, the curvature and $\mathfrak{o}$. To estimate $\mathfrak{U}^1$, we use again Lemma B.24 with $Y = \mathfrak{U}_t\mathfrak{A}^1(\mathfrak{p}(t,\mathbf{w}),$ $\mathfrak{o}(t,\mathbf{w}))$ together with the estimate on $\mathfrak{U}$ to arrive at

$$\|\mathfrak{U}_t^1\| \leqslant \mathfrak{V}_t^1 \exp\left(\int_0^t \phi\big(\mathfrak{p}(\sigma,\mathbf{w})\big)\,d\sigma\right),$$

where $\mathfrak{V}_t^1$ is the sum of norms of product of multiple classical or stochastic integrals with each integrand polynomially bounded in terms of the distance along the path $p(\cdot,\mathbf{w}) = \pi\mathfrak{p}(t,\mathbf{w})$. Similarly, one can estimate higher derivatives. For the kth derivative, we get

$$\|\mathfrak{U}_t^k\| \leqslant \mathfrak{V}_t^k \exp\left(\int_0^t \phi\big(\mathfrak{p}(\sigma,\mathbf{w})\big)\,d\sigma\right),$$

where $\mathfrak{V}_t^k$ is the sum of norms of product of multiple integrals, each integrand being polynomially bounded in terms of the distance along the path $p(\cdot,\pi\mathbf{w})$. From here, (B.20) follows.

Finally, using [8, Theorem 8.46] one can justify the integrability of

$$\exp\left(\int_0^t C\,\mathrm{dist}\big(p(\sigma,\mathbf{w}),o\big)\,d\sigma\right)$$

for any positive constant C. Using Burkholder's inequality one can prove that $\Psi(t) \in \bigcap_{p>1} L^p(\mathcal{W}_d,\mathrm{End}(V))$. The rest is straightforward. □

Next we want to describe the application of this theorem to the conditional expectations and in the next section we will apply these considerations to heat kernels analysis. Take a complete Riemannian manifold M with a compatible connection ∇ which has the same Laplacian on functions as the usual Laplacian. *We assume that the torsion of* ∇ *and the* *curvature of the Levi-Civita connection have at most polynomial growth at infinity in all their derivatives.* Our interest is in the conditional expectations of the form

$$\mathbb{E}^{\mathcal{W}_d}\big[U(t,x,\mathbf{w})\tau^{\nabla}_{p(\cdot,x,\mathbf{w})\restriction[t,0]}\delta_y\big(p(t,x,\mathbf{w})\big)\big], \tag{B.25}$$

where τ^{∇} is the parallel transport with respect to the connection ∇ extended to forms, and U satisfies the differential equation

$$\begin{cases} \dot{U}(t,x,\mathbf{w}) = U(t,x,\mathbf{w})\tau^{\nabla}_{p(\cdot,x,\mathbf{w})\restriction[t,0]}A(p(t,x,\mathbf{w}))\tau^{\nabla}_{p(\cdot,x,\mathbf{w})\restriction[0,t]}, \\ U(0,x,\mathbf{w}) = \mathrm{Id}_{\bigwedge_x(M)}, \end{cases}$$

where $A \in \mathrm{End}(\bigwedge(M))$ (i.e., $A(x): \bigwedge_x(M) \to \bigwedge_x(M)$) is a smooth tensor which has at most polynomial growth at infinity in all its derivatives and there exists $C > 0$ such that for any $x \in M$, $\langle A(x)\xi,\xi\rangle \leqslant C\,\mathrm{dist}(x,o)|\xi|^2$, $\xi \in \bigwedge_x(M)$.

Next we want to interpret the integral in (B.25) as an integral at the frame bundle level. To this end, take the tensor $S_X Y = \nabla_X Y - \nabla^{\mathrm{LC}}_X Y$ and its lift $\mathfrak{S}_{\mathfrak{f}}(\xi) = \mathfrak{f}^{-1}S_{\mathfrak{f}\xi}\mathfrak{f}$, and extend (cf. Definition 1.4) S and $\mathfrak{S}$ to act on forms. Then notice that this tensor is given entirely in terms of the torsion of ∇, therefore they have at most polynomial growth at infinity. Fix $\mathfrak{f} \in \mathcal{O}(M)$ so that $\pi\mathfrak{f} = x$, identify $\mathbb{R}^d$ with $T_x(M)$ by $\mathfrak{f}$, and set

$$\mathfrak{o}(t,\mathbf{w}) = \mathfrak{f}^{-1}\tau^{\nabla}_{p(\cdot,x,\mathbf{w})\restriction[t,0]}\tau_{p(\cdot,x,\mathbf{w})\restriction[0,t]}\mathfrak{f}.$$

Extend this to a map on forms, and observe that

$$d\mathfrak{o}(t,\mathbf{w}) = \mathfrak{o}(t,\mathbf{w})\mathfrak{S}_{\mathfrak{p}(t,\mathbf{w})}(\circ d\mathbf{w}).$$

Now, in distributional sense, one can write δ_y as a linear combinations of the form $X_1X_2\ldots X_d f$, with f a bounded measurable function and X_i, $i = 1,\ldots,d$, compactly supported vector fields. Taking $\mathfrak{X}_1,\mathfrak{X}_2,\ldots,\mathfrak{X}_d$, the horizontal lifts of $X_1,X_2,\ldots,X_d$, we can rewrite the integral (B.25) as

$$\mathbb{E}^{\mathcal{W}_d}\big[\big(\mathfrak{X}_1\mathfrak{X}_2\ldots\mathfrak{X}_d(f\circ\pi)\big)\big(\mathfrak{p}(t,\mathbf{w})\big)\mathfrak{U}(t,\mathbf{w})\mathfrak{o}(t,\mathbf{w})\big],$$

where $\mathfrak{A}(\mathfrak{g}) = \mathfrak{g}^{-1}A(\pi\mathfrak{g})\mathfrak{g}$ and

$$\begin{cases} \dot{\mathfrak{U}}(t,\mathbf{w}) = \mathfrak{U}(t,\mathbf{w})\mathfrak{o}(t,\mathbf{w})\mathfrak{A}(\mathfrak{p}(t,\mathbf{w}))\mathfrak{o}(t,\mathbf{w})^*, \\ \mathfrak{U}(0,\mathbf{w}) = \mathrm{Id}_{\mathbb{R}^d}. \end{cases}$$

Therefore we have,

Corollary B.26. *Under the assumption* (B.1) *on the Ricci curvature of the Levi-Civita connection, and if the torsion of the connection ∇ and the curvature of the Levi-Civita connection have at most polynomial growth at infinity in all their derivatives, then, there exists* *a progressively measurable map* $\Psi(t,x,y,\cdot) \in \bigcap_{p\geqslant 1} L^p(\mathcal{W}_d; \mathrm{Hom}(\bigwedge_y(M), \bigwedge_x(M)))$ *so that,*

$$\mathbb{E}^{\mathcal{W}_d}\big[U(t,x,\mathbf{w})\sigma^{\nabla}_{p(\cdot,x,\mathbf{w})\restriction[t,0]}\delta_y\big(p(t,x,\mathbf{w})\big)\big] = \mathbb{E}^{\mathcal{W}_d}\big[f\big(p(t,x,\mathbf{w})\Psi(t,x,y,\mathbf{w})\big)\big].$$

In addition, if $\Gamma \subset M$ *is an open set,* $x \in \Gamma \subset M$, $\zeta_\Gamma(\mathbf{w}) = \inf\{t > 0,\ p(\cdot,x,\mathbf{w}) \in \Gamma^c\}$ *the first exit time of the path* $p(\cdot,x,\mathbf{w})$ *from* Γ, *then* $\Psi(t,x,y,\mathbf{w})$ *on the set* $\{\zeta_\Gamma \geqslant t\}$ *depends only on the support of* f, *the vectors* $X_1, X_2, \ldots, X_d$ *and* f, S, R (*the curvature*), T (*the torsion of* ∇) *and* A *restricted to* Γ.

Appendix C. About semigroups and heat kernels

In this section we assume that M is a complete Riemannian manifold with the curvature having at most polynomial growth in all its derivatives and satisfying, for a certain constant $C > 0$ and a fixed reference point o,

$$-C\big(1 + \mathrm{dist}(z,o)^2\big)|X|_z^2 \leqslant \mathrm{Ric}_z(X_z, X_z), \quad \text{for all } z \in M,\ X_z \in T_z(M).$$

We will analyze the existence, basic properties and estimates for heat kernels of operators on $\bigwedge(M)$ of the form

$$L = -\Delta^{\nabla} + \sum_{j=1}^{d} B_1(E_j)\nabla_{E_j} + B_2, \tag{C.1}$$

where E_j, $j = 1,\ldots,d$, is any orthonormal basis and the various components satisfy:

(1) ∇ is a connection which is compatible with the metric on M and its Laplacian on functions is the same as the Laplacian of the Levi-Civita connection.
(2) In the notations of Definition 1.4, $B_1 = D^*S$ and $B_2 = D^*T$ where S is an odd tensor and T is an even tensor. Also we assume that $(B_1)_z(X_z)$ is skew-symmetric for all $z \in M$, $X_z \in T_z(M)$.
(3) There exists a constant $K \geqslant 0$ such that

$$\left\langle \left((B_2)_z + \sum_{i=1}^{d} B_1(E_i)_z^2 \right) X_z, X_z \right\rangle \leqslant K|X_z|^2 \quad \text{for any } z \in M,\ X_z \in T_z(M).$$

The next lemma gives estimates on the size of solutions to certain stochastic differential equations.

Lemma C.2. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a finite-dimensional vector space endowed with an inner product and* X_t, Y_t^i, $i = 1,\ldots,d$, *locally bounded progressively measurable* $\mathrm{End}(H)$-

valued processes such that Y_t^i is skew-symmetric for $i = 1, \dots, d$. Let V_t be the solution to the stochastic differential equation

$$\begin{cases} dV_t = V_t\big(X_t\,dt + \sum_{i=1}^d Y_t^i\,d\mathbf{w}_i(t)\big), \\ V_0 = \mathrm{Id}, \end{cases}$$

where $d\mathbf{w}_i$ stands for the Itô stochastic differential. If T_t is the solution to the Stratonovich equation

$$\begin{cases} dT_t = T_t \circ d\sum_{i=1}^d \int_0^t Y_\sigma^i\,d\mathbf{w}_i(\sigma), \\ T_0 = \mathrm{Id}, \end{cases}$$

and W_t is the solution to the ODE

$$\begin{cases} \dot W_t = W_t T_t\big(X_t - \frac{1}{2}\sum_{i=1}^d (Y_t^i)^2\big)T_t^{-1}, \\ W_0 = \mathrm{Id}. \end{cases}$$

Then T_t is unitary for any $t \geqslant 0$ and

$$V_t = W_t T_t.$$

Therefore, estimates on the size of V_t reduces to estimates on the size of W_t.

Proof. Rewrite the equation for V in the Stratonovich form:

$$\begin{cases} dV_t = V_t X_t\,dt + V_t \circ d\sum_{i=1}^d \int_0^t Y_\sigma^i\,d\mathbf{w}_i(\sigma) - \frac{1}{2}d\big\langle\!\big\langle V_t, \int_0^t \sum_{i=1}^d Y_\sigma^i\,d\mathbf{w}_i(\sigma)\big\rangle\!\big\rangle, \\ V_0 = \mathrm{Id}, \end{cases}$$

where for two $\mathrm{End}(H)$-valued martingales A and B, $\langle\!\langle A, B\rangle\!\rangle$ is the $\mathrm{End}(H)$-valued process with bounded variation such that $AB - \langle\!\langle A, B\rangle\!\rangle$ is an $\mathrm{End}(H)$-valued martingale. Notice that if A and B are two $\mathrm{End}(H)$-valued semimartingales, then $\langle\!\langle A, B\rangle\!\rangle$ is the corresponding process associated to their martingale parts. Since

$$V_t = \mathrm{Id} + \int_0^t V_\sigma \sum_{i=1}^d Y_\sigma^i\,d\mathbf{w}_i(\sigma) + \int_0^t V_\sigma X_\sigma\,d\sigma,$$

we obtain

$$\begin{aligned} \left\langle\!\!\left\langle V_t, \int_0^t \sum_{i=1}^d Y_\sigma^i\,d\mathbf{w}_i(\sigma)\right\rangle\!\!\right\rangle &= \left\langle\!\!\left\langle \int_0^t U_\sigma \sum_{i=1}^d Y_\sigma^i\,d\mathbf{w}_i(\sigma), \int_0^t \sum_{j=1}^d Y_\sigma^j\,d\mathbf{w}_j(\sigma)\right\rangle\!\!\right\rangle \\ &= \sum_{i=1}^d \int_0^t V_\sigma Y_\sigma^i Y_\sigma^i\,d\sigma = \sum_{i=1}^d \int_0^t V_\sigma \big(Y_\sigma^i\big)^2\,d\sigma. \end{aligned}$$

Then, the equation for V becomes

$$dV_t = V_t\left(\circ d\sum_{i=1}^{d}\int_0^t Y_\sigma^i\, d\mathbf{w}_i(\sigma) + \left(X_t - \frac{1}{2}\sum_{i=1}^{d}\big(Y_t^i\big)^2\right)dt\right).$$

Now, because $(Y_\sigma^i)^* = -Y_\sigma^i$, an easy computation shows that $\circ d(T_t T_t^*) = 0$, which proves that T_t is an isometry. Further,

$$\begin{aligned}\circ d W_t T_t &= (\circ d W_t)T_t + W_t \circ dT_t \\ &= W_t T_t\left(\circ d\sum_{i=1}^{d}\int_0^t Y_\sigma^i\, d\mathbf{w}_i(\sigma) + \left(X_t - \frac{1}{2}\sum_{i=1}^{d}\big(Y_t^i\big)^2\right)dt\right)\end{aligned}$$

with $W_0 T_0 = \mathrm{Id}$. Hence, by uniqueness $V_t = W_t T_t$. □

Theorem C.3. *The operator L in* (C.1) *determines a semigroup $\{\mathbf{P}_t^L\colon t \geqslant 0\}$ whose action on compactly supported forms is given by*

$$\big(\mathbf{P}_t^L\omega\big)(z) = \mathbb{E}^{\mathcal{W}_d}\big[U(t,z,\mathbf{w})\tau^{\nabla}_{p(\cdot,z,\mathbf{w})\restriction[t,0]}\omega\big(p(t,z,\mathbf{w})\big)\big] \tag{C.4}$$

where the parallel transport τ^{∇} is the parallel transport with respect to the connection ∇ defined in Section 3.3*, and U is the solution to the equation*

$$\begin{cases} dU(t,z,\mathbf{w}) = U(t,z,\mathbf{w})\big(B_2(t,z,\mathbf{w})\,dt + \sum_{j=1}^{d}(B_1)_j(t,z,\mathbf{w})\,d\mathbf{w}_j(t)\big), \\ U(0,z,\mathbf{w}) = \mathrm{Id}_{\bigwedge_z(M)}, \end{cases}$$

with E_j, $j = 1,\dots,d$, an orthonormal basis of $T_z(M)$, and

$$\begin{aligned} B_2(t,z,\mathbf{w}) &= \tau^{\nabla}_{p(\cdot,z,\mathbf{w})\restriction[t,0]}(B_2)_{p(t,z,\mathbf{w})}\tau^{\nabla}_{p(\cdot,z,\mathbf{w})\restriction[0,t]}, \\ (B_1)_j(t,z,\mathbf{w}) &= \tau^{\nabla}_{p(\cdot,z,\mathbf{w})\restriction[t,0]}B_1\big(\tau^{\nabla}_{p(\cdot,z,\mathbf{w})\restriction[0,t]}E_j\big)_{p(t,z,\mathbf{w})}\tau^{\nabla}_{p(\cdot,z,\mathbf{w})\restriction[0,t]}. \end{aligned}$$

Moreover, if the tensors B_1 and B_2 have at most polynomial growth in all their derivatives (cf. Definition B.15*), then the heat kernel $\mathbf{p}^L(t,z_1,z_2)\colon \bigwedge_{z_2}^k M \to \bigwedge_{z_1}^k M$ of L exists and can be written as*

$$\mathbf{p}^L(t,z_1,z_2) = \mathbb{E}^{\mathcal{W}_d}\big[U(t,z_1,\mathbf{w})\tau^{\nabla}_{p(\cdot,z_1,\mathbf{w})\restriction[t,0]}\delta_{z_2}\big(p(t,z_1,\mathbf{w})\big)\big],$$

interpreted via integration by parts on the path space.

Proof. The proof of (C.4) is standard and therefore will be omitted. We point out that in order to justify the integrability in (C.4), one can use Proposition C.2 together with Lemma B.24 to show that U is a bounded map.

For the existence of the heat kernels one can follow the proof outlined in [8, Theorem 6.25], which extends to this case as well since the semigroup here has all the essential properties needed in the proof there. □

The following proposition is a basic tool for heat kernels estimates.

Proposition C.5. *Assume there is a smooth function $\phi : M \to \mathbb{R}$ bounded from above such that*

$$\left\langle (B_2)_z X_z + \sum_{i=1}^{n} B_1(E_i)_z^2 X_z, X_z \right\rangle \leqslant \phi(z)|X_z|^2 \quad \textit{for } z \in M,\ X_z \in T_z(M).$$

Then

$$\left\| \mathbf{p}^L(t, z_1, z_2) \right\| \leqslant \mathbf{p}^{\phi}(t, z_1, z_2)$$

where the last quantity is the heat kernel of the operator $\Delta + \phi$ on functions.

Proof. Using the expression for the semigroups given in Proposition C.3, the estimates in Proposition C.2 and Lemma B.24, we get the bounds on the semigroups. By taking an approximate identity of the delta function in distributional sense, one gets from estimates on semigroups to the estimates on the heat kernels. □

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