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On the differentiable vectors for contragredient representations

Sur les vecteurs différentiables par rapport aux représentations cotragrédientes

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ABSTRACT

We establish a few simple results on contragredient representations of Lie groups, with a view toward applications to the abstract characterization of some spaces of pseudo-differential operators. In particular, this method provides an abstract approach to J. Nourrigat's recent description of the norm closure of the pseudo-differential operators of order zero.

RÉSUMÉ

On obtient quelques résultats sur les représentations contragrédientes des groupes de Lie qui permettent d'aborder, d'une manière abstraite, la caractérisation de l'adhérence normique de l'ensemble des opérateurs pseudodifférentiels d'ordre zero obtenue récemment par J. Nourrigat.

1. Introduction

We study the abstract characterization of some spaces of pseudo-differential operators by using a few simple results on the contragredients of the Banach space representations of Lie groups. The applicability of the method based on a contragredient representation is due to the fact that such a representation may be discontinuous even if the original representation is continuous; see for instance the representation (4) below, which is discontinuous if $r = \infty$. In particular, we provide an abstract approach to J. Nourrigat's recent description [10] of the norm closure of the pseudo-differential operators of order zero (see Example 1 below), and we also bring additional information on some results from the earlier literature.

Preliminaries For any complex Banach space $\mathcal{Y}$, let $\mathcal{Y}^*$ be its topological dual and $\mathcal{B}(\mathcal{Y})^\times$ be the group of invertible elements in the Banach algebra $\mathcal{B}(\mathcal{Y})$ of all bounded linear operators. A Banach space representation of any group G is a group homomorphism $\pi : G \to \mathcal{B}(\mathcal{Y}_\pi)^\times$, where $\mathcal{Y}_\pi$ is a complex Banach space. The contragredient representation of π is $\pi^* : G \to \mathcal{B}(\mathcal{Y}_\pi^*)^\times$, $\pi^*(g) := \pi(g^{-1})^*$, so that $\mathcal{Y}_{\pi^*} := \mathcal{Y}_\pi^*$. If $\sup_{g\in G} \|\pi(g)\| < \infty$, then π is called uniformly bounded, and in this case also π^* is uniformly bounded.

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Now assume that G is a topological group and define $\mathcal{Y}_{\pi_0} := \{x \in \mathcal{Y}_\pi \mid \pi(\cdot)x \in \mathcal{C}(G, \mathcal{Y}_\pi)\}$, where $\mathcal{C}$ indicates the space of continuous mappings. Then $\mathcal{Y}_{\pi_0}$ is a closed linear subspace of $\mathcal{Y}$ since π is uniformly bounded, and moreover $\mathcal{Y}_{\pi_0}$ is invariant under π. Then $\pi_0 : G \to \mathcal{B}(\mathcal{Y}_{\pi_0})$, $\pi_0(g) := \pi(g)|_{\mathcal{Y}_{\pi_0}}$ is a strongly continuous representation. We can similarly define $\mathcal{Y}_{\pi_0^*} := \{\xi \in \mathcal{Y}_\pi^* \mid \pi^*(\cdot)\xi \in \mathcal{C}(G, \mathcal{Y}_\pi^*)\} = \{\xi \in \mathcal{Y}_\pi^* \mid \lim_{g\to\mathbf{1}} \|\pi^*(g)\xi - \xi\| = 0\}$ and $\pi_0^* : G \to \mathcal{B}(\mathcal{Y}_{\pi_0^*})^\times$, $\pi_0^*(g) := \pi^*(g)|_{\mathcal{Y}_{\pi_0^*}}$. If moreover G is a Lie group, then we also define $\mathcal{Y}_\pi^k := \{y \in \mathcal{Y} \mid \pi(\cdot)y \in \mathcal{C}^k(G, \mathcal{Y})\}$ for every integer $k \geqslant 0$, so $\mathcal{Y}_\pi^0 = \mathcal{Y}_{\pi_0}$. If the representation π is strongly continuous, that is, $\mathcal{Y} = \mathcal{Y}_{\pi_0}$, then for every basis $\{X_1, \dots, X_m\}$ in the Lie algebra $\mathfrak{g}$ of G:

$$(\forall k \geqslant 1) \quad \mathcal{Y}_\pi^k = \bigcap_{1\leqslant j_1,\dots,j_k\leqslant m} \mathcal{D}\big(\mathrm{d}\pi(X_{j_1}) \cdots \mathrm{d}\pi(X_{j_k})\big) \tag{1}$$

(see for instance [8, Th. 9.4]). Here we denote by $\mathcal{D}(T)$ the domain of any unbounded operator T.

2. The main abstract results

The following theorem can be regarded as a version of (1) for some discontinuous representations of Lie groups, namely for the contragredient of any uniformly bounded and strongly continuous representation.

Theorem 2.1. *Let G be a Lie group with a strongly continuous representation $\pi : G \to \mathcal{B}(\mathcal{Y})$ which is also assumed to be uniformly bounded. If $\{X_1, \dots, X_m\}$ is any basis in the Lie algebra $\mathfrak{g}$ of G, then for every integer $k \geqslant 1$, we have $\mathcal{Y}_{\pi^*}^k \subseteq \bigcap_{1\leqslant j_1,\dots,j_k\leqslant m} \mathcal{D}(\mathrm{d}\pi(X_{j_1})^* \cdots \mathrm{d}\pi(X_{j_k})^*) \subseteq \mathcal{Y}_{\pi^*}^{k-1}$, and these inclusions could be simultaneously strict.*

It will be convenient to denote $\mathcal{C}^k(\pi^*) := \bigcap_{1\leqslant j_1,\dots,j_k\leqslant m} \mathcal{D}(\mathrm{d}\pi(X_{j_1})^* \cdots \mathrm{d}\pi(X_{j_k})^*)$ for arbitrary $k \geqslant 1$. It is clear that $\mathcal{C}^1(\pi^*) \supseteq \mathcal{C}^2(\pi^*) \supseteq \cdots$. The proof will be based on the following auxiliary result, which should be thought of as an embedding lemma on abstract Sobolev spaces.

Lemma 2.2. *We have $\mathcal{C}^1(\pi^*) \subseteq \mathcal{Y}_{\pi_0^*}$.*

Proof. For every $X \in \mathfrak{g}$ let us denote $\gamma_X : \mathbb{R} \to G$, $\gamma_X(t) := \exp_G(tX)$. It follows by [9, Th. 1.3.1] that:

$$\mathcal{D}\big(\mathrm{d}\pi(X)^*\big) \subseteq \mathcal{Y}_{\pi^*\circ\gamma_X} = \big\{\xi \in \mathcal{Y}^* \,\big|\, \pi^*\big(\gamma_X(\cdot)\big)\xi \in \mathcal{C}\big(\mathbb{R}, \mathcal{Y}^*\big)\big\} \tag{2}$$

for arbitrary $X \in \mathfrak{g}$. On the other hand, the inclusion $\subseteq$ in the following equality is obvious:

$$\mathcal{Y}_{\pi^*} = \mathcal{Y}_{\pi^*\circ\gamma_{X_1}} \cap \cdots \cap \mathcal{Y}_{\pi^*\circ\gamma_{X_m}} \tag{3}$$

while the inclusion $\supseteq$ holds true for the following reason. For all $t_1, \dots, t_m \in \mathbb{R}$ and $\xi \in \mathcal{Y}^*$ we have:

$$\begin{aligned}\big\|\pi^*\big(\gamma_{X_1}(t_1)\cdots\gamma_{X_m}(t_m)\big)\xi - \xi\big\| &\leqslant \sum_{j=1}^m \big\|\pi^*\big(\gamma_{X_1}(t_1)\cdots\gamma_{X_{j-1}}(t_{j-1})\big)\big(\pi^*\big(\gamma_{X_j}(t_j)\big)\xi - \xi\big)\big\| \\ &\leqslant M \sum_{j=1}^m \big\|\pi^*\big(\gamma_{X_j}(t_j)\big)\xi - \xi\big\|\end{aligned}$$

where $M := \sup_{g\in G} \|\pi(g)\|$. Since $\{X_1, \dots, X_m\}$ is a basis in $\mathfrak{g}$, the map $(t_1, \dots, t_m) \mapsto \gamma_{X_1}(t_1)\cdots\gamma_{X_m}(t_m)$ is a local diffeomorphism at $0 \in \mathbb{R}^m$, and then the above estimate shows that, for every $\xi \in \bigcap_{j=1}^m \mathcal{Y}_{\pi^*\circ\gamma_{X_j}}$, we have $\lim_{g\to\mathbf{1}} \|\pi^*(g)\xi - \xi\| = 0$, hence $\xi \in \mathcal{Y}_{\pi^*}$. This completes the proof of (3).

Now the assertion follows by (2) and (3), since $\mathcal{D}(\mathrm{d}\pi(X_1)^*) \cap \cdots \cap \mathcal{D}(\mathrm{d}\pi(X_m)^*) = \mathcal{C}^1(\pi^*)$. □

Proof of Theorem 2.1. By using Lemma 2.2 and [11, Lemma 1.1] we obtain

$$\mathcal{C}^k\big(\pi^*\big) \subseteq \bigcap_{1\leqslant j_1,\dots,j_{k-1}\leqslant m} \mathcal{D}\big(\mathrm{d}\pi_0^*(X_{j_1})\cdots\mathrm{d}\pi_0^*(X_{j_{k-1}})\big) = \mathcal{Y}_{\pi^*}^{k-1}$$

where the latter equality follows by using (1) for the strongly continuous representation π_0^*. The inclusion $\mathcal{Y}_{\pi^*}^k \subseteq \mathcal{C}^k(\pi^*)$ can be easily proved by using (1) and the fact that for every $X \in \mathfrak{g}$ we have $\mathcal{D}(\mathrm{d}\pi_0^*(X)) \subset \mathcal{D}(\mathrm{d}\pi(X)^*)$ and $\mathrm{d}\pi(X)^*|_{\mathcal{D}(\mathrm{d}\pi_0^*(X))} = \mathrm{d}\pi_0^*(X)$.

We now prove by example that the inclusion in the statement can be strict for $k = 1$. Let $G = \mathbb{R}$, $\mathcal{Y}$ be the space of trace-class operators on $L^2(\mathbb{R})$, and $\rho : \mathbb{R} \to \mathcal{B}(L^2(\mathbb{R}))$, $\rho(t)f = f(\cdot + t)$. Then define $\pi : \mathbb{R} \to \mathcal{B}(\mathcal{Y})$, $\pi(t)A = \rho(t)A\rho(t)^{-1}$ and for every $\phi \in L^\infty(\mathbb{R})$ let $\phi(Q)$ be the multiplication-by-ϕ operator on $L^2(\mathbb{R})$, so that $\phi(Q) \in \mathcal{B}(L^2(\mathbb{R})) \simeq \mathcal{Y}^*$. It was noted in [1, Ex. 6.2.7] that $\phi(Q) \in \mathcal{Y}^k_{\pi^*}$ if and only if the first k derivatives of ϕ exist, are bounded, and the k-th derivative is also uniformly continuous on $\mathbb{R}$.

On the other hand, if we denote by $P = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}$ the infinitesimal generator of ρ, then it is easily checked that $\phi(Q) \in \mathcal{C}^1(\pi^*)$ if and only if the commutator $[\phi(Q), P]$ belongs to $\mathcal{B}(L^2(\mathbb{R}))$, hence by using also [1, Prop. 5.1.2(b)] and again [1, Ex. 6.2.7], we see that the latter commutator condition is equivalent to the fact that ϕ is bounded and satisfies the Lipschitz condition globally on $\mathbb{R}$. Therefore there exist $\phi, \psi \in L^\infty(\mathbb{R})$ such that $\phi(Q) \in \mathcal{C}^1(\pi^*) \setminus \mathcal{Y}^1_{\pi^*}$ and $\psi(Q) \in \mathcal{Y}^2_{\pi^*} \setminus \mathcal{C}^1(\pi^*)$. This completes the proof. □

Corollary 2.3. *In Theorem 2.1, the subspace $\bigcap_{k\geqslant 1} \bigcap_{1\leqslant j_1,\dots,j_k\leqslant m} \mathcal{D}(\mathrm{d}\pi(X_{j_1})^* \cdots \mathrm{d}\pi(X_{j_k})^*)$ is dense in $\mathcal{Y}_{\pi_0^*}$.*

Proof. It follows by Theorem 2.1 that this linear subspace is equal to the space of smooth vectors for the strongly continuous representation π_0^*, hence it is dense in the representation space $\mathcal{Y}_{\pi_0^*}$ (see [6]). □

3. Applications

We will develop here a more general version of the example used in the proof of Theorem 2.1. Let G be a Lie group with a continuous unitary representation $\rho : G \to \mathcal{B}(\mathcal{H})$. If $1 \leqslant p < \infty$, denote by $\mathfrak{S}_p(\mathcal{H})$ the p-th Schatten ideal, and let $\mathfrak{S}_\infty(\mathcal{H}) := \mathcal{B}(\mathcal{H})$ and $\mathfrak{S}_0(\mathcal{H})$ be the ideal of all compact operators on $\mathcal{H}$. It is well known that if $p, q \in \{0\} \cup [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $p \neq \infty$, then an isometric linear isomorphism $\mathfrak{S}_p(\mathcal{H})^* \simeq \mathfrak{S}_q(\mathcal{H})$ is defined by the trace duality pairing. The representation $\rho^{(q)}$ can thus be regarded as the contragredient representation of the *strongly continuous* representation $\rho^{(p)}$, where (see also [3]):

$$\big(\forall r \in \{0\} \cup [1,\infty]\big) \quad \rho^{(r)} : G \to \mathcal{B}\big(\mathfrak{S}_r(\mathcal{H})\big), \qquad \rho^{(r)}(g)Y = \rho(g)Y\rho(g)^{-1} \tag{4}$$

In the special case of the Heisenberg group, the following corollary establishes a direct relationship between the classical characterizations of pseudo-differential operators from [2] and [5].

Corollary 3.1. *In the above setting, pick any basis $\{X_1, \dots, X_m\}$ in the Lie algebra $\mathfrak{g}$ of G. Assume $1 \leqslant q \leqslant \infty$ and denote $\Psi_q(\rho) := \{Y \in \mathfrak{S}_p(\mathcal{H}) \mid \rho^{(q)}(\cdot)Y \in \mathcal{C}^\infty(G, \mathfrak{S}_p(\mathcal{H}))\}$. Then we have:*

(i) *The linear subspace $\Psi_q(\rho)$ is precisely the set of all $Y \in \mathfrak{S}_q(\mathcal{H})$ such that for arbitrary $k \geqslant 1$ and $j_1, \dots, j_k \in \{1, \dots, m\}$ we have $[\mathrm{d}\rho(X_{j_1}), \dots, [\mathrm{d}\rho(X_{j_k}), Y] \dots] \in \mathfrak{S}_q(\mathcal{H})$.*
(ii) *If $1 \leqslant q < \infty$, then $\Psi_q(\rho)$ is dense in $\mathfrak{S}_q(\mathcal{H})$. If $q = \infty$, then $\Psi_\infty(\rho)$ contains $\mathfrak{S}_0(\mathcal{H})$ and is dense in the norm-closed subspace $\{Y \in \mathcal{B}(\mathcal{H}) \mid \rho^{(\infty)}(\cdot)Y \in \mathcal{C}(G, \mathcal{B}(\mathcal{H}))\}$ of $\mathcal{B}(\mathcal{H})$.*

Proof. Note that $\mathcal{C}^1(\rho^{(q)}) = \{Y \in \mathfrak{S}_q(\mathcal{H}) \mid [\mathrm{d}\rho(X_j), Y] \in \mathfrak{S}_q(\mathcal{H})$ for $j = 1, \dots, m\}$. Then both assertions are special cases of Theorem 2.1 and Corollary 2.3. □

We can now prove a corollary which shows that the first two conditions in [7, Th. 1] are equivalent, irrespective of the unitary representation involved therein. This also shows that the $\mathcal{C}^\infty$ part of the relation between differentiability and existence of commutators suggested after [5, Eq. (8.4)] holds true, although the $\mathcal{C}^1$ part of that suggestion fails to be true, since the following corollary would be false with the class $\mathcal{C}^\infty$ replaced by $\mathcal{C}^k$ for any $k < \infty$. In fact, recall from the proof of Theorem 2.1 that the corresponding inclusions are strict in a special instance of the present setting, which is precisely the situation of [5].

Corollary 3.2. *If $Y \in \mathcal{B}(\mathcal{H})$ then the above mapping $\rho^{(\infty)}(\cdot)Y : G \to \mathcal{B}(\mathcal{H})$ is of class $\mathcal{C}^\infty$ with respect to the norm operator topology on $\mathcal{B}(\mathcal{H})$ if and only if it is $\mathcal{C}^\infty$ with respect to the strong operator topology.*

Proof. The mapping $\rho^{(\infty)}(\cdot)Y : G \to \mathcal{B}(\mathcal{H})$ is smooth with respect to any topology on $\mathcal{B}(\mathcal{H})$ if and only if it is smooth on any neighborhood of $\mathbf{1} \in G$. On the other hand, just as in the proof of [1, Prop. 5.1.2(b)], one can see that this mapping is smooth with respect to the strong operator topology on $\mathcal{B}(\mathcal{H})$ if and only if the iterated commutator condition in Corollary 3.1(i) is satisfied, hence the conclusion follows by Corollary 3.1(i), where the smoothness of $\rho^{(\infty)}(\cdot)Y$ is understood with respect to the norm operator topology on $\mathfrak{S}_\infty(\mathcal{H}) = \mathcal{B}(\mathcal{H})$. □

Example 1. Let $G = \mathbb{H}_{2n+1}$ be the $(2n+1)$-dimensional Heisenberg group with the Schrödinger representation $\rho : G \to \mathcal{B}(\mathcal{H})$. As recalled in [10], for $1 \leqslant p \leqslant \infty$, the set $\Psi_p(\rho)$ of the above Corollary 3.1 is precisely the set of pseudo-differential operators on $L^2(\mathbb{R}^n)$ corresponding to the space of symbols $\{a \in \mathcal{C}^\infty(\mathbb{R}^{2n}) \mid (\forall \alpha \in \mathbb{N}^{2n})\ \partial^\alpha a \in L^p(\mathbb{R}^{2n})\}$ (see also [4] for similar results on more general nilpotent Lie groups). Thus our Corollary 3.1 leads to the main results of [10].

Example 2. The above Corollary 3.1 also provides additional information on pseudo-differential operators on a compact manifold acted on by a Lie group, as studied for instance in [12] and [7]. Thus, it follows that the notions of U-smoothness and $\mathfrak{A}$-smoothness from [12, Sect. 2] actually coincide.

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