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THE QUADRATIC CONTRIBUTION TO THE BACKSCATTERING TRANSFORM IN THE ROTATION INVARIANT CASE

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Abstract. Considerations of the backscattering data for the Schrödinger operator $H_v = -\Delta + v$ in $\mathbb{R}^n$, where $n \geq 3$ is odd, give rise to an entire analytic mapping from $C_0^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$, the backscattering transformation. The aim of this paper is to give formulas for $B_2(v,w)$ where B_2 is the symmetric bilinear operator that corresponds to the quadratic part of the backscattering transformation and v and w are rotation invariant.

1. **Introduction.** Considerations of the backscattering data for the Schrödinger operator $H_v = -\Delta + v$ in $\mathbb{R}^n$, where $n \geq 3$ is odd, give rise to an entire analytic mapping from $C_0^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$, the backscattering transformation. This is, up to a smooth term given by the negative eigenvalues of H_v and the corresponding eigenfunctions, the real part of the Fourier transform of the anti-diagonal part of the scattering amplitude (see [2] and [5]). The symmetric bilinear operator B_2 that corresponds to the quadratic part of the backscattering transformation is continuous from $C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ to $C_0^\infty(\mathbb{R}^n)$, and it extends by continuity to much larger spaces as shown in [1]. (See also [6] for a related regularity result for the Born approximation of the scattering data in the case $n = 3$.) Let $E(y,z)$ be the unique fundamental solution of the ultra-hyperbolic operator $\Delta_y - \Delta_z$ such that $E(y,z) = -E(z,y)$ and $E(y,z)$ is separately rotation invariant in both variables (see Corollary 10.2 of [5]). An explicit formula for B_2 is provided by Corollary 10.7 of [5], which shows that

$$B_2(v,w)(x) = \iint E(y,z)v(x+\frac{y+z}{2})w(x-\frac{y-z}{2})\,\mathrm{d}y\,\mathrm{d}z, \quad v,\, w \in C_0^\infty(\mathbb{R}^n), \tag{1.1}$$

where the integral is interpreted in the distribution sense.

The aim of this paper is to give explicit formulas for $B_2(v,w)$ when v and w are rotation invariant functions in $\mathbb{R}^n$. Then v and w may be considered as functions in one real variable, and this is also true for $B_2(v,w)$, since rotation invariance is preserved under B_2. This leads to the investigation of a symmetric bilinear operator $\widetilde{B_2}$ acting in spaces of functions on the real line, and we compute its distribution kernel. This enables explicit computation of $B_2(v,w)$ in many cases and illustrates the continuity results obtained in our paper [1]. Moreover, our formula for the

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distribution kernel of $\widetilde{B}_2$ will show that the restriction of $B_2(v,w)$ to any ball $\{x;\, |x| < R\}$ depends on the restriction of v and w to the same ball only. This somewhat surprising result holds for arbitrary odd $n \geq 3$ when v and w are rotation invariant.

Some notation and presentation of results. We assume $n = 2m+3$ is an odd integer ≥ 3 and let c_n denote the volume of the unit sphere in $\mathbb{R}^n$. Thus

$$c_n = 2\pi^{n/2}/\Gamma(n/2). \tag{1.2}$$

We shall also need the constants

$$C_n = \frac{(-1)^m}{c_n(2\pi)^{n-1}} \tag{1.3}$$

and

$$d_n = 2^{3-n}c_{n-1}/c_n,\ e_n = (-1)^m 2^{2m-1} m! d_n^2. \tag{1.4}$$

Points in $(\mathbb{R}^n)^N$ are written $(x_1,\dots,x_N)$ and Δ_j denotes the Laplacian in the variables x_j. The length of x_j is denoted r_j. We use the notation $\vec{s}$ for vectors (s_1,s_2,s_3) in $\mathbb{R}^3$, and by t_+^α we denote the function which is equal to t^α when $t > 0$ and vanishes when $t \leq 0$. Finally, $\mathcal{O}(n)$ denotes the group of orthogonal transformations in $\mathbb{R}^n$.

Consider the distribution

$$\beta(x,y) = \delta^{(n-2)}(\langle x,y\rangle) \tag{1.5}$$

in $\mathbb{R}^n \times \mathbb{R}^n$. The notation needs an explanation. Since the gradient of $\langle x,y\rangle$ is nonvanishing outside the origin the right-hand side of (1.5) is a well-defined distribution outside the origin in $\mathbb{R}^n \times \mathbb{R}^n$. It is homogeneous of degree $2-2n$ and $\beta(x,y)$ is its unique homogeneous extension to $\mathbb{R}^n \times \mathbb{R}^n$. (We refer to [3] for useful background material in distribution theory.) In this notation we have (see [5])

$$E(y,z) = -2^{2m}c_nC_n\beta(y-z,y+z). \tag{1.6}$$

Definition 1.1. We define the distribution $B \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ through

$$B(x_1,x_2,x_3) = C_n\beta(x_2-x_1,x_3-x_1). \tag{1.7}$$

The definition makes sense since the linear mapping which sends $(x_1,x_2,x_3) \in (\mathbb{R}^n)^3$ to $(x_2-x_1,x_3-x_1) \in (\mathbb{R}^n)^2$ is surjective.

A linear change of coordinates in (1.1) together with (1.6), shows that

$$\langle u, B_2(v,w)\rangle = 2^{-1}c_n\langle u\otimes v\otimes w, B\rangle \tag{1.8}$$

when $u,\ v,\ w \in C_0^\infty(\mathbb{R}^n)$. Hence B is a constant times the distribution kernel of B_2.

Assume now that $v,\ w \in C_0^\infty(\mathbb{R}^n)$ are rotation invariant. Then $B_2(v,w)$ is uniquely determined by the values of the right-hand side of (1.8) when u ranges over the set of rotation invariant test functions in $\mathbb{R}^n$. Writing

$$u(x) = f(|x|^2), \quad v(x) = g(|x|^2), \quad w(x) = h(|x|^2)$$

we have

$$\langle u\otimes v\otimes w, B\rangle = \langle f\otimes g\otimes h, \Pi_*B\rangle, \tag{1.9}$$

where $\Pi_*B \in \mathcal{D}'(\mathbb{R}^3)$ is the push-forward of B under the mapping

$$\Pi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ni (x_1,x_2,x_3) \mapsto (|x_1|^2,|x_2|^2,|x_3|^2) \in \mathbb{R}^3. \tag{1.10}$$

This means that

$$\langle \varphi, \Pi_*B\rangle = \langle \Pi^*\varphi, B\rangle = \langle \varphi\circ\Pi, B\rangle$$

when φ is an arbitrary test function in $\mathbb{R}^3$. Since $s_j \geq 0$ in the image of Π it follows that $\Pi_* B$ is supported in $(\overline{\mathbb{R}}_+)^3$.

Definition 1.2. We let $\widetilde{B_2}$ be the bilinear operator from $C_0^\infty(\mathbb{R}) \times C_0^\infty(\mathbb{R})$ to $\mathcal{D}'(\mathbb{R})$ with distribution kernel $\Pi_* B$, that is, given by

$$\langle f, \widetilde{B_2}(g,h)\rangle = \langle f \otimes g \otimes h, \Pi_* B\rangle, \quad f,g,h \in C_0^\infty(\mathbb{R}). \tag{1.11}$$

Since composition by the mapping which sends $x \in \mathbb{R}^n$ to $|x|^2 \in \mathbb{R}$ is continuous from $C_0^\infty(\mathbb{R})$ to $C_0^\infty(\mathbb{R}^n)$ and from $L^2(\mathbb{R})$ to $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ it follows from (1.8) and (1.9) that $\widetilde{B_2}$ is continuous from $C_0^\infty(\mathbb{R}) \times C_0^\infty(\mathbb{R})$ to $L^2(\mathbb{R})$.

A combination of (1.8) and (1.11) then leads to the next proposition.

Proposition 1.3. *Assume $v(x) = g(|x|^2)$, $w(x) = h(|x|^2)$, where $g, h \in C_0^\infty(\mathbb{R})$. Then*

$$B_2(v,w)(x) = |x|^{2-n}\widetilde{B_2}(g,h)(|x|^2). \tag{1.12}$$

It follows that $\widetilde{B_2}$, hence $\Pi_* B$ by (1.11), determines the restriction of B_2 to rotation invariant functions. We need some more notation in order to present a formula for $\Pi_* B$ (and hence also for B_2 on rotation invariant functions).

Definition 1.4. Let $0 \leq j \leq m$. Then we set

$$Q_{m,j}(\vec{s}) = (-1)^{j+1} 2^{j-2-2m} \frac{1}{m!}\binom{m}{j} \psi_{m,j}(s_2+s_3, (s_2-s_3)^2 - 4s_1(s_1-s_2-s_3)), \tag{1.13}$$

where $\psi_{m,j}(s,t)$ is the polynomial defined through

$$\psi_{m,j}(s,t) = (s^2-t)^{-j}\partial_s^{m-j}(s^2-t)^m. \tag{1.14}$$

We notice that $Q_{m,j}$ is a polynomial homogeneous of degree $m-j$. Furthermore, consider the polynomials

$$q_{m,j}(t) = \frac{(-1)^{j+1}}{2^{2+m} j!} \sum_{0 \leq \mu \leq (m-j)/2} \binom{j+2\mu}{\mu}\binom{m}{j+2\mu} t^\mu, \quad 0 \leq j \leq m. \tag{1.15}$$

Then, writing

$$\psi_{m,j}(s,t) = (s^2-t)^{-j}\varphi^{(m-j)}(0),$$

where

$$\varphi(u) = ((s+u)^2-t)^m = (s^2-t+2su+u^2)^m = \sum_{0 \leq \nu \leq m} \binom{m}{\nu} u^{2\nu}(s^2-t+2su)^{m-\nu},$$

we can easily see that

$$Q_{m,j}(\vec{s}) = (s_2+s_3)^{m-j} q_{m,j}\Big(\frac{(s_1-s_2)(s_1-s_3)}{(s_2+s_3)^2}\Big). \tag{1.16}$$

Our main result is the following.

Theorem 1.5. *Let $Q_{m,j}$ be as above. Then*

$$\Pi_* B(\vec{s}) = \sum_{0 \leq j \leq m} (\partial_{s_2} + \partial_{s_3})^j \Big(Q_{m,j}(\vec{s})(s_1-s_2)_+^j (s_1-s_3)_+^j (s_2+s_3-s_1)_+^{-1/2}\Big). \tag{1.17}$$

It follows in particular from the theorem that $0 \leq s_2$, $s_3 \leq s_1$ in the support of $\Pi_* B$. Combining this with (1.12) we obtain the following corollary.

Corollary 1.6. *Assume v, $w \in C_0^\infty$ are rotation invariant. Then $B_2(v,w)(x)$ depends only on the restriction of v and w to the ball with radius $|x|$ and center at the origin.*

It follows from the corollary that B_2 extends to a continuous bilinear mapping in the space of rotation invariant functions in $C^\infty(\mathbb{R}^n)$. In fact, the continuity results in the paper [1] imply that $B_2(v,w)$ is defined when v, w are rotation invariant and locally in L^2 together with the derivatives of order $\leq m$.

Note that, in the case $n = 3$, Theorem 1.5 gives that

$$\Pi_* B(\vec{s}) = -\frac{1}{4} H(s_1 - s_2) H(s_1 - s_3)(s_2 + s_3 - s_1)_+^{-1/2}, \tag{1.18}$$

where H is Heaviside's function. Thus, when v, $w \in C_0^\infty(\mathbb{R}^3)$ are rotation invariant we obtain

$$B_2(v,w)(x) = -\frac{1}{16\pi^2|x|} \iint\limits_{\substack{|x_1|,|x_2|<|x| \\ |x|^2<|x_1|^2+|x_2|^2}} \frac{1}{|x_1|\,|x_2|} \frac{1}{\sqrt{|x_1|^2+|x_2|^2-|x|^2}} v(x_1)w(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

We shall frequently make use of spherical averages when proving the main result and include necessary preparations in Sections 2 and 3. The algebraic computations carried out in Section 4 were necessary for us in order to keep track of the cancellations that take place when taking averages of the distribution B over the group $\mathcal{O}(n)^3$. The final computation of $\Pi_* B$ is taking place in Section 4, in which we first compute $\Pi_* B_\varepsilon$ for a smooth approximation B_ε of B. In the last section we note that $\Pi_* B$ satisfies a differential equation which is hyperbolic in $\mathbb{R}^3_+$, and finally we give some simple examples.

2. Spherical averages.

The elements of the orthogonal group $\mathcal{O}(n)$ are denoted by capital letters Ω and $\mathrm{d}\Omega$ is the normalized Haar measure on $\mathcal{O}(n)$. Then

$$\int\limits_{\mathcal{O}(n)} h(\Omega x)\,\mathrm{d}\Omega = c_n^{-1} \int_{S^{n-1}} h(|x|\omega)\,\mathrm{d}\omega, \quad h \in C(\mathbb{R}^n) \tag{2.1}$$

and

$$\int\limits_{\mathcal{O}(n)} f(\langle \Omega x, y\rangle)\,\mathrm{d}\Omega = \frac{c_{n-1}}{c_n} \int\limits_{-1}^{1} f(s|x|\,|y|)(1-s^2)^m\,\mathrm{d}s \tag{2.2}$$

when $f \in C(\mathbb{R})$, x, $y \in \mathbb{R}^n$.

Definition 2.1. When x, $y \in \mathbb{R}^n \setminus 0$ we let $\mu_{x,y}$ be the probability measure on the real line defined through

$$\int f(t)\,\mathrm{d}\mu_{x,y}(t) = \int\limits_{\mathcal{O}(n)} f(|x + \Omega y|)\,\mathrm{d}\Omega, \quad f \in C_0(\mathbb{R}).$$

In what follows we shall make frequent use of the polynomial

$$q(t; a, b) = -t^2 + 2(a+b)t - (a-b)^2. \tag{2.3}$$

We shall sometimes view q as a polynomial in t with a, b as real parameters. Set

$$q_m(t; a, b) = q^m(t; a, b), \; q_m^{(\nu)}(t; a, b) = \partial_t^\nu q_m(t; a, b).$$

We notice that

$$q(t^2; a^2, b^2) = ((a+b)^2 - t^2)(t^2 - (a-b)^2)$$

has the zeros $\pm(a+b), \pm(a-b)$, and it is nonnegative if and only if $|\,|a| - |b|\,| \leq |t| \leq |a| + |b|$.

Lemma 2.2. *The measure $\mu_{x,y}$ is absolutely continuous with respect to Lebesgue measure. More precisely,*

$$\mathrm{d}\mu_{x,y}/dt = d_n(|x|\,|y|)^{2-n} t q_m(t^2; |x|^2, |y|^2)\chi_{x,y}(t),$$

where $\chi_{x,y}$ is the characteristic function of the interval $[\big||x| - |y|\big|, |x| + |y|]$.

Proof. Writing $|x+\Omega y| = (|x|^2+|y|^2+2\langle \Omega x, y\rangle)^{1/2}$ and applying (2.2) we find when $f \in C_0(\mathbb{R})$ that

$$\int f(s)\,\mathrm{d}\mu_{x,y}(s) = \frac{c_{n-1}}{c_n}\int_{-1}^{1} f((|x|^2 + |y|^2 + 2s|x|\,|y|)^{1/2})(1-s^2)^m\,\mathrm{d}s.$$

We take $t = (|x|^2 + |y|^2 + 2s|x|\,|y|)^{1/2}$ as a new variable of integration and notice that

$$s = \frac{t^2 - |x|^2 - |y|^2}{2|x|\,|y|},\ ds = \frac{tdt}{|x|\,|y|},$$
$$1 - s^2 = \frac{((|x|+|y|)^2 - t^2)(t^2 - (|x|-|y|)^2)}{4|x|^2|y|^2} = \frac{q(t^2; |x|^2, |y|^2)}{4|x|^2|y|^2}.$$

The lemma then follows since the condition $-1 \leq s \leq 1$ is equivalent to $\chi_{x,y}(t) = 1$. ∎

Define

$$L_m(a, u; g) = u^{-2m}\int_{(a-u)^2}^{(a+u)^2} g(t) q_m(t; a^2, u^2)\,\mathrm{d}t \tag{2.4}$$

when $(a, u) \in \mathbb{R}^2$ and $g \in C^\infty(\mathbb{R})$. Then

$$\begin{aligned} L_m(a, u; g) &= u^{-2m}\int_{(a-u)^2}^{(a+u)^2} g(t)\Big(((a+u)^2 - t)(t - (a-u)^2)\Big)^m\,\mathrm{d}t \\ &= (4a)^{2m+1}u\int_0^1 g(4aut + (a-u)^2)(1-t)^m t^m\,\mathrm{d}t \\ &= (2a)^{2m+1}u\int_{-1}^{1} g(2aut + a^2 + u^2)(1-t^2)^m\,\mathrm{d}t. \end{aligned}$$

This shows that $L_m(a, u; g)$ is smooth in (a, u) and it is odd in each of a and u.

Lemma 2.3. *Assume $f \in C^\infty(\mathbb{R})$ and $x_1, x_2, x_3 \in \mathbb{R}^n \setminus 0$. Then*

$$\iiint\limits_{\mathcal{O}(n)^3} f(|\Omega_1 x_1 + \Omega_2 x_2 + \Omega_3 x_3|^2)\,\mathrm{d}\Omega_1\,\mathrm{d}\Omega_2\,\mathrm{d}\Omega_3$$

$$= \frac{1}{2} d_n^2 (|x_1| \cdot |x_2| \cdot |x_3|)^{2-n} \int\limits_{|\,|x_2| - |x_3|\,|}^{|x_2| + |x_3|} L_m(|x_1|, u; f) q_m(u^2; |x_2|^2, |x_3|^2)\,\mathrm{d}u.$$

Proof. Let I denote the left-hand side above. Obviously

$$I = \iint f(|\Omega_1 x_1 + \Omega_2 x_2 + x_3|^2)\, d\Omega_1 d\Omega_2.$$

We may assume for reasons of continuity that $|x_2| \neq |x_3|$ so that $\Omega_2 x_2 + x_3 \neq 0$ for every Ω_2. Applying Lemma 2.2 in the integration with respect to Ω_1 and recalling the definition of L_m above we see that

$$I = \frac{1}{2} d_n |x_1|^{2-n} \int\limits_{\mathcal{O}(n)} |\Omega_2 x_2 + x_3|^{-1} L_m(|x_1|, |\Omega_2 x_2 + x_3|; f)\,\mathrm{d}\Omega_2.$$

The lemma then follows by applying Lemma 2.2 once more. □

Let $\mathcal{R}$ denote the differential operator $\partial_u(u^{-1}\cdot)$ in $\mathbb{R}_+$. The following theorem will be proved in next section.

Theorem 2.4. *Let a be a positive number. Then*

$$\begin{aligned} &L_m(a, u; g^{(2m+1)}) \\ &= 2^m m! \sum_{0 \le k \le m} (-1)^k 2^k \binom{m}{k} \mathcal{R}^{m-k} \Big(g^{(k)}((u+a)^2) - g^{(k)}((u-a)^2) \Big) \end{aligned} \tag{2.5}$$

when $u > 0$ and $g \in C^\infty(\mathbb{R})$.

Theorem 2.5. *Assume $f \in C^\infty(\mathbb{R})$. Then*

$$\begin{aligned} &\iiint\limits_{\mathcal{O}(n)^3} f^{(2m+1)}(|\Omega_1 x_1 + \Omega_2 x_2 + \Omega_3 x_3|^2)\, d\Omega_1 d\Omega_2 d\Omega_3 \\ &= e_n (r_1 r_2 r_3)^{2-n} \sum_{0 \le k \le m} \binom{m}{k} \int\limits_{r_2 - r_3}^{r_2 + r_3} \Big(f^{(k)}((u+r_1)^2) - f^{(k)}((u-r_1)^2) \Big) \cdot \\ &\qquad \cdot q_m^{(m-k)}(u^2; r_2^2, r_3^2)\,\mathrm{d}u \end{aligned} \tag{2.6}$$

for every x_1, x_2, $x_3 \in \mathbb{R}^n \setminus 0$.

Proof. Let $I(r_1, r_2, r_3)$ be the left-hand side of (2.6). Then Lemma 2.3 together with (2.5) gives

$$\begin{aligned} I(r_1, r_2, r_3) &= \frac{1}{2} d_n^2 (r_1 r_2 r_3)^{2-n} \int\limits_{|r_2 - r_3|}^{r_2 + r_3} L_m(r_1, u; f^{(2m+1)}) q_m(u^2; r_2^2, r_3^2)\,\mathrm{d}u \\ &= (-2)^{-m} e_n (r_1 r_2 r_3)^{2-n} \sum_{0 \le k \le m} (-1)^k 2^k \binom{m}{k} J_k(r_1, r_2, r_3), \end{aligned} \tag{2.7}$$

where
(2.8)
$$J_k(r_1,r_2,r_3) = \int\limits_{|r_2-r_3|}^{r_2+r_3} \Big(\mathcal{R}^{m-k}(f^{(k)}((u+r_1)^2) - f^{(k)}((u-r_1)^2))\Big) q_m(u^2;r_2^2,r_3^2)\,\mathrm{d}u.$$

We may assume $r_2 \neq r_3$ for reasons of continuity. Let $\mathcal{R}' = -u^{-1}\partial_u$ be the transpose of $\mathcal{R}$. Then $\mathcal{R}'\varphi(u^2) = -2\varphi'(u^2)$ when $\varphi \in C^\infty(\mathbb{R})$. Hence
$$(\mathcal{R}')^{m-k} q_m(u^2;r_2^2,r_3^2) = (-2)^{m-k} q_m^{(m-k)}(u^2;r_2^2,r_3^2).$$
Since $q(u^2;r_2^2,r_3^2)$ vanishes when $u = |r_2-r_3|$ or $u = r_2+r_3$ we may integrate by parts $m-k$ times in (2.8) without getting any contributions from the boundary of the interval of integration. This gives
$$J_k(r_1,r_2,r_3)$$
$$= (-2)^{m-k} \int\limits_{|r_2-r_3|}^{r_2+r_3} \Big(f^{(k)}((u+r_1)^2) - f^{(k)}((u-r_1)^2)\Big) q_m^{(m-k)}(u^2;r_2^2,r_3^2)\,\mathrm{d}u.$$
Since the integrand in the right hand side is an odd function of u we may instead integrate from r_2-r_3 to r_2+r_3, and the proof is completed by inserting the expression into (2.7). □

3. **Proof of Theorem 2.4.** We notice that it suffices to prove (2.5) when $a = 1$. This is a simple consequence of the identity
$$L_m(a,u;g^{(2m+1)}) = a^{-2m} L_m(1,u/a;g_a^{(2m+1)}), \quad a,u>0,$$
where $g_a(u) = g(a^2u)$, since $\mathcal{R}(h(\cdot/a))(u) = a^{-2}(\mathcal{R}h)(u/a)$ when $h \in C^\infty(\mathbb{R})$. Hence we have to establish the identity
$$L_m(1,u;g^{(2m+1)})$$
(3.1)
$$= 2^m m! \sum_{0\le k\le m} (-1)^k 2^k \binom{m}{k} \mathcal{R}^{m-k}\Big(g^{(k)}((u+1)^2) - g^{(k)}((u-1)^2)\Big).$$

We shall need some preparations before we can prove this theorem.

Lemma 3.1. *We have the identity*
$$u^{2m} L_m(1,u;g^{(2m+1)}) =$$
(3.2)
$$\sum_{0\le j\le m} (-1)^{m-j}(4u)^j \binom{m}{j} (2m-j)! \Big(g^{(j)}((u+1)^2) + (-1)^{j+1} g^{(j)}((u-1)^2)\Big)$$
when $u>0$.

Proof. Set $\alpha = (u-1)^2$, $\beta = (u+1)^2$. Then $0 \le \alpha \le \beta$, and a simple computation shows that
$$u^{2m} L_m(1,u;g^{(2m+1)}) = \int\limits_\alpha^\beta g^{(2m+1)}(t)\Big((\beta-t)(t-\alpha)\Big)^m\,\mathrm{d}t$$
(3.3)
$$= \int\limits_{-1}^{1} h^{(2m+1)}(t)(1-t^2)^m\,\mathrm{d}t = (-1)^m \int\limits_{-1}^{1} h^{(m+1)}(t) q_m^{(m)}(t)\,\mathrm{d}t,$$

where $h(t) = g((\beta-\alpha)t/2+(\alpha+\beta)/2)$ and $q_m(t) = (1-t^2)^m$. Repeated integrations by parts in the right-hand side of (3.3) yield the expression

$$\begin{aligned}&\sum_{0\le j\le m}(-1)^j\Big(h^{(j)}(1)q_m^{(2m-j)}(1) - h^{(j)}(-1)q_m^{(2m-j)}(-1)\Big)\\ &= \sum_{0\le j\le m}(-1)^j q_m^{(2m-j)}(1)\Big(h^{(j)}(1) + (-1)^{j+1}h^{(j)}(-1)\Big).\end{aligned} \tag{3.4}$$

We notice that

$$h^{(j)}(1) = (2u)^j g^{(j)}(\beta), \quad h^{(j)}(-1) = (2u)^j g^{(j)}(\alpha),$$

and writing $q_m(1+2u) = (-4)^m u^m (1+u)^m$ we see that

$$q_m^{(2m-j)}(1) = (-1)^m 2^j \binom{m}{j}(2m-j)!.$$

The lemma then follows by inserting the expressions for $h^{(j)}(\pm 1)$ and $q_m^{(2m-j)}(1)$ in (3.4). □

Next we define some auxiliary functions in $\mathbb{R}_+$:

$$A_{j,k}(g)(u) = 2^j(k-1)!u^{-k}\Big(g^{(j)}((u+1)^2) + (-1)^{j+1}g^{(j)}((u-1)^2)\Big), \quad k\ge 1 \tag{3.5}$$

$$A_{j,0}(g)(u) = 2^j\Big(g^{(j)}((u+1)^2) + (-1)^{j+1}g^{(j)}((u-1)^2)\Big) = uA_{j,1}(g)(u). \tag{3.6}$$

We see that

$$uA_{j,k+1}(g)(u) = kA_{j,k}(g)(u), \tag{3.7}$$

when $k \ge 1$.

It follows by direct computation when $k \ge 2$ that

$$A_{j,k}(g) = -\partial_u A_{j,k-1}(g) + 2uA_{j,k-1}(g') + A_{j+1,k-1}(g). \tag{3.8}$$

Hence, when $k\ge 3$, a combination of (3.7) and (3.8) gives

$$A_{j,k}(g) = -(k-2)\mathcal{R}A_{j,k-2}(g) + 2(k-2)A_{j,k-2}(g') + A_{j+1,k-1}(g), \tag{3.9}$$

and recalling (3.6) we get

$$A_{j,2}(g) = -\mathcal{R}A_{j,0}(g) + 2A_{j,0}(g') + A_{j+1,1}(g). \tag{3.10}$$

We introduce also the function

$$\kappa_{m,N}(g)(u) = \sum_{0\le j\le N}(-1)^{m-j}2^j\binom{m}{j}(2m-j)A_{j,2m-j}(g)(u), \quad u\in\mathbb{R}_+, \tag{3.11}$$

when $m > 0$ and $0 \le N \le m$. When $m = 0$ we set

$$\kappa_{0,0}(g)(u) = g((u+1)^2) - g((u-1)^2). \tag{3.12}$$

With this notation Lemma 3.1 reads

$$L_m(1,u;g^{(2m+1)}) = \kappa_{m,m}(g)(u). \tag{3.13}$$

We are going to compute the $\kappa_{m,N}(g)$ by induction over N.

Define

$$c_{m,N} = (-1)^{m-N}2^{N+1}(m-N)\binom{m}{N}$$

when $0 \leq N \leq m$. We notice that

$$c_{m,m} = 0, \quad c_{m,0} = (-1)^m 2m. \tag{3.14}$$

Lemma 3.2. *The identity*

$$c_{m,N} = \sum_{0 \leq k \leq N} (-1)^{m-k} 2^k \binom{m}{k} (2m-k) \tag{3.15}$$

holds when $0 \leq N < m$.

Proof. Let $c'_{m,N}$ denote the right-hand side of (3.15). A straightforward computation shows that $c_{m,N+1} - c_{m,N} = c'_{m,N+1} - c'_{m,N}$ when $N < m$. The lemma therefore follows since $c_{m,0} = c'_{m,0}$. □

Lemma 3.3. *The identity*

$$\kappa_{m,N}(g) = 2m\mathcal{R}\kappa_{m-1,N}(g) - 4m\kappa_{m-1,N}(g') + c_{m,N} A_{N+1,2m-N-1}(g) \tag{3.16}$$

holds when $0 \leq N < m$.

Proof. We first see that (3.10) implies that (3.16) holds when $m = 1$ and $N = 0$. We may therefore assume that $m \geq 2$.

Assume first that $N = 0$. Then (3.9) gives

$$\begin{aligned}
\kappa_{m,N}(g) = \kappa_{m,0}(g) = (-1)^m 2m A_{0,2m}(g) \\
= (-1)^{m-1} 2m(2m-2)\mathcal{R}A_{0,2m-2}(g) + (-1)^m 4m(2m-2)A_{0,2m-2}(g') \\
+(-1)^m 2m A_{1,2m-1}(g) \\
= 2m\mathcal{R}\kappa_{m-1,0}(g) - 4m\kappa_{m-1,0}(g') + c_{m,0}A_{1,2m-1}(g).
\end{aligned}$$

This shows that (3.16) holds when $N = 0$. Assume now that it holds for some N where $N < m-1$. In this case $2m-N-1 > m \geq 2$. From the induction hypothesis and (3.15) we get that

$$\begin{aligned}
\kappa_{m,N+1}(g) = \kappa_{m,N}(g) \\
+(-1)^{m-N-1} 2^{N+1} \binom{m}{N+1} (2m-N-1) A_{N+1,2m-N-1}(g) \\
= 2m\mathcal{R}\kappa_{m-1,N}(g) - 4m\kappa_{m-1,N}(g') \\
+\Big(c_{m,N} + (-1)^{m-N-1} 2^{N+1} \binom{m}{N+1} (2m-N-1)\Big) A_{N+1,2m-N-1}(g) \\
= 2m\mathcal{R}\kappa_{m-1,N}(g) - 4m\kappa_{m-1,N}(g') + c_{m,N+1}A_{N+1,2m-N-1}(g).
\end{aligned} \tag{3.17}$$

From (3.9) we see that

$$\begin{aligned}
A_{N+1,2m-N-1}(g) = -(2m-N-3)\mathcal{R}A_{N+1,2m-N-3}(g) \\
+2(2m-N-3)A_{N+1,2m-N-3}(g') + A_{N+2,2m-N-2}(g).
\end{aligned} \tag{3.18}$$

Since

$$-c_{m,N+1}(2m-N-3) = 2m(-1)^{m-1-(N+1)} 2^{N+1} \binom{m-1}{N+1} (2m-N-3)$$

it follows that

$$2m\kappa_{m-1,N+1}(h) = 2m\kappa_{m-1,N}(h) - (2m-N-3)c_{m,N+1}A_{N+1,2m-N-3}(h)$$

when $h = g$ or $h = g'$. Then (3.17) and (3.18) yield

$$\begin{aligned}\kappa_{m,N+1}(g) = \mathcal{R}\Big(&2m\kappa_{m-1,N}(g) - (2m-N-3)c_{m,N+1}A_{N+1,2m-N-3}(g)\Big)\\ -2\Big(&2m\kappa_{m-1,N}(g') - (2m-N-3)c_{m,N+1}A_{N+1,2m-N-3}(g')\Big)\\ &+c_{m,N+1}A_{N+2,2m-N-2}(g)\\ = 2m\mathcal{R}&\kappa_{m-1,N+1}(g) - 4m\kappa_{m-1,N+1}(g') + c_{m,N+1}A_{N+2,2m-N-2}(g).\end{aligned}$$

We have therefore proved that (3.16) is true also with N replaced by $N+1$. This completes the proof. □

Lemma 3.4. *We have*

$$\kappa_{m,m}(g) = 2m\mathcal{R}\kappa_{m-1,m-1}(g) - 4m\kappa_{m-1,m-1}(g') \tag{3.19}$$

when $m > 0$.

Proof. It follows from the previous lemma when $N = m-1$ and the fact that

$$\kappa_{m,m}(g) = \kappa_{m,m-1}(g) + m2^m A_{m,m}(g)$$

that

$$\begin{gathered}\kappa_{m,m}(g) = 2m\mathcal{R}\kappa_{m-1,m-1}(g)\\ -4m\kappa_{m-1,m-1}(g') + (c_{m,m-1} + m2^m)A_{m,m}(g).\end{gathered}$$

Since $c_{m,m-1} + m2^m = 0$, this finishes the proof. □

Proof of the Theorem 2.4. We have to prove (3.1). Since this identity is obvious when $m = 0$ we may assume $m \geq 1$. Recalling (3.12) and (3.13) we see that (3.1) is equivalent to

$$\kappa_{m,m}(g) = 2^m m! \sum_{0\leq k\leq m} (-1)^k 2^k \binom{m}{k} \mathcal{R}^{m-k}\kappa_{0,0}(g^{(k)}). \tag{3.20}$$

We make induction over m. When $m = 1$ then (3.20) is equivalent to (3.19). Assume now that $m > 1$ and that (3.20) is proved for lower values of m. We use first (3.19) and then the induction hypothesis to write

$$\begin{aligned}\kappa_{m,m}(g) = 2m2^{m-1}(m-1)! &\sum_{0\leq k\leq m-1} (-1)^k 2^k \binom{m-1}{k} \mathcal{R}^{m-k}\kappa_{0,0}(g^{(k)})\\ -4m2^{m-1}(m-1)! &\sum_{0\leq k\leq m-1} (-1)^k 2^k \binom{m-1}{k} \mathcal{R}^{m-1-k}\kappa_{0,0}(g^{(k+1)})\\ &= 2^m m! \mathcal{R}^m\kappa_{0,0}(g)\\ +2^m m! \sum_{1\leq k\leq m-1} &(-1)^k 2^k \left(\binom{m-1}{k} + \binom{m-1}{k-1}\right) \mathcal{R}^{m-k}\kappa_{0,0}(g^{(k)})\\ &+(-1)^m 2^{2m} m! \kappa_{0,0}(g^{(m)}).\end{aligned}$$

The lemma follows since $\binom{m-1}{k} + \binom{m-1}{k-1} = \binom{m}{k}$. □

4. **Computation of Π_*B.** We prepare the computation of Π_*B by some useful observations.

Lemma 4.1. *The distribution $\Pi_*B(s_1, s_2, s_3)$ is supported in the set where $s_j \ge 0$ and $s_1 \le s_2 + s_3$. It is positively homogeneous of degree $(n-4)/2$ and symmetric in (s_2, s_3).*

Proof. We have already seen that $s_j \ge 0$ in the support of Π_*B. Write

$$X = (x_1, x_2, x_3),\ \ S(X) = x_2^2 + x_3^2 - x_1^2.$$

Then

$$\langle x_2 - x_1, x_3 - x_1\rangle = |x_1 - x_2 - x_3|^2/2 - S(X)/2. \tag{4.1}$$

Since $\langle x_2 - x_1, x_3 - x_1\rangle = 0$ in the support of B it follows that $S(X) \ge 0$ in the support of B, and this implies that $s_2 + s_3 \ge s_1$ in the support of Π_*B. The second assertion follows from the fact that B_2 is homogeneous of degree $2 - 2n$, and the last assertion is a consequence of the symmetry of $B_2(x_1, x_2, x_3)$ in the variables x_2, x_3. ∎

We shall consider smooth approximations of B and choose some function $\chi \in C^\infty(\mathbb{R})$ which is increasing and such that $\chi(t) = 0$ when $t < 0$ while $\chi(t) = 1$ when $t > 1$. Define

$$\chi_\varepsilon(t) = \chi(t/\varepsilon)$$

when $0 < \varepsilon < 1$. This function converges in $\mathcal{D}'(\mathbb{R})$ to the Heaviside function when $\varepsilon \to 0$. We define

$$B_\varepsilon(x_1, x_2, x_3) = 2^{n-1} C_n \chi_\varepsilon^{(n-1)}(|x_1 - x_2 - x_3|^2 - S(X)). \tag{4.2}$$

This will serve to find a useful approximation of Π_*B, as shown in the next lemma.

Lemma 4.2. *The distribution Π_*B_ε converges in the distribution sense to Π_*B as $\varepsilon \to 0$.*

Proof. We notice that

$$\beta(x, y) = \langle x, \partial_y\rangle^{n-1} |x|^{2-2n} Y_+(\langle x, y\rangle)$$

when $x \ne 0$. It follows that

$$\beta(x, y) = (n-1)! \sum_{|\alpha|=n-1} \partial_y^\alpha x^\alpha |x|^{2-2n} Y_+(\langle x, y\rangle)/\alpha!$$

in $\mathbb{R}^n \times \mathbb{R}^n$ since both sides are homogeneous in x of degree $> -n$. If $\beta_\varepsilon(x, y) = 2^{n-1}\chi_\varepsilon^{(n-1)}(2\langle x, y\rangle)$ we also have

$$\beta_\varepsilon(x, y) = (n-1)! \sum_{|\alpha|=n-1} \partial_y^\alpha x^\alpha |x|^{2-2n} \chi_\varepsilon(2\langle x, y\rangle)/\alpha!$$

Hence β_ε converges to β in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ as $\varepsilon \to 0$. Since, by (4.1),

$$B_\varepsilon(x_1, x_2, x_3) = C_n \beta_\varepsilon(x_2 - x_1, x_3 - x_1)$$

it follows that B_ε converges to B in $\mathcal{D}'$ as $\varepsilon \to 0$, and hence Π_*B_ε converges to Π_*B. ∎

In order to derive a formula for $\Pi_* B_\varepsilon$ we need to introduce some more notation. Define

$$\mathcal{W} = -(\partial_{s_2} + \partial_{s_3})/2, \tag{4.3}$$

$$r_{m,j}(u; s_2, s_3) = (\partial_u - \mathcal{W})^{m-j} q_m(u; s_2, s_3). \tag{4.4}$$

Also set

$$U_\varepsilon(u; \vec{s}) = \chi'_\varepsilon((u + \sqrt{s_1})^2 + s_1 - s_2 - s_3) - \chi'_\varepsilon((u - \sqrt{s_1})^2 + s_1 - s_2 - s_3). \tag{4.5}$$

The characteristic function of the interval $(\sqrt{s_2} - \sqrt{s_3}, \sqrt{s_2} + \sqrt{s_3})$, when s_2 and s_3 are nonnegative, is denoted by $\theta(t; s_2, s_3)$.

Lemma 4.3. *We have*

$$\Pi_* B_\varepsilon(\vec{s}) = \frac{2^{1-n}}{m!} \sum_{0 \le j \le m} \binom{m}{j} \mathcal{W}^j \int \theta(u; s_2, s_3) U_\varepsilon(u; \vec{s}) r_{m,j}(u^2; s_2, s_3)\, \mathrm{d}u \tag{4.6}$$

in $\mathbb{R}^3_+$.

Proof. Define

$$\widetilde{B}_\varepsilon(x_1, x_2, x_3) = \iiint_{\mathcal{O}(n)^3} B_\varepsilon(\Omega_1 x_1, \Omega_2 x_2, \Omega_3 x_3)\, \mathrm{d}\Omega_1\, \mathrm{d}\Omega_2\, \mathrm{d}\Omega_3. \tag{4.7}$$

Then

$$\widetilde{B}_\varepsilon(x_1, x_2, x_3) = 2^{n-1} C_n \iiint_{\mathcal{O}(n)^3} \chi_\varepsilon^{(n-1)}(|\Omega_1 x_1 + \Omega_2 x_2 + \Omega_3 x_3|^2 - S(X))\, \mathrm{d}\Omega_1\, \mathrm{d}\Omega_2\, \mathrm{d}\Omega_3.$$

Theorem 2.5 with $f(t) = \chi'_\varepsilon(t - S(X))$ yields

$$\begin{gathered} \widetilde{B}_\varepsilon(x_1, x_2, x_3) = 2^{n-1} C_n e_n (r_1 r_2 r_3)^{2-n} \cdot \\ \cdot \sum_{0 \le k \le m} \binom{m}{k} \int_{r_2 - r_3}^{r_2 + r_3} \Big(\chi_\varepsilon^{(k+1)}((u + r_1)^2 - S(X)) - \chi_\varepsilon^{(k+1)}((u - r_1)^2 - S(X)) \Big) \cdot \\ \cdot q_m^{(m-k)}(u^2, r_2^2, r_3^2)\, \mathrm{d}u. \end{gathered} \tag{4.8}$$

A simple computation shows that

$$\Pi_* \Phi(s_1, s_2, s_3) = (c_n/2)^3 (s_1)_+^{m+1/2} (s_2)_+^{m+1/2} (s_3)_+^{m+1/2} \varphi(\sqrt{s_1}, \sqrt{s_2}, \sqrt{s_3}) \tag{4.9}$$

when $\Phi(x_1, x_2, x_3) = \varphi(|x_1|, |x_2|, |x_3|)$ and φ is a continuous function in $\mathbb{R}^3$. Since $\Pi_* B_\varepsilon = \Pi_* \widetilde{B}_\varepsilon$ it follows from (4.8) that

$$\begin{aligned} \Pi_* B_\varepsilon(\vec{s}) &= 2^{n-1} C_n e_n (c_n/2)^3 \sum_{0 \le k \le m} \binom{m}{k} A_{k,\varepsilon}(\vec{s}) \\ &= \frac{1}{2^{n-1} m!} \sum_{0 \le k \le m} \binom{m}{k} A_{k,\varepsilon}(\vec{s}), \end{aligned} \tag{4.10}$$

in $\mathbb{R}^3_+$, where

$$A_{k,\varepsilon}(\vec{s}) = \int_{\sqrt{s_2}-\sqrt{s_3}}^{\sqrt{s_2}+\sqrt{s_3}} \left(\chi_\varepsilon^{(k+1)}((u+\sqrt{s_1})^2 - \tilde{S}(\vec{s})) - \chi_\varepsilon^{(k+1)}((u-\sqrt{s_1})^2 - \tilde{S}(\vec{s}))\right)\cdot \\ \cdot\, q_m^{(m-k)}(u^2; s_2, s_3)\, du.$$

and $\tilde{S}(\vec{s}) = s_3 + s_2 - s_1$. We write

$$A_{k,\varepsilon}(\vec{s}) = \int \Big(\mathcal{W}^k U_\varepsilon(u;\vec{s})\Big)\Big(\theta(u; s_2, s_3) q_m^{(m-k)}(u^2; s_2, s_3)\Big)\, \mathrm{d}u.$$

Since $q_m^{(m-k)}(u^2; s_2, s_3)$ vanishes to the order k when $u = \sqrt{s_2} \pm \sqrt{s_3}$ it follows after an application of Taylor's formula that
(4.11)

$$A_{k,\varepsilon}(\vec{s}) = \sum_{j\le k} (-1)^{k-j} \binom{k}{j} \mathcal{W}^j \int \theta(u; s_2, s_3) U_\varepsilon(u;\vec{s}) \mathcal{W}^{k-j} q_m^{(m-k)}(u^2; s_2, s_3)\, \mathrm{d}u$$

when the s_j are positive. Inserting this into the right hand side of (4.10) and taking $\nu = k - j$ as a new variable of summation instead of k we get

$$\begin{aligned}\Pi_* B_\varepsilon(\vec{s}) = \frac{2^{1-n}}{m!} \sum_{0\le j\le m} \binom{m}{j} \mathcal{W}^j \int \theta(u; s_2, s_3) U_\varepsilon(u;\vec{s})\cdot \\ \cdot \Big(\sum_{\nu\le m-j} \binom{m-j}{\nu} (-1)^\nu \mathcal{W}^\nu q_m^{(m-j-\nu)}(u^2; s_2, s_3)\Big)\, \mathrm{d}u \\ = \frac{2^{1-n}}{m!} \sum_{0\le j\le m} \binom{m}{j} \mathcal{W}^j \int \theta(u; s_2, s_3) U_\varepsilon(u;\vec{s}) r_{m,j}(u^2; s_2, s_3)\, \mathrm{d}u.\end{aligned}$$

□

We consider now the functions
(4.12)

$$P_{m,j,\varepsilon}(\vec{s}) = \begin{cases} \int \theta(u; s_2, s_3) U_\varepsilon(u;\vec{s}) r_{m,j}(u^2; s_2, s_3)\, \mathrm{d}u & \text{when } s_1, s_2, s_3 \ge 0 \\ 0 \quad \text{when some } s_j \text{ is negative.} \end{cases}$$

Lemma 4.4. *There is $\Pi_{m,j,\varepsilon} \in C^\infty(\mathbb{R}^3)$ such that*

$$P_{m,j,\varepsilon}(\vec{s}) = (s_1)_+^{1/2} (s_2)_+^{1/2} (s_3)_+^{1/2} (s_2 s_3)^j \Pi_{m,j,\varepsilon}(\vec{s})$$

in $\mathbb{R}^3$.

Proof. We consider $P_{m,j,\varepsilon}(\vec{s})$ when the s_j are nonnegative and notice that

$$U_\varepsilon(u;\vec{s}) = 2\sqrt{s_1}\, u V_\varepsilon(u^2, \vec{s}),$$

where $V_\varepsilon \in C^\infty(\mathbb{R}^4)$. Hence

$$U_\varepsilon(u;\vec{s}) r_{m,j}(u^2; s_2, s_3) = 2\sqrt{s_1}\, u W_{m,j,\varepsilon}(u^2, \vec{s})$$

where $W_{m,j,\varepsilon} \in C^\infty(\mathbb{R}^4)$. It follows that

$$(4.13)\qquad \begin{aligned} P_{m,j,\varepsilon}(\vec{s}) &= 2\sqrt{s_1}\int_{\sqrt{s_2}-\sqrt{s_3}}^{\sqrt{s_2}+\sqrt{s_3}} uW_{m,j,\varepsilon}(u^2,\vec{s})\,\mathrm{d}u \\ &= \sqrt{s_1}\int_{(\sqrt{s_2}-\sqrt{s_3})^2}^{(\sqrt{s_2}+\sqrt{s_3})^2} W_{m,j,\varepsilon}(u,\vec{s})\,\mathrm{d}u = \sqrt{s_1}\int_{-2\sqrt{s_2s_3}}^{2\sqrt{s_2s_3}} W_{m,j,\varepsilon}(u+s_1+s_2,\vec{s})\,\mathrm{d}u. \end{aligned}$$

We notice that since

$$r_{m,j}(u;\vec{s}) = (\partial_u - \mathcal{W})^{m-j} q_m(u;s_2,s_3)$$

it follows that $r_{m,j}(u+s_2+s_3;\vec{s})$ is divisible by

$$q^j(u+s_2+s_3;s_2,s_3) = (4s_2s_3-u^2)^j$$

in the ring of polynomials in $(u,\vec{s})$. This in turn implies that $W_{m,j,\varepsilon}(u+s_2+s_3,\vec{s})$ is divisible by $(4s_2s_3-u^2)^j$ in the ring $C^\infty(\mathbb{R}^4)$. Hence $W_{m,j,\varepsilon}(u+s_2+s_3,\vec{s})$ is a finite linear combination of $\phi_{\alpha,\beta}(u,\vec{s})(s_2s_3)^\alpha u^\beta$ where $\phi_{\alpha,\beta} \in C^\infty(\mathbb{R}^4)$ and $\alpha+\beta/2 \geq j$. The assertion of the lemma follows if this observation is combined with (4.13). □

The following corollary is immediate from the previous lemma and Lemma 4.3.

Corollary 4.5. *Let the $P_{m,j,\varepsilon}$ be as in (4.12) Then*

$$(4.14)\qquad \Pi_* B_\varepsilon(\vec{s}) = \frac{2^{1-n}}{m!}\sum_{0\leq j\leq m}\binom{m}{j}\mathcal{W}^j P_{m,j,\varepsilon}(\vec{s})$$

in $\mathbb{R}^3$.

Proof. It follows from Lemma 4.4 that the right-hand side of (4.14) is a continuous function of $\vec{s}$. Since this is also true for the left-hand side in view of (4.9), and since by Lemma 4.3 equality holds when the s_j are positive, it follows that equality holds everywhere. □

We shall finish our computation of $\Pi_* B$ by showing that $P_{m,j,\varepsilon}$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^3)$ when $\varepsilon \to 0$.

Lemma 4.6. *Let K be a compact set in $\mathbb{R}^3$. Then there is a constant C_K such that*

$$|P_{m,j,\varepsilon}(\vec{s})| \leq C_K|s_2+s_3-s_1|^{-1/2} \quad \text{when } \vec{s}\in K,\ 0<\varepsilon<1.$$

Proof. Assume $0 \leq g \in C_0(\mathbb{R})$ and that $a,b\in\mathbb{R}$. Define

$$I(a,b;g) = |b|^{1/2}\int_{-\infty}^{\infty} g((u+a)^2+b)\,\mathrm{d}u.$$

We claim that

$$(4.15)\qquad I(a,b;g) \leq 8(\|g\|_{L^1} + \max_t |tg(t)|).$$

When proving (4.15) we notice that

$$I(a,b;g) = I(0,b;g) = I(0,\mathrm{sgn}(b);g_b),$$

where $g_b(u) = |b|g(|b|u)$, and the right-hand side of (4.15) is invariant under the transformation $g \mapsto g_b$ when $b \neq 0$. We may assume therefore that $|b| = 1$. Then

$$I(0, b; g) \leq \int_{-\infty}^{\infty} g(u+b)|u|^{-1/2}\,\mathrm{d}u = \int_{-\infty}^{\infty} g(u)|u-b|^{-1/2}\,\mathrm{d}u.$$

The estimate (4.15) follows since

$$\int_{|u-b|>1/2} g(u)|u-b|^{-1/2}\,\mathrm{d}u \leq \sqrt{2}\|g\|_{L^1}$$

and

$$\int_{|u-b|<1/2} g(u)|u-b|^{-1/2}\,\mathrm{d}u \leq 2\max|tg(t)| \int_{-1/2}^{1/2} |u|^{-1/2}\,\mathrm{d}u = 4\sqrt{2}\max|tg(t)|.$$

We notice next that there is a constant $C = C(n, K)$ such that

$$|P_{m,j,\varepsilon}(\vec{s})| \leq C \int |U_\varepsilon(u;\vec{s})|\,\mathrm{d}u$$

when $0 < \varepsilon < 1$ and $\vec{s} \in K$. The lemma then follows by combining (4.15) with the observations that

$$|s_1 - s_2 - s_3|^{1/2} \int |U_\varepsilon(u;\vec{s})|\,\mathrm{d}u \leq I(\sqrt{s_1}, s_1 - s_2 - s_3; \chi'_\varepsilon) + I(-\sqrt{s_1}, s_1 - s_2 - s_3; \chi'_\varepsilon)$$

and $\|\chi'_\varepsilon\|_{L_1}$, $\max_t |t\chi'_\varepsilon(t)|$ are independent of ε. □

Combining the previous lemma with the dominated convergence theorem of Lebesgue, and recalling Lemma 4.2 and Corollary 4.5 we have proved that if $P_{m,j,\varepsilon}(\vec{s})$ converges almost everywhere to some function $P_{m,j}(\vec{s})$, then $P_{m,j,\varepsilon}(\vec{s})$ converges to $P_{m,j}(\vec{s})$ in $L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and

$$\Pi_* B(\vec{s}) = \frac{2^{1-n}}{m!} \sum_{0 \leq j \leq m} \binom{m}{j} \mathcal{W}^j P_{m,j}(\vec{s}). \tag{4.16}$$

We recall that the polynomials $\psi_{m,j}$ have been introduced in (1.14).

Lemma 4.7. *Define*

$$\Psi_{m,j}(\vec{s}) = \psi_{m,j}(s_2 + s_3, (s_2 - s_3)^2 - 4s_1(s_1 - s_2 - s_3)). \tag{4.17}$$

Then

$$\lim_{\varepsilon \to 0} P_{m,j,\varepsilon}(\vec{s}) = -4^j \Psi_{m,j}(\vec{s})(s_1 - s_2)_+^j (s_1 - s_3)_+^j (s_2 + s_3 - s_1)_+^{-1/2} \tag{4.18}$$

for almost every $\vec{s} \in \mathbb{R}^3$.

Proof. Since both sides of (4.18) are equal to 0 when $s_j < 0$ for some j it suffices to show that (4.18) holds for almost every $\vec{s} \in \mathbb{R}^3_+$. Assume now that the s_j are positive. Then

$$P_{m,j,\varepsilon}(\vec{s}) = \int \theta(u; s_2, s_3) U_\varepsilon(u;\vec{s}) r_{m,j}(u^2; s_2, s_3)\,\mathrm{d}u. \tag{4.19}$$

Set

$$\rho(t;\vec{s}) = (t + 2s_1 - s_2 - s_3)^2 - 4s_1 t, \tag{4.20}$$

and notice that

$$\rho(u^2;\vec{s}) = ((u+\sqrt{s_1})^2 + s_1 - s_2 - s_3)((u-\sqrt{s_1})^2 + s_1 - s_2 - s_3). \tag{4.21}$$

From this follows that

$$\lim_{\varepsilon\to 0}\int \theta(u;s_2,s_3)U_\varepsilon(u;\vec{s})\psi(u^2;\vec{s})\,\mathrm{d}u = 0$$

for every polynomial $\psi(u;\vec{s})$ which is divisible by $\rho(u;\vec{s})$. We therefore wish to divide out ρ from $r_{m,j}$ in (4.19).

Define

$$\widetilde{r}_{m,j}(t;s_2,s_3) = r_{m,j}(t+s_2+s_3;s_2,s_3)$$

and

$$\widetilde{\rho}(t;\vec{s}) = \rho(t+s_2+s_3;\vec{s}).$$

Then

$$\widetilde{\rho}(t;\vec{s}) = t^2 + 4(s_1-s_2)(s_1-s_3) - 4s_2s_3. \tag{4.22}$$

We notice that

$$\widetilde{r}_{m,j}(t;s_2,s_3) = (-\mathcal{W})^{m-j}q_m(t+s_2+s_3;s_2,s_3) = (-\mathcal{W})^{m-j}(4s_2s_3-t^2)^m, \tag{4.23}$$

since

$$-\mathcal{W}(f(s_2+s_3)) = f'(s_2+s_3)$$

when $f\in C^1$. This also implies that

$$\begin{aligned}(-\mathcal{W})^{m-j}(4s_2s_3-t^2)^m &= (-\mathcal{W})^{m-j}((s_2+s_3)^2-(s_2-s_3)^2-t^2)^m\\ &= \partial_u^{m-j}(u^2-(s_2-s_3)^2-t^2)^m_{u=s_2+s_3}.\end{aligned}$$

Recalling (4.23) and (1.14) we have proved that

$$\widetilde{r}_{m,j}(t;s_2,s_3) = (4s_2s_3-t^2)^j\psi_{m,j}(s_2+s_3,(s_2-s_3)^2+t^2). \tag{4.24}$$

Then a combination of (4.22) and (4.24) shows that

$$\widetilde{r}_{m,j}(t;s_2,s_3) - 4^j(s_1-s_2)^j(s_1-s_3)^j\Psi_{m,j}(\vec{s})$$

is divisible by $\widetilde{\rho}(t;\vec{s})$ in the ring of polynomials in $(t,\vec{s})$. After replacing t by $t-s_1-s_2$ we have proved that

$$r_{m,j}(t;\vec{s}) = T_{m,j}(\vec{s}) + \rho(t;\vec{s})S_{m,j}(t;\vec{s}) \tag{4.25}$$

where $S_{m,j}$ is a polynomial and

$$T_{m,j}(\vec{s}) = 4^j(s_1-s_2)^j(s_1-s_3)^j\Psi_{m,j}(\vec{s}). \tag{4.26}$$

It follows from (4.19) and (4.25) together with the observations after (4.21) that

$$P_{m,j,\varepsilon}(\vec{s}) = T_{m,j}(\vec{s})Y_\varepsilon(s) + R_{m,j,\varepsilon}(\vec{s}), \tag{4.27}$$

where

$$Y_\varepsilon(\vec{s}) = \int \theta(u;s_2,s_3)U_\varepsilon(u,\vec{s})\,\mathrm{d}u \tag{4.28}$$

and

$$\lim_{\varepsilon\to 0}R_{m,j,\varepsilon}(\vec{s}) = 0.$$

We shall complete the proof of the lemma by showing that

$$\lim_{\varepsilon\to 0}Y_\varepsilon(\vec{s}) = Y(\vec{s}) \quad \text{for almost every } \vec{s}\in\mathbb{R}^3_+ \tag{4.29}$$

where

$$Y(\vec{s}) = -(s_2+s_3-s_1)_+^{-1/2}H(s_1-s_2)H(s_1-s_3).$$

If $s_2 + s_3 < s_1$ then $U_\varepsilon(u;\vec{s}) = 0$ for every u when ε is small enough. We may assume therefore that $s_2 + s_3 - s_1 = \sigma^2$, where $\sigma > 0$. Let $N = \{\pm\sqrt{s_1} \pm \sigma\}$. Then there is a constant C depending on $\vec{s}$ such that the support of $u \mapsto U_\varepsilon(u;\vec{s})$ is within distance $C\varepsilon$ from N. We may assume that $\sqrt{s_2} \pm \sqrt{s_3} \notin N$ and then $u \mapsto \theta(u; s_2, s_3)$ is smooth in an open neighbourhood of N. When taking limits of the integral defining Y_ε we can therefore treat θ as a test function in the variable u.

Recalling the definition of U_ε we see that $u \mapsto U_\varepsilon(u;\vec{s})$ converges in $\mathcal{D}'(\mathbb{R})$ to

$$\frac{1}{2\sigma}\Big(\delta(u+\sqrt{s_1}+\sigma) + \delta(u+\sqrt{s_1}-\sigma) - \delta(u-\sqrt{s_1}+\sigma) - \delta(u-\sqrt{s_1}-\sigma)\Big).$$

Hence

$$\begin{aligned} &Y_\varepsilon(\vec{s}) \to Y_0(\vec{s}) \\ &= \frac{1}{2}(s_2+s_3-s_1)^{-1/2}\Big(\theta(-\sqrt{s_1}-\sigma; s_2, s_3) + \theta(-\sqrt{s_1}+\sigma; s_2, s_3) \\ &\quad -\theta(\sqrt{s_1}-\sigma; s_2, s_3) - \theta(\sqrt{s_1}+\sigma; s_2, s_3)\Big). \end{aligned} \tag{4.30}$$

It suffices to show now that $Y_0(\vec{s}) = Y(\vec{s})$ when $s_1 \neq s_2, s_3$. Let $\alpha, \beta = \pm 1$. Then $\alpha\sqrt{s_1} + \beta\sigma$ is not an endpoint of the interval $(\sqrt{s}_2 - \sqrt{s}_3, \sqrt{s}_2 + \sqrt{s}_3)$ and

$$\theta(\alpha\sqrt{s_1} + \beta\sigma; s_2, s_3) = H(t(\alpha, \beta)),$$

where

$$t(\alpha,\beta) = (\sqrt{s_2} + \sqrt{s_3} - \alpha\sqrt{s_1} - \beta\sigma)(\alpha\sqrt{s_1} + \beta\sigma - \sqrt{s_2} + \sqrt{s_3}).$$

We write

$$\begin{aligned} t(\alpha,\beta) &= s_3 - (\alpha\sqrt{s_1} + \beta\sigma - \sqrt{s_2})^2 \\ &= s_3 - s_1 - \sigma^2 - s_2 - 2\alpha\beta\sqrt{s_1}\sigma + 2\alpha\sqrt{s_1}\sqrt{s_2} + 2\beta\sigma\sqrt{s_2} \\ &= -2s_2 + 2\beta\sigma\sqrt{s_2} - 2\alpha\beta\sqrt{s_1}\sigma + 2\alpha\sqrt{s_1}\sqrt{s_2} \\ &= 2(\alpha\sqrt{s_1} - \sqrt{s_2})(\sqrt{s_2} - \beta\sigma). \end{aligned}$$

It is easily verified that

$$H(t(-1,-1)) = 0,\ H(t(-1,1)) = H(s_3 - s_1),\ H(t(1,-1)) = H(s_1 - s_2),$$
$$H(t(1,1)) = H((s_1 - s_2)(s_1 - s_3)).$$

From this follows that

$$\begin{aligned} Y_0(\vec{s}) &= \frac{1}{2}(s_2+s_3-s_1)^{-1/2} \sum_{\alpha,\beta=\pm 1} \alpha H(t(-\alpha,\beta)) \\ &= -(s_2+s_3-s_1)^{-1/2} H(s_1 - s_2) H(s_1 - s_3) \\ &= Y(\vec{s}). \end{aligned}$$

This concludes the proof. □

Proof of Theorem 1.5. It follows from (4.16) and Lemma 4.7 that

$$\begin{aligned} &\Pi_* B(\vec{s}) \\ &= -\frac{2^{1-n}}{m!} \sum_{0\le j\le m} 4^j \binom{m}{j} \mathcal{W}^j \Big(\Psi_{m,j}(\vec{s})(s_1-s_2)_+^j (s_1-s_3)_+^j (s_2+s_3-s_1)_+^{-1/2}\Big) \\ &= \sum_{0\le j\le m} (\partial_{s_2} + \partial_{s_3})^j \Big(Q_{m,j}(\vec{s})(s_1-s_2)_+^j (s_1-s_3)_+^j (s_2+s_3-s_1)_+^{-1/2}\Big), \end{aligned}$$

where we have used (4.3), (4.17) and (1.13). □

5. **Remarks and examples.** We first show that $\Pi_* B$ solves a second order partial differential equation which is hyperbolic in $\mathbb{R}^3_+$.

Theorem 5.1. *Define the differential operator L in $\mathbb{R}^3$ through*

$$L = 2s_2\partial_{s_2}^2 + 2s_3\partial_{s_3}^2 - 2s_1\partial_{s_1}^2 + (n-4)(\partial_{s_1} - \partial_{s_2} - \partial_{s_3}). \tag{5.1}$$

Then

$$L\Pi_* B = (s_1)_+^{n/2-1}\delta(s_1 - s_2)\delta(s_1 - s_3). \tag{5.2}$$

Proof. It follows from (1.6) and (1.7) that

$$B(x_1, x_2, x_3) = -\frac{2^{1+n}}{c_n}E(x_2 + x_3 - 2x_1, x_2 - x_3).$$

We get

$$\begin{aligned}
&(\Delta_1 - \Delta_2 - \Delta_3)B(x_1, x_2, x_3)\\
&= \Big(\partial_{x_1}^2 - (\partial_{x_2} + \partial_{x_3})^2/2 - (\partial_{x_2} - \partial_{x_3})^2/2\Big)B(x_1, x_2, x_3)\\
&= -\frac{2^{1+n}}{c_n}\Big(\frac{1}{2}\partial_{x_1}^2 - \frac{1}{2}(\partial_{x_2} - \partial_{x_3})^2\Big)E(x_2 + x_3 - 2x_1, x_2 - x_3)\\
&= -\frac{2^{2+n}}{c_n}((\Delta_y - \Delta_z)E)(x_2 + x_3 - 2x_1, x_2 - x_3)\\
&= -\frac{2^{2+n}}{c_n}\delta(x_2 + x_3 - 2x_1)\delta(x_2 - x_3)\\
&= -\frac{2^{2+n}}{c_n}\delta(2x_2 - 2x_1)\delta(x_2 - x_3) = -\frac{4}{c_n}\delta(x_2 - x_1)\delta(x_2 - x_3).
\end{aligned}$$

When $f \in C_0^\infty(\mathbb{R}^3)$ we have

$$\begin{aligned}
&\int_0^\infty s^{n/2-1}f(s, s, s)\,\mathrm{d}s = 2\int_0^\infty s^{n-1}f(s^2, s^2, s^2)\,\mathrm{d}s\\
&= \frac{2}{c_n}\int_{\mathbb{R}^n} f(|x|^2, |x|^2, |x|^2)\,\mathrm{d}x = \frac{2}{c_n}\langle \Pi^* f, \delta(x_2 - x_1)\delta(x_2 - x_3)\rangle\\
&= -\frac{1}{2}\langle \Pi^* f, (\Delta_1 - \Delta_2 - \Delta_3)B\rangle = -\frac{1}{2}\langle (\Delta_1 - \Delta_2 - \Delta_3)\Pi^* f, B\rangle.
\end{aligned}$$

Let L' be the formal transpose of L. A simple computation shows that

$$(\Delta_1 - \Delta_2 - \Delta_3) \circ \Pi^* = -2\Pi^* \circ L'.$$

It follows that

$$\int_0^\infty s^{n/2-1}f(s, s, s)\,\mathrm{d}s = \langle \Pi^* L' f, B\rangle = \langle f, L\Pi_* B\rangle.$$

Hence (5.2) holds. □

We shall finally give some examples in the case $n = 3$. In this case Theorem 1.5 gives that (1.18) holds.

In what follows we assume $0 < a \le \infty$, $b \ge 0$, and set

$$g_{a,b}(s) = Y_a(s)\mathrm{e}^{-bs},$$

where Y_a denotes characteristic function of the interval $[0, a]$.

We first compute $\widetilde{B_2}(g_{a,b}, g_{a,b})(s)$.

Theorem 5.2. *Assume $n = 3$. Then*

$$\widetilde{B_2}(g_{a,b}, g_{a,b})(s) = -\frac{1}{2}\mathrm{e}^{-bs}\min(s, 2a-s)_+^{3/2}\int_0^1 (1-t^2)\mathrm{e}^{-bst^2}\,\mathrm{d}t. \tag{5.3}$$

Proof. It follows from (1.11) and (1.18) that

$$\widetilde{B_2}(g_{a,b}, g_{a,b})(s) = -\frac{1}{4}\iint\limits_{t,u\leq s}(t+u-s)_+^{-1/2}Y_a(t)Y_a(u)\mathrm{e}^{-b(t+u)}\,\mathrm{d}t\,\mathrm{d}u.$$

If $\sigma = \min(a, s)$ this means that

$$\begin{aligned}\widetilde{B_2}(g_{a,b}, g_{a,b})(s) &= -\frac{1}{4}\int_s^\infty (t-s)^{-1/2}(Y_\sigma * Y_\sigma)(t)\mathrm{e}^{-bt}\,\mathrm{d}t\\ &= -\frac{\mathrm{e}^{-bs}}{4}\int_0^\infty t^{-1/2}(Y_\sigma * Y_\sigma)(t+s)\mathrm{e}^{-bt}\,\mathrm{d}t.\end{aligned}$$

But $(Y_\sigma * Y_\sigma)(t) = (\sigma - |t-\sigma|)_+$ and hence

$$(Y_\sigma * Y_\sigma)(s+t) = (2\sigma - t - s)_+ = \Big(\min(s, 2a-s) - t\Big)_+.$$

With $\mu = \min(s, 2a-s)$ this yields

$$\widetilde{B_2}(g_{a,b}, g_{a,b})(s) = -\frac{\mathrm{e}^{-bs}}{4}\int_0^\mu t^{-1/2}(\mu - t)\mathrm{e}^{-bt}\,\mathrm{d}t$$

$$= -\frac{\mathrm{e}^{-bs}}{4}\mu^{3/2}\int_0^1 t^{-1/2}(1-t)\mathrm{e}^{-b\mu t}\,\mathrm{d}t = -\frac{\mathrm{e}^{-bs}}{2}\mu^{3/2}\int_0^1 (1-t^2)\mathrm{e}^{-b\mu t^2}\,\mathrm{d}t.$$

□

Corollary 5.3. *Let the function $v_{a,b}$ in $\mathbb{R}^3$ be defined by*

$$v_{a,b}(x) = \begin{cases}\mathrm{e}^{-b|x|^2} & \text{when } |x| < a\\ 0 & \text{when } |x| \geq a.\end{cases}$$

Then

$$B_2(v_{a,b}, v_{a,b})(x) = \begin{cases} -\frac{\mathrm{e}^{-b|x|^2}}{2}|x|^2\int_0^1(1-t^2)\mathrm{e}^{-b|x|^2t^2}\,\mathrm{d}t \quad \text{when } |x| < a\\ -\frac{\mathrm{e}^{-b|x|^2}}{2}(2a^2/|x|^2-1)^{3/2}|x|^2\int_0^1(1-t^2)\mathrm{e}^{-b(2a^2-|x|^2)t^2}\,\mathrm{d}t,\\ \qquad\qquad \text{when } a \leq |x| \leq \sqrt{2}a\\ 0 \quad \text{when } |x| > \sqrt{2}a.\end{cases}$$

Proof. The formula (1.12) gives

$$B_2(v_{a,b}, v_{a,b})(x) = |x|^{-1}\widetilde{B}_2(g_{a^2,b}, g_{a^2,b})(|x|^2),$$

and the result follows by applying Theorem 5.2 □

Remark. We notice that $v_{a,b}$ has a discontinuity at the sphere $S_a = \{x;\, |x| = a\}$ where it is locally in the Sobolev space $H_{(r)}$ (of functions with r derivatives in L^2) for any $r < 1/2$. The function $B_2(v_{a,b}, v_{a,b})$ is singular at the spheres S_a and $S_{\sqrt{2}a}$. The singularity at S_a is the same as that of the function $t \mapsto |t|$ at the origin. Hence $v_{a,b} \in H_{(r)}$ locally at S_a for any $r < 3/2$ but it is not locally in $H_{(3/2)}$. It follows from Theorem 1.1 in the paper [1] that the mapping $v \mapsto B_2(v, v)$ is continous from $H_{(r),\mathrm{loc}}(\mathbb{R}^3)$ to $H_{(r+\varepsilon),\mathrm{loc}}(\mathbb{R}^3)$ for any $r \geq 0$ if $\varepsilon < 1$. The above example shows that it is necessary that $\varepsilon \leq 1$ for this smoothing property to hold for every $r \geq 0$.

We also notice that the singularity of $B_2(v_{a,b}, v_{a,b})$ at $|x| = \sqrt{2}a$ is the same as that of $t_+^{3/2}$ at the origin. Hence $B_2(v_{a,b}, v_{a,b})$ is locally in $H_{(r)}$ at $S_{\sqrt{2}a}$ for any $r < 2$. Finally we notice that $B_2(v_{a,b}, v_{a,b})(x) = B_2(v_{0,b}, v_{0,b})(x)$ when $|x| < a$. This reflects the fact that $s_2, s_3 \leq s_1$ in the support of $\Pi_* B$.

Remark. If $v(x) = e^{-b|x|^2}$ in $\mathbb{R}^n$ where $n = 3$ and $b > 0$ then Corollary 5.3 with $a = \infty$ shows that $B_2(v, v)(x)$ is a (continuous) superposition of Gaussian functions. When n is an arbitrary odd integer ≥ 3 and $v(x)$ is the corresponding Gaussian in $\mathbb{R}^n$ it follows from Theorem 1.5 and (1.16) that

$$B_2(v, v)(x) = \sum_{0 \leq j \leq m} b^j |x|^{2+2j} e^{-b|x|^2} \int_0^1 h_{m,j}(t^2)(1 - t^2)^{1+2j} e^{-bt^2|x|^2}\, \mathrm{d}t,$$

where the $h_{m,j}$ are polynomials.

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