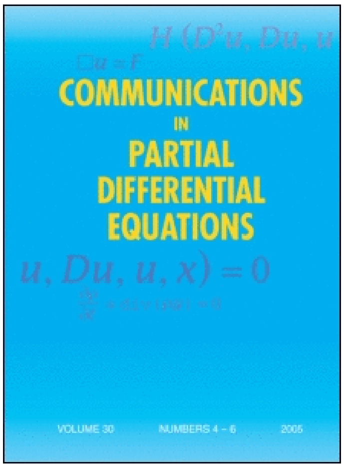

Communications in Partial Differential Equations
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597240

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Ingrid Belti [a]; Anders Melin [b]
[a] Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania [b] Center for Mathematical Sciences, Lund University, Lund, Sweden

Online Publication Date: 01 March 2009

To cite this Article Belti, Ingrid and Melin, Anders(2009)'Local Smoothing for the Backscattering Transform',Communications in Partial Differential Equations,34:3,233 — 256

To link to this Article: DOI: 10.1080/03605300902812384

URL: http://dx.doi.org/10.1080/03605300902812384

Communications in Partial Differential Equations, 34: 233–256, 2009

ISSN 0360-5302 print/1532-4133 online
DOI: 10.1080/03605300902812384

Local Smoothing for the Backscattering Transform

INGRID BELTIŢĂ[1] AND ANDERS MELIN[2]

[1]Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania
[2]Center for Mathematical Sciences, Lund University, Lund, Sweden

An analysis of the backscattering data for the Schrödinger operator in odd dimensions $n \geq 3$ motivates the introduction of the backscattering transform $B : C_0^\infty(\mathbb{R}^n; \mathbb{C}) \to C^\infty(\mathbb{R}^n; \mathbb{C})$. This is an entire analytic mapping and we write $Bv = \sum_1^\infty B_N v$ where $B_N v$ is the Nth order term in the power series expansion at $v = 0$. In this paper we study estimates for $B_N v$ in $H_{(s)}$ spaces, and prove that Bv is entire analytic in $v \in H_{(s)} \cap \mathscr{E}'$ when $s \geq (n-3)/2$.

Keywords Backscattering; Scattering matrix; Ultra-hyperbolic operator; Wave equation.

Mathematics Subject Classification Primary 35R30; Secondary 81U40.

1. Introduction

The present paper is devoted to proving continuity and smoothing properties of the backscattering transform for the Schrödinger operator in odd dimensions $n > 1$.

In order to state the main result a brief description of the mathematical objects involved is necessary. (The reader is referred to Chapter 14 in [4] and [7–9] for details.)

Consider the Schrödinger operator $H_v = -\Delta + v$ in $\mathbb{R}^n$, where $v \in L^2_{\text{cpt}}(\mathbb{R}^n)$ is real. Assume that H_v with domain $H_{(2)}(\mathbb{R}^n)$ is self-adjoint. Then the wave operators

$$W_\pm = \lim_{t \to \pm\infty} e^{\mathrm{i}tH_v} e^{-\mathrm{i}tH_0}$$

exist and are complete.

Let $S = W_+^* W_-$ be the scattering operator and denote $A = S - I$. The distribution kernel of the operator $\mathscr{F} A \mathscr{F}^{-1}$, where $\mathscr{F}$ denotes the Fourier transform, is of the form

$$-2\mathrm{i}\pi \Psi(\xi, \eta)\delta(|\xi|^2 - |\eta|^2),$$

Received December 28, 2007; Accepted January 14, 2009
Address correspondence to Ingrid Beltiţă, Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, Bucharest, Romania; E-mail: Ingrid.Beltita@imar.ro

where $\xi \to \Psi(\xi, \eta)$ is a smooth function of η with values in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. In turn, when $k > 0$, ω, $\theta \in \mathbb{S}^{n-1}$, $\Psi(k\omega, k\theta)$ is a constant times $ka_\infty(k, \omega, \theta)$, where $a_\infty(k, \omega, \theta)$ is the classical far field pattern associated to H_v. Hence $\Psi(-\xi, \xi)$ is given by the classical backscattering amplitude.

It was proved in [9] that when $v \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ the inverse Fourier transform of the distribution $\xi \to (2\pi)^n \overline{\Psi(-\xi/2, \xi/2)}$ is equal to

$$2^n \int v(x-y) W_+(x-y, x+y) \mathrm{d}y, \tag{1.1}$$

where $W_+(x, y)$ is the distribution kernel of W_+. The real part of the expression in (1.1) is equal to

$$\beta v(x) = 2^n \int v(x-y) W(x-y, x+y) \mathrm{d}y,$$

where the operator $W = (W_+ + W_-)/2$ has a real-valued distribution kernel. Hence βv is the real part of the inverse Fourier transform of the distribution $\xi \to (2\pi)^n \overline{\Psi(-\xi/2, \xi/2)}$.

The backscattering transform Bv of $v \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ is a slight modification of βv. Let $K_v(t)$ be the wave group associated to the operator

$$\Box_v = \partial_t^2 - \Delta_x + v,$$

i.e., $u(x, t) = (K_v(t)f)(x)$ is, for every $f \in C_0^\infty(\mathbb{R}^n)$, the unique solution in $C^1([0, \infty), L^2(\mathbb{R}^n))$ to the Cauchy problem

$$\Box_v u(x, t) = 0, \quad u(x, 0) = 0, \quad (\partial_t u)(x, 0) = f(x).$$

Then $K_v(t)$ is a strongly continuous function of t with values in the space of bounded linear operators on $L^2(\mathbb{R}^n)$. (See [8] for details.) We have that $|x-y| \le t$ in the support of the distribution kernel $K_v(x, y; t)$ of K_v and $|x-y| = t$ in the support of $K_0(x, y; t)$. This ensures that the operator

$$G = -\int_0^\infty K_v(t) v \dot{K}_0(t) \mathrm{d}t$$

is well defined and continuous on $L^2_{\mathrm{cpt}}(\mathbb{R}^n)$, where the dot denotes derivative in the variable t. Theorem 7.1 in [9] gives the relation between G and W above: there exists an orthonormal basis $(f_j)_{1 \le j \le \mu}$ of real eigenfunctions corresponding to the negative part of the spectrum of H_v and a set $(g_j)_{1 \le j \le \mu}$ of smooth real-valued functions such that

$$W = I + G + \sum_1^\mu f_j \otimes g_j.$$

It turns out (see below) that $G = G_v$, considered as function of v with values in the space of continuous linear operators in $L^2_{\mathrm{cpt}}(\mathbb{R}^n)$, extends to an entire analytic function of $v \in C_0^\infty(\mathbb{R}^n)$, i.e., to the space of complex-valued v in C_0^∞. Also, if v is sufficiently small (in a sense that we do not make precise here), there are no bound states and $W = I + G$ then. For these reasons it is natural to modify the definition of βv by subtracting the contribution from $\sum_1^\mu f_j \otimes g_j$.

Definition. Assume $v \in C_0^\infty(\mathbb{R}^n)$, not necessarily real-valued. The backscattering transform Bv of v is defined by

$$(Bv)(x) = v(x) + 2^n \int v(x-y)G(x-y, x+y)\mathrm{d}y.$$

Here the integral is taken in distribution sense and $v(x)G(x, y)$ is the distribution kernel of the operator vG.

It was proved in [8] that $G = G_v$ extends to an entire analytic function of $v \in L^q_{\mathrm{cpt}}(\mathbb{R}^n)$ when $q > n$. For such v we can define Bv again as in the previous definition and Bv will be entire analytic in v with values in $\mathscr{D}'(\mathbb{R}^n)$. We write

$$Bv = \sum_1^\infty B_N v \tag{1.2}$$

where $B_N v$ is the Nth order term in the power series expansion at $v = 0$. There are other spaces of v (containing C_0^∞ as a dense subset) to which Bv can be extended analytically. Such extensions can be studied by deriving *a priori* estimates for the $B_N v$ when $v \in C_0^\infty$. In this paper we shall study estimates for $B_N v$ in $H_{(s)}$ spaces, and prove that Bv is entire analytic in $v \in H_{(s)} \cap \mathscr{E}'$ when $s \geq (n-3)/2$.

We remark that the discussion before the definition above shows when $v \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ that βv, i.e., the real part of the inverse Fourier transform of $(2\pi)^n \overline{\Psi(-\xi/2, \xi/2)}$, equals

$$Bv + \sum_1^\mu 2^n \int v(y) f_j(y) g_j(2x - y)\mathrm{d}y.$$

Hence our backscattering transformation is, modulo a correction term associated to the negative spectrum of H_v, the inverse Fourier transformed version of the restricted backscattering mapping introduced in [1]. The latter mapping is real analytic in a dense open set only of potentials, while removing the correction term gives rise to a mapping that can be continued to an entire analytic mapping in appropriate spaces of complex potentials.

We recall some basic ingredients in the construction of Bv when $v \in C_0^\infty(\mathbb{R}^n)$. We recall from [8], or Section 11 in [9], that

$$K_v(t) = \sum_{N \geq 0} (-1)^N K_N(t), \tag{1.3}$$

where K_N are inductively defined by

$$\begin{aligned} K_0(t) &= \frac{\sin t|D|}{|D|}, \\ K_N(t) &= (K_{N-1} * vK_0)(t) = \int_0^t K_{N-1}(s) v K_0(t-s)\mathrm{d}s, \quad N \geq 1. \end{aligned} \tag{1.4}$$

One has the estimate

$$\|K_N(t)\|_{L^2\to L^2} \le \|v\|_{L^\infty}^N t^{2N+1}/(2N+1)!.$$

Since the distribution kernel $K_N(x, y; t)$ of $K_N(t)$ is supported in the set where $|x-y| \le t$, it makes sense to consider

$$G_N = (-1)^N \int_0^\infty K_{N-1}(t) v \dot{K}_0(t) \mathrm{d}t. \tag{1.5}$$

This is a continuous linear operator in $L^2_{\mathrm{cpt}}(\mathbb{R}^n)$, and the estimates for the K_N show that

$$G = \sum_1^\infty G_N$$

is an entire analytic function of v. We see that

$$(B_N v)(x) = 2^n \int v(y) G_{N-1}(y, 2x-y) \mathrm{d}y, \quad N \ge 2. \tag{1.6}$$

The following theorem (Theorem 8, [8]) reveals the smoothing properties of B_N for large N.

Theorem 1.1. *Let $q > n$ and k be a nonnegative integer. Then there is a positive integer $N_0 = N_0(n, q, k)$ such that $\Delta^k B_N v \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ when $v \in L^q(\mathbb{R}^n)$ has compact support and $N \ge N_0$. Moreover, if $R_1, R_2 > 0$, there is a constant C, depending on n, k, R_1, R_2 and q only such that*

$$\|\Delta^k B_N v\|_{L^2(B(0,R_1))} \le C^N \|v\|_{L^q}^N / N!, \quad N \ge N_0,$$

whenever $v \in L^q(\mathbb{R}^n)$ has support in the ball $B(0, R_2)$.

The aim of this paper is to give formulas for B_N and study their (local) continuity properties in $H_{(s)}$ spaces.

Let $\|\cdot\|_{(s)}$ denote the norm in the Sobolev space $H_{(s)}(\mathbb{R}^n)$. Also $H_{(s)}(\Omega)$, $s \ge 0$, is the space of functions which are restrictions to Ω of functions from the Sobolev space $H_{(s)}(\mathbb{R}^n)$, when Ω is an open set with smooth boundary. The norm in $H_{(s)}(\Omega)$, $s \ge 0$, is equivalent to the quotient norm

$$\|f\|_{H_{(s)}(\Omega)} = \inf\left\{\|F\|_{(s)};\, F \in H_{(s)}(\mathbb{R}^n),\, F = f \text{ in } \Omega\right\}.$$

Our main result here is the next theorem.

Theorem 1.2. *Assume a is nonnegative, $s \ge (n-3)/2$, and let $N(a, s)$ be the smallest integer N such that $a < N-1$ and $a \le (N-1)(s-(n-3)/2)$. Then there is a constant C, which depends on n, s and a only, such that*

$$\|B_N v\|_{H_{(s+a)}(B(0,R))} \le C^N R^{(N-1)/2} N^{-N/2} \|v\|_{(s)}^N$$

when $N \ge N(a, s)$, $R > 0$ and $v \in C_0^\infty(B(0, R))$.

A more refined form of this result is given in Theorem 4.1, where the estimates are given for the N-linear operators associated to B_N.

A first corollary is the above-mentioned analyticity of the backscattering transformation.

Corollary 1.3. *The mapping* $C_0^\infty(\mathbb{R}^n) \ni v \to Bv \in C^\infty(\mathbb{R}^n)$ *extends to an entire analytic mapping from* $H_{(s)}(\mathbb{R}^n) \cap \mathscr{E}'(\mathbb{R}^n)$ *to* $H_{(s),\mathrm{loc}}(\mathbb{R}^n)$ *whenever* $s \geq (n-3)/2$.

Summarizing, we have defined a backscattering transformation, which appears as a natural generalization to higher dimensions of a transform arising in the Gelfand–Levitan theory (see Section 4 of [9]). Apart from a contribution from the bound states, it is the real part of the inverse Fourier transform of the anti-diagonal part of the far field pattern viewed as a function of the momentum variable. This transformation is entire analytic between certain spaces of functions. We show that the N-linear operator associated to the Nth order term B_N appearing in the power series expansion (1.2) is a singular integral operator, and its distribution kernel is computed. We give local smoothing estimates for each of these operators. Some other consequences of the main result, concerning the regularity of the difference between v and its backscattering transform and local uniqueness properties of B, are given in the last section of the paper.

The outline of this paper is as follows. In the next section we derive a formula that generalizes to arbitrary $N > 2$ the formula

$$(B_2 v)(x) = \int_{(\mathbb{R}^n)^2} E_2(y_1, y_2) v\left(x - \frac{y_2 - y_1}{2}\right) v\left(x - \frac{y_1 + y_2}{2}\right) \mathrm{d}y_1 \, \mathrm{d}y_2,$$

which appears in Corollary 10.7 of [9]. Here E_2 is the unique fundamental solution of the ultra-hyperbolic operator $\Delta_x - \Delta_y$ such that $E_2(x, y) = -E_2(y, x)$ and E_2 is rotation invariant separately in x and y. When $N > 2$ we have to replace E_2 by a distribution $E_N \in \mathscr{D}'((\mathbb{R}^n)^N)$ which is a fundamental solution of the operator $P_N = (\Delta_{x_N} - \Delta_{x_1})(\Delta_{x_N} - \Delta_{x_2}) \cdots (\Delta_{x_N} - \Delta_{x_{N-1}})$. The distribution E_N is discussed in more detail in Section 3.

Once these formulas have been obtained, the proof of the theorem becomes elementary. The third section contains estimates of the Fourier transforms of (cut-offs of) E_N. These are in turn used in the fourth section when the estimates in Theorem 1.2 are obtained by Fourier transforming the formula for $B_N v$.

We close this presentation with a few words on the existing literature on backscattering problems for the potential scattering in odd dimensions. The classical complex backscattering map was studied in [1] for dimension 3 and in [2] for arbitrary dimensions. The main result in [2] shows that the mapping that associates $\xi \to \Psi(-\xi, \xi)$ to the Fourier transform of v is a continuously Fréchet differentiable, hence analytic, map in a open dense subset $\mathscr{O}_1$ of a weighted Hölder space (with weight of the form $(1 + |\xi|)^N$ with $N > n - 2$) with values in the same space, and its Fréchet derivative at $\hat{v}$ is an isomorphism when $\hat{v}$ belongs to open dense subset $\mathscr{O}_2$ of $\mathscr{O}_1$. No power series expansion is given for the classical backscattering map. For the same map, generic uniqueness was proved in [13] for compactly supported bounded potentials in dimension 3. The problem of recovering the singularities of v from the classical backscattering data was considered in [3, 5, 11, 12]. Let us mention the results in [12], where the cases of dimensions $n = 2, 3$ are considered. In the case

$n = 3$, these results imply that, for L^2 compactly supported potentials, the difference between the potential and the Born series approximation of the Fourier transform of the backscattering data belongs, modulo a smooth function, to the Sobolev space $H_{(\delta)}$ with $\delta < 1/2$, and thus the regularity improves independently of the regularity of the potential. From the point of view of regularity our results are better, in the case $n = 3$, only for potentials in $H_{(s)}$ with $s \geq 1/2$ and compact support. However, no continuity estimates are given. We also mention [10, 14] for an approach using Lax–Phillips scattering. Melrose and Uhlmann consider a generalized backscattering transform for compactly supported potentials in $H_{((n+1)/2)}$ and prove that this is entire analytic and globally Fredholm of index zero. As a result they obtain generic local injectivity for the backscattering operator. The actual backscattering transformation defined as above was considered in [6] for dimension 3. Then it was proved to be analytic when defined on small potentials v such that $\nabla v \in L^1$ and with values in the same space, and consequently uniqueness for the inverse backscattering problem was obtained for small potentials in this space.

Finally, let us fix some notation we use throughout the paper. If $N \geq 2$ we use the notation $\vec{x} = (x_1, \ldots, x_N) \in (\mathbb{R}^m)^N$ where $x_1, \ldots x_N \in \mathbb{R}^m$, for m a positive integer. If $x \in \mathbb{R}^m$ we shall set $\langle x \rangle = (1 + |x|^2)^{1/2}$. The Fourier transform of a distribution u will be denoted either by $\hat{u}$ or by $\mathscr{F}u$.

2. A Formula for B_N

In this section we are going to write $B_N v$ as the value at $(v, \ldots, v)$ of a N-linear operator defined from $C_0^\infty(\mathbb{R}^n) \times \cdots \times C_0^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$, following the procedure in [8]. The key point here is the fact that $K_0(t)$ obeys Huygens' principle, more specifically, that its convolution kernel $k_0(x; t)$ is supported in the set where $|x| = t$.

When $N = 1, 2, \ldots$ we define $Q_N \in \mathscr{D}'((\mathbb{R}^n)^N \times \mathbb{R}_+)$ inductively by

$$Q_1(x; t) = k_0(x; t), \tag{2.1}$$

$$Q_N(x_1, \ldots, x_N; t) = \int_0^t Q_{N-1}(x_1, \ldots x_{N-1}; t - s) Q_1(x_N; s) \mathrm{d}s \quad \text{when } N \geq 2. \tag{2.2}$$

Then the mapping

$$\mathbb{R}_+ \ni t \to Q_N(x_1, \ldots x_N; t) \in \mathscr{D}'\big((\mathbb{R}^n)^N\big)$$

is smooth when $N \geq 1$. It is easily seen that Q_N is symmetric in $x_1, \ldots x_N$, rotation invariant separately in these variables and, since $|x| = t$ in the support of $k_0(x, t)$, it follows that

$$|x_1| + \cdots + |x_N| = t \quad \text{in supp } Q_N. \tag{2.3}$$

Next we define $E_N \in \mathscr{D}'((\mathbb{R}^n)^N)$, $N \geq 2$, by

$$E_N(x_1, \ldots, x_N) = (-1)^{N-1} \int_0^\infty Q_{N-1}(x_1, \ldots, x_{N-1}; t) \dot{k}_0(x_N; t) \mathrm{d}t. \tag{2.4}$$

It follows from (2.3) that

$$|x_1| + \cdots + |x_{N-1}| = |x_N| \quad \text{in supp } E_N, \tag{2.5}$$

E_N is rotation invariant separately in all variables, and symmetric in $x_1, \ldots, x_{N-1}$. We recall here that

$$E_2(x, y) = 4^{-1}(\mathrm{i}\pi)^{1-n}\delta^{n-2}(x^2 - y^2) \quad \text{in } \mathbb{R}^n \times \mathbb{R}^n$$

is the unique fundamental solution of the ultra-hyperbolic operator $\Delta_x - \Delta_y$ such that $E_2(x, y) = -E_2(y, x)$ and E_2 is rotation invariant separately in x and y. (See Theorem 10.4 and Corollary 10.2 in [9].)

The next lemma follows easily from (1.4) and (1.5) by induction and some simple computations.

Lemma 2.1. *Assume* $v \in C_0^\infty(\mathbb{R}^n)$. *Then*

$$K_N(x, y; t) = \int v(x_1)\cdots v(x_N)Q_{N+1}(x - x_1, x_1 - x_2, \ldots, x_{N-1} - x_N, x_N - y; t)\mathrm{d}\vec{x}, \tag{2.6}$$

$$G_N(x, y) = \int_{(\mathbb{R}^n)^N} v(x_1)\cdots v(x_N)E_{N+1}(x - x_1, x_1 - x_2, \ldots, x_{N-1} - x_N, x_N - y)\mathrm{d}\vec{x} \tag{2.7}$$

for every $N \geq 1$.

Proposition 2.2. *For* $N \geq 2$

$$\begin{aligned}(B_N v)(x) = \int_{(\mathbb{R}^n)^N} & E_N(y_1, \ldots, y_N)v\left(x - \frac{y_N}{2} - Y_0\right) \\ & \times v\left(x - \frac{y_N}{2} - Y_1\right)\cdots v\left(x - \frac{y_N}{2} - Y_{N-1}\right)\mathrm{d}\vec{y}\end{aligned}$$

when $v \in C_0^\infty(\mathbb{R}^n)$, *where*

$$Y_0 = \frac{1}{2}\sum_{j=1}^{N-1} y_j \quad \textit{and} \quad Y_k = Y_0 - \sum_{j=1}^{k} y_j, \quad 1 \leq k \leq N-1.$$

Proof. We use (2.7) to express $G_{N-1}(y, 2x - y)$ in (1.6) and get thus

$$\begin{aligned}(B_N v)(x) = 2^n \int_{\mathbb{R}^n \times (\mathbb{R}^n)^{N-1}} & v(y)v(x_1)\cdots v(x_{N-1}) \\ & \times E_N(y - x_1, x_1 - x_2, \ldots, x_{N-2} - x_{N-1}, x_{N-1} + y - 2x)\mathrm{d}y\,\mathrm{d}\vec{x}.\end{aligned}$$

The proposition follows by changing variables $y - x_1 = -y_1,\ x_1 - x_2 = -y_2, \ldots,$ $x_{N-2} - x_{N-1} = -y_{N-1}$, $x_{N-1} + y - 2x = -y_N$, hence

$$y = x - \frac{1}{2}\sum_{j=1}^{N} y_j = x - \frac{y_N}{2} - Y_0$$

$$x_1 = y + y_1 = x - \frac{y_N}{2} - Y_1$$

$$\cdots$$

$$x_{N-1} = x_{N-2} + y_{N-1} = x - \frac{y_N}{2} - Y_{N-1}.$$

Here we have made use of the invariance properties of E_N, which in particular ensure that $E_N(y_1, \ldots, y_N)$ is even in each y_j. □

3. The Distribution E_N

We need some further information on the distribution E_N defined in (2.4).

The first result is a characterization of E_N. We denote

$$P_N = (\Delta_1 - \Delta_N) \cdots (\Delta_{N-1} - \Delta_N),$$

where Δ_j in the Laplacian in the variables x_j.

Lemma 3.1. *The distribution E_N is a fundamental solution of P_N. It has the following properties*:

(i) $E_N(x_1, \ldots, x_N)$ *is rotation invariant in each* x_j;
(ii) $|x_1| + \cdots + |x_{N-1}| = |x_N|$ *in the support of* E_N;
(iii) E_N *is homogeneous of degree* $2(N-1) - nN$.

If E is a fundamental solution of P_N that satisfies (i)–(iii), *then* $E = E_N$.

Proof. We first prove that $P_N E_N = \delta(x_1, \ldots, x_N)$, and when doing this we may assume that $N \geq 3$. Since $\partial_t^2 k_0(x; t) = \Delta_x k_0(x; t)$, it follows easily from (2.2) with N replaced by $N-1$ that

$$\begin{aligned}\partial_t^2 Q_{N-1}(x_1, \ldots, x_{N-1}; t) &= \Delta_{N-1} Q_{N-1}(x_1, \ldots, x_{N-1}; t) \\ &\quad + Q_{N-2}(x_1, \ldots, x_{N-2}; t)\delta(x_{N-1}).\end{aligned}$$

It follows from (2.4) then that

$$\begin{aligned}\Delta_N E_N(x_1, \ldots, x_N) &= (-1)^{N-1} \int_0^\infty Q_{N-1}(x_1, \ldots, x_{N-1}; t)\partial_t^2 \dot{k}_0(x_N; t)\mathrm{d}t \\ &= (-1)^{N-1} \int_0^\infty (\partial_t^2 Q_{N-1}(x_1, \ldots, x_{N-1}; t))\dot{k}_0(x_N; t)\mathrm{d}t \\ &= (-1)^{N-1}\Delta_{N-1} \int_0^\infty Q_{N-1}(x_1, \ldots, x_{N-1}; t)\dot{k}_0(x_N; t)\mathrm{d}t \\ &\quad + (-1)^{N-1} \int_0^\infty Q_{N-2}(x_1, \ldots, x_{N-2}; t)\delta(x_{N-1})\dot{k}_0(x_N; t)\mathrm{d}t \\ &= \Delta_{N-1} E_N(x_1, \ldots, x_N) - E_{N-1}(x_1, \ldots, x_{N-2}, x_N)\delta(x_{N-1}).\end{aligned}$$

We have proved therefore that

$$(\Delta_{N-1} - \Delta_N)E_N(x_1, \ldots, x_N) = E_{N-1}(x_1, \ldots, x_{N-2}, x_N)\delta(x_{N-1}). \tag{3.1}$$

Assuming, as we may, that the assertion has been proved for lower values of N and letting $(\Delta_1 - \Delta_N) \cdots (\Delta_{N-2} - \Delta_N)$ act on both sides of (3.1) we may conclude that $P_N E_N(x_1, \ldots, x_N) = \delta(x_1, \ldots x_N)$.

The conditions (i) and (ii) are simple consequences of the definitions, together with the fact that $k_0(x; t)$ is rotation invariant in x and supported in the set where

$|x| = t$. Since k_0 is homogeneous when considered as a distribution in x and t, it follows that E_N is a homogeneous distribution. Its degree of homogeneity must be equal to the degree of P_N minus the dimension of $(\mathbb{R}^n)^N$. This proves (iii).

It remains to prove that $\Phi = 0$ if $\Phi = \Phi(x_1, \ldots, x_N)$ is a distribution satisfying the conditions in (i)–(iii) and $P_N \Phi = 0$.

Define

$$\Psi(x_1, \ldots, x_N) = (\Delta_1 - \Delta_N) \cdots (\Delta_{N-2} - \Delta_N)\Phi(x_1, \ldots, x_N)$$

(with the interpretation $\Psi = E_2$ if $N = 2$). This a homogeneous distribution of degree $2 - nN$ and

$$(\Delta_{N-1} - \Delta_N)\Psi = 0.$$

Since Ψ is rotation invariant in each x_j, it follows from Theorem 10.1 of [9] that Ψ is symmetric in x_{N-1}, x_N. Since $|x_1| + \cdots + |x_{N-1}| = |x_N|$ in the support of Ψ this implies that $x_1 = \cdots = x_{N-2} = 0$ in its support. Hence

$$\Psi(x_1, \ldots, x_N) = \sum \delta^{(\alpha)}(x_1, \ldots x_{N-2}) u_\alpha(x_{N-1}, x_N),$$

where the $u_\alpha(x, y) \in \mathscr{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ are solutions to the ultra-hyperbolic equation. The rotation invariance of Φ in the x_j implies that the summation takes place over even $|\alpha|$ only and that the $u_\alpha(x, y)$ are rotation invariant separately in x and y. Also, $u_\alpha(x, y) = u_\alpha(y, x)$ and u_α is homogeneous of degree μ_α, where

$$\mu_\alpha = 2 - nN + (N-2)n + |\alpha| = 2 + |\alpha| - 2n$$

is even. Since $\mu_\alpha > -2n$ the proof is completed if we prove that u_α vanishes outside the origin in $\mathbb{R}^n \times \mathbb{R}^n$. In this set we may view u_α as a function $f(s, t)$ in $s = |x|$, $t = |y|$. Since it is supported in the set where $s = t$ we may write

$$f(s, t) = \sum_{0 \le j \le J} c_j \delta^{(j)}(s - t)(s + t)^{j+\nu}$$

where $\nu = 1 + \mu_\alpha$ is odd, and the summation takes place over even j only, since $f(s, t) = f(t, s)$. We assume that $f \neq 0$ and shall see that this leads to a contradiction.

Assume now that $c_J \neq 0$. Expressing the Laplacian in polar coordinates, we get the equation

$$0 = \big(\partial_s^2 - \partial_t^2 + (n-1)(s^{-1}\partial_s - t^{-1}\partial_t)\big) f(s, t).$$

The right-hand side here is a linear combination of $\delta^{(j)}(s - t)(s + t)^{j+\nu-2}$ with $j \le J + 1$, and a simple computation shows that the coefficient in front of $\delta^{(J+1)}(s + t)^{J+\nu-1}$ is equal to $4c_J\kappa$, where

$$\kappa = (J + \nu) + n - 1.$$

This gives us a contradiction, since we know that $\kappa = 0$ while the right-hand side above is an odd integer. We have proved therefore that u_α vanishes outside the origin. $\square$

We need to establish estimates for the Fourier transforms of certain cut-offs of E_N. Namely, we shall consider distributions of the form

$$(-1)^{N-1}\int_0^\infty Q_{N-1}(x_1,\ldots,x_{N-1};t)\dot{k}_0(x_N;t)\chi(t)\mathrm{d}t,\quad N=2,3,\ldots,\tag{3.2}$$

where $\chi\in C_0^\infty(\mathbb{R})$. We notice that $|x_1|+\cdots+|x_{N-1}|=|x_N|<R_0$ in the support of this distribution whenever the support of χ is contained in the interval $(-\infty,R_0)$. Also, if $\chi(t)=1$ when $0\le t\le R_1$, then the restrictions to $(\mathbb{R}^n)^{N-1}\times B(0,R_1)$ of the distribution in (3.2) and of E_N coincide.

We start with some preparatory computations. When $a\in\mathbb{R}$ define

$$\varphi_a(t)=Y_+(t)\frac{\sin(ta)}{a},\quad t\in\mathbb{R},$$

where Y_+ is the Heaviside's function.

Lemma 3.2. *Assume $N\ge 2$ and $a_1,\ldots a_N$ are real numbers such that $a_j^2\ne a_k^2$ when $j\ne k$. Then we have the identity*

$$(\varphi_{a_1}*\cdots*\varphi_{a_N})(t)=\sum_{j=1}^N\prod_{k\ne j}\frac{1}{a_k^2-a_j^2}\varphi_{a_j}(t).\tag{3.3}$$

Proof. Let $\varepsilon>0$ and define $\psi_j(t)=e^{-\varepsilon t}\varphi_{a_j}(t)$. A simple computation shows that

$$\hat{\psi}_j(\tau)=\frac{1}{(\varepsilon+\mathrm{i}\tau)^2+a_j^2}.$$

If $\Psi=\psi_1*\cdots*\psi_N$ it follows that

$$\begin{aligned}\widehat{\Psi}(\tau)&=\prod_1^N\frac{1}{(\varepsilon+\mathrm{i}\tau)^2+a_j^2}=\sum_{j=1}^N\left(\prod_{k\ne j}\frac{1}{a_k^2-a_j^2}\right)\frac{1}{(\varepsilon+i\tau)^2+a_j^2}\\&=\sum_{j=1}^N\left(\prod_{k\ne j}\frac{1}{a_k^2-a_j^2}\right)\hat{\psi}_j(\tau).\end{aligned}$$

Hence

$$\Psi(t)=\sum_{j=1}^N\left(\prod_{k\ne j}\frac{1}{a_k^2-a_j^2}\right)\psi_j(t).$$

The lemma then follows when ε tends to 0. □

Lemma 3.3. *When $N\ge 2$, $a_1,\ldots a_N\in\mathbb{R}$, $\sigma\in\mathbb{C}$, $\operatorname{Re}\sigma>0$, define*

$$F(a_1,\ldots,a_N;\sigma)=\int_0^\infty(\varphi_{a_1}*\cdots*\varphi_{a_{N-1}})(t)\cos(ta_N)\mathrm{e}^{-\sigma t}\,\mathrm{d}t.$$

Then

$$F(a_1, \dots, a_N; \sigma) = \frac{1}{2}\left(\prod_{1\le j\le N-1} \frac{1}{a_j^2 - (a_N - \mathrm{i}\sigma)^2} + \prod_{1\le j\le N-1} \frac{1}{a_j^2 - (a_N + \mathrm{i}\sigma)^2} \right). \quad (3.4)$$

Proof. Since both sides of (3.4) depend continuously on $a_1, \dots, a_N \in \mathbb{R}$ it is no restriction to assume that $a_j^2 \neq a_k^2$ when $j \neq k$.

First notice that when $a, b \in \mathbb{R}$ and $\sigma \in \mathbb{C}$, $\operatorname{Re}\sigma > 0$, one has

$$\int_0^\infty \varphi_a(t)\cos(tb)\mathrm{e}^{-\sigma t}\,\mathrm{d}t = \frac{a^2 - b^2 + \sigma^2}{(a^2 - b^2 + \sigma^2)^2 + 4b^2\sigma^2}. \quad (3.5)$$

When $N = 2$ then (3.4) follows directly from this formula.

Assume $N \geq 3$. The previous lemma and (3.5) give

$$\begin{aligned} F(a_1, \dots, a_N; \sigma) &= \sum_{j=1}^{N-1} \prod_{k\neq j} \frac{1}{a_k^2 - a_j^2} \int_0^\infty \varphi_{a_j}(t)\cos(ta_N)\mathrm{e}^{-\sigma t}\,\mathrm{d}t \\ &= \sum_{j=1}^{N-1} \left(\prod_{k\neq j} \frac{1}{a_k^2 - a_j^2} \right) \frac{a_j^2 - a_N^2 + \sigma^2}{\left(a_j^2 - a_N^2 + \sigma^2\right)^2 + 4a_N^2\sigma^2}. \end{aligned}$$

We can simplify this expression by writing

$$t_j = a_j^2 - a_N^2 + \sigma^2, \quad 0 \le j \le N-1, \quad \text{and} \quad b = 2a_N\sigma.$$

Then

$$\begin{aligned} F(a_1, \dots, a_N; \sigma) &= \sum_{j=1}^{N-1} \left(\prod_{k\neq j} \frac{1}{t_k - t_j} \right) \frac{t_j}{t_j^2 + b^2} \\ &= \frac{1}{2} \sum_{j=1}^{N-1} \left(\prod_{k\neq j} \frac{1}{t_k - t_j} \right) \frac{1}{t_j - \mathrm{i}b} + \frac{1}{2} \sum_{j=1}^{N-1} \left(\prod_{k\neq j, k\le N-1} \frac{1}{t_k - t_j} \right) \frac{1}{t_j + \mathrm{i}b} \\ &= \frac{1}{2} \prod_{1\le j\le N-1} \frac{1}{t_j - \mathrm{i}b} + \frac{1}{2} \prod_{1\le j\le N-1} \frac{1}{t_j + \mathrm{i}b}. \end{aligned}$$

This finishes the proof of the lemma, after noticing that $t_j \pm \mathrm{i}b = a_j^2 - (a_N \mp \mathrm{i}\sigma)^2$. □

The next lemma is a direct consequence of Theorem 1.4.2 in [4].

Lemma 3.4. *There is a sequence* $(\chi_N)_1^\infty$ *in* $C_0^\infty(\mathbb{R})$ *such that* $\chi_N(t) = 1$ *when* $|t| \le 1$, $\chi_N(t) = 0$ *when* $|t| > 2$ *and*

$$|\chi_N^{(k)}(t)| \le C^k N^k, \quad 0 \le k \le 2N + 2.$$

Here $C > 0$ *is independent of* N.

In what follows R is an arbitrary positive number. We set $\chi_{N,R}(t) = \chi_N(t/R)$, so that $\chi_{N,1} = \chi_N$. We define

$$E_{N,R} = (-1)^{N-1} \int_0^\infty Q_{N-1}(x_1, \ldots, x_{N-1}; t)\dot{k}_0(x_N; t)\chi_{N,R}(t)\mathrm{d}t, \quad N = 2, 3 \ldots. \tag{3.6}$$

We notice that

$$|x_1| + \cdots + |x_{N-1}| = |x_N| \leq 2R \quad \text{in } \operatorname{supp}(E_{N,R}) \tag{3.7}$$

and

$$E_{N,R}(x_1, \ldots, x_N) = E_N(x_1, \ldots, x_N) \quad \text{when } |x_N| \leq R. \tag{3.8}$$

We shall derive estimates for the Fourier transform $\mathscr{F}E_{N,R}(\xi_1, \ldots, \xi_N)$ of $E_{N,R}$. We notice here that, due to the homogeneity of $Q_{N-1}(\cdot; t)$ and of $\dot{k}_0(\cdot; t)$ and to the definition of $\chi_{N,R}$, we have

$$E_{N,R}(Rx_1, \ldots, Rx_N) = R^{2N-2}R^{-Nn}E_{N,1}(x_1, \ldots x_N).$$

It follows that

$$(\mathscr{F}E_{N,R})(\xi_1, \ldots, \xi_N) = R^{2N-2}\mathscr{F}E_{N,1}(R\xi_1, \ldots, R\xi_N). \tag{3.9}$$

Therefore it is enough to establish estimates for $\mathscr{F}E_{N,1}$.

The distribution $E_{N,1}(x_1, \ldots., x_N)$ is rotation invariant in the variables $x_1, \ldots, x_N$ and compactly supported. The Fourier transform $\mathscr{F}E_{N,1}(\xi_1, \ldots, \xi_N)$ of $E_{N,1}$ is smooth and rotation invariant in each variable ξ_j. We define $F_N(r_1, \ldots, r_N)$ when $r_j \geq 0$ by

$$(\mathscr{F}E_{N,1})(\xi_1, \ldots, \xi_N) = F_N(r_1, \ldots, r_N) \quad \text{when } r_j = |\xi_j|. \tag{3.10}$$

Hence we need estimates of F_N.

Consider $\gamma > 0$. Let us define the functions $h_\gamma(r, s)$ through

$$h_\gamma(r, s) = (\gamma + |r - s|)^{-1}(\gamma + |r + s|)^{-1}.$$

Lemma 3.5. *When $s, t \in \mathbb{R}$, one has*

$$1 + |s - t| \geq \frac{1 + |s|}{1 + |t|}.$$

Consequently

$$h_\gamma(s, r + t) \leq \gamma^{-2}(\gamma + |t|)^2 h_\gamma(s, r)$$

when $s, t, r \in \mathbb{R}$.

Proof. The lemma follows from the inequalities

$$1 + |s - t| \geq 1 + \frac{|s - t|}{1 + |t|} \geq 1 + \frac{|s| - |t|}{1 + |t|} = \frac{1 + |s|}{1 + |t|}. \qquad \square$$

The estimate of F_N that we need is contained in the next lemma.

Lemma 3.6. *There is a constant C, which does not depend on N and γ, such that*

$$|F_N(r_1,\dots,r_N)| \le C^N N^{2N+1}\gamma^{-(2N+1)}e^{2\gamma} \prod_{1\le j\le N-1} h_\gamma(r_j,r_N). \tag{3.11}$$

Proof. It follows from (3.6) and (2.2) that

$$F_N(r_1,\dots,r_N) = (-1)^{N-1}\int_{-\infty}^{\infty} \Phi_N(r_1,\dots,r_N,t)\chi_N(t)dt \tag{3.12}$$

where

$$\Phi_N(r_1,\dots,r_N,t) = (\varphi_{r_1} * \cdots * \varphi_{r_{N-1}})(t)\cos(tr_N).$$

As a function of t, $\Phi_N(r_1,\dots,r_N,t)$ is supported in $[0,\infty)$ and of polynomial growth at infinity.

Define

$$\widetilde{\Phi}_{N,\gamma}(r_1,\dots,r_N,t) = e^{-\gamma t}\Phi_N(r_1,\dots,r_N,t), \quad \tilde{\chi}_{N,\gamma}(t) = e^{\gamma t}\chi_N(t).$$

Then

$$\begin{aligned} F_N(r_1,\dots,r_N) &= \int_{\mathbb{R}} \widetilde{\Phi}_{N,\gamma}(r_1,\dots,r_N,t)\tilde{\chi}_{N,\gamma}(t)dt \\ &= (2\pi)^{-1}\int_{\mathbb{R}} (\mathcal{F}\widetilde{\Phi}_{N,\gamma})(r_1,\dots,r_N,\tau)(\mathcal{F}\tilde{\chi}_{N,\gamma})(-\tau)d\tau, \end{aligned} \tag{3.13}$$

where the Fourier transform is taken in the variable t. We notice that

$$(\mathcal{F}\widetilde{\Phi}_{N,\gamma})(r_1,\dots,r_N,\tau) = \int \Phi_N(r_1,\dots r_N,t)e^{-\sigma t}\,dt = F(r_1,\dots,r_N;\sigma), \quad \sigma=\gamma+i\tau,$$

with F as in Lemma 3.3. Then an application of that lemma gives the estimate

$$\begin{aligned} |(\mathcal{F}\widetilde{\Phi}_{N,\gamma})(r_1,\dots,r_N,\tau)| &\le \frac{1}{2}\prod_{1\le j\le N-1}|r_j^2-(r_N+i\sigma)^2|^{-1} + \frac{1}{2}\prod_{1\le j\le N-1}|r_j^2-(r_N-i\sigma)^2|^{-1} \\ &= \frac{1}{2}\prod_{1\le j\le N-1}|r_j-(r_N-\tau)-i\gamma|^{-1}|r_j+(r_N-\tau)+i\gamma|^{-1} \\ &\quad + \frac{1}{2}\prod_{1\le j\le N-1}|r_j-(r_N+\tau)-i\gamma|^{-1}|r_j+(r_N+\tau)+i\gamma|^{-1} \\ &\le 2^{N-2}\prod_{1\le j\le N-1}(\gamma+|r_j-(r_N-\tau)|)^{-1}(\gamma+|r_j+(r_N-\tau)|)^{-1} \\ &\quad + 2^{N-2}\prod_{1\le j\le N-1}(\gamma+|r_j-(r_N+\tau)|)^{-1}(\gamma+|r_j+(r_N+\tau)|)^{-1} \\ &= 2^{N-2}\prod_{1\le j\le N-1}h_\gamma(r_j,r_N-\tau) + 2^{N-2}\prod_{1\le j\le N-1}h_\gamma(r_j,r_N+\tau). \end{aligned}$$

Next we see that

$$\begin{aligned}\mathcal{F}\tilde{\chi}_{N,\gamma}(-\tau) &= \int \chi_{N,\gamma}(t)\mathrm{e}^{t(\gamma+\mathrm{i}\tau)}\mathrm{d}t\\ &= (\gamma+\mathrm{i}\tau)^{-(2N+2)}\int \chi_{N,\gamma}(2N+2)(t)\mathrm{e}^{t(\gamma+\mathrm{i}\tau)}\mathrm{d}t.\end{aligned}$$

From this and Lemma 3.4 we deduce that there is a constant C, which is independent of N and γ, such that

$$|\mathcal{F}\tilde{\chi}_{N,\gamma}(-\tau)| \le C^N N^{2N+2}\mathrm{e}^{2\gamma}(\gamma+|\tau|)^{-2N-2}.$$

Then (3.13), (3.14) and the above inequality, together with Lemma 3.5, give

$$\begin{aligned}|F_N(r_1,\ldots,r_N)| &\le C^N N^{2N+2}\mathrm{e}^{2\gamma}\int_{-\infty}^{\infty}(\gamma+|\tau|)^{-2N-2}\Bigg(\prod_{1\le j\le N-1} h_\gamma(r_j, r_N-\tau)\Bigg)\mathrm{d}\tau\\ &\le C^N N^{2N+2}\gamma^{-2(N-1)}\mathrm{e}^{2\gamma}\Bigg(\int_{-\infty}^{\infty}(\gamma+|\tau|)^{-4}\,\mathrm{d}\tau\Bigg)\Bigg(\prod_{1\le j\le N-1} h_\gamma(r_j,r_N)\Bigg)\\ &\le C^N\gamma^{-(2N+1)}N^{2N+2}\mathrm{e}^{2\gamma}\prod_{1\le j\le N-1} h_\gamma(r_j,r_N)\\ &\le (2C)^N\gamma^{-(2N+1)}N^{2N+1}\mathrm{e}^{2\gamma}\prod_{1\le j\le N-1} h_\gamma(r_j,r_N).\end{aligned}$$

This finishes the proof. □

The following theorem gives the estimate we need for the Fourier transform of $E_{N,R}$.

Theorem 3.7. *There is a constant $C>0$, which depends on n only, such that*

$$|(\mathcal{F}E_{N,R})(\xi_1,\ldots,\xi_N)| \le C^N(N/(R\gamma))^{2N+1}e^{2R\gamma}\prod_{1\le j\le N-1} h_\gamma(|\xi_j|,|\xi_N|),\quad \xi_1,\ldots\xi_N\in\mathbb{R}^n$$

for every $N\ge 2$, $R>0$ and $\gamma>0$.

Proof. Let $R>0$. The identity (3.9) and previous lemma show that there is a constant $C>0$, which depends on n only, such that

$$|(\mathcal{F}E_{N,R})(\xi_1,\ldots,\xi_N)| \le C^N(N/\gamma)^{2N+1}R^{2N-2}e^{2\gamma}\prod_{1\le j\le N-1} h_\gamma(R|\xi_j|, R|\xi_N|),$$

when $\xi_1,\ldots,\xi_N\in\mathbb{R}^n$, for every $N\ge 2$, $R>0$ and $\gamma>0$. This in turn shows that, with the same C, one has

$$|(\mathcal{F}E_{N,R})(\xi_1,\ldots,\xi_N)| \le C^N(N/\gamma)^{2N+1}e^{2\gamma}\prod_{1\le j\le N-1} h_{\gamma/R}(|\xi_j|,|\xi_N|).$$

The theorem follows by replacing γ/R by γ. □

4. Proof of the Main Result and Some Consequences

We introduce an N-linear version of B_N, $N \geq 2$. Namely, for $\vec{v} = (v_1, \dots, v_N)$, $v_j \in C_0^\infty(\mathbb{R}^n)$, define

$$\begin{aligned}(\mathbf{B}_N \vec{v})(x) = \int_{(\mathbb{R}^n)^N} & E_N(y_1, \dots, y_N) \\ v_1\left(x - \frac{y_N}{2} - Y_0\right) & v_2\left(x - \frac{y_N}{2} - Y_1\right) \cdots v_N\left(x - \frac{y_N}{2} - Y_{N-1}\right) \mathrm{d}\vec{y}.\end{aligned} \tag{4.1}$$

Here the Y_ks are defined as in Proposition 2.2, that is,

$$Y_0 = \frac{1}{2} \sum_1^{N-1} y_j, \quad Y_k = Y_0 - \sum_{j=1}^{k} y_j, \ \ k = 1, \dots, N-1.$$

Then $\mathbf{B}_N \vec{v}$ is a smooth compactly supported function in $\mathbb{R}^n$ and $B_N v = \mathbf{B}_N(v, \dots, v)$ for every $v \in C_0^\infty(\mathbb{R}^n)$. Therefore the result in Theorem 1.2 is contained in the next theorem. Here and in the rest of the section we use the notation

$$m = \frac{n-3}{2}.$$

Theorem 4.1. *Assume that* $0 < \varepsilon < 1$, $s_j \geq m$ *and* $a_j = \min(s_j - m, 1 - \varepsilon)$, $j = 1, \dots, N$, *where* $N \geq 2$. *Set*

$$\sigma = \min(s_j - a_j) + \sum_{j=1}^{N} a_j.$$

Then there is a constant C which is independent of N and the s_j, but may depend on ε and n, such that

$$\|\mathbf{B}_N \vec{v}\|^2_{H_{(\sigma)}(B(0,R))} \leq C^N N^{2\min(s_j - a_j - m)} (R/N)^{N-1} \prod_1^N \|v_j\|^2_{(s_j)}, \tag{4.2}$$

for every $R > 0$, $v_1, \dots, v_N \in C_0^\infty(B(0, R))$.

The present section is devoted to the proof of the above result and a presentation of some of its consequences. We start with some preparations.

Let $R > 0$ and recall that the distributions $E_{N,R} \in \mathscr{E}'((\mathbb{R}^n)^N)$ were defined in (3.6). When $\vec{v} = (v_1, \dots, v_N)$, $v_j \in C_0^\infty(\mathbb{R}^n)$, we consider

$$\begin{aligned}(\mathbf{B}_{N,R} \vec{v})(x) = \int_{(\mathbb{R}^n)^N} & E_{N,R}(y_1, \dots, y_N) \\ v_1\left(x - \frac{y_N}{2} - Y_0\right) & v_2\left(x - \frac{y_N}{2} - Y_1\right) \cdots v_N\left(x - \frac{y_N}{2} - Y_{N-1}\right) \mathrm{d}\vec{y}.\end{aligned} \tag{4.3}$$

It is easy to see that $\mathbf{B}_{N,R} \vec{v}$ is a smooth compactly supported function in $\mathbb{R}^n$. The following lemma gives the connection between $\mathbf{B}_{N,R} \vec{v}$ and $\mathbf{B}_N \vec{v}$.

Lemma 4.2. *Assume* $v_1, \ldots, v_N \in C_0^\infty(B(0, R))$. *Then* $(\mathbf{B}_{N,4R}\vec{v})(x) = (\mathbf{B}_N\vec{v})(x)$ *in a neighborhood of* $\overline{B(0, R)}$ *and* $\mathbf{B}_{N,2(N-1)R}\vec{v} = \mathbf{B}_N\vec{v}$.

Proof. Choose $\varepsilon > 0$ such that the v_j are supported in $B(0, R-\varepsilon)$ and define

$$V_x(\vec{y}) = v_1(x - y_N/2 - Y_0)\cdots v_N(x - y_N/2 - Y_{N-1}). \tag{4.4}$$

Since $Y_0 + Y_{N-1} = 0$, it follows that

$$|2x - y_N| = |(x - y_N/2 - Y_0) + (x - y_N/2 - Y_{N-1})| \leq 2R - 2\varepsilon$$

when $\vec{y} \in \operatorname{supp}(V_x)$. When $|x| < R + \varepsilon/2$ we see that $|y_N| < 4R$ when $\vec{y}$ is in the support of V_x and, since $E_{N,4R} = E_N$ when $|y_N| < 4R$, it follows from (4.3) that $(\mathbf{B}_{N,4R}\vec{v})(x) = (\mathbf{B}_N\vec{v})(x)$ when $|x| < R + \varepsilon/2$. This proves the first assertion. When proving the second assertion we notice that

$$|y_j| = \left|(x - y_N/2 - Y_{j-1}) - (x - y_N/2 - Y_j)\right| < 2R$$

when $1 \leq j \leq N-1$ and $\vec{y} \in \operatorname{supp}(V_x)$, hence $\sum_1^{N-1} |y_j| < 2(N-1)R$. This shows that the support of V_x does not intersect the support of $E_{N,2(N-1)R} - E_N$, hence $\mathbf{B}_{N,2(N-1)R}\vec{v} = \mathbf{B}_N\vec{v}$. □

Let $\vec{s} = (s_1, \ldots, s_N)$ be a sequence of nonnegative real numbers and let $\sigma \in \mathbb{R}$, $N \geq 2$, $R > 0$. Define

$$\begin{aligned} A(N, R, \vec{s}, \sigma) = \sup_{\xi_N} \int \cdots \int (1 + 4|\xi_N|^2)^\sigma |(\mathcal{F}E_{N,R})(\xi_1, \ldots, \xi_N)|^2 \\ \times M_{\vec{s}}(\xi_1, \ldots, \xi_N)^2 \, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_{N-1}, \end{aligned} \tag{4.5}$$

where

$$M_{\vec{s}}(\xi_1, \ldots, \xi_N) = \langle \xi_1 + \xi_N \rangle^{-s_1} \langle \xi_2 - \xi_1 \rangle^{-s_2} \cdots \langle \xi_N - \xi_{N-1} \rangle^{-s_N}.$$

Then $0 \leq A(N, R, \vec{s}, \sigma) \leq \infty$.

Lemma 4.3. *We have that*

$$\|\mathbf{B}_{N,R}\vec{v}\|_{(\sigma)}^2 \leq (2\pi)^{n(1-N)} A_{N,R}(s_1, \ldots, s_N, \sigma) \prod_1^N \|v_j\|_{(s_j)}^2 \tag{4.6}$$

for every $v_j \in C_0^\infty(\mathbb{R}^n)$, $1 \leq j \leq N$.

Proof. Let V_x be defined as in (4.4). In order to compute the Fourier transform of V_x we introduce the linear map L in $(\mathbb{R}^n)^N$ through $L\vec{z} = \vec{y}$, where

$$y_j = z_j - z_{j+1}, \quad 1 \leq j \leq N-1, \;\; y_N = z_1 + z_N.$$

It is easily seen that $\det(L) = 2^n$ and that $y_N/2 + Y_{j-1} = z_j$ when $1 \le j \le N$. Therefore we may write

$$V_x(\vec{y}) = (v_1 \otimes \cdots \otimes v_N)(-L^{-1}(y_1, \dots y_{N-1}, y_N - 2x)).$$

Hence

$$\mathscr{F}V_x(-\xi_1, \dots, -\xi_N) = 2^n \mathrm{e}^{2\mathrm{i}\langle x, \xi_N\rangle}(\hat{v}_1 \otimes \cdots \otimes \hat{v}_N)(L'\vec{\xi}).$$

Here L' denotes the transpose of L. It is easy to see that $L'\vec{\xi} = \vec{\eta}$, where

$$\eta_1 = \xi_1 + \xi_N, \quad \eta_j = \xi_j - \xi_{j-1}, \ 2 \le j \le N.$$

It follows that

$$(\mathscr{F}V_x)(-\xi_1, \dots, -\xi_N) = 2^n \mathrm{e}^{2\mathrm{i}\langle x, \xi_N\rangle}\hat{v}_1(\xi_1 + \xi_N)\hat{v}_2(\xi_2 - \xi_1)\cdots \hat{v}_N(\xi_N - \xi_{N-1}).$$

Write $w_j = \mathscr{F}\langle D\rangle^{s_j} v_j$ and

$$W(\vec{\xi}) = w_1(\xi_1 + \xi_N)w_2(\xi_2 - \xi_1)\cdots w_N(\xi_N - \xi_{N-1}).$$

It follows from (4.3) and the computations above that

$$\begin{aligned}
(\mathbf{B}_{N,R}\vec{v})(x) &= (2\pi)^{-nN}\int (\mathscr{F}E_{N,R})(\vec{\xi})(\mathscr{F}V_x)(-\vec{\xi})\mathrm{d}\vec{\xi} \\
&= (2\pi)^{-nN}2^n \int \mathrm{e}^{2\mathrm{i}\langle x, \xi_N\rangle}\beta(\xi_N)\mathrm{d}\xi_N \\
&= (2\pi)^{-nN}\int \mathrm{e}^{\mathrm{i}\langle x, \xi_N\rangle}\beta(\xi_N/2)\mathrm{d}\xi_N,
\end{aligned}$$

where

$$\begin{aligned}
\beta(\xi_N) &= \int \cdots \int (\mathscr{F}E_{N,R})(\vec{\xi})\hat{v}_1(\xi_1 + \xi_N)\hat{v}_2(\xi_2 - \xi_1)\cdots \hat{v}_N(\xi_N - \xi_{N-1})\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{N-1} \\
&= \int \cdots \int (\mathscr{F}E_{N,R})(\vec{\xi})M_{\vec{s}}(\vec{\xi})W(\vec{\xi})\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{N-1}.
\end{aligned}$$

This shows that

$$\begin{aligned}
\|\mathbf{B}_{N,R}\vec{v}\|^2_{(\sigma)} &= (2\pi)^{-2n(N-1/2)}\int \langle \xi_N\rangle^{2\sigma}|\beta(\xi_N/2)|^2 \, \mathrm{d}\xi_N \\
&= 2^n(2\pi)^{-2n(N-1/2)}\int (1 + |4\xi_N|^2)^\sigma |\beta(\xi_N)|^2 \, \mathrm{d}\xi_N \\
&\le 2^n(2\pi)^{-2n(N-1/2)}\int \Bigg\{(1 + 4|\xi_n|^2)^\sigma \qquad (4.7) \\
&\quad \times \left(\int \cdots \int |(\mathscr{F}E_{N,R})(\vec{\xi})|^2 M_{\vec{s}}(\xi)^2 \, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{N-1}\right) \\
&\quad \times \left(\int \cdots \int |W(\vec{\xi})|^2 \, \mathrm{d}\xi_1 \cdots d\xi_{N-1}\right)\Bigg\}\mathrm{d}\xi_N \\
&\le 2^n(2\pi)^{-2n(N-1/2)}A(N, R, \vec{s}, \sigma)\int |W(\vec{\xi})|^2 \, \mathrm{d}\vec{\xi}.
\end{aligned}$$

The proof is then completed by the observation that

$$W(\vec{\xi}) = (w_1 \otimes w_2 \otimes \cdots \otimes w_N)(L'\vec{\xi}).$$

It follows that

$$\begin{aligned}\int |W(\vec{\xi})|^2 \,\mathrm{d}\vec{\xi} &= 2^{-n} \int |(w_1 \otimes \cdots \otimes w_N)(\vec{\xi})|^2 \,\mathrm{d}\vec{\xi} \\ &= 2^{-n} \prod_1^N \|w_j\|^2 = 2^{-n}(2\pi)^{nN} \prod_1^N \|\langle D\rangle^{s_j} v_j\|^2 = 2^{-n}(2\pi)^{nN} \prod_1^N \|v_j\|^2_{(s_j)}.\end{aligned}$$

The lemma follows if this is inserted into (4.7). □

We shall arrive at estimates for $B_{N,R}\vec{v}$ by combining the inequality (4.6) with estimates for the expression $A(N, R, \vec{s}, \sigma)$ in (4.5). The following lemma will be needed.

Lemma 4.4. *Assume* $0 < \varepsilon < 1$. *Then there is a constant* $C = C_{n,\varepsilon}$ *such that*

$$\int h_\gamma^2(|\xi|, \rho)\langle \xi - \eta\rangle^{-2s} \,\mathrm{d}\xi \le C\gamma^{-1}\langle \rho\rangle^{2m-2s}, \tag{4.8}$$

when $\eta \in \mathbb{R}^n$, $\rho \ge 0$, $\gamma > 0$, $m \le s \le m + 1 - \varepsilon$.

Proof. Assume $r > 0$ and $\eta \in \mathbb{R}^n \backslash 0$. Set

$$f_s(r, \eta) = \int_{\mathbb{S}^{n-1}} \langle r\theta - \eta\rangle^{-2s} \,\mathrm{d}\theta.$$

If $u = \langle \theta, \eta\rangle/|\eta|$ then a simple computation shows that

$$\langle r\theta - \eta\rangle^2 \ge 1 + r^2(1 - |u|).$$

If f is a continuous function, and c_{n-2} is the area of the $n-2$-dimensional unit sphere, then

$$\int_{\mathbb{S}^{n-1}} f(\langle \theta, \eta\rangle/|\eta|)\mathrm{d}\eta = c_{n-2}\int_{-1}^1 f(t)(1 - t^2)^m \,\mathrm{d}t.$$

This shows that

$$\begin{aligned}f_s(r, \eta) &\le c_{n-2}\int_{-1}^1 (1 + r^2(1 - |t|))^{-s}(1 - t^2)^m \,\mathrm{d}t \\ &\le 2^{m+1}c_{n-2}\int_0^1 (1 + r^2(1 - t))^{-s}(1 - t)^m \,\mathrm{d}t \le 2^{m+1}c_{n-2}\int_0^1 (1 + r^2 t)^{-s} t^m \,\mathrm{d}t \\ &\le 2^{m+1}c_{n-2}\langle r\rangle^{-2s}\int_0^1 t^{m-s} \,\mathrm{d}t.\end{aligned}$$

This gives the estimate

$$f_s(r, \eta) \le C_1\langle r\rangle^{-2s}, \tag{4.9}$$

where $C_1 = 2^{m+1}c_{n-2}/\varepsilon$, for $\eta \in \mathbb{R}^n \backslash 0$. This inequality clearly holds for $\eta = 0$ as well.

Using (4.9) and introducing polar coordinates in the integration one gets

$$\int h_\gamma^2(|\xi|,\rho)\langle\xi-\eta\rangle^{-2s}\,\mathrm{d}\xi \le C_1\int_0^\infty h_\gamma^2(r,\rho)r^{2m+2}\langle r\rangle^{-2s}\,\mathrm{d}r. \tag{4.10}$$

Assume first that $\rho \ge 1$. Then

$$\begin{aligned}\int_0^\infty h_\gamma^2(r,\rho)r^{2m+2}\langle r\rangle^{-2s}\,\mathrm{d}r &= \int_0^\infty \frac{1}{(\gamma+|r-\rho|)^2}\frac{1}{(\gamma+r+\rho)^2}\frac{r^{2m+2}}{(r^2+1)^s}\,\mathrm{d}r\\ &\le \frac{1}{\rho^{2(s-m)}}\int_0^\infty \frac{1}{(\gamma+|r-\rho|)^2}\,\frac{r^{2m+2}}{(r^2+1)^{m+1}}\,\mathrm{d}r \\ &\le \frac{2^{2(s-m)}}{(\rho+1)^{2(s-m)}}\int_{\mathbb{R}}\frac{1}{(\gamma+|r|)^2}\,\mathrm{d}r \le 2^3\gamma^{-1}\langle\rho\rangle^{-2(s-m)}.\end{aligned} \tag{4.11}$$

Assume next that $\rho < 1$. Then

$$\begin{aligned}&\int_0^\infty h_\gamma^2(r,\rho)r^{2m+2}\langle r\rangle^{-2s}\,\mathrm{d}r\\ &\quad\le \int_0^\infty \frac{1}{(\gamma+|r-\rho|)^2}\,\frac{r^{2m}}{(r^2+1)^s}\,\mathrm{d}r \le \int_{\mathbb{R}}\frac{1}{(\gamma+|r|)^2}\,\mathrm{d}r \\ &\quad\le 2^{2(s-m)+1}\gamma^{-1}(\rho^2+1)^{-(s-m)} \le 2^3\gamma^{-1}\langle\rho\rangle^{-2(s-m)}.\end{aligned} \tag{4.12}$$

Combining (4.10)–(4.12) we see that the lemma holds with $C = 2^3C_1$. $\square$

Now we are going to estimate $A(N,R,\vec{s},\gamma)$. Recall that Theorem 3.7 gives that

$$|(\mathcal{F}E_{N,R})(\xi_1,\ldots,\xi_N)| \le C^N(N/(\gamma R))^{2N+1}e^{2R\gamma}\prod_{1\le j\le N-1}h_\gamma(|\xi_j|,|\xi_N|), \tag{4.13}$$

where $\gamma > 0$, $N \ge 2$, $R > 0$ and the constant C is independent of these parameters. We notice that

$$\begin{aligned}M_{\vec{s}}(\xi_1,\ldots,\xi_N) &\le 2^{s_2}\langle\xi_1+\xi_N\rangle^{-s_1}M_{s_2,\ldots,s_N}(\xi_2,\ldots,\xi_N)\\ &\quad + 2^{s_1}\langle\xi_2-\xi_1\rangle^{-s_2}M_{s_1,s_3,\ldots,s_N}(\xi_2,\ldots,\xi_N).\end{aligned} \tag{4.14}$$

In fact, since

$$|\xi_2+\xi_N| \le |\xi_2-\xi_1| + |\xi_1+\xi_N|,$$

either $|\xi_2-\xi_1| \ge |\xi_2+\xi_N|/2$ or $|\xi_1+\xi_N| \ge |\xi_2+\xi_N|/2$. In the first case

$$M_{\vec{s}}(\xi_1,\ldots,\xi_N) \le 2^{s_2}\langle\xi_1+\xi_N\rangle^{-s_1}M_{s_2,\ldots,s_N}(\xi_2,\ldots,\xi_N)$$

and in the second case

$$M_{\vec{s}}(\xi_1,\ldots,\xi_N) \le 2^{s_1}\langle\xi_1-\xi_2\rangle^{-s_2}M_{s_1,s_3,\ldots,s_N}(\xi_2,\ldots,\xi_N).$$

When $N \geq 2$ we define

$$T_{N,\gamma}(\xi, \vec{s}) = \int \cdots \int \left(\prod_{1 \leq j \leq N-1} h_\gamma(|\xi_j|, |\xi|) \right)^2 M_{\vec{s}}^2(\xi_1, \ldots, \xi_{N-1}, \xi) \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \ldots \mathrm{d}\xi_{N-1}.$$

Let $0 < \varepsilon < 1$ and assume that $m \leq s_j \leq m + 1 - \varepsilon$ when $0 \leq j \leq N$. It follows from (4.13) and (4.5) that

$$A(N, R, \vec{s}, \sigma) \leq C^N (N/(\gamma R))^{4N+2} e^{4R\gamma} \sup_\xi (\langle 2\xi \rangle^{2\sigma} T_{N,\gamma}(\xi; \vec{s})). \tag{4.15}$$

Here, and in what follows, C denotes constants that are independent of N, R, $\vec{s}$, σ, γ (but may depend on ε and dimension n).

Assume $N \geq 3$. From (4.14) follows that

$$\begin{aligned} T_{N,\gamma}(\xi; \vec{s}) \leq 2^n & \left(\int h_\gamma^2(|\xi_1|, |\xi|) \langle \xi_1 + \xi \rangle^{-2s_1} \, \mathrm{d}\xi_1 \right) \\ & \int \cdots \int \left(\prod_{2 \leq j \leq N-1} h_\gamma^2(|\xi_j|, |\xi|) \right) M_{s_2, \ldots, s_N}^2(\xi_2, \ldots, \xi_{N-1}, \xi) \mathrm{d}\xi_2 \cdots \mathrm{d}\xi_{N-1} \\ & + 2^n \left(\int h_\gamma^2(|\xi_1|, |\xi|) \langle \xi_1 - \xi_2 \rangle^{-2s_2} \, \mathrm{d}\xi_1 \right) \\ & \int \cdots \int \left(\prod_{2 \leq j \leq N-1} h_\gamma^2(|\xi_j|, |\xi|) \right) M_{s_1, s_3, \ldots, s_N}^2(\xi_2, \ldots, \xi_{N-1}, \xi) \mathrm{d}\xi_2 \cdots \mathrm{d}\xi_{N-1}. \end{aligned}$$

From Lemma 4.4 we get the estimate

$$\begin{aligned} T_{N,\gamma}(\xi; \vec{s}) \leq {} & (C/2)\gamma^{-1} \langle \xi \rangle^{2m-2s_1} T_{N-1,\gamma}(\xi; s_2, \ldots, s_N) \\ & + (C/2)\gamma^{-1} \langle \xi \rangle^{2m-2s_2} T_{N-1,\gamma}(\xi; s_1, s_3, \ldots, s_N). \end{aligned} \tag{4.16}$$

Another application of Lemma 4.4 gives

$$\begin{aligned} T_{2,\gamma}(\xi; s_1, s_2) &= \int h_\gamma^2(|\xi_1|, |\xi|) \langle \xi_1 + \xi \rangle^{-2s_1} \langle \xi_1 - \xi \rangle^{-2s_2} \, \mathrm{d}\xi_1 \\ &\leq \langle \xi \rangle^{-2s_1} \int h_\gamma^2(|\xi_1|, |\xi|) \langle \xi_1 - \xi \rangle^{-2s_2} \, \mathrm{d}\xi_1 \\ &\quad + \langle \xi \rangle^{-2s_2} \int h_\gamma^2(|\xi_1|, |\xi|) \langle \xi_1 + \xi \rangle^{-2s_1} \, \mathrm{d}\xi_1 \\ &\leq C\gamma^{-1} \langle \xi \rangle^{2m-2s_1-2s_2} \end{aligned}$$

where we may assume that C is the same constant as in (4.16). From this we deduce that the inequality

$$T_{N,\gamma}(\xi; \vec{s}) \leq C^{N-1} \gamma^{-(N-1)} \langle \xi \rangle^{2((N-1)m - s_1 - \cdots - s_N)} \tag{4.17}$$

holds when $N = 2$. Applying (4.16) together with an induction argument we obtain that (4.17) holds for every $N \geq 2$. Since $m \leq s_j < m + 1$ we get (with another C)

$$T_{N,\gamma}(\xi; \vec{s}) \leq C^{N-1}\gamma^{-(N-1)}\langle 2\xi\rangle^{2((N-1)m-s_1-\cdots-s_N)}. \tag{4.18}$$

Assume now that $s_j \geq m$, $j = 1, \ldots, N$, but not necessarily $s_j < m + 1$, and let $0 < \varepsilon < 1$. Set $a_j = \min(s_j - m, 1 - \varepsilon)$, $j = 1, \ldots, N$. We notice that

$$|\xi_1 + \xi_N| + |\xi_2 - \xi_1| + \cdots + |\xi_N - \xi_{N-1}| \geq 2|\xi_N|,$$

and therefore

$$\max(|\xi_1 + \xi_N|, |\xi_2 - \xi_1|, \ldots, |\xi_N - \xi_{N-1}|) \geq 2|\xi_N|/N.$$

It follows that

$$\langle \xi_1 + \xi_N\rangle^{-1}\langle \xi_2 - \xi_1\rangle^{-1}\cdots\langle \xi_N - \xi_{N-1}\rangle^{-1} \leq (1 + 4|\xi_N|^2/N^2)^{-1/2} \leq N\langle 2\xi_N\rangle^{-1}.$$

Then we may write

$$M_{\vec{s}}(\vec{\xi}) \leq N^{\min(s_j-(a_j+m))}\langle 2\xi_N\rangle^{-\min(s_j-(a_j+m))}M_{(m+a_1,\ldots,m+a_N)}(\vec{\xi}).$$

This implies that

$$T_{N,\gamma}(\xi; \vec{s}) \leq N^{2\min(s_j-(a_j+m))}\langle 2\xi\rangle^{-2\min(s_j-(a_j+m))}T_{N,\gamma}(\xi; m + a_1, \ldots, m + a_N).$$

Then (4.18) gives

$$T_{N,\gamma}(\xi, \vec{s}) \leq C^{N-1}N^{2\min(s_j-(a_j+m))}\gamma^{-(N-1)}\langle 2\xi\rangle^{-2\min(s_j-a_j)-2\sum_1^N a_j}. \tag{4.19}$$

Combining (4.15) with (4.19) we get the following lemma.

Lemma 4.5. *Assume that* $0 < \varepsilon < 1$, $s_j \geq m$ *and* $a_j = \min(s_j - m, 1 - \varepsilon)$, $j = 1, \ldots, N$. *Set*

$$\sigma = \min(s_j - a_j) + \sum_{j=1}^{N} a_j.$$

Then there is a constant C *which is independent of* N *and the* s_j, *but may depend on* ε *and* n, *such that*

$$A(N, R, \vec{s}, \sigma) \leq C^N N^{2\min(s_j-a_j-m)}(N/(\gamma R))^{4N+2}\gamma^{-(N-1)}e^{4R\gamma} \tag{4.20}$$

for every $\gamma > 0$ *and* $R > 0$.

Next we recall (4.6) which, together with the previous lemma, gives the next proposition.

Proposition 4.6. *Assume that* $0 < \varepsilon < 1$, $s_j \geq m$ *and* $a_j = \min(s_j - m, 1 - \varepsilon)$, $j = 1, \ldots, N$, *where* $N \geq 2$. *Set*

$$\sigma = \min(s_j - a_j) + \sum_{j=1}^{N} a_j.$$

Then there is a constant C which is independent of N and the s_j, but may depend on ε and n, such that

$$\|\mathbf{B}_{N,R}\vec{v}\|_{(\sigma)}^2 \leq C^N N^{2\min(s_j - a_j - m)} (N/(\gamma R))^{4N+2} \gamma^{-(N-1)} e^{4R\gamma} \prod_{1}^{N} \|v_j\|_{(s_j)}^2, \tag{4.21}$$

for every $v_1, \ldots, v_N \in C_0^\infty(\mathbb{R}^n)$, $R > 0$ *and* $\gamma > 0$.

Theorem 4.1 follows from the previous proposition and Lemma 4.2, by replacing R by $4R$ and taking $\gamma = N/(4R)$. When replacing R by $2(N-1)R$ and taking $\gamma = 1/R$, we obtain the following corollary, where we use Lemma 4.2.

Corollary 4.7. *Assume that* $0 < \varepsilon < 1$, $s_j \geq m$ *and* $a_j = \min(s_j - m, 1 - \varepsilon)$, $j = 1, \ldots, N$. *Set*

$$\sigma = \min(s_j - a_j) + \sum_{j=1}^{N} a_j.$$

Then there is a constant C, which depends on n, ε and the s_j only, such that

$$\|\mathbf{B}_N\vec{v}\|_{(\sigma)}^2 \leq C^N R^{N-1} \prod_{1}^{N} \|v_j\|_{(s_j)}^2, \tag{4.22}$$

for every $N \geq 2$, $R > 0$ *and* $v_1, \ldots, v_N \in C_0^\infty(B(0, R))$.

We conclude this section by giving some consequences of our main result. The first one, the analyticity of the backscattering transformation, has already been stated in Corollary 1.3. A second consequence gives the difference of regularity between v and its backscattering transform.

Corollary 4.8. *Assume* $s \geq (n-3)/2$ *and* $0 \leq a < 1$ *satisfy* $a \leq s - (n-3)/2$. *If* $v \in H_{(s)}(\mathbb{R}^n)$ *is compactly supported, then*

$$v - Bv \in H_{(s+a),\mathrm{loc}}(\mathbb{R}^n). \tag{4.23}$$

Let now $R > 0$ be arbitrary fixed. When $s \geq 0$ we denote by $\dot{H}_{(s)}(B(0, R))$ the closure of $C_0^\infty(B(0, R))$ in the $H_{(s)}(B(0, R))$-norm. This is a closed subspace of the Hilbert space $H_{(s)}(B(0, R))$, and let $\Pi_{s,R}\colon H_{(s)}(B(0, R)) \mapsto \dot{H}_{(s)}(B(0, R))$ be the associated orthogonal projection.

The next consequence gives generic local uniqueness for the backscattering transformation. The arguments are quite similar to the arguments in [10, 14], used in the proofs of the corresponding results for the backscattering mapping defined in terms of the Lax–Phillips scattering.

When $s > (n-3)/2$ define

$$\begin{aligned} B_{(R)}v\colon \dot{H}_{(s)}(B(0,R)) &\mapsto \dot{H}_{(s)}(B(0,R)) \\ B_{(R)}v = \Pi_{s,R}(Bv|_{B(0,R)}), &\quad v \in \dot{H}_{(s)}(B(0,R)). \end{aligned} \tag{4.24}$$

Corollary 4.9. *There is a closed subset $G_{s,R}$ of $\dot{H}_{(s)}(B(0,R))$ such that each complex line through the origin of $\dot{H}_{(s)}(B(0,R))$ intersects $G_{s,R}$ in a discrete set only and $B_{(R)}$ is a local isomorphism at every v not in $G_{s,R}$.*

In particular, if $v \in \dot{H}_{(s)}(B(0,R)) \setminus G_{s,R}$, there is a neighborhood U of v in $\dot{H}_{(s)}(B(0,R))$ such that $v_1 = v_2$ if $v_1, v_2 \in U$ and $Bv_1 = Bv_2$ on $B(0,R)$.

Proof. The mapping $v \to B_{(R)}v$ is entire analytic on $\dot{H}_{(s)}(B(0,R))$, according to Theorem 4.1. The same theorem shows that the derivative of this mapping at a point $v \in \dot{H}_{(s)}(B(0,R))$ has the form $I + T_v$, where T_v is a compact operator on $\dot{H}_{(s)}(B(0,R))$. Let $G_{s,R}$ be the set of those v such that $I + T_v$ is not invertible; this is a closed set. By the inverse mapping theorem (see Theorem 1.2.3, [15]), $B_{(R)}$ is a local isomorphism at every v not in $G_{s,R}$. The estimates in Theorem 4.1 show that operator $I + T_{zv}$ is invertible for $|z|$ small enough. Since $z \to T_{zv}$ is entire analytic, the analytic Fredholm alternative (see Theorem 1.8.2 in [16]) ensures that the inversibility can fail only for z in a discrete set. This finishes the proof. □

Acknowledgments

Ingrid Beltiţă was partially supported by SPECT Short Visit Grant 1006 and the Grant 2-CEx05-11-23/2005.

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