# Algebras of symbols associated with the Weyl calculus for Lie group representations 

## Ingrid Beltiță \& Daniel Beltiță

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# Algebras of symbols associated with the Weyl calculus for Lie group representations 

Ingrid Beltiţă • Daniel Beltiţă

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#### Abstract

We develop our earlier approach to the Weyl calculus for representations of infinite-dimensional Lie groups by establishing continuity properties of the Moyal product for symbols belonging to various modulation spaces. For instance, we prove that the modulation space of symbols $M^{\infty, 1}$ is an associative Banach algebra and the corresponding operators are bounded. We then apply the abstract results to two classes of representations, namely the unitary irreducible representations of nilpotent Lie groups, and the natural representations of the semidirect product groups that govern the magnetic Weyl calculus. The classical Weyl-Hörmander calculus is obtained for the Schrödinger representations of the finite-dimensional Heisenberg groups, and in this case we recover the results obtained by J. Sjöstrand (Math Res Lett 1(2):185-192, 1994).


Keywords Weyl calculus • Involutive Banach algebra • Wiener property Lie group • Modulation spaces

Mathematics Subject Classification (2000) Primary 47G30; Secondary 22E25. 22E65 - 47G10

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## 1 Introduction

A quite important class of symbols for pseudo-differential operators was introduced by J. Sjöstrand in [21] (see also [22]). He denoted this class by $S(1)$ and pointed out that it has a number of remarkable properties, such as:
(1) For every symbol $a \in S(1)$ the corresponding operator $\mathrm{Op}(a)$ obtained by the pseudo-differential Weyl calculus is bounded on $L^{2}\left(\mathbb{R}^{n}\right)$.
(2) The class $S(1)$ has a natural structure of unital involutive associative Banach algebra such that the mapping Op: $S(1) \rightarrow \mathcal{B}\left(L^{2}\left(\mathbb{R}^{n}\right)\right)$ is a continuous $*$ homomorphism.
(3) If a symbol $a \in S(1)$ has the property that the operator $\operatorname{Op}(a)$ is invertible in $\mathcal{B}\left(L^{2}\left(\mathbb{R}^{n}\right)\right)$, then there exists $b \in S(1)$ such that $\mathrm{Op}(a)^{-1}=\mathrm{Op}(b)$.
It was later realized that the class $S(1)$ is actually the modulation space $M^{\infty, 1}\left(\mathbb{R}^{2 n}\right)$ in the sense of [6] (see for instance [7] for a broad discussion), and thus the above three properties become as many statements in representation theory of the Heisenberg groups.

The aim of the present paper is to present the deep representation theoretic background of properties (1)-(3), in the sense that we obtain below, in Theorem 3.10, their appropriate versions for some representations of infinite-dimensional Lie groups and their localized Weyl calculus proposed in our earlier papers [1,4]. We then apply these abstract results to two classes of representations:

- Representations of some infinite-dimensional Lie groups constructed as semidirect products of Lie groups and invariant function spaces thereon. As seen in [1,2] the magnetic Weyl calculus developed for instance in $[10,16]$ is in fact a localized Weyl calculus associated to a specific coadjoint orbit and the corresponding representation of such a semidirect product, and we thus find versions of the aforementioned properties in this setting.
- Unitary irreducible representations of arbitrary nilpotent Lie groups. The Weyl correspondence for these representations was developed by [20], and we have later introduced in [3] the modulation spaces in this framework and established continuity properties of the operators constructed by the corresponding Weyl calculus. In particular we found a space of symbols, which for the Heisenberg group reduces to Sjöstrand's class, and gives rise to bounded operators. We now show that space of symbols has all the above properties (1)-(3) in the case of an arbitrary irreducible representation of a nilpotent Lie group.
The present paper is a sequel to [4] and relies on the methods developed there. In addition, we use some ideas contained in the deep analysis of Sjöstrand's class in [8].


## 2 Preliminaries

In this section, we recall some notions introduced in [4], where we refer for a more detailed discussion and proofs.

Let $M$ be a locally convex Lie group with a smooth exponential mapping

$$
\exp _{M}: \mathbf{L}(M)=\mathfrak{m} \rightarrow M
$$

(see [18]). Assume that $\pi: M \rightarrow \mathcal{B}(\mathcal{H})$ is a unitary representation. The space of smooth vectors for the representation $\pi$ is

$$
\mathcal{H}_{\infty}:=\left\{\phi \in \mathcal{H} \mid \pi(\cdot) \phi \in \mathcal{C}^{\infty}(M, \mathcal{H})\right\}
$$

The derived representation $\mathrm{d} \pi: \mathfrak{m} \rightarrow \operatorname{End}\left(\mathcal{H}_{\infty}\right)$ is well defined and given by

$$
(\forall X \in \mathfrak{m})\left(\forall \phi \in \mathcal{H}_{\infty}\right) \quad \mathrm{d} \pi(X) \phi=\left.\frac{\mathrm{d}}{\mathrm{~d} t}\right|_{t=0} \pi\left(\exp _{M}(t X)\right) \phi
$$

The homomorphism of Lie algebras $\mathrm{d} \pi$ extends to a unique homomorphism of unital associative algebras $\mathrm{d} \pi: \mathrm{U}\left(\mathfrak{m}_{\mathbb{C}}\right) \rightarrow \operatorname{End}\left(\mathcal{H}_{\infty}\right)$, where $\mathrm{U}\left(\mathfrak{m}_{\mathbb{C}}\right)$ is the universal enveloping algebra of the complexified Lie algebra $\mathfrak{m}_{\mathbb{C}}$. The space of smooth vectors $\mathcal{H}_{\infty}$ is endowed with the locally convex topology defined by the family of seminorms $\left\{p_{u}\right\}_{u \in \mathrm{U}\left(\mathfrak{m}_{\mathbb{C}}\right)}$, where for every $u \in \mathrm{U}\left(\mathfrak{m}_{\mathbb{C}}\right)$,

$$
p_{u}: \mathcal{H}_{\infty} \rightarrow[0, \infty), \quad p_{u}(\phi)=\|\mathrm{d} \pi(u) \phi\| .
$$

The inclusion mapping $\mathcal{H}_{\infty} \hookrightarrow \mathcal{H}$ is continuous and, for all $u \in \mathrm{U}\left(\mathfrak{m}_{\mathbb{C}}\right)$ and $m \in M$, the linear operators $\mathrm{d} \pi(u): \mathcal{H}_{\infty} \rightarrow \mathcal{H}_{\infty}$ and $\pi(m): \mathcal{H}_{\infty} \rightarrow \mathcal{H}_{\infty}$ are continuous as well. We denote by $\mathcal{H}_{-\infty}$ the strong dual of $\overline{\mathcal{H}_{\infty}}$. Equivalently, $\mathcal{H}_{-\infty}$ can be described as the space of continuous antilinear functionals on $\mathcal{H}_{\infty}$ endowed with the topology of uniform convergence on the bounded subsets of $\mathcal{H}_{\infty}$.

Definition 2.1 In the above setting, the unitary representation $\pi: M \rightarrow \mathcal{B}(\mathcal{H})$ is said to be smooth if the linear subspace of smooth vectors $\mathcal{H}_{\infty}$ is dense in $\mathcal{H}$. If this is the case, then $\pi$ is necessarily continuous, in the sense that the group action $M \times \mathcal{H} \rightarrow \mathcal{H}$, ( $m, f$ ) $\mapsto \pi(m) f$, is continuous.

The representation $\pi$ is said to be nuclearly smooth if the following conditions are satisfied:
(1) $\pi$ is a smooth representation;
(2) $\mathcal{H}_{\infty}$ is a nuclear Fréchet space;
(3) both mappings $M \times \mathcal{H}_{\infty} \rightarrow \mathcal{H}_{\infty},(m, \phi) \mapsto \pi(m) \phi$, and $\mathfrak{m} \times \mathcal{H}_{\infty} \rightarrow \mathcal{H}_{\infty}$, $(X, \phi) \mapsto \mathrm{d} \pi(X) \phi$ are continuous.

Let $\mathcal{B}(\mathcal{H})_{\infty}$ be the space of smooth vectors for the unitary representation

$$
\pi \otimes \bar{\pi}: M \times M \rightarrow \mathcal{B}\left(\mathfrak{S}_{2}(\mathcal{H})\right), \quad(\pi \otimes \bar{\pi})\left(m_{1}, m_{2}\right) T=\pi\left(m_{1}\right) T \pi\left(m_{2}\right)^{-1}
$$

where $\mathfrak{S}_{2}(\mathcal{H})$ denotes the Hilbert-Schmidt ideal.
The representation $\pi: M \rightarrow \mathcal{B}(\mathcal{H})$ is said to be twice nuclearly smooth if it satisfies the following conditions:
(1) The representation $\pi$ is nuclearly smooth.
(2) There exists the commutative diagram

where the vertical arrow on the left is a linear topological isomorphism, while the vertical arrow on the right is the natural unitary operator defined by the condition $\left(\phi_{1}, \phi_{2}\right) \mapsto \phi_{1} \otimes \bar{\phi}_{2}:=\left(\cdot \mid \phi_{2}\right) \phi_{1}$.

Note that if $\pi$ is a smooth representation, then there exist the dense embeddings

$$
\mathcal{H}_{\infty} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{-\infty}
$$

and the duality pairing $(\cdot \mid \cdot): \mathcal{H}_{-\infty} \times \mathcal{H}_{\infty} \rightarrow \mathbb{C}$ extends the scalar product of $\mathcal{H}$.
Until the end of Sect. 3 we keep the following assumption.
Setting 2.2 Let $\Xi$ be a finite-dimensional vector space with a Lebesgue measure and $\Xi^{*}$ be a manifold with a distinguished diffeomorphism onto $\Xi$ (this corresponds to a polynomial structure in the sense of [19]) and a Radon measure, endowed with a function $\langle\cdot, \cdot\rangle: \Xi^{*} \times \Xi \rightarrow \mathbb{R}$ which is linear in the second variable and such that the "Fourier transform"

$$
\widehat{\ddots} L^{1}(\Xi) \rightarrow L^{\infty}\left(\Xi^{*}\right), \quad b(\cdot) \mapsto \widehat{b}(\cdot)=\int_{\Xi} \mathrm{e}^{-\mathrm{i}\langle\cdot, x\rangle} b(x) \mathrm{d} x
$$

gives a linear topological isomorphism $\mathcal{S}(\Xi) \rightarrow \mathcal{S}\left(\Xi^{*}\right)$ and a unitary operator $L^{2}(\Xi) \rightarrow L^{2}\left(\Xi^{*}\right)$. (The space $\mathcal{S}\left(\Xi^{*}\right)$ of Schwartz functions is naturally defined by using the distinguished diffeomorphism of $\Xi^{*}$ onto $\Xi$.) The inverse of this transformation will be denoted by $a \mapsto \check{a}$.
Definition 2.3 Let $\theta: \Xi \rightarrow \mathfrak{m}$ be a linear mapping and $\pi: M \rightarrow \mathcal{B}(\mathcal{H})$ be a twice nuclearly smooth unitary representation.
(a) Orthogonality relations. If either $\phi \in \mathcal{H}_{\infty}$ and $f \in \mathcal{H}_{-\infty}$, or $\phi, f \in \mathcal{H}$, then we define the ambiguity function along the mapping $\theta$,

$$
\mathcal{A}_{\phi}^{\pi, \theta} f: \Xi \rightarrow \mathbb{C}, \quad\left(\mathcal{A}_{\phi}^{\pi, \theta} f\right)(\cdot)=\left(f \mid \pi\left(\exp _{M}(\theta(\cdot))\right) \phi\right)
$$

Note that $\mathcal{A}_{\phi}^{\pi, \theta} f \in \mathcal{C}(\Xi) \cap \mathcal{S}^{\prime}(\Xi)$. We say that the representation $\pi$ satisfies the orthogonality relations along the mapping $\theta$ if

$$
\begin{equation*}
\left(\mathcal{A}_{\phi_{1}}^{\pi, \theta} f_{1} \mid \mathcal{A}_{\phi_{2}}^{\pi, \theta} f_{2}\right)_{L^{2}(\Xi)}=\left(f_{1} \mid f_{2}\right)_{\mathcal{H}} \cdot\left(\phi_{2} \mid \phi_{1}\right)_{\mathcal{H}} \tag{2.2}
\end{equation*}
$$

for arbitrary $\phi_{1}, \phi_{2}, f_{1}, f_{2} \in \mathcal{H}$. In particular, $\mathcal{A}_{\phi}^{\pi, \theta} f \in L^{2}(\Xi)$ for all $\phi, f \in \mathcal{H}$.
(b) Growth condition. We say that the representation $\pi$ satisfies the growth condition along the mapping $\theta$ if

$$
\begin{equation*}
\mathcal{A}_{\phi_{2}}^{\pi, \theta} \phi_{1} \in \mathcal{S}(\Xi), \quad \text { for all } \phi_{1}, \phi_{2} \in \mathcal{H}_{\infty} \tag{2.3}
\end{equation*}
$$

We thus get the sesquilinear map

$$
\mathcal{A}^{\pi, \theta}: \mathcal{H}_{\infty} \times \mathcal{H}_{\infty} \rightarrow \mathcal{S}(\Xi), \quad\left(\phi_{1}, \phi_{2}\right) \mapsto \mathcal{A}_{\phi_{2}}^{\pi, \theta} \phi_{1}
$$

(c) Density condition. The representation $\pi$ is said to satisfy the density condition along the mapping $\theta$ if the set $\left\{\mathcal{A}_{\phi}^{\pi, \theta} f \mid \phi, f \in \mathcal{H}\right\}$ spans a dense linear subspace of $L^{2}(\Xi)$.

Definition 2.4 Let $\theta: \Xi \rightarrow \mathfrak{m}$ be a linear mapping. The localized Weyl calculus for $\pi$ along $\theta$ is the mapping $\mathrm{Op}^{\theta}: \widehat{L^{1}(\Xi)} \rightarrow \mathcal{B}(\mathcal{H})$ given by

$$
\begin{equation*}
\mathrm{Op}^{\theta}(a)=\int_{\Xi} \check{a}(X) \pi\left(\exp _{M}(\theta(X))\right) \mathrm{d} X \tag{2.4}
\end{equation*}
$$

for $a \in \widehat{L^{1}(\Xi)}$, where the integrals are weakly convergent. The localized Weyl calculus for $\pi$ along $\theta$ is said to be regular if

- $\pi$ satisfies the growth condition along the mapping $\theta$,
- $\pi$ is twice nuclearly smooth, and
- $\mathrm{Op}^{\theta}(a) \in \mathcal{B}(\mathcal{H})_{\infty}$ whenever $a \in \mathcal{S}\left(\Xi^{*}\right)$.

Since the representation $\pi$ satisfies the growth condition along the $\theta$, one can extend the localized Weyl calculus $\mathrm{Op}^{\theta}: \mathcal{S}^{\prime}\left(\Xi^{*}\right) \rightarrow \mathcal{L}\left(\mathcal{H}_{\infty}, \mathcal{H}_{-\infty}\right)$ by

$$
\begin{equation*}
\left(\mathrm{Op}^{\theta}(a) \phi \mid \psi\right)=\left\langle\check{a}, \overline{\mathcal{A}_{\phi}^{\pi, \theta} \psi}\right\rangle \tag{2.5}
\end{equation*}
$$

for every $a \in \mathcal{S}^{\prime}\left(\Xi^{*}\right)$ and $\phi, \psi \in \mathcal{H}_{\infty}$, where $\langle\cdot, \cdot\rangle: \mathcal{S}^{\prime}(\Xi) \times \mathcal{S}(\Xi) \rightarrow \mathbb{C}$ is the usual duality pairing.

Remark 2.5 If the localized Weyl calculus for $\pi$ along $\theta$ is regular, then it defines a linear topological isomorphism $\mathrm{Op}^{\theta}: \mathcal{S}\left(\Xi^{*}\right) \rightarrow \mathcal{B}(\mathcal{H})_{\infty} \simeq \mathcal{L}\left(\mathcal{H}_{-\infty}, \mathcal{H}_{\infty}\right)$, its dual topological isomorphism $\operatorname{Op}^{\theta}: \mathcal{S}^{\prime}\left(\Xi^{*}\right) \rightarrow \mathcal{L}\left(\mathcal{H}_{\infty}, \mathcal{H}_{-\infty}\right)$, and the mapping

$$
\begin{equation*}
\mathrm{Op}^{\theta}: L^{2}\left(\Xi^{*}\right) \rightarrow \mathfrak{S}_{2}(\mathcal{H}) \tag{2.6}
\end{equation*}
$$

which is a unitary operator (see [4, Prop. 3.12]).
In the conditions of Remark 2.5, if $a, b \in \mathcal{S}^{\prime}\left(\Xi^{*}\right)$ and the operator product $\mathrm{Op}^{\theta}(a) \mathrm{Op}^{\theta}(b) \in \mathcal{L}\left(\mathcal{H}_{\infty}, \mathcal{H}_{-\infty}\right)$ is well defined, then the Moyal product $a \#^{\theta} b \in$ $\mathcal{S}^{\prime}\left(\Xi^{*}\right)$ is uniquely determined by the condition

$$
\mathrm{Op}^{\theta}\left(a \#^{\theta} b\right)=\mathrm{Op}^{\theta}(a) \mathrm{Op}^{\theta}(b)
$$

Thus the Moyal product defines bilinear mappings $\mathcal{S}\left(\Xi^{*}\right) \times \mathcal{S}\left(\Xi^{*}\right) \rightarrow \mathcal{S}\left(\Xi^{*}\right)$ and $L^{2}\left(\Xi^{*}\right) \times L^{2}\left(\Xi^{*}\right) \rightarrow L^{2}\left(\Xi^{*}\right)$.

Remark 2.6 The setting of [4] is actually slightly narrower in the sense that $\Xi^{*}$ was supposed to be a finite-dimensional linear space and $\langle\cdot, \cdot\rangle: \Xi^{*} \times \Xi \rightarrow \mathbb{R}$ was supposed to be a duality pairing. However, it is easily seen that the above setting ensures that the main results of [4] hold true.

We shall also need the cross-Wigner distribution $\mathcal{W}(f, \phi) \in \mathcal{S}^{\prime}\left(\Xi^{*}\right)$ for $\pi$ along $\theta$, defined by the formula $\widehat{\mathcal{W}(f, \phi)}=\mathcal{A}_{\phi}^{\pi, \theta} f$ for $\phi \in \mathcal{H}_{\infty}$ and $f \in \mathcal{H}_{-\infty}$.

The modulation spaces $M_{\phi}^{p, q}(\pi, \theta)$ are defined for $p, q \in[1, \infty]$ with respect to a decomposition into a direct sum of subspaces $\Xi=\Xi_{1}+\Xi_{2}$ and the window vector $\phi \in \mathcal{H}_{\infty} \backslash\{0\}$. Specifically, for any measurable function $F: \Xi \simeq \Xi_{1} \times \Xi_{2} \rightarrow \mathbb{C}$ set

$$
\|F\|_{L^{p, q}\left(\Xi_{1} \times \Xi_{2}\right)}=\left(\int_{\Xi_{1}}\left(\int_{\Xi_{2}}\left|F\left(X_{1}, X_{2}\right)\right|^{p} \mathrm{~d} X_{1}\right)^{p / q} \mathrm{~d} X_{2}\right)^{1 / q} \in[0, \infty]
$$

with the usual conventions if $p$ or $q$ is infinite. Then

$$
M_{\phi}^{p, q}(\pi, \theta):=\left\{f \in \mathcal{H}_{-\infty} \mid\|f\|_{M_{\phi}^{p, q}(\pi, \theta)}:=\left\|\mathcal{A}_{\phi}^{\pi, \theta} f\right\|_{L^{p, q}\left(\Xi_{1} \times \Xi_{2}\right)}<\infty\right\} .
$$

If $1 \leq p_{1} \leq p_{2} \leq \infty$ and $1 \leq q_{1} \leq q_{2} \leq \infty$, then

$$
\begin{equation*}
M_{\phi}^{p_{1}, q_{1}}(\pi, \theta) \cap M_{\phi}^{\infty, \infty}(\pi, \theta) \subseteq M_{\phi}^{p_{2}, q_{2}}(\pi, \theta) \cap M_{\phi}^{\infty, \infty}(\pi, \theta), \tag{2.7}
\end{equation*}
$$

since $L^{p_{1}, q_{1}}\left(\Xi_{1} \times \Xi_{2}\right) \cap L^{\infty}(\Xi) \subseteq L^{p_{2}, q_{2}}\left(\Xi_{1} \times \Xi_{2}\right) \cap L^{\infty}(\Xi)$.
Definition 2.7 Let us consider the semi-direct product $M \ltimes M$ defined by the action of $M$ on itself by inner automorphisms. Thus $M \ltimes M$ is a locally convex Lie group whose underlying manifold is $M \times M$ and the group operation is

$$
\left(m_{1}, m_{2}\right)\left(n_{1}, n_{2}\right)=\left(m_{1} n_{1}, n_{1}^{-1} m_{2} n_{1} n_{2}\right)
$$

for all $m_{1}, m_{2}, n_{1}, n_{2} \in M$. There exists the natural continuous unitary representation

$$
\pi^{\ltimes}: M \ltimes M \rightarrow \mathcal{B}\left(\mathfrak{S}_{2}(\mathcal{H})\right), \quad \pi^{\ltimes}\left(m_{1}, m_{2}\right) T=\pi\left(m_{1} m_{2}\right) T \pi\left(m_{1}\right)^{-1} .
$$

By using the unitary operator (2.6), we can construct the unitarily equivalent representation

$$
\pi^{\#}: M \ltimes M \rightarrow \mathcal{B}\left(L^{2}\left(\Xi^{*}\right)\right), \quad \pi^{\#}(\cdot)=\left(\mathrm{Op}^{\theta}\right)^{-1} \circ \pi^{\ltimes}(\cdot) \circ \mathrm{Op}^{\theta}
$$

(see [4, Def. 3.13]).

Remark 2.8 For the representation $\pi^{\#}: M \ltimes M \rightarrow \mathcal{B}\left(L^{2}\left(\Xi^{*}\right)\right)$ we have by [4, Remark 3.14(2)]

$$
\begin{equation*}
\pi^{\#}\left(\exp _{M \ltimes M}\left(\theta\left(X_{1}\right), \theta\left(X_{2}\right)\right)\right) f=\mathrm{e}^{\mathrm{i}\left\langle\cdot, X_{1}+X_{2}\right\rangle} \#^{\theta} f \#^{\theta} \mathrm{e}^{-\mathrm{i}\left\langle\cdot, X_{1}\right\rangle} \tag{2.8}
\end{equation*}
$$

for every $X_{1}, X_{2} \in \Xi$ and $f \in L^{2}\left(\Xi^{*}\right)$.
The ambiguity function $\mathcal{A}_{\Phi}^{\pi^{\#}, \theta \times \theta}: \Xi \times \Xi \rightarrow \mathbb{C}$ of the representation $\pi^{\#}$ along the linear mapping $\theta \times \theta: \Xi \times \Xi \rightarrow \mathfrak{m} \ltimes \mathfrak{m}$ (see Definition 2.3) is given by

$$
\left(\mathcal{A}_{\Phi}^{\pi^{\#}, \theta \times \theta} F\right)\left(X_{1}, X_{2}\right)=\left(F \mid \pi^{\#}\left(\exp _{M \ltimes M}\left(\theta\left(X_{1}\right), \theta\left(X_{2}\right)\right) \Phi\right)\right)
$$

for $F \in \mathcal{S}^{\prime}\left(\Xi^{*}\right), \Phi \in \mathcal{S}\left(\Xi^{*}\right)$, and $X_{1}, X_{2} \in \Xi$. Then

$$
\|F\|_{M_{\Phi}^{r, s}\left(\pi^{\#}, \theta \times \theta\right)}=\left(\int_{\Xi}\left(\int_{\Xi}\left|\left(\mathcal{A}_{\Phi}^{\pi^{\#}, \theta \times \theta} F\right)\left(X_{1}, X_{2}\right)\right|^{r} \mathrm{~d} X_{1}\right)^{s / r} \mathrm{~d} X_{2}\right)^{1 / s} \in[0, \infty]
$$

with the usual conventions if $r$ or $s$ is infinite. The space

$$
M_{\Phi}^{r, s}\left(\pi^{\#}, \theta \times \theta\right):=\left\{F \in \mathcal{S}^{\prime}\left(\Xi^{*}\right) \mid\|F\|_{M_{\Phi}^{r, s}\left(\pi^{\#}, \theta \times \theta\right)}<\infty\right\}
$$

is a modulation space of symbols for the localized Weyl calculus $\mathrm{Op}^{\theta}$ associated with the unitary representation $\pi: M \rightarrow \mathcal{B}(\mathcal{H})$ along with the linear mapping $\theta: \Xi \rightarrow \mathfrak{m}$ for the window vector $\Phi \in \mathcal{S}\left(\Xi^{*}\right) \backslash\{0\}$.

## 3 Sjöstrand's algebra of symbols in an abstract setting

The present section gives the main results of the paper, in an abstract setting (Th. 3.10). Sections 4 and 5 are devoted to discussing wide classes of examples satisfying the assumptions of this abstract framework.

Setting 3.1 Throughout this section, in addition to Setting 2.2, we keep the following assumptions:
(1) $M$ is a locally convex Lie group with a smooth exponential mapping

$$
\exp _{M}: \mathbf{L}(M)=\mathfrak{m} \rightarrow M
$$

(2) $\pi: M \rightarrow \mathcal{B}(\mathcal{H})$ is a twice nuclearly smooth unitary representation.
(3) $\theta: \Xi \rightarrow \mathfrak{m}$ is a linear mapping such that
(a) $\pi$ satisfies the orthogonality relations along $\theta$;
(b) $\pi$ satisfies the density condition along $\theta$;
(c) the localized Weyl calculus for $\pi$ along $\theta$ is regular;
(d) for every $u \in \mathrm{U}\left(\mathfrak{m}_{\mathbb{C}}\right)$ and $\phi \in \mathcal{H}_{\infty}$ the function $\left\|\mathrm{d} \pi(u) \pi\left(\exp _{M}(\theta(\cdot))\right) \phi\right\|$ has polynomial growth.
3.1 Ambiguity functions and matrix coefficients

Here is a general version of the usual covariance property of the cross-Wigner distribution (see [7, Prop. 4.3.2]).

Proposition 3.2 For all $f_{1}, f_{2} \in \mathcal{H}$ and $X_{1}, X_{2} \in \Xi$ we have the equation

$$
\begin{aligned}
& \mathcal{W}\left(\pi\left(\exp _{M}\left(\theta\left(X_{1}\right)\right)\right) f_{1}, \pi\left(\exp _{M}\left(\theta\left(X_{2}\right)\right)\right) f_{2}\right) \\
& \quad=\pi^{\#}\left(\exp _{M \ltimes M}\left(\theta\left(X_{2}\right), \theta\left(X_{1}-X_{2}\right)\right)\right) \mathcal{W}\left(f_{1}, f_{2}\right)
\end{aligned}
$$

in $L^{2}\left(\Xi^{*}\right)$.

Proof By using [3, Prop. 3.12] we get the following equalities of continuous linear operators on $\mathcal{H}$,

$$
\begin{aligned}
& \mathrm{Op}^{\theta}\left(\mathcal{W}\left(\pi\left(\exp _{M}\left(\theta\left(X_{1}\right)\right)\right) f_{1}, \pi\left(\exp _{M}\left(\theta\left(X_{2}\right)\right)\right) f_{2}\right)\right) \\
& =\left(\cdot \mid \pi\left(\exp _{M}\left(\theta\left(X_{2}\right)\right)\right) f_{2}\right) \pi\left(\exp _{M}\left(\theta\left(X_{1}\right)\right)\right) f_{1} \\
& =\pi\left(\exp _{M}\left(\theta\left(X_{1}\right)\right)\right)\left(\left(\cdot \mid f_{2}\right) f_{1}\right) \pi\left(\exp _{M}\left(\theta\left(-X_{2}\right)\right)\right) \\
& =\operatorname{Op}^{\theta}\left(\mathrm{e}^{\mathrm{i}\left\langle\cdot, X_{1}\right\rangle}\right) \mathrm{Op}^{\theta}\left(\mathcal{W}\left(f_{1}, f_{2}\right)\right) \mathrm{Op}^{\theta}\left(\mathrm{e}^{-\mathrm{i}\left\langle\cdot, X_{2}\right\rangle}\right) \\
& =\operatorname{Op}^{\theta}\left(\mathrm{e}^{\left.\mathrm{i}\left\langle\cdot, X_{1}\right\rangle \#^{\theta} \mathcal{W}\left(f_{1}, f_{2}\right) \#^{\theta} \mathrm{e}^{-\mathrm{i}\left\langle\cdot, X_{2}\right\rangle}\right)}\right. \\
& =\operatorname{Op}^{\theta}\left(\pi^{\#}\left(\exp _{M \ltimes M}\left(\theta\left(X_{2}\right), \theta\left(X_{1}-X_{2}\right)\right)\right) \mathcal{W}\left(f_{1}, f_{2}\right)\right),
\end{aligned}
$$

where the latter equality relies on (2.8).
Then the assertion follows since $\mathrm{Op}^{\theta}: L^{2}\left(\Xi^{*}\right) \rightarrow \mathfrak{S}_{2}(\mathcal{H})$ is a linear topological isomorphism (see [4, Prop.3.12]).

Theorem 3.3 If $\phi \in \mathcal{H}_{\infty}$ and $a \in \mathcal{S}^{\prime}\left(\Xi^{*}\right)$, then for all $X_{1}, X_{2} \in \Xi$ we have

$$
\left(\mathcal{A}_{\mathcal{W}(\phi, \phi)}^{\pi^{\#}, \theta \times \theta} a\right)\left(X_{1}, X_{2}\right)=\left(\mathrm{Op}^{\theta}(a) \phi_{X_{1}} \mid \phi_{X_{1}+X_{2}}\right)
$$

where we denote $\phi_{X}:=\pi\left(\exp _{M}(\theta(X))\right) \phi \in \mathcal{H}_{\infty}$ for each $X \in \Xi$.

Proof First note that

$$
\begin{aligned}
& \pi\left(\exp _{M}\left(\theta\left(-X_{1}-X_{2}\right)\right)\right) \operatorname{Op}^{\theta}(a) \pi\left(\exp _{M}\left(\theta\left(X_{1}\right)\right)\right) \\
& \quad=\operatorname{Op}^{\theta}\left(\mathrm{e}^{\left.-\mathrm{i}\left\langle\cdot, X_{1}+X_{2}\right\rangle \#^{\theta} a \#^{\theta} \mathrm{e}^{\mathrm{i}\left\langle\cdot, X_{1}\right\rangle}\right)}\right. \\
& \quad=\operatorname{Op}^{\theta}\left(\pi^{\#}\left(\exp _{M \ltimes M}\left(\theta\left(-X_{1}\right), \theta\left(-X_{2}\right)\right)\right) a\right)
\end{aligned}
$$

by (2.8). Therefore

$$
\begin{aligned}
\left(\mathrm{Op}^{\theta}(a) \phi_{X_{1}} \mid \phi_{X_{1}+X_{2}}\right) & =\left(\pi\left(\exp _{M}\left(\theta\left(-X_{1}-X_{2}\right)\right)\right) \mathrm{Op}^{\theta}(a) \pi\left(\exp _{M}\left(\theta\left(X_{1}\right)\right)\right) \phi \mid \phi\right) \\
& =\left(\operatorname{Op}^{\theta}\left(\pi^{\#}\left(\exp _{M \ltimes M}\left(\theta\left(-X_{1}\right), \theta\left(-X_{2}\right)\right)\right) a\right) \phi \mid \phi\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\left(\operatorname{Op}^{\theta}\left(\pi^{\#}\left(\exp _{M \ltimes M}\left(\theta\left(-X_{1}\right), \theta\left(-X_{2}\right)\right)\right) a\right) \mid(\cdot \mid \phi) \phi\right) \\
& =\left(\operatorname{Op}^{\theta}\left(\pi^{\#}\left(\exp _{M \ltimes M}\left(\theta\left(-X_{1}\right), \theta\left(-X_{2}\right)\right)\right) a\right) \mid \operatorname{Op}^{\theta}(\mathcal{W}(\phi, \phi))\right) \\
& =\left(\pi^{\#}\left(\exp _{M \ltimes M}\left(\theta\left(-X_{1}\right), \theta\left(-X_{2}\right)\right)\right) a \mid \mathcal{W}(\phi, \phi)\right) \\
& =\left(a \mid \pi^{\#}\left(\exp _{M \ltimes M}\left(\theta\left(X_{1}\right), \theta\left(X_{2}\right)\right)\right) \mathcal{W}(\phi, \phi)\right) \\
& =\left(\mathcal{A}_{\mathcal{W}(\phi, \phi)}^{\pi^{\#}, \theta \times \theta} a\right)\left(X_{1}, X_{2}\right),
\end{aligned}
$$

and this completes the proof.
Corollary 3.4 Let $\phi \in \mathcal{H}_{\infty}$ and denote $\phi_{X}:=\pi\left(\exp _{M}(\theta(X))\right) \phi \in \mathcal{H}_{\infty}$ for each $X \in \Xi$, and let $p, q \in[1, \infty]$. For $a \in \mathcal{S}^{\prime}\left(\Xi^{*}\right)$ denote by $B_{a}$ the set of all measurable functions $\beta: \Xi \rightarrow[0, \infty]$ satisfying the condition

$$
\begin{equation*}
(\forall X \in \Xi) \quad\left\|\left(\mathrm{Op}^{\theta}(a) \phi_{\bullet} \mid \phi_{\bullet}+X\right)\right\|_{L^{p}(\Xi)} \leq \beta(X) \tag{3.1}
\end{equation*}
$$

Also define

$$
\beta_{a}: \Xi \rightarrow[0, \infty], \quad \beta_{a}(X)=\left\|\left(\mathrm{Op}^{\theta}(a) \phi_{\bullet} \mid \phi_{\bullet}+X\right)\right\|_{L^{p}(\Xi)}
$$

Then we have $a \in M_{\mathcal{W}(\phi, \phi)}^{p, q}\left(\pi^{\#}, \theta \times \theta\right)$ if and only if $B_{a} \cap L^{q}(\Xi) \neq \emptyset$, and in this case $\beta_{a} \in B_{a} \cap L^{q}(\Xi)$ and

$$
\begin{equation*}
\|a\|_{M_{\mathcal{W}(\phi, \phi)}^{p, q}\left(\pi^{\#}, \theta \times \theta\right)}=\inf _{\beta \in B_{a} \cap L^{q}(\Xi)}\|\beta\|_{L^{q}(\Xi)}=\left\|\beta_{a}\right\|_{L^{q}(\Xi)} \tag{3.2}
\end{equation*}
$$

Proof If $a \in M_{\mathcal{W}(\phi, \phi)}^{p, q}\left(\pi^{\#}, \theta \times \theta\right)$, then the function $a_{0}: \Xi \rightarrow[0, \infty]$ defined by $a_{0}(Y):=\left\|\left(\mathcal{A}_{\mathcal{W}(\phi, \phi)}^{\pi^{\#}, \theta \times \theta} a\right)(\cdot, Y)\right\|_{L^{p}(\Xi)}$ has the property $a_{0} \in L^{q}(\Xi)$ and moreover it follows at once by Theorem 3.3 that $\left\|\left(\mathrm{Op}^{\theta}(a) \phi_{\bullet} \mid \phi_{\bullet+X_{2}}\right)\right\|_{L^{p}(\Sigma)} \leq a_{0}\left(X_{2}\right)$ for $X_{2} \in \Xi$. Hence condition (3.3) is satisfied for $\beta:=a_{0}$.

Conversely, if (3.1) holds, then we get by Theorem 3.3 again that for all $Y \in$ $\Xi$ we have $a_{0}(Y) \leq \beta(Y)$, whence $\left\|a_{0}\right\|_{L^{q}(\Xi)} \leq\|\beta\|_{L^{q}(\Xi)}<\infty$, and then $a \in$ $M_{\mathcal{W}(\phi, \phi)}^{p, q}\left(\pi^{\#}, \theta \times \theta\right)$ and $\|a\|_{M_{\mathcal{W}(\phi, \phi)}^{p, q}\left(\pi^{\#}, \theta \times \theta\right)} \leq\|\beta\|_{L^{q}(\Xi)}$.

Equality (3.2) is a by-product of the above reasoning, hence the proof is complete.

Remark 3.5 By using Corollary 3.4 for $p=\infty$ we get the following abstract version of the almost diagonalization theorem established in [8, Th. 3.2]:

Let $\phi \in \mathcal{H}_{\infty}$ and denote as above $\phi_{X}:=\pi\left(\exp _{M}(\theta(X))\right) \phi \in \mathcal{H}_{\infty}$ for each $X \in \Xi$. For $a \in \mathcal{S}^{\prime}\left(\Xi^{*}\right)$ let $B_{a}$ be the set of all measurable functions $\beta: \Xi \rightarrow[0, \infty]$ satisfying the condition

$$
\begin{equation*}
\left(\forall X_{1}, X_{2} \in \Xi\right) \quad\left|\left(\mathrm{Op}^{\theta}(a) \phi_{X_{1}} \mid \phi_{X_{2}}\right)\right| \leq \beta\left(X_{1}-X_{2}\right) \tag{3.3}
\end{equation*}
$$

Also define

$$
\beta_{a}: \Xi \rightarrow[0, \infty], \quad \beta_{a}(X)=\sup _{Y \in \Xi}\left|\left(\mathrm{Op}^{\theta}(a) \phi_{X+Y} \mid \phi_{X}\right)\right| .
$$

Then we have $a \in M_{\mathcal{W}(\phi, \phi)}^{\infty, q}\left(\pi^{\#}, \theta \times \theta\right)$ if and only if $B_{a} \cap L^{q}(\Xi) \neq \emptyset$, and in this case $\beta_{a} \in B_{a} \cap L^{q}(\Xi)$ and

$$
\begin{equation*}
\|a\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, q}\left(\pi^{\#}, \theta \times \theta\right)}=\inf _{\beta \in B_{a} \cap L^{q}(\Xi)}\|\beta\|_{L^{q}(\Xi)}=\left\|\beta_{a}\right\|_{L^{q}(\Xi)} \tag{3.4}
\end{equation*}
$$

whenever $1 \leq q \leq \infty$.
Definition 3.6 Let $\phi \in \mathcal{H}_{\infty}$ with $\|\phi\|=1$ and $\phi_{X}=\pi\left(\exp _{M}(\theta(X))\right) \phi \in \mathcal{H}_{\infty}$ for each $X \in \Xi$. For every $a \in \mathcal{S}^{\prime}\left(\Xi^{*}\right)$ we define

$$
C_{a}: \Xi \times \Xi \rightarrow \mathbb{C}, \quad C_{a}(X, Y):=\left(\mathrm{Op}^{\theta}(a) \phi_{X} \mid \phi_{Y}\right)
$$

and the integral operator in $L^{2}(\Xi)$ defined by the integral kernel $C_{a}$ will be denoted by

$$
T_{a}: \mathcal{D}\left(T_{a}\right) \rightarrow L^{2}(\Xi)
$$

Let us also denote by $V:=\mathcal{A}_{\phi}: \mathcal{H} \rightarrow L^{2}(\Xi)$ the isometry defined by the ambiguity functions. Note that the constant function $1 \in \mathcal{S}^{\prime}\left(\Xi^{*}\right)$ gives rise to the orthogonal projection $T_{1}=T_{1}^{*}=\left(T_{1}\right)^{2} \in \mathcal{B}\left(L^{2}(\Xi)\right)$ with $\operatorname{Ran} T_{1}=\operatorname{Ran} V$ and $T_{1} T_{a}=T_{a} T_{1}=T_{a}$ for every $a \in \mathcal{S}^{\prime}(\Xi)$.

We now present the main idea which allows to use integral operators on $\Xi$ for the study of operators $\mathrm{Op}^{\theta}(a): \mathcal{H}_{\infty} \rightarrow \mathcal{H}_{-\infty}, a \in \mathcal{S}^{\prime}\left(\Xi^{*}\right)$.

Lemma 3.7 For arbitrary $a \in \mathcal{S}^{\prime}(\Xi)$ we have

$$
\mathrm{Op}^{\theta}(a)=V^{*} T_{a} V: \mathcal{D}\left(\mathrm{Op}^{\theta}(a)\right) \rightarrow \mathcal{H}
$$

on the domain

$$
\mathcal{D}\left(\operatorname{Op}^{\theta}(a)\right)=\left\{f \in \mathcal{H} \mid V f \in \mathcal{D}\left(T_{a}\right)\right\} .
$$

In particular, $T_{a} \in \mathcal{B}\left(L^{2}(\Xi)\right)$ if and only if $\mathrm{Op}^{\theta}(a) \in \mathcal{B}(\mathcal{H})$.
Proof Use Definition 3.6 and that the representation $\pi \otimes \bar{\pi}: M \times M \rightarrow \mathcal{B}\left(\mathfrak{S}_{2}(\mathcal{H})\right)$ satisfies the orthogonality relations along $\theta \times \theta: \Xi \times \Xi \rightarrow \mathfrak{m} \times \mathfrak{m}$ (see [4, Lemma 3.8(1)]).

Remark 3.8 If $a_{1}, a_{2} \in \mathcal{S}^{\prime}(\Xi)$ and the operator product $T_{a_{1}} T_{a_{2}}$ is well defined in $L^{2}(\Xi)$, so that $\mathrm{Op}^{\theta}\left(a_{1}\right) \mathrm{Op}^{\theta}\left(a_{2}\right)=V^{*} T_{a_{1}} T_{a_{2}} V \in \mathcal{L}\left(\mathcal{H}_{\infty}, \mathcal{H}_{-\infty}\right)$ is well defined, then the Moyal product $a_{1} \#^{\theta} a_{2} \in \mathcal{S}^{\prime}\left(\Xi^{*}\right)$ makes sense and we have

$$
\begin{equation*}
C_{a_{1} \#^{\theta} a_{2}}(X, Z)=\int_{\Xi} C_{a_{1}}(X, Y) C_{a_{2}}(Y, Z) \mathrm{d} Y \tag{3.5}
\end{equation*}
$$

for all $X, Z \in \Xi$. In fact, it follows by [4, Lemma 3.19(4)] that for $X \in \Xi$ we have the integral $\mathrm{Op}^{\theta}\left(a_{2}\right) \phi_{X}=\int_{\Xi}\left(\mathrm{Op}^{\theta}\left(a_{2}\right) \phi_{X} \mid \phi_{Y}\right) \phi_{Y} \mathrm{~d} Y$ convergent in $\mathcal{H}_{-\infty}$. Therefore, if $\mathrm{Op}^{\theta}\left(a_{2}\right) \phi_{X} \in \mathcal{D}\left(\mathrm{Op}^{\theta}\left(a_{1}\right)\right)$, then

$$
\mathrm{Op}^{\theta}\left(a_{1}\right) \mathrm{Op}^{\theta}\left(a_{2}\right) \phi_{X}=\int_{\Xi}\left(\mathrm{Op}^{\theta}\left(a_{2}\right) \phi_{X} \mid \phi_{Y}\right) \mathrm{Op}^{\theta}\left(a_{1}\right) \phi_{Y} \mathrm{~d} Y
$$

Hence we have

$$
\left(\mathrm{Op}^{\theta}\left(a_{1}\right) \mathrm{Op}^{\theta}\left(a_{2}\right) \phi_{X} \mid \phi_{Z}\right)=\int_{\Xi}\left(\mathrm{Op}^{\theta}\left(a_{2}\right) \phi_{X} \mid \phi_{Y}\right)\left(\mathrm{Op}^{\theta}\left(a_{1}\right) \phi_{Y} \mid \phi_{Z}\right) \mathrm{d} Y
$$

for all $X, Z \in \Xi$.
The next result is a generalization of the version of [9, Prop. 0.1] without weights, which is recovered in the special case when $\pi$ is the Schrödinger representation of the Heisenberg group.

Corollary 3.9 Let $\phi \in \mathcal{H}_{\infty}$ and $p_{1}, p_{2}, p, q_{1}, q_{2}, q \in[1, \infty]$ such that $\frac{1}{p_{1}}+\frac{1}{p_{2}}=\frac{1}{p}$ and $\frac{1}{q_{1}}+\frac{1}{q_{2}}=1+\frac{1}{q}$. Then the Moyal product $\#^{\theta}$ defines a continuous bilinear map

$$
M_{\mathcal{W}(\phi, \phi)}^{p_{1}, q_{1}}\left(\pi^{\#}, \theta \times \theta\right) \times M_{\mathcal{W}(\phi, \phi)}^{p_{2}, q_{2}}\left(\pi^{\#}, \theta \times \theta\right) \rightarrow M_{\mathcal{W}(\phi, \phi)}^{p, q}\left(\pi^{\#}, \theta \times \theta\right) .
$$

In particular, for every $p \in[1, \infty]$, we thus get continuous bilinear maps

$$
M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right) \times M_{\mathcal{W}(\phi, \phi)}^{p, 1}\left(\pi^{\#}, \theta \times \theta\right) \rightarrow M_{\mathcal{W}(\phi, \phi)}^{p, 1}\left(\pi^{\#}, \theta \times \theta\right)
$$

and

$$
M_{\mathcal{W}(\phi, \phi)}^{p, 1}\left(\pi^{\#}, \theta \times \theta\right) \times M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right) \rightarrow M_{\mathcal{W}(\phi, \phi)}^{p, 1}\left(\pi^{\#}, \theta \times \theta\right)
$$

Proof For $j=1,2$ let $a_{j} \in M_{\mathcal{W}(\phi, \phi)}^{p_{j}, q_{j}}\left(\pi^{\#}, \theta \times \theta\right)$. We shall use the notation of Definition 3.6.

Then there exists $\beta_{j} \in L^{q_{j}}(\Xi)$ such that $\left\|C_{a_{j}}(\cdot, \cdot+Y)\right\|_{L^{p_{j}}(\Xi)} \leq \beta_{j}(Y)$ for every $Y \in \Xi$. On the other hand, it follows by (3.5) that for $X_{1}, X_{2} \in \Xi$ we have

$$
\begin{aligned}
C_{a_{1} \#^{\theta} a_{2}}\left(X_{1}, X_{1}+X_{2}\right) & =\int_{\Xi} C_{a_{1}}\left(X_{1}, Y\right) C_{a_{2}}\left(Y, X_{1}+X_{2}\right) \mathrm{d} Y \\
& =\int_{\Xi} C_{a_{1}}\left(X_{1}, X_{1}+Y\right) C_{a_{2}}\left(X_{1}+Y, X_{1}+X_{2}\right) \mathrm{d} Y
\end{aligned}
$$

hence by Minkowski's inequality, and then Hölder's inequality, we get

$$
\begin{aligned}
\left\|C_{a_{1} \#^{\theta} a_{2}}\left(\cdot, \cdot+X_{2}\right)\right\|_{L^{p}(\Xi)} & \leq \int_{\Xi}\left\|C_{a_{1}}(\cdot, \cdot+Y) C_{a_{2}}\left(\cdot+Y, \cdot+X_{2}\right)\right\|_{L^{p}(\Xi)} \mathrm{d} Y \\
& \leq \int_{\Xi}\left\|C_{a_{1}}(\cdot, \cdot+Y)\right\|_{L^{p_{1}}(\Xi)}\left\|C_{a_{2}}\left(\cdot+Y, \cdot+X_{2}\right)\right\|_{L^{p_{2}}(\Xi)} \mathrm{d} Y \\
& \leq \int_{\Xi}\left\|C_{a_{1}}(\cdot, \cdot+Y)\right\|_{L^{p_{1}}(\Xi)}\left\|C_{a_{2}}\left(\cdot, \cdot+X_{2}-Y\right)\right\|_{L^{p_{2}}(\Xi)} \mathrm{d} Y \\
& \leq \int_{\Xi} \beta_{1}(Y) \beta_{2}\left(X_{2}-Y\right) \mathrm{d} Y \\
& \stackrel{\operatorname{not}}{=} \beta\left(X_{2}\right)
\end{aligned}
$$

Since $\beta_{j} \in L^{q_{j}}(\Xi)$ for $j=1,2$, we have $\beta \in L^{q}(\Xi)$ and

$$
\left\|a_{1} \#^{\theta} a_{2}\right\|_{M_{\mathcal{W}(\phi, \phi)}^{p, q}}\left(\pi^{\#}, \theta \times \theta\right) \leq\|\beta\|_{L^{q}(\Xi)} \leq\left\|\beta_{1}\right\|_{L^{q_{1}}(\Xi)}\left\|\beta_{2}\right\|_{L^{q_{2}}(\Xi)} .
$$

By using Corollary 3.4 , it then follows that $a_{1} \#^{\theta} a_{2} \in M_{\mathcal{W}(\phi, \phi)}^{p, q}\left(\pi^{\#}, \theta \times \theta\right)$ and

$$
\left\|a_{1} \#^{\theta} a_{2}\right\|_{M_{\mathcal{W}(\phi, \phi)}^{p, q}\left(\pi^{\#}, \theta \times \theta\right)} \leq\left\|a_{1}\right\|_{M_{\mathcal{W}(\phi, \phi)}^{p_{1}, q_{1}}}\left(\pi^{\#}, \theta \times \theta\right)\left\|a_{2}\right\|_{M_{\mathcal{W}(\phi, \phi)}^{p_{2}, q_{2}}\left(\pi^{\#}, \theta \times \theta\right)},
$$

which ends the proof.
3.2 The abstract version of Sjöstrand's algebra

We can now prove the main result of the paper.
Theorem 3.10 If $\phi \in \mathcal{H}_{\infty} \backslash\{0\}$, then the following assertions hold:
(1) For every $a \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ we have $\mathrm{Op}^{\theta}(a) \in \mathcal{B}(\mathcal{H})$ and moreover $\left\|\mathrm{Op}^{\theta}(a)\right\| \leq\|a\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}}\left(\pi^{\#}, \theta \times \theta\right)$.
(2) The Moyal product $\#^{\theta}$ makes the modulation space $M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ into an involutive associative Banach algebra.
(3) Let

$$
\begin{equation*}
\mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)=\mathbb{C} 1+M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right) \tag{3.6}
\end{equation*}
$$

If $a_{0} \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ and the operator $\mathrm{Op}^{\theta}\left(a_{0}\right)$ is invertible in $\mathcal{B}(\mathcal{H})$, then there exists $b_{0} \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ such that $\mathrm{Op}^{\theta}\left(a_{0}\right)^{-1}=\mathrm{Op}^{\theta}\left(b_{0}\right)$.

Proof To prove Assertion (1), let $a \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ arbitrary and denote $\phi_{X}:=\pi\left(\exp _{M}(\theta(X))\right) \phi \in \mathcal{H}_{\infty}$ for each $X \in \Xi$. It follows by Remark 3.5 that there exists $\beta_{a} \in L^{1}(\Xi)$ such that $\|a\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty}\left(\pi^{\#}, \theta \times \theta\right)}=\left\|\beta_{a}\right\|_{L^{1}(\Xi)}$ and

$$
\left(\forall X_{1}, X_{2} \in \Xi\right) \quad\left|\left(\mathrm{Op}^{\theta}(a) \phi_{X_{1}} \mid \phi_{X_{2}}\right)\right| \leq \beta_{a}\left(X_{1}-X_{2}\right) .
$$

Now let $f \in \mathcal{H}_{\infty}$ and recall from [4, Lemma 3.19] that $f=\int_{\Xi}\left(f \mid \phi_{X}\right) \phi_{X} \mathrm{~d} X$, hence

$$
\begin{aligned}
\left|\left(\mathrm{Op}^{\theta}(a) f \mid \phi_{Y}\right)\right| & \leq \int_{\Xi}\left|\left(f \mid \phi_{X}\right)\right| \cdot\left|\left(\mathrm{Op}^{\theta}(a) \phi_{X} \mid \phi_{Y}\right)\right| \mathrm{d} X \\
& \leq \int_{\Xi}\left|\left(f \mid \phi_{X}\right)\right| \cdot \beta_{a}(X-Y) \mathrm{d} X
\end{aligned}
$$

That is, $\mid\left(\mathcal{A}_{\phi}^{\pi, \theta}\left(\operatorname{Op}^{\theta}(a) f\right)\right)(Y) \leq\left(\left|\mathcal{A}_{\phi}^{\pi, \theta} f\right| * \beta_{a}(-\cdot)\right)(Y)$ for all $Y \in \Xi$. Therefore,

$$
\begin{aligned}
\left\|\mathrm{Op}^{\theta}(a) f\right\| & =\left\|\left(\mathcal{A}_{\phi}^{\pi, \theta}\left(\mathrm{Op}^{\theta}(a) f\right)\right)\right\|_{L^{2}(\Xi)} \leq\left\|\mathcal{A}_{\phi}^{\pi, \theta} f\right\|_{L^{2}(\Xi)}\left\|\beta_{a}\right\|_{L^{1}(\Xi)} \\
& =\|f\| \cdot\|a\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)} .
\end{aligned}
$$

Since $f \in \mathcal{H}_{\infty}$ is arbitrary and $\mathcal{H}_{\infty}$ is dense in $\mathcal{H}$, the assertion follows.
For Assertion (2), to see that $M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ is closed under the Moyal product and the corresponding product is continuous, just use Corollary 3.9 for $p_{1}=$ $p_{2}=p=\infty$ and $q_{1}=q_{2}=q=1$. Next note that if $\|a\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)}=0$, then $\mathcal{A}_{\mathcal{W}(\phi, \phi)}^{\pi^{\#}, \theta \times \theta} a=0$. Since $\pi^{\#}$ satisfies the orthogonality relations along $\theta \times \theta$ (by [4, Lemma 3.8 and Def. 3.12 (2)]), we have

$$
\left\|\mathcal{A}_{\mathcal{W}(\phi, \phi)}^{\pi^{\#}, \theta \times \theta} a\right\|_{L^{2}(\Xi \times \Xi)}=\|\mathcal{W}(\phi, \phi)\|_{L^{2}\left(\Xi^{*}\right)}\|a\|_{L^{2}\left(\Xi^{*}\right)}=\|\phi\|^{2}\|a\|_{L^{2}\left(\Xi^{*}\right)}
$$

Hence $a=0$, and this shows that $\|\cdot\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)}$ is a norm.
To prove that the norm of $M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ is complete, it suffices to check that any Cauchy sequence $\left\{a_{j}\right\}_{j \geq 1}$ has a convergent subsequence. By selecting a suitable subsequence, we may assume that $\left\|a_{j+1}-a_{j}\right\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)}<\frac{1}{2^{j}}$ for every $j \geq 0$, where $a_{0}:=0$. It follows by Remark 3.5 that there exists $\beta_{j+1} \in L^{1}(\Xi)$ such that $\left\|\beta_{j+1}\right\|_{L^{1}(\Xi)}<\frac{1}{2^{j}}$ and

$$
\begin{equation*}
\left(\forall X_{1}, X_{2} \in \Xi\right) \quad\left|\left(\mathrm{Op}^{\theta}\left(a_{j+1}-a_{j}\right) \phi_{X_{1}} \mid \phi_{X_{2}}\right)\right| \leq \beta_{j+1}\left(X_{1}-X_{2}\right) \tag{3.7}
\end{equation*}
$$

Note that $\beta:=\sum_{j=1}^{\infty} \beta_{j} \in L^{1}(\Xi)$ and, by summing up the above inequalities for $j=0, \ldots, k-1$ we get
$\left(\forall X_{1}, X_{2} \in \Xi\right) \quad\left|\left(\operatorname{Op}^{\theta}\left(a_{k}\right) \phi_{X_{1}} \mid \phi_{X_{2}}\right)\right| \leq\left(\beta_{1}+\cdots+\beta_{k}\right)\left(X_{1}-X_{2}\right) \leq \beta\left(X_{1}-X_{2}\right)$.

On the other hand, since $\left\{a_{k}\right\}_{k \geq 1}$ is a Cauchy sequence in $M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$, it follows by Assertion (1) that there exists an operator $T \in \mathcal{B}(\mathcal{H})$ such that $\lim _{k \rightarrow \infty}\left\|\mathrm{Op}^{\theta}\left(a_{k}\right)-T\right\|=0$. It follows by the above inequalities for $k \rightarrow \infty$ that

$$
\left(\forall X_{1}, X_{2} \in \Xi\right) \quad\left|\left(T \phi_{X_{1}} \mid \phi_{X_{2}}\right)\right| \leq \beta\left(X_{1}-X_{2}\right)
$$

Moreover, it follows by Remark 2.5 that $T=\mathrm{Op}^{\theta}(a)$ for some $a \in \mathcal{S}^{\prime}\left(\Xi^{*}\right)$, and then $a \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ by the above inequality along with Remark 3.5. Finally, by summing up the inequalities (3.7) for $j=k, k+1, \ldots$ and using Remark 3.5 again, we get $\left\|a-a_{k}\right\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty}\left(\pi^{\#}, \theta \times \theta\right)} \leq \sum_{j=k}^{\infty} \frac{1}{2^{j}}=\frac{1}{2^{k-1}}$ for arbitrary $k \geq 1$, hence $a=\lim _{k \rightarrow \infty} a_{k}$ in $M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$.

For Assertion (3) let $a_{0} \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ and assume that the operator $\mathrm{Op}^{\theta}\left(a_{0}\right)$ is invertible in $\mathcal{B}(\mathcal{H})$. There exist $\alpha \in \mathbb{C}$ and $a_{00} \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ such that $a_{0}=\alpha+a_{00}$. We shall use the notation of Definition 3.6 and also recall that for the symbol $1 \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ we get the operator $T_{1} \in \mathcal{B}\left(L^{2}(\Xi)\right)$ with the properties $T_{1}=T_{1}^{*}=\left(T_{1}\right)^{2}$ and $\operatorname{Ran} T_{1}=\operatorname{Ran} V$. Moreover, for every $a \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ we have $T_{a} T_{1}=T_{1} T_{a}=T_{a}$, and in particular $T_{a}$ vanishes on $\left(\operatorname{Ran} T_{1}\right)^{\perp}$.

It then follows that if $z \in \mathbb{C} \backslash\{\alpha\}$ and the operator

$$
z \mathbf{1}-\operatorname{Op}^{\theta}\left(a_{0}\right)=\operatorname{Op}^{\theta}\left(z-a_{0}\right)=\operatorname{Op}^{\theta}\left(z-\alpha-a_{00}\right)
$$

is invertible in $\mathcal{B}(\mathcal{H})$, then $(z-\alpha)\left(\mathbf{1}-T_{1}\right)+T_{z-\alpha-a_{00}}=(z-\alpha) \mathbf{1}-T_{a_{00}}$ is invertible in $\mathcal{B}\left(L^{2}(\Xi)\right)$. On the other hand, since $a_{00} \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$, it follows by Remark 3.5 that there exists $\beta_{0} \in L^{1}(\Xi)$ such that the integral kernel $C_{a_{00}}$ of $T_{a_{00}}$ satisfies the estimate $\left|C_{a_{00}}(X-Y)\right| \leq \beta_{0}(X-Y)$ for all $X, Y \in \Xi$. We then get by [13, Th. 5.4.7] (see also [14]) that $\left((z-\alpha) \mathbf{1}-T_{a_{00}}\right)^{-1}=(z-\alpha)^{-1} \mathbf{1}-N_{z}$, where $N_{z} \in \mathcal{B}\left(L^{2}(\Xi)\right)$ is an integral operator whose kernel $K_{N_{z}}$ satisfies a similar estimate $\left|K_{N_{z}}(X-Y)\right| \leq \beta_{z}(X-Y)$ for all $X, Y \in \Xi$ and a suitable function $\beta_{z} \in L^{1}(\Xi)$. Since $T_{a_{00}} T_{1}=T_{1} T_{a_{00}}=T_{a_{00}}$, it follows that $N_{z} T_{1}=T_{1} N_{z}=N_{z}$. By using the fact that $\mathrm{Op}^{\theta}: \mathcal{S}^{\prime}\left(\Xi^{*}\right) \rightarrow \mathcal{L}\left(\mathcal{H}_{\infty}, \mathcal{H}_{-\infty}\right)$ is a linear isomorphism (see [4, Rem. 3.11]) and Lemma 3.7, we then get $b_{z} \in \mathcal{S}^{\prime}\left(\Xi^{*}\right)$ such that $\mathrm{Op}^{\theta}\left(b_{z}\right) \in \mathcal{B}(\mathcal{H})$ and $T_{b_{z}}=N_{z}$. Moreover, the estimates satisfied by the integral kernel of $N_{z}$ show that actually $b_{z} \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ by Remark 3.5 again.

We have thus shown that if $z \in \mathbb{C} \backslash\{\alpha\}$ and $z \mathbf{1}-\operatorname{Op}^{\theta}\left(a_{0}\right)$ (which is equal to $\left.\mathrm{Op}^{\theta}\left(z-a_{0}\right)\right)$ is invertible in $\mathcal{B}(\mathcal{H})$, then there exists $b_{z} \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ such that $\mathrm{Op}^{\theta}\left(z-a_{0}\right)^{-1}=\mathrm{Op}^{\theta}\left((z-\alpha)^{-1}-b_{z}\right)$. Thus we can see that $z-a_{0}$ is invertible in the unital Banach algebra $\mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ and its inverse is $(z-\alpha)^{-1}-b_{z}$. In particular, $z \mapsto b_{z}$ is a holomorphic mapping from the complement of the spectrum of $\mathrm{Op}^{\theta}\left(a_{0}\right)$ into $\mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$. Now, since $\mathrm{Op}^{\theta}\left(a_{0}\right) \in \mathcal{B}(\mathcal{H})$ is an invertible operator, there exists a piecewise smooth closed curve that does not contain $\alpha$ and
surrounds the spectrum of $\mathrm{Op}^{\theta}\left(a_{0}\right)$, and we have by holomorphic functional calculus

$$
\mathrm{Op}^{\theta}\left(a_{0}\right)^{-1}=\frac{1}{2 \pi \mathrm{i}} \int_{\gamma} \frac{1}{z}\left(z-\mathrm{Op}^{\theta}\left(a_{0}\right)\right)^{-1} \mathrm{~d} z
$$

Since $\mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right)$ is a unital Banach algebra, we can define

$$
b_{0}:=\frac{1}{2 \pi \mathrm{i}} \int_{\gamma} \frac{1}{z}\left((z-\alpha)^{-1}-b_{z}\right) \mathrm{d} z \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta \times \theta\right) .
$$

Then

$$
\begin{aligned}
\mathrm{Op}^{\theta}\left(b_{0}\right) & =\frac{1}{2 \pi \mathrm{i}} \int_{\gamma} \frac{1}{z} \mathrm{Op}^{\theta}\left((z-\alpha)^{-1}-b_{z}\right) \mathrm{d} z=\frac{1}{2 \pi \mathrm{i}} \int_{\gamma} \frac{1}{z}\left(z-\mathrm{Op}^{\theta}\left(a_{0}\right)\right)^{-1} \mathrm{~d} z \\
& =\mathrm{Op}^{\theta}\left(a_{0}\right)^{-1}
\end{aligned}
$$

which completes the proof.
Remark 3.11 A more general result on the continuity of the operators $\mathrm{Op}^{\theta}(a)$ on modulation spaces was obtained in [4] by a completely different method based on continuity properties of the cross-Wigner distribution.

## 4 Applications to the magnetic Weyl calculus

In this section, we apply the above abstract results to the magnetic Weyl calculus developed in a series of papers including for instance [1,2,4,10, 11, 16, 17].

Notation 4.1 Let $G$ be a simply connected, nilpotent Lie group with the Lie algebra $\mathfrak{g}$ and the inverse of the exponential map denoted by $\log _{G}: G \rightarrow \mathfrak{g}$. We denote by $\lambda: G \rightarrow \operatorname{End}\left(\mathcal{C}^{\infty}(G)\right), g \mapsto \lambda_{g}$, the left regular representation defined by $\left(\lambda_{g} \phi\right)(x)=\phi\left(g^{-1} x\right)$ for every $x, g \in G$ and $\phi \in \mathcal{C}^{\infty}(G)$. Moreover, we denote by $\mathbf{1}$ the constant function which is identically equal to 1 on $G$. (This should not be confused with the unit element of $G$, which is denoted in the same way.)

If the space of globally defined smooth vector fields on $G$ (that is, global sections in its tangent bundle) is denoted by $\mathfrak{X}(G)$ and the space of globally defined smooth 1 -forms (that is, global sections in its cotangent bundle) is denoted by $\Omega^{1}(G)$, then there exists a natural bilinear map

$$
\begin{equation*}
\langle\cdot, \cdot\rangle: \Omega^{1}(G) \times \mathfrak{X}(G) \rightarrow \mathcal{C}^{\infty}(G) \tag{4.1}
\end{equation*}
$$

defined as usually by evaluations at every point of $G$.
For arbitrary $g \in G$ we denote the corresponding right-translation mapping by $R_{g}: G \rightarrow G, h \mapsto h g$. Then we define the injective linear mapping

$$
\iota^{\mathrm{R}}: \mathfrak{g} \rightarrow \mathfrak{X}(G)
$$

by $\left(\iota^{\mathrm{R}} X\right)(g)=\left(T_{\mathbf{1}}\left(R_{g}\right)\right) X \in T_{g} G$ for all $g \in G$ and $X \in \mathfrak{g}$.

Moreover, we set

$$
\Xi=\Xi^{*}:=\mathfrak{g} \times \mathfrak{g}^{*}
$$

with the symplectic duality pairing

$$
\Xi^{*} \times \Xi \rightarrow \mathbb{R}, \quad\left(\left(X_{1}, \xi_{1}\right),\left(X_{2}, \xi_{2}\right)\right) \mapsto\left\langle\xi_{1}, X_{2}\right\rangle-\left\langle\xi_{2}, X_{1}\right\rangle
$$

where $\langle\cdot, \cdot\rangle: \mathfrak{g}^{*} \times \mathfrak{g} \rightarrow \mathbb{R}$ is the natural duality pairing. This should not be confused with the bilinear mapping (4.1), which is denoted in the same way. In fact, the meaning of $\langle\cdot, \cdot\rangle$ will always be clear from the context.

Setting 4.2 Throughout this section we denote by $\mathcal{F}$ a linear space of real functions on the Lie group $G$ which is endowed with a sequentially complete, locally convex topology and satisfies the following conditions:
(1) The linear space $\mathcal{F}$ is invariant under the representation of $G$ by left translations, that is, if $\phi \in \mathcal{F}$ and $g \in G$ then $\lambda_{g} \phi \in \mathcal{F}$.
(2) There exist the continuous inclusion maps $\mathfrak{g}^{*} \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{C}_{\text {pol }}^{\infty}(G)$, where the embedding $\mathfrak{g}^{*} \hookrightarrow \mathcal{F}$ is given by $\xi \mapsto \xi \circ \log _{G}$.
(3) The mapping $G \times \mathcal{F} \rightarrow \mathcal{F},(g, \phi) \mapsto \lambda_{g} \phi$ is smooth. For every $\phi \in \mathcal{F}$ we denote by $\dot{\lambda}(\cdot) \phi: \mathfrak{g} \rightarrow \mathcal{F}$ the differential of the mapping $g \mapsto \lambda_{g} \phi$ at the point $1 \in G$.

For instance, the function space $\mathcal{C}_{\text {pol }}^{\infty}(G)$ is admissible. Here $\mathcal{C}_{\text {pol }}^{\infty}(G)$ is the space of smooth functions $\phi: G \rightarrow \mathbb{R}$ such that the function $\phi \circ \exp _{G}: \mathfrak{g} \rightarrow \mathbb{R}$ and its partial derivatives have polynomial growth.

Definition 4.3 We define the semidirect product $M=\mathcal{F} \rtimes_{\lambda} G$, which is a locally convex Lie group, and the unitary representation

$$
\pi: M \rightarrow \mathcal{B}\left(L^{2}(G)\right), \quad \pi(\phi, g) f=\mathrm{e}^{\mathrm{i} \phi} \lambda_{g} f \text { for } \phi \in \mathcal{F}, g \in G, \text { and } f \in L^{2}(G)
$$

If we have $A \in \Omega^{1}(G)$ with $\mathcal{F}$-growth, in the sense that $\left\langle A, \iota^{\mathrm{R}} X\right\rangle \in \mathcal{F}$ whenever $X \in \mathfrak{g}$, then we define the linear mapping

$$
\theta^{A}: \mathfrak{g} \times \mathfrak{g}^{*} \rightarrow \mathfrak{m}=\mathcal{F} \ltimes_{\dot{\lambda}} \mathfrak{g}, \quad(X, \xi) \mapsto\left(\xi \circ \log _{G}+\left\langle A, \iota^{\mathrm{R}} X\right\rangle, X\right) .
$$

Remark 4.4 The representation $\pi$ is twice nuclearly smooth and its space of smooth vectors is the Schwartz space $\mathcal{S}(G)$ ([4, Cor.4.5]) and the following assertions hold for every 1-form $A \in \Omega^{1}(G)$ with $\mathcal{F}$-growth:
(1) The representation $\pi$ satisfies the orthogonality relations along the mapping $\theta^{A}$.
(2) The representation $\pi$ satisfies the density condition along $\theta^{A}$.
(3) The localized Weyl calculus for $\pi$ along $\theta^{A}$ is regular and defines a unitary operator $\mathrm{Op}^{\theta^{A}}: L^{2}\left(\mathfrak{g} \times \mathfrak{g}^{*}\right) \rightarrow \mathfrak{S}_{2}\left(L^{2}(G)\right)$.
(4) If $u \in \mathrm{U}\left(\mathfrak{m}_{\mathbb{C}}\right)$ and $\phi \in \mathcal{S}(G)$, the function $\left\|\mathrm{d} \pi\left(\operatorname{Ad}_{\mathrm{U}\left(\mathfrak{m}_{\mathbb{C}}\right)}\left(\exp _{M}\left(\theta^{A}(\cdot)\right)\right) u\right) \phi\right\|$ has polynomial growth on $\mathfrak{g} \times \mathfrak{g}^{*}$. Here for any element $x \in M$ we denote by $\operatorname{Ad}_{\mathrm{U}\left(\mathfrak{m}_{\mathbb{C}}\right)}(x)$ the unique automorphism of the universal enveloping algebra $\mathrm{U}\left(\mathfrak{m}_{\mathbb{C}}\right)$ which extends the Lie algebra automorphism $\operatorname{Ad}_{\mathfrak{m}}(x): \mathfrak{m} \rightarrow \mathfrak{m}$ defined by the adjoint action.

These properties have been established in [4, Cor.4.5], and it thus follows that all of the conditions of Setting 3.1 are satisfied in the present setting provided by the representation $\pi$ and the linear mapping $\theta^{A}$.

Just as in [1] we shall denote the corresponding Moyal product $\#^{\theta^{A}}$ simply by $\#^{A}$ and the localized Weyl calculus for $\pi$ along $\theta^{A}$ is denoted by $\mathrm{Op}^{A}(\cdot)$ and is called the magnetic Weyl calculus associated with the magnetic potential $A \in \Omega^{1}(G)$. The corresponding magnetic field is $B:=\mathrm{d} A \in \Omega^{2}(G)$.

Theorem 4.5 If $\phi \in \mathcal{S}(G)$, then the following assertions hold:
(1) For every $a \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta^{A} \times \theta^{A}\right)$ we have $\operatorname{Op}^{A}(a) \in \mathcal{B}\left(L^{2}(G)\right)$ and moreover $\left\|\mathrm{Op}^{A}(a)\right\| \leq\|a\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta^{A} \times \theta^{A}\right)}$.
(2) The Moyal product $\#^{A}$ makes the modulation space $M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta^{A} \times \theta^{A}\right)$ into an associative Banach algebra.
(3) If $a_{0} \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta^{A} \times \theta^{A}\right)$ and $\mathrm{Op}^{A}\left(a_{0}\right) \in \mathcal{B}\left(L^{2}(G)\right)$ is invertible, then there exists $b_{0} \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}\left(\pi^{\#}, \theta^{A} \times \theta^{A}\right)$ such that $\mathrm{Op}^{A}\left(a_{0}\right)^{-1}=\mathrm{Op}^{A}\left(b_{0}\right)$.

Proof The above Remark 4.4 shows that Corollary 3.9 and Theorem 3.10 apply, and then the assertions follow.

## 5 Applications to representations of nilpotent Lie groups

All of the conditions of Setting 3.1 are satisfied in the case of a finite-dimensional nilpotent Lie group, and a unitary irreducible representation, along the inclusion mapping of a predual of the corresponding coadjoint orbit (in the sense of [3]). This will be the setting of the present section, and our point here is to describe how the abstract results of Sect. 3 can be specialized in this framework, and also to point out how they can be further sharpened in the special case of a square-integrable representation modulo the center.

Setting 5.1 Throughout this section we use the following notation:
(1) Let $G$ be a connected, simply connected, nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then the exponential map $\exp _{G}: \mathfrak{g} \rightarrow G$ is a diffeomorphism with the inverse denoted by $\log _{G}: G \rightarrow \mathfrak{g}$.
(2) We denote by $\mathfrak{g}^{*}$ the linear dual space to $\mathfrak{g}$ and by $\langle\cdot, \cdot\rangle: \mathfrak{g}^{*} \times \mathfrak{g} \rightarrow \mathbb{R}$ the natural duality pairing.
(3) Let $\xi_{0} \in \mathfrak{g}^{*}$ with the corresponding coadjoint orbit $\mathcal{O}:=\operatorname{Ad}_{G}^{*}(G) \xi_{0} \subseteq \mathfrak{g}^{*}$.
(4) Let $\pi: G \rightarrow \mathcal{B}(\mathcal{H})$ be any unitary irreducible representations associated with the coadjoint orbit $\mathcal{O}$ by Kirillov's theorem [12].
(5) The isotropy group at $\xi_{0}$ is $G_{\xi_{0}}:=\left\{g \in G \mid \operatorname{Ad}_{G}^{*}(g) \xi_{0}=\xi_{0}\right\}$ with the corresponding isotropy Lie algebra $\mathfrak{g}_{\xi_{0}}=\left\{X \in \mathfrak{g} \mid \xi_{0} \circ \operatorname{ad}_{\mathfrak{g}} X=0\right\}$. If we denote the center of $\mathfrak{g}$ by $\mathfrak{z}:=\{X \in \mathfrak{g} \mid[X, \mathfrak{g}]=\{0\}\}$, then $\mathfrak{z} \subseteq \mathfrak{g}_{\xi_{0}}$.
(6) Let $n:=\operatorname{dim} \mathfrak{g}$ and fix a sequence of ideals in $\mathfrak{g}$,

$$
\{0\}=\mathfrak{g}_{0} \subset \mathfrak{g}_{1} \subset \cdots \subset \mathfrak{g}_{n}=\mathfrak{g}
$$

such that $\operatorname{dim}\left(\mathfrak{g}_{j} / \mathfrak{g}_{j-1}\right)=1$ and $\left[\mathfrak{g}, \mathfrak{g}_{j}\right] \subseteq \mathfrak{g}_{j-1}$ for $j=1, \ldots, n$.
(7) Pick any $X_{j} \in \mathfrak{g}_{j} \backslash \mathfrak{g}_{j-1}$ for $j=1, \ldots, n$, so that the set $\left\{X_{1}, \ldots, X_{n}\right\}$ will be a Jordan-Hölder basis in $\mathfrak{g}$.

Also consider the set of jump indices of the coadjoint orbit $\mathcal{O}$ with respect to the aforementioned Jordan-Hölder basis,

$$
e:=\left\{j \in\{1, \ldots, n\} \mid \mathfrak{g}_{j} \nsubseteq \mathfrak{g}_{j-1}+\mathfrak{g}_{\xi_{0}}\right\}=\left\{j \in\{1, \ldots, n\} \mid X_{j} \notin \mathfrak{g}_{j-1}+\mathfrak{g}_{\xi_{0}}\right\}
$$

and then define the corresponding predual of the coadjoint orbit $\mathcal{O}$,

$$
\mathfrak{g}_{e}:=\operatorname{span}\left\{X_{j} \mid j \in e\right\} \subseteq \mathfrak{g} .
$$

We note the direct sum decomposition $\mathfrak{g}=\mathfrak{g}_{\xi_{0}}+\mathfrak{g}_{e}$.
This setting is a special instance of the above abstract setting of Sect. 3 with $M=G$, $\Xi=\mathfrak{g}_{e}, \theta=\operatorname{id}_{\mathfrak{g}_{e}}: \mathfrak{g}_{e} \hookrightarrow \mathfrak{g}$, and $\Xi^{*}=\mathcal{O}$. The corresponding operator calculus and Moyal product will be denoted simply by Op and \#, respectively.

Theorem 5.2 If the representation $\pi$ is square integrable modulo the center, then the following assertions hold:
(1) If $p_{1}, p_{2}, p, q_{1}, q_{2}, q \in[1, \infty]$ satisfy the conditions $\frac{1}{p_{1}}+\frac{1}{p_{2}}=\frac{1}{p}$ and $\frac{1}{q_{1}}+\frac{1}{q_{2}}=$ $1+\frac{1}{q}$, then the Moyal product \# defines a continuous bilinear map

$$
M^{p_{1}, q_{1}}\left(\pi^{\#}\right) \times M^{p_{2}, q_{2}}\left(\pi^{\#}\right) \rightarrow M^{p, q}\left(\pi^{\#}\right)
$$

(2) The Moyal product \# makes the modulation space $M^{\infty, 1}\left(\pi^{\#}\right)$ into an associative involutive Banach algebra and the Weyl calculus defines an injective continuous *-homomorphism Op: $M^{\infty, 1}\left(\pi^{\#}\right) \rightarrow \mathcal{B}(\mathcal{H})$.
(3) If $a_{0} \in \mathcal{M}^{\infty, 1}\left(\pi^{\#}\right)$ and $\operatorname{Op}\left(a_{0}\right) \in \mathcal{B}(\mathcal{H})$ is an invertible operator, then there exists $b_{0} \in \mathcal{M}^{\infty, 1}\left(\pi^{\#}\right)$ such that $\mathrm{Op}\left(a_{0}\right)^{-1}=\mathrm{Op}\left(b_{0}\right)$.

Proof Recall that the modulation spaces of symbols $M^{p, q}\left(\pi^{\#}\right)$ are independent on the choice of a window vector by [3, Example 3.4(2)]. Then the assertions follow by the above Corollary 3.9 and Theorem 3.10.

Theorem 5.3 Assume that the representation $\pi$ is square integrable modulo the center of $G$ and let $\mathfrak{g}_{e}=\mathfrak{g}_{e}^{1}+\mathfrak{g}_{e}^{2}$ be any decomposition of the predual into a direct sum of linear subspaces. If $\phi \in \mathcal{H}_{\infty}$ and we have $1 \leq p_{1} \leq p_{2} \leq \infty$ and $1 \leq q_{1} \leq q_{2} \leq \infty$, then $M_{\phi}^{p_{1}, q_{1}}(\pi) \subseteq M_{\phi}^{p_{2}, q_{2}}(\pi)$.

Proof It follows by (2.7) that the proof will be complete as soon as we have proved that if $p, q \in[1, \infty]$ and $f \in M_{\phi}^{p, q}(\pi)$, then $f \in M_{\phi}^{\infty, \infty}(\pi)$, that is, $\mathcal{A}_{\phi} f \in L^{\infty}\left(\mathfrak{g}_{e}\right)$.

In fact, let us define

$$
\begin{aligned}
R_{\phi}: \mathfrak{g}_{e} \times \mathfrak{g}_{e} \rightarrow \mathbb{C}, \quad R_{\phi}(X, Y) & =\left(\pi\left(\exp _{G} X\right) \phi \mid \pi\left(\exp _{G} Y\right) \phi\right) \\
& =\left(\mathcal{A}_{\phi}\left(\pi\left(\exp _{G} \phi\right)\right)\right)(Y) .
\end{aligned}
$$

Let us denote by $*_{e}$ the Baker-Campbell-Hausdorff multiplication on the nilpotent Lie algebra $\mathfrak{g}_{e} \simeq \mathfrak{g} / \mathfrak{z}$. There exists a polynomial map $\alpha: \mathfrak{g}_{e} \times \mathfrak{g}_{e} \rightarrow \mathbb{R}$ such that $\pi\left(\exp _{G}((-X) * Y)\right)=\mathrm{e}^{\mathrm{i} \alpha(-X, Y)} \pi\left(\exp _{G}\left((-X) *_{e} Y\right)\right)$ (see for instance [15]), hence

$$
\begin{aligned}
R_{\phi}(X, Y) & =\mathrm{e}^{-\mathrm{i} \alpha(-X, Y)}\left(\phi \mid \pi\left(\exp _{G}\left((-X) *_{e} Y\right)\right) \phi\right) \\
& =\mathrm{e}^{-\mathrm{i} \alpha(-X, Y)}\left(\mathcal{A}_{\phi} \phi\right)\left((-X) *_{e} Y\right) .
\end{aligned}
$$

Since $\mathcal{A}_{\phi} \phi \in \mathcal{S}\left(\mathfrak{g}_{e}\right)$ (see [20]) and the Lebesgue measure on $\mathfrak{g}_{e}$ coincides with the Haar measure on the nilpotent Lie group ( $\mathfrak{g}_{e}, *_{e}$ ), it then follows that

$$
\begin{equation*}
(\forall r, s \in[1, \infty]) \quad \sup _{X \in \mathfrak{g}_{e}}\left\|R_{\phi}(X, \cdot)\right\|_{L^{r, s}\left(\mathfrak{g}_{e}^{1} \times \mathfrak{g}_{e}^{2}\right)}<\infty \tag{5.1}
\end{equation*}
$$

On the other hand, note that $R_{\phi}(X, Y)=\left(\mathcal{A}_{\phi}\left(\pi\left(\exp _{G} \phi\right)\right)\right)(Y)$, hence

$$
\begin{aligned}
\left(\mathcal{A}_{\phi} f\right)(X) & =\left(f \mid \pi\left(\exp _{G} X\right) \phi\right) \\
& =\left(f \mid \int_{\mathfrak{g}_{e}}\left(\mathcal{A}_{\phi}\left(\pi\left(\exp _{G} \phi\right)\right)\right)(Y) \pi\left(\exp _{G} Y\right) \phi \mathrm{d} Y\right) \\
& =\left(f \mid \int_{\mathfrak{g}_{e}} R_{\phi}(X, Y) \pi\left(\exp _{G} Y\right) \phi \mathrm{d} Y\right) \\
& =\iint_{\mathfrak{g}_{e}} \frac{R_{\phi}(X, Y)}{R^{\prime}}\left(f \mid \pi\left(\exp _{G} Y\right) \phi\right) \mathrm{d} Y,
\end{aligned}
$$

by [3, Cor. 2.9(1)]. Therefore

$$
\begin{equation*}
\left(\forall X \in \mathfrak{g}_{e}\right) \quad\left(\mathcal{A}_{\phi} f\right)(X)=\left(\mathcal{A}_{\phi} f \mid R_{\phi}(X, \cdot)\right), \tag{5.2}
\end{equation*}
$$

where the right-hand side makes sense since $\mathcal{A}_{\phi} f \in \mathcal{C}\left(\mathfrak{g}_{e}\right) \cap \mathcal{S}^{\prime}\left(\mathfrak{g}_{e}\right)$, while $R_{\phi}(X, \cdot) \in$ $\mathcal{S}\left(\mathfrak{g}_{e}\right)$ by [20]. If $f \in M_{\phi}^{p, q}(\pi)$, then it follows by (5.2) along with Hölder's inequality in mixed-norm spaces (see [5]) and (5.1) that

$$
\sup _{X \in \mathfrak{g}_{e}}\left|\left(\mathcal{A}_{\phi} f\right)(X)\right| \leq \sup _{X \in \mathfrak{g}_{e}}\left(\left\|\mathcal{A}_{\phi} f\right\|_{L^{p, q}\left(\mathfrak{g}_{e}^{1} \times \mathfrak{g}_{e}^{2}\right)}\left\|R_{\phi}(X, \cdot)\right\|_{L^{p^{\prime}, q^{\prime}}\left(\mathfrak{g}_{e}^{1} \times \mathfrak{g}_{e}^{2}\right)}\right)<\infty
$$

where $\frac{1}{p}+\frac{1}{p^{\prime}}=\frac{1}{q}+\frac{1}{q^{\prime}}=1$. Thus $\mathcal{A}_{\phi} f \in L^{\infty}\left(\mathfrak{g}_{e}\right)$, and this completes the proof, in view of the beginning remark.

Corollary 5.4 If the representation $\pi$ is square integrable modulo the center of $G$, then for every $p \in[1, \infty]$ the modulation space $M^{p, 1}\left(\pi^{\#}\right)$ is a two-sided ideal of the Banach algebra $M^{\infty, 1}\left(\pi^{\#}\right)$ endowed with the Moyal product \#.

Proof As noted in [3, Example 3.4(2) and Rem. 3.7], the representation $\pi^{\#}: G \ltimes G \rightarrow$ $\mathcal{B}\left(L^{2}(\mathcal{O})\right.$ ) is square integrable (modulo the center) and its modulation spaces are independent on the choice of the window vector. Thus the above Theorem 5.3 applies for the representation $\pi^{\#}$ instead of $\pi$, and it follows that $M^{p_{1}, q_{1}}\left(\pi^{\#}\right) \subseteq M^{p_{2}, q_{2}}\left(\pi^{\#}\right)$ whenever $1 \leq p_{1} \leq p_{2} \leq \infty$ and $1 \leq q_{1} \leq q_{2} \leq \infty$.

In particular we have $M^{p, 1}\left(\pi^{\#}\right) \subseteq M^{\infty, 1}\left(\pi^{\#}\right)$ if $1 \leq p \leq \infty$. Moreover, it follows by Theorem 5.2 (1) that

$$
M^{p, 1}\left(\pi^{\#}\right) \# M^{\infty, 1}\left(\pi^{\#}\right) \subset M^{p, 1}\left(\pi^{\#}\right)
$$

and

$$
M^{\infty, 1}\left(\pi^{\#}\right) \# M^{p, 1}\left(\pi^{\#}\right) \subset M^{p, 1}\left(\pi^{\#}\right)
$$

This completes the proof.
In the special case when $\pi$ is the Schrödinger representation of the Heisenberg group, the above result goes back to [23]; see also [9]. We also note that in this case we have $\mathcal{M}^{\infty, 1}\left(\pi^{\#}\right)=M^{\infty, 1}\left(\pi^{\#}\right)$.

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    I. Beltiță $(\boxtimes) \cdot$ D. Beltiță

    Research Unit 1, Simion Stoilow Institute of Mathematics of the Romanian Academy, P. O. Box 1-764, Bucharest, Romania
    e-mail: Ingrid.Beltita@imar.ro
    D. Beltiță
    e-mail: Daniel.Beltita@imar.ro

