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Modulation Spaces of Symbols for Representations of Nilpotent Lie Groups

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Abstract We investigate continuity properties of operators obtained as values of the Weyl correspondence constructed by Pedersen (Invent. Math. 118:1–36, 1994) for arbitrary irreducible representations of nilpotent Lie groups. To this end we introduce modulation spaces for such representations and establish some of their basic properties. The situation of square-integrable representations is particularly important and in the special case of the Schrödinger representation of the Heisenberg group we recover the classical modulation spaces used in the time-frequency analysis.

Keywords Pseudo-differential Weyl calculus · Modulation space · Nilpotent Lie group · Semidirect product

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1 Introduction

The representation theory of the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}_{2n+1}$ provides a natural background for the pseudo-differential calculus on $\mathbb{R}^n$. It is well known that the representation theoretic approach has led to a deeper understanding of the Weyl calculus, which resulted in simplified proofs and improvements for many basic results. A celebrated example in this connection is the Calderón-Vaillancourt

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theorem on L^2-boundedness for pseudo-differential operators [5]. This classical theorem was strengthened in the paper [18] by using the modulation spaces, which are function (or distribution) spaces defined in terms of the Schrödinger representations of Heisenberg groups. The modulation spaces were introduced in [10] in the framework of harmonic analysis of locally compact abelian groups.

On the other hand, a remarkable Weyl calculus was set up in [31] for arbitrary unitary irreducible representations of any nilpotent Lie group, by replying on [28–30]. We shall call it the *Weyl-Pedersen calculus*. It is a challenging task to understand this interaction of the ideas of pseudo-differential calculus with the representation theory of nilpotent Lie groups.

In the present paper we address the above problem in the shape of the L^2-boundedness theorems. Specifically, we are going to investigate continuity properties of the operators constructed by the Weyl-Pedersen calculus. For this purpose we introduce the modulation spaces $M_\phi^{r,s}(\pi)$ defined in terms of an arbitrary irreducible representation π of a nilpotent Lie group G. One key feature of our representation theoretic approach is that if $\mathcal{O}$ stands for the coadjoint orbit corresponding to π [23], then the symbols of the operators constructed by the Weyl-Pedersen calculus are functions or distributions on the coadjoint orbit $\mathcal{O}$, while the Hilbert space $L^2(\mathcal{O})$ carries a natural irreducible representation $\pi^\#$ of the nilpotent Lie group $G \ltimes G$. Therefore our general notion of modulation spaces for irreducible representations allows us to investigate the modulation spaces of symbols for the operators constructed by the Weyl-Pedersen calculus for the representation π. This approach also reveals the representation theoretic background of the L^2-boundedness theorem of [18].

We find several of the familiar properties of the classical modulation spaces, such as:

- continuity of the operators constructed by the Weyl-Pedersen calculus with symbols in an appropriate modulation space $M_\Phi^{\infty,1}(\pi^\#)$ (Corollary 2.26);
- independence on the choice of a window function, and covariance of the Weyl-Pedersen calculus, in the case of square-integrable representations (Theorems 3.3 and 3.5).

Besides the aforementioned reasons, the present research has also been motivated by the recent interest in the magnetic pseudo-differential Weyl calculus on $\mathbb{R}^n$ (see for instance [22, 25, 26], and the references therein), which was partially extended to nilpotent Lie groups in the papers [1, 2]. Specifically, the results of the present paper apply to the Weyl calculus associated with a polynomial magnetic field on $\mathbb{R}^n$, in particular complementing the L^2-boundedness theorem established in [22] for magnetic fields whose components are bounded and so are also their partial derivatives of arbitrarily high degree.

Notation Throughout the paper we denote by $\mathcal{S}(\mathcal{V})$ the Schwartz space on a finite-dimensional real vector space $\mathcal{V}$. That is, $\mathcal{S}(\mathcal{V})$ is the set of all smooth functions that decay faster than any polynomial together with their partial derivatives of arbitrary order. Its topological dual—the space of tempered distributions on $\mathcal{V}$—is denoted by $\mathcal{S}'(\mathcal{V})$.

We shall also use the convention that the Lie groups are denoted by upper case Latin letters and the Lie algebras are denoted by the corresponding lower case Gothic letters.

For basic notions on Weyl pseudo-differential calculus, we refer to [14, 17, 21].

2 Modulation Spaces for Unitary Irreducible Representations

2.1 Preliminaries on Semidirect Products

Definition 2.1 Let G_1 and G_2 be connected Lie groups and assume that we have a continuous group homomorphism $\alpha : G_1 \to \operatorname{Aut} G_2$, $g_1 \mapsto \alpha_{g_1}$. The corresponding *semidirect product of Lie groups* $G_1 \ltimes_\alpha G_2$ is the connected Lie group whose underlying manifold is the Cartesian product $G_1 \times G_2$ and whose group operation is given by

$$(g_1, g_2) \cdot (h_1, h_2) = \big(g_1 h_1, \alpha_{h_1^{-1}}(g_2) h_2\big) \tag{2.1}$$

whenever $g_j, h_j \in G_j$ for $j = 1, 2$.

Let us denote by $\dot{\alpha} : \mathfrak{g}_1 \to \operatorname{Der} \mathfrak{g}_2$ the homomorphism of Lie algebras defined as the differential of the Lie group homomorphism $G_1 \to \operatorname{Aut} \mathfrak{g}_2$, $g_1 \mapsto \mathbf{L}(\alpha_{g_1})$. Then the *semidirect product of Lie algebras* $\mathfrak{g}_1 \ltimes_{\dot{\alpha}} \mathfrak{g}_2$ is the Lie algebra whose underlying linear space is the Cartesian product $\mathfrak{g}_1 \times \mathfrak{g}_2$ with the Lie bracket given by

$$\big[(X_1, X_2), (Y_1, Y_2)\big] = \big([X_1, Y_1], \dot{\alpha}(X_1)Y_2 - \dot{\alpha}(Y_1)X_2 + [X_2, Y_2]\big) \tag{2.2}$$

if $X_j, Y_j \in \mathfrak{g}_j$ for $j = 1, 2$. One can prove that $\mathfrak{g}_1 \ltimes_{\dot{\alpha}} \mathfrak{g}_2$ is the Lie algebra of the Lie group $G_1 \ltimes_\alpha G_2$ (see for instance Chap. 9 in [20]).

Remark 2.2 Let G_1 and G_2 be nilpotent Lie groups and $\alpha : G_1 \to \operatorname{Aut} G_2$ be a *unipotent automorphism*, in the sense that for every $X_1 \in \mathfrak{g}_1$ there exists an integer $m \geq 1$ such that $\dot{\alpha}(X_1)^m = 0$. Then an inspection of (2.2) shows that $\mathfrak{g}_1 \ltimes_{\dot{\alpha}} \mathfrak{g}_2$ is a nilpotent Lie algebra, hence $G_1 \ltimes_\alpha G_2$ is a nilpotent Lie group.

Example 2.3 For an arbitrary Lie group G with center Z, let us specialize Definition 2.1 for $G_1 = G_2 := G$ and $\alpha : G \to \operatorname{Aut} G$, $g \mapsto \alpha_g$, where $\alpha_g(h) = ghg^{-1}$ whenever $g, h \in G$. Then the corresponding semidirect product will always be denoted simply by $G \ltimes G$ and has the following properties:

(1) If G is nilpotent, then so is $G \ltimes G$.
(2) The Lie algebra of $G \ltimes G$ is $\mathfrak{g} \ltimes_{\mathrm{ad}_\mathfrak{g}} \mathfrak{g}$, which will be denoted simply by $\mathfrak{g} \ltimes \mathfrak{g}$, and the center of $G \ltimes G$ is $Z \times Z$.
(3) The exponential map of the Lie group $G \ltimes G$ is given by

$$\exp_{G \ltimes G}(X, Y) = \big(\exp_G X, \exp_G(-X) \exp_G(X + Y)\big)$$

for every $(X, Y) \in \mathfrak{g} \ltimes_{\mathrm{ad}_\mathfrak{g}} \mathfrak{g}$.

(4) The mapping

$$\mu : G \ltimes G \to G \times G, \qquad (g, h) \mapsto (gh, g)$$

is an isomorphism of Lie groups, and the corresponding isomorphism of Lie algebras is $\mathbf{L}(\mu) : \mathfrak{g} \ltimes \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$, $(X, Y) \mapsto (X + Y, X)$.

In fact, property (1) follows by Remark 2.2. Property (2) is a consequence of the fact that $\dot{\alpha} = \mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \to \mathrm{Der}\,\mathfrak{g}$ along with (2.1).

To prove property (3), note that the mapping $\Pi : G \ltimes G \to G$, $(g_1, g_2) \to g_1 g_2$ is a homomorphism of Lie groups, hence we have the commutative diagram

$$\begin{tikzcd}
\mathfrak{g} \ltimes \mathfrak{g} \arrow[r, "\mathbf{L}(\Pi)"] \arrow[d, "\exp_{G \ltimes G}"'] & \mathfrak{g} \arrow[d, "\exp_G"] \\
G \ltimes G \arrow[r, "\Pi"'] & G
\end{tikzcd}$$

where it is easy to see that the Lie algebra homomorphism $\mathbf{L}(\Pi) : \mathfrak{g} \ltimes \mathfrak{g} \to \mathfrak{g}$ is given by $(X, Y) \mapsto X + Y$. Now let $(X, Y) \in \mathfrak{g} \ltimes \mathfrak{g}$ arbitrary. It is clear that there exists $g \in G$ such that $\exp_{G \ltimes G}(X, Y) = (\exp_G X, g)$, and then the above commutative diagram shows that $\exp_G(X + Y) = \Pi(\exp_{G \ltimes G}(X, Y)) = \Pi(\exp_G X, g) = (\exp_G X)g$, whence $g = \exp_G(-X)\exp_G(X + Y)$.

Finally, property (4) follows by a straightforward computation.

2.2 Weyl-Pedersen Calculus for Unitary Irreducible Representations

Setting 2.4 Throughout the present section we shall use the following notation:

(1) Let G be a connected, simply connected, nilpotent Lie group with the Lie algebra $\mathfrak{g}$. Then the exponential map $\exp_G : \mathfrak{g} \to G$ is a diffeomorphism with the inverse denoted by $\log_G : G \to \mathfrak{g}$.
(2) We denote by $\mathfrak{g}^*$ the linear dual space to $\mathfrak{g}$ and by $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ the natural duality pairing.
(3) Let $\xi_0 \in \mathfrak{g}^*$ with the corresponding coadjoint orbit $\mathcal{O} := \mathrm{Ad}_G^*(G)\xi_0 \subseteq \mathfrak{g}^*$.
(4) The *isotropy group* at ξ_0 is $G_{\xi_0} := \{g \in G \mid \mathrm{Ad}_G^*(g)\xi_0 = \xi_0\}$ with the corresponding *isotropy Lie algebra* $\mathfrak{g}_{\xi_0} = \{X \in \mathfrak{g} \mid \xi_0 \circ \mathrm{ad}_{\mathfrak{g}} X = 0\}$. The *center* $\mathfrak{z} := \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = \{0\}\}$ clearly satisfies $\mathfrak{z} \subseteq \mathfrak{g}_{\xi_0}$.
(5) Let $n := \dim \mathfrak{g}$ and fix a sequence of ideals in $\mathfrak{g}$,

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

such that $\dim(\mathfrak{g}_j/\mathfrak{g}_{j-1}) = 1$ and $[\mathfrak{g}, \mathfrak{g}_j] \subseteq \mathfrak{g}_{j-1}$ for $j = 1, \ldots, n$.
(6) Pick any $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$ for $j = 1, \ldots, n$, so that the set $\{X_1, \ldots, X_n\}$ will be a *Jordan-Hölder basis* in $\mathfrak{g}$.
(7) The set of *jump indices* of the coadjoint orbit $\mathcal{O}$ with respect to the above Jordan-Hölder basis is $e := \{j \in \{1, \ldots, n\} \mid \mathfrak{g}_j \not\subseteq \mathfrak{g}_{j-1} + \mathfrak{g}_{\xi_0}\}$ and does not depend on the

choice of $\xi_0 \in \mathcal{O}$ (see also Proposition 2.4.1 in [28]). The corresponding *predual of the coadjoint orbit* $\mathcal{O}$ is

$$\mathfrak{g}_e := \operatorname{span}\{X_j \mid j \in e\} \subseteq \mathfrak{g}.$$

We shall denote $e = \{j_1, \dots, j_d\}$ with $1 \le j_1 < \cdots < j_d \le n$.

(8) We shall always consider $\mathcal{O}$ endowed with its canonical Liouville measure (see for instance the remark after the statement of the theorem in Sect. 6, Chap. II, Part 2 in [32]).

(9) Let $\pi : G \to \mathcal{B}(\mathcal{H})$ be a fixed unitary irreducible representation associated with the coadjoint orbit $\mathcal{O}$ by Kirillov's theorem [23].

Remark 2.5 The space of *smooth vectors* $\mathcal{H}_\infty := \{v \in \mathcal{H} \mid \pi(\cdot)v \in \mathcal{C}^\infty(G, \mathcal{H})\}$ is a Fréchet space in a natural way and is a dense linear subspace of $\mathcal{H}$ which is invariant under the unitary operator $\pi(g)$ for every $g \in G$. The *derivate representation* $\mathrm{d}\pi : \mathfrak{g} \to \operatorname{End}(\mathcal{H}_\infty)$ is a homomorphism of Lie algebras defined by

$$(\forall X \in \mathfrak{g},\ v \in \mathcal{H}_\infty) \quad \mathrm{d}\pi(X)v = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \pi\big(\exp_G(tX)\big)v.$$

We denote by $\mathcal{H}_{-\infty}$ the space of all continuous antilinear functionals on $\mathcal{H}_\infty$ and the corresponding pairing will be denoted by $(\cdot \mid \cdot) : \mathcal{H}_{-\infty} \times \mathcal{H}_\infty \to \mathbb{C}$ just as the scalar product in $\mathcal{H}$, since they agree on $\mathcal{H}_\infty \times \mathcal{H}_\infty$ if we think of the natural inclusions $\mathcal{H}_\infty \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{-\infty}$. (See for instance [6] for more details.)

Remark 2.6 We now recall a few facts from Sect. 1.2 in [31] for later use. Let us denote by $\mathfrak{S}_p(\mathcal{H})$ the Schatten ideals of operators on $\mathcal{H}$ for $1 \le p \le \infty$. Consider the unitary representation $\pi \otimes \bar{\pi} : G \times G \to \mathcal{B}(\mathfrak{S}_2(\mathcal{H}))$ defined by

$$(\forall g_1, g_2 \in G)\ \big(\forall T \in \mathfrak{S}_2(\mathcal{H})\big) \quad (\pi \otimes \bar{\pi})(g_1, g_2)T = \pi(g_1)T\pi(g_2)^{-1}.$$

It is well-known that $\pi \otimes \bar{\pi}$ is strongly continuous. (See for instance Proposition 4.1.2.4 in [35].) The corresponding space of smooth vectors is denoted by $\mathcal{B}(\mathcal{H})_\infty$ and is called the space of *smooth operators* for the representation π. One can prove that actually $\mathcal{B}(\mathcal{H})_\infty \subseteq \mathfrak{S}_1(\mathcal{H})$. In fact, one may also define similar representations, with $\mathfrak{S}_2(\mathcal{H})$ replaced by a more general norm ideal of compact operators, and find that the corresponding space of smooth vectors is again $\mathcal{B}(\mathcal{H})_\infty$. (See [3].)

For an alternative description of $\mathcal{B}(\mathcal{H})_\infty$ let $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{g}$ with the corresponding universal associative enveloping algebra $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})$. Then the aforementioned homomorphism of Lie algebras $\mathrm{d}\pi$ has a unique extension to a homomorphism of unital associative algebras $\mathrm{d}\pi : \mathrm{U}(\mathfrak{g}_{\mathbb{C}}) \to \operatorname{End}(\mathcal{H}_\infty)$. One can prove that for $T \in \mathcal{B}(\mathcal{H})$ we have $T \in \mathcal{B}(\mathcal{H})_\infty$ if and only if $T(\mathcal{H}) + T^*(\mathcal{H}) \subseteq \mathcal{H}_\infty$ and $\mathrm{d}\pi(u)T, \mathrm{d}\pi(u)T^* \in \mathcal{B}(\mathcal{H})$ for every $u \in \mathrm{U}(\mathfrak{g}_{\mathbb{C}})$.

Since $\{(\cdot \mid f_1)f_2 \mid f_1, f_2 \in \mathcal{H}_\infty\} \subseteq \mathcal{B}(\mathcal{H})_\infty \subseteq \mathfrak{S}_1(\mathcal{H})$ and $\mathcal{H}_\infty$ is dense in $\mathcal{H}$, we get continuous inclusion maps

$$\mathcal{B}(\mathcal{H})_\infty \hookrightarrow \mathfrak{S}_1(\mathcal{H}) \hookrightarrow \mathcal{B}(\mathcal{H}) \hookrightarrow \mathcal{B}(\mathcal{H})^*_\infty, \tag{2.3}$$

where the latter mapping is constructed by using the well-known isomorphism $(\mathfrak{S}_1(\mathcal{H}))^* \simeq \mathcal{B}(\mathcal{H})$ given by the usual semifinite trace on $\mathcal{B}(\mathcal{H})$.

Definition 2.7 The Fourier transform $\mathcal{S}(\mathcal{O}) \to \mathcal{S}(\mathfrak{g}_e)$, $a \mapsto \widehat{a}$, defined by

$$\widehat{a}(X) = \int_{\mathcal{O}} \mathrm{e}^{-\mathrm{i}\langle \xi, X\rangle} a(\xi)\,\mathrm{d}\xi$$

is an isomorphism of Fréchet spaces. The Lebesgue measure on $\mathfrak{g}_e$ can be normalized such that the Fourier transform extends to a unitary operator

$$L^2(\mathcal{O}) \to L^2(\mathfrak{g}_e), \qquad a \mapsto \widehat{a},$$

and its inverse is defined by the usual formula (see Lemma 4.1.1 in [31]). We shall always consider the predual $\mathfrak{g}_e$ endowed with this normalized measure.

If $f \in \mathcal{H}_{-\infty}$ and $\phi \in \mathcal{H}_\infty$, or $f, \phi \in \mathcal{H}$, then we define the corresponding *ambiguity function*

$$\mathcal{A}(f, \phi) = \mathcal{A}_\phi f : \mathfrak{g}_e \to \mathbb{C}, \qquad (\mathcal{A}_\phi f)(X) = \big(f \mid \pi(\exp_G X)\phi\big).$$

For $\phi \in \mathcal{H}_{-\infty}$ and $f \in \mathcal{H}_\infty$ we also define $(\mathcal{A}_\phi f)(X) = \overline{(\phi \mid \pi(\exp_G(-X))f)}$ whenever $X \in \mathfrak{g}_e$.

It follows by Proposition 2.8(1) below that if $f, \phi \in \mathcal{H}$, then $\mathcal{A}_\phi f \in L^2(\mathfrak{g}_e)$, so we can use the aforementioned Fourier transform to define the corresponding *cross-Wigner distribution* $\mathcal{W}(f, \phi) \in L^2(\mathcal{O})$ such that $\widehat{\mathcal{W}(f, \phi)} := \mathcal{A}_\phi f$.

The second equality in Proposition 2.8(1) below could be referred to as the *Moyal identity* since that classical identity (see for instance [17]) is recovered in the special case when G is a simply connected Heisenberg group.

Proposition 2.8 *The following assertions hold*:

(1) *If* $\phi, f \in \mathcal{H}$, *then* $\mathcal{A}_\phi f \in L^2(\mathfrak{g}_e)$. *We have*

$$\begin{aligned}(\mathcal{A}_{\phi_1} f_1 \mid \mathcal{A}_{\phi_2} f_2)_{L^2(\mathfrak{g}_e)} &= (f_1 \mid f_2)_{\mathcal{H}} \cdot (\phi_2 \mid \phi_1)_{\mathcal{H}} \\ &= \big(\mathcal{W}(f_1, \phi_1) \mid \mathcal{W}(f_2, \phi_2)\big)_{L^2(\mathcal{O})}\end{aligned} \tag{2.4}$$

for arbitrary $\phi_1, \phi_2, f_1, f_2 \in \mathcal{H}$.

(2) *If* $\phi_0 \in \mathcal{H}$ *with* $\|\phi_0\| = 1$, *then the operator* $\mathcal{A}_{\phi_0} : \mathcal{H} \to L^2(\mathfrak{g}_e)$, $f \mapsto \mathcal{A}_{\phi_0} f$, *is an isometry and we have*

$$\int_{\mathfrak{g}_e} (\mathcal{A}_{\phi_0} f)(X) \cdot \pi(\exp_G X)\phi\,\mathrm{d}X = (\phi \mid \phi_0) f$$

for every $\phi \in \mathcal{H}$ *and* $f \in \mathcal{H}$. *In particular,*

$$\int_{\mathfrak{g}_e} (\mathcal{A}_{\phi_0} f)(X) \cdot \pi(\exp_G X)\phi_0\,\mathrm{d}X = f$$

for arbitrary $f \in \mathcal{H}$.

Proof (1) We first prove that (2.4) holds for $\phi_1, \phi_2, f_1, f_2 \in \mathcal{H}_\infty$. Since $\mathcal{B}(\mathcal{H}_\infty)$ is contained in the ideal $\mathfrak{S}_1(\mathcal{H})$ of trace-class operators, it makes sense to define

$$\big(\forall A \in \mathcal{B}(\mathcal{H})_\infty\big) \quad f_\pi^A : G \to \mathbb{C}, \quad f_\pi^A(x) = \mathrm{Tr}\big(\pi(x)A\big).$$

It follows by Theorem 2.2.7 in [31] that for the suitably normalized Lebesgue measure on $\mathfrak{g}_e$ we have for every $A, B \in \mathcal{B}(\mathcal{H})_\infty$,

$$\int_{\mathfrak{g}_e} f_\pi^A(\exp_G X)\overline{f_\pi^B(\exp_G X)}\,\mathrm{d}X = \mathrm{Tr}\big(AB^*\big). \tag{2.5}$$

We now denote

$$(\forall f, \phi \in \mathcal{H}) \quad A_{f,\phi} = (\cdot \mid \phi) f \in \mathcal{B}(\mathcal{H})$$

and recall that for arbitrary $f, f_1, f_2, \phi, \phi_1, \phi_2 \in \mathcal{H}$ we have

$$A_{f,\phi}^* = A_{\phi,f}, \qquad \mathrm{Tr}(A_{f,\phi}) = (f \mid \phi), \quad \text{and}$$
$$A_{f_1,\phi_1}A_{f_2,\phi_2} = A_{f_1,(\phi_1 \mid f_2)\phi_2} = A_{(f_2 \mid \phi_1) f_1, \phi_2}.$$

It then easily follows that if $f, \phi \in \mathcal{H}_\infty$, then $A_{f,\phi} \in \mathcal{B}(\mathcal{H})_\infty$ and for arbitrary $X \in \mathfrak{g}_e$ we have

$$\begin{aligned} f_\pi^{A_{f,\phi}}(\exp_G X) &= \mathrm{Tr}\big(\pi(\exp_G X)A_{f,\phi}\big) = \mathrm{Tr}(A_{\pi(\exp_G X)f,\phi}) = \big(\pi(\exp_G X)f \mid \phi\big) \\ &= \big(f \mid \pi\big(\exp_G(-X)\big)\phi\big), \end{aligned}$$

whence

$$(\forall X \in \mathfrak{g}_e) \quad f_\pi^{A_{f,\phi}}(\exp_G X) = (\mathcal{A}_\phi f)(-X). \tag{2.6}$$

Now, by using (2.5) for $A := A_{f_1,\phi_1}$ and $B := A_{f_2,\phi_2}$, we get

$$\begin{aligned} (\mathcal{A}_{\phi_1} f_1 \mid \mathcal{A}_{\phi_2} f_2)_{L^2(\mathfrak{g}_e)} &= \mathrm{Tr}\big(A_{f_1,\phi_1}A_{f_2,\phi_2}^*\big) = \mathrm{Tr}(A_{f_1,\phi_1}A_{\phi_2,f_2}) = \mathrm{Tr}(A_{f_1,(\phi_1 \mid \phi_2)f_2}) \\ &= \big(f_1 \mid (\phi_1 \mid \phi_2) f_2\big) = (f_1 \mid f_2)_{\mathcal{H}} \cdot (\phi_2 \mid \phi_1)_{\mathcal{H}}. \end{aligned}$$

The second part of (2.4) then follows since the Fourier transform $L^2(\mathcal{O}) \to L^2(\mathfrak{g}_e)$ is a unitary operator, as we already mentioned in Definition 2.7.

The extension of (2.4) from $\mathcal{H}_\infty$ to $\mathcal{H}$ proceeds by a density argument. First note that by (2.4) for $\phi_1 = \phi_2 =: \phi \in \mathcal{H}_\infty$ and $f_1 = f_2 =: f \in \mathcal{H}_\infty$ we get $\|\mathcal{A}_\phi f\| = \|\phi\| \cdot \|f\|$. Since $\mathcal{H}_\infty$ is dense in $\mathcal{H}$, it then follows that the sesquilinear mapping $\mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathcal{H}$, $(f, \phi) \mapsto \mathcal{A}_\phi f$ extends uniquely to a mapping $\mathcal{H} \times \mathcal{H} \to \mathcal{H}$ satisfying

$$(\forall f, \phi \in \mathcal{H}) \quad \|\mathcal{A}_\phi f\| = \|\phi\| \cdot \|f\|. \tag{2.7}$$

Now the first part of (2.4) follows as a polarization of (2.7), and then the second part follows by using the Fourier transform $L^2(\mathcal{O}) \to L^2(\mathfrak{g}_e)$ as above.

(2) It follows at once by Assertion (1) that the operator $\mathcal{A}_{\phi_0}\colon \mathcal{H} \to L^2(\mathfrak{g}_e)$ is an isometry if $\|\phi_0\| = 1$. The other properties then follow immediately; see for instance Proposition 2.11 in [15]. □

We now draw some useful consequences of Proposition 2.8. We emphasize that Assertion (3) in the following corollary in the special case of square-integrable representations reduces to a theorem of [7, 9]. One thus recovers Theorem 2.3 in [19] in the case of the Schrödinger representation of the Heisenberg group.

Corollary 2.9 *If $\phi_0 \in \mathcal{H}_\infty$ with $\|\phi_0\| = 1$, then the following assertions hold*:

(1) *For every $f \in \mathcal{H}_{-\infty}$ we have*

$$\int_{\mathfrak{g}_e} (\mathcal{A}_{\phi_0} f)(X) \cdot \pi(\exp_G X)\phi_0 \,\mathrm{d}X = f \tag{2.8}$$

where the integral is convergent in the weak-topology of $\mathcal{H}_{-\infty}$.*

(2) *If $f \in \mathcal{H}_\infty$, then the above integral converges in the Fréchet topology of $\mathcal{H}_\infty$.*

(3) *If $f \in \mathcal{H}_{-\infty}$, then we have $f \in \mathcal{H}_\infty$ if and only if $\mathcal{A}_{\phi_0} f \in \mathcal{S}(\mathfrak{g}_e)$.*

Proof If $f \in \mathcal{H}_{-\infty}$, we have to prove that $\int_{\mathfrak{g}_e} (\mathcal{A}_{\phi_0} f)(X) \cdot (\pi(\exp_G X)\phi_0 \mid \phi)\,\mathrm{d}X = (f \mid \phi)$, for every $\phi \in \mathcal{H}_\infty$, that is,

$$\int_{\mathfrak{g}_e} \bigl(f \mid \pi(\exp_G X)\phi_0\bigr) \cdot \bigl(\pi(\exp_G X)\phi_0 \mid \phi\bigr)\,\mathrm{d}X = (f \mid \phi).$$

Since $(f \mid \cdot)$ is an antilinear continuous functional, the above equation will follow as soon as we have proved that for $\phi \in \mathcal{H}_\infty$ we have

$$\int_{\mathfrak{g}_e} \bigl(\phi \mid \pi(\exp_G X)\phi_0\bigr)\pi(\exp_G X)\phi_0 \,\mathrm{d}X = \phi$$

with an integral that converges in the topology of $\mathcal{H}_\infty$. Note that this is precisely Assertion (2). To prove it, we just have to use Proposition 2.8(2) along with the fact that for $\phi, \phi_0 \in \mathcal{H}_\infty$ the function $X \mapsto (\phi \mid \pi(\exp_G X)\phi_0) = (\mathcal{A}_{\phi_0}\phi)(X)$ belongs to $\mathcal{S}(\mathfrak{g}_e)$ (see Theorem 2.2.6 in [31]) while the function $X \mapsto \pi(\exp_G X)\phi_0$ and all its partial derivatives have polynomial growth.

For Assertion (3), we have just noted that if $f \in \mathcal{H}_\infty$ then $\mathcal{A}_{\phi_0} f \in \mathcal{S}(\mathfrak{g}_e)$ as a direct consequence of Theorem 2.2.6 in [31]. Conversely, if $f \in \mathcal{H}_{-\infty}$ has the property $\mathcal{A}_{\phi_0} f \in \mathcal{S}(\mathfrak{g}_e)$, then the fact that all the partial derivatives of $X \mapsto \pi(\exp_G X)\phi_0$ have polynomial growth implies at once that the integral in (2.8) is convergent in the Fréchet space $\mathcal{H}_\infty$, hence Assertion (1) shows that actually $f \in \mathcal{H}_\infty$. □

Definition 2.10 The *Weyl-Pedersen calculus* $\mathrm{Op}^\pi(\cdot)$ for the unitary representation π is defined for every $a \in \mathcal{S}(\mathcal{O})$ by

$$\mathrm{Op}^\pi(a) = \int_{\mathfrak{g}_e} \widehat{a}(X)\pi(\exp_G X)\,\mathrm{d}X \in \mathcal{B}(\mathcal{H}). \tag{2.9}$$

This definition can be extended to an arbitrary tempered distribution $a \in \mathcal{S}'(\mathcal{O})$ by using Theorems 4.1.4 and 2.2.7 in [31] to define an unbounded operator $\mathrm{Op}^{\pi}(a)$ such that

$$\big(\forall b \in \mathcal{S}(\mathcal{O})\big) \quad \mathrm{Tr}\big(\mathrm{Op}^{\pi}(a)\,\mathrm{Op}^{\pi}(b)\big) = \langle a, b\rangle, \tag{2.10}$$

where we recall that $\langle\cdot,\cdot\rangle : \mathcal{S}'(\mathcal{O}) \times \mathcal{S}(\mathcal{O}) \to \mathbb{C}$ stands for the usual pairing between the tempered distributions and the Schwartz functions. We say that $a \in \mathcal{S}'(\mathcal{O})$ is the *symbol* of the operator $\mathrm{Op}^{\pi}(a)$.

We now record some basic properties of the Weyl-Pedersen calculus constructed in Definition 2.10. These are actually direct consequences of Proposition 2.8(1).

Corollary 2.11 *The following assertions hold*:

(1) *For each $a \in \mathcal{S}(\mathcal{O})$ we have*

$$\big(\mathrm{Op}^{\pi}(a)\phi \mid f\big)_{\mathcal{H}} = (\widehat{a} \mid \mathcal{A}_{\phi} f)_{L^2(\mathfrak{g}_e)} = \big(a \mid \mathcal{W}(f, \phi)\big)_{L^2(\mathcal{O})}$$

whenever $\phi, f \in \mathcal{H}$. Similar equalities hold if $a \in \mathcal{S}'(\mathcal{O})$ and $\phi, f \in \mathcal{H}_{\infty}$.

(2) *If $\phi_1, \phi_2 \in \mathcal{H}_{\infty}$ and $a := \mathcal{W}(\phi_1, \phi_2) \in \mathcal{S}(\mathcal{O})$, then $\mathrm{Op}^{\pi}(a)$ is a rank-one operator, namely $\mathrm{Op}^{\pi}(a) = (\cdot \mid \phi_2)\phi_1$.*

Proof Assertion (1) is a consequence of (2.9) along with Definition 2.7. Then Assertion (2) follows by Assertion (1) along with Proposition 2.8(1). In fact,

$$\begin{aligned}\big(\mathrm{Op}^{\pi}\big(\mathcal{W}(\phi_1, \phi_2)\big) f \mid \phi\big) &= \big(\mathcal{W}(\phi_1, \phi_2) \mid \mathcal{W}(\phi, f)\big) = (\phi_1 \mid \phi) \cdot (f \mid \phi_2) \\ &= \big((f \mid \phi_2)\phi_1 \mid \phi\big)\end{aligned}$$

for arbitrary $\phi \in \mathcal{H}$. □

Remark 2.12 We can define the cross-Wigner distribution $\mathcal{W}(f_1, f_2) \in \mathcal{S}'(\mathcal{O})$ for arbitrary $f_1, f_2 \in \mathcal{H}_{-\infty}$ as follows. An application of Theorem 1.3(b) in [6] shows that if $A \in \mathcal{B}(\mathcal{H})_{\infty}$ and $f \in \mathcal{H}_{-\infty}$, then $Af \in \mathcal{H}_{\infty}$, in the sense that there exists a smooth vector denoted Af such that for every $\phi \in \mathcal{H}_{\infty}$ we have $(f \mid A^*\phi) = (Af \mid \phi)$. Moreover, we thus get a continuous linear map $A : \mathcal{H}_{-\infty} \to \mathcal{H}_{\infty}$ whose restriction to $\mathcal{H}$ is the original operator $A \in \mathcal{B}(\mathcal{H})_{\infty}$. Then for $f_1, f_2 \in \mathcal{H}_{-\infty}$ we can define the continuous antilinear functional

$$T_{f_1, f_2} : \mathcal{B}(\mathcal{H})_{\infty} \to \mathbb{C}, \qquad T_{f_1, f_2}(A) := (f_1 \mid Af_2).$$

That is, $T_{f_1, f_2} \in \mathcal{B}(\mathcal{H})^*_{\infty}$, and then Theorem 4.1.4(5) in [31] shows that there exists a unique distribution $a_{f_1, f_2} \in \mathcal{S}'(\mathcal{O})$ such that $\mathrm{Op}^{\pi}(a_{f_1, f_2}) = T_{f_1, f_2}$. Now define

$$\mathcal{W}(f_1, f_2) := a_{f_1, f_2}.$$

We can consider the rank-one operator $S_{f_1, f_2} := (\cdot \mid f_2) f_1 : \mathcal{H}_{\infty} \to \mathcal{H}_{-\infty}$ and for arbitrary $A \in \mathcal{B}(\mathcal{H})_{\infty}$ thought of as a continuous linear map $A : \mathcal{H}_{-\infty} \to \mathcal{H}_{\infty}$ as above we have

$$\mathrm{Tr}(S_{f_1,f_2}A) = (f_1 \mid Af_2) = T_{f_1,f_2}(A).$$

Thus the trace duality pairing allows us to identify the functional $T_{f_1,f_2} \in \mathcal{B}(\mathcal{H})^*_\infty$ with the rank-one operator $(\cdot \mid f_2)f_1$, and then we can write

$$(\forall f_1, f_2 \in \mathcal{H}_{-\infty}) \quad \mathrm{Op}^\pi\big(\mathcal{W}(f_1, f_2)\big) = (\cdot \mid f_2)f_1. \tag{2.11}$$

In particular, it follows that the above extension of the cross-Wigner distribution to a mapping $\mathcal{W}(\cdot,\cdot): \mathcal{H}_{-\infty} \times \mathcal{H}_{-\infty} \to \mathcal{S}'(\mathcal{O})$ allows us to generalize the assertion of Corollary 2.11(2) to arbitrary $\phi_1, \phi_2 \in \mathcal{H}_{-\infty}$.

Definition 2.13 Recall from Theorem 4.1.4(5) in [31] that the Weyl-Pedersen calculus $\mathrm{Op}^\pi : \mathcal{S}'(\mathcal{O}) \to \mathcal{B}(\mathcal{H})^*_\infty$ is a linear isomorphism and a weak*-homeomorphism. We introduce the linear space

$$\mathcal{S}^0(\mathcal{O}) := \big\{a \in \mathcal{S}'(\mathcal{O}) \mid \mathrm{Op}^\pi(a) \in \mathcal{B}(\mathcal{H})\big\}$$

(see (2.3)). Then the mapping Op^π induces a linear isomorphism $\mathcal{S}^0(\mathcal{O}) \to \mathcal{B}(\mathcal{H})$, hence there exists an uniquely defined bilinear associative *Moyal product*

$$\mathcal{S}^0(\mathcal{O}) \times \mathcal{S}^0(\mathcal{O}) \to \mathcal{S}^0(\mathcal{O}), \qquad (a, b) \mapsto a\#b$$

such that

$$\big(\forall a, b \in \mathcal{S}^0(\mathcal{O})\big) \quad \mathrm{Op}^\pi(a\#b) = \mathrm{Op}^\pi(a)\,\mathrm{Op}^\pi(b).$$

The space of distributions $\mathcal{S}^0(\mathcal{O})$ is thus made into a W^*-algebra such that the mapping $\mathcal{S}^0(\mathcal{O}) \to \mathcal{B}(\mathcal{H})$, $a \mapsto \mathrm{Op}^\pi(a)$ is a $*$-isomorphism.

With Definition 2.13 at hand, we can say that one of the main problems addressed in the present paper is to describe large classes of distributions belonging in the space $\mathcal{S}^0(\mathcal{O})$.

Example 2.14 Here are some examples of distributions in $\mathcal{S}^0(\mathcal{O})$ which are already available.

(1) It follows at once by (2.9) and (2.10) that

$$\big\{a \in \mathcal{S}'(\mathcal{O}) \mid \widehat{a} \in L^1(\mathfrak{g}_e)\big\} \subseteq \mathcal{S}^0(\mathcal{O}).$$

(2) The Schwartz space $\mathcal{S}(\mathcal{O})$ is a $*$-subalgebra of $\mathcal{S}^0(\mathcal{O})$ and the mapping $\mathrm{Op}^\pi : \mathcal{S}(\mathcal{O}) \to \mathcal{B}(\mathcal{H})_\infty$ is an algebra $*$-isomorphism by Theorem 4.1.4 in [31].
(3) The space $L^2(\mathcal{O})$ is a $*$-subalgebra of $\mathcal{S}^0(\mathcal{O})$, and $\mathrm{Op}^\pi : L^2(\mathcal{O}) \to \mathfrak{S}_2(\mathcal{H})$ is a unitary operator and an algebra $*$-isomorphism as an easy consequence of Theorem 4.1.4 in [31]; see also [24].
(4) For every $Y \in \mathfrak{g}$ we have $\mathrm{e}^{\mathrm{i}\langle\cdot,Y\rangle} \in \mathcal{S}^0(\mathcal{O})$ since it follows at once by (2.9) and (2.10) that $\mathrm{Op}^\pi(\mathrm{e}^{\mathrm{i}\langle\cdot,Y\rangle}) = \pi(\exp_G Y)$.

See also Corollary 2.26 for the important example $M^{\infty,1}_\Phi(\pi^\#) \hookrightarrow \mathcal{S}^0(\mathcal{O})$.

2.3 Modulation Spaces

Definition 2.15 Let $\phi \in \mathcal{H}_\infty \setminus \{0\}$ be fixed and assume that we have a direct sum decomposition $\mathfrak{g}_e = \mathfrak{g}_e^1 \dotplus \mathfrak{g}_e^2$.

Then let $1 \le r, s \le \infty$ and for arbitrary $f \in \mathcal{H}_{-\infty}$ define

$$\|f\|_{M_\phi^{r,s}(\pi)} = \left(\int_{\mathfrak{g}_e^2}\left(\int_{\mathfrak{g}_e^1} |(\mathcal{A}_\phi f)(X_1, X_2)|^r \,\mathrm{d}X_1\right)^{s/r} \mathrm{d}X_2\right)^{1/s} \in [0,\infty]$$

with the usual conventions if r or s is infinite. Then we call the space

$$M_\phi^{r,s}(\pi) := \{f \in \mathcal{H}_{-\infty} \mid \|f\|_{M_\phi^{r,s}(\pi)} < \infty\}$$

a *modulation space* for the irreducible unitary representation $\pi : G \to \mathcal{B}(\mathcal{H})$ with respect to the decomposition $\mathfrak{g}_e \simeq \mathfrak{g}_e^1 \times \mathfrak{g}_e^2$ and the *window vector* $\phi \in \mathcal{H}_\infty \setminus \{0\}$.

Remark 2.16 Assume the setting of Definition 2.15 and recall the *mixed-norm space* $L^{r,s}(\mathfrak{g}_e^1 \times \mathfrak{g}_e^2)$ consisting of the (equivalence classes of) Lebesgue measurable functions $\Theta : \mathfrak{g}_e^1 \times \mathfrak{g}_e^2 \to \mathbb{C}$ such that

$$\|\Theta\|_{L^{r,s}} := \left(\int_{\mathfrak{g}_e^2}\left(\int_{\mathfrak{g}_e^1} |\Theta(X_1, X_2)|^r \,\mathrm{d}X_1\right)^{s/r} \mathrm{d}X_2\right)^{1/s} < \infty$$

(cf. [17]). It is clear that $M_\phi^{r,s}(\pi) = \{f \in \mathcal{H}_{-\infty} \mid \mathcal{A}_\phi f \in L^{r,s}(\mathfrak{g}_e^1 \times \mathfrak{g}_e^2)\}$.

Example 2.17 For any choice of $\phi \in \mathcal{H}_\infty \setminus \{0\}$ in Definition 2.15 we have

$$M_\phi^{2,2}(\pi) = \mathcal{H}.$$

Indeed, this equality holds since $\|\mathcal{A}_\phi f\|_{L^2(\mathfrak{g}_e)} = \|\phi\| \cdot \|f\|$ for every $f \in \mathcal{H}$ (see (2.7) in the proof of Proposition 2.8 above).

2.4 Continuity of Weyl-Pedersen Calculus on Modulation Spaces

In the following lemma we use notation introduced in Example 2.3(4) and Remark 2.6.

Lemma 2.18 *Let G be any Lie group with a unitary irreducible representation $\pi : G \to \mathcal{B}(\mathcal{H})$ and define*

$$\pi^{\ltimes} : G \ltimes G \to \mathcal{B}(\mathfrak{S}_2(\mathcal{H})), \qquad \pi^{\ltimes}(g,h)T = \pi(gh)T\pi(g)^{-1}.$$

Then the following assertions hold:

(1) *The diagram*

$$\begin{array}{ccc} G \ltimes G & \xrightarrow{\pi^{\ltimes}} & \mathcal{B}(\mathfrak{S}_2(\mathcal{H})) \\ \downarrow{\scriptstyle \mu} & \nearrow_{\pi\otimes\bar{\pi}} & \\ G \times G & & \end{array}$$

is commutative and $\pi^{\ltimes}$ is a unitary irreducible representation of $G \ltimes G$.

(2) *The space of smooth vectors for the representation $\pi^{\ltimes}$ is $\mathcal{B}(\mathcal{H})_\infty$.*

(3) *Let us denote by $\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathfrak{g}$ and define*

$$\bar{X}_j = \begin{cases} (X_j, 0) & \text{for } j = 1, \dots, n, \\ (X_{j-n}, X_{j-n}) & \text{for } j = n+1, \dots, 2n. \end{cases}$$

Then $\bar{X}_1, \dots, \bar{X}_{2n}$ is a Jordan-Hölder basis in $\bar{\mathfrak{g}}$ and the corresponding predual for the coadjoint orbit $\bar{\mathcal{O}} \subseteq \bar{\mathfrak{g}}^*$ associated with the representation $\pi^{\ltimes}$ is

$$\bar{\mathfrak{g}}_{\bar{e}} = \mathfrak{g}_e \times \mathfrak{g}_e \subseteq \bar{\mathfrak{g}},$$

where $\bar{e}$ is the set of jump indices for $\bar{\mathcal{O}}$.

Proof (1) It is clear that the diagram is commutative, and then the mapping $\pi^{\ltimes}$ is a representation since $\pi \otimes \bar{\pi}$ is a representation and $\mu : G \ltimes G \to G \times G$ is a group isomorphism. It is well-known that the representation $\pi \otimes \bar{\pi}$ is irreducible, hence $\pi^{\ltimes}$ is irreducible as well. For the sake of completeness, we recall the corresponding reasoning. Let arbitrary $\mathcal{A} \in \mathcal{B}(\mathfrak{S}_2(\mathcal{H}))$ satisfying

$$\big(\forall (g, h) \in G \ltimes G\big) \quad \mathcal{A}\pi^{\ltimes}(g, h) = \pi^{\ltimes}(g, h)\mathcal{A}. \tag{2.12}$$

We have to show that $\mathcal{A}$ is a scalar multiple of the identity operator on $\mathfrak{S}_2(\mathcal{H})$. For that purpose, let us define the operators $L_B, R_B : \mathfrak{S}_2(\mathcal{H}) \to \mathfrak{S}_2(\mathcal{H})$ by $L_B X = BX$ and $R_B X = XB$ for $X, B \in \mathcal{B}(\mathcal{H})$. Note that if $h \in G$, then $\pi^{\ltimes}(\mathbf{1}, h) = L_{\pi(h)}$. It then follows by (2.12) that $\mathcal{A}L_{\pi(h)} = L_{\pi(h)}\mathcal{A}$ for every $h \in G$. On the other hand, the representation π is irreducible, the linear space $\mathrm{span}\{\pi(h) \mid h \in G\}$ is dense in $\mathcal{B}(\mathcal{H})$ in the strong operator topology, and then it easily follows that $\mathcal{A}L_B = L_B\mathcal{A}$ for every $B \in \mathcal{B}(\mathcal{H})$. This property implies that there exists $A \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{A} = R_A$ (see for instance [33]). Now, by using (2.12) for $h = \mathbf{1}$, we get $\pi(g)X\pi(g)^{-1}A = \pi(g)XA\pi(g)^{-1}$ for every $g \in G$ and $X \in \mathcal{B}(\mathcal{H})$, which implies $A\pi(g) = \pi(g)A$ for arbitrary $g \in G$. Since π is an irreducible representation, it follows that A is a scalar multiple of the identity operator on $\mathcal{H}$, hence $\mathcal{A} = L_A$ is a scalar multiple of the identity operator on $\mathfrak{S}_2(\mathcal{H})$, as we wished for.

(2) This assertion follows since the space of smooth vectors for $\pi \otimes \bar{\pi}$ is $\mathcal{B}(\mathcal{H})_\infty$, as defined in Remark 2.6.

(3) It is easy to see that the sequence

$$(0, X_1), \dots, (0, X_n), (X_1, 0), \dots, (X_n, 0)$$

is a Jordan-Hölder basis in the direct product $\mathfrak{g} \times \mathfrak{g}$, and the coadjoint orbit corresponding to the representation $\pi \otimes \bar{\pi} : G \times G \to \mathcal{B}(\mathfrak{S}_2(\mathcal{H}))$ is $\mathcal{O} \times \mathcal{O}$. (This follows for instance by the theorem in Sect. 6, Chap. II, Part 2 in [32].) Then the assertion follows by Example 2.3(4) along with the above Assertion (1). $\square$

In the following definition we use an idea similar to one used in [26].

Definition 2.19 Let G be a simply connected, nilpotent Lie group with a unitary irreducible representation $\pi : G \to \mathcal{B}(\mathcal{H})$. Assume that $\mathcal{O} \subseteq \mathfrak{g}^*$ is the coadjoint orbit associated with this representation and define

$$\begin{aligned} &\pi^{\#} : G \ltimes G \to \mathcal{B}\big(L^2(\mathcal{O})\big), \\ &\pi^{\#}(\exp_G X, \exp_G Y) f = \mathrm{e}^{\mathrm{i}\langle\cdot, X\rangle} \# \mathrm{e}^{\mathrm{i}\langle\cdot, Y\rangle} \# f \# \mathrm{e}^{-\mathrm{i}\langle\cdot, X\rangle}, \end{aligned}$$

where # is the Moyal product associated with π (see Definition 2.13). We note the following equivalent expression

$$(\forall X, Y \in \mathfrak{g}) \quad \pi^{\#}\big(\exp_{G \ltimes G}(X, Y)\big) f = \mathrm{e}^{\mathrm{i}\langle\cdot, X+Y\rangle} \# f \# \mathrm{e}^{-\mathrm{i}\langle\cdot, X\rangle} \tag{2.13}$$

which follows by Example 2.3(3). The corresponding *ambiguity function* is given by

$$\mathcal{A}^{\#}_{\Phi} F : \mathfrak{g}_e \times \mathfrak{g}_e \to \mathbb{C}, \qquad \big(\mathcal{A}^{\#}_{\Phi} F\big)(X, Y) = \big(F \mid \pi^{\#}\big(\exp_{G \ltimes G}(X, Y)\big)\Phi\big)$$

for Φ, $F \in L^2(\mathcal{O})$ or for a function $\Phi \in \mathcal{S}(\mathcal{O})$ and a continuous antilinear functional $F : \mathcal{S}(\mathcal{O}) \to \mathbb{C}$ denoted by $\Psi \mapsto (F \mid \Psi)$.

Remark 2.20 To explain the terminology of Definition 2.19, let us see that we really have to do with the ambiguity function of a unitary representation. To this end, recall the unitary operator $\mathrm{Op}^{\pi} : L^2(\mathcal{O}) \to \mathfrak{S}_2(\mathcal{H})$ (see e.g., Example 2.14(3)) and the representation $\pi^{\ltimes} : G \ltimes G \to \mathcal{B}(\mathfrak{S}_2(\mathcal{H}))$ from Lemma 2.18. It follows by Definition 2.13 and Example 2.14(4) that the unitary operator Op^{π} intertwines $\pi^{\#}$ and $\pi^{\ltimes}$, hence we get by Lemma 2.18 that $\pi^{\#}$ is also a unitary irreducible representation. It also follows that $\mathfrak{g}_e \times \mathfrak{g}_e \subseteq \mathfrak{g} \ltimes \mathfrak{g}$ is a predual to the coadjoint orbit $\mathcal{O}^{\#} \subseteq (\mathfrak{g} \ltimes \mathfrak{g})^*$ associated with the representation $\pi^{\#}$.

Let us note that the space of smooth vectors for the representation $\pi^{\#}$ is equal to $\mathcal{S}(\mathcal{O})$, as a consequence of Lemma 2.18(2), since $\mathrm{Op}^{\pi} : \mathcal{S}(\mathcal{O}) \to \mathcal{B}(\mathcal{H})_{\infty}$ is a linear isomorphism by Theorem 4.1.4 in [31].

The next statement points out the representation theoretic background of the computation carried out in the proof of Lemma 14.5.1 in [17].

Proposition 2.21 *Let G be a simply connected, nilpotent Lie group with a unitary irreducible representation $\pi : G \to \mathcal{B}(\mathcal{H})$. Pick any predual $\mathfrak{g}_e \subseteq \mathfrak{g}$ for the coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$ corresponding to the representation π. If either $\phi_1, \phi_2, f_1, f_2 \in \mathcal{H}$ or $\phi_1, \phi_2 \in \mathcal{H}_{\infty}$ and $f_1, f_2 \in \mathcal{H}_{-\infty}$, then*

$$(\forall X, Y \in \mathfrak{g}_e) \quad \mathcal{A}^{\#}_{\Phi}\big(\mathcal{W}(f_1, f_2)\big)(X, Y) = (\mathcal{A}_{\phi_1} f_1)(X + Y) \cdot \overline{(\mathcal{A}_{\phi_2} f_2)(X)},$$

where $\Phi := \mathcal{W}(\phi_1, \phi_2) \in L^2(\mathcal{O})$, *while* $\mathcal{W}(\cdot,\cdot)$ *and* $\mathcal{A}_{\phi_j} f_j : \mathfrak{g}_e \to \mathbb{C}$ *for* $j = 1, 2$ *are cross-Wigner distributions and ambiguity functions for the representation* π, *respectively.*

Proof If we denote $F = \mathcal{W}(f_1, f_2)$, then for arbitrary $X, Y \in \mathfrak{g}_e$ we have by Definition 2.19, Example 2.3(3), and Remark 2.20,

$$
\begin{aligned}
\big(\mathcal{A}^{\#}_{\Phi} F\big)(X,Y) &= \big(F \mid \pi^{\#}\big(\exp_{G\ltimes G}(X,Y)\big)\Phi\big)_{L^2(\mathcal{O})} \\
&= \big(F \mid \pi^{\#}\big(\exp_G X, (\exp_G X)^{-1}\exp_G(X+Y)\big)\Phi\big)_{L^2(\mathcal{O})} \\
&= \big(\mathrm{Op}^{\pi}(F) \mid \pi^{\ltimes}\big(\exp_G X, (\exp_G X)^{-1}\exp_G(X+Y)\big)\,\mathrm{Op}^{\pi}(\Phi)\big)_{\mathfrak{S}_2(\mathcal{H})} \\
&= \big(\mathrm{Op}^{\pi}(F) \mid \pi\big(\exp_G(X+Y)\big)\,\mathrm{Op}^{\pi}(\Phi)\pi(\exp_G X)^{-1}\big)_{\mathfrak{S}_2(\mathcal{H})}.
\end{aligned}
$$

On the other hand Remark 2.12 (particularly (2.11)) shows that

$$\mathrm{Op}^{\pi}(F) = (\cdot \mid f_2) f_1$$

and $\mathrm{Op}^{\pi}(\Phi) = (\cdot \mid \phi_2)\phi_1$, whence

$$\pi\big(\exp_G(X+Y)\big)\,\mathrm{Op}^{\pi}(\Phi)\pi(\exp_G X)^{-1} = \big(\cdot \mid \pi(\exp_G X)\phi_2\big)\pi\big(\exp_G(X+Y)\big)\phi_1.$$

Then the above computation leads to the formula

$$\big(\mathcal{A}^{\#}_{\Phi} F\big)(X,Y) = \big(\pi(\exp_G X)\phi_2 \mid f_2\big)\cdot\big(f_1 \mid \pi\big(\exp_G(X+Y)\big)\phi_1\big),$$

which is equivalent to the equation in the statement. □

We now prove a generalization of Theorem 4.1 in [34] to irreducible representations of nilpotent Lie groups.

Theorem 2.22 *Let* G *be a simply connected, nilpotent Lie group with a unitary irreducible representation* $\pi : G \to \mathcal{B}(\mathcal{H})$. *Let* $\mathcal{O}$ *be the corresponding coadjoint orbit, pick* $\phi_1, \phi_2 \in \mathcal{H}_\infty \setminus \{0\}$, *and denote* $\Phi = \mathcal{W}(\phi_1, \phi_2) \in \mathcal{S}(\mathcal{O})$. *Assume that* $\mathfrak{g}_e$ *is a predual to the coadjoint orbit* $\mathcal{O}$, *and let* $\mathfrak{g}_e = \mathfrak{g}_e^1 \dotplus \mathfrak{g}_e^2$ *be any direct sum decomposition.*

If $1 \le r \le s \le \infty$ *and* $r_1, r_2, s_1, s_2 \in [r, s]$ *satisfy* $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{r} + \frac{1}{s}$, *then the cross-Wigner distribution defines a continuous sesquilinear map*

$$\mathcal{W}(\cdot,\cdot) : M^{r_1,s_1}_{\phi_1}(\pi) \times M^{r_2,s_2}_{\phi_2}(\pi) \to M^{r,s}_{\Phi}\big(\pi^{\#}\big).$$

Proof Let $f_1, f_2 \in \mathcal{H}_{-\infty}$. We have, by Definition 2.15 for the representation $\pi^{\#}$ (see also Remark 2.20), that

$$\|\mathcal{W}(f_1,f_2)\|_{M^{r,s}_{\Phi}(\pi^{\#})} = \left(\int_{\mathfrak{g}_e}\left(\int_{\mathfrak{g}_e} |\mathcal{A}^{\#}_{\Phi}(\mathcal{W}(f_1,f_2))(X,Y)|^r\,\mathrm{d}X\right)^{s/r}\mathrm{d}Y\right)^{1/s}$$

with the usual conventions if r or s is infinite. Note that for every $X \in \mathfrak{g}_e$ we have

$$\overline{(\mathcal{A}_{\phi_2} f_2)(X)} = \overline{\big(f_2 \mid \pi(\exp_G X)\phi_2\big)} = (\mathcal{A}_{f_2}\phi_2)(-X). \tag{2.14}$$

Therefore by Proposition 2.21 we get

$$\|\mathcal{W}(f_1, f_2)\|_{M_\Phi^{r,s}(\pi^\#)} = \left(\int_{\mathfrak{g}_e^2} F(Y_2)\,\mathrm{d}Y_2\right)^{1/s}, \tag{2.15}$$

where

$$F(Y_2) = \int_{\mathfrak{g}_e^1}\left(\int_{\mathfrak{g}_e^2}\int_{\mathfrak{g}_e^1} |(\mathcal{A}_{\phi_1} f_1)(X_1+Y_1, X_2+Y_2) \times (\mathcal{A}_{f_2}\phi_2)(-X_1, -X_2)|^r\,\mathrm{d}X_1\,\mathrm{d}X_2\right)^{s/r}\mathrm{d}Y_1. \tag{2.16}$$

On the other hand, it follows by Minkowski's inequality that for every measurable function $\Gamma\colon \mathfrak{g}_e^1 \times \mathfrak{g}_e^2 \times \mathfrak{g}_e^2 \to \mathbb{C}$ and every real number $t \geq 1$ we have

$$\left(\int_{\mathfrak{g}_e^1}\left(\int_{\mathfrak{g}_e^2} |\Gamma(Y_1, X_2, Y_2)|\,\mathrm{d}X_2\right)^t \mathrm{d}Y_1\right)^{1/t} \leq \int_{\mathfrak{g}_e^2}\left(\int_{\mathfrak{g}_e^1} |\Gamma(Y_1, X_2, Y_2)|^t\,\mathrm{d}Y_1\right)^{1/t}\mathrm{d}X_2 \tag{2.17}$$

whenever $Y_2 \in \mathfrak{g}_e^2$. By (2.16) and (2.17) with $t := s/r$ and

$$\Gamma(Y_1, X_2, Y_2) := \int_{\mathfrak{g}_e^2} |(\mathcal{A}_{\phi_1} f_1)(Y_1 - X_1, Y_2 - X_2)\cdot(\mathcal{A}_{f_2}\phi_2)(X_1, X_2)|^r\,\mathrm{d}X_1$$

we get

$$\begin{aligned} F(Y_2) &\leq \left(\int_{\mathfrak{g}_e^2}\left(\int_{\mathfrak{g}_e^1} \Gamma(Y_1, X_2, Y_2)^{s/r}\,\mathrm{d}Y_1\right)^{r/s}\mathrm{d}X_2\right)^{s/r} \\ &= \left(\int_{\mathfrak{g}_e^2} \|\Gamma(\cdot, X_2, Y_2)\|_{L^{s/r}(\mathfrak{g}_e^1)}\,\mathrm{d}X_2\right)^{s/r}. \end{aligned} \tag{2.18}$$

Now note that $\Gamma(\cdot, X_2, Y_2)$ is equal to the convolution product of the functions $|(\mathcal{A}_{\phi_1} f_1)(\cdot, Y_2 - X_2)|^r$ and $|(\mathcal{A}_{f_2}\phi_2)(\cdot, X_2)|^r$. It follows by Young's inequality that

$$\begin{aligned} \|\Gamma(\cdot, X_2, Y_2)\|_{L^{s/r}(\mathfrak{g}_e^1)} &\leq \||(\mathcal{A}_{\phi_1} f_1)(\cdot, Y_2 - X_2)|^r\|_{L^{t_1}(\mathfrak{g}_e^1)}\cdot\||(\mathcal{A}_{f_2}\phi_2)(\cdot, X_2)|^r\|_{L^{t_2}(\mathfrak{g}_e^1)} \\ &= \|(\mathcal{A}_{\phi_1} f_1)(\cdot, Y_2 - X_2)\|^r_{L^{rt_1}(\mathfrak{g}_e^1)}\cdot\|(\mathcal{A}_{f_2}\phi_2)(\cdot, X_2)\|^r_{L^{rt_2}(\mathfrak{g}_e^1)} \end{aligned}$$

whenever $t_1, t_2 \in [1, \infty]$ satisfy $\frac{1}{t_1} + \frac{1}{t_2} = 1 + \frac{r}{s}$. By using the above inequality with $t_j = \frac{r_j}{r}$ for $j = 1, 2$, and taking into account (2.18), we get

$$\begin{aligned} F(Y_2) &\leq \left(\int_{\mathfrak{g}_e^2} \|(\mathcal{A}_{\phi_1} f_1)(\cdot, Y_2 - X_2)\|^r_{L^{rt_1}(\mathfrak{g}_e^1)}\|(\mathcal{A}_{f_2}\phi_2)(\cdot, X_2)\|^r_{L^{rt_2}(\mathfrak{g}_e^1)}\,\mathrm{d}X_2\right)^{s/r} \\ &=: \theta(Y_2)^{s/r}, \end{aligned} \tag{2.19}$$

where $\theta(\cdot)$ is the convolution product of the functions $X_2 \mapsto \|(\mathcal{A}_{\phi_1} f_1)(\cdot, X_2)\|^r_{L^{rt_1}(\mathfrak{g}_e^1)}$ and $X_2 \mapsto \|(\mathcal{A}_{f_2}\phi_2)(\cdot, X_2)\|^r_{L^{rt_2}(\mathfrak{g}_e^1)}$. It follows by Young's inequality again that

$$\|\theta\|_{L^{s/r}(\mathfrak{g}_e^2)} \le \left(\int_{\mathfrak{g}_e^2} \|(\mathcal{A}_{\phi_1} f_1)(\cdot, X_2)\|^r_{L^{rt_1}(\mathfrak{g}_e^1)}\,\mathrm{d}X_2\right)^{1/m_1} \times \left(\int_{\mathfrak{g}_e^2} \|(\mathcal{A}_{f_2}\phi_2)(\cdot, X_2)\|^r_{L^{rt_2}(\mathfrak{g}_e^1)}\,\mathrm{d}X_2\right)^{1/m_2}$$

provided that $m_1, m_2 \in [1,\infty]$ and $\frac{1}{m_1} + \frac{1}{m_2} = 1 + \frac{r}{s}$. For $m_j = \frac{s_j}{r}$, $j = 1,2$, we get

$$\|\theta\|_{L^{s/r}(\mathfrak{g}_e^2)} \le (\|f_1\|_{M^{r_1,s_1}_{\phi_1}(\pi)})^r (\|f_2\|_{M^{r_2,s_2}_{\phi_2}(\pi)})^r,$$

where we also used (2.14). Then by (2.15) and (2.19) we get

$$\|\mathcal{W}(f_1, f_2)\|_{M^{r,s}_{\Phi}(\pi^\#)} \le \|f_1\|_{M^{r_1,s_1}_{\phi_1}(\pi)} \cdot \|f_2\|_{M^{r_2,s_2}_{\phi_2}(\pi)},$$

and this concludes the proof. □

Remark 2.23 A particularly sharp version of Theorem 2.23 holds for $r_1 = s_1$, $r_2 = s_2$, and $r = s$. That is, let $r, r_1, r_2 \in [1,\infty]$ such that $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$. It follows at once by Proposition 2.21 that for arbitrary $f_1, f_2 \in \mathcal{H}_{-\infty}$ we have

$$\|\mathcal{W}(f_1, f_2)\|_{M^{r,r}_{\Phi}(\pi^\#)} = \|f_1\|_{M^{r_1,r_1}_{\phi_1}(\pi)} \cdot \|f_2\|_{M^{r_2,r_2}_{\phi_2}(\pi)},$$

which in turn implies that $\mathcal{W}(f_1, f_2) \in M^{r,r}_{\Phi}(\pi^\#)$ if and only if for $j = 1,2$ we have $f_j \in M^{r_j,r_j}_{\phi_j}(\pi)$.

Corollary 2.24 *Let G be a simply connected, nilpotent Lie group with a unitary irreducible representation $\pi: G \to \mathcal{B}(\mathcal{H})$, pick $\phi_1, \phi_2 \in \mathcal{H}_\infty \setminus \{0\}$, and denote $\Phi = \mathcal{W}(\phi_1, \phi_2) \in \mathcal{S}(\mathcal{O})$. If $r, r_1, r_2 \in [1,\infty]$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, then the cross-Wigner distribution associated with any predual to the coadjoint orbit of the representation π defines a continuous sesquilinear map*

$$\mathcal{W}(\cdot,\cdot): M^{r_1,r_1}_{\phi_1}(\pi) \times M^{r_2,r_2}_{\phi_2}(\pi) \to M^{r,\infty}_{\Phi}(\pi^\#).$$

Proof One can apply Theorem 2.22 with $r_1 = s_1$, $r_2 = s_2$, and $s = \infty$. Alternatively, a direct proof proceeds as follows. Let $f_1, f_2 \in \mathcal{H}_{-\infty}$. It follows by Proposition 2.21 along with Hölder's inequality that for every $Y \in \mathfrak{g}_e$ we have

$$\left\|\mathcal{A}^\#_{\Phi}\big(\mathcal{W}(f_1, f_2)\big)(\cdot, Y)\right\|_{L^r(\mathfrak{g}_e)} \le \|\mathcal{A}_{\phi_1} f_1\|_{L^{r_1}(\mathfrak{g}_e)} \cdot \|\mathcal{A}_{\phi_2} f_2\|_{L^{r_2}(\mathfrak{g}_e)}$$

whence $\|\mathcal{W}(f_1, f_2)\|_{M^{r,\infty}_{\Phi}(\pi^\#)} \le \|f_1\|_{M^{r_1,r_1}_{\phi_1}(\pi)} \cdot \|f_2\|_{M^{r_2,r_2}_{\phi_2}(\pi)}$, and the conclusion follows. □

The next corollary provides a partial generalization of Theorem 4.3 in [34].

Corollary 2.25 *Let G be a simply connected, nilpotent Lie group with a unitary irreducible representation $\pi : G \to \mathcal{B}(\mathcal{H})$, pick $\phi_1, \phi_2 \in \mathcal{H}_\infty \setminus \{0\}$, and denote $\Phi = \mathcal{W}(\phi_1, \phi_2) \in \mathcal{S}(\mathcal{O})$. Assume that $\mathfrak{g}_e$ is a predual to the coadjoint orbit $\mathcal{O}$ associated with the representation π, and let $\mathfrak{g}_e = \mathfrak{g}_e^1 \dotplus \mathfrak{g}_e^2$ be any direct sum decomposition. If $r, s, r_1, s_1, r_2, s_2 \in [1, \infty]$ satisfy the conditions*

$$r \le s, \qquad r_2, s_2 \in [r, s], \quad \text{and} \quad \frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{s_1} - \frac{1}{s_2} = 1 - \frac{1}{r} - \frac{1}{s},$$

then the following assertions hold:

(1) *For every symbol $a \in M_\Phi^{r,s}(\pi^\#)$ we have a bounded linear operator*

$$\mathrm{Op}^\pi(a) : M_{\phi_1}^{r_1,s_1}(\pi) \to M_{\phi_2}^{r_2,s_2}(\pi).$$

(2) *The linear mapping $\mathrm{Op}^\pi(\cdot) : M_\Phi^{r,s}(\pi^\#) \to \mathcal{B}(M_{\phi_1}^{r_1,s_1}(\pi), M_{\phi_2}^{r_2,s_2}(\pi))$ is continuous.*

Proof We may assume that $\|\phi_1\| = \|\phi_2\| = 1$, hence $\|\Phi\|_{L^2(\mathcal{O})} = 1$.

For every $t \in [1, \infty]$ we are going to define $t' \in [1, \infty]$ by the equation $\frac{1}{t} + \frac{1}{t'} = 1$. With this notation, the hypothesis implies $\frac{1}{r_1} + \frac{1}{r_2'} = \frac{1}{s_1} + \frac{1}{s_2'} = \frac{1}{r'} + \frac{1}{s'}$ and moreover $r_1, s_1, r_2', s_2' \in [r', s']$. Therefore we can apply Theorem 2.22 to obtain

$$\|\mathcal{W}(f_2, f_1)\|_{M_\Phi^{r',s'}(\pi^\#)} \le \|f_1\|_{M_{\phi_1}^{r_1,s_1}(\pi)} \cdot \|f_2\|_{M_{\phi_2}^{r_2',s_2'}(\pi)} \tag{2.20}$$

whenever $f_1, f_2 \in \mathcal{H}_{-\infty}$.

On the other hand, if $a \in M_\Phi^{r,s}(\pi^\#)$, then

$$\big(\mathrm{Op}^\pi(a) f_1 \mid f_2\big) = \big(a \mid \mathcal{W}(f_2, f_1)\big)_{L^2(\mathcal{O})} = \big(\mathcal{A}_\Phi^\# a \mid \mathcal{A}_\Phi^\#\big(\mathcal{W}(f_2, f_1)\big)\big)_{L^2(\mathfrak{g}_e \times \mathfrak{g}_e)}$$

by Corollary 2.11(1) and Proposition 2.8(1). Then Hölder's inequality for mixed-norm spaces (see for instance Lemma 11.1.2(b) in [17]) shows that

$$\begin{aligned} |(\mathrm{Op}^\pi(a) f_1 \mid f_2)| &\le \|\mathcal{A}_\Phi^\# a\|_{L^{r,s}(\mathfrak{g}_e \times \mathfrak{g}_e)} \cdot \|\mathcal{A}_\Phi^\#(\mathcal{W}(f_2, f_1))\|_{L^{r',s'}(\mathfrak{g}_e \times \mathfrak{g}_e)} \\ &= \|a\|_{M_\Phi^{r,s}(\pi^\#)} \cdot \|\mathcal{W}(f_2, f_1)\|_{M_\Phi^{r',s'}(\pi^\#)} \\ &\le \|a\|_{M_\Phi^{r,s}(\pi^\#)} \cdot \|f_1\|_{M_{\phi_1}^{r_1,s_1}(\pi)} \cdot \|f_2\|_{M_{\phi_2}^{r_2',s_2'}(\pi)}, \end{aligned}$$

where the latter inequality follows by (2.20). Now the assertion follows by a straightforward argument that uses the duality of the mixed-norm spaces (see Lemma 11.1.2(d) in [17]). □

Corollary 2.26 *If G be a simply connected, nilpotent Lie group with a unitary irreducible representation $\pi : G \to \mathcal{B}(\mathcal{H})$, then the following assertions hold whenever $\Phi = \mathcal{W}(\phi_1, \phi_2)$ with $\phi_1, \phi_2 \in \mathcal{H}_\infty \setminus \{0\}$:*

(1) *For every $a \in M_\Phi^{\infty,1}(\pi^\#)$ we have $\mathrm{Op}^\pi(a) \in \mathcal{B}(\mathcal{H})$.*
(2) *The linear mapping $\mathrm{Op}^\pi(\cdot) : M_\Phi^{\infty,1}(\pi^\#) \to \mathcal{B}(\mathcal{H})$ is continuous.*

Proof This is the special case of Corollary 2.25 with $r_1 = s_1 = r_2 = s_2 = 2$, $r = 1$, and $s = \infty$, since Example 2.17 shows that $M^{2,2}(\pi) = \mathcal{H}$. □

We conclude this section by a sufficient condition for a pseudo-differential operator to belong to the trace class. In the special case of the Schrödinger representation of a Heisenberg group, a proof for this result can be found for instance in [16] or [18].

Proposition 2.27 *Let G be a simply connected, nilpotent Lie group with a unitary irreducible representation $\pi : G \to \mathcal{B}(\mathcal{H})$, pick $\phi_1, \phi_2 \in \mathcal{H}_\infty$ with $\|\phi_1\| = \|\phi_2\| = 1$, and denote $\Phi = \mathcal{W}(\phi_1, \phi_2) \in \mathcal{S}(\mathcal{O})$. Then for every symbol $a \in M^{1,1}_\Phi(\pi^\#)$ we have $\operatorname{Op}^\pi(a) \in \mathfrak{S}_1(\mathcal{H})$ and $\|\operatorname{Op}^\pi(a)\|_1 \le \|a\|_{M^{1,1}_\Phi(\pi^\#)}$.*

Proof For arbitrary $a \in \mathcal{S}'(\mathcal{O})$ we have by Corollary 2.9(1) and Remark 2.20,

$$a = \iint_{\mathfrak{g}_e \times \mathfrak{g}_e} \big(\mathcal{A}^\#_\Phi a\big)(X, Y) \cdot \pi^\#\big(\exp_{G \ltimes G}(X, Y)\big)\Phi \,\mathrm{d}X \,\mathrm{d}Y,$$

whence by Corollary 2.11

$$\operatorname{Op}^\pi(a) = \iint_{\mathfrak{g}_e \times \mathfrak{g}_e} \big(\mathcal{A}^\#_\Phi a\big)(X, Y) \cdot \operatorname{Op}^\pi\big(\pi^\#\big(\exp_{G \ltimes G}(X, Y)\big)\Phi\big) \,\mathrm{d}X \,\mathrm{d}Y \tag{2.21}$$

where the latter integral is weakly convergent in $\mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty})$. On the other hand, for arbitrary $X, Y \in \mathfrak{g}_e$ we get by (2.13) and Corollary 2.11(2)

$$\begin{aligned}\operatorname{Op}^\pi\big(\pi^\#\big(\exp_{G \ltimes G}(X, Y)\big)\Phi\big) &= \pi\big(\exp_G(X + Y)\big) \circ \operatorname{Op}^\pi(\Phi) \circ \pi(\exp_G X)^{-1}\\ &= \big(\cdot \mid \pi(\exp_G X)\phi_2\big)\pi\big(\exp_G(X + Y)\big)\phi_1.\end{aligned}$$

In particular, $\operatorname{Op}^\pi(\pi^\#(\exp_{G \ltimes G}(X, Y))\Phi) \in \mathfrak{S}_1(\mathcal{H})$ and

$$\|\operatorname{Op}^\pi(\pi^\#(\exp_{G \ltimes G}(X, Y))\Phi)\|_1 = \|\pi(\exp_G(X + Y))\phi_1\| \cdot \|\pi(\exp_G X)\phi_2\| = 1.$$

It then follows that the integral in (2.21) is absolutely convergent in $\mathfrak{S}_1(\mathcal{H})$ for $a \in M^{1,1}_\Phi(\pi^\#)$ and moreover we have

$$\|\operatorname{Op}^\pi(a)\|_1 \le \iint_{\mathfrak{g}_e \times \mathfrak{g}_e} |(\mathcal{A}^\#_\Phi a)(X, Y)| \,\mathrm{d}X \,\mathrm{d}Y = \|a\|_{M^{1,1}_\Phi(\pi^\#)}$$

which concludes the proof. □

3 The Case of Square-Integrable Representations

In this section we focus on square-integrable representations of nilpotent Lie groups. Here and throughout the next sections by square-integrable representations of a nilpotent Lie group we actually mean a representation which is square-integrable modulo the center. In fact, a simply connected nilpotent Lie group has the center of infinite

Haar measure, therefore an irreducible representation cannot be square-integrable on the center. A discussion of the crucial role of these representations along with many examples can be found for instance in [27] and in the monograph [8].

3.1 Independence of the Modulation Spaces on the Window Vectors

Lemma 3.1 *Let G_1 and G_2 be unimodular Lie groups and assume that we have a group homomorphism $\alpha: G_1 \to \operatorname{Aut} G_2$, $g_1 \mapsto \alpha_{g_1}$ such that for every $g_1 \in G_1$, the automorphism α_{g_1} preserves the Haar measure of G_2. Consider the semidirect product $G = G_1 \ltimes_\alpha G_2$ and for every $h \in G$ and $\phi: G \to \mathbb{C}$ define $R_h\phi: G \to \mathbb{C}$, $(R_h\phi)(g) = \phi(gh)$. Fix $r, s \in [1, \infty]$ and consider the mixed-norm space $L^{r,s}(G)$ consisting of the equivalence classes of functions $\phi: G \to \mathbb{C}$ such that*

$$\|\phi\|_{L^{r,s}(G)} := \left(\int_{G_2} \left(\int_{G_1} |\phi(g_1, g_2)|^r \, \mathrm{d}g_1 \right)^{s/r} \mathrm{d}g_2 \right)^{1/s} < \infty,$$

with the usual conventions if r or s is infinite. Then the space $L^{r,s}(G)$ is invariant under the right-translation operator R_h for every $h \in G$, and the mapping

$$\rho: G \to \mathcal{B}\big(L^{r,s}(G)\big), \qquad h \mapsto R_h|_{L^{r,s}(G)}$$

is representation of the Lie group G by isometries on the Banach space $L^{r,s}(G)$.

Proof Let $\phi: G \to \mathbb{C}$ be any measurable function and $h = (h_1, h_2) \in G$. We have $(R_h\phi)(g_1, g_2) = \phi(g_1h_1, \alpha_{h_1^{-1}}(g_2)h_2)$. Since the group G_1 is unimodular, it then follows that for every $g_2 \in G_2$ we have

$$\begin{aligned} \int_{G_1} |(R_h\phi)(g_1, g_2)|^r \, \mathrm{d}g_1 &= \int_{G_1} \big|\phi\big(g_1h_1, \alpha_{h_1^{-1}}(g_2)h_2\big)\big|^r \, \mathrm{d}g_1 \\ &= \int_{G_1} \big|\phi\big(g_1, \alpha_{h_1^{-1}}(g_2)h_2\big)\big|^r \, \mathrm{d}g_1. \end{aligned}$$

By integrating on G_2 both extreme terms in above equality and taking into account that G_2 is unimodular, we get $\|R_h\phi\|_{L^{r,s}(G)} = \|\phi\|_{L^{r,s}(G)}$ for all $h \in G$. □

Remark 3.2 In the setting of Lemma 3.1, the representation ρ is not continuous, in general; see for instance the case $r = s = \infty$.

For latter use, we note that for every $\psi \in L^1(G)$ with compact support and $\chi \in L^{r,s}(G)$ we can define the measurable function

$$\big(\rho(\psi)\chi\big)(g) = \int_G \chi(gh)\psi(h) \, \mathrm{d}h \quad \text{for a.e. } g \in G. \tag{3.1}$$

Let $r', s' \in [1, \infty]$ such that $\frac{1}{r} + \frac{1}{r'} = \frac{1}{s} + \frac{1}{s'} = 1$. If $\varphi \in L^{r',s'}(G)$, then by using Lemma 3.1 we get

$$\int_G |\rho(\psi)\chi(g)\varphi(g)| \, \mathrm{d}g \le \|\psi\|_{L^1(G)} \cdot \|\chi\|_{L^{r,s}(G)} \|\varphi\|_{L^{r',s'}(G)}.$$

By using results in [4] (Sect. 5 Corollary to Theorem 1, Sect. 2 Theorem 1) we get $\rho(\psi)\chi \in L^{r,s}(G)$ and

$$\|\rho(\psi)\chi\|_{L^{r,s}(G)} \le \|\varphi\|_{L^1(G)} \cdot \|\phi\|_{L^{r,s}(G)}. \tag{3.2}$$

Then it is straightforward to see that we may extend (3.1) to arbitrary $\psi \in L^1(G)$, and (3.2) is preserved.

We are now ready to prove a theorem that covers many cases when the modulation spaces for square-integrable representations do not depend on the choice of a window function. The second stage in the proof is inspired by the methods of the theory of coorbit spaces (see [11–13], and also the proof of Proposition 11.3.2(c) in [17]).

Theorem 3.3 *Let G_1 and G_2 be simply connected, nilpotent Lie groups and a unipotent homomorphism $\alpha : G_1 \to \operatorname{Aut} G_2$. Define $G = G_1 \ltimes_\alpha G_2$ and assume that the center $\mathfrak{z}$ of $\mathfrak{g}$ satisfies the condition*

$$\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{g}_1) + (\mathfrak{z} \cap \mathfrak{g}_2). \tag{3.3}$$

Assume the irreducible representation $\pi : G \to \mathcal{B}(\mathcal{H})$ is square-integrable modulo the center of G, and pick any Jordan-Hölder basis in $\mathfrak{g}$ such that for the corresponding predual $\mathfrak{g}_e$ for the coadjoint orbit associated with π we have $\mathfrak{g}_e = (\mathfrak{g}_e \cap \mathfrak{g}_1) + (\mathfrak{g}_e \cap \mathfrak{g}_2)$.

Then the modulation spaces for the representation π with respect to the decomposition $\mathfrak{g}_e \simeq (\mathfrak{g}_e \cap \mathfrak{g}_1) \times (\mathfrak{g}_e \cap \mathfrak{g}_2)$ are independent on the choice of a window vector $\phi \in \mathcal{H}_\infty \setminus \{0\}$.

Proof The proof has two stages.

1° For the sake of simplicity let us identify the Lie group G_j to its Lie algebra $\mathfrak{g}_j$ by means of the exponential map $\exp_{G_j}$, so that G_j will be just $\mathfrak{g}_j$ with the group operation $*$ defined by the Baker-Campbell-Hausdorff series. Let Z be the center of G, whose Lie algebra $\mathfrak{z}$ is the center of $\mathfrak{g}$. Since the representation π is square-integrable, then for arbitrary ξ_0 in the corresponding orbit $\mathcal{O}$, we have $\mathfrak{g}_{\xi_0} = \mathfrak{z}$. (See [8].) Thus, if $\mathfrak{g}_e$ is the predual, as in Setting 2.4, then we have a linear isomorphism $\mathfrak{g}_e \simeq \mathfrak{g}/\mathfrak{z}$, $X \mapsto X + \mathfrak{z}$, and we shall endow $\mathfrak{g}_e$ with the Lie algebra structure which makes this map into an isomorphism of Lie algebras.

If we define $G_e := G/Z$, then G_e is a connected, simply connected nilpotent Lie group, whose Lie algebra is just $\mathfrak{g}_e$. Let $*_e$ denote the multiplication in G_e, which is just the Baker-Campbell-Hausdorff multiplication in $\mathfrak{g}_e$.

Now use assumption (3.3) to see that if $(Y_1, Y_2) \in \mathfrak{z} \subseteq \mathfrak{g} = \mathfrak{g}_1 \ltimes_{\dot\alpha} \mathfrak{g}_2$, then $(Y_1, 0), (0, Y_2) \in \mathfrak{z}$. Now formula (2.2) shows that for every $(X_1, X_2) \in \mathfrak{g}$ we have $0 = [(X_1, X_2), (Y_1, 0)] = ([X_1, Y_1], -\dot\alpha(Y_1)X_2)$, hence Y_1 belongs to the center $\mathfrak{z}_1$ of $\mathfrak{g}_1$ and $\dot\alpha(Y_1) = 0$. This shows that the closed subgroup $Z_1 := Z \cap G_1$ is contained in the center of G_1 and satisfies

$$Z_1 \subseteq \operatorname{Ker} \alpha. \tag{3.4}$$

Also $0=[(X_1,X_2),(0,Y_2)]=(0,\dot\alpha(X_1)Y_2+[X_2,Y_2])$ for every $(X_1,X_2)\in\mathfrak{g}$, whence we see that Y_2 belongs both to the center $\mathfrak{z}_2$ of G_2 and to $\mathrm{Ker}(\dot\alpha(X_1))$ for arbitrary $X_1\in\mathfrak{g}_1$. Therefore the closed subgroup $Z_2:=Z\cap G_2$ is contained in the center of G_2 and we have

$$(\forall g_1\in G_1)\quad \alpha_{g_1}(Z_2)\subseteq Z_2. \tag{3.5}$$

It follows by (3.4) and (3.5) that the group homomorphism $\alpha\colon G_1\to \mathrm{Aut}\,G_2$ induces a group homomorphism $\bar\alpha\colon G_1/Z_1\to \mathrm{Aut}(G_2/Z_2)$ and we have the isomorphisms of Lie groups

$$G_e\simeq G/Z\simeq (G_1/Z_1)\ltimes_{\bar\alpha}(G_2/Z_2).$$

Moreover $Z\simeq Z_1\times Z_2$.

2° We now come back to the proof. Fix $r,s\in[1,\infty]$ and let $\phi_1,\phi_2\in\mathcal{H}_\infty$ be any window functions with $\|\phi_1\|=\|\phi_2\|=1$. For $j=1,2$ and every $f\in\mathcal{H}_{-\infty}$ we have by Corollary 2.9

$$\begin{aligned}(\mathcal{A}_{\phi_2}f)(X)&=\big(f\mid\pi(\exp_G X)\phi_2\big)\\ &=\int_{\mathfrak{g}_e}\chi(Y)\big(\pi(\exp_G Y)\phi_1\mid\pi(\exp_G X)\phi_2\big)\,\mathrm{d}Y\\ &=\int_{\mathfrak{g}_e}\chi(Y)\big(\phi_1\mid\pi\big(\exp_G\big((-Y)*X\big)\big)\phi_2\big)\,\mathrm{d}Y\\ &=\int_{\mathfrak{g}_e}\chi(Y)\mathrm{e}^{\mathrm{i}\alpha(-Y,X)}\big(\phi_1\mid\pi\big(\exp_G\big((-Y)*_e X\big)\big)\phi_2\big)\,\mathrm{d}Y,\\ &=\int_{\mathfrak{g}_e}\chi(Y)\mathrm{e}^{\mathrm{i}\alpha(-Y,X)}(\mathcal{A}_{\phi_2}\phi_1)\big((-Y)*_e X\big)\,\mathrm{d}Y,\\ &=\int_{\mathfrak{g}_e}\chi(X*_e Y)\mathrm{e}^{\mathrm{i}\alpha((-Y)*_e(-X),X)}(\mathcal{A}_{\phi_2}\phi_1)(-Y)\,\mathrm{d}Y,\end{aligned}$$

for every $X\in\mathfrak{g}_e$, where $\chi:=\mathcal{A}_{\phi_1}f\in L^{r,s}(\mathfrak{g}_e\times\mathfrak{g}_e)$ and $\alpha\colon\mathfrak{g}_e\times\mathfrak{g}_e\to\mathbb{R}$ is a suitable polynomial function defined in terms of the central character of the representation π (see e.g., [24]). Now note that $\mathcal{A}_{\phi_2}\phi_1\in\mathcal{S}(\mathfrak{g}_e)$ by Corollary 2.9(3). It then follows by Lemma 3.1 and (3.2) in Remark 3.2 that there exists a constant $C>0$ such that for every $f\in\mathcal{H}_{-\infty}$ we have $\|\mathcal{A}_{\phi_2}f\|_{L^{r,s}(\mathfrak{g}_e\times\mathfrak{g}_e)}\le C\|\mathcal{A}_{\phi_1}f\|_{L^{r,s}(\mathfrak{g}_e\times\mathfrak{g}_e)}$. Thus we get the continuous inclusion map $M^{r,s}_{\phi_1}(\pi)\hookrightarrow M^{r,s}_{\phi_2}(\pi)$. Now the conclusion follows by interchanging ϕ_1 and ϕ_2. □

The previous theorem allows us to omit the window vector in the notation for modulation spaces associated to square-integrable representations.

Example 3.4 Theorem 3.3 applies to a wide variety of situations. Let us mention here just a few of them:

(1) In the case of the Schrödinger representation of the Heisenberg group $\mathbb{H}_{2n+1}=\mathbb{R}^n\ltimes\mathbb{R}^{n+1}$ we recover the well-known property that the classical modulation

spaces used in the time-frequency analysis are independent on the choice of a window function (see for instance Proposition 11.3.2(c) in [17]).

(2) We shall see below (see Sect. 3.3) that one can give sufficient conditions for the continuity of the operators constructed by the Weyl-Pedersen calculus for the square-integrable representation $\pi : G \to \mathcal{B}(\mathcal{H})$ by using spaces of symbols which are modulation spaces $M^{r,s}(\pi^{\#}) \subseteq \mathcal{S}'(\mathcal{O})$. Here $\pi^{\#} : G \ltimes G \to \mathcal{B}(L^2(\mathcal{O}))$ is in turn a square-integrable representation to which Theorem 3.3 applies and ensures that the corresponding modulation spaces do not depend on the choice of a window function.

3.2 Covariance Properties of the Weyl-Pedersen Calculus

We now record the covariance property for the cross-Wigner distributions and its consequence for the Weyl-Pedersen calculus. In the very special case of the Schrödinger representation for the Heisenberg group we recover a classical fact (see e.g., [17]).

Theorem 3.5 *Assume that the representation $\pi : G \to \mathcal{B}(\mathcal{H})$ associated with $\mathcal{O}$ is square-integrable modulo the center of G. Then the following assertions hold*:

(1) *For every $f, h \in \mathcal{H}$ and $X \in \mathfrak{g}$ we have*

$$\mathcal{W}\big(\pi(\exp_G X) f, \pi(\exp_G X) h\big)(\xi) = \mathcal{W}(f, h)\big(\xi \circ \mathrm{e}^{\mathrm{ad}_{\mathfrak{g}} X}\big) \quad \textit{for a.e. } \xi \in \mathcal{O}.$$

(2) *For every symbol $a \in \mathcal{S}'(\mathcal{O})$ and arbitrary $g \in G$ we have*

$$\mathrm{Op}\big(a \circ \mathrm{Ad}_G^*(g^{-1})|_{\mathcal{O}}\big) = \pi(g)\,\mathrm{Op}(a)\pi(g)^{-1}.$$

Proof (1) Let $\xi_0 \in \mathcal{O}$. Note that the following assertions hold:

$$(\forall X \in \mathfrak{z}) \quad \pi(\exp_G X) = \mathrm{e}^{\mathrm{i}\langle \xi_0, X\rangle}\, \mathrm{id}_{\mathcal{H}}, \tag{3.6}$$

$$(\forall \xi \in \mathcal{O}) \quad \xi|_{\mathfrak{z}} = \xi_0|_{\mathfrak{z}}, \tag{3.7}$$

$$\xi_0|_{\mathfrak{g}_e} = 0. \tag{3.8}$$

Also, it easily follows by Definition 2.7 that for arbitrary $f, h \in \mathcal{H}$ we have

$$\mathcal{W}(f, h)(\xi) = \int_{\mathfrak{g}_e} \mathrm{e}^{\mathrm{i}\langle \xi, X\rangle} \big(f \mid \pi(\exp_G X) h\big)\, \mathrm{d}X \quad \text{for a.e. } \xi \in \mathcal{O}.$$

It then follows that for arbitrary $X_0 \in \mathfrak{g}_e$ and a.e. $\xi \in \mathcal{O}$ we have

$$\begin{aligned} &\mathcal{W}\big(\pi(\exp_G X_0) f, \pi(\exp_G X_0) h\big)(\xi) \\ &= \int_{\mathfrak{g}_e} \mathrm{e}^{\mathrm{i}\langle \xi, X\rangle} \big(f \mid \pi\big((\exp_G(-X_0))(\exp_G X)(\exp_G X_0)\big) h\big)\, \mathrm{d}X \\ &= \int_{\mathfrak{g}_e} \mathrm{e}^{\mathrm{i}\langle \xi, X\rangle} \big(f \mid \pi\big(\exp_G\big(\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}(-X_0)} X\big)\big) h\big)\, \mathrm{d}X. \end{aligned}$$

If we denote by $\mathrm{pr}_{\mathfrak{z}}\colon\mathfrak{g}\to\mathfrak{z}$ the natural projection corresponding to the direct sum decomposition $\mathfrak{g}=\mathfrak{z}\dotplus\mathfrak{g}_e$, then we have for every $X\in\mathfrak{g}_e$,

$$\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}(-X_0)}X=\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}_e}(-X_0)}X+\mathrm{pr}_{\mathfrak{z}}\big(\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}(-X_0)}X\big),$$

where we have endowed $\mathfrak{g}_e$ with the Lie algebra structure which makes the linear isomorphism $\mathfrak{g}_e\simeq\mathfrak{g}/\mathfrak{z}$ into an isomorphism of Lie algebras (see also [24]). Therefore, by using (3.6) and (3.8), we get

$$(\forall X\in\mathfrak{g}_e)\quad \pi\big(\big(\exp_G\big(\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}(-X_0)}X\big)\big)\big)=\mathrm{e}^{\mathrm{i}\langle\xi_0,\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}(-X_0)}X\rangle}\pi\big(\exp_G\big(\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}_e}(-X_0)}X\big)\big)$$

and then the above computation leads to

$$\begin{aligned}
&\mathcal{W}\big(\pi(\exp_G X_0)f,\pi(\exp_G X_0)h\big)(\xi)\\
&\quad=\int_{\mathfrak{g}_e}\mathrm{e}^{\mathrm{i}\langle\xi,X\rangle}\mathrm{e}^{-\mathrm{i}\langle\xi_0,\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}(-X_0)}X\rangle}\big(f\mid\pi\big(\exp_G\big(\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}_e}(-X_0)}X\big)\big)h\big)\,\mathrm{d}X\\
&\quad=\int_{\mathfrak{g}_e}\mathrm{e}^{\mathrm{i}\langle\xi,X\rangle}\mathrm{e}^{-\mathrm{i}\langle(\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}(-X_0)})^*\xi_0,X\rangle}\big(f\mid\pi\big(\exp_G\big(\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}_e}(-X_0)}X\big)\big)h\big)\,\mathrm{d}X\\
&\quad=\int_{\mathfrak{g}_e}\mathrm{e}^{\mathrm{i}\langle\xi,\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}_e}X_0}Y\rangle}\mathrm{e}^{-\mathrm{i}\langle(\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}(-X_0)})^*\xi_0,\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}_e}X_0}Y\rangle}\big(f\mid\pi(\exp_G Y)h\big)\,\mathrm{d}Y,
\end{aligned}$$

where we used the change of variables $X\mapsto Y=\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}_e}(-X_0)}X$, which is a measure-preserving diffeomorphism since $\mathfrak{g}_e$ is a nilpotent Lie algebra. Now note that by using (3.7) we get for a.e. $\xi\in\mathcal{O}$ and every $Y\in\mathfrak{g}_e$,

$$\begin{aligned}
&\big\langle\xi,\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}_e}X_0}Y\big\rangle-\big\langle\big(\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}(-X_0)}\big)^*\xi_0,\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}_e}X_0}Y\big\rangle\\
&\quad=\big\langle\xi,\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}X_0}Y\big\rangle-\big\langle\xi,\mathrm{pr}_{\mathfrak{z}}\big(\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}X_0}Y\big)\big\rangle-\big\langle\xi_0,\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}(-X_0)}\big(\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}X_0}Y-\mathrm{pr}_{\mathfrak{z}}\big(\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}X_0}Y\big)\big)\big\rangle\\
&\quad=\big\langle\xi,\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}X_0}Y\big\rangle-\big\langle\xi_0,\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}X_0}Y\big\rangle-\langle\xi_0,Y\rangle+\big\langle\xi_0,\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}X_0}Y\big\rangle\\
&\quad=\big\langle\xi,\mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}X_0}Y\big\rangle
\end{aligned}$$

since $\langle\xi_0,Y\rangle=0$ by (3.8). Thus the conclusion follows by the formula we had obtained above for $\mathcal{W}(\pi(\exp_G X_0)f,\pi(\exp_G X_0)h)(\xi)$, and this completes the proof for $X\in\mathfrak{g}_e$. Then the formula extends to arbitrary $X\in\mathfrak{g}$ by using the fact that $\mathfrak{g}=\mathfrak{g}_e\dotplus\mathfrak{z}$ and taking into account (3.6).

(2) If $a\in\mathcal{S}(\mathcal{O})$, then for every $f,\phi\in\mathcal{H}$ we have

$$\begin{aligned}
\big(\mathrm{Op}\big(a\circ\mathrm{Ad}_G^*\big(g^{-1}\big)|_{\mathcal{O}}\big)\phi\mid f\big)_{\mathcal{H}}&=\big(a\circ\mathrm{Ad}_G^*\big(g^{-1}\big)|_{\mathcal{O}}\mid\mathcal{W}(f,\phi)\big)_{L^2(\mathcal{O})}\\
&=\big(a\mid\mathcal{W}(f,\phi)\circ\mathrm{Ad}_G^*(g)|_{\mathcal{O}}\big)_{L^2(\mathcal{O})}\\
&=\big(a\mid\mathcal{W}\big(\pi(g)^{-1}f,\pi(g)^{-1}\phi\big)\big)_{L^2(\mathcal{O})}\\
&=\big(\mathrm{Op}(a)\pi(g)^{-1}\phi\mid\pi(g)^{-1}f\big)_{\mathcal{H}}\\
&=\big(\pi(g)\,\mathrm{Op}(a)\pi(g)^{-1}\phi\mid f\big)_{\mathcal{H}},
\end{aligned}$$

where the first and the fourth equalities follow by Corollary 2.11(1), the second equality is a consequence of the fact that the coadjoint action preserves the Liouville measure on $\mathcal{O}$, while the third equality follows by Assertion (1) which we already proved.

Thus we obtained the conclusion for $a \in \mathcal{S}(\mathcal{O})$, and then it can be easily extended by duality to any $a \in \mathcal{S}'(\mathcal{O})$ by using equation (2.10) in Definition 2.10. $\square$

3.3 Continuity of Weyl-Pedersen Calculus

In the case of square-integrable representations modulo the center, we now obtain continuity properties of the Weyl-Pedersen calculus in modulation spaces which are independent on the window function.

Lemma 3.6 *Let G be any Lie group with a unitary irreducible representation $\pi : G \to \mathcal{B}(\mathcal{H})$ and*

$$\pi^{\ltimes} : G \ltimes G \to \mathcal{B}\big(\mathfrak{S}_2(\mathcal{H})\big), \qquad \pi^{\ltimes}(g,h)T = \pi(gh)T\pi(g)^{-1}.$$

If G is a unimodular group and π is square-integrable modulo the center of G, then $\pi^{\ltimes}$ is square-integrable modulo the center of $G \ltimes G$.

Proof If π is square-integrable modulo the center Z of G, then there is $\phi_0 \in \mathcal{H} \setminus \{0\}$ such that the function $gZ \mapsto |(\pi(g)\phi_0 \mid \phi_0)|$ is square-integrable on G/Z. Let us define the rank-one projection $T_0 = (\cdot \mid \phi_0)\phi_0$. Then we have

$$\begin{aligned}
&\iint_{(G\ltimes G)/(Z\times Z)} \big|\big(\pi^{\ltimes}(g,h)T_0 \mid T_0\big)\big|^2 \,\mathrm{d}g\,\mathrm{d}h \\
&\quad = \int_{G/Z}\bigg(\int_{G/Z} \big|\big(\pi(gh)T_0\pi(g)^{-1} \mid T_0\big)\big|^2\,\mathrm{d}h\bigg)\,\mathrm{d}g \\
&\quad = \int_{G/Z}\bigg(\int_{G/Z} \big|\big(\pi(h)T_0\pi(g)^{-1} \mid T_0\big)\big|^2\,\mathrm{d}h\bigg)\,\mathrm{d}g.
\end{aligned}$$

Since $T_0 = (\cdot \mid \phi_0)\phi_0$, we get $\pi(h)T_0\pi(g)^{-1} = (\cdot \mid \pi(g)\phi_0)\pi(h)\phi_0$, and then

$$\big(\pi(h)T_0\pi(g)^{-1} \mid T_0\big) = \big(\pi(h)\phi_0 \mid \phi_0\big)\cdot\big(\phi_0 \mid \pi(g)\phi_0\big).$$

Therefore

$$\iint_{(G\ltimes G)/(Z\times Z)} \big|\big(\pi^{\ltimes}(g,h)T_0 \mid T_0\big)\big|^2\,\mathrm{d}g\,\mathrm{d}h = \bigg(\int_{G/Z}\big|\big(\pi(g)\phi_0 \mid \phi_0\big)\big|^2\,\mathrm{d}h\bigg)^2$$

hence the function $(g,h)(Z\times Z) \mapsto |(\pi^{\ltimes}(g,h)T_0 \mid T_0)|$ is square-integrable on the quotient group $(G \ltimes G)/(Z \times Z)$, and this concludes the proof since $Z \times Z$ is the center of $G \ltimes G$ (see Example 2.3). $\square$

Remark 3.7 Assume that $\pi : G \to \mathcal{B}(\mathcal{H})$ is a square-integrable representation of a simply connected, nilpotent Lie group, with the corresponding coadjoint orbit

$\mathcal{O} \subseteq \mathfrak{g}^*$. Recall the representation $\pi^\# : G \ltimes G \to \mathcal{B}(L^2(\mathcal{O}))$ from Definition 2.19 (see also Remark 2.20). The assumption that π is square-integrable modulo the center of G implies by Theorem 3.5(2) that $\pi^\#$ is given by

$$\pi^\# : G \ltimes G \to \mathcal{B}\big(L^2(\mathcal{O})\big), \qquad \pi^\#(g, \exp Y) f = \big(\mathrm{e}^{\mathrm{i}\langle\cdot, Y\rangle} \# f\big) \circ \mathrm{Ad}_G^*\big(g^{-1}\big)|_{\mathcal{O}}.$$

Since the unitary operator $\mathrm{Op}^\pi : L^2(\mathcal{O}) \to \mathfrak{S}_2(\mathcal{H})$ intertwines $\pi^\#$ and $\pi^\ltimes$, we get by Lemma 3.6 that $\pi^\#$ is also a unitary irreducible representation which is square-integrable modulo the center $Z \times Z$ of $G \ltimes G$.

Corollary 3.8 *Let G be a simply connected, nilpotent Lie group with a unitary irreducible representation $\pi : G \to \mathcal{B}(\mathcal{H})$ which is square-integrable modulo the center of G. If $r, r_1, r_2 \in [1, \infty]$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, then the cross-Wigner distribution associated with any predual to the coadjoint of the representation π defines a continuous sesquilinear map*

$$\mathcal{W}(\cdot, \cdot) : M^{r_1, r_1}(\pi) \times M^{r_2, r_2}(\pi) \to M^{r, \infty}\big(\pi^\#\big).$$

Proof Firstly use Corollary 2.24. Then the conclusion follows since both π and $\pi^\#$ are square-integrable representations (see also Remark 3.7), hence Theorem 3.3 shows that the topologies of the modulation spaces $M^{r_1, r_1}(\pi)$, $M^{r_2, r_2}(\pi)$, and $M^{r, \infty}(\pi^\#)$ can be defined by any special choice of window functions. □

Corollary 3.9 *If G be a simply connected, nilpotent Lie group with a unitary irreducible representation $\pi : G \to \mathcal{B}(\mathcal{H})$ which is square-integrable modulo the center of G, then the cross-Wigner distribution associated with any predual to the coadjoint of the representation π defines a continuous sesquilinear map*

$$\mathcal{W}(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \to M^{1, \infty}\big(\pi^\#\big).$$

Proof Just apply Corollary 3.8 with $r_1 = r_2 = 2$ and $r = 1$; and recall from Example 2.17 that $M^{2,2}(\pi) = \mathcal{H}$. □

In the special case of the Schrödinger representation for the Heisenberg group, the following corollary recovers the assertion of Theorem 1.1 in [18] concerning the boundedness of pseudo-differential operators defined by the classical Weyl-Hörmander calculus on $\mathbb{R}^n$.

Corollary 3.10 *Let G be a simply connected, nilpotent Lie group with a unitary irreducible representation $\pi : G \to \mathcal{B}(\mathcal{H})$ which is square-integrable modulo the center of G. If $r, r', r_1, r_2 \in [1, \infty]$ satisfy the equations $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = 1 - \frac{1}{r'}$, then the following assertions hold*:

(1) *For every symbol $a \in M^{r', 1}(\pi^\#)$ we have a bounded linear operator*

$$\mathrm{Op}^\pi(a) : M^{r_1, r_1}(\pi) \to M^{r_2, r_2}(\pi).$$

(2) *The linear mapping* $\mathrm{Op}^{\pi}(\cdot): M^{r',1}(\pi^{\#}) \to \mathcal{B}(M^{r_1,r_1}(\pi), M^{r_2,r_2}(\pi))$ *is continuous*.

Proof Firstly use Corollary 2.25. Then the conclusion follows since Theorem 3.3 shows that the topologies of the modulation spaces involved in the statement can be defined by any special choice of window functions. □

Corollary 3.11 *If* G *be a simply connected, nilpotent Lie group with a unitary irreducible representation* $\pi: G \to \mathcal{B}(\mathcal{H})$ *which is square-integrable modulo the center of* G, *then the following assertions hold*:

(1) *For every* $a \in M^{\infty,1}(\pi^{\#})$ *we have* $\mathrm{Op}^{\pi}(a) \in \mathcal{B}(\mathcal{H})$.
(2) *The linear mapping* $\mathrm{Op}^{\pi}(\cdot): M^{\infty,1}(\pi^{\#}) \to \mathcal{B}(\mathcal{H})$ *is continuous*.

Proof This is the special case of Corollary 3.10 with $r_1 = r_2 = 2$ and $r = 1$, since Example 2.17 shows that $M^{2,2}(\pi) = \mathcal{H}$. □

4 Schrödinger Representations of the Heisenberg Groups

We show in the present section that, in the special case of the Heisenberg group, the modulation spaces of symbols defined in our paper are in fact nothing else than the modulation spaces widely used in time-frequency analysis.

4.1 Schrödinger Representations

Let $\mathcal{V}$ be a finite-dimensional vector space endowed with a nondegenerate bilinear form denoted by $(p, q) \mapsto p \cdot q$. The *Heisenberg algebra* $\mathfrak{h}_{\mathcal{V}} = \mathcal{V} \times \mathcal{V} \times \mathbb{R}$ is the Lie algebra with the bracket

$$\big[(q, p, t), (q', p', t')\big] = \big[(0, 0, p \cdot q' - p' \cdot q)\big].$$

The *Heisenberg group* $\mathbb{H}_{\mathcal{V}}$ is just $\mathfrak{h}_{\mathcal{V}}$ thought of as a group with the multiplication $*$ defined by

$$X * Y = X + Y + \frac{1}{2}[X, Y].$$

The unit element is $0 \in \mathbb{H}_{\mathcal{V}}$ and the inversion mapping given by $X^{-1} := -X$. The *Schrödinger representation* is the unitary representation $\pi_{\mathcal{V}}: \mathbb{H}_{\mathcal{V}} \to \mathcal{B}(L^2(\mathcal{V}))$ defined by

$$\big(\pi_{\mathcal{V}}(q, p, t) f\big)(x) = \mathrm{e}^{\mathrm{i}(p \cdot x + \frac{1}{2} p \cdot q + t)} f(x + q) \quad \text{for a.e. } x \in \mathcal{V} \tag{4.1}$$

for arbitrary $f \in L^2(\mathcal{V})$ and $(q, p, t) \in \mathbb{H}_{\mathcal{V}}$. This is a square-integrable representation (modulo the center, as explained before) and the corresponding coadjoint orbit of $\mathbb{H}_{\mathcal{V}}$ is

$$\mathcal{O} = \big\{\xi : \mathfrak{h}_{\mathcal{V}} \to \mathbb{R} \text{ linear} \mid \xi(0, 0, 1) = 1\big\}. \tag{4.2}$$

Let $\xi_0 \in \mathcal{O}$ be the functional satisfying $\xi_0(q,p,0)=0$ for every $q,p\in\mathcal{V}$. If we denote $\dim\mathcal{V}=n$, then any basis $\{x_1,\dots,x_n\}$ in $\mathcal{V}$ naturally gives rise to the Jordan-Hölder basis

$$(x_1,0,0),\dots,(x_n,0,0),(0,x_1,0),\dots,(0,x_n,0),(0,0,1)$$

in $\mathfrak{h}_{\mathcal{V}}$ and the corresponding predual of $\mathcal{O}$ is

$$(\mathfrak{h}_{\mathcal{V}})_e=\mathcal{V}\times\mathcal{V}\times\{0\}.$$

For the sake of an easier comparison with the previously obtained results we shall denote $G=\mathbb{H}_{\mathcal{V}}$ and $\mathfrak{g}=\mathfrak{h}_{\mathcal{V}}$ from now on, and in particular we shall denote $\mathfrak{g}_e=(\mathfrak{h}_{\mathcal{V}})_e$.

4.2 Computing the Moyal Product Representation

Recall from [24] that for every $f,h\in\mathcal{S}(\mathcal{O})$ we have

$$(\forall\xi\in\mathcal{O})\quad (f\#h)(\xi)=\iint_{\mathfrak{g}_e\times\mathfrak{g}_e}\mathrm{e}^{\mathrm{i}\langle\xi,X+Y\rangle}\mathrm{e}^{(\mathrm{i}/2)\langle\xi_0,[X,Y]\rangle}\widehat{f}(X)\widehat{h}(Y)\,\mathrm{d}X\,\mathrm{d}Y.$$

It then follows by a duality argument that for every $f\in\mathcal{S}(\mathcal{O})$ and $V\in\mathfrak{g}$ we have

$$\big(f\#\mathrm{e}^{-\mathrm{i}\langle\cdot,V\rangle}\big)(\xi)=\int_{\mathfrak{g}_e}\mathrm{e}^{\mathrm{i}\langle\xi,X-V\rangle}\mathrm{e}^{(\mathrm{i}/2)\langle\xi_0,[X,-V]\rangle}\widehat{f}(X)\,\mathrm{d}X,$$

whence

$$(\forall V\in\mathfrak{g})\ (\forall\xi\in\mathcal{O})\quad \big(f\#\mathrm{e}^{-\mathrm{i}\langle\cdot,V\rangle}\big)(\xi)=\mathrm{e}^{-\mathrm{i}\langle\xi,V\rangle}f\big(\xi+(1/2)\xi_0\circ\mathrm{ad}_{\mathfrak{g}}V\big).\tag{4.3}$$

Since $\overline{f\#h}=\bar{h}\#\bar{f}$, we also get

$$(\forall V\in\mathfrak{g})\ (\forall\xi\in\mathcal{O})\quad \big(\mathrm{e}^{\mathrm{i}\langle\cdot,V\rangle}\#f\big)(\xi)=\mathrm{e}^{\mathrm{i}\langle\xi,V\rangle}f\big(\xi+(1/2)\xi_0\circ\mathrm{ad}_{\mathfrak{g}}V\big).\tag{4.4}$$

Now for arbitrary $X,Y\in\mathfrak{g}$, $f\in\mathcal{S}(\mathcal{O})$, and $\xi\in\mathcal{O}$ we get

$$\begin{aligned}\big(\mathrm{e}^{\mathrm{i}\langle\cdot,X+Y\rangle}\#f\#\mathrm{e}^{-\mathrm{i}\langle\cdot,X\rangle}\big)&=\mathrm{e}^{\mathrm{i}\langle\xi,X+Y\rangle}\big(f\#\mathrm{e}^{-\mathrm{i}\langle\cdot,X\rangle}\big)\big(\xi+(1/2)\xi_0\circ\mathrm{ad}_{\mathfrak{g}}(X+Y)\big)\\&=\mathrm{e}^{\mathrm{i}\langle\xi,X+Y\rangle}\mathrm{e}^{-\mathrm{i}\langle\xi+(1/2)\xi_0\circ\mathrm{ad}_{\mathfrak{g}}(X+Y),X\rangle}\\&\quad\times f\big(\xi+(1/2)\xi_0\circ\mathrm{ad}_{\mathfrak{g}}(X+Y)+(1/2)\xi_0\circ\mathrm{ad}_{\mathfrak{g}}X\big)\\&=\mathrm{e}^{\mathrm{i}\langle\xi,Y\rangle}\mathrm{e}^{(\mathrm{i}/2)\langle\xi_0,[X,Y]\rangle}\\&\quad\times f\big(\xi+(1/2)\xi_0\circ\mathrm{ad}_{\mathfrak{g}}\big(X+(1/2)Y\big)\big).\end{aligned}$$

By taking into account (2.13), we now see that the unitary irreducible representation $\pi^{\#}_{\mathcal{V}}\colon G\ltimes G\to\mathcal{B}(L^2(\mathcal{O}))$ is given by

$$\begin{aligned}&\big(\pi^{\#}_{\mathcal{V}}\big(\exp_{G\ltimes G}(X,Y)\big)f\big)(\xi)\\&\quad=\mathrm{e}^{\mathrm{i}(\langle\xi,Y\rangle+\langle\xi_0,[X,Y]\rangle/2)}f\big(\xi+\xi_0\circ\mathrm{ad}_{\mathfrak{g}}\big(X+(1/2)Y\big)\big)\end{aligned}\tag{4.5}$$

where the latter equation follows by (4.2).

4.3 Abstract Unitary Equivalence

Denote the center of G by Z, with the corresponding Lie algebra $\mathfrak{z}$. The above formula yields $\exp_{G\ltimes G}(\mathfrak{z}\times\{0\})\subseteq \operatorname{Ker}\pi^{\#}_{\mathcal{V}}$, hence we get a unitary irreducible representation $\overline{\pi^{\#}_{\mathcal{V}}}\colon (G\ltimes G)/(Z\times\{\mathbf{1}\})\to\mathcal{B}(L^2(\mathcal{O}))$. Also note that there exist the natural isomorphisms of Lie groups

$$(G\ltimes G)/\big(Z\times\{\mathbf{1}\}\big)\simeq (G/Z)\ltimes G\simeq \mathbb{H}_{\mathcal{V}\times\mathcal{V}}. \tag{4.6}$$

By specializing (4.5) for $X,Y\in\mathfrak{z}$ we can see that the representation $\overline{\pi^{\#}_{\mathcal{V}}}$ has the same central character as the Schrödinger representation of the Heisenberg group $\mathbb{H}_{\mathcal{V}\times\mathcal{V}}$, hence they are unitarily equivalent to each other, as a consequence of the Stone-von Neumann theorem.

4.4 Specific Unitary Equivalence

Alternatively, we can exhibit an explicit unitary equivalence as follows. Let us consider the affine isomorphism $\mathcal{O}\to(\mathcal{V}\times\mathcal{V})^*$, $\xi\mapsto\xi|_{\mathcal{V}\times\mathcal{V}\times\{0\}}$, and the natural embedding $\mathcal{V}\times\mathcal{V}\simeq\mathcal{V}\times\mathcal{V}\times\{0\}\hookrightarrow\mathfrak{h}_{\mathcal{V}}$. Now for $X,Y\in\mathcal{V}\times\mathcal{V}$ and $t\in\mathbb{R}$ we have $(X,Y,t)\in\mathbb{H}_{\mathcal{V}\times\mathcal{V}}\simeq(G\ltimes G)/(Z\times\{\mathbf{1}\})$ (see (4.6)) hence

$$\begin{aligned}\big(\overline{\pi^{\#}_{\mathcal{V}}}\big(\exp_{\mathbb{H}_{\mathcal{V}\times\mathcal{V}}}(X,Y,t)\big)f\big)(\xi)&=\big(\pi^{\#}_{\mathcal{V}}\big(\exp_{G\ltimes G}\big((X,0),(Y,t)\big)\big)f\big)(\xi)\\&=\mathrm{e}^{\mathrm{i}(\langle\xi,Y\rangle+t+\omega_{\xi_0}(X,Y)/2)}f\big(\xi+\xi_0\circ\operatorname{ad}_{\mathfrak{g}}\big(X+(1/2)Y\big)\big),\end{aligned}$$

where $\omega_{\xi_0}(V,W):=\langle\xi_0,[V,W]\rangle$ whenever $V,W\in\mathcal{V}\times\mathcal{V}\hookrightarrow\mathfrak{g}$. Thence we get

$$\big(\overline{\pi^{\#}_{\mathcal{V}}}\big(\exp_{\mathbb{H}_{\mathcal{V}\times\mathcal{V}}}\big(X-(1/2)Y,Y,t\big)\big)f\big)(\xi)=\mathrm{e}^{\mathrm{i}(\langle\xi,Y\rangle+t+\omega_{\xi_0}(X,Y)/2)}f(\xi+\xi_0\circ\operatorname{ad}_{\mathfrak{g}}X).$$

Note that $\psi\colon\mathfrak{h}_{\mathcal{V}\times\mathcal{V}}\to\mathfrak{h}_{\mathcal{V}\times\mathcal{V}}$, $(X,Y,t)\mapsto(X-(1/2)Y,Y,t)$ is an automorphism of the Heisenberg algebra $\mathfrak{h}_{\mathcal{V}\times\mathcal{V}}$, hence, by denoting by $\Psi\colon\mathbb{H}_{\mathcal{V}\times\mathcal{V}}\to\mathbb{H}_{\mathcal{V}\times\mathcal{V}}$ the corresponding automorphism of the Heisenberg group $\mathbb{H}_{\mathcal{V}\times\mathcal{V}}$, we get

$$\big(\widetilde{\pi}\big(\exp_{\mathbb{H}_{\mathcal{V}\times\mathcal{V}}}(X,Y,t)\big)f\big)(\xi)=\mathrm{e}^{\mathrm{i}(\langle\xi,Y\rangle+\omega_{\xi_0}(X,Y)/2+t)}f(\xi+\xi_0\circ\operatorname{ad}_{\mathfrak{g}}X),$$

where $\widetilde{\pi}:=\overline{\pi^{\#}_{\mathcal{V}}}\circ\Psi$ is again a representation of the Heisenberg group $\mathbb{H}_{\mathcal{V}\times\mathcal{V}}$. Then for arbitrary $V\in\mathcal{V}\times\mathcal{V}$ we get

$$\begin{aligned}&\big(\widetilde{\pi}\big(\exp_{\mathbb{H}_{\mathcal{V}\times\mathcal{V}}}(X,Y,t)\big)f\big)(\xi_0+\xi_0\circ\operatorname{ad}_{\mathfrak{g}}V)\\&\quad=\mathrm{e}^{\mathrm{i}(\omega_{\xi_0}(V,Y)+\omega_{\xi_0}(X,Y)/2+t)}f\big(\xi_0+\xi_0\circ\operatorname{ad}_{\mathfrak{g}}(V+X)\big).\end{aligned}\tag{4.7}$$

Now let us define the affine isomorphism

$$A\colon\mathcal{V}\times\mathcal{V}\to\mathcal{O},\qquad V\mapsto\xi_0+\xi_0\circ\operatorname{ad}_{\mathfrak{g}}V$$

and consider the unitary operator $U: L^2(\mathcal{V}\times\mathcal{V}) \to L^2(\mathcal{O})$, $f \mapsto f\circ A^{-1}$. It follows by the above equation that if we define the Heisenberg group $\mathbb{H}_{\mathcal{V}\times\mathcal{V}}$ by using the nondegenerate bilinear map

$$(\mathcal{V}\times\mathcal{V})\times(\mathcal{V}\times\mathcal{V}) \to \mathbb{R}, \qquad (V,W)\mapsto -\omega_{\xi_0}(V,W),$$

then the unitary operator U intertwines the representation $\widetilde{\pi}: \mathbb{H}_{\mathcal{V}\times\mathcal{V}} \to \mathcal{B}(L^2(\mathcal{O}))$ and the Schrödinger representation $\pi_{\mathcal{V}\times\mathcal{V}}: \mathbb{H}_{\mathcal{V}\times\mathcal{V}} \to \mathcal{B}(L^2(\mathcal{V}\times\mathcal{V}))$. In other words, the operator U induces a unitary equivalence of the representation $\overline{\pi^{\#}_{\mathcal{V}}}$ with the representation $\pi_{\mathcal{V}\times\mathcal{V}}\circ\Psi^{-1}$.

4.5 Determining the Modulation Spaces of Symbols

It follows by the above discussion that the operator U induces isomorphisms between the modulation spaces for the representations

$$\pi^{\#}_{\mathcal{V}}: G\ltimes G \to \mathcal{B}\big(L^2(\mathcal{O})\big) \quad \text{and} \quad \pi_{\mathcal{V}\times\mathcal{V}}\circ\Psi^{-1}: \mathbb{H}_{\mathcal{V}\times\mathcal{V}} \to \mathcal{B}\big(L^2(\mathcal{V}\times\mathcal{V})\big).$$

Now note that for arbitrary $r,s\in[1,\infty]$ we have $M^{r,s}(\pi_{\mathcal{V}\times\mathcal{V}}\circ\Psi^{-1}) = M^{r,s}(\pi_{\mathcal{V}\times\mathcal{V}})$ since the norm of any measurable function $f: \mathcal{V}\times\mathcal{V}\to\mathbb{C}$ in $L^{r,s}(\mathcal{V}\times\mathcal{V})$ is equal to the norm of the function $(X,Y)\mapsto f(X+(1/2)Y,Y)$ in the same space. Therefore *the operator $f\mapsto f\circ A^{-1}$ actually induces an isomorphism from the modulation space $M^{r,s}(\pi^{\#}_{\mathcal{V}})$ onto the modulation space $M^{r,s}(\pi_{\mathcal{V}\times\mathcal{V}})$ of the Schrödinger representation $\pi_{\mathcal{V}\times\mathcal{V}}: \mathbb{H}_{\mathcal{V}\times\mathcal{V}} \to \mathcal{B}(L^2(\mathcal{V}\times\mathcal{V}))$*. Finally, recall that $M^{r,s}(\pi_{\mathcal{V}\times\mathcal{V}}) = M^{r,s}(\mathcal{V}\times\mathcal{V})$, where the latter is just the classical modulation space on $\mathcal{V}\times\mathcal{V}$ as used for instance in [17].

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