Ann Glob Anal Geom (2009) 36:293–322
DOI 10.1007/s10455-009-9166-8

ORIGINAL PAPER

Magnetic pseudo-differential Weyl calculus on nilpotent Lie groups

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Received: 9 February 2009 / Accepted: 31 March 2009 / Published online: 22 April 2009

Abstract We develop a pseudo-differential Weyl calculus on nilpotent Lie groups, which allows one to deal with magnetic perturbations of right invariant vector fields. For this purpose, we investigate an infinite-dimensional Lie group constructed as the semidirect product of a nilpotent Lie group and an appropriate function space thereon. We single out an appropriate coadjoint orbit in the semidirect product and construct our pseudo-differential calculus as a Weyl quantization of that orbit.

Keywords Weyl calculus · Magnetic field · Lie group · Semidirect product

Mathematics Subject Classification (2000) Primary 47G30 · Secondary 22E25, 22E65, 35S05

1 Introduction

The Weyl calculus of pseudo-differential operators on $\mathbb{R}^n$ initiated in [17] is a central topic in the theory of linear partial differential equations and has been much studied and extended in several directions, among which we mention the pseudo-differential Weyl calculus on nilpotent Lie groups systematically developed in [31]. In the present article, we focus on a circle of ideas with a similar flavor and show that the coadjoint orbits of certain locally convex infinite-dimensional Lie groups (in the sense of [35]) can be employed in order to fill in the gap between two different lines of investigation motivated by the quantum theory:

– the magnetic pseudo-differential Weyl calculus on $\mathbb{R}^n$, initiated independently in [21] and in [28], and further developed in [20] and other works;

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– the program of Weyl quantization for coadjoint orbits of some finite-dimensional Lie groups including the nilpotent ones ([38,36]) and semidirect products involving certain semisimple Lie groups (see [8–10], and the references therein).

Recall that a *magnetic potential* on a Lie group G is simply a 1-form $A \in \Omega^1(G)$, and the corresponding *magnetic field* is $B = dA \in \Omega^2(G)$. The purpose of a magnetic pseudo-differential calculus on G is to facilitate the investigation on first-order linear differential operators of the form

$$-\mathrm{i}P_0 + A(Q)P_0, \tag{1.1}$$

where P_0 is a right invariant vector field on G and $A(Q)P_0$ stands for the operator defined by the multiplication by the function obtained by applying the (non-invariant) 1-form A to the vector field P_0 at every point in G.

In the special case of the abelian Lie group $G = (\mathbb{R}^n, +)$, we have $A = A_1 \mathrm{d}x_1 + \cdots + A_n \mathrm{d}x_n \in \Omega^1(\mathbb{R}^n)$ and the operators (1.1) on $\mathbb{R}^n$ are precisely the linear partial differential operators determined by the vectors $P_0 = (p_1, \ldots, p_n) \in \mathbb{R}^n$,

$$\mathrm{i}\left(p_1 \frac{\partial}{\partial x_1} + \cdots + p_n \frac{\partial}{\partial x_n}\right) + (p_1 A_1(Q) + \cdots + p_n A_n(Q)) = \sum_{j=1}^{n} p_j \left(\mathrm{i}\frac{\partial}{\partial x_j} + A_j(Q)\right) \tag{1.2}$$

where we denote by $A_1(Q), \ldots, A_n(Q)$ the operators of multiplication by the coefficients of the 1-form A. We refer to [20] for the pseudo-differential calculus of the operators (1.2) extending the Weyl calculus constructed in the non-magnetic case (that is, $A = 0$) in the paper [17].

On the other hand, a version of the Weyl calculus for right invariant differential operators on nilpotent Lie groups has been developed in a series of papers including [31–33,27,36,12,13], and there are remarkable applications of this calculus to various problems on partial differential equations on Lie groups. See also [2,19,29,30,14], and [7] for other interesting results related to this circle of ideas.

For these reasons it is quite natural to try to provide a unifying approach to the areas of research mentioned in the preceding two paragraphs. It is one of the aims of the present article to do that by proposing a pseudo-differential calculus on simply connected nilpotent Lie groups which takes into account a given magnetic field. Our strategy is to pick an appropriate left-invariant space $\mathcal{F}$ of functions containing the "coefficients of the magnetic field" on the Lie group G under consideration and then to work within the semidirect product $M = \mathcal{F} \rtimes_\lambda G$. The latter is, in general, an infinite-dimensional Lie group, and yet, we can single out a suitable coadjoint orbit $\mathcal{O}$ of M, which is a *finite-dimensional* symplectic manifold endowed with the Kirillov–Kostant–Souriau 2-form and is actually symplectomorphic to the cotangent bundle T^*G (see Proposition 2.9). The spaces of symbols for our pseudo-differential calculus will be function spaces on the orbit $\mathcal{O}$, which does not depend on the magnetic field. However, we have to take into account a *magnetic predual* $\mathcal{O}_*$ for the orbit $\mathcal{O}$ (Definition 3.4). The set $\mathcal{O}_*$ is just a "copy" of $\mathcal{O}$ contained in the Lie algebra $\mathfrak{m}$ of the infinite-dimensional Lie group M and is the image of $\mathcal{O}$ by a certain mapping θ defined in terms of a magnetic potential $A \in \Omega^1(G)$. In the case $G = (\mathbb{R}^n, +)$, the mapping θ is $(x, \xi) \mapsto (\xi + A(x), x)$.

In the general case, if two magnetic potentials give rise to the same magnetic field, then the corresponding copies of $\mathcal{O}$ in the Lie algebra $\mathfrak{m}$ are moved to each other by the adjoint action of the Lie group M. This leads to the *gauge covariance* of the pseudo-differential calculus, which we are going to attach to the copy $\mathcal{O}_*$ by the formula

$$\mathrm{Op}^A(a)f = \int\limits_{\mathcal{O}_*} \check{a}(v)\pi(\exp_M v) f \,\mathrm{d}\mu(v)$$

for suitable symbols $a\colon \mathcal{O} \to \mathbb{C}$ and functions $f : G \to \mathbb{C}$, where $\check{a}$ stands for the inverse symplectic Fourier transform of a. (It will be actually convenient to work with the above integral after the change of variables $v = \theta(x, \xi)$ with $(x, \xi) \in T^*G$; compare (4.7) and (4.1).) Here, μ is the Liouville measure corresponding to the symplectic structure on the magnetic predual $\mathcal{O}_* \subseteq \mathfrak{m}$ and π is a natural irreducible unitary representation of the infinite-dimensional Lie group M on $L^2(G)$ which corresponds to the coadjoint orbit $\mathcal{O}$ as in the orbit method [22,23]. We show in Theorem 4.4 that the magnetic pseudo-differential Weyl calculus on a nilpotent Lie group G possesses appropriate versions of the basic properties pointed out in the abelian case $G = (\mathbb{R}^n, +)$ in [28], however, the proofs in the present situation are considerably more difficult and require proving properties of the nilpotent Lie algebras, which may also have an independent interest (see for instance Proposition 3.2). We mention that when $G = (\mathbb{R}^n, +)$, if $\mathcal{F}$ is the $(n + 1)$-dimensional vector space of affine functions then one recovers the classical Weyl calculus for pseudo-differential operators, while for $\mathcal{F} = \mathcal{C}^\infty_{\mathrm{pol}}(\mathbb{R}^n)$ the magnetic Weyl calculus of [28] is recovered.

It is noteworthy that, just as in the abelian case, there exists a magnetic Moyal product $\#^A$ on the Schwartz space $\mathcal{S}(\mathcal{O})$, and—as a consequence of the gauge covariance—the isomorphism class of the associative Fréchet algebra $(\mathcal{S}(\mathcal{O}), \#^A)$ depends only on the magnetic field $B = dA \in \Omega^2(G)$. Our Theorem 4.7 records an explicit formula for $\#^A$ in the case when G is a two-step nilpotent Lie group, which extends the corresponding formula established in [28] and [21] and already covers the important situation of the Heisenberg groups. We postpone to forthcoming papers both the formula for magnetic Moyal product in the case of a general (simply connected) nilpotent Lie group and the description and applications of more general classes of symbols for the magnetic pseudo-differential Weyl calculus. We aim to apply these techniques to more general function spaces $\mathcal{F}$ to obtain more general radiation conditions for various Hamiltonian operators appearing in mathematical physics (see for instance [5] and [6]).

Notation Throughout the article, we denote by $\mathcal{S}(\mathcal{V})$, the Schwartz space on a finite-dimensional real vector space $\mathcal{V}$. That is, $\mathcal{S}(\mathcal{V})$ is the set of all smooth functions that decay faster than any polynomial together with their partial derivatives of arbitrary order. Its topological dual—the space of tempered distributions on $\mathcal{V}$—is denoted by $\mathcal{S}'(\mathcal{V})$. We use the notation $\mathcal{C}^\infty_{\mathrm{pol}}(\mathcal{V})$ for the space of smooth functions that grow polynomially together with their partial derivatives of arbitrary order. We use $\langle\cdot,\cdot\rangle$ to denote any duality pairing between finite-dimensional real vector space whose meaning is clear from the context. In particular, this may stand for the self-duality given a symplectic bilinear form.

2 Semidirect products

2.1 One-parameter subgroups in topological groups

Definition 2.1 For an arbitrary topological group G, we define

$$\mathbf{L}(G) = \{X\colon \mathbb{R} \to G \mid X \text{ homomorphism of topological groups}\}$$

and endow this set with the topology of uniform convergence on compact intervals in $\mathbb{R}$. The *adjoint action* of G on $\mathbf{L}(G)$ is the continuous mapping

$$\mathrm{Ad}_G\colon G \times \mathbf{L}(G) \to \mathbf{L}(G), \quad (g, X) \mapsto \mathrm{Ad}_G(g)X := gX(\cdot)g^{-1}.$$

The *exponential function* of G is the continuous mapping

$$\exp_G\colon \mathbf{L}(G) \to G, \quad X \mapsto \exp_G X := X(1).$$

If H is another topological group, then every homomorphism of topological groups $\psi\colon G \to H$ induces a continuous mapping $\mathbf{L}(\psi)\colon \mathbf{L}(G) \to \mathbf{L}(H), \quad X \mapsto \psi \circ X$ and it is easy to see that the diagram

$$\begin{array}{ccc} \mathbf{L}(G) & \xrightarrow{\mathbf{L}(\psi)} & \mathbf{L}(H) \\ \exp_G \downarrow & & \downarrow \exp_H \\ G & \xrightarrow{\psi} & H \end{array} \tag{2.1}$$

is commutative. In fact, $\mathbf{L}(\cdot)$ is a functor from the category of topological groups to the category of topological spaces, and exp is a natural transformation. We refer to [15] and Chap. II in [16] for these concepts and related results.

Remark 2.2 If G is a finite-dimensional Lie group, then every one-parameter group $X \in \mathbf{L}(G)$ is actually smooth and there exists a bijective map

$$\mathbf{L}(G) \simeq T_{\mathbf{1}}G$$

which takes every one-parameter subgroup $X \in \mathbf{L}(G)$ into its infinitesimal generator $\dot{X}(0) \in T_{\mathbf{1}}G$. More generally, this assertion holds if G is a locally exponential Lie group (modeled on a locally convex space); see Def. II.5.1, Def. IV.1.1, and Th. IV.1.18 in [35].

Remark 2.3 Let G be a topological group and $\mathcal{Y}$ a complex Banach space. We denote

$$\mathcal{C}(\mathcal{Y}) = \{T\colon \mathcal{D}(T) \subseteq \mathcal{Y} \to \mathcal{Y} \mid T \text{ closed, densely defined, linear operator}\}.$$

If $\pi\colon G \to \mathcal{B}(\mathcal{Y})$ is a so-continuous representation which is uniformly bounded (that is, $\sup_{g\in G} \|\pi(g)\| < \infty$), then for every $X \in \mathbf{L}(G)$ we get a bounded, so-continuous one-parameter group $\pi \circ X\colon \mathbb{R} \to \mathcal{B}(\mathcal{Y})$. Thus, we can define a mapping

$$\mathbf{L}(\pi)\colon \mathbf{L}(G) \to \mathcal{C}(\mathcal{Y}), \quad X \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi(X(t))$$

by means of the Hille–Yosida theorem, and we have

$$(\forall X \in \mathbf{L}(G)) \quad \pi(\exp_G X) = \exp(\mathbf{L}(\pi)X) \tag{2.2}$$

(which should be compared with (2.1)). Now assume that $\mathcal{V}$ is a linear subspace of $\mathcal{Y}$ and for every $X \in \mathbf{L}(G)$ we have $\mathcal{V} \subseteq \mathcal{D}(\mathbf{L}(\pi)X)$ and $(\mathbf{L}(\pi)X)\mathcal{V} \subseteq \mathcal{V}$. If, moreover, G is a topological group with Lie algebra in the sense of Chap. II in [16], then it follows by the Trotter formulas that $\mathbf{L}(\pi)$ induces a representation of the Lie algebra $\mathbf{L}(G)$ by linear maps on $\mathcal{V}$.

2.2 Semidirect products and their exponential maps

Definition 2.4 Let G be a topological group and $\mathcal{F}$ be a real topological vector space with the unital associative algebra of continuous endomorphisms denoted by End $(\mathcal{F})$. Assume that $\alpha\colon G \to \text{End}\,(\mathcal{F})$, $g \mapsto \alpha_g$, is a continuous representation of G on $\mathcal{F}$, that is, $\alpha_{\mathbf{1}} = \text{id}_{\mathcal{F}}, \alpha_{g_1 g_2} = \alpha_{g_1}\alpha_{g_2}$ for all $g_1, g_2 \in G$, and the mapping $G \times \mathcal{F} \to \mathcal{F}$, $(g, \phi) \mapsto \alpha_g \phi$ is continuous. Then the *semidirect product of groups* denoted $\mathcal{F} \rtimes_\alpha G$ (or $G \ltimes_\alpha \mathcal{F}$) is the topological group whose underlying topological space is $\mathcal{F} \times G$ (respectively, $G \times \mathcal{F}$) with the multiplication

$$(\phi_1, g_1)(\phi_2, g_2) = \big(\phi_1 + \alpha_{g_1}\phi_2, g_1 g_2\big) \tag{2.3}$$

(respectively, $(g_1, \phi_1)(g_2, \phi_2) = (g_1 g_2, \phi_1 + \alpha_{g_1}\phi_2)$) whenever $g_1, g_2 \in G$ and $\phi_1, \phi_2 \in \mathcal{F}$. It is easy to see that $(0, \mathbf{1})$ is the unit element in the group $\mathcal{F} \times_\alpha G$, while the inversion is given by

$$(\phi, g)^{-1} = (-\alpha_{g^{-1}}\phi, g^{-1}) \tag{2.4}$$

for every $\phi \in \mathcal{F}$ and $g \in G$.

Now let $\mathfrak{g}$ be any real topological Lie algebra and assume that $\dot{\alpha}\colon \mathfrak{g} \to \text{End}\,(\mathcal{F})$, $X \mapsto \dot{\alpha}(X)$, is a continuous representation of $\mathfrak{g}$ on $\mathcal{F}$, that is, $\dot{\alpha}$ is a linear mapping such that $\dot{\alpha}([X_1, X_2]) = [\dot{\alpha}(X_1), \dot{\alpha}(X_2)] := \dot{\alpha}(X_1)\dot{\alpha}(X_2) - \dot{\alpha}(X_2)\dot{\alpha}(X_1)$ for all $X_1, X_2 \in \mathfrak{g}$ and the mapping $\mathfrak{g} \times \mathcal{F} \to \mathcal{F}$, $(X, \phi) \mapsto \dot{\alpha}(X)\phi$ is continuous. Then the *semidirect product of Lie algebras* denoted $\mathcal{F} \rtimes_{\dot{\alpha}} \mathfrak{g}$ is the topological Lie algebra whose underlying topological vector space is $\mathcal{F} \times \mathfrak{g}$ with the bracket

$$[(\phi_1, X_1), (\phi_2, X_2)] = (\dot{\alpha}(X_1)\phi_2 - \dot{\alpha}(X_2)\phi_1, [X_1, X_2]) \tag{2.5}$$

for every $X_1, X_2 \in \mathfrak{g}$ and $\phi_1, \phi_2 \in \mathcal{F}$. One can similarly define the semidirect product of Lie algebras $\mathfrak{g} \ltimes_{\dot{\alpha}} \mathcal{F}$.

Remark 2.5 In the setting of Definition 2.4, if G is a locally convex Lie group (see [35]), $\mathcal{F}$ is a complete locally convex vector space and the mapping $G \times \mathcal{F}$, $(g, \phi) \mapsto \alpha_g \phi$ is smooth, then it is straightforward to prove the following assertions:

(1) The semidirect product $M := \mathcal{F} \rtimes_\alpha G$ is a locally convex Lie group whose Lie algebra is $\mathfrak{m} := \mathcal{F} \rtimes_{\dot{\alpha}} \mathfrak{g}$, where $\mathfrak{g} = \mathbf{L}(G)$ is the Lie algebra of G and $\dot{\alpha}\colon \mathfrak{g} \to \text{End}\,(\mathcal{F})$ is defined by the condition that for every $\phi \in \mathcal{F}$ the linear mapping $\mathfrak{g} \to \mathcal{F}$, $X \mapsto \dot{\alpha}(X)\phi$ is the differential of the smooth mapping $G \to \mathcal{F}$, $g \mapsto \alpha_g \phi$ at the point $\mathbf{1} \in G$.

(2) The adjoint action of the Lie group M on its Lie algebra $\mathfrak{m}$ is given by

$$\text{Ad}_M\colon M \times \mathfrak{m} \to \mathfrak{m}, \quad (\text{Ad}_M(\phi, g))(\psi, X) = (\alpha_g \psi - \dot{\alpha}(\text{Ad}_G(g)X)\phi, \text{Ad}_G(g)X)$$

for $(\phi, g) \in \mathcal{F} \rtimes_\alpha G = M$ and $(\psi, X) \in \mathcal{F} \rtimes_{\dot{\alpha}} \mathfrak{g} = \mathfrak{m}$.

(3) The coadjoint action of the Lie group M on the dual of its Lie algebra $\mathfrak{m}^* = \mathcal{F}^* \times \mathfrak{g}^*$ is given by

$$\text{Ad}_M^*\colon M \times \mathfrak{m}^* \to \mathfrak{m}^*, \quad (\text{Ad}_M^*(\phi, g))(\nu, \xi) = (\alpha_{g^{-1}}^* \nu, \text{Ad}_G^*(g)\xi + \dot{\alpha}_\phi^* \alpha_{g^{-1}}^* \nu)$$

for $(\phi, g) \in \mathcal{F} \rtimes_\alpha G = M$ and $(\nu, \xi) \in \mathcal{F}^* \times \mathfrak{g}^* = \mathfrak{m}^*$, where $\alpha_\phi^*\colon \mathcal{F}^* \to \mathfrak{g}^*$ is the dual of the linear mapping $\alpha_\phi := \dot{\alpha}(\cdot)\phi\colon \mathfrak{g} \to \mathcal{F}$ (see item (1) above) and $\alpha_{g^{-1}}^*\colon \mathcal{F}^* \to \mathcal{F}^*$ is the dual of the mapping $\alpha_{g^{-1}}\colon \mathcal{F} \to \mathcal{F}$.

Example 2.6 Let $n \geq 1$ and assume that $\mathcal{F}$ is a linear subspace of the space of real Borel functions $\mathcal{B}_{\mathbb{R}}(\mathbb{R}^n)$ which is invariant under translations and is endowed with a linear topology such that the mapping

$$\mathbb{R}^n \times \mathcal{F} \to \mathcal{F}, \quad (q, f) \mapsto \alpha(q)f := f(q + \cdot)$$

is continuous. If we denote by α the corresponding action of the additive group $(\mathbb{R}^n, +)$ by endomorphisms of the group $(\mathcal{F}, +)$, then we can construct the semidirect product

$$G := \mathcal{F} \rtimes_\alpha \mathbb{R}^n,$$

which is a topological group with the multiplication defined by (2.3). Moreover, G has a natural unitary representation on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^n)$, defined by

$$\pi : G \to \mathcal{B}(\mathcal{H}), \quad \pi(f, q)\phi = \mathrm{e}^{\mathrm{i}f}\phi(q + \cdot) \text{ whenever } \varphi \in \mathcal{H},\ f \in \mathcal{F}, \text{and } q \in \mathbb{R}^n. \quad (2.6)$$

If the topology of the function space $\mathcal{F}$ is stronger than the topology of pointwise convergence, then it follows by Lebesgue's dominated convergence theorem that the representation π is so-continuous.

Here are some special cases of this construction:

(1) For any integer $k \geq 1$, let us consider the following space of polynomial functions on $\mathbb{R}^n$

$$\mathcal{P}_k(\mathbb{R}^n) = \{f \in \mathbb{R}[q_1, \ldots, q_n] \mid \deg f \leq k\}.$$

The linear space $\mathcal{P}_k(\mathbb{R}^n)$ is finite-dimensional and is invariant under translations, hence we can form the semidirect product $G_k := \mathcal{P}_k(\mathbb{R}^n) \rtimes_\alpha \mathbb{R}^n$, which is a finite-dimensional, nilpotent, and simply connected Lie group. The special case $k = 1$ of this construction is particularly important, since G_1 is precisely the $(2n+1)$-dimensional Heisenberg group.

(2) If $\mathcal{F} = \mathcal{C}^\infty_{\mathbb{R}}(\mathbb{R}^n)$ with the natural Fréchet topology, then it follows by Ex. II.5.9 in [35] that $G = \mathcal{C}^\infty_{\mathbb{R}}(\mathbb{R}^n) \rtimes_\alpha \mathbb{R}^n$ is a (Fréchet-)Lie group whose Lie algebra is the semidirect product

$$\mathfrak{g} = \mathcal{C}^\infty_{\mathbb{R}}(\mathbb{R}^n) \rtimes_{\dot{\alpha}} \mathbb{R}^n,$$

where

$$\dot{\alpha} : \mathbb{R}^n \to \mathrm{Der}\left(\mathcal{C}^\infty_{\mathbb{R}}(\mathbb{R}^n)\right), \quad (p_1, \ldots, p_n) \mapsto p_1 \frac{\partial}{\partial q_1} + \cdots + p_n \frac{\partial}{\partial q_n}.$$

The Lie algebra $\mathfrak{g}$ fails to be abelian or even nilpotent; however, it is solvable since $[\mathfrak{g}, \mathfrak{g}] = \mathcal{C}^\infty_{\mathbb{R}}(\mathbb{R}^n)$ and hence $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = \{0\}$. As regards the finite-dimensional Lie groups $G_k := \mathcal{P}_k(\mathbb{R}^n) \rtimes_\alpha \mathbb{R}^n$ for $k \geq 1$, we also note that $G_1 \subset G_2 \subset \cdots \subset \overline{\bigcup_{k \geq 1} G_k} = G$.

The following statement partially extends Theorem 49.6 and Remark 38.9 in [24] and some facts noted in Example II.5.9 in [35]. See also Sect. 3 in [34] for the expression of the exponential map for a semidirect product of finite-dimensional Lie groups.

Proposition 2.7 *Let G be a topological group acting on a topological space D by an action denoted simply by*

$$G \times D \to D, \quad (g, x) \mapsto g \cdot x$$

and assume that $\mathcal{F}$ is a linear subspace of the space of real Borel functions $\mathcal{B}_{\mathbb{R}}(D)$, which is invariant under the translation operators $\alpha_g : \mathcal{B}_{\mathbb{R}}(D) \to \mathcal{B}_{\mathbb{R}}(D)$ defined by $(\alpha_g\phi)(x) =$

$\phi(g^{-1}\cdot x)$ for $g\in G, x\in D$, and $\phi\in\mathcal{B}_{\mathbb{R}}(D)$. Also assume that $\mathcal{F}$ is endowed with a complete, locally convex topology such that the mapping

$$G\times\mathcal{F}\to\mathcal{F},\quad (g,\phi)\mapsto\alpha_g\phi \tag{2.7}$$

is continuous. Then the following assertions hold:

(1) The mapping

$$\mathcal{F}\times\mathbf{L}(G)\to\mathcal{F},\quad (\phi,X)\mapsto\beta(X)\phi:=\int_0^1\alpha_{X(s)}\phi\,\mathrm{d}s \tag{2.8}$$

is well-defined and continuous.

(2) For every pair $(\phi,X)\in\mathcal{F}\times\mathbf{L}(G)$, the function

$$Z_{\phi,X}\colon\mathbb{R}\to\mathcal{F}\times G,\quad Z_{\phi,X}(t)=(t\beta(tX)\phi,X(t))=\left(\int_0^t\alpha_{X(s)}\phi\,\mathrm{d}s,X(t)\right)$$

has the property $Z_{\phi,X}\in\mathbf{L}(\mathcal{F}\rtimes_\alpha G)$. Moreover, $t\mapsto t\beta(tX)\phi$ is a differentiable curve in $\mathcal{F}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(t\beta(tX)\phi)=\phi$.

(3) Let $\psi\in\mathcal{F}$ and $X\in\mathbf{L}(G)$ such that the curve $\mathbb{R}\to\mathcal{F}, t\mapsto\alpha_{X(t)}\psi$ is differentiable, and denote $\dot\alpha(X)\psi:=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\alpha_{X(t)}\psi\in\mathcal{F}$. Then

$$(\forall\phi\in\mathcal{F})\quad\left(\mathrm{Ad}_{\mathcal{F}\rtimes_\alpha G}\psi\right)Z_{\phi,X}=Z_{\phi-\dot\alpha(X)\psi,X}\in\mathbf{L}(\mathcal{F}\rtimes_\alpha G).$$

(4) If we assume that G is a finite-dimensional Lie group acting on itself by left translations (hence $D=G$ and $(\alpha_g\phi)(x)=(\lambda_g\phi)(x)=\phi(g^{-1}x)$ for $g,x\in G$ and $\phi\in\mathcal{F}$) and there exists the continuous inclusion $\mathcal{F}\hookrightarrow\mathcal{C}^\infty(G)$ such that the mapping (2.7) is smooth, then $\mathcal{F}\rtimes_\lambda G$ is a locally convex Lie group with the following properties:

(a) The Lie algebra of $\mathcal{F}\rtimes_\lambda G$ is the semidirect product of Lie algebras $\mathcal{F}\rtimes_{\dot\lambda}\mathfrak{g}$, where $\mathfrak{g}:=\mathbf{L}(G)$, $\mathcal{F}$ is thought of as an abelian Lie algebra and the mapping $\dot\lambda\colon\mathfrak{g}\to\mathrm{Der}\,(\mathcal{F})$ is defined as in (3) above. (That is, $\dot\lambda$ is induced by the natural representation of the elements in $\mathfrak{g}$ as right-invariant vector fields on G.)

(b) The exponential map of the Lie group $\mathcal{F}\rtimes_\lambda G$ is defined by the formula

$$\exp_{\mathcal{F}\rtimes_\lambda G}\colon\mathcal{F}\rtimes_{\dot\lambda}\mathfrak{g}\to\mathcal{F}\rtimes_\lambda G,\quad(\phi,X)\mapsto(\beta(X)\phi,\exp_G X).$$

(c) Assume $G=(\mathbb{R}^n,+)$ with the generic point denoted by $(q_1,\dots,q_n)$. If $A_j,A_j,\psi\in\mathcal{F}$ and $j\in\{1,\dots,n\}$ satisfy $A'_j=A_j+\partial\psi/\partial q_j$, then $\left(\mathrm{Ad}(\exp_{\mathcal{F}\rtimes_\lambda\mathbb{R}^n}\psi)\right)(A'_j,p_j)=(A_j,p_j)\in\mathcal{F}\rtimes_{\dot\lambda}\mathbb{R}^n$.

Proof

(1) For every $(\phi,X)\in\mathcal{F}\times\mathbf{L}(G)$, the function $[0,1]\to\mathcal{F}, s\mapsto\alpha_{X(s)}\phi$ is Riemann integrable, since it is continuous and the locally convex space $\mathcal{F}$ is complete; see for instance, Lemma 2.5 in Chap. I of [24]. The continuity of the mapping $(\phi,X)\mapsto\beta(X)\phi$ follows by the continuity of (2.7) and the continuity properties of the Riemann integral.

(2) The second equality in the definition of $Z_{\phi,X}(t)$ follows by a change of variables in the Riemann integral (Corollary 2.6(3) in Chap. I of [24]), and $Z_{\phi,X}\colon\mathbb{R}\to\mathcal{F}\times G$

is continuous by the previous Assertion (1). Moreover, for arbitrary $t_1, t_2 \in \mathbb{R}$, we have

$$\begin{aligned}Z_{\phi,X}(t_1)Z_{\phi,X}(t_2) &= (t_1\beta(t_1X)\phi, X(t_1))(t_2\beta(t_2X)\phi, X(t_2))\\ &= (t_1\beta(t_1X)\phi + \alpha_{X(t_1)}(t_2\beta(t_2X)\phi), X(t_1)X(t_2))\\ &= (t_1\beta(t_1X)\phi + t_2\alpha_{X(t_1)}\beta(t_2X)\phi, X(t_1+t_2)).\end{aligned}$$

On the other hand,

$$\begin{aligned}t_1\beta(t_1X)\phi + t_2\alpha_{X(t_1)}\beta(t_2X)\phi &= t_1\int_0^1 \alpha_{X(t_1s)}\phi\, \mathrm{d}s + t_2\alpha_{X(t_1)}\int_0^1 \alpha_{X(t_2s)}\phi\, \mathrm{d}s\\ &= \int_0^{t_1} \alpha_{X(s)}\phi\, \mathrm{d}s + \alpha_{X(t_1)}\int_0^{t_2} \alpha_{X(s)}\phi\, \mathrm{d}s\\ &= \int_0^{t_1} \alpha_{X(s)}\phi\, \mathrm{d}s + \int_0^{t_2} \alpha_{X(t_1+s)}\phi\, \mathrm{d}s\\ &= \int_0^{t_1} \alpha_{X(s)}\phi\, \mathrm{d}s + \int_{t_1}^{t_1+t_2} \alpha_{X(s)}\phi\, \mathrm{d}s\\ &= \int_0^{t_1+t_2} \alpha_{X(s)}\phi\, \mathrm{d}s\\ &= (t_1+t_2)\beta((t_1+t_2)X)\phi\end{aligned}$$

and it follows that $Z_{\phi,X}(t_1)Z_{\phi,X}(t_2) = Z_{\phi,X}(t_1+t_2)$. Thus, $Z_{\phi,X} \in \mathbf{L}(\mathcal{F} \rtimes_\alpha G)$. The equality $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(t\beta(tX)\phi) = \phi$ follows by Lemma 2.5 in Chap. I of [24] again.

(3) First, note that $\alpha_{X(t_1+t_2)} = \alpha_{X(t_1)}\alpha_{X(t_2)}$ for every $t_1, t_2 \in \mathbb{R}$, hence, we have

$$(\forall s \in \mathbb{R}) \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=s}\alpha_{X(t)}\psi = \alpha_{X(s)}\dot{\alpha}(X)\psi. \tag{2.9}$$

On the other hand, it follows by (2.3) and (2.4) that $(\psi, \mathbf{1})^{-1} = (-\psi, \mathbf{1})$ and $(\psi, \mathbf{1})(\phi, g)(\psi, \mathbf{1})^{-1} = (\psi + \phi - \alpha_g\psi, g)$ whenever $\phi \in \mathcal{F}$ and $g \in G$. Therefore, for arbitrary $\phi \in \mathcal{F}$ and $t \in \mathbb{R}$ we get

$$\begin{aligned}\left((\mathrm{Ad}_{\mathcal{F}\rtimes_\alpha G}\psi)Z_{\phi,X}\right)(t) &= (\psi, \mathbf{1})Z_{\phi,X}(t)(\psi, \mathbf{1})^{-1}\\ &= (\psi, \mathbf{1})\left(\int_0^t \alpha_{X(s)}\phi\, \mathrm{d}s, X(t)\right)(\psi, \mathbf{1})^{-1}\\ &= \left(\psi + \int_0^t \alpha_{X(s)}\phi\, \mathrm{d}s - \alpha_{X(t)}\psi, X(t)\right)\end{aligned}$$

$$= \left(\int_0^t \alpha_{X(s)}\phi \, \mathrm{d}s - \int_0^t \frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=s} (\alpha_{X(r)}\psi) \, \mathrm{d}s, X(t) \right)$$
$$= \left(\int_0^t \alpha_{X(s)}(\phi - \dot{\alpha}(X)\psi) \, \mathrm{d}s, X(t) \right)$$
$$= Z_{\phi - \dot{\alpha}(X)\psi, X}(t),$$

where the next-to-last equality follows by (2.9).

(4) Let us denote $M = \mathcal{F} \rtimes_\lambda G$ and $\mathfrak{m} = \mathcal{F} \rtimes_{\dot{\lambda}} \mathfrak{g}$. It is clear that $\mathfrak{m} = T_{(0,\mathbf{1})}M$ so in order to prove that $\mathfrak{m} = \mathbf{L}(M)$, we still have to check that the operations of sum and bracket in these spaces agree. The latter fact follows, since for every $(\phi, X) \in \mathfrak{m}$ and $t \in \mathbb{R}$, we have $\exp_M(t(\phi, X)) = Z_{\phi,X}(t)$ by the above Assertion (2). This shows that (4a)–(4b) hold. The remaining property (4c) follows by Assertion (3). □

2.3 Coadjoint orbits of semidirect products

The symplectic structures on coadjoint orbits of semidirect products defined by *finite-dimensional* representations of Lie groups were thoroughly investigated in [3]. As we are interested in semidirect product $M = \mathcal{F} \rtimes_\lambda G$, where $\lambda \colon G \to \mathrm{End}\,(\mathcal{F})$ is a representation on a function space $\mathcal{F}$, which is, in general, infinite-dimensional, in this subsection we shall study a coadjoint orbit $\mathcal{O}$ of M that is not covered by the results in [3]. This orbit will play a central role in our construction of magnetic pseudo-differential operators.

Definition 2.8 Let G be a finite-dimensional Lie group and $\mathcal{F}$ be a linear subspace of $\mathcal{B}_{\mathbb{R}}(G)$ endowed with a locally convex topology. We say that the function space $\mathcal{F}$ is *admissible*, if it satisfies the following conditions:

(1) The linear space $\mathcal{F}$ is invariant under the representation of G by left translations,

$$\lambda \colon G \to \mathrm{End}\,(\mathcal{B}_{\mathbb{R}}(G)), \quad (\lambda_g\phi)(x) = \phi(g^{-1}x).$$

That is, if $\phi \in \mathcal{F}$ and $g \in G$ then $\lambda_g\phi \in \mathcal{F}$. We denote again by $\lambda \colon G \to \mathrm{End}\,(\mathcal{F})$ the restriction to $\mathcal{F}$ of the aforementioned representation of G.

(2) We have $\mathcal{F} \subseteq \mathcal{C}^\infty(G)$ and the topology of $\mathcal{F}$ is stronger than the topology induced from $\mathcal{C}^\infty(G)$. In other words, the inclusion mapping $\mathcal{F} \hookrightarrow \mathcal{C}^\infty(G)$ is continuous.

(3) The mapping $G \times \mathcal{F} \to \mathcal{F}$, $(g, \phi) \mapsto \lambda_g\phi$ is smooth. For every $\phi \in \mathcal{F}$, we denote by $\dot{\lambda}(\cdot)\phi \colon \mathfrak{g} \to \mathcal{F}$ the differential of the mapping $g \mapsto \lambda_g\phi$ at the point $\mathbf{1} \in G$. Thus, for all $X \in \mathfrak{g}$ and $g \in G$, we have

$$(\dot{\lambda}(X)\phi)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \phi(\exp_G(-tX)g) = -(\phi \circ R_g)'_0(X)$$
$$= -({\phi'}_g \circ (R_g)'_0)(X) = -\langle ((R_g)'_0)^*({\phi'}_g), X \rangle, \tag{2.10}$$

where $\langle \cdot, \cdot \rangle \colon \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the canonical duality pairing and $R_g \colon G \to G$, $x \mapsto xg$.

(4) The points in G are separated by the functions in $\mathcal{F}$, that is, for every $g_1, g_2 \in G$ with $g_1 \neq g_2$ there exists $\phi \in \mathcal{F}$ with $\phi(g_1) \neq \phi(g_2)$.

(5) We have $\{{\phi'}_g \mid \phi \in \mathcal{F}\} = T^*_g G$ for every $g \in G$.

It is clear that $\mathcal{C}^\infty(G)$ itself is admissible.

Proposition 2.9 *Let G be a finite-dimensional Lie group and $\mathcal{F} \hookrightarrow \mathcal{C}^\infty(G)$ be an admissible function space on G. Denote $M = \mathcal{F} \rtimes_\lambda G, \mathfrak{m} = \mathbf{L}(M)$ and for every $g \in G$ let $\delta_g : \mathcal{F} \to \mathbb{R}, \phi \mapsto \phi(g)$. Define*

$$\mathcal{O} := \{(\delta_g, \xi) \mid g \in G, \xi \in \mathfrak{g}^*\} \subseteq \mathcal{F}^* \times \mathfrak{g}^* = \mathfrak{m}^*.$$

Then $\mathcal{O}$ is a coadjoint orbit of the locally convex Lie group M, which has the following properties:

(1) The orbit $\mathcal{O}$ is a smooth finite-dimensional manifold such that for every $\mu \in \mathcal{O}$ the coadjoint action defines a trivial smooth bundle $\Pi_\mu : M \to \mathcal{O}, m \mapsto \mathrm{Ad}_M^*(m)\mu$.
(2) There exists a canonical symplectic form $\omega \in \Omega^2(\mathcal{O})$ invariant under the coadjoint action of M on $\mathcal{O}$, such that for every $\mu \in \mathcal{O}$ the pull-back $\Pi_\mu^*(\omega) \in \Omega^2(M)$ is a left invariant 2-form on M whose value at $\mathbf{1} \in M$ is the bilinear functional $(\Pi_\mu^*(\omega))_{\mathbf{1}} : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}, (X, Y) \mapsto -\mu([X, Y])$.
(3) The symplectic manifold $(\mathcal{O}, \omega)$ is symplectomorphic to the cotangent bundle T^*G endowed with its canonical symplectic structure.

Proof Set $\tilde{\delta}_{\mathbf{1}} := (\delta_{\mathbf{1}}, 0) \in \mathcal{O}$. It follows by Remark 2.5(3) that for every $(\phi, g) \in M = \mathcal{F} \rtimes_\lambda G$, we have

$$\Pi_{\tilde{\delta}_{\mathbf{1}}}(\phi, g) = (\mathrm{Ad}_M^*(\phi, g))\tilde{\delta}_{\mathbf{1}} = (\lambda_{g^{-1}}^*(\delta_{\mathbf{1}}), \dot{\lambda}_\phi^*(\lambda_{g^{-1}}^*(\delta_{\mathbf{1}})))$$
$$= (\delta_g, \dot{\lambda}_\phi^*(\delta_g)) = (\delta_g, ((R_g)_0')^*(\phi_g')) \in \mathcal{F}^* \times \mathfrak{g}^* \quad (2.11)$$

since, if we denote again by $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ the canonical duality pairing, then for every $X \in \mathfrak{g}$, we get

$$\langle \dot{\lambda}_\phi^*(\delta_g)), X \rangle = \delta_g(\lambda_\phi(X)) = (\dot{\lambda}(X)\phi)(g) = -\langle ((R_g)_0')^*(\phi_g'), X \rangle,$$

where the latter equality follows by (2.10). Note that $((R_g)_0')^* : T_g^*G \to T_{\mathbf{1}}^*G = \mathfrak{g}^*$ is a linear isomorphism, hence by (2.11) and condition (5) in Definition 2.8, we get $\{(\mathrm{Ad}_M^*(\phi, g))\tilde{\delta}_{\mathbf{1}} \mid (\phi, g) \in M\} = \mathcal{O}$. Thus, the set $\mathcal{O}$ is indeed a coadjoint orbit in $\mathfrak{m}^*$.

We now proceed to proving the other properties of $\mathcal{O}$ mentioned in the statement. Note that the natural surjective mapping

$$T^*G \to \mathcal{O}, \quad (g, \xi) \mapsto (\delta_g, \xi) \quad (2.12)$$

is also injective since points of G are separated by the functions in $\mathcal{F}$ (property (4) in Definition 2.8). We shall endow $\mathcal{O}$ with the structure of smooth finite-dimensional manifold such that the mapping (2.12) is a diffeomorphism. Let $\omega \in \Omega^2(\mathcal{O})$ be the symplectic form obtained by transporting the canonical symplectic form of T^*G by the diffeomorphism (2.12).

Recall that T^*G is a trivial vector bundle over G with the fiber $\mathfrak{g}^*$ and, by using the left trivialization, we may perform the identification $T^*G = G \ltimes_{\mathrm{Ad}_G^*} \mathfrak{g}^*$. This makes T^*G into a finite-dimensional Lie group whose Lie algebra is $\mathbf{L}(T^*G) = \mathfrak{g} \ltimes_{\mathrm{ad}_{\mathfrak{g}}^*} \mathfrak{g}^*$. Then the tangent bundle $T(T^*G) = T^*G \ltimes_{\mathrm{Ad}_{T^*G}} \mathbf{L}(T^*G)$ is a trivial bundle over T^*G with the fiber $\mathbf{L}(T^*G)$, by using again the left trivialization, hence

$$T(T^*G) = T^*G \times (\mathfrak{g} \times \mathfrak{g}^*) = (G \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*)$$

with the natural projection $T(T^*G) \to T^*G$ given by $((g_0, \xi_0), (X, \xi)) \mapsto (g_0, \xi_0)$. Then the Liouville 1-form $\sigma \in \Omega^1(T^*G)$ is $\sigma : T(T^*G) \to \mathbb{R}, ((g_0, \xi_0), (X, \xi)) \mapsto \langle \xi_0, X \rangle$, and the canonical symplectic form on T^*G is $-d\sigma \in \Omega^2(T^*G)$ (see for instance, Chap. V, Sect. 7

in [25], Example 43.9 in [24], or Subsect. 6.5 in [11]). It is easily seen that the value of the 2-form $-d\sigma$ on $T_{(g_0,\xi_0)}(T^*G) \simeq \mathfrak{g} \times \mathfrak{g}^*$ is given by

$$
\begin{aligned}
-\ (d\sigma)_{(g_0,\xi_0)} \colon T_{(g_0,\xi_0)}(T^*G) \times T_{(g_0,\xi_0)}(T^*G) \to \mathbb{R},& \\
((X_1, \xi_1), (X_2, \xi_2)) \mapsto \langle \xi_2, X_1 \rangle - \langle \xi_1, X_2 \rangle.& \qquad (2.13)
\end{aligned}
$$

Note that the symplectic 2-form $-d\sigma$ is invariant under the action of the Lie group T^*G on itself under left translations, while the 1-form η is not. (See also [26].)

For arbitrary $\mu \in \mathcal{O}$ let

$$M_\mu := \{m \in M \mid \mathrm{Ad}^*_M(m)\mu = \mu\}$$

be the corresponding coadjoint isotropy group. It follows by (2.11) that

$$M_{\tilde{\delta}_{\mathbf{1}}} = \{\varphi \in \mathcal{F} \mid \phi'_{\mathbf{1}} = 0\} \times \{\mathbf{1}\} \subseteq \mathcal{F} \rtimes_\lambda G = M. \qquad (2.14)$$

We now prove that the smooth mapping $\Pi_{\tilde{\delta}_{\mathbf{1}}} \colon M \to \mathcal{O}, m \mapsto \mathrm{Ad}^*_M(m)\tilde{\delta}_{\mathbf{1}}$ is a trivial bundle with the fiber $M_{\tilde{\delta}_{\mathbf{1}}}$. In fact, since $\dim \mathfrak{g}^* < \infty$, it easily follows by condition (5) in Definition 2.8 that there exists a linear mapping $\mathfrak{g}^* \to \mathcal{F}, \xi \mapsto \phi_\xi$, such that for every $\xi \in \mathfrak{g}^*$, we have $(\phi_\xi)'_{\mathbf{1}} = \xi$. If $\phi, \chi, \psi \in \mathcal{F}$, $\psi'_{\mathbf{1}} = 0$, and $g \in G$, then the equation $(\phi, g)(\psi, \mathbf{1}) = (\chi, g)$ in $M = \mathcal{F} \rtimes_\lambda G$ is equivalent to $\phi + \lambda_g \psi = \chi$, whence $\lambda_{g^{-1}}\phi + \psi = \lambda_{g^{-1}}\chi$. Since $\psi'_{\mathbf{1}} = 0$, it then follows $(\lambda_{g^{-1}}\phi)'_{\mathbf{1}} = (\lambda_{g^{-1}}\chi)'_{\mathbf{1}}$. This equation is satisfied for $\phi = \lambda_g(\phi_\xi) \in \mathcal{F}$, where $\xi := (\lambda_{g^{-1}}\chi)'_{\mathbf{1}}$. Then, we can take $\psi := \lambda_{g^{-1}}\chi - \phi_\xi = \lambda_{g^{-1}}\chi - \lambda_{g^{-1}}\phi$. This shows that the smooth cross-section of $\Pi_{\tilde{\delta}_{\mathbf{1}}}$ defined by

$$\mathcal{O} \to \mathcal{F} \rtimes_\lambda G, \quad (\delta_g, \xi) \mapsto (\lambda_g(\phi_\xi), g)$$

has the property that every element in $\mathcal{F} \rtimes_\lambda G$ can be uniquely factorized as the product of an element in the image of this cross-section and an element in the isotropy subgroup $M_{\tilde{\delta}_{\mathbf{1}}}$. This implies that $\Pi_{\tilde{\delta}_{\mathbf{1}}} \colon M \to \mathcal{O}$ is a trivial bundle. For an arbitrary element $\mu \in \mathcal{O}$ let $m \in M$ such that $\mathrm{Ad}^*_M(m)\tilde{\delta}_{\mathbf{1}} = \mu$. Then the inner automorphism $\Psi \colon M \to M, n \mapsto mnm^{-1}$ has the property $\Psi(M_{\tilde{\delta}_{\mathbf{1}}}) = M_\mu$, whence we easily get a factorization property in M with respect to M_μ, similar to the one just proved for $M_{\tilde{\delta}_{\mathbf{1}}}$. Thus, the smooth mapping $\Pi_\mu \colon M \to \mathcal{O}, m \mapsto \mathrm{Ad}^*_M(m)\mu$ is a trivial bundle with the fiber M_μ. It then follows that the classical Kirillov–Kostant–Souriau construction of symplectic forms on coadjoint orbits works (see for instance Example 4.31 in [4]) and leads to a symplectic form $\widetilde{\omega} \in \Omega^2(\mathcal{O})$ with the properties mentioned in Assertion (2) in the statement.

In order to complete the proof, we still have to show that the symplectic forms $\omega, \widetilde{\omega} \in \Omega^2(\mathcal{O})$ constructed so far actually coincide. It follows by (2.11) that if we identify $\mathcal{O}$ to T^*G by means of the mapping (2.12), then the differential of the mapping $\Pi_{\tilde{\delta}_{\mathbf{1}}}$ at $(0, \mathbf{1}) \in M$ is the linear map

$$\mathfrak{m} = \mathcal{F} \rtimes_{\dot{\lambda}} \mathfrak{g} \to T_{(\mathbf{1},0)}(T^*G) \simeq \mathfrak{g} \times \mathfrak{g}^*, \quad (\phi, X) \mapsto (X, \phi'_0).$$

Then (2.13) shows that the value of the 2-form $\Pi^*_{\tilde{\delta}_{\mathbf{1}}}(\omega) = \Pi^*_{\tilde{\delta}_{\mathbf{1}}}(-d\sigma)$ at $(0, \mathbf{1}) \in M$ is the bilinear functional

$$
\begin{aligned}
\mathfrak{m} \times \mathfrak{m} \to \mathbb{R}, \quad ((\phi_1, X_1), (\phi_2, X_2)) &\mapsto \langle (\phi_2)'_0, X_1 \rangle - \langle (\phi_1)'_0, X_2 \rangle \\
&= -\tilde{\delta}_{\mathbf{1}}\left([(\phi_1, X_1), (\phi_2, X_2)]\right)
\end{aligned}
$$

(see (2.5)). Thus, $\Pi^*_{\tilde{\delta}_{\mathbf{1}}}(\omega) = \Pi^*_{\tilde{\delta}_{\mathbf{1}}}(\widetilde{\omega})$ on $\mathfrak{m} = T_{(0,\mathbf{1})}M$, and then $\omega = \widetilde{\omega}$ on $T_{\tilde{\delta}_{\mathbf{1}}}\mathcal{O} \simeq T_{(\mathbf{1},0)}(T^*G)$. By using the fact that $\Pi_{\tilde{\delta}_{\mathbf{1}}} \colon M \to \mathcal{O} \simeq T^*G$ is a trivial bundle, it is then straightforward to check that $\omega = \widetilde{\omega}$ (see the proof of Theorem 4.7 in [3]), and we are done.

2.4 Induced representations of semidirect products

This is a classical topic for *locally compact* groups (see for instance, Chap. 5 in [37]). However, in the semidirect product $M = \mathcal{F} \rtimes_\lambda G$, we are working with, the factor $\mathcal{F}$ is generally infinite-dimensional. Therefore, in this section, we shall provide a detailed construction of an appropriate induced representation of M.

In order to construct the unitary representation associated with the coadjoint orbit $\mathcal{O} = \mathrm{Ad}^*_M(M)\tilde{\delta}_{\mathbf{1}}$ in Proposition 2.9, we need to find a real polarization of the functional $\tilde{\delta}_{\mathbf{1}} \in \mathfrak{m}^*$. It is not difficult to check that actually the abelian Lie algebra $\mathcal{F} \simeq \mathcal{F} \times \{0\} \subseteq \mathcal{F} \rtimes_{\dot{\lambda}} \mathfrak{g} = \mathfrak{m}$ is such a polarization, and the corresponding group is $\mathcal{F} \simeq \mathcal{F} \times \{\mathbf{1}\} \subseteq \mathcal{F} \rtimes_\lambda G = M$. Therefore, the representation of the locally convex Lie group M associated with its coadjoint orbit $\mathcal{O}$ should be the one induced from the representation $\mathcal{F} \to \mathbb{C}$, $\phi \mapsto \exp(\mathrm{i}\delta_{\mathbf{1}}(\phi)) = \mathrm{e}^{\mathrm{i}\phi(\mathbf{1})}$. We now describe this induced representation in a more general setting.

Assume the setting of Proposition 2.7 with G an arbitrary topological group and $\mathcal{F} \hookrightarrow \mathcal{B}_{\mathbb{R}}(G)$, which is invariant under the left translation operators, and denote $M := \mathcal{F} \rtimes_\lambda G$. Recall that the multiplication and the inversion in the topological group M are defined by the equations

$$(\phi_1, g_1)(\phi_2, g_2) = (\phi_1 + \lambda_{g_1}\phi_2, g_1 g_2) \quad \text{and} \quad (\phi, g)^{-1} = (-\lambda_{g^{-1}}\phi, g^{-1}),$$

respectively. There exist the embeddings of topological groups $\mathcal{F} \hookrightarrow M, \phi \mapsto (\phi, \mathbf{1})$, and $G \hookrightarrow M, g \mapsto (0, g)$, and the property

$$(\forall\, (\phi, g) \in M) \quad (\phi, g) = (0, g)(\lambda_{g^{-1}}\phi, \mathbf{1}) \tag{2.15}$$

shows that every element in the semidirect product $M = \mathcal{F} \rtimes_\lambda G$ can be uniquely written as a product of elements in the images of G and $\mathcal{F}$ into M.

Now, let $u_0 \colon \mathcal{F} \to \mathbb{R}$ be a linear continuous functional and define $\pi_0 \colon \mathcal{F} \to \mathbb{T}, \phi \mapsto \mathrm{e}^{\mathrm{i}u_0(\phi)}$, which is a character of the abelian topological group $(\mathcal{F}, +)$. We also define

$$M \times_{\mathcal{F}} \mathbb{C} := (M \times \mathbb{C})/\sim$$

where $\sim$ is the equivalence relation on $M \times \mathbb{C}$ defined by

$$(m(\phi, \mathbf{1}), z) \sim (m, \pi_0(\phi) z) \quad \text{whenever } m \in M,\ \phi \in \mathcal{F}, \text{ and } z \in \mathbb{C}. \tag{2.16}$$

We are going to denote by $[(m, z)]$ the equivalence class of any $(m, z) \in M \times \mathbb{C}$. Note that there exists a natural homeomorphism

$$M/\mathcal{F} \to G, \quad (\phi, g)\mathcal{F} \mapsto g$$

(this map is well-defined because of (2.15)) and a continuous surjection

$$\Pi \colon M \times_{\mathcal{F}} \mathbb{C} \to M/\mathcal{F}, \quad [(m, z)] \mapsto m\mathcal{F},$$

which is actually a locally trivial bundle with the fiber $\mathbb{C}$.

There exists a bijective correspondence between the sections $\sigma \colon M/\mathcal{F} \to M \times_{\mathcal{F}} \mathbb{C}$ (that is, functions satisfying $\Pi \circ \sigma = \mathrm{id}_{M/\mathcal{F}}$) and the functions $\widetilde{\sigma} \colon G \to \mathbb{C}$. This correspondence is defined by

$$(\forall x \in G) \quad \sigma\,((0, x)\mathcal{F}) = [((0, x), \widetilde{\sigma}(x))]\,. \tag{2.17}$$

Let us denote by $\Gamma_{\mathrm{Borel}}(M/\mathcal{F}, M \times_{\mathcal{F}} \mathbb{C})$ the space of Borel measurable sections, so that there exists a linear isomorphism from this space onto the space of complex-valued, Borel measurable functions on G,

$$\Gamma_{\mathrm{Borel}}(M/\mathcal{F}, M\times_{\mathcal{F}}\mathbb{C}) \to \mathcal{B}_{\mathbb{C}}(G), \quad \sigma\mapsto\widetilde{\sigma}. \tag{2.18}$$

The representation $\pi := \mathrm{Ind}_{\mathcal{F}}^{M}(\pi_0)$ of M induced by $\pi_0\colon \mathcal{F}\to\mathbb{T}$ is

$$\pi\colon M\to \mathrm{End}\,(\Gamma_{\mathrm{Borel}}(M/\mathcal{F}, M\times_{\mathcal{F}}\mathbb{C}))$$

defined by

$$(\pi(m)\sigma)(\mu) = m\sigma(m^{-1}\mu) \quad \text{for } m\in M,\ \sigma\in\Gamma_{\mathrm{Borel}}(M/\mathcal{F}, M\times_{\mathcal{F}}\mathbb{C}),\ \text{and } \mu\in M/\mathcal{F}.$$

We will denote again by $\pi\colon M\to\mathrm{End}\,(\mathcal{B}_{\mathbb{C}}(G))$ the corresponding representation obtained by (2.18). In order to get a specific description of the latter representation π, note that for every $\phi\in\mathcal{F}$ and $g,x\in G$, we have

$$\begin{aligned}
(\pi(\phi,g)\sigma\,((0,x)\mathcal{F})) &= (\phi,g)\sigma\left((\phi,g)^{-1}(0,x)\mathcal{F}\right)\\
&= (\phi,g)\sigma\left((-\lambda_{g^{-1}}\phi, g^{-1})(0,x)\mathcal{F}\right)\\
&= (\phi,g)\sigma\left((-\lambda_{g^{-1}}\phi, g^{-1}x)\mathcal{F}\right)\\
&= (\phi,g)\sigma\left((0,g^{-1}x)\mathcal{F}\right) \qquad (\text{by } (2.15))\\
&= (\phi,g)\left[\left((0,g^{-1}x),\widetilde{\sigma}(g^{-1}x)\right)\right] \quad (\text{by } (2.17))\\
&= \left[\left((\phi,g)(0,g^{-1}x),\widetilde{\sigma}(g^{-1}x)\right)\right]\\
&= \left[\left((\phi,x),\widetilde{\sigma}(g^{-1}x)\right)\right]\\
&= \left[\left((0,x)(\lambda_{x^{-1}}\phi,\mathbf{1}),\widetilde{\sigma}(g^{-1}x)\right)\right] \quad (\text{by } (2.15))\\
&= \left[\left((0,x),\pi_0(\lambda_{x^{-1}}\phi)\widetilde{\sigma}(g^{-1}x)\right)\right] \quad (\text{by } (2.16))
\end{aligned}$$

whence by (2.17) again we get

$$(\pi(\phi,g)\widetilde{\sigma})(x) = \pi_0(\lambda_{x^{-1}}\phi)\widetilde{\sigma}(g^{-1}x) \text{ for } g,x\in G,\ \phi\in\mathcal{F},\ \text{and } \widetilde{\sigma}\in\mathcal{B}_{\mathbb{C}}(G).$$

For instance, if $u_0=\delta_{\mathbf{1}}\colon\mathcal{F}\to\mathbb{R}$, $\phi\mapsto\phi(\mathbf{1})$, then we get

$$\pi := \pi_{\mathbf{1}}\colon M=\mathcal{F}\rtimes_\lambda G\to\mathrm{End}\,(\mathcal{B}_{\mathbb{C}}(G)), \quad (\pi_{\mathbf{1}}(\phi,g)\widetilde{\sigma})\,(x) = \mathrm{e}^{\mathrm{i}\phi(x)}\widetilde{\sigma}(g^{-1}x).$$

If we define $U\colon\mathcal{B}_{\mathbb{C}}(G)\to\mathcal{B}_{\mathbb{C}}(G)$, $(U\widetilde{\sigma})(x)=\widetilde{\sigma}(x^{-1})$, then we get the equivalent representation $U\pi_{\mathbf{1}}(\cdot)U^{-1}$ with the specific expression

$$\begin{aligned}
(U\pi_{\mathbf{1}}(\phi,g)U^{-1}\widetilde{\sigma})(x) &= (\pi_{\mathbf{1}}(\phi,g)U^{-1}\widetilde{\sigma})(x^{-1}) = \mathrm{e}^{\mathrm{i}\phi(x^{-1})}(U^{-1}\widetilde{\sigma})(g^{-1}x^{-1})\\
&= \mathrm{e}^{\mathrm{i}\phi(x^{-1})}\widetilde{\sigma}(xg)
\end{aligned}$$

for $g,x\in G$, $\phi\in\mathcal{F}$, and $\widetilde{\sigma}\in\mathcal{B}_{\mathbb{C}}(G)$.

3 Magnetic preduals of the coadjoint orbit $\mathcal{O}$

3.1 Auxiliary properties of nilpotent Lie algebras

Definition 3.1 Let $\mathfrak{g}$ be a nilpotent finite-dimensional real Lie algebra of dimension ≥ 1 and define $\mathfrak{g}_0 := \mathfrak{g}$ and

$$(\forall k\geq 1)\quad \mathfrak{g}_k = \mathrm{span}\,\{[X_k,\dots,[X_1,X_0]\dots]\mid X_0,X_1,\dots,X_k\in\mathfrak{g}\}.$$

Then $\mathfrak{g}_0\supseteq\mathfrak{g}_1\supseteq\mathfrak{g}_2\supseteq\cdots$ and, since $\mathfrak{g}$ is a nilpotent Lie algebra, there exists $n\geq 0$ with $\mathfrak{g}_n\neq\{0\}=\mathfrak{g}_{n+1}$. The number $n\geq 0$ is called the *nilpotency index* of $\mathfrak{g}$.

Note that $[\mathfrak{g}, \mathfrak{g}_n] = \mathfrak{g}_{n+1} = \{0\}$, hence $\mathfrak{g}_n$ is contained in the center of $\mathfrak{g}$. In particular, $\mathfrak{g}_n$ is an ideal of $\mathfrak{g}_n$ and then there exists a natural Lie bracket on $\mathfrak{g}/\mathfrak{g}_n$ which makes the quotient map $q\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_n$ into a homomorphism of Lie algebras. It is also easily seen that $\mathfrak{g}/\mathfrak{g}_n$ is a nilpotent Lie algebra whose nilpotency index is $n-1$, provided that $n \geq 1$.

Proposition 3.2 *If $\mathfrak{g}$ is a nilpotent finite-dimensional real Lie algebra, then for every $V \in \mathfrak{g}$, the mapping*

$$\Psi_{\mathfrak{g},V}\colon \mathfrak{g} \to \mathfrak{g}, \quad Y \mapsto \int_0^1 Y * (sV)\,\mathrm{d}s$$

is a polynomial diffeomorphism whose inverse is also polynomial and which preserves the Lebesgue measure.

Proof Recall that the multiplication $*$ defined by the Baker–Campbell–Hausdorff (BCH) formula is a polynomial mapping in the case of the nilpotent Lie algebras, and therefore, the mapping in the statement is polynomial. In order to prove the other properties, we shall proceed by induction on the nilpotency index of the Lie algebra under consideration.

If the nilpotency index of $\mathfrak{g}$ is 0, then this algebra is abelian, so the BCH multiplication $*$ reduces to the vector sum. Then for every $V \in \mathfrak{g}$, we have

$$(\forall Y \in \mathfrak{g}) \quad \Psi_{\mathfrak{g},V}(Y) = \int_0^1 Y + sV\,\mathrm{d}s = Y + \frac{1}{2}V,$$

which clearly has the properties we wish for.

Now let $n \geq 1$ and assume that the assertion holds for the Lie algebras of nilpotency index $< n$. Let $\mathfrak{g}$ be a nilpotent Lie algebra with $\mathfrak{g}_n \neq \{0\} = \mathfrak{g}_{n+1}$ (see the notation in Definition 3.1) and take $V \in \mathfrak{g}$ arbitrary. In order to show that the mapping $\Psi_{\mathfrak{g},V}\colon \mathfrak{g} \to \mathfrak{g}$ is injective, let $Y_1, Y_2 \in \mathfrak{g}$ such that $\Psi_{\mathfrak{g},V}(Y_1) = \Psi_{\mathfrak{g},V}(Y_2)$. If we transform both sides of the latter equation by the Lie algebra homomorphism $q\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_n$ which preserves the BCH multiplication, then we get $\int_0^1 q(Y_1) * (sq(V))\,\mathrm{d}s = \int_0^1 q(Y_2) * (sq(V))\,\mathrm{d}s$. Since the mapping $\Psi_{\mathfrak{g}/\mathfrak{g}_n, q(V)}\colon \mathfrak{g}/\mathfrak{g}_n \to \mathfrak{g}/\mathfrak{g}_n$ is injective by the induction hypothesis, it follows that $q(Y_1) = q(Y_2)$, that is, $Y_0 := Y_1 - Y_2 \in \operatorname{Ker} q = \mathfrak{g}_n$. Then

$$\Psi_{\mathfrak{g},V}(Y_1) = \Psi_{\mathfrak{g},V}(Y_2 + Y_0) = \int_0^1 (Y_0 + Y_2) * (sV)\,\mathrm{d}s = \int_0^1 Y_0 + (Y_2 * (sV))\,\mathrm{d}s$$
$$= Y_0 + \int_0^1 Y_2 * (sV)\,\mathrm{d}s = Y_0 + \Psi_{\mathfrak{g},V}(Y_2),$$

so the assumption $\Psi_{\mathfrak{g},V}(Y_1) = \Psi_{\mathfrak{g},V}(Y_2)$ implies $Y_0 = 0$, whence $Y_1 = Y_2$. We note that the above equalities follow by using the definition of the BCH multiplication $*$ along with the fact that $Y_0 \in \mathfrak{g}_n$, hence $[Y_0, \mathfrak{g}] = \{0\}$.

It remains to check that the mapping $\Psi_{\mathfrak{g},V}\colon \mathfrak{g} \to \mathfrak{g}$ is surjective and its inverse is polynomial. For that purpose, let $\iota\colon \mathfrak{g}/\mathfrak{g}_n \to \mathfrak{g}$ be any linear mapping satisfying $q \circ \iota = \mathrm{id}_{\mathfrak{g}/\mathfrak{g}_n}$. (So ι can be any linear isomorphism of $\mathfrak{g}/\mathfrak{g}_n$ onto a linear complement of $\mathfrak{g}_n$ in $\mathfrak{g}$.) Denote

$$(\forall Z \in \mathfrak{g}) \quad \Delta(Z) := Z - \int_0^1 \iota(\Phi(q(Z))) * (sV)\,\mathrm{d}s, \tag{3.1}$$

where $\Phi := (\Psi_{\mathfrak{g}/\mathfrak{g}_n, q(V)})^{-1} \colon \mathfrak{g}/\mathfrak{g}_n \to \mathfrak{g}/\mathfrak{g}_n$ is a polynomial map which exists because of the induction hypothesis. Note that for every $Z \in \mathfrak{g}$ we have

$$q(\Delta(Z)) = q(Z) - q\left(\int_0^1 \iota(\Phi(q(Z))) * (sV)\,\mathrm{d}s\right) = q(Z) - \int_0^1 q(\iota(\Phi(q(Z)))) * q(sV)\,\mathrm{d}s$$
$$= q(Z) - \int_0^1 \Phi(q(Z)) * (sq(V))\,\mathrm{d}s = 0,$$

where we used the equality $q \circ \iota = \mathrm{id}_{\mathfrak{g}/\mathfrak{g}_n}$ and again the fact that $q \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_n$ is a Lie algebra homomorphism, hence preserves the BCH multiplications. Since $\operatorname{Ker} q = \mathfrak{g}_n$ and $[\mathfrak{g}, \mathfrak{g}_n] = \mathfrak{g}_{n+1} = \{0\}$, we get

$$(\forall Z \in \mathfrak{g}) \quad [\Delta(Z), \mathfrak{g}] = \{0\}.$$

We can use this property to see that (as in the above proof of the fact that $\Psi_{\mathfrak{g},V}$ is injective) we have for every $Z \in \mathfrak{g}$,

$$Z = \Delta(Z) + \int_0^1 \iota(\Phi(q(Z))) * (sV)\,\mathrm{d}s = \int_0^1 \Delta(Z) + (\iota \circ \Phi \circ q)(Z) * (sV)\,\mathrm{d}s$$
$$= \int_0^1 (\Delta(Z) + (\iota \circ \Phi \circ q)(Z)) * (sV)\,\mathrm{d}s = \Psi_{\mathfrak{g},V}(\Delta(Z) + (\iota \circ \Phi \circ q)(Z)).$$

This shows that the mapping $\Psi_{\mathfrak{g},V} : \mathfrak{g} \to \mathfrak{g}$ is indeed surjective and

$$(\forall Z \in \mathfrak{g}) \quad (\Psi_{\mathfrak{g},V})^{-1}(Z) = \Delta(Z) + (\iota \circ \Phi \circ q)(Z). \tag{3.2}$$

In order to conclude the proof, just recall that $\Phi = (\Psi_{\mathfrak{g}/\mathfrak{g}_n, q(V)})^{-1} \colon \mathfrak{g}/\mathfrak{g}_n \to \mathfrak{g}/\mathfrak{g}_n$ is a polynomial map by the induction hypothesis, while the BCH multiplication is a polynomial mapping on every nilpotent Lie algebra. Since both ι and q are linear, it follows by (3.1) that $\Delta \colon \mathfrak{g} \to \mathfrak{g}$ is a polynomial mapping, and then (3.2) shows that so is $(\Psi_{\mathfrak{g},V})^{-1} \colon \mathfrak{g} \to \mathfrak{g}$.

As regards the measure-preserving property, it will be enough to show that for an arbitrary $Y_0 \in \mathfrak{g}$, the differential $(\Psi_{\mathfrak{g},V})'_{Y_0} \colon \mathfrak{g} \to \mathfrak{g}$ is a linear map whose determinant is equal to 1. To this end note that for every $Y \in \mathfrak{g}_n$, we have $[\mathfrak{g}, Y] = \{0\}$ hence $\Psi_{\mathfrak{g},V}(Y) = \int_0^1 Y + sV\,\mathrm{d}s = Y + \frac{1}{2}V$, which implies that $\mathfrak{g}_n$ is invariant under the differential $(\Psi_{\mathfrak{g},V})'_{Y_0}$. Actually, the latter map restricted to $\mathfrak{g}_n$ is equal to the identity map on $\mathfrak{g}_n$, and in particular, the determinant of that restriction is equal to 1. On the other hand, as above in the proof of injectivity of $\Psi_{\mathfrak{g},V}$, we get $q \circ \Psi_{\mathfrak{g},V} = \Psi_{\mathfrak{g}/\mathfrak{g}_n,q(V)} \circ q$. By differentiating this equality at $Y_0 \in \mathfrak{g}$ and taking into account that $q \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_n$ is a linear map, we get $q \circ (\Psi_{\mathfrak{g},V})'_{Y_0} = (\Psi_{\mathfrak{g}/\mathfrak{g}_n,q(V)})'_{q(Y_0)} \circ q$, and then, we get the following commutative diagram

$$\begin{array}{ccccc}
\mathfrak{g}_n & \longrightarrow & \mathfrak{g} & \xrightarrow{q} & \mathfrak{g}/\mathfrak{g}_n \\
\Big\downarrow \mathrm{id}_{\mathfrak{g}_n} & & \Big\downarrow (\Psi_{\mathfrak{g},V})'_{Y_0} & & \Big\downarrow (\Psi_{\mathfrak{g}/\mathfrak{g}_n,q(V)})'_{q(Y_0)} \\
\mathfrak{g}_n & \longrightarrow & \mathfrak{g} & \xrightarrow{q} & \mathfrak{g}/\mathfrak{g}_n
\end{array}$$

whose rows are short exact sequences. Since the determinant of $(\Psi_{\mathfrak{g}/\mathfrak{g}_n,q(V)})'_{q(Y_0)}$ is equal to 1 by the induction hypothesis, it follows that the determinant of the middle vertical arrow is also equal to 1. This completes the induction step and the proof. □

3.2 Magnetic preduals

Here, we work in the following setting:

(1) In order to simplify the notation, the connected and simply connected nilpotent Lie group G is identified with its Lie algebra $\mathfrak{g}$ by means of the exponential map and $*$ denotes the Baker–Campbell–Hausdorff multiplication on $\mathfrak{g}$.
(2) We denote by $\langle\cdot,\cdot\rangle\colon \mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$ the canonical duality pairing.
(3) We also denote by $\mathcal{F}$ an admissible space of functions on $\mathfrak{g}$ which contains both $\mathfrak{g}^*$ and the constant functions. As usual, we denote by $M=\mathcal{F}\rtimes_\lambda\mathfrak{g}$ the corresponding semidirect product of groups, which is a locally convex Lie group with the Lie algebra $T_{(0,0)}M=\mathfrak{m}=\mathcal{F}\rtimes_{\dot\lambda}\mathfrak{g}$. Here we shall distinguish $\mathfrak{m}$ from the set $\mathbf{L}(M)$ of one-parameter subgroups in M.
(4) The *magnetic potential* $A\in\Omega^1(\mathfrak{g})$ is a smooth differential 1-form whose coefficients belong to $\mathcal{F}$. That is, $A\colon\mathfrak{g}\to\mathfrak{g}^*, X\mapsto A_X:=A(X)$, is a smooth mapping such that for every $X\in\mathfrak{g}$ the function $Y\mapsto\langle A_Y,(R_Y)'_0X\rangle$ belongs to $\mathcal{F}$, where $R_Y\colon\mathfrak{g}\to\mathfrak{g}$, $R_Y(W)=W*Y$.
(5) The *magnetic field* is the 2-form $B=dA\in\Omega^2(\mathfrak{g})$. Hence, B is a smooth mapping $X\mapsto B_X$ from $\mathfrak{g}$ into the space of all skew-symmetric bilinear functionals on $\mathfrak{g}$ such that

$$(\forall X,X_1,X_2\in\mathfrak{g})\quad B_X(X_1,X_2)=\langle A'_X(X_1),X_2\rangle-\langle A'_X(X_2),X_1\rangle.$$

Proposition 3.3 *For every $\phi\in\mathcal{F}$ and $X\in\mathfrak{g}$ define $\bar\theta_0^A(\phi,X)\in\mathcal{F}$ by*

$$(\forall Y\in\mathfrak{g})\quad\left(\bar\theta_0^A(\phi,X)\right)(Y)=\phi(Y)+\langle A_Y,(R_Y)'_0X\rangle,$$

and then consider the continuous linear mapping

$$\bar\theta^A\colon\mathcal{F}\rtimes_{\dot\lambda}\mathfrak{g}=\mathfrak{m}\to\mathfrak{m},\quad\bar\theta^A(\phi,X)=\left(\bar\theta_0^A(\phi,X),X\right)$$

and the differential 2-forms

$$\bar\omega\in\Omega^2(\mathcal{F}\times\mathfrak{g}),\quad\bar\omega_{(\phi_0,X_0)}((\phi_1,X_1),(\phi_2,X_2))=\left(\dot\lambda(X_1)\phi_2-\dot\lambda(X_2)\phi_1\right)(X_0)$$

and

$$\bar B\in\Omega^2(\mathcal{F}\times\mathfrak{g}),\quad\bar B_{(\phi_0,X_0)}((\phi_1,X_1),(\phi_2,X_2))=B_{X_0}(X_1,X_2).$$

Then the following assertions hold:

(1) The operator $\bar\theta^A\colon\mathfrak{m}\to\mathfrak{m}$ is invertible and $(\bar\theta^A)^{-1}=\bar\theta^{-A}$.
(2) If $\mathfrak{g}$ is a two-step nilpotent Lie algebra, then

$$\bar\omega\in\Omega^2(\mathcal{F}\times\mathfrak{g}),\quad\bar\omega_{(\phi_0,X_0)}((\phi_1,X_1),(\phi_2,X_2))=(\phi_2)'_{X_0}(X_1)-(\phi_1)'_{X_0}(X_2).$$

Moreover, $d\bar\omega=0$ and $(\bar\theta^A)^*(\bar\omega)=\bar\omega+\bar B$.

Proof The first assertion is easily seen. For the second assertion, note that if $\mathfrak{g}$ is two-step nilpotent, then $(R_Y)'_0=\mathrm{id}_\mathfrak{g}$ for every $Y\in\mathfrak{g}$; hence, the specific expression of $\bar\omega$ follows by (2.10). If we regard $\bar{\omega}$ as a mapping from $\mathcal{F} \times \mathfrak{g}$ into the skew-symmetric bilinear functionals on $\mathcal{F} \times \mathfrak{g}$, then we may differentiate it as such and get

$$\begin{aligned} d\bar{\omega}_{(\phi_0,X_0)}((\phi_1,X_1),(\phi_2,X_2),(\phi_3,X_3)) &= \bar{\omega}'_{(\phi_0,X_0)}(\phi_1,X_1)((\phi_2,X_2),(\phi_3,X_3)) \\ &\quad - \bar{\omega}'_{(\phi_0,X_0)}(\phi_2,X_2)((\phi_1,X_1),(\phi_3,X_3)) \\ &\quad + \bar{\omega}'_{(\phi_0,X_0)}(\phi_3,X_3)((\phi_1,X_1),(\phi_2,X_2)) \\ &= (\phi_3)''_{X_0}(X_1,X_2) - (\phi_2)''_{X_0}(X_1,X_3) \\ &\quad - (\phi_3)''_{X_0}(X_2,X_1) + (\phi_1)''_{X_0}(X_2,X_3) \\ &\quad + (\phi_2)''_{X_0}(X_3,X_1) - (\phi_1)''_{X_0}(X_3,X_2) \\ &= 0, \end{aligned}$$

since the second differentials of the smooth functions $\phi_1, \phi_2, \phi_3 \in \mathcal{F}$ are symmetric. Further, since $\bar{\theta}$ is a linear map, we get

$$\begin{aligned} &\bar{\theta}^*(\bar{\omega})_{(\phi_0,X_0)}\,((\phi_1,X_1),(\phi_2,X_2)) = \bar{\omega}_{\bar{\theta}(\phi_0,X_0)}\left(\bar{\theta}(\phi_1,X_1),\bar{\theta}(\phi_2,X_2)\right) \\ &\quad = \bar{\omega}_{(\phi_0+\langle A(\cdot),X_0\rangle,X_0)}((\phi_1+\langle A(\cdot),X_1\rangle,X_1),(\phi_2+\langle A(\cdot),X_2\rangle,X_2)) \\ &\quad = (\phi_2)'_{X_0}(X_1) + \langle A'_{X_0}(X_1),X_2\rangle - (\phi_1)'_{X_0}(X_2) - \langle A'_{X_0}(X_2),X_1\rangle \\ &\quad = \bar{\omega}_{(\phi_0,X_0)}((\phi_1,X_1),(\phi_2,X_2)) + \bar{B}_{(\phi_0,X_0)}((\phi_1,X_1),(\phi_2,X_2)), \end{aligned}$$

and this completes the proof. □

Definition 3.4 Assume the notation introduced in Proposition 3.3. The set

$$\mathcal{O}_* = \mathcal{O}_*^A := \{(\bar{\theta}_0^A(\xi,X),X) \mid X \in \mathfrak{g}, \xi \in \mathfrak{g}^*\} \subseteq \mathcal{F} \rtimes_{\dot{\lambda}} \mathfrak{g} = \mathfrak{m}$$

will be called the *magnetic predual of the coadjoint orbit* $\mathcal{O}$ (associated with the magnetic potential A). Let $I\colon \mathfrak{g} \times \mathfrak{g}^* \hookrightarrow \mathcal{F} \times \mathfrak{g}$ be the natural embedding $I(X,\xi) = (\xi, X)$. Then the mapping

$$\bar{\theta}^A \circ I\colon \mathfrak{g} \times \mathfrak{g}^* \to \mathcal{O}_* \tag{3.3}$$

is a linear isomorphism which (by Proposition 3.3) takes the canonical symplectic structure of $\mathfrak{g} \times \mathfrak{g}^*$ to a certain symplectic structure on $\mathcal{O}_*$, which will be called the *natural symplectic structure of the magnetic predual* $\mathcal{O}_*$. Thus, (3.3) is an isomorphism of symplectic vector spaces.

Remark 3.5 The magnetic predual $\mathcal{O}_*^A$, essentially, depends only on the magnetic field $B = dA$. Specifically, if $A_1, A_2 \in \Omega^1(\mathfrak{g})$ are magnetic potentials then there exists $m_0 = (\phi_0, X_0) \in M$ such that $\bar{\theta}^{A_1} = \mathrm{Ad}_M(m) \circ \bar{\theta}^{A_2}$ if and only if $dA_1 = dA_2$. This follows by using Remark 2.5(2).

In the following statement, we need some notation from Propositions 2.7 and 2.9. Thus, $\delta_0\colon \mathcal{F} \to \mathbb{R}$ is the functional $\phi \mapsto \phi(0)$.

Proposition 3.6 *Let us define*

$$\theta_0\colon \mathfrak{g} \times \mathfrak{g}^* \to \mathcal{F}, \quad (\theta_0(X,\xi))(Y) = \langle \xi, Y\rangle + \langle A_Y, (R_Y)'_0 X\rangle$$

and

$$\theta\colon \mathfrak{g} \times \mathfrak{g}^* \to \mathbf{L}(M), \quad \theta(X,\xi) = Z_{\theta_0(X,\xi),X}.$$

Then the mapping

$$\Pi_{\tilde{\delta}_0} \circ \exp_M \circ \theta \colon \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{m}^*, \quad (X, \xi) \mapsto \mathrm{Ad}^*_M(\exp_M(\theta(X, \xi)))\tilde{\delta}_0$$

is a diffeomorphism of $\mathfrak{g} \times \mathfrak{g}^*$ *onto the coadjoint orbit* $\mathcal{O}$ *of* $\tilde{\delta}_0 = (\delta_0, 0) \in \mathcal{F}^* \times \mathfrak{g}^* = \mathfrak{m}^*$.

Proof Let $M_{\tilde{\delta}_0}$ be the coadjoint isotropy group at $\tilde{\delta}_0 \in \mathfrak{m}^*$. To prove that $\Phi \colon \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{m}^*$ is a bijection onto $\mathrm{Ad}^*_M(M)\tilde{\delta}_0 \simeq M/M_{\tilde{\delta}_0}$, it is necessary and sufficient to see that the following assertions hold:

(1) The mapping $\mathfrak{g} \times \mathfrak{g}^* \to M$, $(X, \xi) \mapsto \exp_M(\theta(X, \xi))$ is injective.
(2) The multiplication mapping

$$\exp_M(\theta(\mathfrak{g} \times \mathfrak{g}^*)) \times M_{\tilde{\delta}_0} \to M \tag{3.4}$$

is bijective, and additionally, if $m_1, m_2 \in \exp_M(\theta(\mathfrak{g} \times \mathfrak{g}^*))$ satisfy $m_1 \in m_2 M_{\tilde{\delta}_0}$, then necessarily $m_1 = m_2$.

In order to prove these assertions, we shall use fact that by Proposition 2.7, we have

$$(\forall (X, \xi) \in \mathfrak{g} \times \mathfrak{g}^*) \quad \exp_M(\theta(X, \xi)) = (\alpha(X, \xi), X) \in \mathcal{F} \rtimes_\lambda \mathfrak{g} = M. \tag{3.5}$$

Here, the function $\alpha(X, \xi) \in \mathcal{F}$ at an arbitrary point $Y \in \mathfrak{g}$ can be computed in the following way:

$$\begin{aligned}(\alpha(X, \xi)(Y) &= \int_0^1 (\lambda_{sX}(\theta_0(X, \xi))(Y)\mathrm{d}s = \int_0^1 (\theta_0(X, \xi))((-sX) * Y)\mathrm{d}s \\ &= \int_0^1 \xi((-sX) * Y)\mathrm{d}s + \int_0^1 \langle A((-sX) * Y), (R_{(-sX)*Y})'_0 X\rangle \mathrm{d}s\end{aligned}$$

By using the notation introduced in Proposition 3.2, we get

$$\begin{aligned}(\alpha(X, \xi)(Y) &= \left\langle \xi, \int_0^1 (-sX) * Y\mathrm{d}s \right\rangle + \left\langle \int_0^1 A((-sX) * Y)\mathrm{d}s, (R_{(-sX)*Y})'_0 X \right\rangle \\ &= -\langle \xi, \Psi_{\mathfrak{g},X}(-Y)\rangle + \left\langle \int_0^1 A((-sX) * Y)\mathrm{d}s, (R_{(-sX)*Y})'_0 X \right\rangle.\end{aligned} \tag{3.6}$$

Now, to prove assertion (1), just note that if $\exp_M(\theta(X_1, \xi_1)) = \exp_M(\theta(X_2, \xi_2))$, then by (3.5) we get $X_1 = X_2 =: X$ and $\alpha(X, \xi_1) = \alpha(X, \xi_2)$. Then by (3.6), we get $\xi_1 \circ \Psi_{\mathfrak{g},X} = \xi_2 \circ \Psi_{\mathfrak{g},X}$. Since $\Psi_{\mathfrak{g},X} \colon \mathfrak{g} \to \mathfrak{g}$ is a diffeomorphism by Proposition 3.2, it follows that $\xi_1 = \xi_2$.

We now proceed to proving the above assertion (2). In order to prove the second part of that assertion, let us assume that $\exp_M(\theta(X_1, \xi_1)) \in \exp_M(\theta(X_2, \xi_2))M_{\tilde{\delta}_0}$. It then follows by (2.14) and (3.5) that there exists $\phi \in \mathcal{F}$ such that $\phi'_0 = 0$ and $(\alpha(X_1, \xi_1), X_1) = (\alpha(X_2, \xi_2), X_2)(\phi, 0)$. Thence, $X_1 = X_2 =: X$ and $\alpha(X, \xi_1) = \alpha(X, \xi_2) + \lambda_X \phi$, so by (3.6) we get $\langle \xi_2 - \xi_1, \Psi_{\mathfrak{g},X}(-Y)\rangle = \phi((-X) * Y)$ for every $Y \in \mathfrak{g}$. By means of the change of variable $(-X) * Y = -W$ we have $W * (-X) = -Y$, and then $\langle \xi_2 - \xi_1, \Psi_{\mathfrak{g},X}(W * (-X))\rangle = \phi(-W)$ for every $W \in \mathfrak{g}$. Now note that

$$\Psi_{\mathfrak{g},X}(W*(-X)) = \int_0^1 W*(-X)*(sX)\mathrm{d}s = \int_0^1 W*((1-s)X)\mathrm{d}s = \int_0^1 W*(sX)\mathrm{d}s$$
$$= \Psi_{\mathfrak{g},X}(W),$$

hence $\langle \xi_2 - \xi_1, \Psi_{\mathfrak{g},X}(W)\rangle = \phi(-W)$ for every $W \in \mathfrak{g}$. By differentiating the latter equation at $W = 0$ we get $(\xi_2 - \xi_1) \circ (\Psi_{\mathfrak{g},X})'_0 = \phi'_0 = 0$. Now recall that $\Psi_{\mathfrak{g},X} \colon \mathfrak{g} \to \mathfrak{g}$ is a diffeomorphism by Proposition 3.2, hence $(\Psi_{\mathfrak{g},X})'_0 \colon \mathfrak{g} \to \mathfrak{g}$ is a linear isomorphism, and then $\xi_2 - \xi_1 = 0$.

This proves the second part of assertion (2) which, in particular, shows that the multiplication mapping (3.4) is injective. In order to prove that mapping is surjective as well, let $(\phi, X) \in M$ arbitrary. It follows by (2.14) and (3.5) again that it will be enough to find $\xi \in \mathfrak{g}^*$ and $\psi \in \mathcal{F}$ such that $\psi'_0 = 0$ and $(\alpha(X,\xi), X)(\psi, 0) = (\phi, X)$. The latter equation is equivalent to $\alpha(X,\xi) + \lambda_X\psi = \phi$, that is, $\lambda_{-X}(\alpha(X,\xi)) + \psi = \lambda_{-X}\psi$, whence by (3.6) we get

$$(\forall Y \in \mathfrak{g}) \quad \left\langle \xi, \int_0^1 (-sX)*X*Y\mathrm{d}s \right\rangle + \left\langle \int_0^1 A((-sX)*X*Y)\mathrm{d}s, (R_{(-sX)*X*Y})'_0 X \right\rangle$$
$$+\psi(Y) = \phi(X*Y).$$

Since $(-sX)*X = (1-s)X$, the above equation is further equivalent to

$$(\forall Y \in \mathfrak{g}) \quad \left\langle \xi, \int_0^1 (sX)*Y\mathrm{d}s \right\rangle + \left\langle \int_0^1 A((sX)*Y)\mathrm{d}s, (R_{(sX)*Y})'_0 X \right\rangle + \psi(Y) = \phi(X*Y). \tag{3.7}$$

Since the mapping $Y \mapsto \int_0^1 (sX)*Y\mathrm{d}s = -\Psi_{\mathfrak{g},-X}(-Y)$ is a diffeomorphism by Proposition 3.2, it follows that its differential at $Y = 0$ is a linear isomorphism on $\mathfrak{g}$. Now by differentiating (3.7) at $Y = 0$ and using the condition $\psi'_0 = 0$, we see that $\xi \in \mathfrak{g}^*$ can be uniquely determined in terms of the given function $\phi \in \mathcal{F}$. Then, we just have to solve equation (3.7) for ψ. This completes the proof of the fact that the multiplication mapping (3.4) is surjective.

We now know that the mapping $\Pi_{\tilde{\delta}_0} \circ \exp_M \circ\, \theta \colon \mathfrak{g} \times \mathfrak{g}^* \to \mathcal{O}$ in the statement is a bijection. In order to see that it is actually a diffeomorphism, first, note that $\exp_M \circ\, \theta \colon \mathfrak{g} \times \mathfrak{g}^* \to M$ is smooth as an easy consequence of (3.5) and (3.6), and then $\Pi_{\tilde{\delta}_0} \circ \exp_M \circ\, \theta$ is smooth. In order to prove that its inverse is also smooth, we just have to use the fact that the solution ξ of (3.7) depends smoothly on the data $\phi \in \mathcal{F}$ (as a direct consequence of our way to solve Eq. (3.7)).

Corollary 3.7 *Let $\tilde{\delta}_0 = (\delta_0, 0) \in \mathfrak{m}^*$. The mapping*

$$\mathrm{Ad}^*_M(\exp_M(\cdot))\tilde{\delta}_0 \colon \mathcal{O}_* \to \mathcal{O}$$

is a diffeomorphism.

Proof Use Propositions 3.3 and 3.6. □

4 Magnetic Weyl calculus on Lie groups

4.1 Localized Weyl calculus

In this subsection, we sketch a general setting, inspired by [1] and [2], for the Weyl calculus associated with continuous representations of any topological groups, which may be infinite-dimensional Lie groups. We shall apply this construction, in the next subsection, in the case of a semidirect product $M = \mathcal{F} \rtimes_\lambda G$, where $\mathcal{F}$ is a certain function space on the nilpotent Lie group G.

Definition 4.1 Let M be a topological group and $\pi: M \to \mathcal{B}(\mathcal{Y})$ a so-continuous, uniformly bounded representation on the complex separable Banach space $\mathcal{Y}$. Assume the setting defined by the following data:

(1) a duality pairing $\langle\cdot,\cdot\rangle\colon \Xi^* \times \Xi \to \mathbb{R}$ between two real finite-dimensional vector spaces Ξ and Ξ^*;
(2) a map $\theta\colon \Xi \to \mathbf{L}(M)$ which is measurable with respect to the natural Borel structures of Ξ and $\mathbf{L}(M)$.

Denote by

$$\widehat{\cdot}\colon L^1(\Xi) \to L^\infty(\Xi^*), \quad b(\cdot) \mapsto \widehat{b}(\cdot) = \int_\Xi \mathrm{e}^{-\mathrm{i}\langle\cdot,x\rangle} b(x)\,\mathrm{d}x$$

the *Fourier transform with respect to the duality* $\langle\cdot,\cdot\rangle$, and the inverse Fourier transform

$$\check{\cdot}\colon L^1(\Xi^*) \to L^\infty(\Xi), \quad a(\cdot) \mapsto \check{a}(\cdot) = \int_{\Xi^*} \mathrm{e}^{\mathrm{i}\langle\xi,\cdot\rangle} a(\xi)\,\mathrm{d}\xi$$

where the Lebesgue measures on Ξ and Ξ^* are suitably normalized.

Then the corresponding *localized Weyl calculus for* π *along* θ is defined by

$$\mathrm{Op}^\theta\colon \widehat{L^1(\Xi)} \to \mathcal{B}(\mathcal{Y}), \quad \mathrm{Op}^\theta(a)y = \int_\Xi \check{a}(\xi)\pi(\exp_M(\theta(\xi)))y\,\mathrm{d}\xi \text{ for } y \in \mathcal{Y} \text{ and } a \in \widehat{L^1(\Xi)}, \tag{4.1}$$

where we use Bochner integrals of $\mathcal{Y}$-valued functions.

Remark 4.2 In the setting of Definition 4.1, we note the following:

(1) We need the Banach space $\mathcal{Y}$ to be separable to define the Bochner integral. Instead, we could have assumed $\mathcal{Y}$ a reflexive Banach space (for instance a Hilbert space) and defined $\mathrm{Op}^\theta(a) \in \mathcal{B}(\mathcal{Y})$ as a weakly convergent integral.
(2) It follows by (2.2) that

$$\mathrm{Op}^\theta(a)y = \int_{\Xi^*} \check{a}(\xi)\exp\left(\mathbf{L}(\pi)(\theta(\xi))\right) y\,\mathrm{d}\xi \tag{4.2}$$

for $y \in \mathcal{Y}$ and $a \in \widehat{L^1(\Xi)}$, hence the localized Weyl functional calculus actually depends on the mapping $\mathbf{L}(\pi)\colon \mathbf{L}(M) \to \mathcal{C}(\mathcal{Y})$, rather than on the representation $\pi\colon M \to \mathcal{B}(\mathcal{Y})$ itself. If $\mathcal{Y}$ is a Hilbert space, π is a unitary representation, and $\theta\colon \Xi \to \mathbf{L}(M)$ is continuous, it easily follows that (4.2) makes sense for every bounded continuous function $a\colon \Xi^* \to \mathbb{C}$ whose inverse Fourier transform $\check{a}$ is a finite Radon measure on Ξ. It thus follows that for every $\xi_0 \in \Xi^*$, we get the usual functional calculus of the self-adjoint operator $\mathbf{L}(\pi)\theta(\xi_0)$ by suitably extending Op^θ to functions of the form $\xi \mapsto b(\langle \xi_0, \xi\rangle)$ with $b\colon \mathbb{R} \to \mathbb{C}$.

(3) The localized Weyl functional calculus for π along θ has the following covariance property: If $\theta'\colon \Xi^* \to \mathbf{L}(M)$ is another measurable map such that there exists $m \in M$ satisfying $\mathrm{Ad}_M(m) \circ \theta' = \theta$, then

$$\left(\forall a \in \widehat{L^1(\Xi)}\right) \quad \mathrm{Op}^\theta(a) = \pi(m)\mathrm{Op}^{\theta'}(a)\pi(m)^{-1}. \tag{4.3}$$

In fact, for every $\xi \in \Xi^*$, we have

$$\begin{aligned}\exp_M(\theta(\xi)) &= (\theta(\xi))(1) = ((\mathrm{Ad}_M(m))\,\theta'(\xi))(1) = m(\theta'(\xi)(1))m^{-1}\\ &= m\exp_M(\theta'(\xi))m^{-1},\end{aligned}$$

hence, $\pi(\exp_M(\theta(\xi))) = \pi(m)\pi(\exp_M(\theta'(\xi)))\pi(m)^{-1}$, and now (4.3) follows by (4.1).

4.2 Magnetic pseudo-differential calculus on nilpotent Lie groups

We are going to specialize here the ideas of Subsect. 4.1 in order to construct a magnetic Weyl calculus. We work in the setting of Subsect. 3.2, keeping, however, for the moment the distinction between the Lie group and its Lie algebra. Thus, G is a (connected and) simply connected nilpotent Lie group with $\mathbf{L}(G) = \mathfrak{g}$. Then the exponential map $\exp_G\colon \mathfrak{g} \mapsto G$ is a diffeomorphism, and we use the notation $\log_G = \exp_G^{-1}$. We recall that the Haar measure on the group G is taken by $\log_G$ into the Lebesgue measure on $\mathfrak{g}$, consequently, the Lebesgue measure on $\mathfrak{g}$ is invariant under the transformations $Y \mapsto Y * X$ and $Y \mapsto (-Y)$.

Assume $\mathcal{F}$ an admissible space of real continuous functions on G, which is invariant under the left regular action, hence the mapping

$$\lambda\colon G \times \mathcal{F} \to \mathcal{F}, \quad (\lambda_g\varphi)(x) = \varphi(g^{-1}x)$$

is well-defined. Since $\mathcal{F}$ is endowed with a topology such that λ is continuous, we may consider the semidirect product $M = \mathcal{F} \rtimes_\lambda G$. Proposition 2.7 shows that the Lie algebra of M is the semidirect product $\mathcal{F} \rtimes_{\dot\lambda} \mathfrak{g}$ and the exponential map $\exp_M$ is given by

$$\exp_M(\varphi, X) = \left(\int_0^1 \lambda_{\exp_G(sX)}\varphi\,\mathrm{d}s,\ \exp_G(X)\right).$$

We denote the duality between $\mathfrak{g}$ and $\mathfrak{g}^*$ also by

$$\mathfrak{g}^* \times \mathfrak{g} \ni (\xi, X) \mapsto \langle \xi, X\rangle \in \mathbb{R}.$$

We set $\Xi = \mathfrak{g} \times \mathfrak{g}^*$. The mapping

$$\langle\cdot,\cdot\rangle : \Xi \times \Xi \to \mathbb{R}, \quad \langle (X_1, \xi_1), (X_2, \xi_2)\rangle = \langle \xi_1, X_2\rangle - \langle \xi_2, X_1\rangle$$

defines a symplectic structure on Ξ. This is, in particular, a duality pairing, Ξ being self-dual with respect to this pairing. The Fourier transform associated to $\langle\cdot,\cdot\rangle$ is given by

$$(F_\Xi a)(X, \xi) = \hat{a}(X, \xi) = \int_\Xi \mathrm{e}^{-\mathrm{i}\langle (X,\xi),(Y,\eta)\rangle} a(Y, \eta)\,\mathrm{d}(Y, \eta), \quad a \in L^1(\Xi).$$

It extends to an invertible operator $\mathcal{S}'(\Xi) \to \mathcal{S}'(\Xi)$, $F_\Xi^{-1} = F_\Xi$ and we denote $\check{a} = F_\Xi^{-1} a$. Note that if $F_\mathfrak{g} \colon \mathcal{S}'(\mathfrak{g}^*) \to \mathcal{S}'(\mathfrak{g})$ is the Fourier transform associated to the duality between $\mathfrak{g}$ and $\mathfrak{g}^*$ (normalized such that it is unitary $L^2(\mathfrak{g}) \to L^2(\mathfrak{g}^*)$) then

$$F_\Xi = \iota^* \left(F_\mathfrak{g} \otimes F_\mathfrak{g}^{-1}\right) = \left(F_\mathfrak{g}^{-1} \otimes F_\mathfrak{g}\right) \left(\iota^{-1}\right)^* \tag{4.4}$$

where ι^* is the pull-back by $\iota \colon \mathfrak{g}^* \times \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}^*$, $\iota(\xi, X) = (X, \xi)$.

We need a natural representation on M by unitary operators in $\mathcal{Y} = L^2(\mathfrak{g})$, given by the natural induced representation described in Subsect. 2.4. Namely, $\pi : M \to \mathcal{B}(\mathcal{Y})$ is given by

$$\pi(\varphi, g) f(X) = \mathrm{e}^{\mathrm{i}\varphi(\exp_G X)} f((-\log_G g) * X), \qquad f \in \mathcal{Y}. \tag{4.5}$$

Then $\pi(\varphi, g)$ is unitary for every $(\varphi, g) \in M$.

Consider now $\theta_0 \colon \Xi \to \mathcal{F}$, a Borel measurable function. Then, we set

$$\theta \colon \Xi \to \mathbf{L}(M), \qquad \theta(X, \xi) = Z_{\theta_0(X,\xi),X}, \quad (X, \xi) \in \Xi = \mathfrak{g} \times \mathfrak{g}^*, \tag{4.6}$$

where $Z_{\varphi,X}$ with $(\varphi, X) \in \mathcal{F} \times \mathfrak{g}$ has been defined in Proposition 2.7.

We consider the Weyl calculus for π along θ above. Recall that when $a \in F_\Xi L^1(\Xi)$

$$\mathrm{Op}^\theta(a) f = \int_\Xi \check{a}(X, \xi) \pi \left(\exp_M \theta(X, \xi)\right) f \,\mathrm{d}(X, \xi) \qquad f \in \mathcal{Y}. \tag{4.7}$$

We see that here

$$\exp_M \theta(X, \xi) = \theta(X, \xi)(1) = \left(\int_0^1 \lambda_{\exp_G(sX)} \theta_0(X, \xi) \,\mathrm{d}s,\ \exp_G X \right),$$

hence

$$\begin{aligned} \pi \left(\exp_M \theta(X, \xi)\right) f(Y) &= \mathrm{e}^{\mathrm{i} \int_0^1 \theta_0(X,\xi)\left(\exp_G(-sX) \exp_G Y\right) \mathrm{d}s} f((-X) * Y) \\ &= \mathrm{e}^{\mathrm{i} \int_0^1 \theta_0(X,\xi)\left(\exp_G((-sX)*Y)\right) \mathrm{d}s} f((-X) * Y) \end{aligned} \tag{4.8}$$

when $f \in \mathcal{Y}$.

We have, thus, obtained

$$\mathrm{Op}^\theta(a) f(Y) = \int_\Xi \check{a}(X, \xi) \mathrm{e}^{\mathrm{i} \int_0^1 \theta_0(X,\xi)(\exp_G((-sX)*Y)) \,\mathrm{d}s} f((-X) * Y) \,\mathrm{d}(X, \xi). \tag{4.9}$$

By changing variables, we get that

$$\mathrm{Op}^\theta(a) f(Y) = \int_\Xi \check{a}(Y * (-Z), \xi) \mathrm{e}^{\mathrm{i} \int_0^1 \theta_0(Y*(-Z),\xi)(\exp_G((s(Z*(-Y)))*Y)) \,\mathrm{d}s} f(Z) \,\mathrm{d}(Z, \xi). \tag{4.10}$$

We may use Fubini's theorem to see that the operator $\mathrm{Op}^\theta(a)$ is an integral operator with kernel

$$K_a(Y,Z)=\int_{\mathfrak{g}^*}\check{a}(Y*(-Z),\xi)\mathrm{e}^{\mathrm{i}\int_0^1\theta_0(Y*(-Z),\xi)(\exp_G((s(Z*(-Y)))*Y))\,\mathrm{d}s}\mathrm{d}\xi. \tag{4.11}$$

In the case where θ_0 is of the form

$$\theta_0(X,\xi)(x)=\langle\xi,\log_G x\rangle+\langle A(\log_G x),(R_{\log_G x})'_0X\rangle \tag{4.12}$$

where $A\colon\mathfrak{g}\to\mathfrak{g}^*$ is continuous and $x\mapsto\langle A(\log_G x),(R_{\log_G x})'_0X\rangle$ belongs to $\mathcal{F}$ for every $X\in\mathfrak{g}$, the expressions above can be further simplified. Denote

$$\alpha_A(Y,Z)=\exp\left(\mathrm{i}\int_0^1\langle A((s(Z*(-Y)))*Y),\left(R_{(s(Z*(-Y)))*Y}\right)'_0(Y*(-Z))\rangle\,\mathrm{d}s\right). \tag{4.13}$$

This is a continuous complex valued function on $\mathfrak{g}\times\mathfrak{g}$. With this notation (4.11) becomes

$$K_a(Y,Z)=\alpha_A(Y,Z)\int_{\mathfrak{g}^*}\check{a}(Y*(-Z),\xi)\mathrm{e}^{\mathrm{i}\int_0^1\langle\xi,(s(Z*(-Y)))*Y\rangle\,\mathrm{d}s}\mathrm{d}\xi.$$

Hence, in the case where θ_0 is as in (4.12), we get

$$\begin{aligned}K_a(Y,Z)&=\alpha_A(Y,Z)\int_{\mathfrak{g}^*}(F_{\mathfrak{g}}\otimes F_{\mathfrak{g}^*}a(\xi,Y*(-Z))\mathrm{e}^{\mathrm{i}\int_0^1\langle\xi,(s(Z*(-Y)))*Y\rangle\,\mathrm{d}s}\mathrm{d}\xi\\&=\alpha_A(Y,Z)(1\otimes F_{\mathfrak{g}}^{-1})a\left(\int_0^1(s(Z*(-Y)))*Y\right)\mathrm{d}s,Y*(-Z)).\end{aligned} \tag{4.14}$$

Definition 4.3 In the setting of Subsect. 3.2, the simply connected nilpotent Lie group G is identified with its Lie algebra $\mathfrak{g}$ by means of the exponential map. Let $\mathcal{F}$ an admissible space of functions on $\mathfrak{g}$ which contains both $\mathfrak{g}^*$ and the constant functions. Assume that $A\in\Omega^1(\mathfrak{g})$ is a magnetic potential such that for every $X\in\mathfrak{g}$ the function $Y\mapsto\langle A_Y,(R_Y)'_0X\rangle$ belongs to $\mathcal{F}$ and define $\theta_0\colon\mathfrak{g}\times\mathfrak{g}^*\to\mathcal{F}$ as in (4.12) (or Proposition 3.6). Then for every $a\in\mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*)$ there exists a linear operator $\mathrm{Op}^\theta(a)$ in $L^2(\mathfrak{g})$ defined by (4.10). We will denote $\mathrm{Op}^A(a):=\mathrm{Op}^\theta(a)$ and will call it a *magnetic pseudo-differential operator* with respect to the magnetic potential A. The function a is the *magnetic Weyl symbol* of the pseudo-differential operator $\mathrm{Op}^A(a)$, and the *Weyl calculus with respect to the magnetic potential A* is the mapping Op^A which takes a function $a\in\mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*)$ into the corresponding pseudo-differential operator.

Theorem 4.4 *Assume that $A\in\Omega^1(\mathfrak{g})$ is a magnetic potential such that for every $X\in\mathfrak{g}$ the function $Y\mapsto\langle A_Y,(R_Y)'_0X\rangle$ belongs to $\mathcal{F}$. Then, the Weyl calculus Op^A has the following properties:*

(1) For $P_0\in\mathfrak{g}$ let $A(Q)P_0$ be the multiplication operator defined by the function $Y\mapsto\langle A_Y,(R_Y)'_0P_0\rangle$. Then the usual functional calculus for the self-adjoint operator $-\mathrm{i}\dot{\lambda}(P_0)+A(Q)P_0$ in $L^2(\mathfrak{g})$ can be recovered from Op^A.

(2) *Gauge covariance with respect to the magnetic potential A: If* $A_1 \in \Omega^1(\mathfrak{g})$ *is another magnetic potential with* $dA = dA_1 \in \Omega^2(\mathfrak{g})$ *and the function* $Y \mapsto \langle A_Y, (R_Y)'_0 X\rangle$ *belongs to* $\mathcal{F}$ *for every* $X \in \mathfrak{g}$*, then there exists* $\psi \in \mathcal{F}$ *such that unitary operator* $U\colon L^2(\mathfrak{g}) \to L^2(\mathfrak{g})$ *defined by the multiplication by* $\mathrm{e}^{\mathrm{i}\psi}$ *satisfies* $U\mathrm{Op}^A(a)U^{-1} = \mathrm{Op}^{A_1}(a)$ *for every symbol* $a \in \mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*)$.

(3) *If* $\mathcal{C}^\infty_{\mathrm{pol}}(\mathfrak{g}) \subseteq \mathcal{F}$ *and the function* $Y \mapsto \langle A_Y, (R_Y)'_0 X\rangle$ *belongs to* $\mathcal{C}^\infty_{\mathrm{pol}}(\mathfrak{g})$ *for every* $X \in \mathfrak{g}$*, then for every* $a \in \mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*)$ *the magnetic pseudo-differential operator* $\mathrm{Op}^A(a)$ *is bounded linear on* $L^2(\mathfrak{g})$ *and is defined by an integral kernel* $K_a \in \mathcal{S}(\mathfrak{g}\times\mathfrak{g})$ *given by formula* (4.14).

(4) *Under the hypothesis of* (3) *the correspondence* $a \mapsto K_a$ *is an isomorphism of Fréchet spaces* $\mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*) \to \mathcal{S}(\mathfrak{g}\times\mathfrak{g})$ *and extends to a unitary operator* $L^2(\mathfrak{g}\times\mathfrak{g}^*) \to L^2(\mathfrak{g}\times\mathfrak{g})$.

Proof Assertion (1) follows by Remark 4.2(2) along with the fact that for the representation (4.5) we get, by (4.8)

$$\pi\left(\exp_M(\theta(tP_0,0))\right) f(Y) = \mathrm{e}^{\mathrm{i}\int_0^1 \theta_0(tP_0,0)((-stP_0)*Y)\,\mathrm{d}s} f((-tP_0)*Y)$$

where

$$\int_0^1 \theta_0(tP_0,0)((-stP_0)*Y)\,\mathrm{d}s = \int_0^1 \left\langle A((-stP_0)*Y), (R_{(-stP_0)*Y})'_0(tP_0)\right\rangle \mathrm{d}s$$
$$= \int_0^t \left\langle A((-sP_0)*Y), (R_{(-sP_0)*Y})'_0 P_0\right\rangle \mathrm{d}s.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi\left(\exp_M(\theta(tP_0,0))\right) f = \dot{\lambda}(P_0)f + \mathrm{i}(A(Q)P_0)f$$

for $f \in L^2(\mathfrak{g})$ such that the right-hand side belongs to $L^2(\mathfrak{g})$. See Remark 4.2(2) for the way the functional calculus of the self-adjoint operator $-\mathrm{i}\dot{\lambda}(P_0) + A(Q)P_0$ can be recovered.

For assertion (2) note that if $d(A - A_1) = 0$ on $\mathfrak{g}$, then $d\psi = A - A_1$ for the function $\psi\colon \mathfrak{g} \to \mathbb{R}$ defined by $\psi(X) = \int_0^1 \langle (A - A_1)_{tX}, X\rangle\,\mathrm{d}t$. In particular, $\psi \in \mathcal{F}$ and it follows by Proposition 2.7(3) that in the group $M = \mathcal{F} \rtimes_\lambda \mathfrak{g}$, we have

$$(\mathrm{Ad}_M \psi)\theta(X,\xi) = \theta_1(X,\xi)$$

for every $X \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, where $\theta_1(X,\xi)$ is obtained as in (4.6) with $\theta_0(X,\xi)$ replaced by the function $Y \mapsto \xi(Y) + \langle (A_1)_Y, (R_Y)'_0 X\rangle$. Now Remark 4.2(3) shows that the assertion holds with $U = \pi(\psi)\colon L^2(\mathfrak{g}) \to L^2(\mathfrak{g})$. Also note that, according to (4.5), U is actually the multiplication operator by the function $\mathrm{e}^{\mathrm{i}\psi}$.

Now assume the hypothesis of assertions (3) and (4) and remember that the first of these properties have been already proved in the discussion preceding Definition 4.3. Further, note that $\alpha_A(\cdot), \alpha_A(\cdot)^{-1} \in \mathcal{C}^\infty_{\mathrm{pol}}(\mathfrak{g}\times\mathfrak{g})$ by (4.13). Since, moreover $|\alpha(\cdot)| = 1$, we see from formula (4.14) that in order to show prove the asserted properties of the correspondence $a \mapsto K_a$ it will be enough to check that the mapping

$$\Sigma\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}\times\mathfrak{g}, \quad \Sigma(Y,Z) = \left(\int_0^1 (s(Z*(-Y)))*Y)\,\mathrm{d}s,\ Y*(-Z)\right)$$

is a polynomial diffeomorphism whose inverse is polynomial and which preserves the Lebesgue measure on $\mathfrak{g} \times \mathfrak{g}$. For this purpose, let us note that $\Sigma = \Sigma_2 \circ \Sigma_1$, where the mappings $\Sigma_1, \Sigma_2 \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ are defined by

$$\Sigma_1(Y, Z) = (-Y, Y * (-Z)) \quad \text{and} \quad \Sigma_2(V, W) = \left(-\int_0^1 V * (sW)\, \mathrm{d}s,\, W \right).$$

By using the fact that $\mathfrak{g}$ is a nilpotent Lie algebra it is straightforward to prove that Σ_1 is a measure-preserving polynomial diffeomorphism whose inverse is polynomial, and so is Σ_2 because of Proposition 3.2. This completes the proof. □

Definition 4.5 If we assume that $\mathcal{C}^\infty_{\mathrm{pol}}(\mathfrak{g}) \subseteq \mathcal{F}$ and $A \in \Omega^1(\mathfrak{g})$ has the property that the function $Y \mapsto \langle A_Y, (R_Y)'_0 X\rangle$ belongs to $\mathcal{C}^\infty_{\mathrm{pol}}(\mathfrak{g})$ for every $X \in \mathfrak{g}$, then it follows by Theorem 4.4(3) that for every $a_1, a_2 \in \mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*)$ there exists a unique function $a_1 \#^A a_2 \in \mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*)$ such that $\mathrm{Op}^A(a_1)\mathrm{Op}^A(a_2) = \mathrm{Op}^A(a_1\#^A a_2)$ and the *magnetic Moyal product*

$$\mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*) \times \mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*) \to \mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*), \quad (a_1, a_2) \mapsto a_1\#^A a_2$$

is a bilinear continuous mapping. For the sake of simplicity, we denote $a_1\# a_2 := a_1\#^A a_2$, whenever the magnetic potential A had been already specified.

4.3 The magnetic Moyal product for two-step nilpotent Lie algebras

In this subsection, we shall assume that $\mathfrak{g}$ is a *two-step nilpotent Lie algebra*, that is, $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = \{0\}$, and moreover, $\mathcal{C}^\infty_{\mathrm{pol}}(\mathfrak{g}) \subseteq \mathcal{F}$ and $A \in \Omega^1(\mathfrak{g})$ is a magnetic potential such that $\langle A(\cdot), X\rangle \in \mathcal{C}^\infty_{\mathrm{pol}}(\mathfrak{g})$ for every $X \in \mathfrak{g}$.

Lemma 4.6 *The following assertions hold in the two-step nilpotent Lie algebra* $\mathfrak{g}$*:*

(1) *For every* $X, Y \in \mathfrak{g}$*, we have* $(s(X * (-Y))) * Y = sX + (1-s)Y$ *for arbitrary* $s \in \mathbb{R}$ *and* $\int_0^1 (s(X * (-Y))) * Y\, \mathrm{d}s = \frac{1}{2}(X+Y)$.

(2) *For arbitrary* $X, Y, Z, T \in \mathfrak{g}$*, we have*

$$\begin{cases} X = \frac{1}{2}(Y+Z) \\ T = Y * (-Z) \end{cases} \iff \begin{cases} Y = (\frac{1}{2}T) * X \\ Z = (-\frac{1}{2}T) * X. \end{cases}$$

Moreover, the diffeomorphism $\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}\times\mathfrak{g}$, $(Y, Z) \mapsto (\frac{1}{2}(Y+Z), Y*(-Z))$ *preserves the Lebesgue measure.*

(3) *For arbitrary* $X, Z, T, z, t \in \mathfrak{g}$*, we have*

$$\begin{cases} \frac{1}{2}((\frac{1}{2}T) * X) + Z) = z \\ \frac{1}{2}(Z + ((-\frac{1}{2}T) * X)) = t \end{cases} \iff \begin{cases} T = 2(z-t) + [X, z-t] \\ Z = z + t - X. \end{cases}$$

Moreover, the diffeomorphism $\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}\times\mathfrak{g}$, $(Z, T) \mapsto (z, t)$ *preserves the Lebesgue measure.*

Proof

(1) Indeed, for arbitrary $s \in \mathbb{R}$, we have

$$\begin{aligned}(s(X*(-Y)))*Y &= (s(X-Y-\frac{1}{2}[X,Y]))*Y\\ &= sX-sY-\frac{s}{2}[X,Y]+Y+\frac{s}{2}[X,Y]\\ &= sX+(1-s)Y.\end{aligned}$$

(2) For the implication "⇒" note that $T=Y*(-Z)$ actually means $T=Y-Z-\frac{1}{2}[Y,Z]$. If we apply $-\mathrm{ad}_{\mathfrak{g}}Z$ to both sides of the latter equation, then we get $[T,Z]=[Y,Z]$, and then

$$T=Y-Z-\frac{1}{2}[T,Z]. \tag{4.15}$$

On the other hand, the first of the assumed equations implies $2X=Y+Z$, and then, we can eliminate Y between this equation and (4.15). We, thus, get $2X-T=2Z+\frac{1}{2}[T,Z]$, and then by applying $\mathrm{ad}_{\mathfrak{g}}T$ to both sides of this equality we get $[T,X]=[T,Z]$. It then follows by (4.15) that $T=Y-Z-\frac{1}{2}[T,X]$. Since $2X=Y+Z$, we get

$$\begin{cases} Y=X+\frac{1}{2}T+\frac{1}{4}[T,X]=(\frac{1}{2}T)*X\\ Z=X-\frac{1}{2}T-\frac{1}{4}[T,X]=(-\frac{1}{2}T)*X.\end{cases}$$

This concludes the proof of the implication "⇒", and the converse implication can be easily proved in a similar manner. The assertion regarding the measure-preserving property can be easily checked by computing the Jacobian of the diffeomorphism.

(3) We have

$$\begin{cases}\frac{1}{2}(((\frac{1}{2}T)*X)+Z)=z\\ \frac{1}{2}(Z+((-\frac{1}{2}T)*X))=t\end{cases} \iff \begin{cases} X+\frac{1}{2}T-\frac{1}{4}[X,T]+Z=2z\\ X-\frac{1}{2}T+\frac{1}{4}[X,T]+Z=2t,\end{cases}$$

and now the conclusion follows at once. □

Theorem 4.7 *Assume that $\mathfrak{g}$ is a two-step nilpotent Lie algebra, $\mathcal{C}^\infty_{\mathrm{pol}}(\mathfrak{g})\subseteq\mathcal{F}$, and $A\in\Omega^1(\mathfrak{g})$ is a magnetic potential such that $\langle A(\cdot),X\rangle\in\mathcal{C}^\infty_{\mathrm{pol}}(\mathfrak{g})$ for every $X\in\mathfrak{g}$. Then the following assertions hold:*

(1) For every $a\in\mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*)$ the integral kernel of the operator $\mathrm{Op}^A(a)\colon L^2(\mathfrak{g})\to L^2(\mathfrak{g})$ *is given by the formula*

$$K_a(Y,Z)=\alpha_A(Y,Z)(1\otimes F_{\mathfrak{g}}^{-1})a\left(\frac{1}{2}(Y+Z),Y*(-Z)\right), \tag{4.16}$$

where

$$\alpha_A(Y,Z)=\exp\left(-\mathrm{i}\int_0^1\langle A(sZ+(1-s)Y),Z*(-Y)\rangle\,\mathrm{d}s\right), \tag{4.17}$$

for every $Y,Z\in\mathfrak{g}$.

(2) Set

$$\beta_A\colon\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}\to\mathbb{C},\quad \beta_A(X,Y,Z)=\alpha_A^{-1}(X,Y)\alpha_A(Y,Z)\alpha_A(Z,X).$$

If $a, b \in \mathcal{S}(\mathfrak{g} \times \mathfrak{g}^*)$ *then*

$$(a\#b)(X,\xi) = \iiiint\limits_{\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}^*\times\mathfrak{g}^*} a(Z,\zeta)b(T,\tau)\mathrm{e}^{2\mathrm{i}\langle(Z-X,\zeta-\xi),(T-X,\tau-\xi)\rangle} \times \mathrm{e}^{-\mathrm{i}(\langle\xi+\zeta,[X,Z]\rangle+\langle\zeta+\tau,[Z,T]\rangle+\langle\tau+\xi,[T,X]\rangle)} \times \beta_A(Z-T+X, T-Z+X, Z+T-X)\,\mathrm{d}\zeta\mathrm{d}\tau\mathrm{d}Z\mathrm{d}T, \quad (4.18)$$

for every $(X, \xi) \in \mathfrak{g} \times \mathfrak{g}^*$.

Proof Formulas (4.16) and (4.17) follow at once by using (4.14) and (4.13), respectively, and taking into account Lemma 4.6(1).

In order to prove (4.18), note first that by (4.16) and Lemma 4.6(2), we have for every $c \in \mathcal{S}(\mathfrak{g} \times \mathfrak{g}^*)$ and $X, T \in \mathfrak{g}$ the equation

$$K_c((\frac{1}{2}T) * X, (-\frac{1}{2}T) * X) = \alpha_A((\frac{1}{2}T) * X, (-\frac{1}{2}T) * X)(1 \otimes F_{\mathfrak{g}}^{-1})c(X,T),$$

whence

$$c(X,\xi) = \int\limits_{\mathfrak{g}} \mathrm{e}^{-\mathrm{i}\langle\xi,T\rangle}\left(\alpha_A^{-1}K_c\right)\left(\left(\frac{1}{2}T\right) * X, \left(-\frac{1}{2}T\right) * X\right)\mathrm{d}T.$$

for every $c \in \mathcal{S}(\mathfrak{g} \times \mathfrak{g}^*)$. Hence, by using the well-known formula for the integral kernel of the product of two operators defined by integral kernels, we get

$$\begin{aligned}(a\#b)(X,\xi) &= \int\limits_{\mathfrak{g}} \mathrm{e}^{-\mathrm{i}\langle\xi,T\rangle}(\alpha_A^{-1}K_{a\#b})\left(\left(\frac{1}{2}T\right) * X, \left(-\frac{1}{2}T\right) * X\right)\mathrm{d}T \\ &= \int\limits_{\mathfrak{g}}\int\limits_{\mathfrak{g}} \mathrm{e}^{-\mathrm{i}\langle\xi,T\rangle}\alpha_A^{-1}\left(\left(\frac{1}{2}T\right) * X, \left(-\frac{1}{2}T\right) * X\right)K_a\left(\left(\frac{1}{2}T\right) * X, Z\right) \\ &\quad\times K_b\left(Z, \left(-\frac{1}{2}T\right) * X\right)\mathrm{d}Z\mathrm{d}T.\end{aligned}$$

On the other hand, by (4.16) we get

$$\begin{aligned}K_a\left(\left(\frac{1}{2}T\right) * X, Z\right) &= \alpha_A\left(\left(\frac{1}{2}T\right) * X, Z\right) \\ &\quad\times \left(1 \otimes F_{\mathfrak{g}}^{-1}\right)a\left(\frac{1}{2}\left(\left(\left(\frac{1}{2}T\right) * X\right) + Z\right), \left(\frac{1}{2}T\right) * X * (-Z)\right) \\ &= \alpha_A\left(\left(\frac{1}{2}T\right) * X, Z\right)\int\limits_{\mathfrak{g}^*} \mathrm{e}^{\mathrm{i}\left\langle\zeta, \left(\frac{1}{2}T\right)*X*(-Z)\right\rangle} \\ &\quad\times a\left(\frac{1}{2}\left(\left(\left(\frac{1}{2}T\right) * X\right) + Z\right), \zeta\right)\mathrm{d}\zeta,\end{aligned}$$

and also by (4.16) we have similarly

$$K_b\left(Z,\left(-\frac{1}{2}T\right)*X\right)=\alpha_A\left(Z,\left(-\frac{1}{2}T\right)*X\right)$$
$$\times\left(1\otimes F_{\mathfrak{g}}^{-1}\right)b\left(\frac{1}{2}(Z+\left(\left(-\frac{1}{2}T\right)*X\right)\right),Z*(-X)*\left(\frac{1}{2}T\right)\right)$$
$$=\alpha_A(Z,\left(-\frac{1}{2}T\right)*X)\int\limits_{\mathfrak{g}^*}\mathrm{e}^{\mathrm{i}\left\langle\tau,Z*(-X)*\left(\frac{1}{2}T\right)\right\rangle}$$
$$\times b\left(\frac{1}{2}\left(Z+\left(\left(-\frac{1}{2}T\right)*X\right)\right),\tau\right)\mathrm{d}\tau.$$

We plug in these formulas in the above expression of the magnetic Moyal product $a\#b$ and get

$$(a\#b)(X,\xi)=\int\limits_{\mathfrak{g}}\int\limits_{\mathfrak{g}}\int\limits_{\mathfrak{g}^*}\int\limits_{\mathfrak{g}^*}\alpha_A^{-1}\left(\left(\frac{1}{2}T\right)*X,\left(-\frac{1}{2}T\right)*X\right)$$
$$\times\alpha_A\left(\left(\frac{1}{2}T\right)*X,Z\right)\alpha_A\left(Z,\left(-\frac{1}{2}T\right)*X\right)$$
$$\times\mathrm{e}^{\mathrm{i}E(\zeta,\tau,Z,T)}a\left(\frac{1}{2}\left(\left(\left(\frac{1}{2}T\right)*X\right)+Z\right),\zeta\right)$$
$$\times b\left(\frac{1}{2}\left(Z+\left(\left(-\frac{1}{2}T\right)*X\right)\right),\tau\right)\mathrm{d}\zeta\mathrm{d}\tau\mathrm{d}Z\mathrm{d}T$$

where

$$E(\zeta,\tau,Z,T)=-\langle\xi,T\rangle+\left\langle\zeta,\left(\frac{1}{2}T\right)*X*(-Z)\right\rangle+\left\langle\tau,Z*(-X)*\left(\frac{1}{2}T\right)\right\rangle.$$

We change of variables $(Z,T)\mapsto(z,t)$ of Lemma 4.6(3). In these new variables, we have

$$\left(\frac{1}{2}T\right)*X=2z-Z=z-t+X,$$
$$\left(-\frac{1}{2}T\right)*X=2t-Z=t-z+X.$$

It follows that

$$(a\#b)(X,\xi)=\int\limits_{\mathfrak{g}}\int\limits_{\mathfrak{g}}\int\limits_{\mathfrak{g}^*}\int\limits_{\mathfrak{g}^*}\beta_A(z-t+X,t-z+X,z+t-X)$$
$$\times\mathrm{e}^{\mathrm{i}E(\zeta,\tau,z+t-x,2(z-t)+[X,z-t])}a(z,\zeta)b(t,\tau)\,\mathrm{d}\zeta\mathrm{d}\tau\mathrm{d}z\mathrm{d}t.$$

Note that in the change of variables above we have

$$\left(\frac{1}{2}T\right)*X*(-Z)=(2z-Z)*(-Z)$$
$$=2(z-Z)+\frac{1}{2}[2z-Z,-Z]$$
$$=2(X-t)+[z,X-t],$$

and similarly

$$\begin{aligned} Z * (-X) * \left(\frac{1}{2}T\right) &= Z * (Z - 2t) \\ &= 2(Z - t) + \frac{1}{2}[Z, Z - 2t] \\ &= 2(z - X) + [t, z - X], \end{aligned}$$

Thus,

$$\begin{aligned} E(\zeta, \tau, z + t - x, 2(z - t) + [X, z - t]) &= -\langle \xi, 2(z - t) + [X, z - t] \rangle \\ &\quad + \langle \tau, 2(X - t) + [z, X - t] \rangle \\ &\quad + \langle \tau, 2(z - X) + [t, z - X] \rangle \\ &= \langle 2(\tau - \xi), z - X \rangle - \langle 2(\zeta - \xi), t - X \rangle \\ &\quad - \mathrm{i}(\langle \xi + \zeta, [X, z] \rangle \\ &\quad + \langle \zeta + \tau, [z, t] \rangle + \langle \tau + \xi, [t, X] \rangle \end{aligned}$$

and this completes the proof of (4.18). □

It is clear that in the case when $\mathfrak{g}$ is an abelian Lie algebra, formula (4.18) specializes to the formula for the magnetic Moyal product on $\mathbb{R}^n$; see [21] and [28]. If, moreover, the magnetic potential $A \in \Omega^1(\mathfrak{g})$ vanishes, then one recovers the formula for the composition of pseudo-differential operators in the framework of the Weyl calculus; see Sect. 18.5 in [18].

Acknowledgements We are grateful to Anders Melin, Benjamin Cahen, and Horia Cornean for their kind help. We also thank the Referee for interesting comments and carefully reading the manuscript. Partial financial support from the grant PNII—Programme "Ideas" (code 1194) is acknowledged.

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