Journal of Geometry and Physics 60 (2010) 81–95

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Uncertainty principles for magnetic structures on certain coadjoint orbits

Ingrid Beltiţă, Daniel Beltiţă *

Institute of Mathematics, "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, Bucharest, Romania

ARTICLE INFO

Article history:
Received 13 June 2009
Received in revised form 25 August 2009
Accepted 20 September 2009
Available online 25 September 2009

2000 Mathematics Subject Classification:
primary 81S30
secondary 22E25
22E65
35S05
47G30

Keywords:
Weyl calculus
Magnetic field
Lie group
Semidirect product

ABSTRACT

By building on our earlier work, we establish uncertainty principles in terms of Heisenberg inequalities and of the ambiguity functions associated with magnetic structures on certain coadjoint orbits of infinite-dimensional Lie groups. These infinite-dimensional Lie groups are semidirect products of nilpotent Lie groups and invariant function spaces thereon. The recently developed magnetic Weyl calculus is recovered in the special case of function spaces on abelian Lie groups.

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1. Introduction

The relationship between the Weyl calculus of pseudo-differential operators on $\mathbb{R}^n$ and the Heisenberg group $\mathbb{R}^{n+1} \rtimes \mathbb{R}^n$ is a classical topic (see for instance [1,2], or [3]). In fact, the Weyl calculus provides a quantization of a nontrivial coadjoint orbit for the Heisenberg group. On the other hand, a magnetic gauge-covariant pseudo-differential calculus on $\mathbb{R}^n$ has also been recently developed by using techniques of hard analysis; see [4,5]. As our alternative approach has shown [6], this magnetic calculus can be set up for any nilpotent Lie group G and can be understood as a quantization of a certain coadjoint orbit for some Lie group $\mathcal{F} \rtimes G$, which is infinite dimensional unless the magnetic field is polynomial. More specifically, by adapting ideas of [7], the cotangent bundle T^*G has been symplectomorphically realized as a coadjoint orbit of $\mathcal{F} \rtimes G$ and the pseudo-differential calculus has been constructed as a Weyl quantization of that orbit. (We refer to [8] for a discussion on Weyl quantizations.) In our case, the semidirect product is needed in order to deal with rather general perturbations of invariant differential operators on G. The semidirect products have also turned out to be an important tool in mechanics; see for instance [9].

In the present paper we investigate some uncertainty principles for the magnetic Weyl calculus developed in [6]. The uncertainty principles have been an active area of research. We refer to the survey [10] for a comprehensive introduction to this circle of ideas, to [11] for the case of families of pseudo-differential operators, and to [12,13] for Hardy's uncertainty principles on Lie groups. The main point of the present approach is that the aforementioned Weyl quantization allows us to

* Corresponding author.
E-mail addresses: Ingrid.Beltita@imar.ro (I. Beltiţă), Daniel.Beltita@imar.ro, beltita@gmail.com (D. Beltiţă).

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doi:10.1016/j.geomphys.2009.09.007

obtain versions of Heisenberg's inequality —taking into account magnetic momenta— and Lieb's uncertainty principle [14] for a certain wavelet transform associated with the coadjoint orbit T^*G of $\mathcal{F} \rtimes G$.

Let us describe the contents of our paper in some more detail. Section 2 is devoted to establishing Heisenberg's uncertainty inequality in the magnetic setting on nilpotent Lie groups. In Section 2.1, after describing the necessary notation used throughout the paper, we introduce the ambiguity function and the cross-Wigner distribution in the present framework and prove some of their main properties including Moyal's identity (Theorem 2.8). Preliminary material on magnetic Weyl calculus from [6] is provided in Section 2.2 along with additional properties in connection with the Wigner distribution. Thus, in Proposition 2.18 we indicate the significance of its marginal distributions for the functional calculus with both the position operators and the "noncommutative magnetic momentum" operators. Let us point out that using the usual Fourier transform does not seem very natural in the present context. This is due both to the presence of the magnetic potential and to the fact that the invariant vector fields on a nilpotent Lie group may not have constant coefficients (see Example 4.2). Versions for Heisenberg's inequality are established in Theorem 2.19 and Corollary 2.20.

Section 3 deals with a version of Lieb's uncertainty principle in the present setting. The main result is Theorem 3.5 and is stated in terms of magnetic ambiguity functions and mixed-norm Lebesgue spaces on the cotangent bundle of a nilpotent Lie group. In the case of abelian Lie groups and no magnetic potential we recover one of the results of [15]. (See also [16,17] for related results in this classical case.) Among the consequences of Theorem 3.5 we mention an embedding theorem for the natural versions of the modulation spaces in our setting (Corollary 3.6).

Finally, in Section 4 we illustrate the main ideas by considering the special case of two-step nilpotent Lie algebras.

2. Heisenberg's uncertainty inequality in the magnetic setting on nilpotent Lie groups

2.1. Moyal's identity on nilpotent Lie groups

In this subsection we introduce the ambiguity function and the cross-Wigner distribution in the present setting and prove some of their main properties including Moyal's identity (Theorem 2.8). This property occurs in connection with a finite-dimensional coadjoint orbit of a semidirect product which is in general an infinite-dimensional Lie group (see Prop. 2.9 in [6]). It corresponds to the orthogonality relations proved in [1] for the matrix coefficients of *any* irreducible representation of a nilpotent Lie group. Let us also note that other wavelet transforms associated with semidirect products of locally compact (or finite-dimensional Lie) groups appeared in [18,19].

Setting 2.1. Throughout the present paper we work in the setting of Section 4 in [6]. Let us briefly recall the main notation involved therein.

- A connected, simply connected, nilpotent Lie group G is identified to its Lie algebra $\mathfrak{g}$ by means of the exponential map. We denote by $*$ the Baker-Campbell-Hausdorff multiplication on $\mathfrak{g}$, so that $G = (\mathfrak{g}, *)$.
- The cotangent bundle T^*G is a trivial bundle and we perform the identification
$$T^*G \simeq \mathfrak{g} \times \mathfrak{g}^* \tag{2.1}$$
by using the trivialization by left translations.
- $\mathcal{F}$ is an admissible function space on the Lie group G (see Def. 2.8 in [6]); in particular, $\mathcal{F}$ is invariant under translations to the left on G,
$$\lambda_X : \mathcal{F} \to \mathcal{F}, \quad \phi \mapsto \phi((-X) * \cdot)$$
for all $X \in \mathfrak{g}$, and $\mathcal{F}$ is endowed with a locally convex topology such that we have continuous inclusions $\mathfrak{g}^* \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{C}^\infty(G)$. For instance $\mathcal{F}$ can be the whole space $\mathcal{C}^\infty(G)$ or the space $\mathcal{C}^\infty_{\mathrm{pol}}(G)$ of smooth functions with polynomial growth. See however Example 4.1 below for specific situations when $\dim \mathcal{F} < \infty$.
- The semidirect product $M = \mathcal{F} \rtimes_\lambda G$ is an infinite-dimensional Lie group in general, whose Lie algebra is $\mathfrak{m} = \mathcal{F} \rtimes_{\dot{\lambda}} \mathfrak{g}$. We refer to [20] or [21] for basic facts on infinite-dimensional Lie groups.
- We endow $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$ with Lebesgue measures suitably normalized such that the Fourier transform $L^2(\mathfrak{g}) \to L^2(\mathfrak{g}^*)$ is a unitary operator, and we denote $\mathcal{H} = L^2(\mathfrak{g})$.
- We define a unitary representation $\pi : M \to \mathcal{B}(\mathcal{H})$ by
$$(\pi(\phi, X)f)(Y) = \mathrm{e}^{\mathrm{i}\phi(Y)} f((-X) * Y)$$
for $(\phi, X) \in M$, $f \in \mathcal{H}$, and $Y \in \mathfrak{g}$.
- The *magnetic potential* is a smooth mapping $A : \mathfrak{g} \to \mathfrak{g}^*$, $X \mapsto A_X$, with polynomial growth such that for every $X \in \mathfrak{g}$ we have $\langle A_\bullet, (R_\bullet)'_0 X \rangle \in \mathcal{F}$.
- We also need the mappings
$$\theta_0 : \mathfrak{g} \times \mathfrak{g}^* \to \mathcal{F}, \quad \theta_0(X, \xi) = \xi + \langle A_\bullet, (R_\bullet)'_0 X \rangle$$
and
$$\theta : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{m}, \quad (X, \xi) \mapsto (\theta_0(X, \xi), X).$$
Here $R_Y : \mathfrak{g} \to \mathfrak{g}$, $Z \mapsto Z * Y$, is the translation to the right defined by any $Y \in \mathfrak{g}$. $\square$

Remark 2.2. We wish to explain here the relationship between the objects introduced in Setting 2.1 and the *magnetic field* $B := \mathrm{d}A \in \Omega^2(\mathfrak{g})$, where the magnetic potential is thought of as a 1-form $A \in \Omega^1(\mathfrak{g})$. Specifically, for arbitrary $\phi \in \mathcal{F}$ and $X \in \mathfrak{g}$, let us define the function $\bar{\theta}_0^A(\phi, X) \in \mathcal{F}$ by

$$(\forall Y \in \mathfrak{g}) \quad (\bar{\theta}_0^A(\phi, X))(Y) = \phi(Y) + \langle A_Y, (R_Y)'_0 X\rangle.$$

Now consider the continuous linear mapping

$$\bar{\theta}^A : \mathcal{F} \rtimes_{\dot{\lambda}} \mathfrak{g} = \mathfrak{m} \to \mathfrak{m}, \quad \bar{\theta}^A(\phi, X) = (\bar{\theta}_0^A(\phi, X), X)$$

and the differential 2-forms $\bar{\omega}, \bar{B} \in \Omega^2(\mathcal{F} \times \mathfrak{g})$ defined by

$$\bar{\omega}_{(\phi_0, X_0)}((\phi_1, X_1), (\phi_2, X_2)) = (\phi_2)'_{X_0}(X_1) - (\phi_1)'_{X_0}(X_2)$$

and

$$\bar{B}_{(\phi_0, X_0)}((\phi_1, X_1), (\phi_2, X_2)) = B_{X_0}(X_1, X_2),$$

respectively. Then the following assertions hold:

(1) The operator $\bar{\theta}^A : \mathfrak{m} \to \mathfrak{m}$ is invertible and $(\bar{\theta}^A)^{-1} = \bar{\theta}^{-A}$.
(2) We have $d\bar{\omega} = 0$ and $(\bar{\theta}^A)^*(\bar{\omega}) = \bar{\omega} + \bar{B}$. If we restrict the latter equality to $\mathfrak{g}^* \times \mathfrak{g}$ ($\subseteq \mathcal{F} \times \mathfrak{g}$), then we see that the *magnetic perturbation* $(\bar{\omega} + \bar{B})|_{\mathfrak{g}^* \times \mathfrak{g}}$ of the canonical symplectic form $\bar{\omega}|_{\mathfrak{g}^* \times \mathfrak{g}} \in \Omega^2(\mathfrak{g} \times \mathfrak{g}^*)$ can be recovered as the pull-back of the universal form $\bar{\omega} \in \Omega^2(\mathcal{F} \times \mathfrak{g})$ by a mapping constructed in terms of a magnetic potential, namely $\bar{\theta}^A|_{\mathfrak{g}^* \times \mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathcal{F} \times \mathfrak{g}$. (The 2-form $\bar{\omega}$ is universal in the sense that it does not depend on the magnetic field.) Note that $\bar{\theta}^A(\xi, X) = \theta(X, \xi)$ whenever $\xi \in \mathfrak{g}^* \subseteq \mathcal{F}$ and $X \in \mathfrak{g}$.
(3) Here is a way to describe the mutual relationship between the aforementioned mappings constructed out of different potentials of the same magnetic field: If $A_1, A_2 \in \Omega^1(\mathfrak{g})$ are magnetic potentials then there exists $m_0 \in M$ such that $\bar{\theta}^{A_1} = \mathrm{Ad}_M(m_0) \circ \bar{\theta}^{A_2}$ if and only if $dA_1 = dA_2$.

We refer to Prop. 3.3 and Rem. 3.5 in [6] for some more details. □

Notation 2.3. We shall denote for every $X \in \mathfrak{g}$,

$$\Psi_X : \mathfrak{g} \to \mathfrak{g}, \quad \Psi_X(Y) = \int_0^1 Y * (sX)\mathrm{d}s$$

(see Prop. 3.2 in [6]) and also

$$\tau_A(X, Y) = \exp\Big(\mathrm{i} \int_0^1 \langle A_{(-sX)*Y}, (R_{(-sX)*Y})'_0 X\rangle \mathrm{d}s\Big)$$

for $X, Y \in \mathfrak{g}$. □

Remark 2.4. We note that for arbitrary $(\phi, X) \in \mathfrak{m}$ and $f \in L^2(\mathfrak{g})$ we have

$$\pi(\exp_M(\bar{\theta}^A(\phi, X)))f(\cdot) = \tau_A(X, \cdot)\pi(\exp_M(\phi, X))f(\cdot),$$

since

$$\exp_M(\bar{\theta}^A(\phi, X)) = \Big(\int_0^1 \lambda_{sX}\phi \mathrm{d}s + \int_0^1 \lambda_{sX}\langle A_\bullet, (R_\bullet)'_0 X\rangle \mathrm{d}s, X\Big)$$

while $\exp_M(\phi, X) = (\int_0^1 \lambda_{sX}\phi\mathrm{d}s, X)$. It follows by associativity of the multiplication and the above formula that the mapping $X \mapsto \tau_A(X, \cdot)$ gives rise to a cocycle with values in the multiplicative group $\{\mathrm{e}^{\mathrm{i}\phi} \mid \phi \in \mathcal{F}\}$. We refer to [4] for a discussion of this circle of ideas in the case when $\mathfrak{g}$ is abelian. □

Lemma 2.5. *For every $(X, \xi) \in \mathfrak{g} \times \mathfrak{g}^*$ and $f \in L^2(\mathfrak{g})$ we have*

$$(\pi(\exp_M(\theta(X, \xi)))f)(Y) = \tau_A(X, Y)\mathrm{e}^{-\mathrm{i}\langle \xi, \Psi_X(-Y)\rangle} f((-X) * Y), \tag{2.2}$$

$$(\pi(\exp_M(-\theta(X, \xi)))f)(Y) = \tau_A(X, X * Y)^{-1}\mathrm{e}^{\mathrm{i}\langle \xi, \Psi_X(-(X*Y))\rangle} f(X * Y) \tag{2.3}$$

for arbitrary $Y \in \mathfrak{g}$.

Proof. Formula (2.2) follows at once by Remark 2.4, Notation 2.3, and the fact that

$$(\forall \xi \in \mathfrak{g}^*)(\forall X, Y \in \mathfrak{g}) \quad \int_0^1 (\lambda_{sX}\xi)(Y)\mathrm{d}s = -\langle \xi, \Psi_X(-Y)\rangle.$$

In order to prove the second formula, note that for $\phi \in L^2(\mathfrak{g})$ we have by (2.2)

$$\begin{aligned}(\pi(\exp_M(\theta(X,\xi)))f)(Y) = {} & \phi(Y)\\ \iff {} & \tau_A(X,Y)\exp(-\mathrm{i}\langle \xi, \Psi_X(-Y)\rangle)f((-X)*Y) = \phi(Y)\\ \iff {} & f((-X)*Y) = \tau_A(X,Y)^{-1}\exp(\mathrm{i}\langle \xi, \Psi_X(-Y)\rangle)\phi(Y)\end{aligned}$$

for arbitrary $Y \in \mathfrak{g}$, which is further equivalent to

$$(\forall Y \in \mathfrak{g}) \quad f(Y) = \tau_A(X, X*Y)^{-1}\exp(\mathrm{i}\langle \xi, \Psi_X(-(X*Y))\rangle)\phi(X*Y)$$

and this concludes the proof. □

Definition 2.6. For arbitrary $\phi, f \in L^2(\mathfrak{g})$ we define the function

$$\mathcal{A}_\phi f \colon \mathfrak{g}\times\mathfrak{g}^* \to \mathbb{C}, \quad (\mathcal{A}_\phi f)(X,\xi) = (f \mid \pi(\exp_M(\theta(X,\xi)))\phi).$$

We shall call $\mathcal{A}_\phi f$ the *ambiguity function* defined by $\phi, f \in L^2(\mathfrak{g})$. By using the canonical symplectic structure on $\mathfrak{g}\times\mathfrak{g}^*$ given by

$$(\mathfrak{g}\times\mathfrak{g}^*)\times(\mathfrak{g}\times\mathfrak{g}^*) \to \mathbb{R}, \quad ((X_1,\xi_1),(X_2,\xi_2)) \mapsto \langle \xi_1, X_2\rangle - \langle \xi_2, X_1\rangle$$

we also define the symplectic Fourier transform of the ambiguity function

$$\mathcal{W}(f,\phi) := \widehat{\mathcal{A}_\phi f} \in L^2(\mathfrak{g}\times\mathfrak{g}^*)$$

and we call it the *cross-Wigner distribution (function)* of $\phi, f \in L^2(\mathfrak{g})$. The definition of $\mathcal{W}(f,\phi)$ makes sense since it follows by Theorem 2.8 below that $\mathcal{A}_\phi f \in L^2(\mathfrak{g})$. □

Remark 2.7. Let $\phi \in \mathcal{S}(\mathfrak{g})$. Formula (2.2) shows that for every $(X,\xi) \in \mathfrak{g}\times\mathfrak{g}^*$ we have $\pi(\exp_M(\theta(X,\xi)))\phi \in \mathcal{S}(\mathfrak{g})$. Moreover, the mapping

$$\mathfrak{g}\times\mathfrak{g}^* \to \mathcal{S}(\mathfrak{g}), \quad (X,\xi) \mapsto \pi(\exp_M(\theta(X,\xi)))\phi$$

is continuous. Thus we can extend the definition of $\mathcal{A}_\phi f$ for every $f \in \mathcal{S}'(\mathfrak{g})$ to obtain the continuous function

$$\mathcal{A}_\phi f \colon \mathfrak{g}\times\mathfrak{g}^* \to \mathbb{C}, \quad (\mathcal{A}_\phi f)(X,\xi) = \langle f, \overline{\pi(\exp_M(\theta(X,\xi)))\phi}\rangle,$$

where $\langle\cdot,\cdot\rangle\colon \mathcal{S}'(\mathfrak{g})\times\mathcal{S}(\mathfrak{g}) \to \mathbb{C}$ is the usual duality pairing.

We also note that if $f, \phi \in \mathcal{S}(\mathfrak{g})$, then $\mathcal{A}_\phi f \in \mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*)$ as an easy consequence of Lemma 2.5. □

The second equality in Theorem 2.8(1) below will be referred to as *Moyal's identity* just as in the classical situation when the Lie algebra $\mathfrak{g}$ is abelian (see for instance [2]).

Theorem 2.8. *The following assertions hold:*

(1) *For every $\phi, f \in L^2(\mathfrak{g})$ we have $\mathcal{A}_\phi f \in L^2(\mathfrak{g}\times\mathfrak{g}^*)$ and*

$$\begin{aligned}(\mathcal{A}_{\phi_1}f_1 \mid \mathcal{A}_{\phi_2}f_2)_{L^2(\mathfrak{g}\times\mathfrak{g}^*)} &= (f_1 \mid f_2)_{L^2(\mathfrak{g})}\cdot(\phi_2 \mid \phi_1)_{L^2(\mathfrak{g})}\\ &= (\mathcal{W}(f_1,\phi_1) \mid \mathcal{W}(f_2,\phi_2))_{L^2(\mathfrak{g}\times\mathfrak{g}^*)}\end{aligned}$$

whenever $\phi_1, f_1, \phi_2, f_2 \in L^2(\mathfrak{g})$.

(2) *If $\phi_0 \in L^2(\mathfrak{g})$ with $\|\phi_0\| = 1$, then the operator $\mathcal{A}_{\phi_0}\colon L^2(\mathfrak{g}) \to L^2(\mathfrak{g}\times\mathfrak{g}^*)$, $f \mapsto \mathcal{A}_{\phi_0}f$, is an isometry and we have*

$$\iint_{\mathfrak{g}\times\mathfrak{g}^*} (\mathcal{A}_{\phi_0}f)(X,\xi)\cdot\pi(\exp_M(\theta(X,\xi)))\phi\,\mathrm{d}(X,\xi) = (\phi \mid \phi_0)f$$

for every $\phi, f \in L^2(\mathfrak{g})$. In particular,

$$\iint_{\mathfrak{g}\times\mathfrak{g}^*} (\mathcal{A}_{\phi_0}f)(X,\xi)\cdot\pi(\exp_M(\theta(X,\xi)))\phi_0\,\mathrm{d}(X,\xi) = f$$

for arbitrary $f \in L^2(\mathfrak{g})$.

Proof. (1) We may assume $f_1, f_2, \phi_1, \phi_2 \in \mathcal{S}(\mathfrak{g})$. Let $X \in \mathfrak{g}$ be fixed for the moment. We have by Lemma 2.5

$$\begin{aligned}\int_{\mathfrak{g}^*} \mathcal{A}_{\phi_1}f_1(X,\xi)\cdot\overline{\mathcal{A}_{\phi_2}f_2(X,\xi)}\mathrm{d}\xi &= \lim_{\varepsilon\to 0}\int_{\mathfrak{g}^*} \mathrm{e}^{-\epsilon|\xi|^2}\cdot\mathcal{A}_{\phi_1}f_1(X,\xi)\cdot\overline{\mathcal{A}_{\phi_2}f_2(X,\xi)}\mathrm{d}\xi \\ &= \lim_{\varepsilon\to 0}\iiint_{\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}^*} \mathrm{e}^{-\epsilon|\xi|^2}\cdot f_1(Y_1)\cdot\overline{\tau_A(X,Y_1)}\cdot\overline{\phi_1((-X)*Y_1)}\cdot\overline{f_2(Y_2)} \\ &\quad\times\tau_A(X,Y_2)\cdot\phi_1((-X)*Y_2)\cdot\mathrm{e}^{\mathrm{i}\langle\xi,\Psi_X(-Y_1)-\Psi_X(-Y_2)\rangle}\mathrm{d}Y_1\mathrm{d}Y_2\mathrm{d}\xi \\ &= \lim_{\varepsilon\to 0}\langle U_\varepsilon, F_X\rangle\end{aligned}$$

where $\langle\cdot,\cdot\rangle\colon \mathcal{S}'(\mathfrak{g}\times\mathfrak{g})\times\mathcal{S}(\mathfrak{g}\times\mathfrak{g})\to\mathbb{C}$ stands for the usual duality between the tempered distributions and the Schwartz space. Here we think of the function

$$U_\varepsilon(Y_1,Y_2) = \int_{\mathfrak{g}^*}\mathrm{e}^{-\epsilon|\xi|^2}\cdot\mathrm{e}^{\mathrm{i}\langle\xi,\Psi_X(-Y_1)-\Psi_X(-Y_2)\rangle}\mathrm{d}\xi$$

as a tempered distribution on $\mathfrak{g}\times\mathfrak{g}$, while the function

$$F_X(Y_1,Y_2) = f_1(Y_1)\cdot\overline{\tau_A(X,Y_1)}\cdot\overline{\phi_1((-X)*Y_1)}\cdot\overline{f_2(Y_2)}\cdot\tau_A(X,Y_2)\cdot\phi_1((-X)*Y_2)$$

belongs to $\mathcal{S}(\mathfrak{g}\times\mathfrak{g})$. Since $\Psi_X\colon\mathfrak{g}\to\mathfrak{g}$ is a polynomial diffeomorphism of $\mathfrak{g}$ whose inverse is again a polynomial diffeomorphism (by Prop. 3.2 in [6]), it follows by standard reasoning that for a certain constant $C_0 > 0$ depending on the choice of the Lebesgue measure on $\mathfrak{g}$ we have

$$\lim_{\varepsilon\to 0} U_\varepsilon = C_0\,\delta(Y_1-Y_2) \tag{2.4}$$

in the weak topology of the space $\mathcal{S}'(\mathfrak{g}\times\mathfrak{g})$, where $\delta(\cdot)$ is the Dirac distribution at $0\in\mathfrak{g}$. Specifically, let $\phi\in\mathcal{S}(\mathfrak{g}\times\mathfrak{g})$ arbitrary. Then a change of variables shows that

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle U_\varepsilon,\phi\rangle &= \lim_{\varepsilon\to 0}\iiint_{\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}^*}\mathrm{e}^{-\varepsilon|\xi|^2}\mathrm{e}^{-\mathrm{i}\langle\xi,Z_1\rangle+\mathrm{i}\langle\xi,Z_2\rangle}\psi(Z_1,Z_2)\mathrm{d}Z_1\mathrm{d}Z_2\mathrm{d}\xi \\ &= \lim_{\varepsilon\to 0}\int_{\mathfrak{g}^*}\mathrm{e}^{-\varepsilon|\xi|^2}(\mathcal{F}\psi)(\xi,-\xi)\mathrm{d}\xi = \int_{\mathfrak{g}^*}(\mathcal{F}\psi)(\xi,-\xi)\mathrm{d}\xi,\end{aligned}$$

where $\mathcal{F}\colon\mathcal{S}(\mathfrak{g}\times\mathfrak{g})\to\mathcal{S}(\mathfrak{g}^*\times\mathfrak{g})$ is the usual Fourier transform. Here we have set $\psi(Z_1,Z_2) =: \phi(-\Psi_X^{-1}(-Z_1), -\Psi_X^{-1}(-Z_1))$ and used Lebesgue's convergence theorem. On the other hand,

$$\int_{\mathfrak{g}^*}(\mathcal{F}\psi)(\xi,-\xi)\mathrm{d}\xi = C_0\int_{\mathfrak{g}}\psi(Y,Y)\mathrm{d}Y.$$

This can be seen by applying the equality $\int_{\mathfrak{g}^*}(\mathcal{F}\chi)(\xi,0)\mathrm{d}\xi = C_0\int_{\mathfrak{g}}\chi(0,Y)\mathrm{d}Y$ with the function $\chi(X,Y) := \psi(Y+X, Y-X)$. Now (2.4) follows at once by the above computations.

We then obtain by (2.4)

$$\begin{aligned}\int_{\mathfrak{g}^*}\mathcal{A}_{\phi_1}f_1(X,\xi)\cdot\overline{\mathcal{A}_{\phi_2}f_2(X,\xi)}\mathrm{d}\xi &= C_0\int_{\mathfrak{g}}F_X(Y,Y)\,\mathrm{d}Y \\ &= C_0\int_{\mathfrak{g}}f_1(Y)\cdot\overline{\phi_1((-X)*Y)}\cdot f_2(Y)\cdot\phi_2((-X)*Y)\,\mathrm{d}Y\end{aligned}$$

since $|\tau_A(X,Y)| = 1$. By integrating the above equality with respect to $X\in\mathfrak{g}$ and taking into account our convention on the relationship between the Lebesgue measures on $\mathfrak{g}$ and $\mathfrak{g}^*$, we eventually get

$$(\mathcal{A}_{\phi_1}f_1 \mid \mathcal{A}_{\phi_2}f_2)_{L^2(\mathfrak{g}\times\mathfrak{g}^*)} = (f_1\mid f_2)_{L^2(\mathfrak{g})}\cdot(\phi_2\mid\phi_1)_{L^2(\mathfrak{g})}.$$

This is just the first equation we wished for. The second equality in the assertion follows from this one by using the well-known fact that the symplectic Fourier transform $L^2(\mathfrak{g}\times\mathfrak{g}^*)\to L^2(\mathfrak{g}\times\mathfrak{g}^*)$ is a unitary operator.

(2) It follows at once by Assertion (1) that the operator $\mathcal{A}_{\phi_0}\colon L^2(\mathfrak{g})\to L^2(\mathfrak{g}\times\mathfrak{g}^*)$ is an isometry if $\|\phi_0\| = 1$. The other properties then follow by general arguments; see for instance Proposition 2.11 in [22]. □

Proposition 2.9. *If $f,\phi\in\mathcal{S}(\mathfrak{g})$, then the following assertions hold:*

(1) *For every $(X,\xi)\in\mathfrak{g}\times\mathfrak{g}^*$ we have*

$$(\mathcal{A}_\phi f)(X,\xi) = \int_{\mathfrak{g}}\mathrm{e}^{-\mathrm{i}\langle\xi,Y\rangle}\overline{\tau_A(X,-\Psi_X^{-1}(-Y))}f(-\Psi_X^{-1}(-Y))\overline{\phi((-X)*(-\Psi_X^{-1}(-Y)))}\,\mathrm{d}Y.$$

(2) *For every* $(Y, \eta) \in \mathfrak{g} \times \mathfrak{g}^*$ *we have*

$$\mathcal{W}(f, \phi)(Y, \eta) = \int_{\mathfrak{g}} \mathrm{e}^{-\mathrm{i}\langle \eta, X\rangle} \overline{\tau_A(X, -\Psi_X^{-1}(-Y))} f(-\Psi_X^{-1}(-Y)) \overline{\phi((-X) * (-\Psi_X^{-1}(-Y)))}\, \mathrm{d}X.$$

Proof. It follows by Definition 2.6 and Lemma 2.5 that

$$(\mathcal{A}_\phi f)(X, \xi) = \int_{\mathfrak{g}} f(Z) \overline{\tau_A(X, Z)} \mathrm{e}^{\mathrm{i}\langle \xi, \Psi_X(-Z)\rangle} \overline{\phi((-X) * Z)}\, \mathrm{d}Z.$$

Since $\Psi_X\colon \mathfrak{g} \to \mathfrak{g}$ is a diffeomorphism with the Jacobian function equal to 1 everywhere, we can change variables and set $Y = -\Psi_X(-Z)$ in the above integral. Then $Z = -\Psi_X^{-1}(-Y)$ and we get the formula in Assertion (1). Then recall from Definition 2.6 that

$$\mathcal{W}(f, \phi)(Y, \eta) = \iint_{\mathfrak{g} \times \mathfrak{g}^*} \mathrm{e}^{-\mathrm{i}(\langle \eta, X\rangle - \langle \xi, Y\rangle)} (\mathcal{A}_\phi f)(X, \xi)\, \mathrm{d}\xi\, \mathrm{d}X$$

If we plug in the formula of Assertion (1) in the above equation and use the Fourier inversion formula, then we get the formula for $\mathcal{W}(f, \phi)(Y, \eta)$ as claimed. □

Remark 2.10. It follows by Proposition 2.9(1) that the function $\mathcal{A}_\phi f(X, \cdot)\colon \mathfrak{g}^* \to \mathbb{C}$ is equal to the inverse Fourier transform of the function

$$\overline{\tau_A(X, -\Psi_X^{-1}(\cdot))} f(-\Psi_X^{-1}(\cdot)) \overline{\phi((-X) * (-\Psi_X^{-1}(\cdot)))}\colon \mathfrak{g} \to \mathbb{C}. \quad \square$$

Remark 2.11. We can use the above Proposition 2.9 along with Prop. 3.2 in [6] to check that the sesquilinear mappings

$$\mathcal{A}(\cdot, \cdot),\ \mathcal{W}(\cdot, \cdot)\colon \mathcal{S}(\mathfrak{g}) \times \mathcal{S}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{g} \times \mathfrak{g}^*)$$

are continuous. □

2.2. Magnetic pseudo-differential operators and Wigner distributions

This subsection includes background material from [6] together with some new properties of the magnetic Weyl calculus on nilpotent Lie groups.

Definition 2.12. For every $a \in \mathcal{S}(\mathfrak{g} \times \mathfrak{g}^*)$ the corresponding *magnetic pseudo-differential operator* is defined by

$$\mathrm{Op}(a)f = \iint_{\mathfrak{g} \times \mathfrak{g}^*} \widehat{a}(X, \xi) \cdot \pi(\exp_M(\theta(X, \xi))) f\, \mathrm{d}(X, \xi) \tag{2.5}$$

for every $f \in \mathcal{S}(\mathfrak{g})$, where $\theta\colon \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{m}$ is described in Setting 2.1. □

We record in the following proposition some immediate properties of the magnetic pseudo-differential operators constructed in Definition 2.12.

Proposition 2.13. *The following assertions hold:*

(1) *For each* $a \in \mathcal{S}(\mathfrak{g} \times \mathfrak{g}^*)$ *we have*

$$(\mathrm{Op}(a)f \mid \phi)_{L^2(\mathfrak{g})} = (\widehat{a} \mid \mathcal{A}_f \phi)_{L^2(\mathfrak{g} \times \mathfrak{g}^*)} = (a \mid \mathcal{W}(\phi, f))_{L^2(\mathfrak{g} \times \mathfrak{g}^*)}$$

whenever $f, \phi \in \mathcal{S}(\mathfrak{g})$.

(2) *If* $\phi_1, \phi_2 \in \mathcal{S}(\mathfrak{g})$ *and* $a := \mathcal{W}(\phi_1, \phi_2) \in \mathcal{S}(\mathfrak{g} \times \mathfrak{g}^*)$, *then* $\mathrm{Op}(a)$ *is a rank-one operator, namely*

$$\mathrm{Op}(a)f = (f \mid \phi_2)_{L^2(\mathfrak{g})} \cdot \phi_1 \text{ for every } f \in \mathcal{S}(\mathfrak{g}).$$

Proof. Assertion (1) is a consequence of formula (2.5) along with Definition 2.6. Then Assertion (2) follows by Assertion (1) by taking into account Moyal's identity (Theorem 2.8(1)). In fact, we get

$$\begin{aligned}(\mathrm{Op}(\mathcal{W}(\phi_1, \phi_2))f \mid \phi) &= (\mathcal{W}(\phi_1, \phi_2) \mid \mathcal{W}(\phi, f)) = (\phi_1 \mid \phi) \cdot (f \mid \phi_2) \\ &= ((f \mid \phi_2)\phi_1 \mid \phi)\end{aligned}$$

for arbitrary $\phi \in \mathcal{S}(\mathfrak{g})$, and the conclusion follows since $\mathcal{S}(\mathfrak{g})$ is dense in $L^2(\mathfrak{g})$. □

Remark 2.14. We can use the equations in above Proposition 2.13(1) and Remark 2.11 to define for every $a \in \mathcal{S}'(\mathfrak{g} \times \mathfrak{g}^*)$ the corresponding magnetic pseudo-differential operator as a continuous linear operator $\mathrm{Op}(a)\colon \mathcal{S}(\mathfrak{g}) \to \mathcal{S}'(\mathfrak{g})$. It follows by this definition that the following assertions hold:

(1) If $\lim_{j\in J} a_j = a$ in the weak*-topology in $\mathcal{S}'(\mathfrak{g}\times\mathfrak{g}^*)$, then for every $f\in\mathcal{S}(\mathfrak{g})$ we have $\lim_{j\in J}\mathrm{Op}(a_j)f = \mathrm{Op}(a)f$ in the weak*-topology in $\mathcal{S}'(\mathfrak{g})$.

(2) The distribution kernel $K_a\in\mathcal{S}'(\mathfrak{g}\times\mathfrak{g})$ of the operator $\mathrm{Op}(a)\colon\mathcal{S}(\mathfrak{g})\to\mathcal{S}'(\mathfrak{g})$ is given by the formula

$$K_a = \alpha_A\cdot(((1\otimes F_{\mathfrak{g}}^{-1})a)\circ\Sigma), \tag{2.6}$$

where the function α_A multiplies the composition between partial inverse Fourier transform $(1\otimes F_{\mathfrak{g}}^{-1})a\in\mathcal{S}'(\mathfrak{g}\times\mathfrak{g})$ and the polynomial diffeomorphism

$$\Sigma\colon\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}\times\mathfrak{g},\quad \Sigma(X,Y)=\Bigl(\int_0^1(s(Y*(-X)))*X\,\mathrm{d}s,\,X*(-Y)\Bigr)$$

whose inverse is again polynomial. In fact, this follows by Th. 4.4 and eq. (4.14) in [6] for $a\in\mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*)$. Then the general case can be obtained by the preceding continuity property, since $\mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*)$ is weakly*-dense in $\mathcal{S}'(\mathfrak{g}\times\mathfrak{g}^*)$.

For the sake of completeness, let us write (2.6) explicitly as

$$K_a(X,Y)=\alpha_A(X,Y)\int_{\mathfrak{g}^*}\mathrm{e}^{\mathrm{i}\langle\xi,X*(-Y)\rangle}a\Bigl(\int_0^1(s(Y*(-X)))*X\,\mathrm{d}s,\,\xi\Bigr)\,\mathrm{d}\xi \tag{2.7}$$

which makes sense whenever $a\in\mathcal{S}'(\mathfrak{g}\times\mathfrak{g}^*)$ is defined by a function such that the right-hand side is well defined. Here we have used the notation

$$\alpha_A(X,Y)=\exp\Bigl(\mathrm{i}\int_0^1\langle A((s(Y*(-X)))*X),\,(R_{(s(Y*(-X)))*X})'_0(X*(-Y))\rangle\,\mathrm{d}s\Bigr) \tag{2.8}$$

for every $X,Y\in\mathfrak{g}$ (see eq. (4.13) in [6]). $\square$

Example 2.15. We wish to use Remark 2.14 in order to compute the magnetic pseudo-differential operators defined by some special types of symbol.

(1) Let $a\colon\mathfrak{g}\to\mathbb{C}$ be a smooth function of polynomial growth and look at it as a symbol in $\mathcal{S}'(\mathfrak{g}\times\mathfrak{g}^*)$ depending only on the variable in $\mathfrak{g}$. Since $\alpha_A(X,X)=1$, it the follows at once from (2.6) that $\mathrm{Op}(a)$ is the multiplication operator in $L^2(\mathfrak{g})$ defined by the function a.

(2) Let $X_0\in\mathfrak{g}$ and define $a_{X_0}\colon\mathfrak{g}\times\mathfrak{g}^*\to\mathbb{C}$, $a_{X_0}(X,\xi)=\langle\xi,X_0\rangle$. Then it follows by Th. 4.4(1) (and its proof) in [6] that

$$\mathrm{Op}(a_{X_0})=-\mathrm{i}\dot\lambda(X_0)+A(Q)X_0$$

and this operator is the infinitesimal generator of a 1-parameter group of unitary operators, hence it is essentially self-adjoint in $L^2(\mathfrak{g})$. Here $\dot\lambda(X_0)$ is the first-order differential operator defined by the right-invariant vector field $\overline{X}_0$ on the nilpotent Lie group $(\mathfrak{g},*)$ whose value at $0\in\mathfrak{g}$ is X_0. On the other hand, $A(Q)X_0$ stands for the multiplication operator given by the function whose value at an arbitrary point is obtained by applying the 1-form $A\in\Omega^1(\mathfrak{g})$ to the aforementioned vector field $\overline{X}_0$. Let us note that an explicit formula for $\dot\lambda(X_0)$ can be easily obtained by Lemma 5 in [23], namely for every $f\in\mathcal{C}^\infty(\mathfrak{g})$ and $Y\in\mathfrak{g}$ we have $(\dot\lambda(X_0)f)(Y)=\langle f'_Y,\overline{X}_0(Y)\rangle$, which is the first-order differential operator defined by the vector field $\overline{X}_0\colon\mathfrak{g}\to\mathfrak{g}$,

$$\overline{X}_0(Y)=\mathcal{R}(\mathrm{ad}_{\mathfrak{g}}Y)X_0=X_0-\frac{1}{2}[Y,X_0]+\frac{1}{12}[Y,[Y,X_0]]+\cdots. \tag{2.9}$$

Here we use the holomorphic function $\mathcal{R}\colon\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}^*\to\mathbb{C}$, $\mathcal{R}(z)=z/(e^z-1)$ whose power series around 0 is $1-\frac{1}{2}z+\frac{1}{12}z^2+\cdots$.

(3) Now assume that the magnetic potential A vanishes. Let $a\in L^1(\mathfrak{g}^*)$ and think of it as a symbol in $\mathcal{S}'(\mathfrak{g}\times\mathfrak{g}^*)$ depending only on the variable in $\mathfrak{g}^*$. If we denote by $b\in L^\infty(\mathfrak{g})$ the inverse Fourier transform of a, then it follows by (2.7) that $K_a(X,Y)=b(X*(-Y))$, hence

$$(\forall f\in\mathcal{S}(\mathfrak{g}))\quad(\mathrm{Op}(a)f)(X)=\int_{\mathfrak{g}}b(X*(-Y))f(Y)\,\mathrm{d}X.$$

Thus $\mathrm{Op}(a)$ is a convolution operator on the nilpotent Lie group $(\mathfrak{g},*)$. $\square$

Our next aim is to show that the Weyl calculus with real symbols gives rise to symmetric pseudo-differential operators; see Proposition 2.17 below.

Lemma 2.16. *If we define*

$$\Sigma_1\colon\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g},\quad\Sigma_1(X,Y)=\int_0^1(s(Y*(-X)))*X\,\mathrm{d}s,$$

then for every $X,Y\in\mathfrak{g}$ we have $\Sigma_1(X,Y)=\Sigma_1(Y,X)$.

Proof. Note that for every $X, Y \in \mathfrak{g}$ we have

$$\Psi_X(Y * (-X)) = \int_0^1 Y * (-X) * sX \, \mathrm{d}s = \int_0^1 Y * (-(1-s)X) \, \mathrm{d}s = \Psi_{-X}(Y).$$

If we replace X by $(-X) * Y$, then we get $\Psi_{(-X)*Y}(Y * (-Y) * X) = \Psi_{(-Y)*X}(Y)$, that is,

$$(\forall X, Y \in \mathfrak{g}) \quad \Psi_{(-X)*Y}(X) = \Psi_{(-Y)*X}(Y).$$

Now the conclusion follows since

$$\Sigma_1(X, Y) = -\Psi_{X*(-Y)}(-X)$$

for every $X, Y \in \mathfrak{g}$. □

Proposition 2.17. *Let $a \in \mathcal{S}'(\mathfrak{g} \times \mathfrak{g}^*)$ be a real distribution, in the sense that its values on real valued functions are real numbers. Then the distribution kernel $K_a \in \mathcal{S}'(\mathfrak{g} \times \mathfrak{g})$ has the following symmetry property:*

$$(\forall f, \phi \in \mathcal{S}(\mathfrak{g})) \quad \langle K_a, f \otimes \bar{\phi} \rangle = \overline{\langle K_a, \phi \otimes \bar{f} \rangle}.$$

Proof. First note that for every $X, Y \in \mathfrak{g}$ we have by (2.8)

$$\begin{aligned}
\alpha_A(Y, X) &= \exp\Big(\mathrm{i} \int_0^1 \langle A((s(X * (-Y))) * Y), (R_{(s(X*(-Y)))*Y})'_0(Y * (-X)) \rangle \, \mathrm{d}s\Big) \\
&= \exp\Big(\mathrm{i} \int_0^1 \langle A((-sZ) * Z * X), (R_{(-sZ)*Z*X})'_0 Z \rangle \, \mathrm{d}s\Big) \\
&= \exp\Big(\mathrm{i} \int_0^1 \langle A(((1-s)Z) * X), (R_{((1-s)Z)*X})'_0 Z \rangle \, \mathrm{d}s\Big) \\
&= \exp\Big(\mathrm{i} \int_0^1 \langle A((sZ) * X), (R_{(sZ)*X})'_0 Z \rangle \, \mathrm{d}s\Big) \\
&= \exp\Big(\mathrm{i} \int_0^1 \langle A((s(Y * (-X))) * X), (R_{(s(Y*(-X)))*X})'_0(Y * (-X)) \rangle \, \mathrm{d}s\Big) \\
&= \exp\Big(-\mathrm{i} \int_0^1 \langle A((s(Y * (-X))) * X), (R_{(s(Y*(-X)))*X})'_0(X * (-Y)) \rangle \, \mathrm{d}s\Big) \\
&= \overline{\alpha(X, Y)},
\end{aligned}$$

where we used the notation $X * (-Y) = -Z$, hence $Y = Z * X$. Now the assertion follows at once by using Lemma 2.16 and formula (2.7). □

The next result shows the significance of the marginal distributions of the cross-Wigner function in our setting. It is worth pointing out that this is a natural extension of the similar property in the classical case of the Schrödinger representation. Actually, the functional calculus with both the position operators (see Assertion (1)) and the *noncommutative magnetic momentum* operators $-\mathrm{i}\dot{\lambda}(X_0) + A(Q)X_0$ (Assertion (2)) can thus be read off with the cross-Wigner distribution.

Proposition 2.18. *If $f, \phi \in \mathcal{S}(\mathfrak{g})$, then the following assertions hold:*

(1) *For every $Y \in \mathfrak{g}$ we have*

$$f(Y)\overline{\phi(Y)} = \int_{\mathfrak{g}^*} \mathcal{W}(f, \phi)(Y, \eta) \, \mathrm{d}\eta.$$

(2) *If we define*

$$\Gamma_{f,\phi} : \mathfrak{g}^* \to \mathbb{C}, \quad \Gamma_{f,\phi}(\eta) = \int_{\mathfrak{g}} \mathcal{W}(f, \phi)(Y, \eta) \, \mathrm{d}Y,$$

then for every $X_0 \in \mathfrak{g}$ and $a_0 \in \mathcal{S}(\mathbb{R})$ we have

$$(a_0(-\mathrm{i}\dot{\lambda}(X_0) + A(Q)X_0)f \mid \phi) = \int_{\mathfrak{g}^*} \Gamma_{f,\phi}(\eta) a_0(\langle \eta, X_0 \rangle) \, \mathrm{d}\eta,$$

where the left-hand side involves the Borel functional calculus for the essentially self-adjoint operator $-\mathrm{i}\dot{\lambda}(X_0) + A(Q)X_0$ in $L^2(\mathfrak{g})$.

Proof. For Assertion (1) use Proposition 2.9(2) along with the Fourier inversion formula to get

$$\int_{\mathfrak{g}^*} \mathcal{W}(f,\phi)(Y,\eta)\,\mathrm{d}\eta = \overline{\tau_A(0,-\Psi_0^{-1}(-Y))}\cdot f(-\Psi_0^{-1}(-Y))\cdot\overline{\phi(0*(-\Psi_0^{-1}(-Y)))}$$
$$= f(Y)\overline{\phi(Y)}.$$

The latter equality follows at once by the formulas in Notation 2.3.

In order to prove Assertion (2), let us denote by $\mathbf{1}\otimes a_0(\langle\cdot,X_0\rangle)$ the function defined on $\mathfrak{g}\times\mathfrak{g}^*$ by $(X,\xi)\mapsto a_0(\langle\xi,X_0\rangle)$. It then follows by Example 2.15(2) that $\mathbf{1}\otimes a_0(\langle\cdot,X_0\rangle) = a_0\circ a_{X_0}$ and then

$$a_0(-\mathrm{i}\dot{\lambda}(X_0)+A(Q)X_0) = \mathrm{Op}(\mathbf{1}\otimes a_0(\langle\cdot,X_0\rangle))$$

(see also Sect. 5.1 in [21]). By using this equality along with Remark 2.14 and the formula in Proposition 2.13(1), we get

$$\begin{aligned}(a_0(-\mathrm{i}\dot{\lambda}(X_0)+A(Q)X_0)f\mid\phi)_{L^2(\mathfrak{g})} &= (\mathbf{1}\otimes a_0(\langle\cdot,X_0\rangle)\mid\mathcal{W}(\phi,f))_{L^2(\mathfrak{g}\times\mathfrak{g}^*)}\\ &= \iint_{\mathfrak{g}\times\mathfrak{g}^*} a_0(\langle\eta,X_0\rangle)\cdot\mathcal{W}(f,\phi)(Y,\eta)\,\mathrm{d}Y\mathrm{d}\eta\\ &= \int_{\mathfrak{g}^*} a_0(\langle\eta,X_0\rangle)\Big(\int_{\mathfrak{g}}\mathcal{W}(f,\phi)(Y,\eta)\,\mathrm{d}Y\Big)\mathrm{d}\eta,\end{aligned}$$

and this leads to the asserted formula. □

2.3. Heisenberg's inequality

In the following statement we shall use the symbols

$$a_{X_0}\colon\mathfrak{g}\times\mathfrak{g}^*\to\mathbb{C},\quad a_{X_0}(X,\xi)=\langle\xi,X_0\rangle,$$
$$a_{\xi_0}\colon\mathfrak{g}\times\mathfrak{g}^*\to\mathbb{C},\quad a_{\xi_0}(X,\xi)=\langle\xi_0,X\rangle.$$

for arbitrary $X_0\in\mathfrak{g}$ and $\xi_0\in\mathfrak{g}^*$.

Theorem 2.19. *Let $X_0\in\mathfrak{g}$ and $c_0\in\mathbb{R}$. Assume that the coadjoint orbit $\mathcal{O}\subseteq\mathfrak{g}^*$ is contained in the affine hyperplane $\{\xi\in\mathfrak{g}^*\mid\langle\xi,X_0\rangle=c_0\}$. Then*

$$[\mathrm{Op}(a_{X_0}),\mathrm{Op}(a_{\xi_0})] = \mathrm{i}c_0\cdot\mathrm{id}_{L^2(\mathfrak{g})} \tag{2.10}$$

and

$$\|\mathrm{Op}(a_{X_0})f\|\cdot\|\mathrm{Op}(a_{\xi_0})f\|\ge\frac{1}{2}|c_0| \tag{2.11}$$

for every $\xi_0\in\mathcal{O}$, whenever $f\in L^2(\mathfrak{g})$ with $\|f\|=1$ belongs to the domains of both operators $\mathrm{Op}(a_{X_0})$ *and* $\mathrm{Op}(a_{\xi_0})$.

Proof. For the sake of simplicity we shall use the convention that the operator of multiplication by some function will be denoted by the same symbol as that function. Then, according to Example 2.15(1)–(2) we have $\mathrm{Op}(a_{\xi_0})=\xi_0$ and $\mathrm{Op}(a_{X_0}) = -\mathrm{i}\dot{\lambda}(X_0)+A(Q)X_0$. Since $\mathrm{i}\dot{\lambda}(X_0)$ is a first-order linear differential operator on $\mathcal{C}^\infty(\mathfrak{g})$, hence a derivation on $\mathcal{C}^\infty(\mathfrak{g})$, it easily follows that

$$[\mathrm{Op}(a_{X_0}),\mathrm{Op}(a_{\xi_0})] = -\mathrm{i}\dot{\lambda}(X_0)\xi_0. \tag{2.12}$$

Now, by using eq. (2.10) in [6] we get for every $X\in\mathfrak{g}$

$$(\dot{\lambda}(X_0)\xi_0)(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\langle\xi_0,(-tX_0)*X\rangle. \tag{2.13}$$

Note that the Baker–Campbell–Hausdorff formula gives that for every $t\in\mathbb{R}$

$$(-tX_0)*X = -tX_0+X+t\sum_{j\ge1}b_j(\mathrm{ad}_{\mathfrak{g}}X)^jX_0+t^2P(t,X,X_0), \tag{2.14}$$

where $\{b_j\}_{j\ge1}$ is sequence of real numbers while $P\colon\mathbb{R}\times\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is a certain polynomial mapping.

On the other hand, for every $X\in\mathfrak{g}$ and $t\in\mathbb{R}$ we have $\xi_0\circ\mathrm{e}^{t\,\mathrm{ad}_{\mathfrak{g}}X}\in\mathcal{O}$ hence

$$c_0 = \langle\xi_0\circ\mathrm{e}^{t\,\mathrm{ad}_{\mathfrak{g}}X},X_0\rangle = \sum_{j\ge0}\frac{t^j}{j!}\langle\xi_0\circ(\mathrm{ad}_{\mathfrak{g}}X)^j,X_0\rangle.$$

Thence $\langle\xi_0\circ(\mathrm{ad}_{\mathfrak{g}}X)^j,X_0\rangle=0$ for every $j\ge1$ and $X\in\mathfrak{g}$. By combining this with (2.12)–(2.14), we get (2.10). Then the inequality (2.12) follows by general arguments; see for instance Prop. 2.1 in [10]. □

In the following statement we use the notation δ_{jk} for Kronecker's delta.

Corollary 2.20. *Let $\{X_1, \dots, X_n\}$ be a Jordan–Hölder basis in $\mathfrak{g}$ and denote by $\{\xi_1, \dots, \xi_n\}$ the dual basis in $\mathfrak{g}^*$. If $1 \le k \le j \le n$, then we have*

$$[\mathrm{Op}(a_{X_j}), \mathrm{Op}(a_{\xi_k})] = \mathrm{i}\delta_{jk}\mathrm{id}_{L^2(\mathfrak{g})} \tag{2.15}$$

and

$$\|\mathrm{Op}(a_{X_j})f\| \cdot \|\mathrm{Op}(a_{\xi_k})f\| \ge \frac{\delta_{jk}}{2} \tag{2.16}$$

whenever $f \in L^2(\mathfrak{g})$ with $\|f\| = 1$ belongs to the domains of both operators $\mathrm{Op}(a_{X_j})$ and $\mathrm{Op}(a_{\xi_k})$.

Proof. The hypothesis that $\{X_1, \dots, X_j\}$ is a Jordan–Hölder basis in $\mathfrak{g}$ implies that for $j = 1, \dots, n$ we have $[X_j, \mathfrak{g}] \subseteq \mathrm{span}\,\{X_l \mid j < l \le n\}$. Then we can apply Theorem 2.19 to get the conclusion. □

Remark 2.21. It is often the case that a coadjoint orbit of a nilpotent Lie group is contained in an affine subspace, like in Theorem 2.19. Here are a few specific situations:

(1) When $\xi_0 \in \mathfrak{g}^*$ vanishes on $[\mathfrak{g}, \mathfrak{g}]$, its coadjoint orbit reduces to $\{\xi_0\}$, hence it is clearly contained in many affine subspaces.
(2) If $\mathfrak{g}$ is a two-step nilpotent Lie algebra, then every coadjoint orbit is an affine subspace.
(3) For every coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$ and every Z_0 in the center of $\mathfrak{g}$ there exists $c_0 \in \mathbb{R}$ such that $\mathcal{O} \subseteq \{\xi \in \mathfrak{g}^* \mid \langle \xi, Z_0 \rangle = c_0\}$.

We also note that the hypothesis $\mathcal{O} \subseteq \{\xi \in \mathfrak{g}^* \mid \langle \xi, X_0 \rangle = c_0\}$ in Theorem 2.19 is equivalent to the fact that some (actually, every) $\xi_0 \in \mathcal{O}$ vanishes on the ideal generated by $[X_0, \mathfrak{g}]$ in $\mathfrak{g}$. This implies that if $\mathfrak{g}$ is a two-step nilpotent Lie algebra in Corollary 2.20 then the conclusion holds for every j and k. □

3. Uncertainty principles for magnetic ambiguity functions

In this section we establish a version of Lieb's uncertainty principle [14] along with some of its consequences in the present setting. The main result is Theorem 3.5 and is stated in terms of magnetic ambiguity functions and mixed-norm Lebesgue spaces on the cotangent bundle of a nilpotent Lie group.

3.1. Magnetic modulation spaces

We first introduce the magnetic modulation spaces on a simply connected nilpotent Lie group G. The natural tool for that purpose proves to be the ambiguity function and not a short-time Fourier transform, as it is customary in the classical case when the nilpotent Lie group G is the additive group $(\mathbb{R}^n, +)$ (see for instance [2]). Nevertheless, our notion of modulation space agrees with the classical one because of the well-known relationship between the ambiguity function and the short-time Fourier transform.

Definition 3.1. Assume $1 \le p, q \le \infty$ and let $\phi \in \mathcal{S}(\mathfrak{g})$. For every tempered distribution $f \in \mathcal{S}'(\mathfrak{g})$ define

$$\|f\|_{M^{p,q}_\phi} = \Big(\int_{\mathfrak{g}}\Big(\int_{\mathfrak{g}^*} |(\mathcal{A}_\phi f)(X, \xi)|^q \mathrm{d}\xi\Big)^{p/q} \mathrm{d}X\Big)^{1/p} \in [0, \infty]$$

with the usual conventions if p or q is infinite. Then the space

$$M^{p,q}_\phi(\mathfrak{g}) := \{f \in \mathcal{S}'(\mathfrak{g}) \mid \|f\|_{M^{p,q}} < \infty\}$$

will be called a *magnetic modulation space* on the Lie group $G = (\mathfrak{g}, *)$. □

Remark 3.2. In the setting of Definition 3.1 let us introduce the *mixed-norm space* $L^{p,q}(\mathfrak{g} \times \mathfrak{g}^*)$ consisting of the (equivalence classes of) Lebesgue measurable functions $\Theta\colon \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{C}$ such that

$$\|\Theta\|_{L^{p,q}} := \Big(\int_{\mathfrak{g}}\Big(\int_{\mathfrak{g}^*} |(\Theta(X, \xi))|^q \mathrm{d}\xi\Big)^{p/q} \mathrm{d}X\Big)^{1/p} < \infty$$

(cf. [2]). It is clear that $M^{p,q}_\phi(\mathfrak{g}) = \{f \in \mathcal{S}'(\mathfrak{g}) \mid \mathcal{A}_\phi f \in L^{p,q}(\mathfrak{g} \times \mathfrak{g}^*)\}$. □

Example 3.3. For any choice of $\phi \in \mathcal{S}(\mathfrak{g})$ in Definition 3.1 we have

$$M^{2,2}(\mathfrak{g}) := M^{2,2}_\phi(\mathfrak{g}) = L^2(\mathfrak{g}).$$

To see this, just note that the operator $\mathcal{A}_\phi\colon L^2(\mathfrak{g}) \to L^2(\mathfrak{g}\times\mathfrak{g}^*)$ satisfies

$$\|\mathcal{A}_\phi f\|_{L^2(\mathfrak{g}\times\mathfrak{g}^*)} = \|\phi\|_{L^2(\mathfrak{g})}\cdot\|f\|_{L^2(\mathfrak{g})}$$

for every $f\in L^2(\mathfrak{g})$, by Theorem 2.8(1). Therefore

$$\|f\|_{M^{2,2}} = \|\phi\|_{L^2(\mathfrak{g})}\cdot\|f\|_{L^2(\mathfrak{g})}\in[0,\infty]$$

for each $f\in\mathcal{S}'(\mathfrak{g})$. $\square$

Notation 3.4. For every real number $p\in(1,\infty)$ we shall denote

$$p' := p/(p-1)\in(1,\infty),$$

so that $\frac{1}{p}+\frac{1}{p'}=1$. $\square$

3.2. Uncertainty principles for ambiguity functions

In the following theorem we extend Lieb's uncertainty principle [14] to the present setting that takes into account a magnetic potential on a nilpotent Lie group G. In the special case when the magnetic potential vanishes, the Lie group is the abelian group $(\mathbb{R}^n,+)$, and the estimate for $\mathcal{A}_{\phi_1}f_1\cdot\overline{\mathcal{A}_{\phi_2}f_2}$ is an ordinary L^p one instead of a mixed-norm one, we recover Th. 4.1 in [15], due to the simple relationship between the ambiguity functions and the short-time Fourier transforms on abelian groups.

Theorem 3.5. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with the corresponding simply connected Lie group $G=(\mathfrak{g},*)$.*

(1) *If the following conditions are satisfied:*
  (a) $p_1,p_2\in(1,\infty)$;
  (b) $r_j,s_j\ge\max\{p_j,p_j'\}(\ge 2)$ *for* $j=1,2$;
  (c) $p=(\frac{1}{r_1}+\frac{1}{r_2})^{-1}$ *and* $q=(\frac{1}{s_1}+\frac{1}{s_2})^{-1}$;
  (d) $t_j=(\frac{1}{r_j}+\frac{1}{s_j'}-\frac{1}{p_j})^{-1}$ *for* $j=1,2$,

  then for every $f_j\in L^{p_j}(\mathfrak{g})$ and $\phi_j\in L^{t_j}(\mathfrak{g})$ for $j=1,2$ we have

$$\|\mathcal{A}_{\phi_1}f_1\cdot\overline{\mathcal{A}_{\phi_2}f_2}\|_{L^{p,q}(\mathfrak{g}\times\mathfrak{g}^*)}\le C\cdot\|f_1\|_{L^{p_1}(\mathfrak{g})}\cdot\|f_2\|_{L^{p_2}(\mathfrak{g})}\cdot\|\phi_1\|_{L^{t_1}(\mathfrak{g})}\cdot\|\phi_2\|_{L^{t_2}(\mathfrak{g})},$$

  where $C\in(0,1)$ is a certain constant depending only on p_1,p_2,r_1,r_2,s_1,s_2, and $\dim\mathfrak{g}$.

(2) *For every $p\ge 1$ and $f_1,f_2,\phi_1,\phi_2\in L^2(\mathfrak{g})$ we have*

$$\|\mathcal{A}_{\phi_1}f_1\cdot\overline{\mathcal{A}_{\phi_2}f_2}\|_{L^{p}(\mathfrak{g}\times\mathfrak{g}^*)}\le (p^{-1/p})^{\dim\mathfrak{g}}\cdot\|f_1\|_{L^2(\mathfrak{g})}\cdot\|f_2\|_{L^2(\mathfrak{g})}\cdot\|\phi_1\|_{L^2(\mathfrak{g})}\cdot\|\phi_2\|_{L^2(\mathfrak{g})}.$$

Proof. It is enough to prove these inequalities for $f_1,f_2,\phi_1,\phi_2\in\mathcal{S}(\mathfrak{g})$. Note that

$$\|\mathcal{A}_{\phi_1}f_1\cdot\overline{\mathcal{A}_{\phi_2}f_2}\|_{L^{p,q}(\mathfrak{g}\times\mathfrak{g}^*)} = \Bigl(\int_{\mathfrak{g}}\|\mathcal{A}_{\phi_1}f_1(X,\cdot)\cdot\overline{\mathcal{A}_{\phi_2}f_2(X,\cdot)}\|_{L^q(\mathfrak{g}^*)}\mathrm{d}X\Bigr)^{1/p}. \tag{3.1}$$

Since $\frac{1}{q}=\frac{1}{s_1}+\frac{1}{s_2}$, we can use Hölder's inequality to get

$$\|\mathcal{A}_{\phi_1}f_1(X,\cdot)\cdot\overline{\mathcal{A}_{\phi_2}f_2(X,\cdot)}\|_{L^q(\mathfrak{g}^*)}\le\|\mathcal{A}_{\phi_1}f_1(X,\cdot)\|_{L^{s_1}(\mathfrak{g}^*)}\cdot\|\mathcal{A}_{\phi_2}f_2(X,\cdot)\|_{L^{s_2}(\mathfrak{g}^*)} \tag{3.2}$$

for almost every $X\in\mathfrak{g}$. Now note that Proposition 2.9 implies that

$$\mathcal{A}_\phi f(X,\xi)=\int_{\mathfrak{g}}\mathrm{e}^{-\mathrm{i}\langle\xi,Z\rangle}\overline{\tau_A(X,-\Psi_X^{-1}(-Z))}f(-\Psi_X^{-1}(-Z))\overline{\phi((-X)*(-\Psi_X^{-1}(-Z)))}\mathrm{d}Z$$

for $f,\phi\in\mathcal{S}(\mathfrak{g})$. Therefore, since $s_j\ge 2$, we can apply the Hausdorff-Young inequality for the Fourier transform $L^{s_j'}(\mathfrak{g})\to L^{s_j}(\mathfrak{g}^*)$ to obtain

$$\begin{aligned}\|\mathcal{A}_{\phi_j}f_j(X,\cdot)\|_{L^{s_j}(\mathfrak{g}^*)} &\le \Bigl(\int_{\mathfrak{g}}\Bigl|\overline{\tau_A(X,-\Psi_X^{-1}(-Z))}f_j(-\Psi_X^{-1}(-Z))\overline{\phi_j((-X)*(-\Psi_X^{-1}(-Z)))}\Bigr|^{s_j'}\mathrm{d}Z\Bigr)^{1/s_j'}\\ &= \Bigl(\int_{\mathfrak{g}}|f_j(-\Psi_X^{-1}(-Z))\phi_j((-X)*(-\Psi_X^{-1}(-Z)))|^{s_j'}\mathrm{d}Z\Bigr)^{1/s_j'}\\ &= \Bigl(\int_{\mathfrak{g}}|f_j(Y)\phi_j((-X)*Y)|^{s_j'}\mathrm{d}Y\Bigr)^{1/s_j'}\end{aligned}$$

where we have performed the change of variables $Y = -\Psi_X^{-1}(-Z)$ and used the fact that $\Psi_X: \mathfrak{g} \to \mathfrak{g}$ is a diffeomorphism with the Jacobian equal to 1 everywhere on $\mathfrak{g}$ by Proposition 3.2 in [6]. If we define $\widetilde{\phi}_j(v) := \phi_j(-v)$ for every $v \in \mathfrak{g}$, it then follows that for almost every $X \in \mathfrak{g}$ we have

$$\|\mathcal{A}_{\phi_j} f_j(X, \cdot)\|_{L^{s_j}(\mathfrak{g}^*)} \le ((|f_j|^{s'_j} \star |\widetilde{\phi}_j|^{s'_j})(X))^{1/s'_j}, \tag{3.3}$$

where $\star$ stands for the usual convolution product of functions on the nilpotent Lie group G.

On the other hand, by (3.1) and (3.2) we get

$$\begin{aligned}\|\mathcal{A}_{\phi_1} f_1 \cdot \overline{\mathcal{A}_{\phi_2} f_2}\|_{L^{p,q}(\mathfrak{g}\times\mathfrak{g}^*)} &\le \left(\int_{\mathfrak{g}} \|\mathcal{A}_{\phi_1} f_1(X, \cdot)\|^p_{L^{s_1}(\mathfrak{g}^*)} \|\mathcal{A}_{\phi_2} f_2(X, \cdot)\|^p_{L^{s_2}(\mathfrak{g}^*)} \mathrm{d}X\right)^{1/p} \\ &\le \left(\int_{\mathfrak{g}} \|\mathcal{A}_{\phi_1} f_1(X, \cdot)\|^{r_1}_{L^{s_1}(\mathfrak{g}^*)} \mathrm{d}X\right)^{1/r_1} \left(\int_{\mathfrak{g}} \|\mathcal{A}_{\phi_2} f_2(X, \cdot)\|^{r_2}_{L^{s_2}(\mathfrak{g}^*)} \mathrm{d}X\right)^{1/r_2},\end{aligned} \tag{3.4}$$

where the latter inequality follows by Hölder's inequality since $\frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2}$. Now note that by (3.3) we get

$$\begin{aligned}\left(\int_{\mathfrak{g}} \|\mathcal{A}_{\phi_j} f_j(X, \cdot)\|^{r_j}_{L^{s_j}(\mathfrak{g}^*)} \mathrm{d}X\right)^{1/r_j} &\le \left(\int_{\mathfrak{g}} ((|f_j|^{s'_j} \star |\widetilde{\phi}_j|^{s'_j})(X))^{r_j/s'_j} \mathrm{d}X\right)^{1/r_j} \\ &= \||f_j|^{s'_j} \star |\widetilde{\phi}_j|^{s'_j}\|^{1/s'_j}_{L^{r_j/s'_j}(\mathfrak{g})}.\end{aligned} \tag{3.5}$$

The Hausdorff inequality on the connected, simply connected, nilpotent Lie group G (see Corollary 2.5′in [24] or Corollary to Th. 3 in [25]) implies that for a certain constant $C_j \in (0, 1)$ depending only on p_j, r_j, s_j, and $\dim \mathfrak{g}$ we have

$$\begin{aligned}\||f_j|^{s'_j} \star |\widetilde{\phi}_j|^{s'_j}\|^{1/s'_j}_{L^{r_j/s'_j}(\mathfrak{g})} &\le C_j \cdot \||f_j|^{s'_j}\|^{1/s'_j}_{L^{\alpha_j}(\mathfrak{g})} \cdot \||\widetilde{\phi}_j|^{s'_j}\|^{1/s'_j}_{L^{\beta_j}(\mathfrak{g})} \\ &= C_j \cdot \|f_j\|_{L^{s'_j\alpha_j}(\mathfrak{g})} \cdot \|\phi_j\|_{L^{s'_j\beta_j}(\mathfrak{g})} \\ &= C_j \cdot \|f_j\|_{L^{p_j}(\mathfrak{g})} \cdot \|\phi_j\|_{L^{t_j}(\mathfrak{g})},\end{aligned}$$

where $\alpha_j := p_j/s'_j$ while β_j is chosen such that $\frac{s'_j}{r_j} + 1 = \frac{1}{\alpha_j} + \frac{1}{\beta_j}$. It is easily checked that $s'_j\beta_j = t_j$. It then follows by (3.4) and (3.5) that the asserted estimate holds for the constant $C := C_1C_2$ that depends only on $p_1, p_2, r_1, r_2, s_1, s_2$, and $\dim \mathfrak{g}$.

To prove Assertion (2), recall from Corollary 2.5′in [24] or Corollary to Th. 3 in [25] that if we denote

$$(\forall l \in (1, \infty)) \quad A_l = \left(\frac{l^{1/l}}{l'^{1/l'}}\right)^{1/2},$$

then for $j = 1, 2$ we have $C_j = (A_{\alpha_j} A_{\beta_j} A_{\gamma_j})^{\dim \mathfrak{g}}$, where $\gamma_j := \frac{r_j/s'_j}{(r_j/s'_j)-1}$. By considering the special case $p_1 = p_2 = 2$ and $r_1 = r_2 = s_1 = s_2 = 2p = 2q \ge 2$, a careful analysis of the constants (which are the same as in the case when $\mathfrak{g}$ is abelian) then leads to the conclusion we wish for; see the proof of Th. 4.1 and Cor. 4.2 in [15] for details. □

With Theorem 3.5 at hand, one can obtain several versions of the uncertainty principle for the ambiguity function on the nilpotent Lie group G in the present magnetic setting; see Corollaries 3.7 and 3.8 below. Before stating these consequences, we note the relationship between the magnetic modulation spaces and the L^p spaces on the Lie group G. We refer to [26] for more general properties of this type in the case when G is the abelian Lie group $(\mathbb{R}^n, +)$.

Corollary 3.6. *Let G be a connected, simply connected, nilpotent Lie group with the Lie algebra $\mathfrak{g}$ and assume that the following conditions are satisfied:*

(1) $p \in (1, \infty)$;
(2) $r, s \ge \max\{p, p'\}$ (≥ 2);
(3) $t = (\frac{1}{r} + \frac{1}{s'} - \frac{1}{p})^{-1}$.

Then for every $f \in L^p(\mathfrak{g})$ and $\phi \in L^t(\mathfrak{g})$ for $j = 1, 2$ we have

$$\|\mathcal{A}_\phi f\|_{L^{r,s}(\mathfrak{g}\times\mathfrak{g}^*)} \le C \cdot \|f\|_{L^p(\mathfrak{g})} \cdot \|\phi\|_{L^t(\mathfrak{g})},$$

where $C \in (0, 1)$ is a certain constant depending only on p, r, s, and $\dim \mathfrak{g}$. In particular, we have a continuous embedding

$$L^p(\mathfrak{g}) \hookrightarrow M^{r,s}_\phi(\mathfrak{g}) \quad \text{if } r, s \ge \max\{p, p'\}$$

for every $\phi \in \mathcal{S}(\mathfrak{g})$.

Proof. Just consider the special case of Theorem 3.5 with $p_1 = p_2$, $r_1 = r_2$, $s_1 = s_2$, $\phi_1 = \phi_2$, and $f_1 = f_2$. □

The next corollary is the version in the present setting for Th. 4.2 and Remark 4.4 in [15] or Th. 3.3.3 in [2], which were stated in terms of the short-time Fourier transforms on $\mathbb{R}^n$.

Corollary 3.7. *Assume the setting of Theorem 3.5 (1) with $r_1 = s_1$ and $r_2 = s_2$ and denote $h = (\frac{1}{\max\{p_1,p_1'\}} + \frac{1}{\max\{p_2,p_2'\}})^{-1}$. If the number $\varepsilon > 0$ and the Borel subset $U \subseteq \mathfrak{g} \times \mathfrak{g}^*$ satisfy the inequality*

$$\iint_U |(\mathcal{A}_{\phi_1} f_1 \cdot \overline{\mathcal{A}_{\phi_2} f_2})(X,\xi)|\, \mathrm{d}X \mathrm{d}\xi \geq (1-\varepsilon)\|f_1\|_{L^{p_1}(\mathfrak{g})} \cdot \|f_2\|_{L^{p_2}(\mathfrak{g})} \cdot \|\phi_1\|_{L^{p_1'}(\mathfrak{g})} \cdot \|\phi_2\|_{L^{p_2'}(\mathfrak{g})},$$

then the Lebesgue measure of U is at least $\sup_{p>h}((1-\varepsilon)C)^{p/(p-1)}$. If moreover $p_1 = p_2 = 2$, then the measure of U is greater than $\sup_{p>2}(1-\varepsilon)^{p/(p-2)}(p/2)^{(2\dim\mathfrak{g})/(p-2)}$.

Proof. Use the method of proof of Th. 4.2 in [15] or Th. 3.3.3 in [2], by relying on our Theorem 3.5. □

We now record an estimate for the entropy of the ambiguity function. This is obtained by a method similar to the one indicated for obtaining (6.9) in [10].

Corollary 3.8. *Let $f, \phi \in L^2(\mathfrak{g})$ such that $\|f\|_{L^2(\mathfrak{g})} \cdot \|\phi\|_{L^2(\mathfrak{g})} = 1$, and denote*

$$\rho_{f,\phi}(\cdot) := |(\mathcal{A}_\phi f)(\cdot)|^2 \in \bigcap_{p\geq 1} L^p(\mathfrak{g} \times \mathfrak{g}^*).$$

Then we have

$$-\iint_{\mathfrak{g}\times\mathfrak{g}^*} \rho_{f,\phi} \log \rho_{f,\phi} \geq \dim \mathfrak{g} \geq 1.$$

Proof. For every $p \geq 1$ denote

$$\gamma(p) = \iint_{\mathfrak{g}\times\mathfrak{g}^*} (\rho_{f,\phi}(\cdot))^p \quad \text{and } \chi(p) = p^{-\dim\mathfrak{g}}.$$

Then Theorem 3.5(2) implies that $\gamma(p) \leq \chi(p)$ for every $p \geq 1$. On the other hand, it follows at once by Proposition 2.9(1) that $\rho_{f,\phi}(\cdot) \leq 1$ on $\mathfrak{g} \times \mathfrak{g}^*$, hence $\gamma(\cdot)$ is a nonincreasing function on $[1,\infty)$. Since so is the function $\chi(\cdot)$, and $\gamma(1) = \chi(1)$ by Theorem 2.8(1), it then follows that $\gamma'(1) \leq \chi'(1)$, which is just the inequality we wish for. □

4. The case of two-step nilpotent Lie algebras

In this section we are going to point out some specific features of the above constructions in the special case of a *two-step nilpotent Lie algebra* $\mathfrak{g}$ (that is, $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] = \{0\}$). The importance of this situation is partially motivated by the fact that it covers the Heisenberg algebras, which are characterized by the property $\dim[\mathfrak{g},\mathfrak{g}] = 1$. On the other hand, this class of Lie algebras contains many algebras which are neither abelian nor Heisenberg. In fact, the classification of two-step nilpotent Lie algebras is still an open problem although it was raised a long time ago (see [27,28], and the references therein). To emphasize the richness of the class of two-step nilpotent Lie algebras, let us just mention that in every dimension ≥ 9 there exist infinitely many algebras of this type which are nonisomorphic to each other (see [29]). By contrast, there exists precisely one abelian Lie algebra and at most one Heisenberg algebra in each dimension.

Example 4.1. Here we show that nilpotent Lie algebras with arbitrarily high nilpotency index can be constructed as semidirect products of two-step nilpotent Lie algebras and appropriate function spaces thereon. These algebras were considered in several papers for the study of Schrödinger operators with polynomial magnetic fields; see for instance [30,31] and the references therein.

Let $\mathfrak{g}$ be a two-step nilpotent Lie algebra and $N \geq 1$ a fixed integer. Denote by $\mathcal{P}_N(\mathfrak{g})$ the finite-dimensional linear space of real polynomial functions of degree $\leq N$ on $\mathfrak{g}$. Then $\mathcal{F} := \mathcal{P}_N(\mathfrak{g})$ is an admissible function space in the sense of Def. 2.8 in [6] (see also Setting 2.1 above). Note that if we think of $\mathfrak{g}$ as a Lie group with respect to the Baker-Campbell-Hausdorff multiplication

$$(\forall X, Y \in \mathfrak{g}) \quad X * Y = X + Y + \frac{1}{2}[X,Y],$$

then $\mathcal{P}_N(\mathfrak{g})$ is invariant under the left translations on $\mathfrak{g}$ since every left translation $Y \mapsto X * Y$ is a polynomial mapping of degree ≤ 1.

By using the formula for the bracket in the semidirect product of Lie algebras $\mathfrak{m} = \mathcal{F} \rtimes_{\dot\lambda} \mathfrak{g}$,

$$[(f_1, X_1), (f_2, X_2)] = (\dot\lambda(X_1)f_2 - \dot\lambda(X_2)f_1, [X_1, X_2])$$

it is easy to see that $\mathfrak{m}$ is a nilpotent Lie algebra whose nilpotency index is at least $\max\{N, 2\}$. It also follows that the center of $\mathfrak{m}$ is $\mathbb{R} \cdot \mathbf{1} \times \{0\}$. Here we have denoted

$$(\dot{\lambda}(X)f)(Y) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f((-tX) * Y) = f'_Y(-X - \frac{1}{2}[X, Y])$$

(compare formula (2.10) in [6]). $\square$

Example 4.2. Let $\mathfrak{g}$ be a two-step nilpotent Lie algebra again and denote the center of $\mathfrak{g}$ by $\mathfrak{z}$. It follows by Example 2.15(2) that for every $X_0 \in \mathfrak{g}$ the corresponding right-invariant vector field on $\mathfrak{g}$ is

$$\overline{X}_0 \colon \mathfrak{g} \to \mathfrak{g}, \quad \overline{X}_0(Y) = X_0 - \frac{1}{2}[Y, X_0].$$

(In particular, if $X_0 \in \mathfrak{z}$, then $\overline{X}_0$ defines a first-order differential operator $\dot{\lambda}(X_0)$ with constant coefficients in any coordinate system on $\mathfrak{g}$.)

On the other hand, for every $\xi_0 \in \mathfrak{g}^*$ we have

$$(\forall Y \in \mathfrak{g}) \quad (\dot{\lambda}(X_0)\xi_0)(Y) = \langle \xi_0, \overline{X}_0(Y)\rangle = \langle \xi_0, X_0\rangle - \frac{1}{2}\langle \xi_0, [Y, X_0]\rangle.$$

Since $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = \{0\}$, it follows that $[X_0, \mathfrak{g}]$ is an ideal in $\mathfrak{g}$. $\square$

Corollary 4.3. *Let $\mathfrak{g}$ be a two-step nilpotent Lie algebra and $f, \phi \in \mathcal{S}(\mathfrak{g})$ be arbitrary.*

(1) *For every $(X, \xi) \in \mathfrak{g} \times \mathfrak{g}^*$ we have*

$$(\mathcal{A}_\phi f)(X, \xi) = \int_{\mathfrak{g}} \mathrm{e}^{-\mathrm{i}\langle \xi, Y\rangle} \cdot \overline{\tau_A(X, (X/2) * Y)} \cdot f((X/2) * Y) \cdot \overline{\phi((-X/2) * Y)}\mathrm{d}Y.$$

(2) *For every $(Y, \eta) \in \mathfrak{g} \times \mathfrak{g}^*$ we have*

$$\mathcal{W}(f, \phi)(Y, \eta) = \int_{\mathfrak{g}} \mathrm{e}^{-\mathrm{i}\langle \eta, X\rangle} \cdot \overline{\tau_A(X, (X/2) * Y)} \cdot f((X/2) * Y) \cdot \overline{\phi((-X/2) * Y)}\mathrm{d}X.$$

Proof. Since $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = \{0\}$, it follows at once that for every $X, Y \in \mathfrak{g}$ we have $\Psi_X(Y) = Y * (X/2)$ and $\tau_A(X, Y) = \exp\Big(\mathrm{i}\int_0^1 \langle A_{(-sX)*Y}, X + \frac{1}{2}[X, Y]\rangle \mathrm{d}s\Big)$. Then use Proposition 2.9. $\square$

Acknowledgments

We wish to thank Professor José Galé for his kind help. We also thank the Referee for several suggestions that helped improving our presentation.

Partial financial support from the grant PNII - Programme "Idei" (code 1194) is acknowledged.

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