INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY
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DOCTORAL THESIS

APPLICATIONS OF POTENTIAL THEORY:

FINE CARRIER ON CONES OF POTENTIALS

ADVISOR:
Prof. Dr. BUCUR Gheorghe

PhD STUDENT
BENFRIHA Habib

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Introduction

The origins of potential theory may be situated in the 18th century when Lagrange remarked in 1773 that the gravitational forces have as components the partial derivatives of function(which was called *potential function* or *potential* by Green in (1828) respectively Gauss (1840) and when Laplace (1782) showed that outside the mass generating these forces, this function satisfies the partial differential equation $\Delta = 0$, so called *Laplace equation*

The fundamental principles of potential theory were elaborated during the 19th century and constitutes so called "*Classical potential theory*". Essential contributions to this classical potential theory were made by

1. S. D. Poisson (1823) with his famous formula which solve the Dirichlet Problem in the sphere
2. G. Green who introduce so called "*Green Function* " for domains with sufficiently smooth boundary and applied such kind of functions to solve the same Dirichlet Problem for much more complicated open subset of $\mathbb{R}^n$
3. S. Earnshow (1839) who discovered the minimum principal for the harmonic function, solutions of the Laplace equation
4. C. F. Gauss who solved (1840) the equilibrium problem
5. W. Thomson , H.A. Schwarz (1870) , L. Dirichlet and B. Riemann: studied the behaviour at the boundary of the Poisson Integral in the plane
6. A. Poincaré who invented the balayage method (1887) for solving the Dirichlet problem in $\mathbb{R}^3$ and A. Harnak (1986) who discovered his famous inequality and applied it in the studied of convergence for monotone sequences of harmonic functions

These three principles: minimum principles, existence of a sufficiently large class (for instance the ball) of the solutions for Dirichlet problem and the principle of convergence for monotone sequences of harmonic functions turned out to be axioms that allow obtaining the main results from the study of harmonic functions and these principles are also fulfilled by the solutions of any elliptic or parabolic equation.

So in the middle of the 20th century G. Tautz, J.L. Doob, M. Brelot and H. Bauer developed their axiomatic potential theory starting with the above three principles. The difference between them consists essentially in formulating the axiom of convergence for monotone sequences. The stronger of them is Brelot convergence which asserts that an increasing sequence of harmonic functions which converges only in one point of a domain, has as limit a harmonic function on this domain. For the same result we arrive if we suppose the sequence converging uniformly on compact subsets (Bauer axiomatic) or on a dense subset (Doob axiomatic).

In this line of development of potential theory we remember the remarkable contribution of Romanian mathematician who elaborated the theory of poliharmonic and policaloric functions (M. Nicolescu) which are solutions of the $n-$ iterated Laplace operator or heat operator and also constructed their own axiomatic theory (N. Boboc, C. Constantinescu, A. Cornea) and published the Springer monograph" Potential theory on harmonic spaces (1972). In order to cover the harmonic theory for the parabolic equation their starting point was a sheaf of hyperharmonic functions instead of a sheaf of harmonic functions.

Besides the studies of potential theory on harmonic spaces, starting with 6th decade of 19th century and based essentially on the new achievements in mathematics as for example : The Choquet theory of capacity, Choquet theory of convexity allowing the integral representation on the set of extremal points of a compact convex sets, Hunt potential theory for Markov processes, the achievements of R-M Hervé in the carrier theory, etc… many mathematicians began to study some type of ordered convex cones which have similar properties as the convex cone of positive superharmonic functions or supermedinan functions with respect to a kernel or with respect to a supermedian function associated with a resolvent family of kernels. The cones of potentials introduced by G. Mokobodzki turned out to be useful tool for developing an abstract potential theory on the line of above potential theory on harmonic spaces of N. Boboc, C. Constantinescu and A. Cornea.

Among the mathematicians who contributed at this new axiomatic potential theory we mention G. Mokobodzki, D. Sibony, A. de la Pradel, D. Feyel, W. Hansen, I. Bliedtner, K. Jansen, Sewinking, I. Netuke, J. Vessely, J. Lukes, B. Fuglede and others .

Again the Romanian School in potential theory has an important role. We remark here the theory of duality they managed to build and publish it in the springer monograph " order and convexity: H-cone" 1982 written by N. Boboc, Gh. Bucur and A. Cornea that illustrate their remarkable contribution. Later, in a more general frame N. Boboc and L. Beznea published a kluwer academic publishers monograph "Potential Theory and Right Processes" 2004.

In this new frame of potential theory, the réduite (resp. balayage) operator play an important role which may be justified just from its origin method for solving Dirichlet problem. Also the regular elements in this theory which replace the continuous potentials in the theory of harmonic spaces are important since they may be the origin of some associated kernels with complete maximum principle.

To those elements we may associate a carrier theory and to build a specific multiplication i.e. an integral on a measurable space with respect to a measure with values in the cone of regular elements.

The thesis presented here mainly concerns with this objects.

Chapter 1. In the first chapter we present a summary of the well known results in the theory of "Cones of Potentials" or more precisely cones of excessive or supermedian functions with respect to a sub-Markovian resolvent family of kernels on a measurable space. Among the results obtained here we mention:

1. A Choquet type assertion in the frame of supermedian functions as well as in the frame of excessive functions with respect to this resolvent.
2. A decomposition of supermedian function as a sum of two supermedians which are stable with respect to the réduite operation on a borel set respectively its complement.
3. For any finite supermedian function s we associate a map B_s from $\mathcal{S}$ into $\mathcal{S}$ given by:
   $$B_s t = \sup\{u \in \mathcal{S} / u \leq t \ \ and\ u \preccurlyeq \alpha s \ \ for\ some\ \alpha > 0\}$$
   and we prove that this map is a *pseudo-balayage* i.e. it is increasing, additive, idempotent, contractive and$B_s s = s$.
   We prove that B_s is the smallest increasing, additive, idempotent and contractive map T from $\mathcal{S}$ into $\mathcal{S}$ such that $Ts = s$.
   Similar assertions are valid replacing $\mathcal{S}$ by the cone $\mathcal{E}$ of excessive functions.
4. We develop a fine carrier theory on the set $\mathcal{E}$ of excessive functions in the following conditions: $(X, \mathcal{B})$ is a measurable space, $\mathcal{V} = (V_\alpha)_{\alpha>0}$ is a proper sub-Markovian resolvent, the convex cone $\mathcal{E}$ is min-stable, contains the positive constant functions, and the space X is supposed to be nearly saturated i.e. any balayage B on $\mathcal{E}$ is representable.
5. Among others we characterize the regular excessive elements as being those excessives for which the pseudo-balayages associated with are balayages.
   Many assertions presented in this chapterhave similar versions in varius monographs, such as, e.g. [4], [5], [8], [10],[16],[17], [19], [20], [21],[22],[23]

<u>Chapter 2</u>. In the second chapter we present a theory of Lebesgue integral on a measurable space with respect to a measure having its values in the cone of regular excessive functions with respect to a standard resolvent on this measurable space i.e. $\mathcal{V} = (V_\alpha)_{\alpha>0}$ is a resolvent family of kernels on $(X, \mathcal{B})$ such that:

a. $\mathcal{V}$ is a proper sub-Markovian resolvent
b. The convex cone $\mathcal{E}$ of all excessive functions with respect to $\mathcal{V}$ is min-stable and contains the positive constant functions
c. There exists a distance d on X such that the associated topology τ_d is smaller than the fine topology on X (i.e. the coarsest topology τ on X making continuous all functions of $\mathcal{E}$)
d. The Borel structure associated with the distance d coincides with $\mathcal{B}$
e. The space $(X, \mathcal{B})$ is nearly saturated with respect to $\mathcal{E}$ i.e. any $\sigma -$balayage on $\mathcal{E}$ is representable on X.

We construct a map $(f, s) \mapsto f.s$ from $pb\mathcal{B} \times \mathcal{E}^r \rightarrow \mathcal{E}^r$ such that it is bilinear and has the following properties:

1. The sequence $(f_n . s)_n$ is specifically increasing (resp. decreasing) to $f.s$ whenever $(f_n)_n$ increases (resp. decreases) to f

2. If $(f_n)_n$ is uniformly bounded and $(f_n)_n$ is pointwisely convergent to f then the sequence $(f_n.\mathscr{s})_n$ converges to $f.\mathscr{s}$ with respect to the specific order relation in $\mathcal{E}$.
3. If $p_n \in \mathcal{E}$ and $\sum p_n \in \mathcal{E}^r$ then we have $f.\mathscr{s} = \sum_n f.p_n \; \forall f \in pb\mathcal{B}$
4. $carr\ f.\mathscr{s} \subset \overline{[f > 0]} \cap carr\ \mathscr{s}$ ($\bar{A}$ means the fine closure of A)
5. $t \geq f.\mathscr{s}$ whenever $t \in \mathcal{E}$ and $t \geq f.\mathscr{s}$ on the set $[f > 0]$
6. If $\mathscr{s} \in \mathcal{E}^0$ and $f_0 > 0$ is such that Vf_0 is bounded then the kernel $W_{\mathscr{s}}: pb\mathcal{B} \to \mathcal{E}^r$ by $W_{\mathscr{s}}(f) = f.(\mathscr{s} + Vf_0)$ has *the complete maximum principle*, $\mathcal{E}_{W_{\mathscr{s}}} \equiv \mathcal{E}_{\mathscr{v}}$ and $\mathscr{s}$ is a $W - potential.$
7. $f.(g.\mathscr{s}) = (fg).\mathscr{s}$ for any $f, g \in pb\mathcal{B}$ and any $\mathscr{s} \in \mathcal{E}^r$

This construction generalize a similar one given in [7] , [11]
Similar results presented in this chapter may be found in [6], [7], [8], [9],[10],[11],[12],[16],[17].

Chapter 3. In this chapter we present a study of so called *Darboux-Stieltjes* integral for the functions taking values in a Banach space with respect to a real function. For the scalar case this notion was introduce by [15] and deepened by I. Bucur [13], [14] , [15]

Here we give the proof of some well known assertions in the particular case of Riemann-Stieltjes integrability but which are not so trivial in our case. At the end of this chapter we suggest by an example how to use this type of integration in potential theory (the duality in a particular case of potential theory)

CHAPTER 1

NEARLY SATURATION, BALAYAGE AND FINE CARRIER IN EXCESSIVE STRUCTURES

In this chapter, we give minimal conditions on the space X, such that a good part of potential theory in the frame of excessive structure, associated with a proper submarkovian resolvent family of kernels on X , may be developed. We characterize the regular excessive elements as being those excessive functions for which the pseudo-balayages associated with, are balayages and we construct a fine carrier theory without using any kind of compactification.

1.1. PRELIMINARIES AND FIRST RESULTS

In this paragraph $\mathcal{V} = (\mathrm{V}_\alpha)_{\alpha \geq 0}$ is a proper sub-Markovian resolvent of kernels on a measurable space$(\mathrm{X}, \mathcal{B})$. As usually we denote by $\mathcal{S} = \mathcal{S}_{\mathcal{V}}$ the set of all positive $\mathcal{B}-$ measurable functions $\mathcal{s} : \mathrm{X} \longrightarrow [0, +\infty]$ such that $\alpha \mathrm{V}_\alpha \mathcal{s} \leq \mathcal{s}$ for all $\alpha > 0$ and by $\mathcal{S}^f$ the set of all finite elements of $\mathcal{S}$. In addition, we denote by $\mathcal{E} = \mathcal{E}_{\mathcal{V}}$ the set off all excessive, $\mathcal{B}$ −measurable functions, which are finite $\mathcal{V} - a.e.$ i.e.

$$\mathcal{E} = \{\mathcal{s} \in \mathcal{S} / \sup_\alpha \alpha \mathrm{V}_\alpha \mathcal{s} = \mathcal{s} \;\; and \;\; \mathrm{V}_\alpha(\mathbb{I}_{[\mathcal{s}=\infty]}) = 0 \quad \forall \alpha \in \mathbb{R}_+\}$$

It is known that for any $\mathcal{s} \in \mathcal{S}$ the family $(\alpha \mathrm{V}_\alpha \mathcal{s})_{\alpha \in \mathbb{R}_+}$is increasing and the function $\hat{\mathcal{s}}$ defined by:

$$\hat{\mathcal{s}} = \lim_{\alpha \longrightarrow \infty} \alpha \mathrm{V}_\alpha \mathcal{s} = \lim_{n \longrightarrow \infty} n \mathrm{V}_n \mathcal{s} = \sup_n n \mathrm{V}_n \mathcal{s}$$

called the regularized of $\mathcal{s}$ (w.r to $\mathcal{V}$) is dominated by $\mathcal{s}$ and the set $[\hat{\mathcal{s}} < s]$ is $\mathcal{V} -$ negligible i.e. $\mathrm{V}_\alpha(\mathbb{I}_{[\hat{\mathcal{s}} < s]}) = 0$ for any $\alpha \in \mathbb{R}_+$ or for some $\alpha_0 \in \mathbb{R}_+$. Moreover, the following properties hold:

1. $\widehat{\alpha s_1 + \beta s_2} = \alpha\,\widehat{s_1} + \beta\,\widehat{s_2}\quad \forall\, \alpha, \beta \in \mathbb{R}_+,\ s_1, s_2 \in \mathcal{S}$ whenever the algebraic operations make sense
2. $s_1 \leq s_2 \implies \widehat{s_1} \leq \widehat{s_2}\ \ \forall\, s_1, s_2 \in \mathcal{S}$
3. $\widehat{\widehat{s}} = \widehat{s}\quad \forall s \in \mathcal{S}$ and $s = \widehat{s}\ \forall s \in \mathcal{E}$
4. $\widehat{s_n} \uparrow \widehat{s}\quad \forall s, s_n \in \mathcal{S}$ such that $s_n \uparrow s$
5. If $s_1, s_2 \in \mathcal{S}$ and $s_1 + s_2 \in \mathcal{E}^f$ then $s_1, s_2 \in \mathcal{E}$
6. For any increasing sequence $(s_n)_n$ from $\mathcal{S}$ (respectively $\mathcal{E}$) the function $sup_n\, s_n$ belongs to $\mathcal{S}$(respectively $\mathcal{E}, if\ sup_n\, s_n < \infty$ $v.a.e.$)
7. For any sequence $(s_n)_n$from $\mathcal{S}$, the function $inf_n\, s_n$ belongs to $\mathcal{S}$
8. For any sequences $(s_n)_n$ from $\mathcal{E}$ the function $\widehat{inf_n\, s_n}$ is the infimum in $\mathcal{E}$ of the set $\{s_n /\ n \in \mathbb{N}\}$ and will be denoted by $\bigwedge_n s_n$
   We have $\quad s + \bigwedge_n s_n = \bigwedge_n (s + s_n)\qquad \forall s \in \mathcal{E}$
9. For any $\mathcal{B} -$ measurable function f on X the set
   $$\{s \in \mathcal{S}\ /\ s \geq f\}$$
   possesses the smallest element denoted by ${}^{\mathcal{S}}Rf$ or Rf
   Particularly if f is of the form $s_2 - s_1$ with $s_1, s_2 \in \mathcal{S}$ then we have:
   $$R(s_2 - s_1) = {}^{\mathcal{S}}R(s_2 - s_1) \underset{\mathcal{S}}{\preccurlyeq} s_2$$
   where we have written $u \underset{\mathcal{S}}{\preccurlyeq} v$ if there exists $s \in \mathcal{S}$ such that:
   $v = u + s, u$ and v being positive functions on X.
   The relation $\underset{\mathcal{S}}{\preccurlyeq}$ is so called the *specific order given by* $\mathcal{S}$
   If $A \in \mathcal{B}$ and $s \in \mathcal{S}$ then the element $R(\mathbb{I}_A . s)$ is called *the réduite of* s *on the set* A and will be noted by $R^A s$. The following properties of the reduite operation are well known:
   a. The map $s \longrightarrow R^A s$ from $\mathcal{S}$ to $\mathcal{S}$ is a $\sigma - balayage$ on $\mathcal{S}$ i.e. it is *additive*, *increasing*, σ −continous in order from below $(Rs_n)_{\mathrm{n}} \uparrow Rs$ whenever $s_n \in \mathcal{S}, s_n \uparrow s$)
   b. If $(A_n)_n$ is an increasing sequence from $\mathcal{B}$ and $A = \bigcup_n A_n$ then we have:
   $$R^{A_n} s \ \uparrow R^A s\ \ \forall s \in \mathcal{S}$$
   c. $R^A s = s$ on A and $R^{A_1 \cup A_2} + R^{A_1 \cap A_2} \leq R^{A_1} s + R^{A_2} s$ for any $A_1, A_1 \in \mathcal{B}$ and any $s \in \mathcal{S}$
10. If $s_1, s_2 \in \mathcal{E}_v$ then $R^A(s_2 - s_1) \in \mathcal{E}_v$ and $R^A(s_2 - s_1) \preccurlyeq_{\mathcal{E}_v} s_2$ where $\underset{\mathcal{E}_v}{\preccurlyeq}$ is the specific order given by $\mathcal{E}_v$
11. The set $(\mathcal{E}_v, \underset{\mathcal{E}_v}{\preccurlyeq})$ is a conditionally $\sigma - complete\ lattice\ i.e.$ for any sequence $(s_n)_n \in \mathcal{E}_v$ there exists *the greatest lower bound* noted by $\curlywedge_n s_n$ and we have:
   $$s + \underset{n}{\curlywedge} s_n = \underset{n}{\curlywedge}(s + s_n)\ \ \forall s \in \mathcal{E}$$
   If $(s_n)_n \in \mathcal{E}_v$ is specifically dominated in $\mathcal{E}$ there exists *the smaller upper bound* noted by $\curlyvee_n s_n$ and we have:
   $$s + \underset{n}{\curlyvee} s_n = \underset{n}{\curlyvee}(s + s_n)\ \ \forall s \in \mathcal{E}$$
   Moreover, if the sequence $(s_n)_n$ is specifically increasing (resp. decreasing) then we have:
   $$\underset{n}{\curlyvee} s_n = \underset{n}{sup}\, s_n\ \ (resp. \underset{n}{\curlywedge} s_n = \underset{n}{inf}\, s_n)$$

where $sup_n s_n$ $(resp. inf_n s_n)$ is the pointwise supremum (resp. infimum) of the sequence of functions $(s_n)_n$ on X.

Particularly the *Riesz decomposition property holds in* $\mathcal{E}$ and $\mathcal{S}$ i.e. for any s, t_1, t_2 belonging to $\mathcal{E}$ (resp. $\mathcal{S}$) with $s \preccurlyeq t_1 + t_2$ there exist s_1, s_2 in $\mathcal{E}$ (resp. $\mathcal{S}$) such that $s_1 \preccurlyeq t_1$, $s_2 \preccurlyeq t_2$, $s = s_1 + s_2$. In fact the same $\sigma -$*Riesz decomposition property* may be immediately shown

$$s \preccurlyeq \sum_{i=1}^{\infty} t_i \Rightarrow s = \sum_{i=1}^{\infty} s_i \ , s_i \preccurlyeq t_i \quad \forall i \in \mathbb{N}$$

Other well known assertion from the $\sigma -$ vector lattices may be restated in the convex cones $\mathcal{E}$ and $\mathcal{S}$.

Among them the following one will be used in the sequel: For any s_1, s_2 in $\mathcal{E}$ (resp. $\mathcal{S}$) we have

$$s_1 \curlywedge s_2 + s_1 \curlyvee s_2 = s_1 + s_2$$

12. The Riesz decomposition property with respect to the pointwise order relation holds in $\mathcal{S}$(respectively $\mathcal{E}$) i.e. for any s, t_1, t_2 *in* $\mathcal{S}$ (resp. $\mathcal{E}$) with $s \leq t_1 + t_2$ there exist s_1, s_2 in $\mathcal{S}$ (resp. $\mathcal{E}$) such that $s = s_1 + s_2, s_1 \leq t_1, \ s_2 \leq t_2$.

The following decomposition property is inspired by similar one used by Mokobodzki in the study of subordination resolvents (see [22])

Lemma1.1.1. *For any* $s \in \mathcal{S}^f$, *and any* $A \in \mathcal{B}$ *there exist* s_A *and* s'_A *in* $\mathcal{S}$ *such that*

$$s = s_A + s'_A \ \ and \ \ R^A s_A = s_A, R^{X \setminus A} s'_A = s'_A$$

Proof: We define inductively two sequences $(s'_n)_n$ and $(s''_n)_n$ in $\mathcal{S}$ as follows:

$$s''_1 = R(s - R^A s), \quad s'_1 = s - R(s - R^A s)$$

$$s''_{n+1} = R(s'_n - R^A s'_n), \quad s'_{n+1} = s'_n - R(s'_n - R^A s'_n)$$

Obviously, we have $s'_n = s'_{n+1} + s''_{n+1}$ and one may show that $s'_{n+1} \leq R^A s'_n \leq s'_n$ and $s''_{n+1} = R^{X \setminus A} s''_{n+1}$. So the sequence $(s'_n)_n$ is specifically decreasing in $\mathcal{S}$ and the sequence $(\sum_{i=1}^{n} s''_i)_n$ is specifically increasing in $\mathcal{S}$ and we have

$$s = s'_n + \sum_{i=1}^{n} s''_i \qquad \forall n \in \mathbb{N}^*$$

Therefore, $s = s_A + s'_A$ where we have noted

$$s_A = \inf_n s'_n = \underset{\mathcal{S}}{\curlywedge} s'_n \ , s'_A = \sum_{i=1}^{\infty} s''_i := \sup_n \sum_{i=1}^{n} s''_i = \underset{n}{\curlyvee} \sum_{i=1}^{n} s''_i$$

From the preceding considerations we deduce

$$R^A\left(\underset{n}{\curlywedge} s'_n\right) = \underset{n}{\curlywedge} R^A s'_n = \underset{n}{\curlywedge} s'_n \quad , R^A s_A = s_A$$

$$R^{X\setminus A}\left(\sum_{i=1}^{\infty} s''_i\right) = \sum_{i=1}^{\infty} R^{X\setminus A} s''_i = \sum_{i=1}^{\infty} s''_i \, , R^{X\setminus A} s'_A = s'_A \qquad \blacksquare$$

1.2. A CHOQUET TYPE LEMMA

Lemma 1.2.1. *Let $(s_n)_n$ be a sequence in $\mathcal{S}$ and for any $n \in \mathbb{N}$ let $(s_{nm})_{nm}$ be a sequence in $\mathcal{S}$ which is specifically increasing to s_n .*

1. *We have*
$$\underset{\mathcal{S}}{\curlyvee} \{s_n / n \in \mathbb{N}\} = \underset{\mathcal{S}}{\curlyvee} \{t_n / n \in \mathbb{N}\}$$
where
$$t_n =: \underset{i,j \le n}{\curlyvee} s_{i,j}$$
2. *If $s_n < \infty$ and for any sequence $\sigma = (m_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ we denote*
$$s_\sigma = \underset{\mathcal{S}}{\curlywedge} \{s_{nm_n} / n \in \mathbb{N}\}$$
then we have
$$\underset{\mathcal{S}}{\curlywedge}\{s_n / n \in \mathbb{N}\} = \sup\{s_\sigma / \sigma \in \textstyle\sum\mathbb{N}\}$$
where sup stands for the pointwise supremum and $\sum\mathbb{N}$ for the set of all sequences of natural numbers.

Proof. 1. Obviously we have

$$s_n = \underset{\mathcal{S}}{\curlyvee}\{s_{nm} / m \in \mathbb{N}\} \preccurlyeq \underset{\mathcal{S}}{\curlyvee}\{t_k / k \in \mathbb{N}\} \preccurlyeq \underset{\mathcal{S}}{\curlyvee}\{s_k / k \in \mathbb{N}\}$$

and therefore

$$\underset{\mathcal{S}}{\curlyvee}\{s_n / n \in \mathbb{N}\} = \underset{\mathcal{S}}{\curlyvee}\{t_n / n \in \mathbb{N}\}$$

2. Let $x \in X$ and let ε be a real number, $\varepsilon > 0$. Since the sequence $(s_{nm})_m$ is specifically increasing (in $\mathcal{S}$) to the element s_n of $\mathcal{S}$ we have

$$s_n(x) = \sup_m s_{nm}(x) = \lim_{m\to\infty} s_{nm}(x)$$

and therefore we may consider $m_n \in \mathbb{N}$ such that

$$s_n(x) \le s_{nm_n}(x) + \frac{\varepsilon}{2^n} \quad or \ t_n(x) < \frac{\varepsilon}{2^n}$$

where $t_n \in \mathcal{S}$ is such that $s_n = s_{nm_n} + t_n$

If we denote $s_0 = \curlywedge_{\mathcal{S}}\{s_n / n \in \mathbb{N}\}$, from the preceding consideration we have

$$s_0 \preccurlyeq s_{nm_n} + s_0 \curlywedge \left(\sum_{i=1}^{k} t_i\right) \quad \forall n \in \mathbb{N},$$

$$s_0 \preccurlyeq s_{nm_n} + \curlyvee_{\mathcal{S}}\{s_0 \curlywedge \left(\sum_{i=1}^{k} t_i\right) \ /k \in \mathbb{N}\},$$

$$s_0 \preccurlyeq \curlywedge_{\mathcal{S}}\{s_{nm_n} /n \in \mathbb{N}\} + \curlyvee_{\mathcal{S}}\{s_0 \curlywedge \left(\sum_{i=1}^{k} t_i\right)/k \in \mathbb{N}\}$$

On the other hand at the point $x \in X$ the following inequality holds

$$\curlyvee_{\mathcal{S}}\{s_0 \curlywedge \left(\sum_{i=1}^{k} t_i\right) \ /k \in \mathbb{N}\}(x) = \lim_{k\to\infty}\left(s_0 \curlywedge \left(\sum_{i=1}^{k} t_i\right)\right)(x)$$

$$\leq \lim_{k\to\infty}\sum_{i=1}^{k} t_i \ (x) \leq \ \varepsilon$$

and therefore

$s_0(x) \leq s_\sigma(x) + \ \varepsilon$ where $\sigma = (m_n)_{n\in\mathbb{N}}$.

The number ε being arbitrary we get

$$s_0(x) = \sup_{\sigma\in\sum\mathbb{N}} s_\sigma(x) \quad \forall x \in X \qquad \blacksquare$$

Lemma 1.2.2. *Let $(s_n)_n$ be a sequence in $\mathcal{E}$ and for any $n \in \mathbb{N}$ let $(s_{nm})_m$ be a sequence in $\mathcal{E}$ which is $\mathcal{E}$- specifically increasing to s_n*

1. *If the sequence $(s_n)_n$ has a specific majorant in $\mathcal{E}$ then*

$$\curlyvee_{\mathcal{E}} s_n = \curlyvee_{\mathcal{E}} t_n$$

where

$$t_n =: \curlyvee_{\mathcal{E}}\{s_{ij} /i,j \leq n\}$$

2. *If $s_n < \infty$ and for any sequence $\sigma = (m_n)_{n\in\mathbb{N}}$* , in $\mathbb{N}$, *we denote*

$$s_\sigma = \curlywedge_{\mathcal{E}}\{s_{n m} /m \in \mathbb{N}\}$$

then we have

$$\curlywedge_{\mathcal{E}}\{s_n \ / \ n \in \mathbb{N}\} = \sup\{s_\sigma /\sigma \in \textstyle\sum\mathbb{N}\}$$

where sup stands for the pointwise supremum and $\sum\mathbb{N}$ for the set of all sequences of natural numbers.

Proof. We apply Lemma 1.2.2. we have

1. $\curlyvee_{\mathcal{E}} s_n = \curlyvee_{\mathcal{S}} s_n$, $\quad \curlyvee_{\mathcal{E}} t_n = \curlyvee_{\mathcal{S}} t_n$

2. $\curlywedge_{\mathcal{E}}\{s_{n m} /m \in \mathbb{N}\} = \curlywedge_{\mathcal{S}}\{s_{n m} /m \in \mathbb{N}\}$ $\blacksquare$

We remember also the following definition

A map $\mu: \mathcal{E} \longrightarrow \overline{\mathbb{R}}_+$ which is *additive ,increasing, σ −continuous in order from below and for any $s \in \mathcal{E}$ there exists a sequence $(s_n)_n$ in $\mathcal{E}$, increasing to s such that $\mu(s_n) < \infty$ for all n, is called $\sigma - H - integral$.*

1.3. PSEUDO-BALAYAGES ASSOCIATED WITH SUPERMEDIAN FUNCTIONS

We remember that a map $B: \mathcal{S} \longrightarrow \mathcal{S}$ is called *pseudo-balayage* on $\mathcal{S}$ if it is *increasing* (with respect to the pointwise order relation), *additive*, *contractive* ($Bs \leq s$) and *idempotent* ($B^2 s = B(Bs) = Bs$ for all $\in \mathcal{S}$.
A pseudo-balayage B is called *balayage* if it is *$\sigma -$ continuous in order from below* i.e. the sequence $(Bs_n)_n$ increases to Bs whenever the sequence $(s_n)_n$ increases to s.
A typical example of balayage on $\mathcal{S}$ is the map:

$$s \longrightarrow R^A s$$

where $A \in \mathcal{B}$.

In the sequel, for any element $s \in \mathcal{S}^f$ we associate a pseudo-balayage $B_s$such that $B_s s = s$. The procedure is inspired from a similar one developed in the frame of *satandard H-cones.*

Proposition 1.3.1. *Let $s \in \mathcal{S}$ be a finite element. Then for any $t \in \mathcal{S}$ the set*

$$D_t := \{u \in \mathcal{S}/u \leq t \ \ and\ u \preccurlyeq \alpha s \ \ for\ some\ \alpha > 0\}$$

has an upper bound in $\mathcal{S}$ with respect to the pointwise order relation and the map

$$t \longrightarrow \sup_t D_t := B_s t$$

Is a pseudo-balayage with $B_s(s) = s$. Moreover if B is a pseudo-balayage with $B(s) = s$ we have $B_s \leq B$ i.e. $B_s t \leq Bt \quad \forall t \in \mathcal{S}$

Proof. We consider the subset D_t^0 of D_t given by

$$D_t^0 = \{ns - R(ns - t)/n \in \mathbb{N}^*\}$$

The set D_t^0 is countable an co-final in D_t i.e. for any $u \in D_t$ there exists $n \in \mathbb{N}$ such that

$$u \leq n\mathcal{s} - R(n\mathcal{s} - u)$$

Indeed, let $\alpha \in \mathbb{R}_+$such that $u \preccurlyeq \alpha\mathcal{s}$ and $\leq t$. We have $u \preccurlyeq n\mathcal{s}$ for $n \in \mathbb{N}, n \geq \alpha$ and we remark that

$$u = n\mathcal{s} - R(n\mathcal{s} - u)$$

On the other hand we notice that the sequence $(n\mathcal{s} - R(n\mathcal{s} - t))_n$ is increasing. Hence, the supremum of the set D_t^0 belongs to $\mathcal{S}$ and we have

$$B_{\mathcal{s}}t = supD_t = supD_t^0 \leq t$$

If $= \mathcal{s}$, obviously $\mathcal{s} \in D_{\mathcal{s}}$ and therefore $B_{\mathcal{s}}\mathcal{s} = \mathcal{s}$

The fact that the map $B_{\mathcal{s}}$ is increasing follows from the definition of $B_{\mathcal{s}}$ because if $t_1 \leq t_2$ then $D_{t_1} \subset D_{t_2}$

Using the definition of the sets $D_{t_1}, D_{t_2}$and $D_{t_1+t_2}$for $t_1, t_2 \in \mathcal{S}$ we deduce, using Riesz decomposition property(with respect to the pointwise order relation) that

$$D_{t_1} + D_{t_2} = D_{t_1+t_2}$$

So we have

$$B_{\mathcal{s}}(t_1 + t_2) = supD_{t_1+t_2} = supD_{t_1} + supD_{t_2} = B_{\mathcal{s}}(t_1) + B_{\mathcal{s}}(t_2)$$

For any $t \in \mathcal{S}$ and any $u \in D_t$ we have $u \leq B_{\mathcal{s}}t$ and by the definition of $D_{B_{\mathcal{s}}t}$ we have $u \in D_{B_{\mathcal{s}}t}$. Hence

$$u \leq B_{\mathcal{s}}(B_{\mathcal{s}}t), \quad B_{\mathcal{s}}(t) \leq B_{\mathcal{s}}\big(B_{\mathcal{s}}(t)\big), \ B_{\mathcal{s}}(t) = B_{\mathcal{s}}^2 t$$

If B is a pseudo-balayage on $\mathcal{S}$ such that $B\mathcal{s} = \mathcal{s}$, then for any $u \in \mathcal{S}, u \preccurlyeq \alpha\mathcal{s}$ for some $\alpha > 0$ we have

$$\begin{aligned}
&B(\alpha\mathcal{s}) = \alpha B\mathcal{s} = \alpha\mathcal{s} \,, \\
&B(u) + B(\alpha\mathcal{s} - u) = B(\alpha\mathcal{s}) = \alpha\mathcal{s} = u + (\alpha\mathcal{s} - u), \\
&Bu \leq u, \qquad B(\alpha\mathcal{s} - u) \leq \alpha u - \mathcal{s}
\end{aligned}$$

And therefore $Bu = u, B(\alpha\mathcal{s} - u) = \alpha\mathcal{s} - u$

Let now $t \in \mathcal{S}$ and $u \in D_t$. From the preceding consideration we deduce

$$Bu = u \quad \forall u \in D_t, B_{\mathcal{s}}t = \sup_{u \in D_t} u = \sup_{u \in D_t} Bu \leq Bt \qquad \blacksquare$$

Remark 1.3.2. For the convex cone $\mathcal{E}$ we have similar definition of the pseudo-balayge or balayage operator $B: \mathcal{E} \longrightarrow \mathcal{E}$

Corollary 1.3.3. *For any element $\mathcal{s} \in \mathcal{E}^f$, the restriction of the map $B_{\mathcal{s}}$ defined as above is a pseudo-balayage on $\mathcal{E}$.*

Proof. We remark that for any $t \in \mathcal{S}$ which is finite $\mathcal{v} - a.e$ we have $B_{\mathcal{s}}t \in \mathcal{E}$. Indeed, we have $D_t \subset \mathcal{E}$ and therefore the supremum of the increasing and dominated sequence $(n\mathcal{s} - R(n\mathcal{s} - t))_n$ is an element of $\mathcal{E}$ $\blacksquare$

1.4. FINE CARRIER FOR EXCESSIVE FUNCTIONS

In the sequel we shall denote by $\mathcal{E}^0$ the set of all finite excessive functions s on X such that for any specific minorant $u \in \mathcal{E}$ $(u \preccurlyeq s)$ the associated pseudo-balayage B_u is a balayage on $\mathcal{E}$.

As in the introduction of this chapter for any subset A of X and any element $t \in \mathcal{E}$ we denote

$$^{\mathcal{E}}R^A t := \inf\{t' \in \mathcal{E} \,/ t' \geq t \;\; on\; A\}$$

Generally, the function $^{\mathcal{E}}R^A t$ is not $\mathcal{B}-$ measurable but if it is then this function belongs to $\mathcal{S}_{\mathcal{V}}$ and the function

$$x \longmapsto \sup \alpha V_\alpha \left({}^{\mathcal{E}}R^A t\right)(x)$$

is denoted by $B^A t$. Obviously $B^A t \in \mathcal{E}$.

Definition 1.4.1. The set A is called *subbasic* if the function $B^A t$ is defined for all $s \in \mathcal{E}$ and we have $B^A s = s$ on A.

A subbasic set M is called *a basic set* if we have

$$M = \{x \in \mathrm{X} \,/\, B^M s(x) = s(x), \forall s \in \mathcal{E}\}$$

Remark 1.4.2. It is obvious that a subset M of X is a subbasic if and only if the function $^{\mathcal{E}}R^A s$ belongs to $\mathcal{E}$ and therefore $^{\mathcal{E}}R^A t = B^A s$ for all $s \in \mathcal{E}$.

Remark 1.4.3. If M is subbasic then the map on $\mathcal{E}$

$$s \longmapsto B^M s$$

is a balayage on $\mathcal{E}$.

Remark 1.4.4. If M is a subbasic set and $b(M)$ is given by

$$b(M) = \{x \in \mathrm{X} \,/\, B^M s(x) = s(x) \;\; \forall s \in \mathcal{E}\}$$

then $B^{b(M)} s = s$ for all $s \in \mathcal{E}$ and $b(M) \in \mathcal{B}$.

The last assertion follows immediately from the fact that

$$b(M) = [B^M V f_0 = \mathrm{V} f_0]$$

where f_0 is a $\mathcal{B}$ $-$measurable, $0 < f_0 < 1$ and $V f_0 < \infty$.

On the space X, we consider as usually *the fine topology* i.e. the coarsest topology τ on X making continuous all functions of the vector lattice $\mathcal{E}_b - \mathcal{E}_b$ of bounded functions on X. We suppose here that $\mathcal{E}$ is min-stable and $\mathbb{I}_X \in \mathcal{E}$.

We remember that all elements $s \in \mathcal{E}$ are continuous with respect to τ and any point $x_0 \in X$ has a base of neighbourhoods of the form $x_0 \in [s - t > 0]$ with $s, t \in \mathcal{E}$, $t \leq s \leq 1$. Obviously, the elements of this base belong to $\mathcal{B}$.

Definition 1.4.5. We say that a balayage B on $\mathcal{E}$ is *representable* if there exists a basic set in X denoted by $b(B)$ such that

$$Bs = B^{b(B)}s$$

for all $s \in \mathcal{E}$.

The space X is called *nearly saturated* if all balayages on $\mathcal{E}$ are representable.

From now on, we suppose that X is nearly saturated and the convex cone $\mathcal{E}$ is min-stable and contains the constant functions.

Definition 1.4.6. For any element $s \in \mathcal{E}^0$ we associate the subset $b(B_s)$ the base of the balayage B_s. We shall denote it by $carr\ s$ and we shall call it the (fine) *carrier of* s (with respect to $\mathcal{E}$).

From the preceding remark 1.4.4., we deduce that the set $carr\ s$ is fine closed and we have

$$carr\ s = \emptyset \Leftrightarrow s = 0$$

Proposition 1.4.7. *The following assertions hold*

1. $\mathcal{E}^0$ *is a solid convex sub-cone of* $\mathcal{E}$ *with respect to the specific order*
2. $carr\ (s_1 + s_2) = carr\ s_1 \cup carr\ s_2 \qquad \forall s_1, s_2 \in \mathcal{E}^0$
3. *If* $(s_n)_n$ *is a sequence in* $\mathcal{E}^0$ *such that the function* $\sum_{n=1}^{\infty} s_n$ *is finite then this function belongs to* $\mathcal{E}^0$ *and the set* $carr(\sum_{n=1}^{\infty} s_n)$ *is the closure (with respect to* τ*) of the set* $\bigcup_{n=1}^{\infty} carr\ s_n$

Proof. 1. and 2. We remark firstly that $M_1 \cup M_2$ is also a basic set and for any element $t \in \mathcal{E}$ we have

$$B^{M_1 \cup M_2} t = B^{M_1} t \vee B^{M_2} t$$

Hence if we take $M_1 = carr\ s_1, M_2 = carr\ s_2$ then

$$s_1 + s_2 = B^{M_1} s_1 + B^{M_1} s_2 \leq B^{M_1 \cup M_2} s_1 + B^{M_1 \cup M_2} s_2 = B^{M_1 \cup M_2}(s_1 + s_2) \leq s_1 + s_2,$$

$$B^{M_1 \cup M_2}(s_1 + s_2) = s_1 + s_2$$

And therefore for any $u \in \mathcal{E}, u \preccurlyeq \alpha(s_1 + s_2)$ we have

$$B^{M_1 \cup M_2} u = u$$

Hence for any $t \in \mathcal{E}$ and any $u \in \mathcal{E}, u \leq t, u \preccurlyeq \alpha(s_1 + s_2)$ for some $\alpha > 0$ we have

$$u = B^{M_1 \cup M_2} u \leq B^{M_1 \cup M_2} t, B_{s_1 + s_2} t \leq B^{M_1 \cup M_2} t$$

We have also $B^{M_1 \cup M_2} t \leq B_{s_1 + s_2} t$ because $B_{s_i} \leq B_{s_1 + s_2}$ for i =1, 2 .

Hence the map on $\mathcal{E}$

$$t \longmapsto B_{s_1+s_2}t = B^{M_1 \cup M_2}t$$

Is a balayage on $\mathcal{E}$. The preceding considerations show that $s_1 + s_2 \in \mathcal{E}^0$ for all $s_1, s_2 \in \mathcal{E}^0$ and

$$carr\ (s_1 + s_2) = b\left(B_{s_1+s_2}\right) = b(B_{s_1}) \cup b\left(B_{s_2}\right) = carr\ s_1 \cup carr\ s_2$$

The last assertion may be proved using the proposition 1.3.1 and the fact that a countable union of basic set is a subbasic set ■

Proposition 1.4.8. *For any element* $u \in \mathcal{E}^0$ *we have*

a. $s \geq u$ *on* $carr\ u \Longrightarrow s \geq u$ *on* X
b. *The set* $carr\ u$ *is fine closed and* $\mathcal{B}$*-measurable subset of* X
c. *If* F *is a fine closed subset of* X *such that*
   $s \in \mathcal{E}, s \geq_F u \Longrightarrow s \geq u$ *on* X
   then we have $carr\ u \subset F$
d. $carru = \{x \in X / \mu\ \sigma - H - integral, \mu \leq_{\mathcal{E}} \varepsilon_x, \mu(u) = u(x) \Rightarrow \mu =_{\mathcal{E}} \varepsilon_x\}$

Proof. a) We have

$$u = B_u u = B^{carr\ u}u \leq B^{carr\ u}s \leq s \ \text{ if } \ t \in \mathcal{E}, s \geq_{carr\ u} u$$

The assertion b) follows from the fact that

$$\begin{aligned} carr\ u = b(B_u) &= \{x \in X / B_u s(x) = s(x) \quad \forall s \in \mathcal{E}\} \\ &= \{x \in X / B_u Vf(x) = Vf(x)\} \end{aligned}$$

where f is a $\mathcal{B}$ $-$measurable $0 < f < 1, Vf < \infty$.

c) Let us denote for simplicity by $R_\circ{}^A s = {}^{\mathcal{E}}R^A s$.
Using the hypothesis, we have

$$u \geq R_\circ{}^F u \geq u, R_\circ{}^F u = u, R_\circ{}^F \alpha u = \alpha u \qquad \forall \alpha \in \mathbb{R}_+$$

Since generally we have $R_\circ{}^F s \leq s$ we deduce that

$$R_\circ{}^F v = v \qquad \forall v \in \mathcal{E}, v \preccurlyeq u$$

and therefore for any $v \in \mathcal{E}$, $v \leq s, v \preccurlyeq \alpha u$ for some $\alpha > 0$ we have

$$R_\circ{}^F s \geq R_\circ{}^F v = v$$

The element v being arbitrary we get $B_u s \leq B^F s$ for any $s \in \mathcal{E}$.

Let now $x_0 \in carr\ u \setminus F$ and let $s_1, s_2 \in \mathcal{E}, s_i \leq 1$ be such that

$$s_1 \leq s_2, s_1(x_0) < s_2(x_0), s_1 = s_2 \ \ on\ F$$

From the preceding considerations, we have the contradictory relations:

$$R^F s_1 = R^F s_2 \quad \text{on } X$$

$$0 < s_2(x_0) - s_1(x_0) = B^{carr\, u} s_2(x_0) - B^{carr\, u} s_1(x_0)$$

$$s_i \geq R^F s_i \geq B_u s_i \;\; i = 1,2 \quad s_i(x_0) \geq R^F s_i(x_0) \geq B_u s_i(x_0) = s_i(x_0)$$

Hence $carr\, u \setminus F = \emptyset$.

d) let $x_0 \in X$ such that if μ is an $\sigma - H -$ integral on $\mathcal{E}$ with $\mu \leq \varepsilon_{x_0}$ on $\mathcal{E}$ and $\mu(u) = u(x_0)$ then $\mu = \varepsilon_{x_0}$ on $\mathcal{E}$.

If $x_0 \notin carr\, u$ then using b) we may consider two functions

$$s_1, s_2 \in \mathcal{E}, s_1 \leq s_2 \text{ on } X, s_1(x_0) < s_2(x_0) \text{ and } s_1 = s_2 \text{ on } carr\, u$$

We take as a $\sigma - H - integral\ \mu$ on $\mathcal{E}$ the map

$$s \longmapsto B_u s(x_0) = \mu(s)$$

Obviously we have $\mu(s) \leq s(x_0)$ for all $s \in \mathcal{E}$ and $\mu(u) = u(x_0)$ and therefore, using the hypothesis $\mu(s) = s(x_0)$ for all $s \in \mathcal{E}$

The last assertion gives us

$$B_u s_1 = B_u s_2 \;\; on\;\; X, B_u s_i = \mu(s_i) = s_i(x_0) \quad i = 1,2$$

$$s_1(x_0) = s_2(x_0),$$

This contradicts the choice of s_1 and s_2.

Let now $x_0 \in carr\, u$ and let μ be an $\sigma - h -$ integral such that $\mu \leq \varepsilon_{x_0}$ on $\mathcal{E}$, $\mu(u) = u(x_0)$. We get the relation

$$\mu(v) = v(x_0) \text{ for all } v \in \mathcal{E}, v \preccurlyeq \alpha u \text{ for some } \alpha \in \mathbb{R}_+$$

Hence taking $s \in \mathcal{E},\ v \in \mathcal{E},\ v \leq s, v \preccurlyeq \alpha u$ for some $\alpha \in \mathbb{R}_+$ we have

$$\mu(s) \geq \mu(v) = v(x_0)$$

The element v being arbitrary we get

$$\mu(s) \geq B_u s(x_0) = s(x_0), \mu = \varepsilon_{x_0} \;\; on\ \mathcal{E} \qquad \blacksquare$$

1.5. THE REGULAR EXCESSIVE ELEMENTS

Definition 1.5.1. An element $s \in \mathcal{E}^f$ is called *regular* if for any increasing sequence $(s_n)_n$ with $\sup s_n = s$ we have

$$\bigwedge_n R(s - s_n) = 0$$

The potentials are regular elements.

The following result is well known in standard H-cones .A similar result may be found in [6] (therem 3.2.9)

Proposition 1.5.2. *If $\mathscr{s}$ is a regular element of $\mathcal{E}$ then the associated pseudo-balayage $B_{\mathscr{s}}$ is a balayage.*

Proof. Let $(\mathscr{s}_n)_n$ be a sequence in $\mathcal{E}$ increasing to $\mathscr{s}$ and for any $n \in \mathbb{N}$ let $u_n \in \mathcal{E}$ be such that

$$R(\mathscr{s}-\mathscr{s}_n)+u_n=\mathscr{s}$$

The sequence $(R(\mathscr{s}-\mathscr{s}_n))_n$ is decreasing and the sequence $(u_n)_n$ is increasing with respect to the pointwise order relation.

Therefore, we have

$$u \coloneqq \sup_n u_n \in \mathcal{E} \quad \text{and} \quad \inf_{\mathrm{n}} \mathrm{R}(\mathscr{s}-\mathscr{s}_n) \in \mathcal{S}$$

But since

$$u+\inf_{\mathrm{n}} \mathrm{R}(\mathscr{s}-\mathscr{s}_n)=\mathscr{s}$$

We deduce that $\inf_{\mathrm{n}} \mathrm{R}(\mathscr{s}-\mathscr{s}_n) \in \mathcal{E}$. Hence using the regularity of $\mathscr{s}$ we have

$$\inf_n R(\mathscr{s}-\mathscr{s}_n)=\bigwedge_n R(\mathscr{s}-\mathscr{s}_n)=0,\;\; u=\mathscr{s}$$

With the above notations we have

$$u_n \leq \mathscr{s}_n \leq \mathscr{s}, \qquad u_n=B_{\mathscr{s}}u_n \leq B_{\mathscr{s}}\mathscr{s}_n \;\; \forall n \in \mathbb{N}$$

Therefore $\sup B_{\mathscr{s}}\,\mathscr{s}_n=\mathscr{s}$. Obviously for any $\alpha \in \mathbb{R}_+$ and any sequence $(\mathscr{s}_n)$ in $\mathcal{E}$ increasing to $\alpha\mathscr{s}$ we have $\sup B_{\mathscr{s}}\,\mathscr{s}_n=\alpha\mathscr{s}$

Now if $u \in \mathcal{E}, u \preccurlyeq \mathscr{s}$ and $(u_n)_n$ is a sequence in $\mathcal{E}$ increasing to u we have

$$\sup B_{\mathscr{s}}u_n=u$$

Indeed if we denote $v=\mathscr{s}-u$ then the sequence $(u_n+v)_n$ increases to $\mathscr{s}$ and therefore

$$\sup_n B_{\mathscr{s}}(u_n+v)=\mathscr{s}, \sup_n B_{\mathscr{s}}u_n+v=\mathscr{s},\; \sup_n B_{\mathscr{s}}u_n=u$$

To finish the proof we consider an arbitrary element t of $\mathcal{E}$ and a sequence $(t_n)_n$ in $\mathcal{E}$ increasing to t. Let $u \in D_t$ where

$$D_t \coloneqq \{u \in \mathcal{S}/u \leq t \;\; and\; u \preccurlyeq \alpha\mathscr{s} \;\; for\; some\; \alpha>0\}$$

Since $u \leq t$ then the sequence $(\inf(u,t_n))_n$is in $\mathcal{E}$ and increases to $\inf(u,t)=u$

But $u \preccurlyeq \alpha\mathscr{s}$ for some $\alpha \in \mathbb{R}_+$. From the preceding considerations we have

$$\sup_n B_{\mathscr{s}}(\inf(u,t_n))=B_{\mathscr{s}}u=u$$

Hence

$$\sup_n B_s t_n \geq \sup_n B_s(\inf(u, t_n)) = B_s u = u$$

But u being arbitrary we get

$$\sup_n B_s t_n \geq B_s t, \ \sup B_s t_n = B_s t \qquad \blacksquare$$

Theorem 1.5.3. *The element $s \in \mathcal{E}^f$ is regular if and if for any $u \in \mathcal{E}$, $u \preccurlyeq s$ the pseudo-balayage B_u is a balayage on $\mathcal{E}$ i.e. $s \in \mathcal{E}^0$*

Proof. let $s \in \mathcal{E}^0$ and let $(s_n)_n$ be a sequence in $\mathcal{E}$ increasing to s. For $\epsilon \in \mathbb{R}, \varepsilon > 0$ and $n \in \mathbb{N}, n > 0$ we denote by A_n the subset of X given by

$$A_n = \left[s < s_n + \left(1 - \frac{1}{n}\right)\epsilon\right] = \{x \in X / s(x) < s_n(x) + \left(1 - \frac{1}{n}\right)\epsilon\}$$

Obviously we have $\overline{A}_n \subset A_{n+1}$ and A_n is fine open for every $n \in \mathbb{N}$, $n > 0$. Moreover $\bigcup_{n=1}^{\infty} A_n = X$.

Let us denote $u_n = R\left(s - s_n - \left(1 - \frac{1}{n}\right)\epsilon\right)$ and $v_n = s - u_n$, obviously

$$u_n = R^{X \setminus A_n} u_n = {}^{\mathcal{E}}R^{X \setminus A_n} u_n \ \text{ since } \left[s > s_n + \left(1 - \frac{1}{n}\right)\epsilon\right] \subset X \setminus A_n$$

And therefore

$$u_{n+m} = R^{X \setminus A_{n+m}} u_{n+m} \leq R^{X \setminus A_n} u_{n+m} \qquad \forall n, m \in \mathbb{N}^*$$

$$u_{n+m} = R^{X \setminus A_n} u_{n+m} \qquad \forall n, m \in \mathbb{N}^*$$

Since $u_n + v_n = s$, and the sequence $(u_n)_n$ is decreasing it follows that the sequence $(v_n)_n$ is increasing to an element $v \in \mathcal{E}$, and if we denote $u = \inf_n u_n$, we have

$$u \in \mathcal{S}, u + v = s, \hat{u} + \hat{v} = \hat{s}, \hat{u} + v = s, u = \hat{u}$$

Hence $u \in \mathcal{E}^0$ and from the preceding consideration, it follows

$$R^{X \setminus A_n}(u_{n+m} + v_{n+m}) = R^{X \setminus A_n} s \quad \forall n, m \in \mathbb{N}^*,$$

$$u_{n+m} + R^{X \setminus A_n}(v_{n+m}) = R^{X \setminus A_n} s \quad \forall n, m \in \mathbb{N}^*,$$

Making $m \longrightarrow \infty$ we obtain

$$u + R^{X \setminus A_n} v = R^{X \setminus A_n} s$$

But on the other hand

$$R^{X \setminus A_n} u + R^{X \setminus A_n} v = R^{X \setminus A_n} s$$

And therefore $R^{X\setminus \mathrm{A}_n}u = u$. The set $X \setminus \mathrm{A}_n$ being finely closed we deduce $carr\ u \subset X \setminus \mathrm{A}_n$for any $n \in \mathbb{N}$. But $\bigcap_{n=1}^{\infty} X \setminus \mathrm{A}_n = \emptyset$ and therefore $u = 0$, $\inf_n R\left(\mathscr{s} - \mathscr{s}_n - \left(1 - \frac{1}{n}\right)\epsilon\right) = 0$

The relations

$$^{\mathcal{E}}R(\mathscr{s} - \mathscr{s}_n) \leq {}^{\mathcal{E}}R(\mathscr{s} - \mathscr{s}_n - (1 - \tfrac{1}{n})\epsilon) + (1 - \tfrac{1}{n})\epsilon \quad \forall n \geq 1\ and\ \epsilon > 0$$

give us the relations

$$\inf_n {}^{\mathcal{E}}R(s - \mathscr{s}_n) \leq \epsilon,\ \inf_n {}^{\mathcal{E}}R(s - \mathscr{s}_n) = 0$$

that is $\mathscr{s}$ is a regular element of $\mathcal{E}$

Conversely, if $\mathscr{s}$ is regular then any element $u \in \mathcal{E}, u \preccurlyeq \mathscr{s}$ is regular and by proposition 1.5.2. the pseudo-balayage B_u is a balayage i.e. $\mathscr{s} \in \mathcal{E}^0$ ∎

Remark 1.5.4. In their papers concerning the semi-polar sets and regular excessive functions respectively balayages on excessive measures L. Beznea and N. Boboc (see [4] and [5]) show that for any basic set M which is analytic there exists a bounded regular excessive function $\mathscr{q}$, such that its fine carrier, is contains in M

Remark 1.5.5. We may prove the following assertion :
If $\mathscr{v} = (V_\alpha)_{\alpha > 0}$ is a standard resolvent family of kernels on a measurable space $(X, \mathcal{B})$ i.e.

a. $\mathscr{v}$ is a proper sub-Markovian resolvent
b. The convex cone $\mathcal{E}$ of all excessive functions with respect to $\mathscr{v}$ is min-stable and contains the positive constant functions
c. There exists a distance d on X such that the associated topology τ_d is smaller than the fine topology on X (i.e. the coarsest topology τ on X making continuous all functions of $\mathcal{E}$)
d. The Borel structure associated with the distance d coincides with $\mathcal{B}$

and if the space $(X, \mathcal{B})$ is such that for any regular and bounded excessive function p with respect to the resolvent $\mathscr{v}$, the balayage associated as above to p is representable then all balayages on $\mathcal{E}$ are representable.

CHAPTER 2

THE SPECIFIC MULTIPLICATION IN EXCESSIVE STRUCTURE

In this chapter, we construct so called *specific multiplication* in the frame of excessive structure i.e. if $\mathcal{V} = (V_\alpha)_{\alpha>0}$ is a standard resolvent on a measurable space $(X, \mathcal{B})$ then noting by $\mathcal{E}^r$ the set of all regular excessive elements of the convex cone $\mathcal{E}$ of all excessive functions we construct a map $(f, s) \mapsto f.s$ from $pb\mathcal{B} \times \mathcal{E}^r \to \mathcal{E}^r$ such that it is bilinear and has the following properties:

1. The sequence $(f_n.s)_n$ is specifically increasing (resp. decreasing) to $f.s$ whenever $(f_n)_n$ increases (resp. decreases) to f
2. If $(f_n)_n$ is uniformly bounded and $(f_n)_n$ is pointwisely convergent to f then the sequence $(f_n.s)_n$ converges to $f.s$ with respect to the specific order relation in $\mathcal{E}$.
3. If $p_n \in \mathcal{E}$ and $\sum p_n \in \mathcal{E}^r$then we have $f.s = \sum_n f.p_n \ \forall f \in pb\mathcal{B}$
4. $carr\ f.s \subset \overline{[f > 0]} \cap carr\ s$ ($\bar{A}$ means the fine closure of A)
5. $t \geq f.s$ whenever $t \in \mathcal{E}$ and $t \geq f.s$ on the set $[f > 0]$
6. If $s \in \mathcal{E}^0$ and $f_0 > 0$ is such that Vf_0 is bounded then the kernel $W_s: pb\mathcal{B} \to \mathcal{E}^r$ by $W_s(f) = f.(s + Vf_0)$ has *the complete maximum principle*, $\mathcal{E}_{W_s} \equiv \mathcal{E}_{\mathcal{V}}$ and s is a $W-potential.$
7. $f.(g.s) = (fg).s$ for any $f, g \in pb\mathcal{B}$ and any $s \in \mathcal{E}^r$

2.1. PRELIMINARIES AND FIRST RESULTS

Definition 2.1.1. If $(X, \mathcal{B})$ is a measurable space and $\mathcal{V} = (V_\alpha)_{\alpha>0}$ is a resolvent family of kernels on $(X, \mathcal{B})$ we shall say that $\mathcal{V}$ is *a standard resolvent* if

a. $\mathcal{V}$ is a proper sub-Markovian resolvent
b. The convex cone $\mathcal{E}$ of all excessive functions with respect to $\mathcal{V}$ is min-stable and contains the positive constant functions
c. There exists a distance d on X such that the associated topology τ_d is smaller than the fine topology on X (i.e. the coarsest topology τ on X making continuous all functions of $\mathcal{E}$)
d. The Borel structure associated with the distance d coincides with $\mathcal{B}$
e. The space $(X, \mathcal{B})$ is nearly saturated with respect to $\mathcal{E}$ i.e. any $\sigma-$balayage on $\mathcal{E}$ is representable.

Remark 2.1.2. In the case where the above properties a), b) are fulfilled and $\mathcal{B}$ is separable and $\mathcal{B}$ coincides with the σ −algebra generated by $\mathcal{E}$ then there exists a distance d as before.

Throughout this chapter $\mathcal{v}$ will be a standard resolvent on $(X, \mathcal{B})$ and we shall mark by $\mathcal{S}$ or $\mathcal{S}_{\mathcal{v}}$ the convex cone of all $\mathcal{v}$-supermedian functions i.e. all positive $\mathcal{B}-$ measurable functions $s : X \longrightarrow [0, +\infty]$ such that $\alpha v_\alpha s \leq s$ for all $\alpha > 0$ and by $\mathcal{S}^f$ the set of all finite elements of $\mathcal{S}$.

We remember that an element $s_0 \in \mathcal{E}$ is called *weak unit* in $\mathcal{E}$ if for any $s \in \mathcal{E}$ the sequence $s_n = s \wedge (n s_0)$ increases to s. Obviously s_0 is a weak unit in $\mathcal{E}$ if and only if $s_0 > 0$ on X.

If s_0 is a weak unit then an element s of $\mathcal{E}$ is called s_0 −*bounded* if there exists $\alpha \in \mathbb{R}_+$ such that $s \leq \alpha s_0$.

Lemma 2.1.3. *For any element $s \in \mathcal{E}^f$ and any subset A of $\mathcal{B}$ there exist s_A and s'_A in $\mathcal{E}$ such that*

$$s = s_A + s'_A, \quad R_\circ^A s_A = R^A s_A = s_A, \qquad R_\circ^{X\setminus A} s'_A = R^{X\setminus A} s'_A = s'_A$$

where for any $t \in \mathcal{E}$ and $A \in \mathcal{B}$ we have noted

$$R_\circ^A t = \inf\{u \in \mathcal{E} / u \geq t \quad on\ A\}$$

$$R^A t = \inf\{u \in \mathcal{S}_{\mathcal{v}} / u \geq t \quad on\ A\}$$

Proof. Using lemma 1.1.1. we deduce that for the function $s \in \mathcal{E}^f$ we may construct s_A and s'_A in $\mathcal{S}_{\mathcal{v}}$ with the following properties

$$s = s_A + s'_A, \quad R^A s_A = s_A, \qquad R^{X\setminus A} s'_A = s'_A$$

Since $s \in \mathcal{E}$ and s_A, s'_A are in $\mathcal{S}_{\mathcal{v}}$ we deduce that $s_A = \widehat{s_A}$, $s'_A = \widehat{s'_A}$ and therefore $s_A \in \mathcal{E}, s'_A \in \mathcal{E}$. Obviously we have

$$s_A = R^A s_A \leq R_\circ^A s_A \leq s_A, R_\circ^A s_A = s_A = R^A s_A$$

$$s'_A = R^{X\setminus A} s'_A \leq R_\circ^{X\setminus A} s'_A \leq s'_A, R_\circ^{X\setminus A} s'_A = s'_A = R^{X\setminus A} s'_A \quad \blacksquare$$

We remember that an element $p \in \mathcal{E}$ is called *regula*r if for any sequence $(s_n)_n$ in $\mathcal{E}$ which increases to p we have

$$\bigwedge_n R(p - s_n) = 0$$

We denote by $\mathcal{E}^r$ the convex cone of all regular elements of $\mathcal{E}$ and by $\mathcal{E}^f$ the convex cone of all regular finite element of $\mathcal{E}$.

The following statement shows that generally in the study of regular elements we may restrict our self to the finite regular elements.

Lemma 2.1.4. *For any weak unit s_0 in $\mathcal{E}$ and any regular element p of $\mathcal{E}$ there exists a sequence $(p_n)_n \subset \mathcal{E}^r$ such that p_n is s_0 −bounded for any $n \in \mathbb{N}$ and* $p = \sum_{n=1}^{\infty} p_n$

Proof. For any $n \in \mathbb{N}$, $n \geq 1$ we consider the regular element s_n which is s_0 −bounded given by

$$s_n + R(p - ns_0) = p, \qquad s_n \leq ns_0$$

Since the sequence $(t_n)_n$ of $\mathcal{S}$ given by

$$t_n = p \wedge (ns_0)$$

increases to p, the sequence

$$R(p - ns_0) = R(p - t_n)$$

decreases to zero in $\mathcal{E}$.
For any $n \in \mathbb{N}, n \geq 1$ we put

$$u_n = \curlyvee_{i=1}^{n} s_i, \quad v_n = \curlywedge_{i=1}^{n} R(p - t_i)$$

Obviously, $u_i + v_i = p$ for any $i \geq 1$, the sequence $(u_n)_n$ is specifically increasing, the sequence $(v_n)_n$ is specifically decreasing and since
$v_n \leq R(p - t_n)$ we get $\curlywedge_n v_n = 0$
Hence the sequence $(u_n)_n$ is specifically increasing to p. We put now

$$p_1 = u_1 = s_1$$
$$p_{n+1} = u_{n+1} - u_n$$

We have

$$\sum_{i=1}^{n} p_i = u_n, \sum_{n=1}^{\infty} p_n = \curlyvee_n u_n = p$$

and for any $n \in \mathbb{N}$ we have

$$p_n \leq u_n \leq \sum_{i=1}^{n} p_i \leq \sum_{i=1}^{n} i s_0 = \frac{n(n+1)}{2} s_0 \quad \blacksquare$$

Corollary2.1.5. *Any regular element p of $\mathcal{E}$ is a sum of sequence $(p_n)_n$ of $\mathcal{E}^r$ with p_n bounded for any $n \in \mathbb{N}$*

We choose a weak unit the constant function $\mathbb{1}$.

2.2 THE SPECIFIC RESTRICTION TO THE OPEN SETS OF REGULAR ELEMENTS

Lemma 2.2.1. *For any $p \in \mathcal{E}^r$ and any $\mathrm{G} \in \tau_d$ there exist p_{G} and p'_{G} in $\mathcal{E}^r$ such that*

$$p_{\mathrm{G}} + p'_{\mathrm{G}} = p, \qquad carr\, p'_{\mathrm{G}} \subset X \setminus \mathrm{G}$$

$$p_{\mathrm{G}} \curlywedge q = 0 \quad \forall q \in \mathcal{E}^r \ \ with\ \ carr\, q \subset X \setminus \mathrm{G}$$

Such a decomposition is unique. Moreover there exists a sequence $(p_n)_n$ in $\mathcal{E}^r$ specifically increasing and a sequence $(F_n)_n$ of τ_d −closed subsets of X such that

$$F_n \subset \overset{\circ}{F}_{n+1} \subset \mathrm{G} \quad \text{ and } \ \mathrm{R}^{F_n} p_n = p_n.$$

Proof. We consider the sequence $(\mathrm{G}_n)_n$ in τ_d defined by

$$\mathrm{G}_n = \{x \in \mathrm{G} / \ d(x, \mathrm{X} \setminus \mathrm{G}) > \tfrac{1}{n}\}$$

One can easily verify that $(\mathrm{G}_n)_n$ is increasing to G and $\overline{\mathrm{G}}_n \subset \mathrm{G}_{n+1}$ for all $n \in \mathbb{N}, n \geq 1$. (we denoted by $\overline{A}$ the closer of the set A with respect to the topology τ_d).

Using lemma 2.1.3. there exist p_n and p'_n in $\mathcal{E}^r$ such that

$$p = p_n + p'_n, \qquad R^{\mathrm{G}_n} p_n = p_n, \ \ R^{X \setminus \mathrm{G}_n} p'_n = p'_n$$

Since $p_n \geq R^{\overline{\mathrm{G}}_n} p_n \geq R^{\mathrm{G}_n} p_n = p_n$ we deduce, using lemma 2.1.3. , that $carr\, p_n \subset \overline{\mathrm{G}}_n \subset \mathrm{G}_{n+1}$. Analogously we deduce that $carr\, p'_n \subset X \setminus \mathrm{G}_n$.

The sequence $(p_n)_n$, respectively $(p'_n)_n$, is specifically increasing, respectively specifically decreasing. Indeed, from the equality

$$p_n + p'_n = p_{n+1} + p'_{n+1} = p$$

we deduce that there exist u, v in $\mathcal{E}^r$ such that

$$p_n = u + v, \qquad u \preccurlyeq p_{n+1}, \qquad v \preccurlyeq p'_{n+1}$$

Since $carr\, v \subset X \setminus \mathrm{G}_{n+1} \subset X \setminus \overline{\mathrm{G}}_n$ and $carr\, v \subset carr\, p_n \subset \overline{\mathrm{G}}_n$ we get $carr\, v = \emptyset$ and therefore $v = 0, p_n = u \preccurlyeq p_{n+1}$

We denote

$$p_{\mathrm{G}} := \curlyvee_n p_n = \bigvee_n p_n = \sup_n p_n$$
$$p'_{\mathrm{G}} := \curlywedge_n p'_n = \bigwedge_n p'_n = \inf_n p'_n$$

We have

$$p = p_{\mathrm{G}} + p'_{\mathrm{G}}, \qquad carr p'_{\mathrm{G}} \subset \bigcup_{n=1}^{\infty} p'_n \subset X \setminus \mathrm{G}$$

Let now $q \in \mathcal{E}^r$ be such that $carr\ q \subset X \setminus \mathrm{G}$
For any $n \in \mathbb{N}$ we have

$$carr\ (q \curlywedge p_n) \subset carr q \cap carr\ p_n \subset (X \setminus \mathrm{G}) \cap \overline{\mathrm{G}}_n = \emptyset$$

Hence $q \curlywedge p_n = 0$ for any $n \in \mathbb{N}$ and therefore

$$q \curlywedge\ p_{\mathrm{G}} = \curlyvee_n (q \curlywedge p_n) = 0$$

As for the uniqueness, we consider $t, t' \in \mathcal{E}^r$ such that $p = t + t', carr\ t' \subset X \setminus \mathrm{G}$ (or $R^{X \setminus \mathrm{G}} t' = t'$) and $t \curlywedge q = 0$ for any $q \in \mathcal{E}^r$ with $carr\ q \subset X \setminus \mathrm{G}$.

From the equality

$$p_{\mathrm{G}} + p'_{\mathrm{G}} = p = t + t'$$

and from the hypotheses we get

$$p_{\mathrm{G}} \curlywedge t' = 0, p_{\mathrm{G}} \preccurlyeq t;$$

$$p'_{\mathrm{G}} \curlywedge t = 0, t \preccurlyeq p_{\mathrm{G}}$$

and therefore $p_{\mathrm{G}} = t, p'_{\mathrm{G}} = t'$.
We finish the proof taking as F_n the set $\overline{\mathrm{G}}_n$ ■

2.3. CALCULUS WITH SPECIFIC RESTRICTION OF A REGULAR ELEMENT

In the sequel, for any element $p \in \mathcal{E}^r$, any closed subset F of X we shall denote by p_{F} the element p'_{G} where $\mathrm{G} = X \setminus \mathrm{F}$ from lemma 2.2.1. uniquely determined by the properties :

$p = p_{\mathrm{G}} + p_{\mathrm{F}}$; $p_{\mathrm{G}}, p_{\mathrm{F}} \in \mathcal{E}, carr\ p_{\mathrm{F}} \subset \mathrm{F}, p_{\mathrm{G}} \curlywedge q = 0$ for $q \in \mathcal{E}^r$ with $carr\ q \subset \mathrm{F}$.

Proposition 2.3.1.

a. *Let $p \in \mathcal{E}^r$ and $\mathrm{G}_1, \mathrm{G}_2$ be open subsets of X such that $\mathrm{G}_1 \subset \mathrm{G}_2$. We have*

$$p_{\mathrm{G}_1} \preccurlyeq p_{\mathrm{G}_2}$$

b. *If $\mathrm{F}_1, \mathrm{F}_2$ are closed subsets of X such that $\mathrm{F}_1 \subset \mathrm{F}_2$, we have*

$$p_{\mathrm{F}_1} \preccurlyeq p_{\mathrm{F}_2}$$

c. *For any sequence $(\mathrm{G}_n)_n$ of open sets of X (resp. any sequence $(\mathrm{F}_n)_n$ of closed subset of X) we have*

$$p_{\bigcup_1^\infty \mathrm{G}_n} = \curlyvee_n p_{\mathrm{G}_n} \ \ (resp. \, p_{\bigcap_1^\infty \mathrm{F}_n} = \curlywedge_n p_{\mathrm{F}_n})$$

d. *For any* $\mathrm{G} \in \tau_d$ *we have*

$$p_{\mathrm{G}} = \curlyvee \{q \in \mathcal{E}/q \preccurlyeq p, carr\ q \subset \mathrm{G}\} = \curlyvee \{p_{\mathrm{F}}/\mathrm{F} = \overline{\mathrm{F}} \subset \mathrm{G}\}$$

and there exists an increasing sequence $(\mathrm{F}_n)_n$ *of closed sets of* X, $\mathrm{F}_n \subset \mathrm{G}$ *such that the sequence* $(p_{\mathrm{F}_n})_n$ *is specifically increasing to* p_{G}

e. *For any closed set* F *of* X *we have*

$$p_{\mathrm{F}} = \curlywedge \{p_{\mathrm{G}}/\mathrm{G} \text{ open}, \mathrm{F} \subset \mathrm{G}\}$$

and there exists a decreasing sequence $(\mathrm{G}_n)_n$ *of open sets in* X *such that* $\mathrm{F} \subset \mathrm{G}_n$ *for any* $n \in \mathbb{N}$ *and such that the sequence* $(p_{\mathrm{G}_n})_n$ *is specifically decreasing to* p_{F}

f. *If* $\mathrm{G}_1, \mathrm{G}_2$ *are open and if* $\mathrm{F}_1, \mathrm{F}_2$ *are closed then we have*

$$p_{\mathrm{G}_1} \curlywedge p_{\mathrm{G}_2} = p_{\mathrm{G}_1 \cap \mathrm{G}_2}, \qquad p_{\mathrm{F}_1} \curlyvee p_{\mathrm{F}_2} = p_{\mathrm{F}_1 \cup \mathrm{F}_2}$$

Proof. a. From the relations

$$p = p_{\mathrm{G}_1} + p'_{\mathrm{G}_1} = p_{\mathrm{G}_2} + p'_{\mathrm{G}_2}$$

and using the fact that for any $q \in \mathcal{E}, q \preccurlyeq p$ and $carr\ q \subset X \setminus \mathrm{G}$, we have $p_{\mathrm{G}_1} \curlywedge q = 0$ (lemma 2.2.1)

we deduce that $p_{\mathrm{G}_1} \curlywedge p'_{\mathrm{G}_1} = 0$ and therefore $p_{\mathrm{G}_1} \preccurlyeq p_{\mathrm{G}_2}$

b. This assertion follows from a. because if $\mathrm{F}_1 \subset \mathrm{F}_1$ then taking $\mathrm{G}_1 = X \setminus \mathrm{F}_1$, $\mathrm{G}_2 = X \setminus \mathrm{F}_2$ we have $\mathrm{G}_2 \subset \mathrm{G}_1$ and therefore $p_{\mathrm{G}_2} \preccurlyeq p_{\mathrm{G}_1} \preccurlyeq p$. Hence we have

$$p - p_{\mathrm{G}_1} \preccurlyeq p - p_{\mathrm{G}_2} \text{ i.e. } p_{\mathrm{F}_1} \preccurlyeq p_{\mathrm{F}_2}$$

c. Since $\mathrm{G}_k \subset \bigcup_{n=1}^\infty \mathrm{G}_n$ for any $k \in \mathbb{N}$ we deduce $p_{\mathrm{G}_k} \preccurlyeq p_{\bigcup_1^\infty \mathrm{G}_n}$ and therefore $\curlyvee_n p_{\mathrm{G}_n} \preccurlyeq p_{\bigcup_1^\infty \mathrm{G}_n}$. On the other hand using the relation

$$p = p_{\mathrm{G}_n} + p'_{\mathrm{G}_n} \ \ \forall n \in \mathbb{N}$$

we deduce

$$p = \curlyvee_n p_{\mathrm{G}_n} + \curlywedge_n p'_{\mathrm{G}_n}$$

Since for any $k \in \mathbb{N}$ we have $carr\ \curlywedge_n p'_{\mathrm{G}_n} \subset carr\ p'_{\mathrm{G}_k} \subset X \setminus \mathrm{G}_k$ we deduce

$$carr\ \curlywedge_n p'_{\mathrm{G}_n} \subset \bigcap_{k=1}^\infty (X \setminus \mathrm{G}_k) = X \setminus (\bigcup_n \mathrm{G}_n)$$

Using again lemma 2.2.1. and the preceding considerations, from the relation

$$p_{\cup_n \mathrm{G}_n} \preccurlyeq p = \curlyvee_n p_{\mathrm{G}_n} + \curlywedge_n p'_{\mathrm{G}_n}$$

we deduce

$$p_{\cup_n \mathrm{G}_n} \curlywedge \left(\curlywedge_n p'_{\mathrm{G}_n}\right) = 0, \quad p_{\cup_n \mathrm{G}_n} \preccurlyeq \curlyvee_n p_{\mathrm{G}_n}$$

and finally $p_{\cup_n \mathrm{G}_n} = \curlyvee_n p_{\mathrm{G}_n}$

If $(F_n)_n$ is a sequence of closed subsets of X and if we denote $\mathrm{G}_n = X \setminus F_n$ we have

$$p_{\cup_n \mathrm{G}_n} + p_{\cap_n F_n} = p = p_{\mathrm{G}_n} + p_{\mathrm{F}_n} \quad \forall n \in \mathbb{N}$$

and therefore

$$p_{\cup_n \mathrm{G}_n} + p_{\cap_n F_n} = \curlyvee_n p_{\mathrm{G}_n} + \curlywedge_n p_{\mathrm{F}_n}$$

The equality $p_{\cap_n F_n} = \curlywedge_n p_{\mathrm{F}_n}$ follows now from the equality $p_{\cup_n \mathrm{G}_n} = \curlyvee_n p_{\mathrm{G}_n}$

d. let $q \in \mathcal{E}$, $q \preccurlyeq p$ with $carr\ q \subset \mathrm{G}$. From the relations

$$q \preccurlyeq p = p_{\mathrm{G}} + p'_{\mathrm{G}} \ \ and\ carr\ q \cap (X \setminus \mathrm{G}) = \emptyset, \mathrm{carr}\ p'_{\mathrm{G}} \subset X \setminus \mathrm{G}$$

we get $q \preccurlyeq p_{\mathrm{G}}$

From lemma 2.2.1. there exists a sequence $(p_n)_n$ in $\mathcal{E}^r$, a sequence $(\mathrm{F}_n)_n$of closed subsets of X , $\mathrm{F}_n \subset \mathrm{G}$ such that $carr\ p_n \subset \mathrm{F}_n$ for any $n \in \mathbb{N}$ and $\curlyvee_n p_n = p_{\mathrm{G}}$. Much more than this we have, using again the same lemma,

$$p_n \preccurlyeq p_{F_n} \preccurlyeq p_{\mathrm{G}}$$

and therefore $\curlyvee_n p_{F_n} = p_{\mathrm{G}}$.

e. Let $F = \bar{F}$ be a closed subset of X and $\mathrm{G} = X \setminus F$.
We consider as in the preceding point d. a sequence $(F_n)_n$ of closed subset of X, $F_n \subset \mathrm{G}$ for all $n \in \mathbb{N}$ and such that $\curlyvee_n p_{F_n} = p_{\mathrm{G}}$. From the relations

$$p_{F_n} + p_{X\setminus F_n} = p = p_{\mathrm{G}} + p_F \quad \forall n \in \mathbb{N}$$

we deduce

$$\curlyvee_n p_{F_n} + \curlywedge_n p_{X\setminus F_n} = p_{\mathrm{G}} + p_F, p_F = \curlywedge_n p_{\mathrm{G}_n} \ \ where\ \mathrm{G}_n \coloneqq X \setminus F_n$$

Obviously, For any $\mathrm{G} \in \tau_d$ with $F \subset \mathrm{G}$ we have, using the point d. $p_F \preccurlyeq p_{\mathrm{G}}$ and from the preceding considerations

$$p_F = \curlywedge_n p_{\mathrm{G}_n} = \curlywedge \{p_{\mathrm{G}}/\mathrm{G}\ \text{open}, F \subset \mathrm{G}\}.$$

f. Since $p_{\mathrm{G}_1 \cap \mathrm{G}_2} \preccurlyeq p_{\mathrm{G}_i}\ i = 1,2$ we get $p_{\mathrm{G}_1 \cap \mathrm{G}_2} \preccurlyeq p_{\mathrm{G}_1} \curlywedge p_{\mathrm{G}_2}$

Let now $(p_n)_n$ be a sequence in $\mathcal{E}^r$, $p_n \preccurlyeq p, carr\ p_n \subset \mathrm{G}_1$, such that $(p_n)_n$is specifically increasing to p_{G_1} and let $(q_n)_n$ be a sequence in $\mathcal{E}^r$, $q_n \preccurlyeq p, carr\ q_n \subset \mathrm{G}_2$ such that $(q_n)_n$ is specifically increasing to p_{G_2}. We have

$$p_{\mathrm{G}_1} \curlywedge p_{\mathrm{G}_2} = \curlyvee_n (p_n \curlywedge q_n)$$

Since for any $n \in \mathbb{N}$ we have $carr\ (p_n \curlywedge q_n) \subset \mathrm{G}_1 \cap \mathrm{G}_2$, using the point d., we have $p_n \curlywedge q_n \preccurlyeq p_{\mathrm{G}_1 \cap \mathrm{G}_2}$ and therefore

$$p_{\mathrm{G}_1} \curlywedge p_{\mathrm{G}_2} \preccurlyeq p_{\mathrm{G}_1 \cap \mathrm{G}_2}, \qquad p_{\mathrm{G}_1} \curlywedge p_{\mathrm{G}_2} = p_{\mathrm{G}_1 \cap \mathrm{G}_2}$$

If F_1, F_2 are two closed subsets of X, noting $\mathrm{G}_1 = X \setminus F_1, \mathrm{G}_2 = X \setminus F_2$ and using the first part of the point f. and the following relations

$$p_{\mathrm{G}_1} + p_{F_1} = p = p_{\mathrm{G}_2} + p_{F_2}$$

we get

$$p_{\mathrm{G}_1} \curlywedge p_{\mathrm{G}_2} + p_{F_1} \curlyvee p_{F_2} = p = p_{\mathrm{G}_1 \cap \mathrm{G}_2} + p_{F_1 \cup F_2}, p_{F_1} \curlyvee p_{F_2} = p_{F_1 \cup F_2} \qquad \blacksquare$$

Preposition 2.3.2.

a. *If $p, q \in \mathcal{E}^r$ and $G \in \tau_d$ respectively $F = \overline{F}$ then*

$$(p+q)_{\mathrm{G}} = p_{\mathrm{G}} + q_{\mathrm{G}}\ (respectively. (p+q)_F = p_F + q_F$$

b. *if $G_i \in \tau_d$* (resp. $F_i = \overline{F_i}$) *i=1,2 and $p \in \mathcal{E}^r$ then we have*

$$(p_{\mathrm{G}_1})_{\mathrm{G}_2} = p_{\mathrm{G}_1} \curlywedge p_{\mathrm{G}_2} = p_{\mathrm{G}_1 \cap \mathrm{G}_2},$$

$$(p_{F_1})_{F_2} = p_{F_1} \curlywedge p_{F_2} = p_{F_1 \cap F_2}, (p_{\mathrm{G}_1})_{F_1} = p_{\mathrm{G}_1} \curlywedge p_{F_1} = (p_{F_1})_{\mathrm{G}_1}$$

c. *If* $(p_n)_n$ is a sequence in $\mathcal{E}^r$ such that $\sum_n p_n \in \mathcal{E}^r$then for any $\mathrm{G} \in \tau_d$ *(resp. $F = \overline{F}$) we have*

$$(\textstyle\sum_{n=1}^{\infty} p_n)_{\mathrm{G}} = \sum_{n=1}^{\infty} p_{n_{\mathrm{G}}}\ (resp. (\sum_{n=1}^{\infty} p_n)_F = \sum_{n=1}^{\infty} p_{n_F}$$

d. *If a sequence $(p_n)_n$ is in $\mathcal{E}^r$ specifically increasing and dominated in $\mathcal{E}$ then for any* $\mathrm{G} \in \tau_d$ *(resp. $F = \overline{F}$) we have*

$$\curlyvee_n p_{n_{\mathrm{G}}} = (\curlyvee_n p_n)_{\mathrm{G}}\ (resp. \curlyvee_n p_{n_F} = (\curlyvee_n p_n)_F$$

e. *If a sequence $(p_n)_n$ is in $\mathcal{E}^r$ specifically decreasing then for any* $\mathrm{G} \in \tau_d$ *(resp. $F = \overline{F}$) we have*

$$\curlywedge_n p_{n_{\mathrm{G}}} = (\curlywedge_n p_n)_{\mathrm{G}}\ (resp. \curlywedge_n p_{n_F} = (\curlywedge_n p_n)_F$$

Proof. a. Let s, t be elements in $\mathcal{E}^r$ such that $s \preccurlyeq p, t \preccurlyeq q$ and

$$carr\ s \subset\ carr\ t\ \subset \mathrm{G}$$

Obviously we have

$$s + t \preccurlyeq p + q, carr(s + t) \subset \mathrm{G}, s + t \preccurlyeq (p + q)_{\mathrm{G}}$$

From the proposition 2.3.1 we deduce that

$$p_{\mathrm{G}} + q_{\mathrm{G}} \preccurlyeq (p + q)_{\mathrm{G}}$$

Let now $u \in \mathcal{E}, u \preccurlyeq p + q, carr\ u \subset \mathrm{G}$. Using Riesz decomposition property with respect to the specific order, we may choose $s, t \in \mathcal{E}$, such that

$$s \preccurlyeq p, t \preccurlyeq q, u = s + t$$

Since $carr\ s \subset carr\ u \subset \mathrm{G}, carr\ t \subset carr\ u \subset \mathrm{G}$ we get

$$s \preccurlyeq p_{\mathrm{G}}, t \preccurlyeq q_{\mathrm{G}}$$

and therefore, s and t being arbitrary

$$(p + q)_{\mathrm{G}} \preccurlyeq p_{\mathrm{G}} + q_{\mathrm{G}}\ , (p + q)_{\mathrm{G}} = p_{\mathrm{G}} + q_{\mathrm{G}}$$

If $F = \overline{F}$ we take $F = \mathrm{X} \setminus \mathrm{G}$ and we remark that that we have

$$\begin{gathered}(p + q)_{\mathrm{G}} + (p + q)_F = p + q = (p_{\mathrm{G}} + p_F) + (q_{\mathrm{G}} + q_F) \\ = (p_{\mathrm{G}} + q_{\mathrm{G}}) + (p_F + q_F)\end{gathered}$$

and therefore

$$(p + q)_F = p_F + q_F$$

b. Using the point a. we have successively

$$p_{\mathrm{G}_1} \preccurlyeq p, (p_{\mathrm{G}_1})_{\mathrm{G}_2} \preccurlyeq p_{\mathrm{G}_2}, (p_{\mathrm{G}_1})_{\mathrm{G}_2} \preccurlyeq p_{\mathrm{G}_1}, (p_{\mathrm{G}_1})_{\mathrm{G}_2} \preccurlyeq p_{\mathrm{G}_1} \curlywedge p_{\mathrm{G}_2}$$

Since $p_{\mathrm{G}_1} \curlywedge p_{\mathrm{G}_2} \preccurlyeq p_{\mathrm{G}_1}$and using a. we get

$$(p_{\mathrm{G}_1} \curlywedge p_{\mathrm{G}_2})_{\mathrm{G}_2} \preccurlyeq (p_{\mathrm{G}_1})_{\mathrm{G}_2}$$

But since $(p_{\mathrm{G}_2})_{\mathrm{G}_2} = p_{\mathrm{G}_2}$ we deduce that for any $q \in \mathcal{E}^r, q \preccurlyeq p_{\mathrm{G}_2}$ we have $q + q' = p_{\mathrm{G}_2}$ for some $q' \in \mathcal{E}^r$ and therefore using a. we get

$$q + q' = (q + q')_{\mathrm{G}} = q_{\mathrm{G}} + {q'}_{\mathrm{G}}\ , q_{\mathrm{G}} = q, {q'}_{\mathrm{G}} = q'$$

Particularly we have

$$(p_{\mathrm{G}_1} \curlywedge p_{\mathrm{G}_2})_{\mathrm{G}_2} = p_{\mathrm{G}_1} \curlywedge p_{\mathrm{G}_2}\ , p_{\mathrm{G}_1} \curlywedge p_{\mathrm{G}_2} \preccurlyeq (p_{\mathrm{G}_1})_{\mathrm{G}_2}, p_{\mathrm{G}_1} \curlywedge p_{\mathrm{G}_2} = (p_{\mathrm{G}_1})_{\mathrm{G}_2}$$

The relations $(p_{F_1})_{F_2} = p_{F_1} \curlywedge p_{F_2} = p_{F_1 \cap F_2}$ and $(p_{\mathrm{G}_1})_{F_1} = p_{\mathrm{G}_1} \curlywedge p_{F_1} = (p_{F_1})_{\mathrm{G}_1}$ may be similarly shown.

c. Since for any $k \in \mathbb{N}$ we have $\sum_{n=1}^{k} p_n \preccurlyeq \sum_{n=1}^{\infty} p_n$ we get $\left(\sum_{n=1}^{k} p_n\right)_{\mathrm{G}} \preccurlyeq (\sum_{n=1}^{\infty} p_n)_{\mathrm{G}}$,

$$\sum_{n=1}^{k} p_{n_{\mathrm{G}}} \preccurlyeq (\sum_{n=1}^{\infty} p_n)_{\mathrm{G}}, \sum_{n=1}^{\infty} p_{n_{\mathrm{G}}} \preccurlyeq (\sum_{n=1}^{\infty} p_n)_{\mathrm{G}}$$

On the other hand, for any $k \in \mathbb{N}$ we have

$$(\sum_{n=1}^{\infty} p_n)_{\mathrm{G}} = (\sum_{n=1}^{k} p_n)_{\mathrm{G}} + (\sum_{n=k+1}^{\infty} p_n)_{\mathrm{G}} = \sum_{n=1}^{k} p_{n_{\mathrm{G}}} + (\sum_{n=k+1}^{\infty} p_n)_{\mathrm{G}} ;$$

$$(\sum_{n=1}^{\infty} p_n)_{\mathrm{G}} \preccurlyeq \sum_{n=1}^{\infty} p_{n_{\mathrm{G}}} + \sum_{n=k+1}^{\infty} p_n$$

The last inequality holds for all $k \in \mathbb{N}$ and therefore

$$(\sum_{n=1}^{\infty} p_n)_{\mathrm{G}} \preccurlyeq \sum_{n=1}^{\infty} p_{n_{\mathrm{G}}} + \curlywedge_{k\in\mathbb{N}} (\sum_{n=k+1}^{\infty} p_n) = \sum_{n=1}^{\infty} p_{n_{\mathrm{G}}}$$

$$(\sum_{n=1}^{\infty} p_n)_{\mathrm{G}} = \sum_{n=1}^{\infty} p_{n_{\mathrm{G}}}$$

A similar arguments may be used for showing that

$$(\sum_{n=1}^{\infty} p_n)_F = \sum_{n=1}^{\infty} p_{n_F}$$

for any closed subset F of X.

The assertion d. follows from c. since $\curlyvee_1^{\infty} p_n = \sum_{n=1}^{\infty}(p_n - p_{n-1})$ where $p_0 = 0$

and therefore

$$(\curlyvee_1^{\infty} p_n)_{\mathrm{G}} = \sum_{n=1}^{\infty}(p_n - p_{n-1})_{\mathrm{G}} = \curlyvee_1^{\infty} (\sum_{k=1}^{n}(p_k - p_{k-1})_{\mathrm{G}}) = \curlyvee_1^{\infty} ((\sum_{k=1}^{n} p_k - p_{k-1})_{\mathrm{G}})$$

$$= \curlyvee_n p_{n_{\mathrm{G}}}$$

$$(\curlyvee_1^{\infty} p_n)_F = \sum_{n=1}^{\infty}(p_n - p_{n-1})_F = \curlyvee_1^{\infty} (\sum_{k=1}^{n}(p_k - p_{k-1})_F) = \curlyvee_1^{\infty} ((\sum_{k=1}^{n} p_k - p_{k-1})_F)$$

$$= \curlyvee_n p_{n_F}$$

f. If we denote $p = \curlywedge_{n=1}^{\infty} p_n$ and $q_n = p_1 - p_n$ for any $n \in \mathbb{N}$ then $(q_n)_n$ is a sequence in $\mathcal{E}^r$ specifically increasing to$p_1 - p$. Using the point d. we have successively

$$\curlyvee_n q_{n_{\mathrm{G}}} = (\curlyvee_n q_n)_{\mathrm{G}} \ , \curlyvee_n (p_{1_{\mathrm{G}}} - p_{n_{\mathrm{G}}}) = p_{1_{\mathrm{G}}} - p_{\mathrm{G}}, \curlywedge_n p_{n_{\mathrm{G}}} = p_{\mathrm{G}} = (\curlywedge_n p_n)_{\mathrm{G}}$$

$$\left(resp. \curlyvee_n q_{n_F} = \left(\curlyvee_n q_n\right)_F \ , \curlyvee_n (p_{1_F} - p_{n_F}) = p_{1_F} - p_{\mathrm{G}}, \curlywedge_n p_{n_F} = p_F = (\curlywedge_n p_n)_F\right) \ \blacksquare$$

2.4. Extension of specific restriction to the σ −algebra of measurable sets

Lemma 2.4.1. *let us denote by $\mathcal{A}(\tau_d)$ the algebra of subsets of X generated by τ_d. Then for any $A \in \mathcal{A}(\tau_d)$ there exists an increasing sequence $(F_n)_n$ of closed subsets, $F_n \subset A$ for any $n \in \mathbb{N}$ and there exists a decreasing sequence $(G_n)_n$ in τ_d, $A \subset G_n$ for any $n \in \mathbb{N}$, such that*

$$\curlyvee_n p_{F_n} = \sup_n p_{F_n} = \curlywedge_n p_{G_n} = \inf_n p_{G_n}$$

Proof. If we denote by $\mathcal{A}_0$ the set of all subsets A of X for which there exists an increasing sequence $(F_n)_n$ of closed subsets, $F_n \subset A$ for any $n \in \mathbb{N}$, and there exists a decreasing sequence $(D_n)_n$ in τ_d, $A \subset D_n$ for any $n \in \mathbb{N}$ such that

$$\curlyvee_n p_{F_n} = \sup_n p_{F_n} = \curlywedge_n p_{D_n} = \inf_n p_{D_n}$$

then $\mathcal{A}_0$ is an algebra of subsets of X. Indeed, we remark that if $A \in \mathcal{A}_0$ and the sequences $(F_n)_n$ and $(D_n)_n$ are as before then the sequences $(F'_n)_n$, $(D'_n)_n$ given by $F'_n = X \setminus D_n$, $D'_n = X \setminus F_n$ are such that $F'_n = \overline{F'_n} \subset X \setminus A \subset D'_n \in \tau_d$, the sequence $(F'_n)_n$ is increasing, the sequence $(D'_n)_n$ is decreasing and from the relations

$$p_{F_n} + p_{D'_n} = p = p_{F'_n} + p_{D_n} \quad \forall n \in \mathbb{N}$$

we obtain

$$\curlyvee_n p_{F_n} + \curlywedge_n p_{D'_n} = p = \curlyvee_n p_{F'_n} + \curlywedge_n p_{D_n}, \curlyvee_n p_{F_n} = \curlywedge_n p_{D'_n}$$

i.e. $X \setminus A \in \mathcal{A}_0$. We show now that for $A, B \in \mathcal{A}_0$ we have $A \cup B \in \mathcal{A}_0$.

Let $(F'_n)_n$, $(F''_n)_n$ two increasing sequences of closed subsets with $F'_n \subset A$, $F''_n \subset B$ for any $n \in \mathbb{N}$, let $(D'_n)_n$, $(D''_n)_n$ be two decreasing sequences in τ_d with $A \subset D'_n$, $B \subset D''_n$ for any $n \in \mathbb{N}$ and such that

$$\curlyvee_n p_{F'_n} = \curlywedge_n p_{D'_n}, \curlyvee_n p_{F''_n} = \curlywedge_n p_{D''_n}$$

Using proposition 2.3.1 c., f. we have successively

$$\left(\curlyvee_n p_{F'_n}\right) \curlyvee \left(\curlyvee_n p_{F''_n}\right) = \left(\curlywedge_n p_{D'_n}\right) \curlyvee \left(\curlywedge_n p_{D''_n}\right)$$

$$\curlyvee_n \left(p_{F'_n} \curlyvee p_{F''_n}\right) = \curlywedge_{i,j\in\mathbb{N}} \left(p_{D'_i} \curlyvee p_{D''_j}\right) = \curlywedge_n \left(p_{D'_n} \curlyvee p_{D''_n}\right),$$

$$\curlyvee_n p_{F'_n \cup F''_n} = \curlywedge_n p_{D'_n \cup D''_n}$$

The fact that the set $A \cup B$ belongs to $\mathcal{A}_0$ follows now from the definition of $\mathcal{A}_0$ and from the fact that the sequence $(F'_n \cup F''_n)_n$ of closed subsets of X is increasing $F'_n \cup F''_n \subset A \cup B$ for any $n \in \mathbb{N}$ and the sequence $(D'_n \cup D''_n)_n$ from τ_d is decreasing and $A \cup B \subset D'_n \cup D''_n$ for all $n \in \mathbb{N}$.

The fact that $\tau_d \subset \mathcal{A}_0$ follows from Proposition 2.3.1. d. Hence $\mathcal{A}(\tau_d) \subset \mathcal{A}_0$.

Lemma 2.4.2. *If for any subset* A of X we denote

$$\underline{p}(A) = \sup\{p_F/F = \bar{F} \subset A\}$$

$$\overline{p}(A) = \inf\{p_D/D \in \tau_d, A \subset D\}$$

where $p \in \mathcal{E}^r, p < \infty$ *we have*

a. $\underline{p}(A) \le \overline{p}(A) \quad \forall A \subset X$

b. $\overline{p}(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \overline{p}(A_n)$ *for any sequence* $(A_n)_n$ *of pairwise disjoint subsets* A_n *in* X

c. $\underline{p}(\bigcup_{n=1}^{\infty} A_n) \ge \sum_{n=1}^{\infty} \underline{p}(A_n)$ *for any sequence* $(A_n)_n$ *of pairwise disjoint subsets* A_n *in* X

d. $\underline{p}(A) + \overline{p}(X \setminus A) = p$ *for all* $A \subset X$

e. *If* $p_n \in \mathcal{E}^r$ *and* $p = \sum p_n$ *then* $\underline{p}(A) = \sum_{n=1}^{\infty} \underline{p_n}(A), \overline{p}(A) = \sum_{n=1}^{\infty} \overline{p}_n(A)$

Proof. This assertion follows from the definition of $\underline{p}, \overline{p}$ and using proposition 2.3.1 .i.e. $p_F \preccurlyeq p_D$ for any $F = \bar{F} \subset D \in \tau_d$

d. Let $F = \bar{F}$ be such that $F \subset A$. Obviously $X \setminus A \subset X \setminus F$ and therefore $p_F + p_{X \setminus F} = p$. Hence if we fix $D_0 \in \tau_d$ with $X \setminus A \subset D_0$ and we denote $F_0 = X \setminus D_0$ we have

$$p_{F_0} + p_{D_0} = p, \sup\{p_F/F = \bar{F} \subset A\} + p_{D_0} \ge p, \underline{p}(A) + p_{D_0} \ge p$$

and since D_0 is arbitrary we get $\underline{p}(A) + \overline{p}(X \setminus A) \ge p$.

If we fix $F_0 = \overline{F_0} \subset A$ we have

$$p_{F_0} + p_{X \setminus F_0} = p, p_{F_0} + \inf\{p_D/D \in \tau_d, X \setminus A \subset D\} \le p, p_{F_0} + \overline{p}(X \setminus A) \le p$$

Since F_0 is arbitrary we deduce

$$\underline{p}(A) + \overline{p}(X \setminus A) \le p,. \quad \underline{p}(A) + \overline{p}(X \setminus A) = p$$

b. Let x be a point in X, let $\epsilon \in \mathbb{R}, \epsilon > 0$ and for any G_n be an open set such that $A_n \subset G_n$ and $p_{G_n}(x) \le \overline{p}(A_n) + \frac{\epsilon}{2^n}$

Obviously, $\bigcup_n A_n \subset \bigcup_n G_n \in \tau_d$ and therefore using PROPOSITION 2.3.1. c.

$$\overline{p}(\bigcup_n A_n) \le p_{\bigcup_n G_n} = \curlyvee_n p_{G_n} = \curlyvee_n(\curlyvee_{k \le n} p_{G_k})$$

But for any $n \in \mathbb{N}$ we have

$$\curlyvee_{k \le n} p_{G_k} \preccurlyeq \sum_{k=1}^{n} p_{G_k}, (\curlyvee_{k \le n} p_{G_k})(x) \le \sum_{k=1}^{n} p_{G_k}(x) \le \sum_{k=1}^{\infty} p_{G_k}(x)$$

and therefore

$$p_{\bigcup_{n=1}^{\infty} G_n}(x) = \sup_n p_{\bigcup_{k=1}^{n} G_k}(x) = \sup_n \left(\curlyvee_{k\le n} p_{G_k}\right)(x) \le \sum_{k=1}^{\infty} p_{G_k}(x)$$

$$\le \sum_{n=1}^{\infty} \overline{p}(A_n)(x) + \epsilon$$

Hence we have

$$\overline{p}(\bigcup_n A_n)(x) \le \sum_n \overline{p}\,(A_n)(x) + \epsilon$$

The number ϵ being arbitrary we get

$$\overline{p}(\bigcup_n A_n) \le \sum_n \overline{p}\,(A_n), \overline{p}(\bigcup_n A_n) = \sum_n \overline{p}\,(A_n)$$

c. This assertion is equivalent with the inequality

$$\sum_{n=1}^{k} \underline{p}\,(A_n) \le \underline{p}(\bigcup_{n=1}^{\infty} A_n)\ \forall k \in \mathbb{N}$$

Let now $k \in \mathbb{N}$ and for any $n \le k$ let F_n be a closed subset of X, $F_n \subset A_n$. We remark now that if F', F'' are two disjoint closed subsets of X we have

$$p_{F' \cup F''} = p_{F'} + p_{F''}$$

Indeed, Using proposition 2.3.1 c. f. we have

$$p_{F'} + p_{F''} = p_{F'} \curlywedge p_{F''} + p_{F'} \curlyvee p_{F''} = p_{F' \cap F''} + p_{F' \cup F''} = p_{F' \cup F''}$$

since $F' \cap F'' = \emptyset$.

In our case the subsets $F_n, n \le k$ are pairwise disjoint since $A_n, n \le k$ are pairwise disjoint. Hence

$$\sum_{n=1}^{k} p_{F_n} = p_{\bigcup_{n=1}^{k} F_n} \le \underline{p}(\bigcup_{n=1}^{\infty} A_n)$$

Since the closed subset F_n was arbitrary with $F_n \subset A_n$. We conclude that we have

$$\sum_{n=1}^{k} \underline{p}\,(A_n) \le \underline{p}\left(\bigcup_{k=1}^{\infty} A_k\right), \sum_{n=1}^{\infty} \underline{p}\,(A_n) \le \underline{p}\left(\bigcup_{n=1}^{\infty} A_n\right)$$

The assertion e. Follows from the definition and using Proposition 2.3.2. c.

Theorem 2.4.3. *With the above notation we have*
a. The set $\mathcal{M}_p$ defined by

$$\mathcal{M}_p = \{A \subset X / \underline{p}(A) = \overline{p}(A)\}$$

is a σ $-$algebra of subsets of X and the map

$$A \to p_A := \underline{p}(A) = \overline{p}(A)$$

from $\mathcal{M}_p$ into the convex cone of positive real functions on X is countable additive.

b. If p, q are finite regular elements then $\mathcal{M}_{p+q} = \mathcal{M}_p \cap \mathcal{M}_q$ and for any $A \in \mathcal{M}_{p+q}$ we have

$$(p+q)_A = p_A + q_A$$

c. If $p_n \in \mathcal{E}$ are such that $p = \sum_{n=1}^{\infty} p_n$ then $\mathcal{M}_p = \bigcap_{n=1}^{\infty} \mathcal{M}_{p_n}$ and for any $A \in \mathcal{M}_p$ we have

$$p_A = \sum_{n=1}^{\infty} p_{n_A}$$

Proof. a. First we remark that $A \in \mathcal{M}_p$ if and only if $X \setminus A \in \mathcal{M}_p$ because we have

$$\underline{p}(X \setminus A) + \overline{p}(A) = p = \overline{p}(X \setminus A) + \underline{p}(A)$$

We show that if $A, B \in \mathcal{M}_p$ then $A \cup B \in \mathcal{M}_p$. Let $x \in X$ be arbitrary and for $\epsilon \in \mathbb{R},\ \epsilon > 0$ let F', F'' be closed subset of X, D', D'' open subsets of X such that

$$F' \subset A \subset D', F'' \subset B \subset D''$$

$$p_{D'}(x) - p_A(x) < \epsilon, p_A(x) < p_{F'}(x) + \epsilon, p_{D''}(x) - p_B(x) < \epsilon, p_B(x) < p_{F''}(x) + \epsilon$$

We have

$$p_{D' \cup D''}(x) - p_{F' \cup F''}(x) = (p_{D'} \curlyvee p_{D''} - p_{F'} \curlyvee p_{F''})(x)$$

$$\leq \left((p_{D'} - p_{F'}) + (p_{D''} - p_{F''})\right)(x) \leq 4\epsilon$$

And therefore

$$\overline{p}(A \cup B)(x) - \underline{p}(A \cup B)(x) \leq p_{D' \cup D''}(x) - p_{F' \cup F''}(x) < 4\epsilon$$

Hence ϵ being arbitrary we deduce $A \cup B \in \mathcal{M}_p$. From the preceding consideration it follows that $\mathcal{M}_p$ is an algebra of subsets of X. To finish the point *a.* we consider a sequence $(A_n)_n$ in$\mathcal{M}_p$ of pairwise disjoint subsets. We have

$$\overline{p}(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \overline{p}(A_n) = \sum_{n=1}^{\infty} \underline{p}(A_n) \leq \underline{p}(\bigcup_{n=1}^{\infty} A_n) \leq \overline{p}(\bigcup_{n=1}^{\infty} A_n)$$

$$\sum_{n=1}^{\infty} \overline{p}(A_n) = \underline{p}(\bigcup_{n=1}^{\infty} A_n) = \overline{p}(\bigcup_{n=1}^{\infty} A_n)$$

So $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}_p$ and $p_{\bigcup_{n=1}^{\infty} A_n} = \overline{p}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \overline{p}(A_n) = \sum_{n=1}^{\infty} p_{A_n}$

b. From the definition of the maps $\overline{p}, \underline{p}$ we get

$$\underline{p+q}(A) = \underline{p}(A) + \underline{q}(A), \overline{p+q}(A) = \overline{p}(A) + \overline{q}(A)$$

So we have

$$\overline{p+q}(A) - \underline{p+q}(A) = \left(\overline{p}(A) - \underline{p}(A)\right) + (\overline{q}(A) - \underline{q}(A))$$

for any subsets A of X and therefore

$$A \in \mathcal{M}_{p+q} \Leftrightarrow A \in \mathcal{M}_p \cap \mathcal{M}_q$$

Moreover we have obviously

$$A \in \mathcal{M}_{p+q} \Longrightarrow (p+q)_A = \overline{p+q}(A) = \overline{p}(A) + \overline{q}(A) = p_A + q_A$$

c. From the point b. We get $\mathcal{M}_p \subset \mathcal{M}_q$ whenever $p, q \in \mathcal{E}^r, q \preccurlyeq p$ and therefore

$$\mathcal{M}_p \subset \bigcap_{n=1}^{\infty} \mathcal{M}_{p_n}$$

Let now A be an element belonging to $\mathcal{M}_{p_n}$ for all $n \in \mathbb{N}$.
Using lemma 2.4.2., e. We have

$$\underline{p}(A) = \sum_{n=1}^{\infty} \underline{p_n}(A) = \sum_{n=1}^{\infty} \overline{p}_n(A) = \overline{p}(A)$$

Hence $A \in \mathcal{M}_p$ and moreover

$$p_A = \sum\nolimits_{n=1}^{\infty} p_{n_A}$$

Theorem 2.4.4. *If $p \in \mathcal{E}^r, p < \infty$ then the σ –algebra $\mathcal{B}$ (generated by τ_d) is included in $\mathcal{M}_p$ and for any $A \in \mathcal{B}$ the element p_A belongs to $\mathcal{E}^r$ and we have*

$$p_A + p_{X\setminus A} = p, p_A = \sup\{p_F / F = \overline{F}, F \subset A\} = \inf\{p_G / G \in \tau_d, A \subset G\}$$

Proof. First we show that the set $\mathcal{M}_p^0$ defined by

$$\mathcal{M}_p^0 = \{A \in \mathcal{M}_p / p_A \in \mathcal{E}^r, p_{X\setminus A} \in \mathcal{E}^r\}$$

is a monotone class of subsets of X. Indeed, let $(A_n)_n$ be an increasing sequence in $\mathcal{M}_p^0$. Since $p_{A_n} \in \mathcal{E}^r$ and the sequence $(p_{A_n})_n$ increases to $p_{\cup_{n=1}^{\infty} A_n}$ it follows that $p_{\cup_{n=1}^{\infty} A_n} \in \mathcal{E}^r$. On the other hand the sequence $(p_{X\setminus A_n})_n$ of excessive functions decreases to $p_{X\setminus \cup_n A_n}$. Hence $p_{X\setminus \cup_n A_n}$ belongs to $\mathcal{S}_v$. But for any $n \in \mathbb{N}$ we have

$$p_{A_n} + p_{X\setminus A_n} = p$$

and therefore

$$p_{\cup_{n=1}^{\infty} A_n} + p_{X\setminus \cup_{n=1}^{\infty} A_n} = p, p_{X\setminus \cup_{n=1}^{\infty} A_n} \in \mathcal{E}^r,\ \cup_{n=1}^{\infty} A_n \in \mathcal{M}_p^0$$

Using now theorem 2.4.3. we deduce that the algebra $\mathcal{A}(\tau_d)$ generated by τ_d is a part of $\mathcal{M}_p^0$. But $\mathcal{M}_p^0$ being a monotone class of subsets of X we deduce that the σ −algebra generated by τ_d (or $\mathcal{A}(\tau_d)$) is included in $\mathcal{M}_p^0$.

Corollary 2.4.5. *a. If* $(p_n)_n$ *is a sequence in* $\mathcal{E}^r$ *such that the function* $p := \sum_n p_n$ *is finite. Then for any* $A \in \mathcal{B}$ *we have*

$$p_A = \sum_{n=1}^{\infty} p_{n_A}$$

b. For any A, B *in* $\mathcal{B}$ *and any* $p \in \mathcal{E}^r, p < \infty$ *we have*

$$(p_A)_B = (p_B)_A = p_{A\cap B} = p_A \curlywedge p_B$$

Proof. The point a. was already shown. For the assertion b. We start by showing that for any $A \in \mathcal{B}$ we have

$$p_A = \curlyvee \{p_F / F = \overline{F}, F \subset A\} = \curlywedge \{p_G / G \in \tau_d, A \subset G\}$$

Indeed, for any $F = \overline{F} \subset A$ we have $p_F + p_{A\setminus F} = p_A$ and therefore $p_F \preccurlyeq p_A$. Let now $q \in \mathcal{E}$ a specific majorant of p_F for any $F = \overline{F} \subset A$. We shall have $p_F \preccurlyeq p_A \curlywedge q \preccurlyeq p_A$.
But

$$\sup_{F=\overline{F}\subset A} p_F = \underline{p}\,(A) = p_A$$

by the definition of $\underline{p}(A)$ and p_A , and therefore we get

$$p_A \leq p_A \curlywedge q \leq p_A, p_A \curlywedge q = p_A, p_A \preccurlyeq q$$

Now for any $A, B \in \mathcal{B}$ we have

$$p_A \preccurlyeq p, (p_A)_B \preccurlyeq p_B, (p_A)_B \preccurlyeq p_A, (p_A)_B \preccurlyeq p_A \curlywedge p_B$$

Let F', F'' be two arbitrary closed subsets of X such that $F' \subset A, F'' \subset B$. We have $F' \cap F' \subset A \cap B$ and therefore

$$p_{F'} \curlywedge p_{F''} = p_{F'\cap F'}$$

On the other hand if $F = \overline{F} \subset A \cap B$ we have

$$p_F \preccurlyeq p_A, p_F = (p_F)_F \preccurlyeq (p_A)_F \preccurlyeq (p_A)_B$$

and therefore

$$p_{F'} \curlywedge p_{F''} \preccurlyeq (p_A)_B, \curlyvee_{\substack{F'\subset A\\ F''\subset B}} (p_{F'} \curlywedge p_{F''}) \preccurlyeq (p_A)_B$$

$$p_A \curlywedge p_B = \Big(\curlyvee_{F'\subset A} p_{F'}\Big) \curlywedge \Big(\curlyvee_{F''\subset B} p_{F''}\Big) = \curlyvee (p_{F'} \curlywedge p_{F''}) \preccurlyeq (p_A)_B$$

Hence

$$p_A \curlywedge p_B = (p_A)_B = (p_B)_A$$

The relation $p_{A\cap B} \preccurlyeq p_A \curlywedge p_B$ is obvious .

If $F' = \overline{F'} \subset A, F'' = \overline{F''} \subset B$ we have $F' \cap F'' \subset A \cap B$ and therefore $p_{F'\cap F''} \preccurlyeq p_{A\cap B}, p_{F'} \curlywedge p_{F''} \preccurlyeq p_{A\cap B}$

Passing now to the supremum with respect to the specific order, we get

$$\left(\curlyvee_{F'\subset A} p_{F'}\right) \curlywedge \left(\curlyvee_{F''\subset B} p_{F''}\right) \preccurlyeq p_{A\cap B} , p_A \curlywedge p_B \preccurlyeq p_{A\cap B}, p_A \curlywedge p_B = p_{A\cap B}$$

2.5. THE SPECIFIC MULTIPLICATION WITH POSITIVE BOREL FUNCTIONS OF REGULAR ELEMENT

Theorem 2.5.1. *Let $p \in \mathcal{E}^r, p < \infty$ and let f be a positive $\mathcal{B}$ −measurable and bounded function. For any $x \in X$ we consider the finite positive measure p^x on $\mathcal{B}$* given by

$$p^x(A) = p_A(x)$$

The positive function $f.p$ on X defined by

$$(f.p)(x) = \int f dp^x$$

belongs to $\mathcal{E}^r$ and the map $f \longmapsto f.p$ defined on $pb\mathcal{B}$ with values in $\mathcal{E}^r$ is a kernel with the complete maximum principle and $1.p = 1$.

Any element $s \in \mathcal{E}$ is a dominant function with respect to this kernel.

Moreover, if $p = \sum_n p_n$, $p_n \in \mathcal{E}$ we have

$$f.p = \sum_n f.p_n \quad \forall f \in pb\mathcal{B}$$
$$f(g.p) = (fg).p \quad \forall f,g \in pb\mathcal{B}$$

Proof. let f be a positive, bounded $\mathcal{B}$ −measurable function on X. We consider an increasing sequence $(f_n)_n$ of positive $\mathcal{B}$ −measurable functions of the form

$$f_n = \sum_{j=1}^{k_n} \alpha_j^n \mathbb{1}_{A_j^n}, \alpha_j^n \in \mathbb{R}_+, A_j^n \in \mathcal{B}$$

which increases to f. The functions $f_n.p$ given by

$$f_n.p(x) = \int f_n dp^x = \sum_{j=1}^{k_n} \alpha_j^n p_{A_j^n}(x)$$

are element of $\mathcal{E}^r$ and this sequence increases to

$$f.p(x) := \lim_n \int f_n dp^x = \sup_n f_n . p(x)$$

And therefore $f.p$ is a regular element of $\mathcal{E}$. We mention that f being dominated by the constant function $\|f\|$ we have

$$f.p \preccurlyeq M.p$$

Let s be an element of $\mathcal{E}$ such that $f.p \leq s$ on the $[f > 0]$. Certainly, using the above notations, we have

$$s \geq f_n.p \text{ on } [f_n > 0],\ s \geq \sum_{j=1}^{k_n} \alpha_j^n p_{A_j^n}(x) \ \ on\ \bigcup_j A_j^n$$

and therefore the function $s' = \min(s, \sum_{j=1}^{k_n} \alpha_j^n p_{A_j^n})$ belongs to $\mathcal{E}$ and we have and we have $s' = \sum_{j=1}^{k_n} \alpha_j^n p_{A_j^n} \quad on\ \bigcup_j A_j^n$.

Since s' is dominated by $\sum_{j=1}^{k_n} \alpha_j^n p_{A_j^n}$ we decompose s' under the form

$$s' = \sum_j s_j', s_j' \leq \alpha_j^n p_{A_j^n} \quad on\ X, \quad \forall j = 1,2, \dots, k_n$$

From the preceding considerations we have $s_j' = \alpha_j^n p_{A_j^n}\ \ on\ \ A_j^n$. If we consider a closed subset F of A_j^n we have

$$\frac{1}{\alpha_j^n} s_j' \geq p_F \ \ on\ F$$

and therefore from lemma 2.2.1 we get $\frac{1}{\alpha_j^n} s_j' \geq p_F\ \ on\ X$. Hence F being arbitrary we deduce $s_j' \geq \alpha_j^n p_F, s_j' \geq \alpha_j^n p_{A_j^n}\ \ on\ X, s_j' = \alpha_j^n p_{A_j^n}\ \ on\ X,$

$$s' \geq s = \sum_{j=1}^{k_n} \alpha_j^n p_{A_j^n} = f_n.p \ \ on\ X$$

The number $n \in \mathbb{N}$ being arbitrary we have $s \geq f_n.p \quad on\ X.$

The fact that

$$f.(p + q) = f.p + f.q \quad \forall p, q \in \mathcal{E}^r, f \in pb\mathcal{B}$$

Follows from the equality $(p + q)^x = p^x + q^x$

If $p_n \in \mathcal{E}$ and $\sum_n p_n \in \mathcal{E}$, noting $\mathrm{P}_n = \sum_{j=1}^n p_j$, $\mathrm{Q}_n = \sum_{j>n} p_j$

we have

$$f.\left(\sum_1^\infty p_n\right) = f.\mathrm{P}_n + f.\mathrm{Q}_n \geq \sum_{j=1}^n f.p_j, \sum_{j=1}^\infty f.p_j \leq f.\left(\sum_1^\infty p_n\right)$$

Since $\wedge_n \mathrm{Q}_n = \wedge_n \mathrm{Q}_n = 0$ we have

$$f.\left(\sum_1^\infty p_n\right) \leq f.\mathrm{P}_n + \|f\|.\mathrm{Q}_n \ \ \forall n \in \mathbb{N},$$

$$f.\left(\sum_{1}^{\infty} p_n\right) \leq \sup_n f.\mathrm{P}_n + \|f\|.\bigwedge_n \mathrm{Q}_n = \sup_n f.p_n \leq \sum_{j=1}^{\infty} f.p_j$$

We finish the proof by the following remark

If A and B are in $\mathcal{B}$ w have

$$\mathbb{1}_A(\mathbb{1}_B.p) = (\mathbb{1}_A\mathbb{1}_B).p = \mathbb{1}_{A\cap B}.p$$

(see corollary 2.4.5) ■

CHAPTER 3

DARBOUX-STIELTJES CALCULUS ON BANACH SPACES

The purpose of this chapter is to extend in the vector case the study of Darboux-Stieltjes integrability as it was initiated in [18] , then studied by I. Bucur [13], [14] and to give some new results. Among them we note: the symmetry principle, the formula of integration by parts, the extension integrability principle, a convergence theorem. At the end we give an application of this type of integration in describing the duality in a particular case of $H-cones$.

3.1. PRELIMINARIES AND FIRST RESULTS

For a given interval $[a,b]$ of $\mathbb{R}$ we denote by $D[a,b]$ the set of all divisions $d = (a = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n = b)$ of this interval. The norm of this division is denoted by $\nu(d)$, i.e. $\nu(d) = max\,\{x_{i+1} - x_i |\, i = 0,1,\ldots,n-1\}$.

By *intermediary system* of d we shall understand a new divison ξ of $[a,b]$,

$$\xi = (a = \xi_0 \leq \xi_1 \leq \xi_2 \leq \ldots \leq \xi_n \leq \xi_{n+1} = b)$$

where $\xi_1 \in [x_0,x_1], \xi_2 \in [x_1,x_2], \ldots \xi_n \in [x_{n-1},x_n]$. The set of all intermediary systems of d will be noted by $\mathcal{L}$ *(d)*. Obviously we have $\xi \in \mathcal{L}(d) \Rightarrow d \in \mathcal{L}(\xi)$ and $\nu(\xi) \leq 2\nu(d), \nu(d) \leq 2\nu(\xi)$.

Let *X* be a Banach space over $\mathbb{R}$, let $f : [a,b] \to X$, $g : [a,b] \to \mathbb{R}$ be two arbitrary bounded functions. For $d \in D[a,b]$, $\xi \in \mathcal{L}$ *(d)* as below we denote by $\sigma(f,g;d,\xi)$ the element of X given by

$$\sigma(f,g;d,\xi) = \sum_{i=1}^{n} f(\xi_i)\,(g(x_i) - g(x_{i-1}))$$

The following *reciprocity formula* may be easily verified

$$\sigma(f,g;d,\xi) = f(b)g(b) - f(a)g(a) - \sigma(f,g;\ \xi,d)$$

If $d_1, d_2 \in D[a,b]$ we say that d_2 *is finer than* d_1 and we write $d_1 \leq d_2$ if for any element of d_1 belongs to d_2

Definition 3.1.1. We say that the function f is *Riemann-Stieltjes integrable* with respect to the function g if there exists an element $I \in X$ such that for any $\epsilon > 0$ there exists $\eta_\epsilon > 0$ such that

$$\|\sigma(f, g; d, \xi) - I\| < \epsilon, \forall d \in D[a, b] \text{ with } v(d) \leq \eta_\epsilon, \forall \xi \in \mathcal{L}(d)$$

The element I of the Banach space X is uniquely determined and it is called the *Riemann-Stieltjes integral* of f with respect to g.

We write $f \in RS(g)$ instead of "f *is Riemann-Stieltjes integrable with respect to g*" and

$$I = (RS)\int_a^b f\, dg = RS\int_a^b f\, dg$$

It is well known that the function f is Riemann-Stieltjes integrable with respect to g if and only if for any sequence $(d_n)_n$ in D[a, b] with $\lim_{n\to\infty} v(d_n) = 0$ and any $\xi^n \in \mathcal{L}(d_n)$, the sequence $(\sigma(f, g; d_n, \xi^n))_n$ is convergent in X. One can see that the above limit does not depend on the sequences $(d_n)_n, (\xi^n)_n, \xi^n \in \mathcal{L}(d_n)$ with $\lim_{n\to\infty} v(d_n) = 0$.

Also we have the following

Proposition 3.1.2. (*Cauchy criterion*). *One has* $f \in RS(g)$ *if and only if* $\epsilon > 0$ *there exists* $\eta_\epsilon > 0$ *such that*

$$\|\sigma(f, g; d', \xi') - \sigma(f, g; d'', \xi'')\| < \epsilon,$$

for all d', d'' *with* $v(d') < \eta_\epsilon, v(d'') < \eta_\epsilon, \xi' \in \mathcal{L}(d'), \xi'' \in \mathcal{L}(d'')$.

As in the real case one can show that the set $RS(g)$ is a linear vector space over $\mathbb{R}$ and we have: $\alpha, \beta \in \mathbb{R}, f_1, f_2 \in RS(g) \Longrightarrow \alpha f_1 + \beta f_2 \in RS(g)$ and

$$(RS)\int_a^b (\alpha f_1 + \beta f_2)\, dg = \alpha (RS)\int_a^b f_1\, dg + \beta (RS)\int_a^b f_2\, dg.$$

Using the above reciprocity formula, we have $f \in RS(g) \Longrightarrow g \in RS(f)$ and

$$(RS)\int_a^b f\, dg = f(b)g(b) - f(a)g(a) - (RS)\int_a^b g df$$

and moreover, if $f \in RS(g_1) \cap RS(g_2)$ and $\alpha, \beta \in \mathbb{R}$ we have:

$$f \in RS(\alpha g_1 + \beta g_2)$$

and

$$(RS)\int_a^b f d(\alpha g_1 + \beta g_2) = \alpha (RS)\int_a^b f\, dg_1 + \beta (RS)\int_a^b f\, dg_1.$$

Definition 3.1.3. The function f is called *Darboux-Sieltjes integrale with respect to* g *if there exists* $I \in X$ *and for any* $\epsilon > 0$ *there exists* $d_\epsilon \in D[a, b]$ *such that*

$$\|\sigma(f, g; d, \xi) - I\| \leq \epsilon, \forall d \in D[a, b], d_\epsilon \leq d, \forall \xi \in \mathcal{L}(d).$$

It is not difficult to show that the element $I \in X$ in the above definition is uniquely determined and it will be called the *Darboux-Stieltjes integral of* f *with respect to* g.

We write $f \in DS(g)$ instead of " the function f is Darboux-Stieltjes integrable with respect to g " and we denote the element I as follows

$$I = (DS)\int_a^b f\,dg \;\; or \; I = \int_a^b f\,dg$$

The following assertion generalizes a well-known Riemann-Stieltjes integrability criterion using sequences of divisions.

Proposition 3.1.4. *The function* f is a *Darboux-Stieltjes integrable with respect to* g *if and only if there exists a sequence* $(d_n)_n$ *in* $D[a,b]$*such that for any sequence* $(d'_n)_n$ *in* $D[a,b]$, *with* $d_n \leq d'_n$ ($\forall n \in \mathbb{N}$) *and any* $\xi'_n \in \mathcal{L}(d'_n)$ ($\forall n \in \mathbb{N}$) *the sequence* $(\sigma(f,g;d'_n,\xi'_n))_n$ *converges in the Banach space* X.

Proof. We suppose that $f \in DS(g)$. Just from the definition we deduce that there exists $I \in X$ such that for any $n \in \mathbb{N}^*$, there exists $d_n \in D[a,b]$ for which we have

$$\|\sigma(f,g;d'_n,\xi'_n) - I\| < \tfrac{1}{n}, \forall d'_n \in D[a,b], d_n \leq d'_n, \forall \xi'_n \in \mathcal{L}(d'_n).$$

Hence, we deduce

$$\lim_{n\to\infty} \sigma(f,g;d'_n,\xi'_n) = I.$$

Conversely, we suppose the existence of a sequence $(d_n)_n$ in $D[a,b]$ such that for any sequence $(d'_n)_n$ in $D[a,b]$, with $d_n \leq d'_n$, for any $\forall n \in \mathbb{N}$ and any $\xi'_n \in \mathcal{L}(d'_n)$ the sequence $(\sigma(f,g;d'_n,\xi'_n))_n$ converges in X. Using a mixing procedure we deduce that that the element $\lim_{n\to\infty}\sigma(f,g;d'_n,\xi'_n)$ of X does not depend on the above sequences $(d'_n)_n$ and $(\xi'_n)_n$. We denote by I this limit and we show that for any $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$\|\sigma(f,g;d',\xi') - I\| < \epsilon, \forall d' \in D[a,b], d' \leq d_{n_\epsilon}, \forall \xi' \in \mathcal{L}(d')$$

In the contrary case ther exists $\epsilon_0 > 0$ such that for any $n \in \mathbb{N}$ there exists $d'_n \in D[a,b]$, $d_n \leq d'_n$ and $\xi'_n \in \mathcal{L}(d'_n)$ such that

$$\|\sigma(f,g;d',\xi') - I\| \geq \epsilon_0.$$

The contradiction we have arrived shows that $f \in DS(g)$ ■

The following assertion, the Cauchy criterion, is almost obvious:

Proposition 3.1.5. *The function* f *is Darboux-Stieltjes integrable with respect to* g *if and only if for any* $\epsilon > 0$ *there exists a division* d_ϵ *of* $[a,b]$ *such that*

$$\|\sigma(f,g;d',\xi') - \sigma(f,g;d'',\xi'')\| < \epsilon$$

for any $d', d'' \in D[a,b]$, $d_\epsilon \leq d', d_\epsilon \leq d''$ *and for any* $\xi' \in \mathcal{L}(d'), \xi'' \in \mathcal{L}(d'')$.

Remark 3.1.6. It is easy to see that for $X = \mathbb{R}$ and g an increasing function on $[a, b]$ the fact that $f \in DS(g)$ is equivalent with the relation

$$\int_{\underline{a}}^{b} f\, dg = \int_{a}^{\bar{b}} f\, dg$$

where $\int_{\underline{a}}^{b} f\, dg$ (respectively $\int_{a}^{\bar{b}} f\, dg$) means the lower (respectively upper) Darboux-Stieltjes integral of f with respect to g; that is we get the well known classical situation.

3.2. RELATION BETWEEN *RS* AND *DS* INTEGRABILITY

The function f and g will be as before. If the function f is Riemann-Stieltjes integrable with respect to g then we consider an arbitrary sequence $(d_n)_n$in $D[a, b]$ such that $\lim_{n\to\infty} \nu(d_n) = 0$.

If we consider another sequence $(d'_n)_n$ in $D[a, b]$ with $d_n \leq d'_n$ for any $n \in \mathbb{N}$ then we have $\nu(d'_n) \leq \nu(d_n)$ and therefore $\lim_{n\to\infty} \nu(d'_n) = 0$. In this case we have

$$\lim_{n\to\infty} \sigma(f, g; d'_n, \xi'_n) = (RS)\int_{a}^{b} f\, dg, \qquad \forall \xi'_n \in \mathcal{L}(d'_n)$$

and therefore, using proposition 3.1.4 we deduce $f \in DS(g)$.

Hence we have the following assertion

Proposition 3.2.1. if $f \in RS(g)$ then $f \in DS(g)$ and

$$(DS)\int_a^b f\, dg = (RS)\int_a^b f\, dg.$$

Remark 3.2.2. The converse of thee above proposition is not always true. Indeed, we consider an element $y \in X, y \neq 0_X$ and the functions $f: [0,2] \to X$, $f: [0,2] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} y, & if\ 1 \leq x \leq 2 \\ 0_X, & if\ 0 \leq x \leq 1 \end{cases}, g(x) = \begin{cases} 1, & if\ 1 \leq x \leq 2 \\ 0, & if\ 0 \leq x \leq 1 \end{cases}$$

Let $d_0 \in D[0, 2], d_0 = (0 = x_0 < x_1 < x_2 = 2)$ be such that $x_1 = 1$ and let $d' \in D[0, 2]$, $d' \geq d_0$ be of the form

$$d' = (0 = x'_0 < x'_1 < x'_2 < \cdots < x'_k < 1 < x'_{k+2} < \cdots x'_n = 2)$$

If we consider

$$\xi' \in \mathcal{L}(d'), \xi' = (0 = \xi'_0 \leq \xi'_1 \leq \xi'_2 \leq \cdots \leq \xi'_k \leq \cdots \leq \xi'_{n+1} = 2)$$

where

$\sigma(f, g; d', \xi') = \sum_{i=1}^{n} f(\xi'_i)(g(x'_i) - g(x'_{i-1})) = f(\xi'_{k+2})\big(g(x'_{k+2}) - g(1)\big) = y,$

$\|\sigma(f, g; d', \xi') - y\| = 0.$

From the definition we deduce $f \in DS(g)$ and (DS) $(DS)\int_a^b f\,dg = y$. On the other hand, if we consider an arbitrary division d of [0, 2],

$$d = (0 = x_0 < x_1 < x_2 < \cdots < x_p < x_{p+1} < \cdots < x_m = 2$$

such that $x_p < 1 < x_{p+1}$ and $\xi' \in \mathcal{L}(d), \xi'' \in \mathcal{L}(d)$,

$$\xi' = (0 = \xi'_0 \le \xi'_1 \le \xi'_2 \le \cdots \le \xi'_{m+1} = 2), \xi'_i \in [x_{i-1}, x_i], i \in \overline{1,m},$$

$$\xi'' = (0 = \xi''_0 \le \xi''_1 \le \xi''_2 \le \cdots \le \xi''_{m+1} = 2), \xi''_i \in [x_{i-1}, x_i], i \in \overline{1,m}$$

and $\xi'_{p+1} \in (x_p, 1)$, $\xi''_{p+1} \in (1, x_{p+1})$, we have

$$\sigma(f,g;d,\xi') = 0_X, \sigma(f,g;d,\xi'') = y, \qquad y \neq 0_X.$$

Using now the Cauchy criterion of Riemann-Stieltjes integrability we deduce that f is not Riemann-Stieltjes integrable with respect to g.

The non-(RS)-integrability in our previous remark is an immediate consequence of the next result. The interested reader can easily find more examples using our technique. The following statement shows how far is Riemann-Stieltjes integrability from the Darboux- Stieltjes integrability

Proposition 3.2.3 a. If the functions f and g have a common point of discontinuity on the left hand side (or on the right hand side) then the function f is not Darboux-Stieltjes integrable with respect to g.

b. If the functions f and g have a common point of discontinuity then the function f is not Riemann-Stieltjes integrable with respect to g.

Proof. a. We suppose that f and g are discontinuous on the left at the point $c \in (a,b]$. In this case there exists $r' > 0, r'' > 0$ and two sequences $(x'_n)_n$, $(x''_n)_n$which increase to c and such that for any $n \in \mathbb{N}$ we have

$$x'_n < x''_n < x'_{n+1} < c, \|f(x'_n) - f(c)\| > r', \|g(x''_n) - g(c)\| > r''.$$

Let now d be an arbitray division of $[a,b]$,

$$d = \left(a = x_0 < x_1 < \cdots < x_{k_0} < \cdots < x_n = b\right),$$

such that $x_{k_0} = c$. For n sufficiently large we have

$$x_{k_0-1} < x''_n < x'_{n+1} < c$$

and we consider the division d' of $[a,b]$ obtained from d adding the point x''_{n_0}

with $x_{k_0-1} < x''_{n_0} < c$. We consider now ξ, ξ'in $\mathcal{L}(d')$ which differ between them

only by the points $\xi_{k_0} = x'_{n_0+1}, \xi'_{k_0} = c$ of the interval $[x''_{n_0}, c]$of the division d'.

We have

$$\sigma(f,g;d',\xi)-\sigma(f,g;d',\xi')=f\left(x'_{n_0+1}\right)\left(g\left(x''_{n_0}\right)-g(c)\right)-f(c)(g\left(x''_{n_0}\right)-g(c)),$$

$$\|\sigma(f,g;d',\xi)-\sigma(f,g;d',\xi')\|=\left\|f\left(x'_{n_0+1}\right)-f(c)\right\|.\left|g\left(x''_{n_0}\right)-g(c)\right|>r'.r''.$$

Now, using proposition 3.1.5. and the fact that the division d of $[a,b]$ was arbitrary we deduce that the function f is not Darboux-Stieltjes integrable with respect to the function g.

An analogous treatement may be done for the case where f and g are discontinuous on the right at a point $c\in[a,b)$.

b. The function f and g are both discontinuous on the same side of a point $c\in[a,b]$; this is a trivial consequence of the assertion a. So, let $c\in(a,b)$ such that f is discontinuous on the left but it is continuous on the right at the point c whereas the function g is continuous on the left, but it is discontinuous on the right at the point c. In this case there exists $r'>0, r''>0$ and there exist two sequences: $(x'_n)_n$ strictly increasing to c and $(x''_n)_n$ strictly decreasing to c such that we have

$$\|f(x'_n)-f(c)\|>r', \|g(x''_n)-g(c)\|>r'', \forall n\in\mathbb{N}.$$

Let now $d\in D[a,b]$ be an arbitrary division such that c is not a point of d

$$d=\left(a=x_0<x_1<\cdots<x_{k_0}<x_{k_0+1}<\cdots<x_m=b\right), x_{k_0}<c<x_{k_0+1}.$$

For n_0 sufficiently large we have $x_{k_0}<x'_{n_0}<c<x''_{n_0}<x_{k_0+1}$. We add to the division d the point x'_n, x''_n with $n\geq n_0$ and we denote by $d_n$this new division of $[a,b]$. Further we consider ξ',ξ''in $\mathcal{L}(d_n)$ which differ between them only intermediary point ξ'_n, respectively ξ''_1 in the interval $[x'_n,x''_n]$, namely $\xi'_n=x'_n$, $\xi''=x''_n$. We shall have

$$\sigma(f,g;d_n,\xi'')-\sigma(f,g;d_n,\xi')=\left(f(\xi'_n)-f(\xi''_n)\right).\left(g(x'_n)-g(x''_n)\right)=$$

$$=\left(f(x'_n)-f(x''_n)\right).\left(g(x'_n)-g(x''_n)\right).$$

Since f is continuous ion the right and g is continuous on the left at the point c and $\|f(x'_n)-f(c)\|>r', \|g(x''_n)-g(c)\|>r''$,for all $n\in\mathbb{N}$, we deduce that

$$\|f(x'_n)-f(x''_n)\|>\frac{r'}{2}, \|g(x''_n)-g(x'_n)\|>\frac{r''}{2}$$

if n is sufficiently large. So we have

$$\|\sigma(f,g;d_n,\xi'')-\sigma(f,g;d_n,\xi')\|>(\frac{r'.r''}{4})$$

for n is sufficiently large. Using the fact that the division d is arbitrary we can deduce that $f\notin RS(g)$ from the Cauchy criterion. ■

Proposition 3.2.4. *If* $f \in DS(g)$ *and the function* f *and* g *have no common point of discontinuity then we have* $f \in RS(g)$.

Proof. Let us denote

$$\|f\| = sup\{\|f(x)\|; x \in [a,b]\}, \|g\| = \sup\{|g(x)|; x \in [a,b]\}$$

and $\epsilon > 0$ be arbitrary. We consider $d_\epsilon \in D[a,b]$ such that for any $d \in D[a,b]$, $d_\epsilon \le d$ and any $\xi \in \mathcal{L}(d)$ we have

$$\left\|\sigma(f,g;d,\xi) - (DS)\int_a^b f\,dg\right\| < \epsilon .$$

If $d_\epsilon = (a = x_0 < x_1 < x_2 < \cdots < x_k = b)$, then using hypothesis concerning the continuity we may consider $\eta > 0$ such that for any $i \in \{0,1,2,\dots,k\}$ we have at least one of the relations

$$|z - x_i| < \eta \Longrightarrow \|f(z) - f(x_i)\| \le \frac{\epsilon}{r} \text{ or } |g(z) - g(x_i)| \le \frac{\epsilon}{r},$$

where $r := 4k(\|f\|.\|g\|)$.
Let now $d_0 \in D[a,b]$, $d_0 = (a = y_0 < y_1 < y_2 < \cdots < y_n = b)$ with $v(d_0) < \eta$ and let $\xi = (a = \xi_0 < \xi_1 \le \xi_2 \le \cdots \le \xi_n \le \xi_{n+1} = b)$, $\xi \in \mathcal{L}(d_0)$ with $\xi_i \in [y_{i-1}, y_i]$ for all $i \in \{0,1,2,\dots,n\}$.

Suppose that one point x_i of the division d_ϵ belongs to the interval $[y_{j_0}, y_{j_0+1}]$.

We choose $\xi'_{j_0} \in [y_{j_0}, x_i], \xi''_{j_0} \in [x_i, y_{j_0+1}]$ and we consider the division $d_{x_i} \in D[a,b]$ obtained by adding the point x_i to the division d_0. As an intermediary system $\xi^* \in \mathcal{L}(d_{x_i})$ we take the following one

$$\xi^* = \{a = \xi_0 \le \xi_1 \le \cdots \le \xi_{j_0} \le \xi'_{j_0} \le \xi''_{j_0} \le \xi_{j_0+2} \le \xi_{j_0+3} \le \cdots \le \xi_{n+1} = b\}.$$

We shall have

$$\sigma(f,g;d_0,\xi) - \sigma(f,g;d_{x_i},\xi^*) =$$

$$= f(\xi_{j_0+1})(g\left(y_{j_0+1}\right) - g(y_{j_0}) - f(\xi'_{j_0})\left(g(x_i) - g(y_{j_0})\right) - f(\xi''_{j_0})\left(g(y_{j_0+1}) - g(x_i)\right) =$$

$$= \left(f(\xi_{j_0+1}) - f(\xi'_{j_0})\right)\left(g(x_i) - g(y_{j_0})\right) + \left(f(\xi_{j_0+1}) - f(\xi''_{j_0})\right)(g(y_{j_0+1}) - g(x_i)).$$

$$\|\sigma(f,g;d_0,\xi) - \sigma(f,g;d_{x_i},\xi^*)\| \le \|f(\xi_{j_0+1}) - f(\xi'_{j_0})\|.|g(x_i) - g(y_{j_0})| +$$

$$+ \left\|f\left(\xi_{j_0+1}\right) - f(\xi''_{j_0})\right\|\left|g\left(y_{j_0+1}\right) - g(x_i)\right| \le 4(\|f\|.\|g\|).\frac{\epsilon}{r}$$

We start with the divisions d and ξ as before and taking $i = 1$ we construct as above the division $d_1 = d_0 \cup \{x_i\}$ and the division $\xi^1 := \xi^*$. We have

$$\left\|\sigma(f,g;d_0,\xi) - \sigma\left(f,g;d_1,\xi^1\right)\right\| \le 4(\|f\|.\|g\|).\frac{\epsilon}{r}$$

Then starting with the divisions d_1, ξ^1 we construct in a similar manner $d_2 = d_1 \cup \{x_2\}, \xi^2 = (\xi^1)^* \in \mathcal{L}(d_2)$. We have

$$\left\|\sigma\left(f,g;d_1,\xi^1\right) - \sigma\left(f,g;d_2,\xi^2\right)\right\| \leq 4(\|f\|.\|g\|).\frac{\epsilon}{r}$$

We continue this procedure $(k-1)-$ times and we construct the divisions $(d_1,\xi^1),(d_2,\xi^2),(d_3,\xi^3),\dots,(d_{k-1},\xi^{k-1})$ such that $d_{i+1} = d_i \cup \{x_{i+1}\}, \xi^{i+1} = (\xi^i)^*$. By construction we have

$$\left\|\sigma\left(f,g;d_i,\xi^i\right) - \sigma\left(f,g;d_{i+1},\xi^{i+1}\right)\right\| \leq 4(\|f\|.\|g\|).\frac{\epsilon}{r}, \quad i+1 \leq k-1$$

and therefore, applying this $k-$times and taking into account the fact that $r = 4k(\|f\|\|g\|)$, we get

$$\left\|\sigma(f,g;d_0,\xi) - \sigma\left(f,g;d_{k-1},\xi^{k-1}\right)\right\| \leq 4(\|f\|.\|g\|).\frac{\epsilon}{r} = \epsilon.$$

But $d_\epsilon \leq d_{k-1}$ and therefore we have

$$\left\|\sigma(f,g;d_{k-1},\xi^{k-1}) - (DS)\int_a^b f\,dg\right\| \leq \epsilon$$

From the last two inequalities follows

$$\left\|\sigma(f,g;d_0,\xi) - (DS)\int_a^b f\,dg\right\| \leq 2\epsilon$$

for any $d_0 \in \mathcal{D}[a,b]$ with $v(d_0) < \eta$ and any $\xi \in \mathcal{L}(d_0)$

Corollary 3.2.5. *If we have $f \in DS(g)$ and one of the functions f or g is continuous on $[a,b]$, then $f \in RS(g)$.*

Remark 3.2.6. the PROPOSITION 3.2.4 and the COROLLARY 3.2.5 were previously considered in the scalar case for g increasing ([13], [14])

The concept of Darboux-Stieltjes integrability is much more related with the concept of Lebesgue(or Bochner) integrability than the Riemann-Stieltjes concept is.

Let g be increasing and continuous on the left and let μ_g be the measure on $([a,b],\mathcal{B})$ where $\mathcal{B}$ is the set of all Borel subsets of $[a,b]$, for which we have

$$\mu_g([c,d]) = g(d) - g(c), \forall c,d \in \mathbb{R}, a \leq c < d \leq b.$$

If $f:[a,b] \to \mathbb{R}$ is a bounded function then, proceeding as in [14] and [15] we can prove the following results:

Proposition 3.2.7. *If the function f is Darboux-Stieltjes integrable with respect to g then the function g is Bochner integrable and we have*

$$(DS)\int_a^b f\,dg = \int_a^b f\,d\mu_g$$

Proposition 3.2.8. *If $(f_n)_n$ is a sequence of uniformly bounded real functions on $[a,b]$ such that $f_n \in DS(g)$, for all n and this sequence is pointwise convergent to a function f such that $f \in DS(g)$, then we have*

$$\lim_{n\to\infty}(DS)\int_a^b f_n\,dg = DS\int_a^b f dg.$$

3.3 HEREDITARY PROPERTIES AND THE FORMULA OF INTEGRATION BY PARTS

It is well known that if a bounded real function f on the interval $[a,b]$ is Riemann Stieltjes integrable with respect to the function g defined on the same interval, then, for any $c,d \in [a,b], c < d,$ the restriction of f to $[c,d]$ is Riemann-Stieltjes integrable with respect to the restriction of g to $[c,d]$. Generally the converse assertion is not true i.e. the Riemann-Stieltjes integrability of f with respect to g on the intervals $[a,c]$ and $[c,d]$ does not imply the RS-integrability of the function f with respect to g on the whole interval $[a,b]$. From this point of view the Darboux-Stieltjes integrale is more convenient.

Proposition 3.3.1. *if $f:[a,b]\to X$ and $g:[a,b]\to\mathbb{R}$ are bounded functions then we have:*

a. *If f is Darboux-Stieltjes integrable with respect to g on $[a,b]$then f is Darboux-Stieltjes integrable with respect to g on many subinterval $[c,d]$of $[a,b]$ i.e. the restriction of f to $[c,d]$ is Darboux-Stieltjes integrable with respect to the restriction of g to $[c,d]$.*
   Moreover, we have
   $$(DS)\int_a^b f dg = (DS)\int_a^c f dg + (DS)\int_c^d f dg + (DS)\int_d^b f dg.$$
b. *If c is a point in $[a,b]$ and the function f is Darboux-Stieltjes integrable with respect to g on the interval $[a,c]$ and $[c,b]$, then f is Darboux-Stieltjes integrable with respect to g on $[a,b]$.*

Proof. For any divisions $d' \in \mathcal{D}[a,b], d'' \in \mathcal{D}[a,b]$ we denote by $d'\vee d''$ the division of $[a,b]$ given by

$$d'\vee d'' = (a = x_0 < x_1 < x_2 < \cdots < x_n = c = y_0 < y_1 < y_2 < \cdots < y_n = b)$$

where

$$d' = (a = x_0 < x_1 < x_2 < \cdots < x_n = c),\ d'' = (a = y_0 < y_1 < y_2 < \cdots < y_n = b)$$

We use an analogous notation $\xi'\vee\xi''$ for $\xi' \in \mathcal{L}\left(d'\right), \xi'' \in \mathcal{L}(d'')$
Obviously we have

$$\sigma(f,g;d'\vee d'',\xi'\vee\xi'') = \sigma(f,g;d',\xi') + \sigma(f,g;d'',\xi'').$$

a. The proof follows using Cauchy criterion of Darboux-Stieltjes integrability.

b. Let $(d_n'^0)_n, (d_n''^0)_n$ be two sequences in $[a,c]$, respectively $\mathcal{D}[c,b]$ such that for any sequences $(d_n')_n \in \mathcal{D}[a,c], d_n'^0 \le d_n', (d_n'')_n \in \mathcal{D}[c,b],\ \ d_n''^0 \le d_n''$ and for any $\xi_n' \in \mathcal{L}(d_n')$, respectively any $\xi_n'' \in \mathcal{L}(d_n'')$, we have

$$\lim_{n\to\infty} \sigma(f,g;d_n',\xi_n') = \int_a^c fdg\,, \lim_{n\to\infty} \sigma(f,g;d_n'',\xi_n'') = \int_c^b fdg$$

Let now $(d_n'^0)_n$ be a sequence in $\mathcal{D}[a,b]$ such that $d_n^0 = d_n'^0 \vee d_n''^0, \forall n \in \mathbb{N}$ and let $(d_n)_n$ be a sequence in $\mathcal{D}[a,b]$ such that $d_n^0 \le d_n$ for any n. If we choose $\xi_n \in \mathcal{L}(d_n)$ and we denote $d_n' = d_n \cap [a,c], \xi_n' = \xi_n \cap [a,c]$ $d_n'' = d_n \cap [c,b]$ $\xi_n'' = \xi_n \cap [c,b]$ we have $d_n'^0 \le d_n'$, $\xi_n' \in \mathcal{L}(d_n')$, $d_n''^0 \le d_n''$, $\xi_n'' \in \mathcal{L}(d_n'')$ and therefore

$$\lim_{n\to\infty} \sigma(f,g;d_n',\xi_n') = \int_a^c fdg\,, \lim_{n\to\infty} \sigma(f,g;d_n'',\xi_n'') = \int_c^b fdg$$

It is obvious that $d_n' \vee d_n'' = d_n$, $\xi_n' \vee \xi_n'' = \xi_n$. We have

$$\sigma(f,g;d_n,\xi_n) = \sigma(f,g;d_n',\xi_n') + \sigma(f,g;d_n'',\xi_n''), \forall n \in \mathbb{N},$$

$$\lim_{n\to\infty} \sigma(f,g;d_n,\xi_n) = \int_a^c fdg + \int_c^b fdg$$

Hence using proposition 3.1.4, the function f is Darboux-Stieltjes integrable with respect to g.

Definition 3.3.2 If $g:[a,b] \to \mathbb{R}$ and $f:[a,b] \to X$ we say that g is *D-S* integrable with respect to f there exists an element $I^* \in X$ such that for any $\epsilon > 0$ there exists $d_\epsilon \in \mathcal{D}[a,b]$ with the property

$$\|I^* - \sigma(g,f;d,\xi)\| < \epsilon$$

whenever $d_\epsilon \le d$ and for any intermediary system ξ in d. I^* is called the Darboux-Stieltjes integral of g with respect to f.

Proposition 3.3.3. *(Symmetry principle) If the function f is Darboux-Stieltjes integrable with respect to g is Darboux-Stieltjes integrable with respect to f and we have*

$$\int_a^b gdf = f.g\Big|_a^b - \int_a^b fdg = f(b)g(b) - f(a)g(a) - \int_a^b fdg$$

(Integration by parts)

Proof. For $\epsilon > 0$ we consider $d_\epsilon \in \mathcal{D}[a,b]$, $d_\epsilon = (a = y_0 < y_1 < y_2 < \cdots < y_k = b)$ such that for any $d \in \mathcal{D}[a,b], d_\epsilon \le d$ and any $\xi \in \mathcal{L}(d)$ we have

$$\left\|\sigma(f,g;d,\xi) - \int_a^b fdg\right\| < \epsilon$$

Using hypothesis and proposition 3.2.3- b. we deduce that for any y_i , $i \in \{1,2,\dots,k\}$ at least one of the functions f and g is continuous on the left at the point y_i. Hence we may choose $\eta > 0$ such that, for any $z \in [a,b]$, $z \in [y_i - \eta, y_i]$ we have

$$\|f(y_i) - f(z)\| < \frac{\epsilon}{M} \quad or \quad |g(y_i) - g(z)| < \frac{\epsilon}{M},$$

where $M := k(\|f\| + 1)(\|g\| + 1)$.

Let now d'_ϵ be a division of $[a,b]$ such that $v(d'_\epsilon) < \eta$ and such that $d'_\epsilon \le d_\epsilon$. We take an arbitrary division d of $[a,b]$ such that $d'_\epsilon \le d$ and we consider an arbitrary $\xi \in \mathcal{L}(d)$. We have

$d = (a = x_0 = y_0 < x_1 < x_2 < \cdots < x_{j_1} < y_1 < x_{j_1+1} < \cdots < x_{j_2} < y_2 < x_{j_2+1} < \cdots < x_{j_k} < y_k = b$, $\xi = (a = \xi_0 \le \xi_1 \le \cdots \le \xi_{j_k+k} = b)$, $\xi \in \mathcal{L}(d)$ and we modify ξ replacing the element ξ_{j_p} in the interval $\left[x_{j_p}, y_p\right]$ by the element y_p, for all $p = 1,2,\dots,k$. We obtain a new intermediary division ξ' of d and we have

$$\left\|f\left(\xi_{j_p}\right)(f(y_p) - f\left(x_{j_p}\right) - g(y_p)(f(y_p) - f(x_{j_p})\right\| =$$

$$\left\|f(y_p) - f(x_{j_p})\right\| . \left|g(y_p) - g(\xi_{j_p})\right| \le 2(\|f\| + 1)(\|g\| + 1).\frac{\epsilon}{M}$$

We deduce the relation

$$\|\sigma(f,g;d,\xi) - \sigma(f,g;d,\xi')\| \le \sum_{p=1}^{k} \left\|f(y_p) - f(x_{j_p})\right\| \left|g(y_p) - g(\xi_{j_p})\right|$$

$$\|\sigma(f,g;d,\xi) - \sigma(f,g;d,\xi')\| \le 2(\|f\| + 1)(\|g\| + 1).\frac{\epsilon}{M} = \epsilon$$

We remark that $d_\epsilon \le \xi'$ and therefore we have

$$\left\|\sigma(f,g;d,\xi) - \int_a^b f dg\right\| < \epsilon$$

On the other hand, using the reciprocity formula, we get

$$\left\|\sigma(f,g;d,\xi) - (f.g\,\Big|_a^b - \int_a^b f dg)\right\| \le \|\sigma(f,g;d,\xi) - \sigma(f,g;d,\xi')\| +$$

$$+\left\|\sigma(f,g;d,\xi') - (f.g\,\Big|_a^b - \int_a^b f dg)\right\| \le \epsilon + \left\|\int_a^b f dg - \sigma(f,g;d,\xi')\right\| \le 2\epsilon.$$

Hence the function g is Darboux-Stieltjes integrable with respect to f and we have the following rule

$$\int_a^b g df = f.g\Big|_a^b - \int_a^b f dg \qquad \blacksquare$$

3.4. APPLICATIONS IN POTENTIAL THEORY

Let $\mathcal{E}$ be the convex cone of all increasing and lower semi-continuous positive functions on the space $X = (0,1)$. It is known that $\mathcal{E}$ is a standard H −cone (see [7]) and its dual $\mathcal{E}^*$ may be identified with the convex cone of all positive, decreasing and lower semi-continuous on $(0,1)$. We may extend any function $s \in \mathcal{E}$ by $s(0) = 0$ and $s(1) = \sup_{x<1} s(x)$ and also we extend the element $s^* \in \mathcal{E}^*$ by

$$s^*(0) = \sup_{x\in(0,1)} s(x) \quad , s^*(1) = 0$$

The duality between $\mathcal{E}$ and $\mathcal{E}^*$ is the following one

$$[s, s^*] = (DS)\int_0^1 s^*\, ds$$

Generally, the function s^* is not (RS) integrable with respect to s. But since the functions s and s^* has no one side common discontinuous points, the function s^* is (DS) integrable with respect to s.

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