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Classification of pseudo-Riemannian submersions with totally geodesic fibres from pseudo-hyperbolic spaces

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Abstract

We classify pseudo-Riemannian submersions with connected totally geodesic fibres from a real pseudo-hyperbolic space onto a pseudo-Riemannian manifold. Also, we obtain the classification of the pseudo-Riemannian submersions with (para-)complex connected totally geodesic fibres from a (para-)complex pseudo-hyperbolic space onto a pseudo-Riemannian manifold.

1. *Introduction and the main theorem*

Riemannian submersions, introduced by O'Neill [**37**] and Gray [**24**], have been used by many authors to construct new specific Riemannian metrics, like Einstein or positively curved ones [**8**, **27**], and to study various geometric structures of Riemannian manifolds [**17**]. In this paper, we show that the pseudo-Riemannian submersions with connected, totally geodesic fibres from a pseudo-hyperbolic onto a pseudo-Riemannian manifold are equivalent to the Hopf ones; see below. First, we give a short review of well-known classification results of Riemannian submersions and of their nice applications in Riemannian geometry and then we discuss the pseudo-Riemannian case.

In early work, Escobales [**15**, **16**] and Ranjan [**39**] classified Riemannian submersions with connected totally geodesic fibres from a sphere, and with complex connected totally geodesic fibres from a complex projective space. Using a topological argument, Ucci [**44**] showed that there are no Riemannian submersions with fibres $\mathbb{C}P^3$ from the complex projective space $\mathbb{C}P^7$ onto $S^8(4)$, and with fibres $\mathbb{H}P^1$ from the quaternionic projective space $\mathbb{H}P^3$ onto $S^8(4)$. A major advance obtained by Gromoll and Grove [**26**] is that, up to equivalence, the only Riemannian submersions from spheres with connected fibres are the Hopf fibrations, except possibly for fibrations of the 15-sphere by homotopy 7-spheres. This classification was invoked in the proofs of the Diameter Rigidity Theorem in Gromoll and Grove [**25**] and of the Radius Rigidity Theorem in Wilhelm [**45**]. Using Morse theory, Wilking [**46**] ruled out the Gromoll and Grove unsettled case by showing that any Riemannian submersion $\pi : S^{15} \to B^8$ is equivalent to a Riemannian submersion with totally geodesic fibres, which by Escobales' classification must be equivalent to a Hopf Riemannian submersion. A nice consequence of this classification is the improved version of the Diameter Rigidity Theorem due to Wilking [**46**].

In the pseudo-Riemannian set-up, the pioneering work is due to Magid [**33**], who proved that the pseudo-Riemannian submersions with connected totally geodesic fibres from an anti-de Sitter space onto a Riemannian manifold are equivalent to the Hopf pseudo-Riemannian submersions $H_1^{2m+1} \to \mathbb{C}H^m$. Generalizing Magid's result, Stere Ianuş and I showed that any pseudo-Riemannian submersion with connected totally geodesic fibres from a pseudo-hyperbolic space onto a Riemannian manifold is equivalent to one of the Hopf

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pseudo-Riemannian submersions: $H_1^{2m+1} \to \mathbb{C}H^m$, $H_3^{4m+3} \to \mathbb{H}H^m$ or $H_7^{15} \to \mathbb{H}^8(-4)$, and as a consequence we classified the pseudo-Riemannian submersions with connected complex totally geodesic fibres from a complex pseudo-hyperbolic space onto a Riemannian manifold (see [4]). In [3], I extended these results to the case of a pseudo-Riemannian base under the assumption that either (i) the base space is isotropic or (ii) the dimension of fibres is less than or equal to 3, and the metrics induced on the fibres are negative definite. I also proved that condition (ii) implies (i) (see [3]). In this paper, we drop these assumptions and we prove the following main result.

THEOREM 1.1. *Let $\pi : H_l^a \to B$ be a pseudo-Riemannian submersion with connected totally geodesic fibres from a real pseudo-hyperbolic space H_l^a of curvature -1 onto a pseudo-Riemannian manifold. Then π is equivalent to one of the following Hopf pseudo-Riemannian submersions*:

(a) $\pi_{\mathbb{C}} : H_{2t+1}^{2m+1} \to \mathbb{C}H_t^m$, $0 \leqslant t \leqslant m$,
(b) $\pi_{\mathbb{A}} : H_m^{2m+1} \to \mathbb{A}P^m$,
(c) $\pi_{\mathbb{H}} : H_{4t+3}^{4m+3} \to \mathbb{H}H_t^m$, $0 \leqslant t \leqslant m$,
(d) $\pi_{\mathbb{B}} : H_{2m+1}^{4m+3} \to \mathbb{B}P^m$,
(e) $\pi_{\mathbb{O}}^1 : H_{15}^{15} \to H_8^8(-4)$,
(f) $\pi_{\mathbb{O}'} : H_7^{15} \to H_4^8(-4)$,
(g) $\pi_{\mathbb{O}}^2 : H_7^{15} \to H^8(-4)$,

where $\mathbb{C}H_t^m$ and $\mathbb{H}H_t^m$ are the indefinite complex and quaternionic pseudo-hyperbolic spaces of holomorphic, respectively, quaternionic curvature -4; $\mathbb{A}P^m$ is the para-complex projective space of real dimension $2m$, signature (m, m) and para-holomorphic curvature -4; $\mathbb{B}P^m$ is the para-quaternionic projective space of real dimension $4m$, signature $(2m, 2m)$ and para-quaternionic curvature -4.

The plan of the paper can be summarized as follows. Section 2 presents some known definitions and results in the theory of pseudo-Riemannian submersions. In § 3, we exhibit the construction of the Hopf pseudo-Riemannian submersions from pseudo-hyperbolic spaces, which ensures the existence of at least one pseudo-Riemannian submersion in each class (a)–(g) of Theorem 1.1. In § 4, we see that the base space B is isometric to either a pseudo-hyperbolic space or a complete, simply connected, special Osserman pseudo-Riemannian manifold, which was classified in [10]. To exclude the Cayley planes of octonions, and of para-octonions from the list of possible base spaces, we prove that the curvature tensor of B has a Clifford structure. For the remaining cases, we establish that the dimension and the index of the total space are, in fact, those claimed in Theorem 1.1. This reduces the equivalence problem of two pseudo-Riemannian submersions to the one of the same base space, which we resolve in § 5. Section 6 features consequences of Theorem 1.1: (a) the classification of the pseudo-Riemannian submersions with totally geodesic fibres from complex pseudo-hyperbolic spaces or from para-complex projective spaces under the assumption that the fibres are, respectively, complex or para-complex submanifolds and (b) the non-existence of the pseudo-Riemannian submersions with quaternionic or para-quaternionic fibres from $\mathbb{H}H_t^m$ or $\mathbb{B}P^m$, respectively.

2. *Preliminaries*

In this section, we recall several notions and results that will be used throughout the paper.

DEFINITION 2.1. A smooth surjective submersion $\pi : (M, g) \to (B, g')$ between two pseudo-Riemannian manifolds is said to be a *pseudo-Riemannian submersion* (see [38]) when π_* preserves scalar products of vectors normal to fibres and when the metric induced on every fibre $F_b = \pi^{-1}(b)$, where $b \in B$, is non-degenerate.

The vectors tangent to fibres are called vertical and those normal to fibres are called horizontal. We denote the vertical distribution by $\mathcal{V}$ and the horizontal distribution by $\mathcal{H}$. The geometry of pseudo-Riemannian submersions is characterized in terms of the O'Neill tensors T, A (see [37, 38]) defined for every vector fields E, F on M by

$$A_E F = h\nabla_{hE} vF + v\nabla_{hE} hF, \quad T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF, \tag{2.1}$$

where ∇ is the Levi-Civita connection of g, and v and h denote the orthogonal projections on $\mathcal{V}$ and $\mathcal{H}$, respectively. We assume that the fibres are totally geodesic, which is equivalent to $T_E F = 0$ for every E, F. The O'Neill tensor A is alternating, that is, $A_X Y = -A_Y X$ for any horizontal vectors X, Y, and skew-symmetric with respect to g, that is, $g(A_E F, G) = -g(F, A_E G)$ for every vector fields E, F, G (see [8, 17, 37, 38]). Throughout the paper, X, Y, Z, Z' will always be horizontal vector fields, while U, V, W, W' will be vertical vector fields. We assume that $\dim M > \dim B$ and that M is connected.

We denote by R, R' and $\hat{R}$ the Riemann curvature tensors of M, B and of a fibre F_b, respectively. We choose the convention for the curvature tensor $R(E, F) = \nabla_E \nabla_F - \nabla_F \nabla_E - \nabla_{[E,F]}$. By $R'(X, Y)Z$ we shall also denote the horizontal lift of $R'(\pi_* X, \pi_* Y)\pi_* Z$. The structure equations of a pseudo-Riemannian submersion, usually called the O'Neill equations, are stated next in a totally geodesic fibre set-up.

PROPOSITION 2.2 [8, 17, 24, 37]. *If $\pi : M \to B$ is a pseudo-Riemannian submersion with totally geodesic fibres, then*

(a) $R(X, Y, Z, Z') = R'(X, Y, Z, Z') - 2g(A_X Y, A_Z Z') + g(A_Y Z, A_X Z') - g(A_X Z, A_Y Z')$;
(b) $R(X, Y, Z, U) = g((\nabla_Z A)_X Y, U)$;
(c) $R(X, U, Y, V) = g((\nabla_U A)_X Y, V) + g(A_X U, A_Y V)$;
(d) $R(U, V, W, W') = \hat{R}(U, V, W, W')$ *and* (e) $R(U, V, W, X) = 0$.

COROLLARY 2.3. *If $\pi : M \to B$ is a pseudo-Riemannian submersion with totally geodesic fibres, then*

(a) $R(X, Y, X, Y) = R'(X, Y, X, Y) - 3g(A_X Y, A_X Y)$;
(b) $R(X, U, X, U) = g(A_X U, A_X U)$.

DEFINITION 2.4. A vector field X on M is said to be *basic* if X is horizontal and π-related to a vector field X' on B. A vector field X along the fibre $\pi^{-1}(b)$, $b \in B$ is said to be *basic along* $\pi^{-1}(b)$ if X is horizontal and $\pi_{*p} X(p) = \pi_{*q} X(q)$ for every p, $q \in \pi^{-1}(b)$.

We note that each vector field X' on B has a unique horizontal lift X to M which is basic. For a vertical vector field V and a basic vector field X, we have $h\nabla_V X = A_X V$ (see [37]).

DEFINITION 2.5. Two pseudo-Riemannian submersions $\pi, \pi' : (M, g) \to (B, g')$ are said to be *equivalent* if there exists an isometry f of M that induces an isometry $\tilde{f}$ of B so that $\pi' \circ f = \tilde{f} \circ \pi$.

3. *The construction of the Hopf pseudo-Riemannian submersions*

In this section, we exhibit the constructions of the real, complex, quaternionic pseudo-hyperbolic spaces, of the para-complex and para-quaternionic projective spaces and of the Hopf pseudo-Riemannian submersions from the real pseudo-hyperbolic spaces.

DEFINITION 3.1. Let $\langle\cdot,\cdot\rangle_{\mathbb{R}^{m+1}_{t+1}}$ be the inner product of signature $(m-t,t+1)$ on $\mathbb{R}^{m+1}$ given by

$$\langle x,y\rangle_{\mathbb{R}^{m+1}_{t+1}} = -\sum_{i=0}^{t} x_iy_i + \sum_{i=t+1}^{m} x_iy_i \tag{3.1}$$

for $x=(x_0,\dots,x_m), y=(y_0,\dots,y_m)\in\mathbb{R}^{m+1}$. For any $c<0$ and any positive integer t, let $H^m_t(c)=\{x\in\mathbb{R}^{m+1}\,|\,\langle x,x\rangle_{\mathbb{R}^{m+1}_{t+1}}=1/c\}$ be the pseudo-Riemannian submanifold of

$$\mathbb{R}^{m+1}_{t+1}=(\mathbb{R}^{m+1},ds^2=-dx_0\otimes dx_0-\cdots-dx_t\otimes dx_t+dx_{t+1}\otimes dx_{t+1}+\cdots+dx_m\otimes dx_m).$$

The space $H^m_t(c)$ is called the m-dimensional *pseudo-hyperbolic space* of index t. The hyperbolic space $H^m(c)$ is the hypersurface $\{x=(x_0,x_1,\dots,x_m)\in\mathbb{R}^{m+1}\,|\,x_0>0,\ \langle x,x\rangle_{\mathbb{R}^{m+1}_1}=1/c\}$ endowed with the metric induced from $\mathbb{R}^{m+1}_1$.

It is well-known that $H^m_t(c)$ has constant sectional curvature c, and we shall define simply $H^m_t=H^m_t(-1)$.

Throughout the paper, we use the notation: $\mathbb{H}$ for the field of quaternions; $\mathbb{A}$ and $\mathbb{B}$ for the algebras of para-complex and para-quaternionic numbers, respectively; $\mathbb{O}$ for the algebra of octonions (Cayley numbers) and $\mathbb{O}'$ for that of para-octonions [29] (split octonions). For $F\in\{\mathbb{C},\mathbb{A},\mathbb{H},\mathbb{B},\mathbb{O},\mathbb{O}'\}$, and for $z\in F$, we denote by $\bar z$ the conjugate of z in F and, as usual, $|z|^2_F=\bar zz=z\bar z\in\mathbb{R}$.

3.1. *The indefinite Hopf pseudo-Riemannian submersions*

When $K\in\{\mathbb{C},\mathbb{H}\}$, let $\langle\cdot,\cdot\rangle_{K^{m+1}_{t+1}}$ be the inner product on K^{m+1} given by

$$\langle z,w\rangle_{K^{m+1}_{t+1}}=\mathrm{Re}\left(-\sum_{i=0}^{t}z_i\bar w_i+\sum_{i=t+1}^{m}z_i\bar w_i\right), \tag{3.2}$$

where $z=(z_0,\dots,z_m), y=(w_0,\dots,w_m)\in K^{m+1}$. We set $d=\dim_{\mathbb{R}}K$ and assume $c<0$. We simply note that $S^{d-1}=\{z\in K\,|\,z\bar z=1\}$, and

$$H^{d(m+1)-1}_{d(t+1)-1}(c/4)=\{z\in K^{m+1}\,|\,\langle z,z\rangle_{K^{m+1}_{t+1}}=4/c\}. \tag{3.3}$$

The restriction of the projection

$$\{z\in K^{n+1}\,|\,\langle z,z\rangle_{K^{m+1}_{t+1}}<0\}\longrightarrow\{z\in K^{n+1}\,|\,\langle z,z\rangle_{K^{m+1}_{t+1}}<0\}/K^*,\quad z\mapsto zK^* \tag{3.4}$$

to $H^{d(m+1)-1}_{d(t+1)-1}(c/4)$ is a submersion

$$\pi_K:H^{d(m+1)-1}_{d(t+1)-1}(c/4)\longrightarrow KH^m_t(c)=H^{d(m+1)-1}_{d(t+1)-1}(c/4)/S^{d-1},\quad z\mapsto zS^{d-1}, \tag{3.5}$$

called the indefinite Hopf fibration of $H^{d(m+1)-1}_{d(t+1)-1}(c/4)$. There is a unique pseudo-Riemannian metric on $KH^m_t(c)$ such that $\pi_K:H^{d(m+1)-1}_{d(t+1)-1}(c/4)\to KH^m_t(c)$ is a pseudo-Riemannian submersion with totally geodesic fibres. We shall simply define $KH^m_t=KH^m_t(-4)$. For $c=-4$, and for $K=\mathbb{C}$ and $K=\mathbb{H}$, respectively, the Hopf pseudo-Riemannian submersions are:

(a) $\pi_{\mathbb{C}}:H^{2m+1}_{2t+1}\to\mathbb{C}H^m_t$ with the fibres isometric to $H^1_1=(S^1,-g_{S^1})$ and
(b) $\pi_{\mathbb{H}}:H^{4m+3}_{4t+3}\to\mathbb{H}H^m_t$ with the fibres $H^3_3=(S^3,-g_{S^3})$.

A nice reference for the construction of $\pi_{\mathbb{C}}$ is [7]. Note that $\mathbb{C}H_t^m$ has holomorphic sectional curvature -4 (see [7]), and that $\mathbb{H}H_t^m$ has quaternionic sectional curvature -4.

3.2. *The para-Hopf pseudo-Riemannian submersions*

There are several models of para-complex and of para-quaternionic projective spaces [9, 11, 14, 18]. Following [14, 18], we present a para-complex model of a para-complex projective space, $\mathbb{A}P^m$, which is simply connected for $m \geqslant 2$, and a simply connected, para-quaternionic model for the para-quaternionic projective space, $\mathbb{B}P^m$; see [9].

For $D \in \{\mathbb{A}, \mathbb{B}\}$, let $d = \dim_{\mathbb{R}} D$. We consider the inner product of signature $((m+1)d/2, (m+1)d/2)$ on D^{m+1} given by

$$\langle z, w\rangle = \operatorname{Re}\left(\sum_{i=0}^{m} z_i \bar{w}_i\right) \tag{3.6}$$

for $z = (z_0, \dots, z_m), y = (w_0, \dots, w_m) \in D^{m+1}$. Identifying $D^{m+1} = \mathbb{R}^{d(m+1)}_{d(m+1)/2}$, via $(z_0, \dots, z_m) \simeq (z_0^1, \dots, z_m^1, \dots, z_0^d, \dots, z_m^d)$, where $z_i = (z_i^1, \dots, z_i^d)$, $0 \leqslant i \leqslant m$, we simply have $\langle z, w\rangle = -\langle z, w\rangle_{\mathbb{R}^{d(m+1)}_{d(m+1)/2}}$ for any z, w. In particular, we can write $H_m^{2m+1} = \{z \in \mathbb{A}^{m+1} \mid \langle z, z\rangle = 1\}$ and $H_{2m+1}^{4m+3} = \{z \in \mathbb{B}^{m+1} \mid \langle z, z\rangle = 1\}$.

We set $\mathbb{A}_0^{m+1} = \{z \in \mathbb{A}^{m+1} \mid \langle z, z\rangle > 0\}$ and $\mathbb{A}_+ = \{t = x + \varepsilon y \in \mathbb{A} \mid t\bar{t} > 0, x > 0\}$. The para-complex projective space $\mathbb{A}P^m$ is defined to be the quotient of $\mathbb{A}_0^{m+1}$ under the equivalence relation: $Z \simeq W$ if $Z = tW$ for some $t \in \mathbb{A}_+$ (see [14, 18]).

We note that $H^1 = \{t \in \mathbb{A}_+ \mid t\bar{t} = 1\}$. The restriction of the projection $\mathbb{A}_0^{m+1} \to \mathbb{A}P^m = \mathbb{A}_0^{m+1}/\mathbb{A}_+$ to H_m^{2m+1} gives the Hopf submersion

$$\pi_{\mathbb{A}} :: H_m^{2m+1} \longrightarrow \mathbb{A}P^m = H_m^{2m+1}/H^1. \tag{3.7}$$

Moreover, there exists a unique pseudo-Riemannian metric g' on $\mathbb{A}P^m$ such that $\pi_{\mathbb{A}}$ is a pseudo-Riemannian submersion with totally geodesic fibres [14]. The space $(\mathbb{A}P^m, g')$ is a complete para-holomorphic space form and its para-holomorphic curvature is -4.

The construction of $\mathbb{B}P^m$ is analogous to the para-complex projective space. We have

$$\mathbb{B}P^m = \{z \in \mathbb{B}^{m+1} \mid \langle z, z\rangle = 1\}/\{t \in \mathbb{B} \mid t\bar{t} = 1\} = H_{2m+1}^{4m+3}/H_1^3, \tag{3.8}$$

and there exists a unique pseudo-Riemannian metric g' on $\mathbb{B}P^m$ such that the projection

$$\pi_{\mathbb{B}} : H_{2m+1}^{4m+3} \longrightarrow \mathbb{B}P^m = H_{2m+1}^{4m+3}/H_1^3 \tag{3.9}$$

is a pseudo-Riemannian submersion with totally geodesic fibres [9]. Moreover, $(\mathbb{B}P^m, g')$ is a complete, simply connected, para-quaternionic space form of para-quaternionic curvature -4 (see [9]).

3.3. *The Hopf pseudo-Riemannian submersions between pseudo-hyperbolic spaces: the Hopf construction*

All Hopf pseudo-Riemannian submersions between (real) pseudo-hyperbolic spaces can explicitly be obtained by the Hopf construction.

A bilinear map $G : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ is said to be an orthogonal multiplication if G is norm-preserving, that is, $|G(x, y)| = |x|\,|y|$ for any $x \in \mathbb{R}^p, y \in \mathbb{R}^q$ (see [5, 41]). A *Hopf construction* is a map $\varphi : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{n+1}$ defined by $\varphi(x, y) = (|x|^2 - |y|^2, 2G(x, y))$ for some orthogonal multiplication G (see [5, 41]). The Hopf construction can provide several examples of harmonic morphisms (see [31, 41]), and we would like to refer the reader to the beautiful book [5] due to Baird and Wood for other nice results on this topic. Since the sectional curvatures K, K' of the total and base spaces of any pseudo-Riemannian submersion between real space forms must obey $K' = 4K$, we are forced to consider the map $\varphi(x, y)/2$ instead.

Let $F \in \{\mathbb{C}, \mathbb{A}, \mathbb{H}, \mathbb{B}, \mathbb{O}, \mathbb{O}'\}$, and let $\varphi_1, \varphi_2 : F \times F \to \mathbb{R} \times F$ be, respectively, the maps given by

$$\varphi_1(x,y) = ((|x|^2 - |y|^2)/2, \bar{x}y) \quad \text{and} \quad \varphi_2(x,y) = ((|x|^2 + |y|^2)/2, \bar{x}y) \tag{3.10}$$

for any $x, y \in F$, where $\bar{x}$ denotes the conjugation of x in F and as usual $|x|^2 = x\bar{x}$, $|y|^2 = y\bar{y}$. For convenience, we denote $t_1 = (|x|^2 - |y|^2)/2 \in \mathbb{R}$, $t_2 = (|x|^2 + |y|^2)/2 \in \mathbb{R}$ and $w = \bar{x}y \in F$. Since $|w|^2 = |\bar{x}y|^2 = |x|^2|y|^2$ for any $x, y \in F$, it is easy to see that

(i) if $|x|^2 + |y|^2 = 1$, then $t_1^2 + |w|^2 = \frac{1}{4}$;
(ii) if $|x|^2 - |y|^2 = 1$, then $t_2^2 - |w|^2 = \frac{1}{4}$.

Setting $d = \dim_{\mathbb{R}} F$, we identify $F \times F \simeq \mathbb{R}^{2d}$ via

$$((x^1, \dots, x^d), (y^1, \dots, y^d)) \simeq (x^1, y^1, \dots, x^d, y^d). \tag{3.11}$$

When $F \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$, we consider the following restrictions of φ_1 and φ_2 to H_{2d-1}^{2d-1} and to H_{d-1}^{2d-1}, respectively:

$$\varphi_1 : H_{2d-1}^{2d-1} = \{(x,y) \in F^2 \,|\, |x|^2 + |y|^2 = 1\} \longrightarrow H_d^d(-4) = \{(t_1, w) \in \mathbb{R} \times F \,|\, t_1^2 + |w|^2 = \tfrac{1}{4}\};$$
$$\varphi_2 : H_{d-1}^{2d-1} = \{(x,y) \in F^2 \,|\, |x|^2 - |y|^2 = 1\} \longrightarrow H^d(-4) = \{(t_2, w) \in \mathbb{R} \times F \,|\, t_2^2 - |w|^2 = \tfrac{1}{4}\}.$$

This simple construction gives six Hopf pseudo-Riemannian submersions with totally geodesic fibres:

$$\pi_1 : H_3^3 \longrightarrow H_2^2(-4) = \mathbb{C}H_1^1, \quad \pi_2 : H_7^7 \longrightarrow H_4^4(-4) = \mathbb{H}H_1^1, \quad \pi_3 : H_{15}^{15} \longrightarrow H_8^8(-4),$$
$$\pi_4 : H_1^3 \longrightarrow H^2(-4) = \mathbb{C}H^1, \quad \pi_5 : H_3^7 \longrightarrow H^4(-4) = \mathbb{H}H^1, \quad \pi_6 : H_7^{15} \longrightarrow H^8(-4) = \mathbb{O}H^1.$$

The first three submersions are the well-known Hopf fibrations between spheres.

When $F \in \{\mathbb{A}, \mathbb{B}, \mathbb{O}'\}$, the restriction of φ_1 to H_{d-1}^{2d-1},

$$\varphi_1 : H_{d-1}^{2d-1} = \{(x,y) \in F^2 \,|\, |x|^2 + |y|^2 = 1\} \longrightarrow H_{d/2}^d(-4) = \{(t_1, w) \in \mathbb{R} \times F \,|\, t_1^2 + |w|^2 = \tfrac{1}{4}\},$$

gives another three Hopf pseudo-Riemannian submersions with totally geodesic fibres between pseudo-hyperbolic spaces:

$$\pi_7 : H_1^3 \longrightarrow H_1^2(-4) = \mathbb{A}H^1, \quad \pi_8 : H_3^7 \longrightarrow H_2^4(-4) = \mathbb{B}H^1, \quad \pi_9 : H_7^{15} \longrightarrow H_4^8(-4).$$

Note that, for $F \in \{\mathbb{A}, \mathbb{B}, \mathbb{O}'\}$, the restriction of φ_2 to H_{d-1}^{2d-1} will give the same π_7, π_8, π_9. In [**31**], Konderak constructed the harmonic morphisms $2\pi_7$ and $2\pi_8$ via the Hopf construction (see also [**5**, Examples 14.6.5–6]). For identification (3.11) of $\mathbb{O}' \times \mathbb{O}' \simeq \mathbb{R}^{16}$, the Hopf pseudo-Riemannian submersion $\pi_9 : H_7^{15} \to H_4^8(-4)$ can be written explicitly as

$$\begin{aligned}
&\pi_9(x_1, y_1, \dots, x_8, y_8)\\
&\quad = ((x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 - x_6^2 - x_7^2 - x_8^2 - y_1^2 - y_2^2 - y_3^2 - y_4^2 + y_5^2\\
&\qquad + y_6^2 + y_7^2 + y_8^2)/2,\ x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7 - x_8y_8, -x_2y_1\\
&\qquad + x_1y_2 + x_4y_3 - x_3y_4 - x_6y_5 + x_5y_6 + x_8y_7 - x_7y_8,\ -x_3y_1 - x_4y_2 + x_1y_3 + x_2y_4\\
&\qquad - x_7y_5 - x_8y_6 + x_5y_7 + x_6y_8,\ -x_4y_1 + x_3y_2 - x_2y_3 + x_1y_4 - x_8y_5 + x_7y_6 - x_6y_7\\
&\qquad + x_5y_8,\ -x_5y_1 - x_6y_2 - x_7y_3 - x_8y_4 + x_1y_5 + x_2y_6 + x_3y_7 + x_4y_8,\ -x_6y_1 + x_5y_2\\
&\qquad - x_8y_3 + x_7y_4 - x_2y_5 + x_1y_6 - x_4y_7 + x_3y_8,\ -x_7y_1 + x_8y_2 + x_5y_3 - x_6y_4 - x_3y_5\\
&\qquad + x_4y_6 + x_1y_7 - x_2y_8,\ -x_8y_1 - x_7y_2 + x_6y_3 + x_5y_4 - x_4y_5 - x_3y_6 + x_2y_7 + x_1y_8).
\end{aligned}$$

Note that π_1, π_2, π_4, π_5, π_7, π_8 fall, respectively, in the categories $\pi_{\mathbb{C}}$, $\pi_{\mathbb{H}}$, $\pi_{\mathbb{C}}$, $\pi_{\mathbb{H}}$, $\pi_{\mathbb{A}}$, $\pi_{\mathbb{B}}$ of §§3.1 and 3.2. Define $\pi_{\mathbb{O}}^1 = \pi_3$, $\pi_{\mathbb{O}}^2 = \pi_6$, $\pi_{\mathbb{O}'} = \pi_9$. To the best of our knowledge, $\pi_{\mathbb{O}'}$ does not appear in the literature.

The construction of the Hopf pseudo-Riemannian submersions solves the existence problem for each class (a) –(g) of Theorem 1.1. In the following sections, we approach the uniqueness.

REMARK 3.2. The Hopf pseudo-Riemannian submersions are homogeneous, that is, of the form $\pi : G/K \to G/H$ with $K \subset H$ closed Lie subgroups:

$$\pi_{\mathbb{C}} : H^{2m+1}_{2t+1} = SU(m-t,t+1)/SU(m-t,t) \longrightarrow \mathbb{C}H^m_t = SU(m-t,t+1)/S(U(1)U(m-t,t)),$$
$$\pi_{\mathbb{H}} : H^{4m+3}_{4t+3} = \mathrm{Sp}(m-t,t+1)/\mathrm{Sp}(m-t,t) \longrightarrow \mathbb{H}H^m_t = \mathrm{Sp}(m-t,t+1)/\mathrm{Sp}(1)\mathrm{Sp}(m-t,t),$$
$$\pi_{\mathbb{A}} : H^{2m+1}_m = SU(m+1,\mathbb{A})/SU(m,\mathbb{A}) \longrightarrow \mathbb{A}P^m = SU(m+1,\mathbb{A})/S(U(1,\mathbb{A})U(m,\mathbb{A})),$$
$$\pi_{\mathbb{B}} : H^{4m+3}_{2m+1} = \mathrm{Sp}(m+1,\mathbb{B})/\mathrm{Sp}(m,\mathbb{B}) \longrightarrow \mathbb{B}P^m = \mathrm{Sp}(m+1,\mathbb{B})/\mathrm{Sp}(1,\mathbb{B})\mathrm{Sp}(m,\mathbb{B}),$$
$$\pi^1_{\mathbb{O}} : H^{15}_{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7) \longrightarrow H^8_8(-4) = \mathrm{Spin}(9)/\mathrm{Spin}(8),$$
$$\pi^2_{\mathbb{O}} : H^{15}_7 = \mathrm{Spin}(8,1)/\mathrm{Spin}(7) \longrightarrow H^8(-4) = \mathrm{Spin}(8,1)/\mathrm{Spin}(8),$$
$$\pi_{\mathbb{O}'} : H^{15}_7 = (\mathrm{Spin}(5,4)/\mathrm{Spin}(3,4))_0 \longrightarrow H^8_4(-4) = (\mathrm{Spin}(5,4)/\mathrm{Spin}(4,4))_0.$$

By Harvey's book [**28**, p. 312], each of $\mathrm{Spin}(5,4)/\mathrm{Spin}(3,4)$ and $\mathrm{Spin}(5,4)/\mathrm{Spin}(4,4)$ has two connected components: a pseudo-sphere and a pseudo-hyperbolic space. Here $(\cdot)_0$ denotes the pseudo-hyperbolic component.

By analogy to Hopf Riemannian submersions from spheres [**8**], each of the canonical variations of $\pi_{\mathbb{B}}$, $\pi_{\mathbb{H}}$, $\pi^1_{\mathbb{O}}$, $\pi^2_{\mathbb{O}}$ and $\pi_{\mathbb{O}'}$ gives a new homogeneous Einstein metric on the pseudo-hyperbolic space. The classification problem of homogeneous Einstein metrics on pseudo-hyperbolic spaces shall be discussed somewhere else.

4. *The geometry of the base space*

An important step of the proof of Theorem 1.1 is to establish that the base space is either a real space form or a special Osserman pseudo-Riemannian manifold. By the classification of complete, simply connected, special Osserman pseudo-Riemannian manifolds [**10**, **19**], we explicitly get the geometry of the base space, and then we see that the dimensions and the indices of the total space and of the base are those claimed in Theorem 1.1. First, we recall [**3**, Proposition 3.8], which provides the completeness and the simply connectedness of the base space.

PROPOSITION 4.1. *Let $\pi : M \to B$ be a pseudo-Riemannian submersion with connected, totally geodesic fibres from a complete connected pseudo-Riemannian manifold M onto a pseudo-Riemannian manifold B. Then B is complete. Moreover, if M is simply connected, then B is also simply connected.*

Let $\pi : M \to B$ be a pseudo-Riemannian submersion. We use the following notation throughout the paper: $n = \dim B$, $s = \mathrm{index}\, B$, $F_b = \pi^{-1}(b)$ for some $b \in B$, $r = \dim F_b$ and $r' = \mathrm{index}\, F_b$.

4.1. *The construction of a special basis $\mathcal{B}$ of $\mathcal{H}$ along a fibre*

A key ingredient for understanding the geometry of the base and of the fibres is the construction of a special orthonormal basis $\mathcal{B}$ of $\mathcal{H}$ along a fibre, which we recall from [**3**]. First, we state the following lemma, which provides useful properties of O'Neill's integrability tensor for a constant curvature total space.

LEMMA 4.2 [**3**]. *Let $\pi : M \to B$ be a pseudo-Riemannian submersion with connected, totally geodesic fibres from a pseudo-Riemannian manifold M with constant curvature $c \neq 0$. Then the following assertions are true.*

(a) *If X is a horizontal vector such that $g(X,X) \neq 0$, then the map $A_X : \mathcal{V} \to \mathcal{H}$ given by $A_X(V) = A_X V$ is injective and the map $A^*_X : \mathcal{H} \to \mathcal{V}$ given by $A^*_X(Y) = A_X Y$ is surjective.*

(b) *If X, Y are the horizontal lifts along the fibre $\pi^{-1}(\pi(p))$, $p \in M$, of two vectors $X', Y' \in T_{\pi(p)}B$, respectively, $g'(X', X') \neq 0$ and $(A_X Y)(p) = 0$, then $A_X Y = 0$ along the fibre $\pi^{-1}(\pi(p))$.*

The proof of Lemma 4.2 relies on the O'Neill equations. Corollary 2.3(b) simply gives

$$A_X^* A_X V = -cg(X, X)V \tag{4.1}$$

for every vertical vector field V, which implies (i). By Corollary 2.3(a), we get (ii).

Let $p \in M$ and let $\{v_{1p}, \ldots, v_{rp}\}$ be an orthonormal basis in $\mathcal{V}_p$. Let $X' \in T_{\pi(p)}B$ such that $g'(X', X') = \pm 1$ and let X be the horizontal lift along the fibre $\pi^{-1}(\pi(p))$ of X'. Let $Y_1, Y_2, \ldots, Y_r$ be the horizontal lifts along the fibre $\pi^{-1}(\pi(p))$ of

$$\frac{1}{cg(X,X)}\pi_* A_X v_{1p}, \quad \frac{1}{cg(X,X)}\pi_* A_X v_{2p}, \ldots, \frac{1}{cg(X,X)}\pi_* A_X v_{rp},$$

respectively. For each $i \in \{1, \ldots, r\}$, we consider the vector $v_i = A_X Y_i$ defined along the fibre $\pi^{-1}(\pi(p))$. By Corollary 2.3(a), $\{v_1, v_2, \ldots, v_r\}$ is an orthonormal basis of $\mathcal{V}_q$ at any $q \in \pi^{-1}(\pi(p))$ (see [3]), which can be restated as the following lemma.

LEMMA 4.3 [3]. *In the set-up of Lemma 4.2, the fibres are parallelizable.*

Set $L_0 = X$. For every integer α with $1 \leqslant \alpha < n/(r+1)$, let L_α be a horizontal vector field along the fibre $\pi^{-1}(\pi(p))$ such that

(1) L_α is the horizontal lift of some unit vector (that is, $g(L_\alpha, L_\alpha) \in \{-1, 1\}$), and
(2) L_α is orthogonal to $L_0, L_1, \ldots, L_{\alpha-1}$, and

$$L_\alpha(p) \in \ker A^*_{L_0(p)} \cap \ker A^*_{L_1(p)} \cap \cdots \cap \ker A^*_{L_{\alpha-1}(p)}. \tag{4.2}$$

Condition (4.2) is nothing but the statement that $L_\alpha(p)$ is orthogonal to any vector in the system $\{L_0(p), A_{L_0}v_1(p), \ldots, A_{L_0}v_r(p), \ldots, L_{\alpha-1}(p), A_{L_{\alpha-1}}v_1(p), \ldots, A_{L_{\alpha-1}}v_r(p)\}$. Moreover, by Lemma 4.2(b), $L_\alpha(q)$ belongs to $\ker A^*_{L_0(q)} \cap \ker A^*_{L_1(q)} \cap \cdots \cap \ker A^*_{L_{\alpha-1}(q)}$ for every $q \in \pi^{-1}(\pi(p))$. In the set-up of Lemma 4.2, Proposition 2.2(c) implies that

$$\mathcal{B} = \{L_0, A_{L_0}v_1, \ldots, A_{L_0}v_r, \ldots, L_{k-1}, A_{L_{k-1}}v_1, \ldots, A_{L_{k-1}}v_r\} \tag{4.3}$$

is an orthonormal basis of $\mathcal{H}_q$ for any $q \in \pi^{-1}(\pi(p))$ (see [3]). It is worth pointing out that any element in $\mathcal{B}$ is basic along the fibre $\pi^{-1}(\pi(p))$ by (4.2) and Proposition 2.2(a) (see [3]). Such a basis $\mathcal{B}$ is said to be a *special basis*.

Counting the time-like vectors of $\mathcal{B}$, we get the following proposition.

PROPOSITION 4.4 [3]. *In the set-up of Lemma 4.2, we have $n = k(r+1)$ for some positive integer k and $s = q_1(r'+1) + q_2(r-r')$ for some non-negative integers q_1, q_2 with $q_1 + q_2 = k$.*

The following corollary will be needed later.

COROLLARY 4.5 [3]. *If $s \in \{0, n\}$, then $r' = r$ (that is, the metrics induced on fibres are negative definite).*

We now split the problem of identifying the geometry of B into two cases: (i) $n = r+1$ (that is, $k = 1$) and (ii) $n \neq r+1$ (that is, $k > 1$).

4.2. *Case* $n = r + 1$

This case features a constant curvature base space.

PROPOSITION 4.6. *In the set-up of Theorem* 1.1, $n = r + 1$ *if and only if* B *has constant curvature* -4.

Proof. Let $b \in B$, $X' \in T_bB$ such that $g'(X', X') = \pm 1$ and $p \in \pi^{-1}(b)$. Let $X \in \mathcal{H}_p$ be the horizontal lift of X'.

Assuming $n = r + 1$, that is, $\dim \mathcal{H}_p = \dim \mathcal{V}_p + 1$, we see that $A_X : \mathcal{V}_p \to X^\perp = \{Y \in \mathcal{H}_p \mid g(X, Y) = 0\}$ is bijective, and thus, for every $Y \in X^\perp$, we can write $Y = A_X V$ for some vertical vector V. By (4.1), we get

$$g(A_X Y, A_X Y) = g(A_X A_X V, A_X A_X V) = g(X, X)^2 g(V, V). \tag{4.4}$$

On the other hand, by Corollary 2.3(b), we have

$$g(Y, Y) = g(A_X V, A_X V) = -g(X, X)g(V, V). \tag{4.5}$$

Combining equations (4.4) and (4.5), we simply get $g(A_X Y, A_X Y) = -g(X, X)g(Y, Y)$ for every $Y \in X^\perp$, which implies that $A_X A_X Z = g(X, X)Z - g(X, Z)X$ for any horizontal vector Z. Now, by Corollary 2.3(a), we obtain

$$\begin{aligned} R'(X, Y, X, Y) &= -g(X, X)g(Y, Y) + g(X, Y)^2 + 3g(A_X Y, A_X Y) \\ &= -4(g(X, X)g(Y, Y) - g(X, Y)^2), \end{aligned} \tag{4.6}$$

which means that B has constant curvature -4.

Conversely, if B has constant curvature -4, then, by (4.6), we get $g(A_X Y, A_X Y) = -g(X, X)g(Y, Y)$ for every $Y \in X^\perp$, which implies $A_X A_X Y = g(X, X)Y$ for every $Y \in X^\perp$. Therefore, by (4.1), $A_X : \mathcal{V} \to X^\perp$ is bijective with its inverse given by $(A_X)^{-1}(Y) = (1/(g(X, X)))A_X Y$ for $Y \in X^\perp$. As a consequence, $n - 1 = \dim X^\perp = \dim \mathcal{V}_p = r$. □

THEOREM 4.7. *In the set-up of Theorem* 1.1, *if* $n = r + 1$ *and* $0 < s < n$, *then* π *falls into one of the following cases*:

(a) $\pi : H_1^3 \to H_1^2(-4) = \mathbb{A}H^1$,
(b) $\pi : H_3^7 \to H_2^4(-4) = \mathbb{B}H^1$,
(c) $\pi : H_7^{15} \to H_4^8(-4)$.

Proof. First, recall that B has constant curvature -4 by Proposition 4.6. Let X, $Y \in \mathcal{H}_p$ such that $g(X, X) = 1$ and $g(Y, Y) = -1$. Let $\mathcal{B} = \{X, A_X v_1, \ldots, A_X v_r\}$, $\mathcal{B}' = \{Y, A_Y v_1', \ldots, A_Y v_r'\}$ be two special bases of $\mathcal{H}_p$. The index of $\mathcal{B}$, the number of time-like vectors, is $r - r'$, while the index of $\mathcal{B}'$ is $r' + 1$. Therefore, $r = 2r' + 1$, $s = r' + 1$ and $n = 2(r' + 1)$. The pseudo-Riemannian submersion π is of the form $\pi : H_{2r'+1}^{4r'+3} \to B_{r'+1}^{2r'+2}$.

By a theorem due to Reckziegel [**40**], the horizontal distribution $\mathcal{H}$ of a pseudo-Riemannian submersion with totally geodesic fibres is an Ehresmann connection, and thus, by Ehresmann [**13**], π is a locally trivial fibration, which always comes with a long exact homotopy sequence

$$\cdots \longrightarrow \pi_2(B) \longrightarrow \pi_1(F_{\pi(p)}) \longrightarrow \pi_1(H_{2r'+1}^{4r'+3}) \longrightarrow \pi_1(B) \longrightarrow \pi_0(F_{\pi(p)}) \to \cdots. \tag{4.7}$$

Now, we proceed in two cases: (i) $r' = 0$ and (ii) $r' \geqslant 1$.

Case $r' = 0$. Since the fibres are connected, totally geodesic, one-dimensional submanifolds (when $r' = 0$), any fibre is the image of a space-like geodesic in $H_{2r'+1}^{4r'+3}$. Thus, the fibres are

diffeomorphic to the real line (see [**38**, p. 113]) and $\pi_1(F_{\pi(p)}) = 0$. The long exact homotopy sequence (4.7) gives $\pi_1(B) = \pi_1(H_1^3) = \mathbb{Z}$. Because B is of constant curvature -4, and, by Proposition 4.1, is also complete, it simply follows that B is isometric to the pseudo-hyperbolic space $H_1^2(-4)$, and that corresponds to (a).

Case $r' \geqslant 1$. By the long exact homotopy sequence (4.7), and by $\pi_1(H_{2r'+1}^{4r'+3}) = \pi_1(S^{2r'+1}) = 0$, we get $\pi_1(B) = 0$. The manifold B is additionally complete and of constant curvature -4. Therefore, B must be isometric to $H_{r'+1}^{2r'+2}(-4)$. The case $r' = 1$ corresponds to (b).

We now assume that $r' \geqslant 2$. Since, for $r' \geqslant 2$, $\pi_2(B) = \pi_2(H_{r'+1}^{2r'+2}(-4)) = \pi_2(S^{r'+1} \times \mathbb{R}^{r'+1}) = 0$ and $\pi_1(H_{2r'+1}^{4r'+3}) = \pi_1(S^{2r'+1} \times \mathbb{R}^{2r'+2}) = 0$, the long exact homotopy sequence (4.7) gives $\pi_1(F_{\pi(p)}) = 0$. On the other hand, since the fibres are totally geodesic in $H_{2r'+1}^{4r'+3}$, the fibres are complete and of curvature -1. Therefore, the fibres must be isometric to $H_{r'}^{2r'+1}$. By Lemma 4.3, the fibres are also parallelizable, and that restricts the choices of $r' \geqslant 2$ to $r' \in \{3, 7\}$. The value $r' = 3$ corresponds to the cases (c).

We now show that the case $r' = 7$ is not possible, namely, we see that there is no pseudo-Riemannian submersion $\pi : H_{15}^{31} \to H_8^{16}(-4)$ with connected totally geodesic fibres. By Ranjan [**39**], the linear map $\mathcal{U} : \mathcal{V}_p \to \mathrm{Hom}(\mathcal{H}_p, \mathcal{H}_p)$ given by $\mathcal{U}(V)(X) = A_X V$ extends to a Clifford representation $\mathcal{U} : \mathrm{Cl}(\mathcal{V}_p, -\hat{g}) \to \mathrm{Hom}(\mathcal{H}_p, \mathcal{H}_p)$, namely, $\mathcal{U}(v)\mathcal{U}(w) + \mathcal{U}(w)\mathcal{U}(v) = 2g(v, w)\mathrm{Id}$ for every $v, w \in \mathcal{V}_p$, because of Corollary 2.3(b). This makes the sixteen-dimensional space $\mathcal{H}_p$ a $\mathrm{Cl}(\mathcal{V}_p)$-module, which, as usual, decomposes into irreducible $\mathrm{Cl}(\mathcal{V}_p)$-modules. On the other hand, the signature of the inner product $-\hat{g}(v, w) = -g(v, w)$ of $\mathcal{V}_p$ is $(7, 8)$, and from the Classification Table of the Clifford algebras [**32**, p. 29], we see that $\mathrm{Cl}(\mathcal{V}_p, -\hat{g}) = \mathrm{Cl}_{(7,8)} = \mathbb{R}(128) \oplus \mathbb{R}(128)$. In consequence, any irreducible $\mathrm{Cl}(\mathcal{V}_p)$-module is of dimension 128, and thus the dimension of $\mathcal{H}_p$ is too small to allow a non-trivial Clifford representation $\mathcal{U} : \mathrm{Cl}(\mathcal{V}_p) \to \mathrm{Hom}(H_p, H_p)$ as above. □

The case $s = 0$ corresponds to a Riemannian base space which was completely classified in [**4**], while the case $s = n$ is of a Riemannian submersion from spheres (classified in [**15**, **39**]) when we apply a change of signs of the metrics of the total and of the base spaces. By Corollary 4.5, the metrics induced on fibres are negative definite if $s \in \{0, n\}$.

THEOREM 4.8 [**4**, **15**, **39**]. *In the set-up of Theorem* 1.1, *we assume* $n = r + 1$. *Then the following assertions are true.*

(i) *If* $s = 0$, *then* π *is one of the following*:
  (a) $\pi : H_1^3 \to H^2(-4)$,
  (b) $\pi : H_3^7 \to H^4(-4)$,
  (c) $\pi : H_7^{15} \to H^8(-4)$.

(ii) *If* $s = n$, *then* π *is one of the following*:
  (a′) $\pi : H_3^3 \to H_2^2(-4)$,
  (b′) $\pi : H_7^7 \to H_4^4(-4)$,
  (c′) $\pi : H_{15}^{15} \to H_8^8(-4)$.

4.3. *Case* $n \neq r + 1$

We show that B is a complete, simply connected, special Osserman pseudo-Riemannian manifold.

4.3.1. *Special Osserman manifolds.* Following [**19**], we recall the definitions of a Jacobi operator and of a special Osserman pseudo-Riemannian manifold.

DEFINITION 4.9. Let (B, g') be a pseudo-Riemannian manifold and let R' be the Riemann curvature tensor of (B, g'). For $x \in T_bB$, we consider the linear map $R'(\cdot, x)x : T_bB \to T_bB$. Since $g'(R'(z, x)x, x) = 0$, we have $\mathrm{Im}(R'(\cdot, x)x) \subset x^\perp$, where $x^\perp = \{y \in T_bB \,|\, g'(y, x) = 0\}$. For $x \in S_bB = \{x \in T_bB \,|\, g'(x, x) = \pm 1\}$, the restriction $R'_x : x^\perp \to x^\perp$ of $R'(\cdot, x)x$ to $x^\perp$ is called the *Jacobi operator* with respect to x, that is, $R'_x(z) = R'(z, x)x$ for $z \in x^\perp$.

DEFINITION 4.10. A pseudo-Riemannian manifold (B, g') is called *special Osserman* if the following two conditions are satisfied at each $b \in B$:

(I) For every $x \in S_bB$, the Jacobi operator $R'_x : x^\perp \to x^\perp$ is diagonalizable with exactly two distinct eigenvalues $\varepsilon_x\lambda$ and $\varepsilon_x\mu$, where $\varepsilon_x = g'(x, x)$ and $\lambda, \mu \in \mathbb{R}$.
(II) Let $E_\lambda(x) = \mathrm{span}\{x\} \oplus \ker(R'_x - \varepsilon_x\lambda\mathrm{Id})$. For each $x \in S_bB$, if $z \in E_\lambda(x) \cap S_bB$, then $E_\lambda(x) = E_\lambda(z)$, and moreover if $y \in S_bB \cap \ker(R'_x - \varepsilon_x\mu\mathrm{Id})$, then $x \in \ker(R'_y - \varepsilon_y\mu\mathrm{Id})$.

The values λ and μ involved in the previous definition are not interchangeable, for example, if (B, g', J) is the complex or the para-complex pseudo-hyperbolic space of real dimension $2n > 2$, then $\mu = \lambda/4$ and $\ker(R'_x - \varepsilon_x\lambda\mathrm{Id}) = \mathrm{span}\{Jx\}$ is one-dimensional, while $\ker(R'_x - \varepsilon_x\mu\mathrm{Id}) = \{x, Jx\}^\perp = \{z \,|\, g'(z, x) = g'(z, Jx) = 0\}$ is $(2n-2)$-dimensional.

4.3.2. *The base space is Special Osserman.* For a pseudo-Riemannian submersion $\pi : (M, g) \to (B, g')$, we denote by $R'_{X'}$ the Jacobi operator of (B, g') with respect to a vector $X' \in T_bB$ and, for $X, Y \in \mathcal{H}_p$, we also denote by R'_XY the horizontal lift of $R'_{\pi_*(X)}(\pi_*Y)$ and we consider R'_X as an operator $R'_X : X^\perp \to X^\perp$, with $X^\perp = \{Y \in \mathcal{H}_p \,|\, g(Y, X) = 0\}$.

THEOREM 4.11. *In the set-up of Theorem* 1.1, *if* $n \neq r + 1$, *then* B *is special Osserman.*

Proof. Let $b \in B$, $X' \in S_bB$, $Z' \in T_bB$ and $p \in \pi^{-1}(b)$. Let $X, Z \in \mathcal{H}_p$ be the horizontal lifts of X' and Z', respectively. By Corollary 2.3(a), R'_X is given by

$$R'_X(Z) = R'(Z, X)X = R(Z, X)X - 3A_XA_XZ = R_XZ - 3A_XA_XZ. \tag{4.8}$$

Let $\{v_1, v_2, \dots, v_r\}$ be an orthonormal basis in $\mathcal{V}_p$, that is, $g(v_i, v_j) = \varepsilon_i\delta_{i,j}$ with $\varepsilon_i \in \{-1, 1\}$. Let

$$\mathcal{B} = \{L_0, A_{L_0}v_1, \dots, A_{L_0}v_r, \dots, L_{k-1}, A_{L_{k-1}}v_1, \dots, A_{L_{k-1}}v_r\}$$

be a special basis of $\mathcal{H}_p$, that is, an orthonormal basis $\mathcal{B}$ with $L_0 = X$ and $A_{L_\alpha}L_\beta = 0$ for every $\alpha, \beta \in \{0, \dots, k-1\}$. We show that $R'_X : X^\perp \to X^\perp$ is diagonalizable with respect to $\mathcal{B}$ and R'_X has exactly two eigenvalues. By (4.8) and (4.1), we have

$$\begin{aligned} R'_X(A_Xv_i) &= R_X(A_Xv_i) - 3A_XA_XA_Xv_i \\ &= -g(X, X)A_Xv_i - 3g(X, X)A_Xv_i = -4\varepsilon_XA_Xv_i, \end{aligned} \tag{4.9}$$

which gives $R'_{X'}(\pi_*(A_Xv_i)) = \pi_*(R'_X(A_Xv_i)) = -4\varepsilon_{X'}\pi_*(A_Xv_i)$. Since

$$0 = g(A_Xv_j, A_{L_\alpha}v_i) = -g(v_j, A_XA_{L_\alpha}v_i)$$

for every i, j and every $\alpha \geqslant 1$, we get $A_XA_{L_\alpha}v_i = 0$, which implies that

$$R'_X(A_{L_\alpha}v_i) = R_X(A_{L_\alpha}v_i) - 3A_XA_XA_{L_\alpha}v_i = -g(X, X)A_{L_\alpha}v_i = -\varepsilon_XA_{L_\alpha}v_i. \tag{4.10}$$

Projecting (4.10) to the base, we have $R'_{X'}(\pi_*(A_{L_\alpha}v_i)) = -\varepsilon_{X'}\pi_*(A_{L_\alpha}v_i)$. Since $A_XL_\alpha = 0$ by construction, we see that

$$R'_X(L_\alpha) = R_X(L_\alpha) - 3A_XA_XL_\alpha = -g(X, X)L_\alpha = -\varepsilon_XL_\alpha \tag{4.11}$$

for every $\alpha \geqslant 1$ and every i. Therefore, $R'_{X'}(\pi_*(L_\alpha)) = -\varepsilon_{X'}\pi_*(L_\alpha)$. Summarizing, the Jacobi operator $R'_{X'}$ is diagonalizable with the eigenvalues $-4\varepsilon_{X'}$ and $-\varepsilon_{X'}$, and moreover their eigenspaces are

$$\ker(R'_{X'} + 4\varepsilon_{X'}\mathrm{Id}) = \{\pi_*(A_X v_1), \dots, \pi_*(A_X v_r)\} \quad \text{and,} \tag{4.12}$$

$$\begin{aligned}\ker(R'_{X'} + \varepsilon_{X'}\mathrm{Id}) = \{&\pi_*(L_1), \pi_*(A_{L_1} v_1), \dots, \pi_*(A_{L_1} v_r), \dots,\\ &\pi_*(L_{k-1}), \pi_*(A_{L_{k-1}} v_1), \dots, \pi_*(A_{L_{k-1}} v_r)\}.\end{aligned} \tag{4.13}$$

Now, we check that Condition (II) of Definition 4.10 holds.

LEMMA 4.12. *If $Y' \in E_{-4}(X')$, $g'(X', X') = \pm 1$ and $g'(Y', Y') = \pm 1$, then $X' \in E_{-4}(Y')$.*

Proof of Lemma 4.12. By (4.12),

$$E_{-4}(X') = \mathrm{span}\{X'\} \oplus \ker(R'_{X'} + 4\varepsilon_{X'}\mathrm{Id}) = \mathrm{span}\{\pi_* X, \pi_*(A_X v_1), \dots, \pi_*(A_X v_r)\},$$

and, thus, the horizontal lift Y of Y' satisfies

$$Y = aX + A_X U \tag{4.14}$$

for some $a \in \mathbb{R}$ and some vertical vector U. By (4.14),

$$g(A_X U, A_X U) = g(Y, Y) - a^2 g(X, X). \tag{4.15}$$

To prove $X' \in E_{-4}(Y')$, it is sufficient to show that X can be written as

$$X = bY + A_Y W \tag{4.16}$$

for some $b \in \mathbb{R}$ and some vertical vector W. Applying A_Y to (4.16), we get $A_Y X = bA_Y Y + A_Y A_Y W = g(Y, Y)W$, which gives $W = -A_X Y/(g(Y, Y))$. Similarly, applying A_X to (4.14), we obtain $A_X Y = A_X A_X U = g(X, X)U$. Substituting Y and W into (4.16), we obtain an equation in $b \in \mathbb{R}$

$$X = b(aX + A_X U) - \frac{g(X, X)}{g(Y, Y)} A_{aX + A_X U} U, \quad \text{which is equivalent to} \tag{4.17}$$

$$X = baX - \frac{g(X, X)}{g(Y, Y)} A_{A_X U} U + \left(b - \frac{ag(X, X)}{g(Y, Y)}\right) A_X U. \tag{4.18}$$

By Corollary 2.3(b),

$$g(A_X U, A_Z U) = -g(X, Z)g(U, U) \tag{4.19}$$

for every horizontal vectors X, Z and for every vertical vector U. Since A is skew-symmetric with respect to g and alternating, we have $g(A_X U, A_Z U) = -g(A_Z A_X U, U) = g(A_{A_X U} Z, U) = -g(Z, A_{A_X U} U)$, which, by (4.19), implies that $A_{A_X U} U = g(U, U)X$. Then

$$\begin{aligned} baX - \frac{g(X, X)}{g(Y, Y)} A_{A_X U} U &= \left(ba - \frac{g(X, X)g(U, U)}{g(Y, Y)}\right) X = \left(ba + \frac{g(A_X U, A_X U)}{g(Y, Y)}\right) X \\ &= \left(ba + \frac{g(Y, Y) - a^2 g(X, X)}{g(Y, Y)}\right) X = X - a\left(b - \frac{ag(X, X)}{g(Y, Y)}\right) X, \end{aligned}$$

by (4.15). Therefore, (4.18) has the unique solution $b = ag(X, X)/g(Y, Y)$. □

LEMMA 4.13. *If $Y' \in \ker(R'_{X'} + \varepsilon_{X'}\mathrm{Id})$, $g'(X', X') = \pm 1$ and $g'(Y', Y') = \pm 1$, then $X' \in \ker(R'_{Y'} + \varepsilon_{Y'}\mathrm{Id})$.*

Proof of Lemma 4.13. Let X and Y be the horizontal lifts of X' and Y', respectively. The Jacobi operator $R'_{X'}$ satisfies

$$R'_{X'}(Y') = \pi_*(R_X(Y) - 3A_XA_XY) = -g'(X',X')Y' - 3\pi_*(A_XA_XY) \tag{4.20}$$

for any $Y' \in X'^{\perp}$. Therefore, $Y' \in \ker(R'_{X'} + \varepsilon_{X'}\mathrm{Id})$ if and only if $A_XA_XY = 0$. Since, by Lemma 4.2(a), $A_X : \mathcal{V} \to \mathcal{H}$ is injective, $A_XY = 0$, hence $A_YX = 0$, which implies that $R'_{Y'}(X') = \pi_*(-3A_YA_YX + R_Y(X)) = -g'(Y',Y')X' = -\varepsilon_{Y'}X'$. □

These conclude that B is a special Osserman pseudo-Riemannian manifold. □

In the next theorem, we identify the geometry of the base space and we find the dimension and the index of the total space in terms of the geometry of the base space.

THEOREM 4.14. *Let $\pi : H^{n+r}_{s+r'} \to B^n_s$ be a pseudo-Riemannian under the assumptions of Theorem* 1.1. *If $n \neq r+1$, then π falls in one of the following cases*:

(a) $H^{2m+1}_{2t+1} \to \mathbb{C}H^m_t$,
(b) $H^{2m+1}_{m} \to \mathbb{A}H^m$,
(c) $H^{4m+3}_{4t+3} \to \mathbb{H}H^m_t$,
(d) $H^{4m+3}_{2m+1} \to \mathbb{B}H^m$,
(e) $H^{23}_{7} \to \mathbb{O}H^2$,
(f) $H^{23}_{15} \to \mathbb{O}H^2_1$,
(g) $H^{23}_{23} \to \mathbb{O}H^2_2$,
(h) $H^{23}_{q} \to \mathbb{O}'P^2$,

for $0 \leqslant t \leqslant m$ and $m \geqslant 2$, and for some $8 \leqslant q \leqslant 15$.

Proof. We first prove that B is simply connected. When $s + r' > 1$, $H^{n+r}_{s+r'}$ is simply connected and thus, by Proposition 4.1, B is also simply connected. If $s + r' = 1$, then either (i) $s = 0$ and $r' = 1$, or (ii) $s = 1$ and $r' = 0$.

In the case (i) $s = 0$ and $r' = 1$, the base space is Riemannian, which, by Magid [**33**], must be isometric to $\mathbb{C}H^m$, and thus B is simply connected.

In the case (ii) $s = 1$ and $r' = 0$, B is Lorentzian Osserman at the point p, which by García-Río, Kupeli and Vázquez-Lorenzo [**19**], must be of constant curvature at the point p. On the other hand, B has constant curvature if and only if $n = r + 1$. This contradicts our working assumption $n \neq r + 1$. These conclude that B is simply connected.

By the classification theorem of simply connected, complete special Osserman pseudo-Riemannian manifolds [**10**, **19**], B is isometric to one of the following:

(a) a definite or indefinite complex space form of signature $(2m - 2s, 2s)$, $0 \leqslant s \leqslant m$;
(b) a definite or indefinite quaternionic space form of signature $(4m - 4s, 4s)$, $0 \leqslant s \leqslant m$;
(c) a para-complex space form of signature (m, m);
(d) a para-quaternionic space form of signature $(2m, 2m)$;
(e) a Cayley plane of octonions with definite or indefinite metric, or a Cayley plane of para-octonions with indefinite metric of signature $(8, 8)$.

Any non-flat complete, simply connected, para-complex space form is isometric to the symmetric space $\mathrm{SL}(m+1,\mathbb{R})/(\mathrm{SL}(m,\mathbb{R}) \times \mathbb{R}) = \mathbb{A}P^m$ (see [**10**, **11**, **19**]), and any non-flat complete, simply connected, para-quaternionic space form is isometric to the symmetric space $\mathrm{Sp}(m+1,\mathbb{B})/(\mathrm{Sp}(m,\mathbb{B})\mathrm{Sp}(1,\mathbb{B})) = \mathrm{Sp}(2m+2,\mathbb{R})/(\mathrm{Sp}(2m,\mathbb{R})SL(2,\mathbb{R})) = \mathbb{B}P^m$ (see [**10**, **12**, **19**, **20**]).

By the proof of Theorem 4.11, the values λ and μ of Definition 4.10 are negative, namely $\lambda = -4$ and $\mu = -1$. Then B must be isometric to one of the following spaces:

$$\mathbb{C}H_t^m,\ \mathbb{H}H_t^m,\ \mathbb{A}P^m,\ \mathbb{B}P^m,\ \mathbb{O}H^2,\ \mathbb{O}H_1^2,\ \mathbb{O}H_2^2 \quad \text{or } \mathbb{O}'P^2, \tag{4.21}$$

with $m \geqslant 2$ and $0 \leqslant t \leqslant m$. By (4.12), we simply have $\dim\ker(R'_{X'} + 4\varepsilon_{X'}\mathrm{Id}) = r$, and in particular the following conditions are satisfied.

(a) If $B \in \{\mathbb{C}H_t^m, \mathbb{A}P^m\}$, then $\ker(R'_{X'} + 4\varepsilon_{X'}\mathrm{Id}) = \mathrm{span}\{IX'\}$, where I is a complex or para-complex structure. Thus, $r = 1$ and $n + r = 2m + 1$.

(b) If $B \in \{\mathbb{H}H_t^m, \mathbb{B}P^m\}$, then $\ker(R'_{X'} + 4\varepsilon_{X'}\mathrm{Id}) = \mathrm{span}\{IX', JX', KX'\}$, with $\{I, J, K\}$ a local quaternionic or para-quaternionic structure. Therefore, $r = 3$ and $n + r = 4m + 3$.

(c) If $B \in \{\mathbb{O}H_i^2, \mathbb{O}'P^2\}_{0\leqslant i\leqslant 2}$, then $\dim\ker(R'_{X'} + 4\varepsilon_{X'}\mathrm{Id}) = 7$. Thus, $r = 7$ and $n + r = 23$.

Now, we find the index of the total space for the choices of B in (4.21).

Case 1: $B \in \{\mathbb{C}H_t^m, \mathbb{H}H_t^m, \mathbb{O}H_i^2\}_{0\leqslant t\leqslant m,\ 0\leqslant i\leqslant 2}$. In this case, the Riemann tensor satisfies

$$R'(X', Y', X', Y') \leqslant -(g(X', X')g(Y', Y') - g(X', Y')^2) \tag{4.22}$$

for any X', Y' vectors on B. Let $\{v_i\}_{i\in\{1,\dots,r\}}$ be an orthonormal basis of $\mathcal{V}_p$ and let X be the horizontal lift of a non-null vector $X' \in T_{\pi(p)}B$. Taking $Y' = \pi_*(A_X v_i)$, inequality (4.22) becomes

$$R'(\pi_*X, \pi_*(A_X v_i), \pi_*X, \pi_*(A_X v_i)) \leqslant -g(X, X)g(A_X v_i, A_X v_i). \tag{4.23}$$

On the other hand, by Corollary 2.3(a) and by (4.1),

$$R'(\pi_*X, \pi_*(A_X v_i), \pi_*X, \pi_*(A_X v_i)) = -4g(X, X)g(A_X v_i, A_X v_i).$$

Now, (4.23) implies $0 \leqslant 3g(X, X)g(A_X v_i, A_X v_i) = -g(X, X)^2 g(v_i, v_i)$ for any i. Thus, the fibres are negative definite. Therefore, in Case 1, π should be in one of (a), (c), (e)–(g) of Theorem 4.14. Note that, in Case 1, B is isotropic, which means that, for any $b \in B$ and any $t \in \mathbb{R}$, the group of isometries of B preserving b acts transitively on $\{Z \in T_bB \,|\, g'(Z, Z) = t,\ Z \neq 0\}$ (see [**47**, p. 367]).

Case 2: $B = \mathbb{A}P^m$. Since $B = \mathbb{A}P^m$ is a para-quaternionic space form of para-holomorphic curvature $\lambda = -4$,

$$R'(X', Y', X', Y') \geqslant -(g(X', X')g(Y', Y') - g(X', Y')^2). \tag{4.24}$$

By a similar argument to Case 1, specializing (4.24) for a non-null vector X' and $\pi_*(A_X v_1)$, we get $0 \geqslant 3g(X, X)g(A_X v_1, A_X v_1) = -g(X, X)^2 g(v_1, v_1)$ and thus the fibres are positive definite and π falls in (b).

Case 3: $B = \mathbb{B}P^m$. We shall show that the fibres have signature $(2, 1)$. Note that $(\mathbb{B}P^m, g')$ has a natural para-quaternionic Kähler structure and its curvature tensor satisfies the relation

$$\begin{aligned} R'(X', Y', X', Y') = &-(g'(X', X')g'(Y', Y') - g'(X', Y')^2 \\ &- 3g'(J_1X', Y')^2 - 3g'(J_2X', Y')^2 + 3g'(J_3X', Y')^2), \end{aligned} \tag{4.25}$$

where $\{J_1, J_2, J_3\}$ is a local para-quaternionic structure, a triple of $(1, 1)$-tensors satisfying $J_1J_2 = -J_2J_1 = J_3$, $J_i^2 = \varepsilon_i\mathrm{Id}$, $g'(J_iX', Y') + g'(X', J_iY') = 0$ and $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1$. Obviously, for any X', Y' such that $g'(J_3X', Y') = 0$, we have

$$R'(X', Y', X', Y') \geqslant -(g'(X', X')g'(Y', Y') - g'(X', Y')^2). \tag{4.26}$$

Let $X' \in T_b\mathbb{B}P^m$ such that $g'(X', X') = \pm 1$ and let X be its horizontal lift at $p \in \pi^{-1}(b)$. Let $J_3X \in \mathcal{H}_p$ be the horizontal lift of J_3X'. By (4.25), $R'(X', J_3X', X', J_3X') = -4g'(X', X')g'(J_3X', J_3X')$, and thus

$$g(A_XJ_3X, A_XJ_3X) = -g(X, X)g(J_3X, J_3X) = -g(X, X)^2 = -1,$$

by Corollary 2.3(a). Let $\{v_1, v_2, v_3\}$ be an orthonormal basis of $\mathcal{V}_p$ such that $v_3 = A_X J_3 X$. We simply note that $g(v_3, v_3) = -1$. For $i \in \{1, 2\}$, taking $Y' = \pi_*(A_X v_i)$ in (4.26), we get

$$R'(X', \pi_*(A_X v_i), X', \pi_*(A_X v_i)) \geqslant -g'(X', X')g'(\pi_*(A_X v_i), \pi_*(A_X v_i)). \tag{4.27}$$

On the other hand, $R'(X, A_X v_i, X, A_X v_i) = -4g(X, X)g(A_X v_i, A_X v_i)$. Thus, (4.27) becomes $0 \geqslant 3g(X, X)g(A_X v_i, A_X v_i) = -3g(X, X)^2 g(v_i, v_i)$ for $i \in \{1, 2\}$. Therefore, $g(v_i, v_i) > 0$ for $i \in \{1, 2\}$. □

To see that the cases (e)–(h) of Theorem 4.14 never occur, we first recall the notion of Clifford structure.

4.3.3. *Clifford structures.* We adapt the definition of Clifford structure introduced by Gilkey [**21**] and Gilkey, Swann and Vanhecke [**23**] to pseudo-Riemannian geometry.

Definition 4.15. Let (B, g') be a pseudo-Riemannian manifold and let R' be its curvature tensor. The space (B, g') has a Cliff(ν)-structure if at every point b there exist (1,1)-tensors $J_1, J_2, \ldots, J_\nu$ such that

$$\begin{aligned} R'(x, y)z = \lambda_0(g'(y, z)x - g'(x, z)y) + \frac{1}{3}\sum_{s=1}^{\nu} \varepsilon_s(\lambda_s - \lambda_0)(g'(J_s y, z)J_s x \\ - g'(J_s x, z)J_s y - 2g'(J_s x, y)J_s z) \end{aligned} \tag{4.28}$$

for any $x, y, z \in T_b B$, where $\lambda_0, \lambda_1, \ldots, \lambda_\nu : B \to \mathbb{R}$, $\lambda_s(b) \neq \lambda_0(b)$ for $s \geqslant 1$, and $g'(J_s x, y) = -g'(x, J_s y)$ and $J_s J_t + J_t J_s = -2\varepsilon_s \delta_{s,t}\mathrm{Id}$, with $\varepsilon_s = \pm 1$.

The Jacobi operator at the point b of a manifold with a Cliff(ν)-structure is given by

$$R'_y(x) = \lambda_0 g'(y, y)x + \sum_{s=1}^{\nu} \varepsilon_s(\lambda_s - \lambda_0)g'(x, J_s y)J_s y \tag{4.29}$$

for any $x \in y^\perp$. Moreover,

$$R'_y(J_s y) = \lambda_s g'(y, y)J_s y \quad \text{for any } s \in \{1, \ldots, \nu\} \quad \text{and} \tag{4.30}$$

$$R'_y(x) = \lambda_0 g'(y, y)x \quad \text{for any } x \in \{y, J_1 y, \ldots, J_r y\}^\perp, \tag{4.31}$$

and thus a pseudo-Riemannian manifold with a Cliff(ν)-structure is pointwise Osserman (see [**22**]).

In the Riemannian set-up, Clifford structures turned out to be a very valuable tool for the Osserman Conjecture. In [**23**], Gilkey, Swann and Vanhecke suggested a two-step approach: (i) show that the pointwise Osserman condition implies the existence of a Clifford structure with (4.30), (4.31) and (ii) find the manifolds having the curvature tensors of (i). Using this approach, Nikolayevsky proved the Osserman conjecture in dimension $n \neq 16$; see [**35**, **36**]. In dimension $n = 16$, the Cayley planes $\mathbb{O}H^2$, $\mathbb{O}P^2$ do not admit Clifford structures [**36**, p. 510] and the Osserman Conjecture remains open.

Since the curvature tensor formulae of the Cayley planes of octonions or of para-octonions are similar to that of $\mathbb{O}P^2$, in particular, the eigenspace of the Jacobi operator for $\lambda = -4$ satisfies

$$\ker(R'_{(a,b)} + 4\varepsilon_{(a,b)}\mathrm{Id}) = \begin{cases} \left\{ \left(c,\ \dfrac{1}{|a|^2}(b\bar{a})c \right) \mid \mathrm{Re}\,(c\bar{a}) = 0 \right\}, & \text{if } |a|^2 \neq 0, \\ \left\{ \left(\dfrac{1}{|b|^2}(a\bar{b})d,\ d \right) \mid \mathrm{Re}\,(d\bar{b}) = 0 \right\}, & \text{if } |b|^2 \neq 0, \end{cases} \tag{4.32}$$

for any $(a,b) \in S_bB$ (see [29]), one can easily see, by analogy to [36, p. 510], that $\mathbb{O}H_2^2, \mathbb{O}H_1^2, \mathbb{O}H^2$, $\mathbb{O}'P^2$ do not admit Cliff(7)-structures. To exclude (e)–(h) of Theorem 4.14, it is now sufficient to establish the following theorem.

THEOREM 4.16. *Let $\pi : M \to B$ be a pseudo-Riemannian submersion with connected, totally geodesic fibres. If M has constant curvature $c \neq 0$, then B has a* Cliff(r)*-structure.*

Proof. Without loss of the generality, we may assume $c = \pm 1$. Let $p \in M$ and $b = \pi(p) \in B$. Let $\{v_1, \ldots, v_r\}$ be an orthonormal basis of $\mathcal{V}_p$. For any $1 \leqslant s \leqslant r$, let $\varepsilon_s = cg(v_s, v_s) \in \{-1, 1\}$ and let $J_s(X') = \pi_*(A_X v_s)$, where $X \in T_pM$ is the horizontal lift of $X' \in T_bB$. For any vertical vector $v \in \mathcal{V}_p$, we define the linear map $A^v : \mathcal{H}_p \to \mathcal{H}_p$ given by $A^v(x) = A_x v$ for $x \in \mathcal{H}_p$. Since M has constant curvature c, by Ranjan's paper [39] we have

$$A^v A^w + A^w A^v = -2cg(v,w)\mathrm{Id} \tag{4.33}$$

for any v, w vertical vectors. Thus, $J_sJ_t + J_tJ_s = -2cg(v_s, v_t)\mathrm{Id} = -2\varepsilon_s\delta_{s,t}\mathrm{Id}$. Also, by Ranjan's paper [39] we have $g(A^vX, Y) = -g(X, A^vY)$ for any $X, Y \in \mathcal{H}_p$, which simply implies $g'(J_sX', Y') = -g'(X', J_sY')$ for every $X', Y' \in T_bB$ and every $1 \leqslant s \leqslant r$.

Now, we show that the Jacobi operator of B satisfies (4.29). Let $X', Y' \in T_bB$ with $g'(Y', Y') = \pm 1$, and $g(X', Y') = 0$. Let X and Y be the horizontal lifts of X' and Y', respectively. Let

$$\mathcal{B} = \{L_0, A_{L_0}v_1, \ldots, A_{L_0}v_r, \ldots, L_{k-1}, A_{L_{k-1}}v_1, \ldots, A_{L_{k-1}}v_r\}$$

be a special basis of $\mathcal{H}_p$ such that $L_0 = Y$. We recall that $\mathcal{B}$ is orthonormal and that $A_{L_\alpha}L_\beta = 0$ for every $\alpha, \beta \in \{0, \ldots, k-1\}$, by construction. X can be written as

$$\begin{aligned} X = g(X,Y)Y + \sum_\alpha \frac{g(X, L_\alpha)}{g(L_\alpha, L_\alpha)} L_\alpha \\ + \sum_i \frac{g(X, A_Y v_i)}{cg(Y,Y)g(v_i,v_i)} A_Y v_i + \sum_{i,\alpha} \frac{g(X, A_{L_\alpha} v_i)}{cg(L_\alpha, L_\alpha)g(v_i,v_i)} A_{L_\alpha} v_i. \end{aligned} \tag{4.34}$$

Since $\mathcal{B}$ is orthonormal, $A_Y A_{L_\alpha} v_i = 0$ by the proof of Theorem 4.11. Applying $A_Y A_Y$ to (4.34), we get

$$\begin{aligned} A_Y A_Y X &= \sum_i \frac{g(X, A_Y v_i)}{cg(Y,Y)g(v_i,v_i)} A_Y A_Y A_Y v_i \\ &= -c \sum_i \varepsilon_i g(X, A_Y v_i) A_Y v_i = -\sum_i \varepsilon_i cg(X, J_iY) J_iY. \end{aligned}$$

Then

$$R'_{Y'}(X') = \pi_*(R_Y X - 3A_Y A_Y X) = cg'(Y', Y')X' + 3c \sum_i \varepsilon_i g'(X', J_iY')J_iY'. \tag{4.35}$$

Polarizing (4.35), we get

$$\begin{aligned} R'(X', Y')Z' = c(g'(Y', Z')X' - g'(X', Z')Y') \\ + c \sum_{i=1}^r \varepsilon_i (g'(J_iY', Z')J_iX' - g'(J_iX', Z')J_iY' - 2g'(J_iX', Y')J_iZ'). \end{aligned}$$

□

COROLLARY 4.17. *There are no pseudo-Riemannian submersions $\pi : H_t^{23} \to B$ with connected, totally geodesic fibres from a 23-dimensional pseudo-hyperbolic space H_t^{23} onto any of the Cayley pseudo-hyperbolic planes of octonions $\mathbb{O}H_2^2, \mathbb{O}H_1^2, \mathbb{O}H^2$ or onto the Cayley projective plane of para-octonions $\mathbb{O}'P^2$.*

REMARK 4.18. Ranjan [39] proved that there are no Riemannian submersions $\pi : S^{23} \to \mathbb{O}P^2$ with connected, totally geodesic fibres (that is, (g) of Theorem 4.14). For a topological proof of this fact we refer the reader to [42].

5. *The theorem of uniqueness*

To prove Theorem 1.1, we need the following Theorem of Uniqueness.

THEOREM 5.1. *Let $\pi_1, \pi_2 : H_l^a \to B$ be two pseudo-Riemannian submersions with connected, totally geodesic fibres from a pseudo-hyperbolic space onto a pseudo-Riemannian manifold. Then there exists an isometry $f : H_l^a \to H_l^a$ such that $\pi_2 \circ f = \pi_1$. In particular, π_1 and π_2 are equivalent.*

Proof. The main ideas of the proof are: (1) for a given basepoint b construct special bases $\mathcal{B}^1$ and $\mathcal{B}^2$ for the fibres F_b^1 and F_b^2, respectively, such that $\mathcal{B}^1$ and $\mathcal{B}^2$ have the same projections to the base B and (2) show that the unique isometry sending $\mathcal{B}^1$ into $\mathcal{B}^2$ preserves the integrability tensors everywhere and sends fibres into fibres.

Let $b \in B$ and $p, q \in H_l^a$ such that $\pi_1(p) = \pi_2(q) = b$. We denote by $\mathcal{V}^1$ and $\mathcal{V}^2$ the vertical distributions of π_1 and π_2, and by $\mathcal{H}^1$ and $\mathcal{H}^2$ the horizontal distributions of π_1 and π_2, respectively.

Let $\{v_{1p}, \dots, v_{rp}\}$ be an orthonormal basis of $\mathcal{V}_p^1$ and let $X' \in T_bB$ such that $g'(X', X') = \pm 1$. We denote by X^1 and X^2 the π_1- and π_2-horizontal lifts of X' along the fibres $F_b^1 = \pi_1^{-1}(b)$ and $F_b^2 = \pi_2^{-1}(b)$, respectively. Let $(Y_1^1, Y_2^1, \dots, Y_r^1)$ and $(Y_1^2, Y_2^2, \dots, Y_r^2)$ be the π_1- and π_2-horizontal lifts of

$$\left(\frac{1}{-g(X^1, X^1)} \pi_{1*} A_{X^1} v_{1p}, \frac{1}{-g(X^1, X^1)} \pi_{1*} A_{X^1} v_{2p}, \dots, \frac{1}{-g(X^1, X^1)} \pi_{1*} A_{X^1} v_{rp} \right),$$

along F_b^1 and F_b^2, respectively. For each $i \in \{1, \dots, r\}$, we consider the vectors $v_i = A^1_{X^1} Y_i^1$, defined along F_b^1, and $w_i = A^2_{X^2} Y_i^2$ along F_b^2. By § 4, $\{v_1, \dots, v_r\}$ is a global orthonormal basis of vector fields on F_b^1, and we claim that so is $\{w_1, \dots, w_r\}$. Indeed, by Corollary 2.3(a), we see that

$$\begin{aligned} g(w_i, w_j) &= g(A^2_{X^2} Y_i^2, A^2_{X^2} Y_j^2) \\ &= (\tfrac{1}{3})(R'(\pi_{2*}X^2, \pi_{2*}Y_j^2, \pi_{2*}X^2, \pi_{2*}Y_j^2) - g(X^2, X^2)g(Y^2, Y^2) + g(X^2, Y^2)^2) \\ &= (\tfrac{1}{3})(R'(\pi_{1*}X^1, \pi_{1*}Y_j^1, \pi_{1*}X^1, \pi_{1*}Y_j^1) - g(X^1, X^1)g(Y^1, Y^1) + g(X^1, Y^1)^2) \\ &= g(A^1_{X^1} Y_i^1, A^1_{X^1} Y_j^1) = g(v_i, v_j) = \varepsilon_i \delta_{ij} \end{aligned}$$

along F_b^2. Let $\mathcal{B}^1 = \{L_0^1, A^1_{L_0^1} v_1, \dots, A^1_{L_0^1} v_r, \dots, L_{k-1}^1, A^1_{L_{k-1}^1} v_1, \dots, A^1_{L_{k-1}^1} v_r\}$ be a special basis of $\mathcal{H}^1$ along F_b^1 such that $L_0^1 = X^1$ (and $A^1_{L_\alpha^1} L_\beta^1 = 0$). Let $L_1^2, \dots, L_{k-1}^2$ be the π_2-horizontal lifts of $\pi_{1*}L_1^1, \dots, \pi_{1*}L_{k-1}^1$ along F_b^2. We take $L_0^2 = X^2$. Let

$$\mathcal{B}^2 = \{L_0^2, A^2_{L_0^2} w_1, \dots, A^2_{L_0^2} w_r, \dots, L_{k-1}^2, A^2_{L_{k-1}^2} w_1, \dots, A^2_{L_{k-1}^2} w_r\}. \qquad \square$$

LEMMA 5.2. (i) *The vector field $A^2_{X^2} w_i$ is basic along F_b^2 and $\pi_{1*} A^1_{X^1} v_i = \pi_{2*} A^2_{X^2} w_i$ for every i.*

(ii) *We have $A^2_{X^2} L_\alpha^2 = 0$ and $A^2_{L_\alpha^2} L_\beta^2 = 0$ for every α and β.*

(iii) *The basis $\mathcal{B}^2$ is a special basis of $\mathcal{H}^2$ along F_b^2 and $\pi_{1*} A^1_{L_\alpha^1} v_i = \pi_{2*} A^2_{L_\alpha^2} w_i$ for every i and α.*

Proof. Let $Z' \in T_bB$, and let Z^1 and Z^2 be the π_1- and π_2-horizontal lifts of Z' along F_b^1 and F_b^2, respectively. By Corollary 2.3(a), we get

$$\begin{aligned} g(A^2_{X^2}w_i, Z^2) &= -g(A^2_{X^2}Y_i^2, A^2_{X^2}Z^2) = (\tfrac{1}{3})(R(X^2,Y_i^2,X^2,Z^2) - R'(X^2,Y_i^2,X^2,Z^2)) \\ &= (\tfrac{1}{3})(R(X^1,Y_i^1,X^1,Z^1) - R'(X^1,Y_i^1,X^1,Z^1)) = -g(A^1_{X^1}Y_i^1, A^1_{X^1}Z^2) \\ &= g(A^1_{X^1}v_i, Z^1), \end{aligned}$$

which simply implies (i). By (i), we see that

$$\begin{aligned} g(A^2_{X^2}L^2_\alpha, w_i) &= -g(L^2_\alpha, A^2_{X^2}w_i) = -g'(\pi_{2*}L^2_\alpha, \pi_{2*}A^2_{X^2}w_i) \\ &= -g'(\pi_{1*}L^1_\alpha, \pi_{1*}A^1_{X^1}v_i) = g(A^1_{X^1}L^1_\alpha, v_i) = 0, \end{aligned} \tag{5.1}$$

for every i and α. Thus, $A^2_{X^2}L^2_\alpha = 0$. Therefore, by Proposition 2.2(a), we obtain that

$$\begin{aligned} 2g(A^2_{L^2_\alpha}L^2_\beta, w_i) &= 2g(A^2_{L^2_\alpha}L^2_\beta, A^2_{X^2}Y_i^2) = R'(L^2_\alpha, L^2_\beta, X^2, Y_i^2) \\ &\quad - R(L^2_\alpha, L^2_\beta, X^2, Y_i^2) + g(A^2_{L^2_\beta}X^2, A^2_{L^2_\alpha}Y_i^2) - g(A^2_{L^2_\alpha}X^2, A^2_{L^2_\beta}Y_i^2) \\ &= R'(L^1_\alpha, L^1_\beta, X^1, Y_i^1) - R(L^1_\alpha, L^1_\beta, X^1, Y_i^1) = 2g(A^1_{L^1_\alpha}L^1_\beta, v_i) = 0, \end{aligned}$$

for every i. Thus, $A^2_{L^2_\alpha}L^2_\beta = 0$ and hence $\mathcal{B}^2$ is a special basis of $\mathcal{H}^2$.

By Proposition 2.2(c), $A^2_{L^2_\alpha}w_i$ is basic along F_b^2 (for details see [**3**, Lemma 3.4]), and by an argument similar to [**3**, Lemma 3.4] one can see that $\pi_{1*}A^1_{L^1_\alpha}v_i = \pi_{2*}A^2_{L^2_\alpha}w_i$. □

Since $\mathcal{B}^1$ and $\mathcal{B}^2$ are special bases, they are orthonormal, by § 4. Let $F : T_pH_l^a \to T_qH_l^a$ be the linear isometry given by $F(v_i) = w_i$, $F(A^1_{L^1_\alpha}v_i) = A^2_{L^2_\alpha}w_i$, $F(L^1_\alpha) = L^2_\alpha$ for any $1 \leqslant i \leqslant r$, $0 \leqslant \alpha \leqslant k-1$. Since H_l^a is a frame-homogeneous space, there exists an isometry $f : H_l^a \to H_l^a$ such that $f(p) = q$ and $f_{*p} = F$ (see [**38**, **47**]). It remains to prove that $\pi_2 \circ f = \pi_1$.

We say that the condition $(\star)$ is satisfied at $x \in H_l^a$ if

$$(\star) \qquad \pi_2(f(x)) = \pi_1(x), \quad f_{*x}(\mathcal{H}^1_x) = \mathcal{H}^2_{f(x)}, \quad f_*(A^1_E F) = A^2_{f_*E} f_* F \quad \text{for any } E, F \in T_xH_l^a.$$

We will proceed in four steps.

Step 1. $(\star)$ holds at p.

Step 2. $(\star)$ holds at every $z \in F_b^1$.

Step 3. If $\tilde{\gamma} : [0,1] \to H_l^a$ is a π_1-horizontal geodesic with $\tilde{\gamma}(0) \in F_b^1$, then $(\star)$ holds at any point $\tilde{\gamma}(t)$, where $t \in [0,1]$.

Step 4. $\pi_2(f(x)) = \pi_1(x)$ for any $x \in H_l^a$.

Proof of Step 1. From the definition of F, we simply have $\pi_2(f(p)) = \pi_1(p)$ and

$$f_{*p}(\mathcal{H}^1_p) = \mathcal{H}^2_{f(p)}. \tag{5.2}$$

We recall that the vectors of $\mathcal{B}^1$ are basic along F_b^1. Since

$$A^1_{A^1_{L^1_\alpha}v_i}A^1_{L^1_\beta}v_j = g(L^1_\alpha, L^1_\beta)\hat{\nabla}^1_{v_i}v_j \tag{5.3}$$

along F_b^1 (see [**3**]) and since A^1 is alternating, we see that $\hat{\nabla}^1_{v_i}v_j = (\frac{1}{2})[v_i, v_j]$. Similar relations hold for π_2, and, at p, we simply have $f_*[v_i, v_j] = [f_*v_i, f_*v_j] = [w_i, w_j]$. Therefore,

$$f_*(A^1_{A^1_{L^1_\alpha}v_i}A^1_{L^1_\beta}v_j) = A^2_{f_*(A^1_{L^1_\alpha}v_i)}f_*(A^1_{L^1_\beta}v_j). \tag{5.4}$$

By the definition of f and (5.4), we get $f_{*p}(A^1_E F) = A^2_{f_{*p}E} f_{*p} F$ for any $E, F \in T_pH_l^a$. □

Proof of Step 2. The following lemma shall be needed right away. □

LEMMA 5.3 [38, p. 105]. *Let N_1, N_2 be two complete, connected, totally geodesic pseudo-Riemannian submanifolds of a pseudo-Riemannian manifold M. If $p \in N_1 \cap N_2$ and $T_pN_1 = T_pN_2$, then $N_1 = N_2$.*

Since $f(F_b^1)$, F_b^2 are totally geodesic in a complete manifold, they are complete. By the definition of f, $f(p) = q$, $f(p) \in f(F_b^1) \cap F_b^2$. By (5.2), $T_{f(p)}(f(F_b^1)) = T_{f(p)}F_b^2$, which, by Lemma 5.3, implies that $f(F_b^1) = F_b^2$. It follows that $(\pi \circ f)(z) = \pi_2(z)$ for every $z \in F_b^1$ and that $T_{f(z)}f(F_b^1) = T_{f(z)}F_b^2$ for every $z \in F_b^1$. Hence, $f_{*z}(\mathcal{H}_z^1) = \mathcal{H}_{f(z)}^2$ for every $z \in F_b^1$. Since $f_{*p} = (\pi_{2*q}|_{\mathcal{H}^2})^{-1} \circ (\pi_{1*p}|_{\mathcal{H}^1})$ and since every vector of $\mathcal{B}^1$ and $\mathcal{B}^2$ is basic along F_b^1 and F_b^2, respectively, $f_{*z}(A_E^1 F) = A_{f_{*z}E}^2 f_{*z}F$ for every $E, F \in T_zH_l^a$ and every $z \in F_b^1$.

Proof of Step 3. Let $\gamma : [0,1] \to B$ be a geodesic in B starting from $b = \gamma(0)$. Let $c = \gamma(1)$. For any $z \in F_b^1$, $w \in F_b^2$ we denote by $\gamma_z^1 : [0,1] \to H_l^a$ and $\gamma_w^2 : [0,1] \to H_l^a$ the π_1- and π_2-horizontal lifts of γ starting from $z = \gamma_z^1(0)$ and from $w = \gamma_w^1(0)$, respectively. Note that the global existence of the horizontal lifts is ensured by the Ehresmann-completeness of $\mathcal{H}$. Let $\tau_\gamma^1 : F_b^1 \to F_c^1$ and $\tau_\gamma^2 : F_b^2 \to F_c^2$ be the holonomy diffeomeorphisms of γ given by $\tau_\gamma^1(z) = \gamma_z^1(1)$ and $\tau_\gamma^2(w) = \gamma_w^2(1)$, respectively (see [8, 27]). A nice fact to point out is that τ_γ^1 and τ_γ^2 are isometries since the fibres are totally geodesic [8, 30]. Now, we prove that $f \circ \tau_\gamma^1(z) = \tau_\gamma^2 \circ f(z)$ for any $z \in F_b^1$.

The geodesic $f \circ \gamma_z^1$ is π_2-horizontal if its initial velocity is (cf. [8, 15]). We see that

$$\frac{d}{dt}(f \circ \gamma_z^1)\Big|_{t=0} = f_{*z}(\dot{\gamma}_z^1(0)) \in f_{*z}(\mathcal{H}_z^1) = \mathcal{H}_{f(z)}^2. \tag{5.5}$$

Thus, $\gamma_{f(z)}^2 = f \circ \gamma_z^1$ for any $z \in F_b^1$, which can be reinterpreted as $f \circ \tau_\gamma^1(z) = \tau_\gamma^2 \circ f(z)$. Therefore, $f(F_c^1) = F_c^2$, hence $f_{*z}(\mathcal{H}_z^1) = \mathcal{H}_{f(z)}^2$ and $\pi_2 \circ f(z) = \pi_1(z)$ for any $z \in F_c^1$.

We now check that f preserves the O'Neill integrability tensors. Let $X'(t), Y_1'(t), \dots, Y_r'(t)$, $L_1'(t), \dots, L_{k-1}'(t)$ be the parallel transports along γ of $\pi_{1*}X^1$, $\pi_{1*}Y_1^1, \dots, \pi_{1*}Y_r^1$, $\pi_{1*}L_1^1, \dots, \pi_{1*}L_{k-1}^1$. Let $(X^1(t), Y_1^1(t), \dots, Y_r^1(t), L_1^1(t), \dots, L_{k-1}^1(t))$ and $(X^2(t), Y_1^2(t), \dots, Y_r^2(t), L_1^2(t), \dots, L_{k-1}^2(t))$ be the π_1- and π_2-horizontal lifts of

$$(X'(t), Y_1'(t), \dots, Y_r'(t), L_1'(t), \dots, L_{k-1}'(t))$$

along $F_{\gamma(t)}^1$ and $F_{\gamma(t)}^2$, respectively. Set $v_i(t) = A_{X^1(t)}^1 Y_i^1(t)$ and $w_i(t) = A_{X^2(t)}^2 Y_i^2(t)$. Fixing $z \in F_b^1$, we simply define $\gamma^1 = \gamma_z^1$. We need to establish the following technical lemma. □

LEMMA 5.4. (i) *We have $v^1(\nabla_{\dot{\gamma}^1(t)} A_{X^1(t)}^1 Y_i^1(t)) = 0$ and $v^1(\nabla_{\dot{\gamma}^1(t)} A_{L_\alpha^1(t)}^1 L_\beta^1(t)) = 0$ for any i, α, β.*

(ii) *The basis $\{v_1(t), \dots, v_r(t)\}$ is an orthonormal basis of vector fields on the fibre $F_{\gamma(t)}^1$.*

(iii) *We have $h^1(\nabla_{\dot{\gamma}^1(t)} A_{L_\alpha^1(t)}^1 v_i(t)) = 0$.*

(iv) *The vector field $\pi_{1*}(A_{L_\alpha^1(t)}^1 v_i(t))$ is the parallel transport of $\pi_{1*}(A_{L_\alpha^1} v_i)$.*

(v) *The basis $\mathcal{B}^1(t) = \{L_0^1(t), A_{L_0^1(t)}^1 v_1(t) \dots, A_{L_0^1(t)}^1 v_r(t), \dots L_{k-1}^1(t), A_{L_{k-1}^1(t)}^1 v_1(t) \dots, A_{L_{k-1}^1(t)}^1 v_r(t)\}$ is an orthonormal basis of $\mathcal{H}_{\gamma^1(t)}^1$, and moreover $A_{L_\alpha^1(t)}^1 L_\beta^1(t) = 0$ for any α and β.*

Proof of Lemma 5.4. (i) Since H_l^a has constant curvature, by Proposition 2.2(b), we get

$$\begin{aligned} 0 &= R(X^1(t), Y_i^1(t), \dot{\gamma}^1, U) = g((\nabla_{\dot{\gamma}^1} A^1)_{X^1(t)} Y_i^1(t), U) \\ &= g(\nabla_{\dot{\gamma}^1} A_{X^1(t)}^1 Y_i^1(t), U) - g(A_{\nabla_{\dot{\gamma}^1} X^1(t)}^1 Y_i^1(t), U) - g(A_{X^1(t)}^1 \nabla_{\dot{\gamma}^1} Y_i^1(t), U) \\ &= g(\nabla_{\dot{\gamma}^1} A_{X^1(t)} Y_i^1(t), U). \end{aligned}$$

Therefore, $v^1(\nabla_{\dot{\gamma}^1(t)} A_{X^1(t)}^1 Y_i^1(t)) = 0$. Similarly, we get $v^1(\nabla_{\dot{\gamma}^1(t)} A_{L_\alpha^1(t)}^1 L_\beta^1(t)) = 0$.

(ii) We simply have

$$\dot\gamma^1(t)g(v_i(t),v_j(t)) = g(v^1\nabla_{\dot\gamma^1(t)}v_i, v_j) + g(v_i, v^1\nabla_{\dot\gamma^1(t)}v_j) = 0, \tag{5.6}$$

which implies that $g(v_i(t),v_j(t))$ is constant along $\gamma^1(t)$ and thus $\{v_i(t)\}_{1\leqslant i\leqslant r}$ is an orthonormal basis.

(iii) Using the fact that $(\nabla_{E_1}A)_{E_2}$ is skew-symmetric with respect to g (see [8]), and that the total space has constant curvature, by Proposition 2.2(b), we have

$$\begin{aligned}0 &= R(L^1_\alpha(t), Z, \dot\gamma^1, v_i(t)) = g((\nabla_{\dot\gamma^1}A^1)_{L^1_\alpha(t)}Z, v_i(t)) = -g(Z, (\nabla_{\dot\gamma^1}A^1)_{L^1_\alpha(t)}v_i(t))\\ &= -g(Z, \nabla_{\dot\gamma^1}A^1_{L^1_\alpha(t)}v_i(t)) + g(Z, A^1_{\nabla_{\dot\gamma^1}L^1_\alpha(t)}v_i(t)) + g(Z, A^1_{L^1_\alpha(t)}v^1\nabla_{\dot\gamma^1}v_i(t))\\ &= -g(Z, \nabla_{\dot\gamma^1}A^1_{L^1_\alpha(t)}v_i(t)),\end{aligned}$$

which implies (iii).

(iv) By (iii), we simply have $\nabla'_{\dot\gamma(t)}\pi_{1*}(A^1_{L^1_\alpha(t)}v_i(t)) = 0$.

(v) By (iv), we have that $\mathcal{B}^1(t)$ is an orthonormal basis of $\mathcal{H}^1_{\gamma^1(t)}$. By (i), we get

$$\dot\gamma^1(t)g(A^1_{L^1_\alpha(t)}L^1_\beta(t), v_i(t)) = g(v^1\nabla_{\dot\gamma^1(t)}A^1_{L^1_\alpha(t)}L^1_\beta(t), v_i) + g(A^1_{L^1_\alpha(t)}L^1_\beta(t), v^1\nabla_{\dot\gamma^1(t)}v_i) = 0,$$

which implies that $g(A^1_{L^1_\alpha(t)}L^1_\beta(t), v_i(t)) = g(A^1_{L^1_\alpha(0)}L^1_\beta(0), v_i(0)) = 0$ for any i. Therefore, $A^1_{L^1_\alpha(t)}L^1_\beta(t) = 0$. □

Similar results hold for π_2. In particular, $\pi_{2*}(A^2_{L^2_\alpha(t)}w_i(t))$ is the parallel transport of $\pi_{2*}(A^2_{L^2_\alpha}w_i)$. From Step 2, $\pi_{1*}(A^1_{L^1_\alpha}v_i) = \pi_{2*}(A^2_{L^2_\alpha}w_i)$, and therefore their parallel transports must be equal to each other:

$$\pi_{1*z}(A^1_{L^1_\alpha(t)}v_i(t)) = \pi_{2*f(z)}(A^2_{L^2_\alpha(t)}w_i(t)), \tag{5.7}$$

and that can be rewritten as $f_{*z}(A^1_{L^1_\alpha(t)}v_i(t)) = A^2_{L^2_\alpha(t)}w_i(t)$. Using an argument similar to Step 2 for the special bases $\mathcal{B}^1(t)$ and $\mathcal{B}^2(t)$, we simply get $f_{*z}(A^1_EF) = A^2_{f_*E}f_*F$ for any $E, F \in \mathcal{B}^1(t)$.

Proof of Step 4. Let x be an arbitrary point in H^a_l. Since H^a_l is connected, there exists a broken geodesic $\gamma(t)$ in B connecting b and $\pi_1(x)$ (see [38, p. 72]). Applying successively Step 3 to each smooth piece of the broken geodesic, we see that $(\star)$ is satisfied at every point $z \in F_{\gamma(t)}$ for every t; in particular, $(\star)$ holds at x. □

Remark 5.5. A very important result due to Escobales is the criterion of equivalence of two Riemannian submersions, which states that if $\pi_1, \pi_2 : M \to B$ are Riemannian submersions with connected totally geodesic fibres from a connected complete Riemannian manifold onto a Riemannian manifold, and if, for some isometry $f : M \to M$, the condition $(\star)$ holds at a given point $p \in M$, then there exists an isometry $\tilde f : B \to B$ such that $\pi_2 \circ f = \tilde f \circ \pi_1$. Although the proof of Lemma 5.4(i) invokes $R(X,Y,Z,U) = 0$, a usual hypothesis in the geometry of transversally symmetric (pseudo-)Riemannian foliations (see [43]), the proof of Theorem 5.1 relies on the construction of a special basis, which is specific to a pseudo-Riemannian submersion with totally geodesic fibres of a non-flat real space form. In Theorems 6.1 and 6.2, we shall see that Theorem 5.1 can be adapted to the case of pseudo-Riemannian submersions with (para-)complex, connected, totally geodesic fibres from a (para-)complex pseudo-hyperbolic space.

6. *Applications of the main theorem*

We summarize the results proved in the previous sections.

Proof of Theorem 1.1. By Theorems 4.7, 4.8, 4.14 and Corollary 4.17, B is isometric to one of the following spaces: $H_4^8(-4), H^8(-4), H_8^8(-4), \mathbb{C}H_t^m, \mathbb{A}P^m, \mathbb{H}H_t^m, \mathbb{B}P^m$, denoted simply by B'. There exists an isometry $\tilde{f} : B \to B'$. Let $\pi' : M' \to B'$ be the Hopf pseudo-Riemannian submersion with the base space B' and with M' a pseudo-hyperbolic space. Also, by Theorems 4.7, 4.8, 4.14, we see that $a = \dim(M')$, $l = \operatorname{index}(M')$, and thus $M' = H_l^a$. By Theorem 5.1, $\pi' : H_l^a \to B'$ is equivalent to $\tilde{f} \circ \pi : H_l^a \to B'$, namely, there exists an isometry $f : H_l^a \to H_l^a$ such that $\pi \circ f = \tilde{f} \circ \pi$. Therefore, π and π' are equivalent. □

As a consequence of Theorem 1.1, we now obtain classification results for pseudo-Riemannian submersions with totally geodesic fibres from (a) $\mathbb{C}H_t^m$, (b) $\mathbb{H}H_t^m$, (c) $\mathbb{A}P^m$, (d) $\mathbb{B}P^m$. First, we define the following Hopf pseudo-Riemannian submersions with totally geodesic fibres:

(a) $\pi_{\mathbb{C},\mathbb{H}} : \mathbb{C}H_{2t+1}^{2m+1} = H_{4t+3}^{4m+3}/H_1^1 \to \mathbb{H}H_t^m = H_{4t+3}^{4m+3}/H_3^3$, given by $\pi_{\mathbb{C},\mathbb{H}}([zH_1^1]) = [zH_3^3]$;
(b) $\pi_{\mathbb{C},\mathbb{B}} : \mathbb{C}H_m^{2m+1} = H_{2m+1}^{4m+3}/H_1^1 \to \mathbb{B}P^m = H_{2m+1}^{4m+3}/H_1^3$, given by $\pi_{\mathbb{C},\mathbb{B}}([zH_1^1]) = [zH_1^3]$;
(c) $\pi_{\mathbb{A},\mathbb{B}} : \mathbb{A}P^{2m+1} = H_{2m+1}^{4m+3}/H^1 \to \mathbb{B}P^m = H_{2m+1}^{4m+3}/H_1^3$, given by $\pi_{\mathbb{A},\mathbb{B}}([zH^1]) = [zH_1^3]$.

The fibres of $\pi_{\mathbb{C},\mathbb{H}}$, $\pi_{\mathbb{C},\mathbb{B}}$, $\pi_{\mathbb{A},\mathbb{B}}$ are isometric to $\mathbb{C}H_1^1$, $\mathbb{C}H^1$, $\mathbb{A}P^1$, respectively.

THEOREM 6.1. *If $\pi : \mathbb{C}H_b^a \to B$ is a pseudo-Riemannian submersion with connected, totally geodesic fibres from a complex pseudo-hyperbolic space onto a pseudo-Riemannian manifold and if the fibres are complex submanifolds, then π is equivalent to one of the following Hopf pseudo-Riemannian submersions:*

$$\text{(a) } \pi_{\mathbb{C},\mathbb{H}} : \mathbb{C}H_{2t+1}^{2m+1} \longrightarrow \mathbb{H}H_t^m \quad \text{(b) } \pi_{\mathbb{C},\mathbb{B}} : \mathbb{C}H_m^{2m+1} \longrightarrow \mathbb{B}P^m.$$

Proof. Let $\theta : H_{2b+1}^{2a+1} \to \mathbb{C}H_b^a$ be the Hopf pseudo-Riemannian submersion over $\mathbb{C}H_b^a$. Now, π and θ are pseudo-Riemannian submersions with totally geodesic fibres, and by Escobales [**16**, Theorem 2.5] so is $\pi \circ \theta$, to which we can apply Theorem 1.1. By our usual assumption $\dim \mathbb{C}H_b^a > \dim B$, we see that the dimension of the fibres of $\pi \circ \theta$ is greater than 1. Therefore, $\pi \circ \theta$ is equivalent to the Hopf pseudo-Riemannian submersions (c)–(g) of Theorem 1.1, which implies that π must be of the following forms:

(i) $\mathbb{C}H_{2t+1}^{2m+1} \to \mathbb{H}H_t^m$,
(ii) $\mathbb{C}H_m^{2m+1} \to \mathbb{B}P^m$,
(iii) $\mathbb{C}H_3^7 \to H^8(-4)$,
(iv) $\mathbb{C}H_3^7 \to H_4^8(-4)$,
(v) $\mathbb{C}H_7^7 \to H_8^8(-4)$.

By Nagy [**34**, Proposition 4.2], the dimension of the fibres must be 2; thus (iii)–(v) are not possible. We refer the reader to [**39**] for a different proof of the non-existence of (v), and to [**4**] for that of (iii). Let $\pi_1, \pi_2 : \mathbb{C}H_{2t+1}^{2m+1} \to \mathbb{H}H_s^m$ be two pseudo-Riemannian submersions with totally geodesic fibres. By Theorem 5.1, $\pi_1 \circ \theta$ and $\pi_2 \circ \theta$ are equivalent, and, by the proof of Theorem 5.1, there exists an isometry $f : H_{4t+3}^{4m+3} \to H_{4t+3}^{4m+3}$ depending on the choice of an orthonormal basis $\{v_{1p}, v_{2p}, v_{3p}\}$ of $\mathcal{V}_p^1 = \operatorname{Ker}(\pi_1 \circ \theta)$, $p \in H_{4s+3}^{4m+3}$, such that

$$\pi_2 \circ \theta \circ f = \pi_1 \circ \theta. \tag{6.1}$$

If we choose this orthonormal basis such that v_{3p} is θ-vertical, then, by a similar argument to the proof of Theorem 5.1, we see that f sends any θ-fibre into a θ-fibre, and thus there exists an isometry $\tilde{f} : \mathbb{C}H_{2s+1}^{2m+1} \to \mathbb{C}H_{2s+1}^{2m+1}$ such $\tilde{f} \circ \theta = \theta \circ f$. By (6.1), we get $\pi_2 \circ f = \pi_1$.

A similar argument can be used to show the equivalence of two pseudo-Riemannian submersions $\pi_1, \pi_2 : \mathbb{C}H_m^{2m+1} \to \mathbb{B}P^m$. □

THEOREM 6.2. *If $\pi : \mathbb{A}P^a \to B$ is a pseudo-Riemannian submersion with connected, totally geodesic fibres from a para-complex projective space onto a pseudo-Riemannian manifold and if the fibres are para-complex submanifolds, then π is equivalent to the Hopf pseudo-Riemannian submersions*:

$$\pi_{\mathbb{A},\mathbb{B}} : \mathbb{A}P^{2m+1} \longrightarrow \mathbb{B}P^m.$$

Proof. Let $\pi_{\mathbb{A}} : H_a^{2a+1} \to \mathbb{A}P^a$ be the Hopf pseudo-Riemannian submersion over $\mathbb{A}P^a$. One can show by an analogous argument to [**34**, Proposition 4.2] that in the para-case, the fibres are also of dimension 2. Applying Theorem 1.1 to $\pi \circ \pi_{\mathbb{A}}$, we obtain that π should be of the form

$$\text{(i) } \mathbb{A}P^{2m+1} \longrightarrow \mathbb{B}P^m \quad \text{or} \quad \text{(ii) } \mathbb{A}P^{4m+3} \longrightarrow \mathbb{H}H_m^{2m+1}.$$

Since the signatures of $\mathbb{H}H_m^{2m+1}$ and $\mathbb{A}P^{4m+3}$ are $(4m+4, 4m)$ and $(4m+3, 4m+3)$, respectively, (ii) is not possible. The uniqueness of (i) follows analogously to the proof of Theorem 6.1. □

REMARK 6.3. The two twistor spaces $\pi : (Z^\varepsilon, g) \to \mathbb{B}P^m$, $\varepsilon = \pm 1$ (see [**1**]) of the para-quaternionic Kähler manifold $\mathbb{B}P^n$ are equivalent to the Hopf pseudo-Riemannian submersions $\pi_{\mathbb{C},\mathbb{B}} : \mathbb{C}H_m^{2m+1} \to \mathbb{B}P^m$ (when $\varepsilon = -1$) and $\pi_{\mathbb{A},\mathbb{B}} : \mathbb{A}P^{2m+1} \to \mathbb{B}P^m$ (when $\varepsilon = 1$). Here g is the canonical Kähler–Einstein (when $\varepsilon = -1$) or para-Kähler–Einstein (when $\varepsilon = 1$) metric of Z^ε (see [**1**]). By Alekseevsky and Cortés [**1**, Theorem 3], there are two Einstein metrics in the canonical variation on Z^ε and only one of them is ε- Kähler–Einstein. Another nice fact is that the twistor space $\pi : Z \to \mathbb{H}H_t^m$ of the quaternionic Kähler manifold $\mathbb{H}H_t^m$ is equivalent to $\pi_{\mathbb{C},\mathbb{H}} : \mathbb{C}H_{2t+1}^{2m+1} \to \mathbb{H}H_t^m$.

COROLLARY 6.4. (i) *There are no pseudo-Riemannian submersions $\pi : \mathbb{H}H_t^m \to B$ with connected quaternionic fibres.*

(ii) *There are no pseudo-Riemannian submersions $\pi : \mathbb{B}P^m \to B$ with connected para-quaternionic fibres.*

Proof. First, we recall that any (para-)quaternionic submanifold of a (para-) quaternionic manifold is totally geodesic [**2**].

(i) To obtain a contradiction, suppose that such a submersion π exists. Let $\pi_{\mathbb{H}} : H_{4t+3}^{4m+3} \to \mathbb{H}H_t^m$ be the Hopf pseudo-Riemannian submersion over $\mathbb{H}H_t^m$. By Theorem 1.1, $\pi \circ \pi_{\mathbb{H}}$ is equivalent to one of the following: $H_7^{15} \to H^8(-4)$, $H_7^{15} \to H_4^8(-4)$ or $H_{15}^{15} \to H_8^8(-4)$, thus π must be of the form

$$\text{(a) } \mathbb{H}H_1^3 \longrightarrow H^8(-4), \quad \text{(b) } \mathbb{H}H_1^3 \longrightarrow H_4^8(-4) \quad \text{or} \quad \text{(c) } \mathbb{H}H_3^3 \longrightarrow H_8^8(-4). \tag{6.2}$$

We conclude that the fibres are four-dimensional and that $\pi \circ \pi_{\mathbb{C},\mathbb{H}} : \mathbb{C}H_{2t+1}^7 \to H_s^8(-4)$, $(t,s) \in \{(1,0),(1,4),(3,8)\}$ are pseudo-Riemannian submersions with complex, totally geodesic, six-dimensional fibres, which contradicts Theorem 6.1.

The proof of (ii) is analogous to (i). □

REMARK 6.5. The Ucci topological proof [**44**] of the non-existence of (6.2(c)) cannot be extended to (6.2(a)) and (6.2(b)), because $\mathbb{H}H_1^3$, $H^8(-4)$, $H_4^8(-4)$ have the homotopy types of S^4, a point and S^4, respectively.

REMARK 6.6. Unlike the Riemannian submersions from spheres, the pseudo-Riemannian ones from pseudo-hyperbolic spaces feature less rigidity when we drop the condition of totally geodesic fibres. Particularly, while any Riemannian submersion from a sphere is equivalent to a Hopf one [46], this is no longer true for the pseudo-Riemannian submersions from pseudo-hyperbolic spaces. Indeed (cf. [6]) any pseudo-hyperbolic space H^a_l can simply be written as a warped product $H^a_l = (H^{a-l} \times_f S^l, g_{H^a_l})$, via the identification $\phi : H^{a-l} \times S^l \to H^a_l$, given by $\phi((x_0, x), u) = (x_0 u, x)$ for every $u \in S^l$, $(x_0, x) \in H^{a-l}$, $x_0 \in \mathbb{R}_+$, $x \in \mathbb{R}^{a-l}$. Here $f : H^{a-l} \to \mathbb{R}_+$ is given by $f(x_0, (x_1, \ldots, x_{a-l})) = x_0$, and the metric of the warped product is $g_{H^{a-l}} - f^2 g_{S^l}$. Now the projection

$$\pi : H^a_l = H^{a-l} \times_f S^l \longrightarrow H^{a-l}$$

is a pseudo-Riemannian submersion (with totally umbilical fibres [8]), which is not equivalent to a Hopf one, except possibly when $(a, l) \in \{(3, 1), (7, 3), (15, 7)\}$. The classification problem of pseudo-Riemannian submersions from pseudo-hyperbolic spaces remains open.

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