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SEMI-RIEMANNIAN SUBMERSIONS WITH TOTALLY UMBILIC FIBRES

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We get obstructions to existence of semi-Riemannian submersions with totally umbilic fibres.

Introduction.

The theory of Riemannian submersions was initiated by O'Neill [13] and Gray [8]. Presently, there is an extensive literature on Riemannian submersions with different conditions imposed on the total space and on the fibres. A systematic exposition could be found in the A.Besse's book [4]. Semi-Riemannian submersions were introduced by O'Neill in his book [14]. In this paper we study the semi-Riemannian submersions with totally umbilic fibres.

The main purpose of section §2 is to obtain obstructions to the existence of the semi-Riemannian submersions with totally umbilic fibres and with compact and orientable total space, in terms of sectional and scalar curvature. Using a formula of Ranjan [15], we obtain an integral formula for mixed scalar curvature τ^{HV}, which give us obstructions to existence of the semi-Riemannian submersions in some special cases. Then we establish other integral formula of scalar curvature of total, base and fibres spaces and another obstruction to existence of semi-Riemannian submersions is obtained.

In section § 3 we study the semi-Riemannian submersions $\pi: M \to B$ with totally umbilic fibres, when the mean curvature vector field H is parallel in the horizontal bundle along fibres and $R(X, Y, X, Y)$ is constant along fibres for every X, Y basic vector fields. If moreover we assume

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that B is a Riemannian manifold and M is a semi-Riemannian manifold of index $\dim M - \dim B$ we deduce that the mean curvature vector field H is basic if and only if the horizontal projection of $R(X,Y)A_XY$, denoted by $hR(X,Y)A_XY$ (see page 2), is basic for every X, Y basic vector fields. We get there are no semi-Riemannian submersions $\pi\colon M \to B$ with totally umbilic fibres, M a constant positive curvature semi-Riemannian manifold of index $\dim M - \dim B \geq 2$ and B a compact and orientable Riemannian manifold. Then we find that a semi-Riemannian submersion with totally umbilic fibres, with $R(X,Y,X,Y)$ constant along fibres for every X, Y basic vector fields and with basic mean curvature vector field from an m-dimensional semi-Riemannian manifold of index $r = m-n$ with non-negative mixed curvature onto an n-dimensional compact and orientable Riemannian manifold, has totally geodesic fibres, integrable horizontal distribution and null mixed curvature. Therefore a semi-Riemannian submersion $\pi\colon M \to B$ with totally umbilic fibres and with sectional curvature of the fibres non-vanishing anywhere, from a constant curvature semi-Riemannian manifold M of index $\dim M - \dim B \geq 2$ is a Clairaut semi-Riemannian submersion. Also we study the case of positive sectional curvature fibres. We give a sufficient condition to have every fibre with zero sectional curvature, when the total space has constant curvature.

1. Preliminaries.

In this section we recall some notions and results which will be needed.

DEFINITION. *Let (M,g) be an m-dimensional connected semi-Riemannian manifold of index s ($0 \leq s \leq m$), let (B,g') be an n-dimensional connected semi-Riemannian manifold of index $s' \leq s$, ($0 \leq s' \leq n$). A semi-Riemannian submersion (see O'Neill [14]) is a smooth map $\pi\colon M \to B$ which is onto and satisfies the following three axioms:*

(a) $\pi_*|_p$ is onto for all $p \in M$;

(b) The fibres $\pi^{-1}(b)$, $b \in B$ are semi-Riemannian submanifolds of M;

(c) π_* preserves scalar products of vectors normal to fibres.

We shall always assume that the dimension of the fibres $\dim M - \dim B$ is positive and the fibres are connected.

The vectors tangent to fibres are called vertical and those normal to fibres are called horizontal. We denote by $\mathcal{V}$ the vertical distribution and by $\mathcal{H}$ the horizontal distribution.

B.O'Neill [13] has characterized the geometry of a Riemannian submersion in terms of the field tensors T , A defined for E, $F \in \Gamma(TM)$ by

$$A_E F = h\nabla_{hE} vF + v\nabla_{hE} hF$$
$$T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF$$

where ∇ is the Levi-Civita connection of g. Here the symbols v and h are the orthogonal projections on $\mathcal{V}$ and $\mathcal{H}$ respectively. For basic properties of Riemannian submersions and examples see [4], [8], [13]. The letters U, V, W, W' will always denote vertical vector fields, X, Y, Z, Z' horizontal vector fields and E,F,G,G' arbitrary vector fields on M. A vector field X on M is said to be basic if X is horizontal and $\pi-$related to a vector field X' on B. It is easy to see that every vector field X' on B has a unique horizontal lift X to M and X is basic. The following lemmas are well known (see [13], [6]).

LEMMA 1.1. *Let X be a horizontal vector field. If $g_p(X, Z) = g_{p'}(X, Z)$ for all Z basic vector fields on M, for all $p, p' \in \pi^{-1}(b)$ and for all $b \in B$ then $\pi_* X$ is a well defined vector field on B and X is basic.*

LEMMA 1.2. *We suppose X and Y are basic vector fields on M which are π-related to X' and Y', and V is a vertical vector field. Then*

a) $h[X, Y]$ is basic and π-related to $[X', Y']$;

b) $h\nabla_X Y$ is basic and π-related to $\nabla'_{X'} Y'$, where ∇' is the Levi-Civita connection on B;

c) $h\nabla_V X = A_X V$.

Let $\{e_1,...,e_m\}$ be a local field of orthonormal frames on M such that $e_1, ..., e_r$ are vertical vector fields and $e_{r+1}, ..., e_m$ are basic vector fields, where $r = \dim M - \dim B$ denotes the dimension of fibres. We have $g(e_a, e_b) = \varepsilon_a \delta_{ab}$ for every a, b (where $\varepsilon_a \in \{-1, 1\}$). We shall always denote *vertical* indices by $i, j, k, l, ... = 1, ..., r$ and *horizontal* indices by $\alpha, \beta, \gamma, \delta, ... = r + 1, ..., m$. The summation $\sum$ is taken over all repeated indices, unless otherwise stated.

The convention for the Riemann tensor used is

$$R(E, F)G = \nabla_E \nabla_F G - \nabla_F \nabla_E G - \nabla_{[E,F]} G$$

and

$$R(E, F, G, G') = -g(R(E, F)G, G').$$

Let $\hat{g}$ be the semi-Riemannian metric of a fibre $\pi^{-1}(b)$, $b \in B$. We make the following notations:

R, R', $\hat{R}$ for the Riemann tensors, K, K', $\hat{K}$ for the sectional curvatures, and s, s', $\hat{s}$ for the scalar curvatures of the metrics g, g', $\hat{g}$, respectively;

$$\tau^{HV} = \sum \varepsilon_i \varepsilon_\alpha R(e_\alpha, e_i, e_\alpha, e_i)\ ,\ H = \sum \varepsilon_i T_{e_i} e_i,$$

$$g(A, A) = \sum_{\alpha,\beta} \varepsilon_\alpha \varepsilon_\beta g(A_{e_\alpha} e_\beta, A_{e_\alpha} e_\beta) = \sum_{\alpha,i} \varepsilon_i \varepsilon_\alpha g(A_{e_\alpha} e_i, A_{e_\alpha} e_i),$$

$$g(T, T) = \sum_{i,j} \varepsilon_i \varepsilon_j g(T_{e_i} e_j, T_{e_i} e_j) = \sum_{\alpha,i} \varepsilon_i \varepsilon_\alpha g(T_{e_i} e_\alpha, T_{e_i} e_\alpha),$$

$$\operatorname{div}(E) = \sum_i \varepsilon_i g(\nabla_{e_i} E, e_i) + \sum_\alpha \varepsilon_\alpha g(\nabla_{e_\alpha} E, e_\alpha).$$

If X is an unitary horizontal vector and V is an unitary vertical vector the sectional curvature of the 2-plane $\{X, V\}$ is called the *mixed sectional curvature*.

DEFINITION. *A semi-Riemannian submanifold F of a semi-Riemannian manifold (M, g) is said to be totally umbilic submanifold if the second fundamental form Π of F is given by $\Pi(U, V) = g(U, V)\dfrac{H}{r}$ for every U, V tangent vector fields to F.*

Notice that $T_U V$ is the second fundamental form of the fibres and $A_X Y$ is a natural *obstruction* to integrability of horizontal distribution.

If the fibres of the semi-Riemannian submersion are totally umbilic submanifolds then $T_U V = \dfrac{1}{r} g(U, V) H$ for every U, V vertical vectors fields and $g(T, T) = \dfrac{1}{r} g(H, H)$.

By O'Neill's equations [13] we get the following lemma.

LEMMA 1.3. *If $\pi : (M, g) \to (B, g')$ is a semi-Riemannian submersion with totally umbilic fibres then:*

a) $R(U, V, U, V) = \hat{R}(U, V, U, V) + [g(U, V)^2 - g(U, U)g(V, V)]g\left(\dfrac{H}{r}, \dfrac{H}{r}\right)$

b) $R(X, U, X, U) = g(U, U)\left[g\left(\nabla_X \dfrac{H}{r}, X\right) - g\left(X, \dfrac{H}{r}\right)^2\right] + g(A_X U, A_X U)$

c) $R(X, Y, X, Y) = R'(\pi_* X, \pi_* Y, \pi_* X, \pi_* Y) - 3g(A_X Y, A_X Y)$

Using a relation of R. Escobales and Ph. Parker [7], we have the following proposition.

PROPOSITION 1.4. *Let $\pi: (M, g) \to (B, g')$ be a semi-Riemannian submersion with totally umbilic fibres and X, Y be basic vector fields. Then $A_X Y$ is a Killing vector field along fibres if and only if $g(\nabla_Y H, X) = g(\nabla_X H, Y)$.*

Proof. Let U,V be vertical vector fields. Since the fibres are totally umbilic, for every X, Y basic vector fields we have (see [7])

$$g(\nabla_U(A_X Y), V)+g(\nabla_V(A_X Y), U) = \frac{g(U, V)}{r}(g(\nabla_Y H, X)-g(\nabla_X H, Y)).\square$$

Corollary 1.5. *Let $\pi: (M, g) \to (B, g')$ be a semi-Riemannian submersion with totally umbilic fibres. We suppose that the mean curvature H is a basic vector field. Then $A_X Y$ is a Killing vector field along fibres for every X,Y basic vector fields if and only if $(\pi_* H)^\flat$ is a closed $1-$form on B.*

Proof. Let $\omega = (\pi_* H)^\flat$, X',Y' be vector fields on B. We have

$$2\mathrm{d}\omega)(X', Y') = g'(\nabla_{X'}\pi_* H, Y') - g'(\nabla_{Y'}\pi_* H, X').$$

Let X,Y be basic vector fields such that $\pi_* X = X'$, $\pi_* Y = Y'$. By lemma 1.2, we have

$$(g'(\nabla_{X'}\pi_* H, Y') - g'(\nabla_{Y'}\pi_* H, X')) \circ \pi = g(\nabla_X H, Y) - g(\nabla_Y H, X).$$

Applying proposition 1.4 we get the conclusion. $\square$

In [3] we proved the following result.

PROPOSITION 1.6. *Let $\pi: (M, g) \to (B, g')$ be a semi-Riemannian submersion from an $(n + s)$-dimensional semi-Riemannian manifold of index $s \geq 1$ onto an n-dimensional Riemannian manifold. If M is geodesically complete and simply connected then*

1) *B is complete and simply connected;*
2) *we have an exact homotopy sequence*

$$\cdots \to \pi_2(B) \to \pi_1(fibre) \to \pi_1(M) \to \pi_1(B) \to 0;$$

3) *If moreover B has non-positive curvature then the fibres are simply connected.*

The following proposition is a semi-Riemannian version of Ranjan's formula (see [15]) in the case with totally umbilic fibres.

THEOREM 1.7. *If the semi-Riemannian submersion* $\pi:(M,g)\to(B,g')$ *has totally umbilic fibres, then*

$$\tau^{HV}=div(H)+\left(1-\frac{1}{r}\right)g(H,H)+g(A,A). \tag{1.1}$$

Proof. From O'Neill [13] , we have the following formula

$$R(e_\alpha,e_i,e_\alpha,e_i)=g(\,(\nabla_{e_\alpha}T)_{e_i}e_i,e_\alpha)+g(A_{e_\alpha}e_i,A_{e_\alpha}e_i)-g(T_{e_i}e_\alpha,T_{e_i}e_\alpha)$$

$$g(\,(\nabla_{e_\alpha}T)_{e_i}e_i,e_\alpha)=g(\nabla_{e_\alpha}(T_{e_i}e_i),e_\alpha)-2g(T_{e_i}(v\nabla_{e_\alpha}e_i),e_\alpha)$$

If we denote by $q^\alpha_{ij}=g(v\nabla_{e_\alpha}e_i,e_j)$, it is easy to see that $q^\alpha_{ji}+q^\alpha_{ij}=0$. Since $T_{e_i}e_j=T_{e_j}e_i$ we get $\sum\limits_j\varepsilon_j g(T_{e_j}(v\nabla_{e_\alpha}e_j),e_\alpha)=\sum\limits_{i,j}\varepsilon_i\varepsilon_j q^\alpha_{ji}T_{e_j}e_i=0$.

Then

$$\tau^{HV}=\sum_{\alpha,i}\varepsilon_i\varepsilon_\alpha R(e_\alpha,e_i,e_\alpha,e_i)=\sum_\alpha\varepsilon_\alpha g\left(\nabla_{e_\alpha}\left(\sum_i\varepsilon_i T_{e_i}e_i\right),e_\alpha\right)+$$
$$\sum_{\alpha,i}\varepsilon_\alpha\varepsilon_i g(A_{e_\alpha}e_i,A_{e_\alpha}e_i)-\sum_{\alpha,i}\varepsilon_\alpha\varepsilon_i g(T_{e_i}e_\alpha,T_{e_i}e_\alpha)$$

We get the semi-Riemannian version of Ranjan's formula

$$\tau^{HV}=\mathrm{div}(H)+g(H,H)+g(A,A)-g(T,T).$$

Since the fibres of the semi-Riemannian submersion π are totally umbilic we have $g(T,T)=\dfrac{1}{r}g(H,H)$. □

2. Integral formulae.

As a consequence of theorem 1.7, we have the following proposition.

PROPOSITION 2.1. *If the semi-Riemannian submersion* $\pi:(M,g)\to(B,g')$ *has totally umbilic fibres and M is a compact and orientable manifold, then*

$$\int_M\tau^{HV}\mathrm{dv}_g=\left(1-\frac{1}{r}\right)\int_M g(H,H)\mathrm{dv}_g+\int_M g(A,A)\mathrm{dv}_g \tag{2.2}$$

Proof. We use the relation (1.1) and $\int_M \operatorname{div}(H) = 0$. □

COROLLARY 2.2 *If $\pi: M \to B$ is a Riemannian submersion with totally umbilic fibres and M is a compact and orientable manifold then* $\int_M \tau^{HV} \mathrm{dv}_g \geq 0$.

By proposition 2.1, we have the following splitting theorem, which is a generalization of proposition 3.1. of R. Escobales [6].

THEOREM 2.3. *Let $\pi: M \to B$ be a Riemannian submersion with totally umbilic fibres. We suppose that M is a compact and orientable manifold with non-positive mixed sectional curvature (i.e. $K(X, V) \leq 0$ for every X horizontal vector field and for every V vertical vector field). Then*

a) $R(X, U, Y, V) = 0$ for every X, Y horizontal vector fields and for every U , V vertical vector fields;
b) the horizontal distribution is integrable (this is equivalent with $A \equiv 0$);
c) the fibres are totally geodesic.

Proof. Since $\tau^{HV} \leq 0$, $g(A, A) \geq 0$ and $g(H, H) \geq 0$, we get the relations $\tau^{HV} = \sum_{i,\alpha} R(e_\alpha, e_i, e_\alpha, e_i) = 0$, $g(A, A) = 0$ and $\left(1 - \frac{1}{r}\right) g(H, H) = 0$, by formula (2.2). Hence $A \equiv 0$.

Let X be a horizontal vector and V be a vertical vector, $X \neq 0$, $V \neq 0$. We can choose a local field $e_1, ..., e_m$ of orthonormal frames adapted to the Riemannian submersion such that $e_1 = \frac{V}{\|V\|}$, $e_{r+1} = \frac{X}{\|X\|}$. Since $\tau^{HV} = 0$ and $R(e_\alpha, e_i, e_\alpha, e_i) \leq 0$ for all i, α we get $R(e_\alpha, e_i, e_\alpha, e_i) = 0$ for all i, α. Therefore $R(X, V, X, V) = 0$ for every X horizontal vector field and for every V vertical vector field.

We shall prove that the fibres are totally geodesic.

Since $A \equiv 0$, $R(X, V, X, V) = 0$, we have, by lemma 1.3,

$$g\left(\nabla_X \frac{H}{r}, X\right) = g\left(X, \frac{H}{r}\right)^2 \tag{2.3}$$

for every X horizontal vector field. Let p be an arbitrary point in M and $\gamma: \mathbb{R} \to M$ a horizontal geodesic in M, $\gamma(0) = p$. We denote by $h(t) =$

$g\left(\frac{H}{r}, \dot{\gamma}(t)\right)$. Rewriting formula (2.3), for every $t \in \mathbb{R}$ we get

$$\frac{\mathrm{d}h}{\mathrm{d}t}(t) = h(t)^2 \tag{2.4}$$

The differential equation (2.4) has the solution $h(t) \equiv 0$ or $h(t) = -\frac{1}{t + A}$ for some $A \in \mathbb{R}$. But the domain of the maximal solution is the entire real line, only for the null solution. Hence $g\left(\frac{H}{r}, \dot{\gamma}(t)\right) \equiv 0$ for every γ horizontal geodesic. Therefore $H \equiv 0$. □

PROPOSITION 2.4. *Let B be an n-dimensional Riemannian manifold, $\pi: M \to B$ be a semi-Riemannian submersion with totally umbilic fibres and M be an m-dimensional compact, orientable semi-Riemannian manifold of index $r = m - n$. Then*

$$\int_M \left(\tau^{HV} - \left(1 - \frac{1}{r}\right) g(H, H)\right) \mathrm{dv}_g \leq 0, \tag{2.5}$$

we have equality in (2.5) if and only if the horizontal distribution is integrable;

$$\int_M (\tau^{HV} - g(A, A)) \mathrm{dv}_g \geq 0, \tag{2.6}$$

we have equality in (2.6) if and only if either $r = 1$ or the fibres are totally geodesic.

Proof. Since π sends isometrically the horizontal spaces into the tangent space of B, and B is a Riemannian manifold, it follows that the fibres are semi-Riemannian manifolds of indices r and $g(H, H) \geq 0$. Since $g(A, A) = \sum_{\alpha,\beta} \varepsilon_\alpha \varepsilon_\beta g(A_{e_\alpha} e_\beta, A_{e_\alpha} e_\beta)$, $\varepsilon_\alpha = 1$ for all α , and the induced metrics on fibres are negative definite, we obtain $g(A, A) \leq 0$. By formula (2.2) we have the conclusion. □

Applying proposition (2.4) for a Lorentzian total space we get the following obstruction.

THEOREM 2.5. *Let B be a n-dimensional Riemannian manifold. If M is a compact, orientable $(n + 1)$-dimensional semi-Riemannian space of index 1, with positive mixed curvature then there are no $\pi: M \to B$ semi-Riemannian submersions.*

Proof. We suppose that there is such a semi-Riemannian submersion. Since $r = 1$ the inequality (2.5) implies $\int_M \tau^{HV} \leq 0$, which is a contradiction with the stated condition of positive mixed curvature. □

PROPOSITION 2.6. *If M is a $(r+1)$-dimensional semi-Riemannian manifold of index r, with negative mixed curvature and B is a one dimensional Riemannian manifold then there are no $\pi: M \to B$ semi-Riemannian submersions with compact, orientable total space and totally umbilic fibres.*

Proof. We suppose that there is such a semi-Riemannian submersion. Since dim $\mathcal{H}$ =1 we have $A \equiv 0$. By relation (2.6) we obtain $\int_M \tau^{HV} \geq 0$, which is a contradiction with negative mixed curvature condition. □

DEFINITION. *The total scalar curvature of a compact manifold (M, g) is $S_g = \int_M s_g \mathrm{dv}_g$, where s_g is the scalar curvature of (M, g).*

In what follows we give some obstructions to the existence of semi-Riemannian submersions in terms of total scalar curvatures of the total space M, the base space B and of the fibres.

We denote by S, S', $\hat{S}$ the total scalar curvature of (M, g), (B, g') and $(\pi^{-1}(x), \hat{g})$ respectively.

By lemma 1.3 and formula (1.1), we get immediately the following proposition.

PROPOSITION 2.7. *If $\pi: M \to B$ is a semi-Riemannian submersion with totally umbilic fibres, then*

$$s = s' \circ \pi + \hat{s} + 2div(H) + \left(1 - \frac{1}{r}\right) g(H, H) - g(A, A) \tag{2.7}$$

Integrating formula (2.7) we get

PROPOSITION 2.8. *If $\pi: M \to B$ is a semi-Riemannian submersion with compact and orientable total space M and with totally umbilic fibres, then*

$$S - (S' \circ \pi + \hat{S}) = \left(1 - \frac{1}{r}\right) \int_M g(H, H) \mathrm{dv}_g - \int_M g(A, A) \mathrm{dv}_g \tag{2.8}$$

COROLLARY 2.9. *Let B be an n-dimensional Riemannian manifold, $\pi: M \to B$ be a semi-Riemannian submersion with totally umbilic fibres and M be an m-dimensional compact and orientable semi-Riemannian manifold of index $r = m - n$. Then $S \geq S' \circ \pi + \hat{S}$. We have equality if and only if the horizontal distribution is integrable and either $r = 1$ or the fibres are totally geodesic.*

Proof. We have $g(H, H) \geq 0$, $g(A, A) \leq 0$. Hence, by formula (2.8), we get the conclusion. □

PROPOSITION 2.10. *If $\pi: M \to B$ is a Riemannian submersion with compact and orientable total space M and with fibres of dimension 1 then $S \leq S' \circ \pi$ We have equality if and only if the horizontal distribution is integrable.*

Proof. We have $g(A, A) \geq 0$, $S - (S' \circ \pi + \hat{S}) = -\int_M g(A, A)\mathrm{dv}_g$ and $\hat{s} = 0$. Hence, by (2.8), we get $S \leq S' \circ \pi$. □

Let $s^{\mathcal{H}} = \sum_\alpha \varepsilon_\alpha \rho(e_\alpha, e_\alpha)$, where ρ is the Ricci tensor of M (see [12]).

PROPOSITION 2.11. *If $\pi: (M, g) \to (B, g')$ is a semi-Riemannian submersion with totally umbilic fibres then*

$$s^{\mathcal{H}} - s' \circ \pi = div(H) + \left(1 - \frac{1}{r}\right) g(H, H) - 2g(A, A) \tag{2.9}$$

Proof. By lemma 1.3 we have

$$\begin{aligned} s^{\mathcal{H}} - s' \circ \pi &= \sum_{\alpha,i} \varepsilon_\alpha \varepsilon_i R(e_\alpha, e_i, e_\alpha, e_i) + \sum_{\alpha,\beta} \varepsilon_\alpha \varepsilon_\beta R(e_\alpha, e_\beta, e_\alpha, e_\beta) \\ &= \sum_\alpha \varepsilon_\alpha (g(\nabla_{e_\alpha} H, e_\alpha) - \frac{1}{r} g(e_\alpha, H)^2) + g(A, A) - 3g(A, A) \\ &= \mathrm{div}(H) - \sum_i \varepsilon_i g(\nabla_{e_i} H, e_i) \\ &\quad - \frac{1}{r} g\left(H, \sum_\alpha \varepsilon_\alpha g(e_\alpha, H) e_\alpha\right) - 2g(A, A) \\ &= \mathrm{div}(H) + \left(1 - \frac{1}{r}\right) g(H, H) - 2g(A, A) \end{aligned}$$ □

Integrating formula (2.9) we get

PROPOSITION 2.12. *If $\pi\colon (M, g) \to (B, g')$ is a semi-Riemannian submersion with totally umbilic fibres and if M is a compact and orientable manifold then*

$$\int_M (s^{\mathcal{H}} - s' \circ \pi)\mathrm{dv}_g = \left(1 - \frac{1}{r}\right) \int_M g(H, H)\mathrm{dv}_g - 2\int_M g(A, A)\mathrm{dv}_g \tag{2.10}$$

COROLLARY 2.13. *Let B be an n-dimensional Riemannian manifold, $\pi\colon M \to B$ be a semi-Riemannian submersion with totally umbilic fibres and M be an m-dimensional compact and orientable semi-Riemannian manifold of index $r = m - n$. Then*

$$\int_M (s^{\mathcal{H}} - s' \circ \pi)\mathrm{dv}_g \geq 0$$

We have equality if and only if the horizontal distribution is integrable and either $r = 1$ or the fibres are totally geodesic.

Proof. We have $g(H, H) \geq 0$ and $g(A, A) \leq 0$. Therefore, by (2.10), we get the conclusion. □

3. Mean curvature vector field.

We denote by ρ, ρ' and $\hat{\rho}$ the Ricci tensors of the manifolds M, B and of the fibre $\pi^{-1}(b)$, $b \in B$. The letters U, V denote vertical vector fields and X, Y horizontal vector fields. We introduce the following notations (cf. [4]):

$$g(A_X, T_U) = \sum_\alpha \varepsilon_\alpha g(A_X e_\alpha, T_U e_\alpha) = \sum_i \varepsilon_i g(A_X e_i, T_U e_i),$$

$$\check{\delta} A = -\sum_\alpha \varepsilon_\alpha (\nabla_{e_\alpha} A)_{e_\alpha},\ \hat{\delta} T = -\sum_i \varepsilon_i (\nabla_{e_i} T)_{e_i}$$

where $\varepsilon_i \delta_{ik} = g(e_i, e_k)$ for all i, k , $\varepsilon_\alpha \delta_{\alpha\beta} = g(e_\alpha, e_\beta)$ for all α, β.

PROPOSITION 3.1. *Let $\pi\colon (M, g) \to (B, g')$ be a semi-Riemannian submersion with totally umbilic fibres. Assume that the dimension of fibres $r \geq 2$. Then the following conditions are equivalent:*

i) the mean curvature vector field H is a basic vector field;

ii) $g((\hat{\delta}T)U, X) - g(A_X, T_U) = 0$ for every U vertical vector field and for every X horizontal vector field;

iii) $\rho(X, U) + g((\check{\delta}A)(X), U) + (r+1)g(T_U, A_X) = 0$ *for every* U *vertical vector field and for every* X *horizontal vector field.*

Proof. i)$\Leftrightarrow$ii)

Since the fibres are totally umbilic we have

$$g((\hat{\delta}T)U, X) = -\frac{1}{r}g(\nabla_U H, X)$$

and

$$g(A_X, T_U) = \frac{1}{r}g(A_X U, H)$$

for every U vertical vector field and for every X horizontal vector field.

Let X be a basic vector field. By lemma 1.2, we have $A_X U = h\nabla_U X$. Then

$$\begin{aligned} g((\hat{\delta}T)U, X) - g(A_X, T_U) &= -\frac{1}{r}g(\nabla_U H, X) - \frac{1}{r}g(h\nabla_U X, H) \\ &= -\frac{1}{r}Ug(H, X). \end{aligned}$$

Hence, by lemma 1.1, H is a basic vector field if and only if the function $g(H, X)$ is constant along fibres for every X basic vector field.

i)$\Leftrightarrow$ iii)

Let X be a basic vector field. Using (9.36b) in [4], we compute

$$\begin{aligned} \rho(U, X) &= g((\hat{\delta}T)U, X) + g(\nabla_U H, X) - g((\check{\delta}A)(X), U) - 2g(T_U, A_X) \\ &= g((\hat{\delta}T)U, X) - g(T_U, A_X) + Ug(H, X) - g(H, A_X U) \\ &\quad - g((\check{\delta}A)(X), U) - \frac{1}{r}g(H, A_X U) \\ &= \left(1 - \frac{1}{r}\right)Ug(H, X) - \frac{r+1}{r}g(H, A_X U) - g((\check{\delta}A)(X), U). \end{aligned}$$

Then

$$\rho(X, U) + g((\check{\delta}A)(X), U) + (r+1)g(T_U, A_X) = (1 - \frac{1}{r})Ug(H, X) \tag{3.11}$$

Applying again lemma 1.1, we get the conclusion. □

Remark 3.2. By proposition 3.1 and theorem 2.2 in [2], H is a basic vector field if and only if the contractions $(1, 3)$ and $(2, 4)$ of the Riemann tensor of the Vrânceanu connection are equal.

The following proposition is well known (see Proposition 9.104 in [4]).

PROPOSITION 3.3. *If B, F are semi-Riemannian manifolds and $f: B \to \mathbb{R}$ is a positive function then $\pi: B \times_f F \to B$ is a semi-Riemannian submersion with totally umbilic fibres, $A \equiv 0$ and H a basic vector field.*

Conversely, if $\pi: (M, g) \to (B, g')$ is a semi-Riemannian submersion with totally umbilic fibres $A \equiv 0$ and H a basic vector field then M is a locally warped product.

COROLLARY 3.4. *If $\pi: (M, g) \to (B, g')$ is a semi-Riemannian submersion with totally umbilic fibres, if M is an Einstein manifold, $r \geq 2$ and the horizontal distribution $\mathcal{H}$ is integrable then M is a locally warped product.*

Proof. Since $A \equiv 0$, $\rho(X, U) = 0$, we have H is a basic vector field, by proposition 3.1. By proposition 9.104 in [4] we have M is a locally warped product. □

DEFINITION. *A semi-Riemannian submanifold is said to be an extrinsic sphere if it is totally umbilic and the mean curvature vector field H is nonzero anywhere and parallel in the normal bundle.*

COROLLARY 3.5. *Let $\pi: (M, g) \to (B, g')$ be a semi-Riemannian submersion. If the mean curvature vector field H is basic,* $\dim B = 2$, *and the fibres are extrinsic spheres then the horizontal distribution $\mathcal{H}$ is integrable and M is a locally warped product.*

Proof. Since H is a basic vector field and $h\nabla_U H = 0$ for every U vertical vector field, it follows $A_H \equiv 0$, by lemma 1.2. By $\dim \mathcal{H} = 2$ and $H_p \neq 0$ for every $p \in M$ we have $A \equiv 0$. Therefore, by proposition 9.104 in [4], M is a locally warped product. □

We would like to know how much a semi-Riemannian submersion with totally umbilic fibres is different to be a locally warped product. For this purpose we assume that $R(X, Y, X, Y)$ is constant along fibres for every X, Y basic vector fields and the mean curvature vector field H is parallel in the horizontal bundle along fibres.

First, we give equivalent conditions to these assumptions.

PROPOSITION 3.7. *Let $\pi: (M, g) \to (B, g')$ be a semi-Riemannian submersion. Then the following conditions are equivalent:*

i) $R(X, Y, X, Y)$ *is constant along fibres for every* X, Y, Z *basic vector fields;*

ii) the function $g(A_X Y, A_X Z)$ *is constant along fibres for every* X, Y, Z *basic vector fields;*

iii) $hR(X, Y)Z$ *is a basic vector field for every* X, Y, Z *basic vector fields.*

Proof. $i) \Rightarrow ii)$ Let X, Y be basic vector fields. By lemma 1.3, we have

$$R(X, Y, X, Y) = R'(\pi_* X, \pi_* Y, \pi_* X, \pi_* Y) \circ \pi - 3g(A_X Y, A_X Y)$$

If $R(X, Y, X, Y)$ is constant along fibres then $g(A_X Y, A_X Y)$ is constant along fibres. By polarization, we get $g(A_X Y, A_X Z)$ is constant along fibres.

$ii) \Rightarrow iii)$ If $g(A_X Y, A_X Z)$ is constant along fibres for every X, Y, Z basic vector fields then $A_X A_X Y$ is basic for every X, Y basic vector fields, by lemma 1.1. Therefore, by polarization, $A_X A_Y Z + A_Y A_X Z$ is basic for every X, Y, Z basic vector fields. By O'Neill's equations (see [13]), we have $hR(X, Y)Z = R'(X, Y)Z - 2A_Z A_X Y + A_X A_Y Z - A_Y A_X Z$, where $R'(X, Y)Z$ is the horizontal lifting of $R'(\pi_* X, \pi_* Y)\pi_* Z$. Rewriting this formula we get

$$hR(X, Y)Z = R'(X, Y)Z - (A_Z A_X Y + A_X A_Z Y) + (A_Z A_Y X + A_Y A_Z X).$$

Hence $hR(X, Y)Z$ is a basic vector field for every X, Y, Z basic vector fields □.

Let $\rho^{\mathcal{V}}(E) = \sum_i \varepsilon_i R(E, e_i)e_i$ for every E tangent vector field to M.

PROPOSITION 3.7. *Let $\pi: (M, g) \to (B, g')$ be a semi-Riemannian submersion with totally umbilic fibres. If $r \geq 2$ then the following conditions are equivalent:*

i) the mean curvature vector field H is parallel in the horizontal bundle along fibres,

ii) $\rho^{\mathcal{V}}(U)$ *is vertical for every* U *vertical vector field.*

Proof. By O'Neill's equation (see [13]), we have

$$R(e_i, U, e_i, X) = g((\nabla_U T)_{e_i} e_i, X) - g((\nabla_{e_i} T)_U e_i, X).$$

Then

$$\begin{aligned}\sum_i \varepsilon_i g((\nabla_U T)_{e_i} e_i, X) =& \sum_i \varepsilon_i [g(\nabla_U T_{e_i} e_i, X) - g(T_{\nabla_U e_i} e_i, X) \\ & - g(T_{e_i} \nabla_U e_i, X)] \\ =& g(\nabla_U H, X) - \sum_i 2\varepsilon_i g(\nabla_U e_i, e_i) g\left(\frac{H}{r}, X\right) \\ =& g(\nabla_U H, X) - \sum_i \varepsilon_i U(g(e_i, e_i))\, g\left(\frac{H}{r}, X\right) \\ =& g(\nabla_U H, X)\end{aligned}$$

Since the fibres are totally umbilic, we also have

$$\sum_i \varepsilon_i g((\nabla_{e_i} T)_{e_i} U, X) = \frac{1}{r} g(\nabla_U H, X).$$

Replacing these in O'Neill's equation, we obtain

$$g(\rho^{\mathcal{V}}(U), X) = \sum_i \varepsilon_i R(e_i, U, e_i, X) = \left(1 - \frac{1}{r}\right) g(\nabla_U H, X).$$

Therefore $\rho^{\mathcal{V}}(U)$ is a vertical vector field for every U vertical vector field if and only if H is parallel in the horizontal bundle along fibres. □

THEOREM 3.8. *Let $\pi: (M, g) \to (B, g')$ be a semi-Riemannian submersion with totally umbilic fibres. We suppose that $R(X, Y, X, Y)$ is constant along fibres, H is parallel in the horizontal bundle along fibres, $s - s' \in \{0, r\}$. Then H is a basic vector field if and only if $hR(X, Y)A_X Y$ is a basic vector field for every X, Y basic vector fields.*

Proof. a) We suppose that $hR(X, Y)A_X Y$ is a basic vector field for every X, Y basic vector fields. Let X, Y, Z be basic vector fields.

Using O'Neill's equation (see [13]) we get

$$\begin{aligned}R(X, Y, X, A_X Y) =& g((\nabla_X A)_X Y, A_X Y) + 2g(A_X Y, T_{A_X Y} X) \\ =& g(\nabla_X A_X Y, A_X Y)) - g(A_{\nabla_X X} Y, A_X Y) - g(A_X \nabla_X Y, A_X Y) \\ & + 2g(A_X Y, T_{A_X Y} X) \\ =& \frac{1}{2} X(g(A_X Y, A_X Y)) - g(A_Y h\nabla_X X, A_Y X) \\ & - g(A_X h\nabla_X Y, A_X Y) + 2g(A_X Y, T_{A_X Y} X).\end{aligned}$$

By proposition 3.6, the function $g(A_X Y, A_X Z)$ is constant along fibres. If $hR(X, Y)A_X Y$ is basic then $g(A_X Y, T_{A_X Y} X)$ is constant along fibres. Let U be a vertical vector field. Then

$$0 = U(g(T_{A_X Y} X, A_X Y)) = -\frac{1}{r} U(g(X, H))\, g(A_X Y, A_X Y) - \frac{1}{r} g(X, H)\, U(g(A_X Y, A_X Y)),$$

thus, we get

$$0 = U(g(X, H))\, g(A_X Y, A_X Y) \tag{3.12}$$

for every X, Y basic vector fields.

Let p in M be an arbitrary point. Let Y be a basic vector field such that $Y_p = H_p$.

The relation (3.12) in point p become

$$0 = U_p(g(X, H))\, g_p(A_{X_p} H_p, A_{X_p} H_p) \tag{3.13}$$

We have two possible situations:

Case. 1. $A_{X_p} H_p \neq 0$.

Since the metrics of fibres are negative definite for $s - s' = r$ or positive definite for $s - s' = 0$ we have $g_p(A_{X_p} H_p, A_{X_p} H_p) \neq 0$. By relation (3.13), we get $U_p(g(X, H)) = 0$.

Case. 2. $A_{X_p} H_p = 0$

Since H is parallel in the horizontal bundle along fibres we have $U(g(H, X)) = -g(A_X H, U)$. Using the hypothesis of *Case* 2, $A_{X_p} H_p = 0$, we get $U_p(g(H, X)) = 0$

In both cases we proved that $U_p g(H, X) = 0$ for an arbitrary $p \in M$ and for every X basic vector field. By lemma 1.1, it follows that H is a basic vector field.

b) We suppose H is a basic vector field and we shall prove $hR(X, Y)A_X Y$ is basic for every X, Y basic vector fields.

By O'Neill's equation, we get

$$\begin{aligned} R(X, Y, Z, A_X Y) = &\frac{1}{2} Z(g(A_X Y, A_X Y)) - g(A_Y h\nabla_Z X, A_Y X) \\ &- g(A_X h\nabla_Z Y, A_X Y) - g(A_X Y, A_X Y) g(\frac{H}{r}, Z) \\ &- g(A_Y Z, A_Y X) g(\frac{H}{r}, X) - g(A_X Z, A_X Y) g(\frac{H}{r}, Y). \end{aligned}$$

Hence $g(hR(X, Y)A_X Y, Z)$ is constant along fibres for every X, Y, Z basic vector fields. Therefore, by lemma 1.1, $hR(X, Y)A_X Y$ is basic. □

We apply the theorem 3.8 in two particular cases, when the total space is either a constant curvature semi-Riemannian manifold or a generalized complex space form.

COROLLARY 3.9. *Let $\pi: (M, g) \to (B, g')$ be a semi-Riemannian submersion with totally umbilic fibres. If M is a semi-Riemannian manifold with constant curvature and if $r \geq 2$, $s - s' \in \{0, r\}$ then H is a basic vector field and $A_H \equiv 0$.*

Proof. If M has constant sectional curvature c then for every X, Y basic vector fields and for every U vertical vector field we get:

1) $R(X, Y, X, Y) = c(g(X, X)g(Y, Y) - g(X, Y)^2)$ is constant along fibres;
2) $R(X, Y)A_X Y = c(g(A_X Y, X)X - g(A_X Y, Y)Y) = 0$;
3) $\rho^{\mathcal{V}}(U) = \sum_i \varepsilon_i R(U, e_i)e_i = \sum_i \varepsilon_i c(g(e_i, e_i)U - g(U, e_i)e_i) = c(r-1)U$ is a vertical vector field. Therefore, by theorem 3.8 and proposition 3.7, H is a basic vector field. Hence, by lemma 1.2, $A_H \equiv 0$ □.

The totally umbilic submanifolds of generalized complex space forms was classified by L. Vanhecke in [16] (see also survey paper [10]). Applying theorem 3.8 we get

COROLLARY 3.10. *Let $\pi: (M, g) \to (B, g')$ be a semi-Riemannian submersion with totally umbilic fibres from a generalized complex space form onto an almost hermitian manifold. If π is a holomorphic map, $r \geq 2$, $s - s' \in \{0, r\}$ then H is a basic vector field and $A_H \equiv 0$.*

Proof. If (M, J) is a generalized complex space form of constant holomorphic sectional curvature μ and of type α then the curvature tensor field

satisfies (see [10])

$$\begin{aligned} R(E, F, G, G') =& \frac{1}{4}(\mu + 3\alpha)\{g(E, G)g(F, G') - g(E, G')g(F, G)\} \\ &+ \frac{1}{4}(\mu - \alpha)\{g(E, JG)g(F, JG') - g(E, JG')g(F, JG)\} \\ &+ \frac{1}{2}(\mu - \alpha)g(E, JF)g(G, JG'). \end{aligned}$$

Since π is a holomorphic map, we get JX is basic for every X basic vector field. Hence $R(X, Y, X, Y)$ is constant along fibres, $g(\rho^{\mathcal{V}}(U), X) = 0$ and $R(X, Y, Z, U) = 0$ for every X, Y, Z basic vector fields and for every U vertical vector field. Therefore, by theorem 3.8 and proposition 3.7, H is a basic vector field. □

PROPOSITION 3.11. *Let $\pi : (M, g) \to (B, g')$ be a semi-Riemannian submersion with totally umbilic fibres. If $R(X, Y, X, Y)$ is constant along fibres for every X, Y basic vector fields, H is basic and B is a compact and orientable manifold then τ^{HV} is constant along fibres and*

$$\int_B \tau'^{HV} = -\frac{1}{r}\int_B g'(\pi_* H, \pi_* H)dv'_g + \int_B g'(A, A)dv'_g \tag{3.14}$$

where $g'(A, A)$, τ'^{HV} are the functions on B satisfying $g'(A, A) \circ \pi = g(A, A)$ and $\tau'^{HV} \circ \pi = \tau^{HV}$.

Proof. By theorem 1.7 we have

$$\tau^{HV} = \operatorname{div}(H) + \left(1 - \frac{1}{r}\right) g(H, H) + g(A, A).$$

Using lemma 1.2 we get

$$\begin{aligned} \operatorname{div}(H) + g(H, H) &= \sum_\alpha \varepsilon_\alpha g(\nabla_{e_\alpha} H, e_\alpha) + \sum_i \varepsilon_i g(\nabla_{e_i} H, e_i) + g(H, H) \\ &= \sum_\alpha \varepsilon_\alpha g'(\nabla'_{\pi_* e_\alpha} \pi_* H, \pi_* e_\alpha) \circ \pi \\ &\quad + \sum_i \varepsilon_i [g(T_{e_i} H, e_i) + g(H, T_{e_i} e_i)] \\ &= \operatorname{div}'(\pi_* H) \circ \pi \end{aligned}$$

Since the function $g(A_{e_\alpha}e_\beta, A_{e_\alpha}e_\beta)$ is constant along fibres we can consider the function $g'(A, A)$ on B given by $g'(A, A) \circ \pi = g(A, A)$. Then

$$\tau^{HV} = \left[\operatorname{div}'(\pi_* H) - \frac{1}{r} g'(\pi_* H, \pi_* H) + g'(A, A)\right] \circ \pi. \tag{3.15}$$

and τ^{HV} is constant along fibres. Let τ'^{HV} be the function on B such that $\tau'^{HV} \circ \pi = \tau^{HV}$. Since $\int_B \operatorname{div}'(\pi_* H) = 0$, it follows the formula (3.14). □

COROLLARY 3.12. *Let $\pi: (M, g) \to (B, g')$ be a semi-Riemannian submersion with totally umbilic fibres. If M is a semi-Riemannian manifold with constant curvature c, $r \geq 2$, $s - s' \in \{0, r\}$ and B is a compact and orientable manifold then*

$$rnc\, vol\,(B) = -\frac{1}{r} \int_B g'(\pi_* H, \pi_* H) dv'_g + \int_B g'(A, A) dv'_g \tag{3.16}$$

where $g'(A, A)$ is the function on B satisfying $g'(A, A) \circ \pi = g(A, A)$.

Proof. Since M is a semi-Riemannian manifold with constant curvature c, we get $R(e_\alpha, e_i, e_\alpha, e_i) = \varepsilon_i \varepsilon_\alpha c$. So $\tau^{HV} = rnc$.

THEOREM 3.13. *Let B be an n-dimensional compact and orientable Riemannian manifold, $\pi: M \to B$ be a semi-Riemannian submersion with totally umbilic fibres. We suppose that M is an m-dimensional semi-Riemannian manifold of index $r = m - n$ with non-negative mixed curvature, $R(X, Y, X, Y)$ is constant along fibres, for every X, Y basic vector fields, and H is basic. Then*

i) $K(X, V) = 0$ for every X horizontal vector field and for every V vertical vector field;

ii) the fibres are totally geodesic and the horizontal distribution is integrable, hence M is a locally warped product.

Proof. Since $\varepsilon_\alpha = 1$, the metrics induced on fibres are negative definite, we have

$$g(A, A) = \sum_{\alpha,\, \beta} \varepsilon_\alpha \varepsilon_\beta g(A_{e_\alpha} e_\beta, A_{e_\alpha} e_\beta) \leq 0.$$

By proposition 3.11, $\int_B \tau'^{HV} \leq 0$.

But the mixed curvature is non-negative, hence $\tau'^{HV} \geq 0$. Therefore $K(X, V) = 0$ for every X horizontal vector field and for every V vertical vector field and $A \equiv 0$, $T \equiv 0$. □

COROLLARY 3.14. *Let B be an n-dimensional compact and orientable Riemannian manifold, $\pi: M \to B$ be a semi-Riemannian submersion with totally umbilic fibres. We suppose M is an m-dimensional semi-Riemannian manifold of index $r = m - n$ with constant curvature c and $r \geq 2$. Then*

i) $c \leq 0$;

ii) If $c = 0$ then the fibres are totally geodesic and the horizontal distribution $\mathcal{H}$ is integrable.

Proof. If we suppose $c > 0$ then, by theorem 3.13, we get $c = K(X, V) = 0$, which is a contradiction with our assumption. Therefore $c \leq 0$ □.

COROLLARY 3.15. *Let B be an n-dimensional simply connected Riemannian manifold, $\pi: M \to B$ be a semi-Riemannian submersion with totally umbilic fibres. If M is an m-dimensional semi-Riemannian manifold of index $r = m - n \geq 2$ with constant curvature c then B is not compact.*

Proof. If we suppose that B is a compact Riemannian manifold then B is complete. By corollary 3.14, we get $c \leq 0$. It follows $K' \leq 0$, by lemma 1.3. By Hadamard's theorem, we have B is diffeomorphic to $\mathbb{R}^n$. Therefore B is not compact.

We introduce the notion of Clairaut semi-Riemannian submersion (see [1]).

DEFINITION. *Let B be an n-dimensional Riemannian manifold, $\pi: (M, g) \to (B, g')$ be a semi-Riemannian submersion, M be an m-dimensional semi-Riemannian manifold of index $r = m - n$ and γ be a timelike geodesic in M. We denote the velocity vector field of γ by $E = \gamma'$ and its vertical part by V. At each point $\gamma(s)$ we define $\varphi(s)$ to be the hyperbolic angle between E and V, i.e. $\varphi \geq 0$ is the number satisfying:*

$$g(E, V) = -\|E\| \cdot \|V\| \cosh \varphi,$$

where $\|E\|^2 = -g(E, E)$ and $\|V\|^2 = -g(V, V)$.

$\pi: (M, g) \to (B, g')$ is said to be a Clairaut semi-Riemannian submersion if there is a positive function $w: M \to \mathbb{R}$ such that for every timelike geodesic γ in M, $w \cosh \varphi$ is constant along γ. We call r the girth of the submersion.

The following characterization of Clairaut semi-Riemannian submersion given by D.Allison [1] for 1-dimensional fibres has a similar proof for r-dimensional case

PROPOSITION 3.16. *Let B be an n-dimensional Riemannian manifold, $\pi:(M,g)\to(B,g')$ be a semi-Riemannian submersion and M be an m-dimensional semi-Riemannian manifold of index $r=m-n$. Then π is a Clairaut semi-Riemannian submersion with girth $w=\exp f$ if and only if the fibres are totally umbilic with a gradient $H=-grad\ f$ as mean curvature vector field. Furthermore for a Clairaut semi-Riemannian submersion having connected fibres $f=f_*\circ\pi$ for some $f_*:B\to\mathbb{R}$ and H is a basic vector field obtained by lifting $H_*=-grad\ f_*$ horizontally.*

LEMMA 3.17. *Let $\pi:(M,g)\to(B,g')$ be a semi-Riemannian submersion with totally umbilic fibres. We suppose that H is a basic vector field and parallel in the horizontal bundle along fibres and there is a constant $c\in\mathbb{R}$ such that $g(\rho^{\mathcal{V}}(\frac{H}{r}),X)=cg(H,X)$ for every X horizontal vector field. Then:*

$$\frac{1}{2}grad\left(c+g\left(\frac{H}{r},\frac{H}{r}\right)\right)=\left(c+g\left(\frac{H}{r},\frac{H}{r}\right)\right)\frac{H}{r} \tag{3.17}$$

Proof. Let X,Y be horizontal vector fields and U be a vertical vector field. By O'Neill's equation [13] we have

$$\begin{aligned}R(X,U,Y,U)=&g(U,U)\left[g\left(\nabla_X\frac{H}{r},Y\right)-g\left(X,\frac{H}{r}\right)g\left(Y,\frac{H}{r}\right)\right]\\&+g((\nabla_UA)_XY,U)+g(A_XU,A_YU)\end{aligned}$$

Let $Y=\dfrac{H}{r}$. Since H is a basic vector field and parallel in the horizontal bundle along fibres we get $A_H\equiv 0$. Therefore

$$g((\nabla_UA)_XH,U)+g(A_XU,A_HU)=0$$

Since

$$\sum\varepsilon_iR\left(X,e_i,\frac{H}{r},e_i\right)=g\left(\rho^{\mathcal{V}}\left(\frac{H}{r}\right),X\right)=cg(H,X)$$

we obtain

$$g\left(X,\frac{H}{r}\right)c=g\left(\nabla_X\frac{H}{r},\frac{H}{r}\right)-g\left(X,\frac{H}{r}\right)g\left(\frac{H}{r},\frac{H}{r}\right);$$

$$g\left(X, \frac{H}{r}\right)\left(c + g\left(\frac{H}{r}, \frac{H}{r}\right)\right) = g\left(\nabla_X \frac{H}{r}, \frac{H}{r}\right) = \frac{1}{2} X g\left(\frac{H}{r}, \frac{H}{r}\right);$$

$$g\left(X, \frac{1}{2}\text{grad}\, g\left(\frac{H}{r}, \frac{H}{r}\right) - \left(c + g\left(\frac{H}{r}, \frac{H}{r}\right)\right)\frac{H}{r}\right) = 0$$

for every X horizontal vector field.

This implies

$$\frac{1}{2} h\left(\text{grad}\, g\left(\frac{H}{r}, \frac{H}{r}\right)\right) = \left(c + g\left(\frac{H}{r}, \frac{H}{r}\right)\right)\frac{H}{r}.$$

Since H is parallel in the horizontal bundle along fibres, it follows that $g(H, H)$ is constant along fibres. Hence

$$v\text{grad}\left(g\left(\frac{H}{m}, \frac{H}{m}\right) + c\right) = 0. \quad \square$$

LEMMA 3.18. *Let $\pi : M \to B$ be a semi-Riemannian submersion. If either*

i) M has constant curvature, or

ii) M and B are Einstein manifolds and $A_H \equiv 0$

then there is $c \in \mathbb{R}$ such that

$$g\left(\rho^{\mathcal{V}}\left(\frac{H}{r}\right), X\right) = c g(H, X)$$

for every X horizontal vector field.

Proof. i) If M has constant curvature c then

$$g\left(\rho^{\mathcal{V}}\left(\frac{H}{r}\right), X\right) = \sum_i \varepsilon_i R\left(\frac{H}{r}, e_i, X, e_i\right) = c g(H, X)$$

for every X horizontal vector field.

ii) We define

$$\rho^{\mathcal{H}}(E) = \sum_\alpha \varepsilon_\alpha R(E, e_\alpha) e_\alpha$$

for every E vector field on M. We have

$$\rho\left(\frac{H}{r}, X\right) = g\left(\rho^{\mathcal{V}}\left(\frac{H}{r}\right), X\right) + g\left(\rho^{\mathcal{H}}\left(\frac{H}{r}\right), X\right).$$

Since M and B are Einstein manifolds, for some $\lambda, \lambda' \in \mathbb{R}$ we have

$$\rho\left(\frac{H}{r}, X\right) = \lambda g\left(\frac{H}{r}, X\right)$$

and

$$\rho'\left(\pi_*\frac{H}{r}, \pi_* X\right) = \lambda' g\left(\pi_*\frac{H}{r}, \pi_* X\right)$$

for every X horizontal vector.

By lemma 1.3, we get

$$\begin{aligned}\rho'\left(\pi_*\frac{H}{r}, \pi_* X\right) &= \sum_\alpha \varepsilon_\alpha \left(R\left(\frac{H}{r}, e_\alpha, X, e_\alpha\right) + 3g\left(A_{e_\alpha}\frac{H}{r}, A_{e_\alpha} X\right)\right) \\ &= g\left(\rho^{\mathcal{H}}\left(\frac{H}{r}\right), X\right)\end{aligned}$$

It follows

$$g\left(\rho^{\mathcal{V}}\left(\frac{H}{r}\right), X\right) = (\lambda - \lambda')g\left(\frac{H}{r}, X\right)$$

for every X horizontal vector field. □

THEOREM 3.19. *Let $\pi: (M, g) \to (B, g')$ be a semi-Riemannian submersion with totally umbilic fibres. We suppose that H is a basic vector field and parallel in the horizontal bundle along fibres and there is a constant $c \in \mathbb{R}$ such that*

$$g\left(\rho^{\mathcal{V}}\left(\frac{H}{r}\right), X\right) = cg(H, X)$$

for every X horizontal vector field. If

$$g\left(\frac{H_p}{r}, \frac{H_p}{r}\right) \neq -c$$

for all $p \in M$ then

$$\frac{H}{r} = \frac{1}{2} grad\left(ln\left|c + g\left(\frac{H}{r}, \frac{H}{r}\right)\right|\right).$$

If moreover B is an n-dimensional Riemannian manifold, and M be an m-dimensional semi-Riemannian manifold of index $r = m - n$ then π is a Clairaut semi-Riemannian submersion.

Proof. Since $g\left(\frac{H_p}{r}, \frac{H_p}{r}\right) + c \neq 0$ for every point $p \in M$, we have, by formula (3.17),

$$\frac{H}{r} = \frac{1}{2}\text{grad}\left(ln\left|g\left(\frac{H}{r}, \frac{H}{r}\right) + c\right|\right).$$

By proposition 3.16, we get π is a Clairaut semi-Riemannian submersion. □

COROLLARY 3.20. *Let B be an n-dimensional Riemannian manifold, $\pi: (M, g) \to (B, g')$ be a semi-Riemannian submersion with totally umbilic fibres and M be an m-dimensional semi-Riemannian manifold of index $r = m - n \geq 2$ with constant curvature c. If $g\left(\frac{H_p}{r}, \frac{H_p}{r}\right) \neq -c$ for all $p \in M$ then*

$$\frac{H}{r} = \frac{1}{2}grad\left(ln\left|c + g\left(\frac{H}{r}, \frac{H}{r}\right)\right|\right)$$

and π is a Clairaut semi-Riemannian submersion.

Proof. By lemma 3.18 and by theorem 3.19 we get the conclusion. □

By corollaries 3.9 and 3.12 and use of [11], we get more information about curvature of total space, in the constant curvature case.

PROPOSITION 3.21. *Let $\pi: (M, g) \to (B, g')$ be a semi-Riemannian submersion with totally umbilic fibres. We assume that B is an n-dimensional Riemannian manifold, M is an m-dimensional semi-Riemannian manifold of index $r = m - n \geq 2$ with constant curvature c. If $g\left(\frac{H}{r}, \frac{H}{r}\right) + c \geq 0$ everywhere, $g\left(\frac{H_{p_0}}{r}, \frac{H_{p_0}}{r}\right) + c > 0$ at some point $p_0 \in M$, and if each fibre is a compact manifold then the horizontal distribution is integrable and M is a locally warped product. If moreover B is a compact and orientable manifold then $c < 0$, $n \neq 1$ and M is not a compact and orientable manifold.*

Proof. If M has constant curvature, then for every U vertical vector field we have

$$\hat{\rho}(U, U) = (r - 1)g(U, U)\left(g\left(\frac{H}{r}, \frac{H}{r}\right) + c\right).$$

Since the metrics of the fibres are negative definite we have $\hat{\rho}(U, U) \leq 0$ everywhere and $\hat{\rho}$ is negative definite in p_0. By corollary 3.9, we have H is a

basic vector field. Therefore, by the theorem in [11], the horizontal distribution is integrable and M is a locally warped product. If B is compact and orientable then, by corollary 3.12,

$$(n-1)c\,\mathrm{vol}\,(B) = -\int_B \left(g'\left(\frac{\pi_* H}{r}, \frac{\pi_* H}{r}\right) + c \right) \mathrm{dv}_g.$$

Hence $(n-1)c < 0$. If we suppose M is a compact and orientable manifold then by proposition 2.1 we have

$$(n+r-1)c\,\mathrm{vol}\,(M) = (r-1)\int_M \left(g\left(\frac{H}{r}, \frac{H}{r}\right) + c \right) \mathrm{dv}_g + \frac{1}{r}\int_M g(A, A)\mathrm{dv}_g$$

By $g\left(\frac{H}{r}, \frac{H}{r}\right) + c > 0$ in p_0 and $g(A, A) \equiv 0$ we get $c > 0$, which is a contradiction with corollary 3.14. □

PROPOSITION 3.22. *Let B be an n-dimensional compact and orientable Riemannian manifold, $\pi\colon (M, g) \to (B, g')$ be a semi-Riemannian submersion with totally umbilic fibres and M be an m-dimensional semi-Riemannian manifold of index $r = m - n$ with constant curvature c. If each fibre is a compact manifold, $r \geq 2$, $n \geq 3$, and if*

$$g\left(\frac{H_p}{r}, \frac{H_p}{r}\right) + c \geq 0 \tag{3.18}$$

$$3g\left(\frac{H_p}{r}, \frac{H_p}{r}\right) + nc - \frac{1}{r} g_p(A, A) \geq 0 \tag{3.19}$$

for all $p \in M$ then $g\left(\frac{H}{r}, \frac{H}{r}\right) + c \equiv 0$.

Proof. We denote by $\Delta_B f = \mathrm{div}(\mathrm{grad}\, f)$ the Laplacian of B defined on functions $f\colon B \to \mathbb{R}$.

Let f be the function on B given by

$$f = g'\left(\frac{\pi_* H}{r}, \frac{\pi_* H}{r}\right) + c$$

By lemma 3.17 we have the equation $\frac{1}{2}\text{grad } f = f\frac{\pi_* H}{r}$. We get

$$\begin{aligned}\frac{1}{2}\Delta_B f =& \frac{1}{2}\text{div}(\text{grad } f) = \text{div}\left(f\frac{\pi_* H}{r}\right)\\ =& \left(\frac{\pi_* H}{r}\right)(f) + f\,\text{div}\left(\frac{\pi_* H}{r}\right)\\ =& g'\left(\text{grad} f, \frac{\pi_* H}{r}\right) + f\,\text{div}\left(\frac{\pi_* H}{r}\right)\\ =& 2fg'\left(\frac{\pi_* H}{r}, \frac{\pi_* H}{r}\right) + f\,\text{div}\left(\frac{\pi_* H}{r}\right)\end{aligned}$$

Using relation (3.15) we have

$$\text{div}\left(\frac{\pi_* H}{r}\right) = nc + g'\left(\frac{\pi_* H}{r}, \frac{\pi_* H}{r}\right) - \frac{1}{r}g'(A, A),$$

where $g'(A, A)$ is the function satisfying $g(A, A) = g'(A, A) \circ \pi$. We obtain

$$\begin{aligned}\frac{1}{2}\Delta_B f =& f(3(f - c) + nc - \frac{1}{r}g'(A, A))\\ =& \left(g'\left(\frac{\pi_* H}{r}, \frac{\pi_* H}{r}\right) + c\right)\left(3g'\left(\frac{\pi_* H}{r}, \frac{\pi_* H}{r}\right)\right.\\ & \left. + nc - \frac{1}{r}g'(A, A)\right) \geq 0\end{aligned}$$

Since $\Delta_B(f) \geq 0$ and B is a compact and orientable manifold, we have f is a constant function and $\Delta_B(f) \equiv 0$, by Hopf's lemma.

If we suppose

$$f = g'\left(\frac{\pi_* H}{r}, \frac{\pi_* H}{r}\right) + c > 0$$

then, by corollary 3.20, $A \equiv 0$, $c < 0$. $\Delta_B f \equiv 0$ implies $3g'\left(\frac{\pi_* H}{r}, \frac{\pi_* H}{r}\right) + nc \equiv 0$. From corollary 3.12, we obtain

$$\int_B \left(g'\left(\frac{\pi_* H}{r}, \frac{\pi_* H}{r}\right) + nc\right) \text{dv}_{g'} = 0.$$

Since $g'\left(\frac{\pi_* H}{r}, \frac{\pi_* H}{r}\right)$ is a nonnegative constant function we have $g'\left(\frac{\pi_* H}{r}, \frac{\pi_* H}{r}\right) = 0$. It follows $H \equiv 0$. Therefore $c > 0$. This is a contradiction with condition $c \leq 0$ given by corollary 3.14.

All these imply $g'\left(\frac{\pi_* H}{r}, \frac{\pi_* H}{r}\right) + c \equiv 0$. □

LEMMA 3.23. *The condition* $3g\left(\frac{H_p}{r}, \frac{H_p}{r}\right) + nc \geq 0$ *for every* $p \in M$ *implies the conditions (3.18) and (3.19) in proposition 3.22.*

Proof. Since $g(A, A) \leq 0$ and $(n-3)g\left(\frac{H_p}{r}, \frac{H_p}{r}\right) \geq 0$ for every $p \in M$ we have the conditions (3.18) and (3.19) in proposition 3.22.

By lemma 3.23 and proposition 3.22, we have the following corollary.

COROLLARY 3.24. *Let B be an 3-dimensional compact and orientable Riemannian manifold, $\pi : (M, g) \to (B, g')$ be a semi-Riemannian submersion with totally umbilic fibres and M be an m-dimensional semi-Riemannian manifold of index $r = m-3$ with constant curvature c. If $r \geq 2$, if each fibre is a compact manifold and if $g\left(\frac{H}{r}, \frac{H}{r}\right) + c \geq 0$ then $g\left(\frac{H}{r}, \frac{H}{r}\right) + c \equiv 0$.*

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