CLASSIFICATION OF HOMOGENEOUS EINSTEIN METRICS ON PSEUDO-HYPERBOLIC SPACES

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ABSTRACT. We classify the effective and transitive actions of a Lie group G on an n-dimensional non-degenerate hyperboloid (also called real pseudo-hyperbolic space), under the assumption that G is a closed, connected Lie subgroup of an indefinite special orthogonal group. Under the same assumption on G, we also obtain that any G-homogeneous Einstein pseudo-Riemannian metric on a real, complex or quaternionic pseudo-hyperbolic space, or on a para-complex or para-quaternionic projective space is homothetic to either the canonical metric or the Einstein metric of the canonical variation of a Hopf pseudo-Riemannian submersion.

1. INTRODUCTION AND THE MAIN THEOREM

The homogeneous Einstein Riemannian metrics on spheres and projective spaces are, up to homothety, the canonical metrics or the Einstein metrics of the canonical variations of the Hopf fibrations (see Ziller [22]). Essentially, up to a scaling factor, S^{15} has 3 homogeneous Einstein Riemannian metrics, $\mathbb{C}P^{2n+1}$ and S^{4n+3} have 2 homogeneous Einstein Riemannian metrics, and each of the remaining spaces S^{2n}, S^{4n+1}, $\mathbb{C}P^{2n}$, $\mathbb{H}P^n$ has only one homogeneous Einstein metric (see Besse [3, Theorem 9.86] and Ziller [22]).

Motivated by the recent classification of the pseudo-Riemannian submersions with totally geodesic fibres from pseudo-hyperbolic spaces (see Bădițoiu [1]), in this paper we obtain a pseudo-Riemannian generalization of Ziller's classification mentioned above (see Ziller [22]), and we prove the following main result.

Theorem 1.1. *Let G be a connected, closed Lie subgroup of $SO_0(n-r, r+1)$. Any G-homogeneous Einstein pseudo-Riemannian metric on one of the following sets: H_r^n, $\mathbb{C}H_{r/2}^{n/2}$, $\mathbb{H}H_{r/4}^{n/4}$, $\mathbb{A}P^{n/2}$ (with $r = (n+1)/2$), $\mathbb{B}P^{n/4}$ (with $r = (n+1)/4$) is homothetic to either the canonical metric or the Einstein metric g_{t_0} ($t_0 \neq 1$) of the canonical variation of a Hopf pseudo-Riemannian submersion. Therefore, under the same assumption on G, the following hold:*

   (i) *H_s^{2m}, H_s^{4m+1}, $\mathbb{C}H_s^{2m}$, $\mathbb{C}H_{2s}^{2m+1}$ (with $m \neq 2s$), $\mathbb{A}P^{2m}$, $\mathbb{B}P^m$ have only one homogeneous Einstein metric;*

   (ii) *H_{4s+3}^{4m+3} (with $m \neq 3$ and $m \neq 2s+1$), $\mathbb{C}H_{2s}^{4s+1}$, $\mathbb{C}H_{2s+1}^{2m+1}$ (with $m \neq 2s+1$) and $\mathbb{A}P^{2m+1}$ have 2 homogeneous Einstein metrics.*

   (iii) *H_{15}^{15}, H_{4s+3}^{8s+7} (with $s \neq 1$) and $\mathbb{C}H_{2s+1}^{4s+3}$ have 3 homogeneous Einstein metrics.*

   (iv) *H_7^{15} has 5 homogeneous Einstein metrics.*

The key ingredient of the proof of Theorem 1.1 is the classification of effective transitive actions of a Lie group G on a real pseudo-hyperbolic space under the assumption of Theorem 1.1. Now, we give a short review of well-known classification results of effective and transitive actions.

The pioneering work is due to Montgomery and Samelson (see [14]) and Borel (see [5]), who classified the compact Lie groups acting effectively and transitively on spheres. Using homotopic methods, Onishchik obtained the classification of the connected compact Lie groups G acting transitively on simply connected manifolds of rank 1 (see Onishchik [16, 18]).

The case of transitive actions on non-compact spaces is a lot more challenging than the compact case and therefore, one has to impose additional assumptions on the Lie group G. The case G reductive was investigated by Onishchik in [17], where he studied the equivalent problem of finding decompositions $G = G'G''$ into two proper Lie subgroups G' and G''. Of special interest for us is his classification of semisimple decompositions of $so(n-r, r+1)$, simply because it solves our problem of finding all transitive and effective actions on H_s^n in the case of a semisimple $G \subset SO_0(n-r, r+1)$. Using the Borel-Montgomery-Samelson classification of effective transitive actions on spheres, Wolf obtained a classification of the connected, closed Lie subgroups of $SO_0(n - r, r + 1)$ acting transitively both on (a) a component of a non-empty quadric $\{x \in \mathbb{R}_{r+1}^{n+1} \mid ||x||^2 = a\}$ ($a \neq 0$) and (b) the light cone $\{x \in \mathbb{R}_{r+1}^{n+1} \mid ||x||^2 = 0, x \neq 0\}$ (see Wolf [21, Theorem 3.1]). In our Theorem 3.1, we drop (b) and the assumption on the semisimplicity of G.

2. The Hopf pseudo-Riemannian submersions and their canonical variations

First, we introduce some standard definitions and notation that shall be needed throughout the paper.

Definition 2.1. Let $\langle \cdot, \cdot \rangle_{\mathbb{R}_{r+1}^{n+1}}$ be the standard inner product of signature $(n - r, r + 1)$ on $\mathbb{R}^{n+1}$ given by

$$(2.1) \qquad \langle x, y \rangle_{\mathbb{R}_{r+1}^{n+1}} = -\sum_{i=0}^{r} x_i y_i + \sum_{i=r+1}^{n} x_i y_i$$

for $x = (x_0, \cdots, x_n), y = (y_0, \cdots, y_n) \in \mathbb{R}^{n+1}$. For any $c < 0$ and any positive integer $r \leq n$, the set $H_r^n(c) = \{x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle_{\mathbb{R}_{r+1}^{n+1}} = 1/c\}$ is called the *real pseudo-hyperbolic space* of index r and dimension n. The hyperbolic space is defined as $H_0^n(c) = \{x = (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} \mid x_0 > 0, \ \langle x, x \rangle_{\mathbb{R}_1^{n+1}} = 1/c\}$. For convenience, we write $H_r^n = H_r^n(-1)$.

Notation 1. We define

$$SO(n - r, r + 1) = \{g \in SL(n + 1, \mathbb{R}) \mid \langle gx, gy \rangle_{\mathbb{R}_{r+1}^{n+1}} = \langle x, y \rangle_{\mathbb{R}_{r+1}^{n+1}}\}.$$

When K is a Lie group, we shall always denote by K_0 its connected component of the identity.

Let $\mathbb{C}$, $\mathbb{H}$, $\mathbb{A}$, $\mathbb{B}$ be the algebras of complex, quaternionic, para-complex and para-quaternionic numbers, respectively. For $F \in \{\mathbb{A}, \mathbb{B}\}$, we denote by $\bar{z}$, as usual, the conjugate of $z \in F$. For any $z = (z_1, \cdots, z_m), w = (w_1, \cdots, w_m) \in F^m$, we define the standard inner product $\langle z, w \rangle_{F^m}$ on F^m by

$$(2.2) \qquad \langle z, w \rangle_{F^m} = \text{Re}(\sum_{i=1}^{m} \bar{z}_i w_i).$$

The group $U^\pi(m) = \{g \in \text{GL}(m, \mathbb{A}) \mid \langle gz, gw \rangle_{\mathbb{A}^m} = \langle z, w \rangle_{\mathbb{A}^m}\}$ is called the para-unitary group (see [6, Prop. 4] or [12, p. 508]). Let $Sp^\pi(m) = \{g \in \text{GL}(m, \mathbb{B}) \mid \langle gz, gw \rangle_{\mathbb{B}^m} = \langle z, w \rangle_{\mathbb{B}^m}\}$ be the para-symplectic group (see [12, p. 510]). We have a natural inclusion $Sp^\pi(m) \subset U^\pi(2m)$ and some identifications $U^\pi(m) = \text{GL}(m, \mathbb{R})$ (see [12, p. 508]), and $Sp^\pi(m) \cong Sp(m, \mathbb{R})$ (see [12, p. 510, Prop. 1.4.3]). Here, our convention is that $Sp(m, \mathbb{R})$ denotes the group of $2m \times 2m$-symplectic matrices with entries in $\mathbb{R}$.

Definition 2.2. We define (see [1], [7, 8] for $\mathbb{A}P^m$, [4] for $\mathbb{B}P^m$):

$$\mathbb{C}H_s^m(c) = H_{2s+1}^{2m+1}(c/4)/U(1), \ \mathbb{C}H_s^m = \mathbb{C}H_s^m(-4), \ \mathbb{H}H_s^m(c) = H_{4s+3}^{4m+3}(c/4)/Sp(1), \ \mathbb{H}H_s^m = \mathbb{H}H_t^m(-4);$$

$$\mathbb{A}P^m = \{z \in \mathbb{A}^{m+1} \mid \langle z, z \rangle_{\mathbb{A}^{m+1}} = 1\}/\{t = x + ey \in \mathbb{A} \mid t\bar{t} = 1, x > 0\} = H_m^{2m+1}/H^1, \text{(with } e^2 = 1);$$

$$\mathbb{B}P^m = \{z \in \mathbb{B}^{m+1} \mid \langle z, z \rangle_{\mathbb{B}^{m+1}} = 1\}/\{t \in \mathbb{B} \mid t\bar{t} = 1\} = H_{2m+1}^{4m+3}/H_1^3.$$

Any Hopf pseudo-Riemannian submersions can be written as a homogeneous map $\pi : G/K \to G/H$ with $K \subset H$ closed Lie subgroups in G (see [1], for (10) see also Krahe [12, p. 518, Example 2.2.1], for (7) see also Harvey [9, p. 312]):

(1) $\pi_{\mathbb{C}} : H_{2s+1}^{2m+1} = SU(m-s, s+1)/SU(m-s, s) \to \mathbb{C}H_s^m = SU(m-s, s+1)/S(U(1)U(m-s, s))$,

(2) $\pi_{\mathbb{A}} : H_m^{2m+1} = SU^\pi(m+1)/SU^\pi(m) \to \mathbb{A}P^m = SU^\pi(m+1)/S(U^\pi(1)U^\pi(m))$,

(3) $\pi_{\mathbb{H}} : H_{4s+3}^{4m+3} = Sp(m-s, s+1)/Sp(m-s, s) \to \mathbb{H}H_s^m = Sp(m-s, s+1)/Sp(1)Sp(m-s, s)$,

(4) $\pi_{\mathbb{B}} : H_{2m+1}^{4m+3} = Sp^\pi(m+1)/Sp^\pi(m) \to \mathbb{B}P^m = Sp^\pi(m+1)/(Sp^\pi(1)Sp^\pi(m))$,

(5) $\pi_{\mathbb{O}}^1 : H_{15}^{15} = Spin(9)/Spin(7) \to H_8^8(-4) = Spin(9)/Spin(8)$,

(6) $\pi_{\mathbb{O}}^2 : H_7^{15} = Spin_0(8, 1)/Spin(7) \to H^8(-4) = Spin_0(8, 1)/Spin(8)$,

(7) $\pi_{\mathbb{O}'} : H_7^{15} = Spin_0(5, 4)/Spin_0(4, 3) \to H_4^8(-4) = Spin_0(5, 4)/Spin_0(4, 4)$,

(8) $\pi_{\mathbb{C}, \mathbb{H}} : \mathbb{C}H_{2s+1}^{2m+1} = Sp(m-s, s+1)/Sp(m-s, s)U(1) \to \mathbb{H}H_s^m = Sp(m-s, s+1)/Sp(m-s, s)Sp(1)$,

(9) $\pi_{\mathbb{C}, \mathbb{B}} : \mathbb{C}H_m^{2m+1} = Sp^\pi(m+1)/Sp^\pi(m)U(1) \to \mathbb{B}P^m = Sp^\pi(m+1)/Sp^\pi(m)Sp^\pi(1)$,

(10) $\pi_{\mathbb{A}, \mathbb{B}} : \mathbb{A}P^{2m+1} = Sp^\pi(m+1)/Sp^\pi(m)U^\pi(1) \to \mathbb{B}P^m = Sp^\pi(m+1)/Sp^\pi(m)Sp^\pi(1)$,

here the pseudo-Riemannian metrics on H_s^m and $H_s^m(-4)$ are the ones with constant curvature c, with $c = -1$ for H_s^m, and $c = -4$ for $H_s^m(-4)$; the pseudo-Riemannian metrics on $\mathbb{C}H_s^m$, $\mathbb{H}H_s^m$, $\mathbb{A}P^m$, $\mathbb{B}P^m$ are the ones with constant holomorphic, quaternionic, para-holomorphic or para-quaternionic curvature -4; and we call these metrics the canonical ones.

2.1. The Einstein metrics of the canonical variation.

Let $\pi : (M, g) \to (B, g')$ be a pseudo-Riemannian submersion. We denote by $\hat{g}$ the metrics induced on fibres. The family of metrics g_t, with $t \in \mathbb{R} \setminus \{0\}$ and g_t given by

$$g_t = \pi^* g' + t\hat{g},$$

is called the *canonical variation* of π. To find the values of t for which g_t is an Einstein metric, we use the following pseudo-Riemannian version of a theorem obtained in the Riemannian case by Matsuzawa [13], and independently by Berard-Bergery, see Besse [3, Lemma 9.74]. First, we introduce the notation: $\lambda' = s'/n$ and $\hat{\lambda} = \hat{s}/p$, where s' and $\hat{s}$ are the scalar curvatures of g' and $\hat{g}$, respectively, and $n = \dim M$ and $p = \dim fibre$.

Lemma 2.3. *Let $\pi : (M, g) \to (B, g')$ be a pseudo-Riemannian submersion with totally geodesic fibres. Assume that g, g' and $\hat{g}$ are Einstein and the O'Neill integrability tensor $A \not\equiv 0$. Then the following two conditions are equivalent:*

(i) $t_0 = \frac{\hat{\lambda}}{\lambda' - \hat{\lambda}}$ *is the unique nonzero value different from 1 such that g_t is also Einstein*

(ii) $\hat{\lambda} \neq \frac{1}{2}\lambda'$ *and* $\hat{\lambda} \neq 0$.

Remark 2.4. Note that $\hat{\lambda} = 0$ when the fibres are one-dimensional. Therefore, the canonical variations of $\pi_{\mathbb{C}}$ and $\pi_{\mathbb{A}}$ do not provide any non-canonical Einstein metrics on the real pseudo-hyperbolic space.

Remark 2.5. For the Hopf pseudo-Riemannian submersions (3-10), the value $t_0 \neq 1$ for which g_{t_0} is an Einstein metric is the following:

(a) For $\pi_{\mathbb{H}}$ and $\pi_{\mathbb{B}}$, we see that $\lambda' = -(4m + 8)$, $\hat{\lambda} = -2$, and hence $t_0 = \frac{1}{2m+3}$.

(b) For $\pi_{\mathbb{O}}^1$, $\pi_{\mathbb{O}}^2$ and $\pi_{\mathbb{O}'}$, we have $\lambda' = -28$, $\hat{\lambda} = -6$ which gives $t_0 = \frac{3}{11}$.

(c) For $\pi_{\mathbb{C}, \mathbb{H}}$, $\pi_{\mathbb{C}, \mathbb{B}}$, $\pi_{\mathbb{A}, \mathbb{B}}$ we have $\lambda' = -(4m + 8)$, $\hat{\lambda} = -4$ and thus $t_0 = \frac{1}{m+1}$.

Clearly, the Einstein metrics g_{t_0} (with $t_0 \neq 1$) of the canonical variations of the Hopf pseudo-Riemannian submersions (3-10) are neither isometric to each other, nor to the canonical metrics.

Definition 2.6. The pseudo-Riemannian manifold (M, g) is called a *G-homogeneous manifold* if G is a closed Lie subgroup of the isometry group $I(M, g)$.

Note that (M, g) is a G-homogeneous manifold if and only if G acts effectively and transitively on M and g is a G-invariant metric on M. To show that all G-homogeneous Einstein metrics are the ones

claimed in Theorem 1.1, we shall first classify the closed, connected groups $G \subset SO_0(n-r, r+1)$ acting effectively and transitively on those spaces.

3. The classification of the groups acting transitively on pseudo-hyperbolic spaces.

Throughout this section, we shall denote by H the isotropy group of an action of G on M.

Theorem 3.1. *A closed, connected subgroup G of $SO_0(n-r, r+1)$ acts effectively and transitively on H_s^n if and only if G is contained in Table 1.*

Table 1

No.	G	H	G/H
(1)	$SO_0(n-r, r+1)$	$SO_0(n-r, r)$	H_r^n
(2)	$Spin(9)$	$Spin(7)$	$H_{15}^{15} = S^{15}$
(3)	$Spin(7)$	G_2	$H_7^7 = S^7$
(4)	G_2	$SU(3)$	$H_6^6 = S^6$
(5)	$Spin_0(8,1)$	$Spin(7)$	H_7^{15}
(6)	$Spin_0(5,4)$	$Spin_0(4,3)$	H_7^{15}
(7)	$Spin_0(4,3)$	G_2^*	H_3^7
(8)	G_2^*	$SU(2,1)$	H_2^6
(9)	G_2^*	$SL(3,\mathbb{R}) = SU^\pi(3)$	H_3^6
(10)	$SU(m-s, s+1)$	$SU(m-s, s)$	H_{2s+1}^{2m+1}
(11)	$Sp(m-s, s+1)$	$Sp(m-s, s)$	H_{4s+3}^{4m+3}
(12)	$Sp(m-s, s+1)Sp(1)$	$Sp(m-s, s)Sp(1)$	H_{4s+3}^{4m+3}
(13)	$SL(m+1, \mathbb{R}) = SU^\pi(m+1)$	$SL(m, \mathbb{R}) = SU^\pi(m)$	H_m^{2m+1}
(14)	$Sp(m+1, \mathbb{R}) = Sp^\pi(m+1)$	$Sp(m, \mathbb{R}) = Sp^\pi(m)$	H_{2m+1}^{4m+3}
(15)	$Sp(m+1, \mathbb{R})Sp(1, \mathbb{R}) = Sp^\pi(m+1)Sp^\pi(1)$	$Sp(m, \mathbb{R})Sp(1, \mathbb{R}) = Sp^\pi(m)Sp^\pi(1)$	H_{2m+1}^{4m+3}
(16)	$SU(m-s, s+1)U(1) = U(m-s, s+1)$	$SU(m-s, s)U(1) = U(m-s, s)$	H_{2s+1}^{2m+1}
(17)	$Sp(m-s, s+1)U(1)$	$Sp(m-s, s)U(1)$	H_{4s+3}^{4m+3}
(18)	$GL_+(m+1) = SU^\pi(m+1)U_0^\pi(1) = U_0^\pi(m+1)$	$GL_+(m) = SU^\pi(m)U_0^\pi(1) = U_0^\pi(m)$	H_m^{2m+1}
(19)	$Sp(m+1, \mathbb{R})\mathbb{R}_+^* = Sp^\pi(m+1)U_0^\pi(1)$	$Sp(m, \mathbb{R})\mathbb{R}_+^* = Sp^\pi(m)U_0^\pi(1)$	H_{2m+1}^{4m+3}
(20)	$Sp^\pi(m+1)U(1)$	$Sp^\pi(m)U(1)$	H_{2m+1}^{4m+3}

Proof. If G is compact, then so is H and $G/H = H_r^n$. Hence $n = r$ and G/H is a sphere. By the Borel-Montgomery-Samelson classification of the compact groups acting effectively and transitively on spheres, we have that G is one of the following groups: $Spin(9)$, $Spin(7)$, G_2, $SO(m)$, $SU(m)$, $U(m)$, $Sp(m)$, $Sp(m)U(1)$, $Sp(m)Sp(1)$ (see [5, 14, 16]) and these correspond to the cases (2-4), (1, with $n=r$), (10-12, 16-17, with $m=s$) in Table 1.

When G is not compact (that is equivalent to $r < n$), we split the proof into two cases: (a) G semi-simple and (b) G non-semisimple.

3.1. **G semisimple.** In the case G semisimple, we shall obtain the cases (5–15) of Table 1 from a classification theorem due to Onishchik (see [17, Theorem 4.1]). We first recall some facts on transitive actions from Onishchik [17]. Let K' and K'' be two closed Lie subgroups of K, and let $\mathfrak{k}'$, $\mathfrak{k}'' \subset \mathfrak{k}$ be their associated Lie algebras. The subgroup K' acts transitively on K/K'' if and only if K can be written as a product $K = K'K''$. In the case of a semisimple triple $(\mathfrak{k}, \mathfrak{k}', \mathfrak{k}'')$, that is also equivalent to $\mathfrak{k} = \mathfrak{k}' + \mathfrak{k}''$. Additionally, one has $K/K'' = K'/(K' \cap K'')$. Specializing to our case, a closed, connected subgroup G of $SO_0(n-r, r+1)$ acts transitively on $H_r^n = SO_0(n-r, r+1)/SO_0(n-r, r)$ if and only if $so(n-r, r+1) = so(n-r, r) + \mathfrak{g}$. By Onishchik's classification of the semisimple decompositions of $so(n-r, r+1)$ with $r < n$ (see [17, Theorem 4.1 and Table 1]), we get the cases (5–15) in our table.

3.2. **G non-semisimple.** We proceed by splitting this case into two subcases: (b1) G acts irreducibly on $\mathbb{R}^{n+1}$ and (b2) G does not act irreducibly on $\mathbb{R}^{n+1}$.

3.2.1. *G acts irreducibly on $\mathbb{R}^{n+1}$.* In [2, p. 321, Theorem 6], Berger obtained that the subgroups G of $\mathrm{GL}(n+1, \mathbb{R})$ acting effectively and transitively on H_r^n, with G acting irreducible on $\mathbb{R}^{n+1}$, are, except a finite number, in the cases (1), (10-12), (16-17) of Table 1. To see that all excepted Lie groups are semisimple, we shall now recall from Wolf [21, Proof of Theorem 3.1] the construction of the compact form G^* associated to G and his proof of the fact that G^* acts transitively on a sphere.

If G acts irreducibly on $\mathbb{R}^{n+1}$, then so does its Lie algebra $\mathfrak{g} \subset \mathrm{gl}(n+1, \mathbb{R})$. Hence, by [10, Proposition 19.1], $\mathfrak{g}$ is reductive and $\dim Z(\mathfrak{g}) \leq 1$, which correlated to our working assumptions: (i) $G \subset SO_0(n-r, r+1)$, and (ii) G non-semisimple, it gives $G = (G, G)U(1)$ (see [21, Lemma 1.2.1]). There exists a Cartan involution T of $so(n-r, r+1)$ such that $\mathfrak{g}$ is T-invariant (see [15, Theorem 6]).

Let $H_x = \{g \in SO_0(n-r, r+1) \mid gx = x\} \equiv SO_0(n-r, r)$ be the isotropy group at $x \in H_s^n$ and let $\mathfrak{h}$ be its Lie algebra. Changing x, we may assume that $\mathfrak{h}$ is also T-invariant (see [15, Theorem 6], or [21]). The transitivity of G on $H_r^n = SO_0(n-r, r+1)/H_x$ simply implies that $so(n-r, r+1) = \mathfrak{g} + \mathfrak{h}$. Let

$$\mathfrak{s}_- = \{X \in so(n-r, r+1) \mid T(X) = -X\}, \quad \mathfrak{s}_+ = \{X \in so(n-r, r+1) \mid T(X) = X\},$$

$$\mathfrak{g}_\pm = \mathfrak{s}_\pm \cap \mathfrak{g}, \quad \mathfrak{h}_\pm = \mathfrak{s}_\pm \cap \mathfrak{h}.$$

The associated compact forms of $so(n-r, r+1)$, $\mathfrak{g}$ and $\mathfrak{h}$, defined by

$$so(n-r, r+1)^* = \mathfrak{s}_+ + i\mathfrak{s}_-, \quad \mathfrak{g}^* = \mathfrak{g}_+ + i\mathfrak{g}_-, \quad \mathfrak{h}^* = \mathfrak{h}_+ + i\mathfrak{h}_-,$$

naturally satisfy the relation

$$so(n+1) = so(n-r, r+1)^* = \mathfrak{g}^* + \mathfrak{h}^* = \mathfrak{g}^* + so(n).$$

Hence, the connected compact Lie group G^* (with $\mathrm{Lie}(G^*) = \mathfrak{g}^*$) acts transitively and effectively on the sphere $S^n = SO(n+1)/SO(n)$ (see Wolf [21, Proof of Theorem 3.1]), and thus, the non-semisimple Lie groups G^* belong the infinite families $U(m)$ or $Sp(m)U(1)$. It follows that G must be one of the groups in the cases (16-17).

3.2.2. *G does not act irreducibly on $\mathbb{R}_{r+1}^{n+1}$.* Let V be a proper G-invariant subspace of $\mathbb{R}_{r+1}^{n+1}$. By Wolf [20, Lemma 8.2], we have that $2(r+1) \leq n+1$ and $W_1 = V \cap V^\perp$ is a G-invariant maximal totally isotropic subspace of dimension $r+1$.

Let W_2 be a totally isotropic space such that $W_1 \oplus W_2 \oplus U = \mathbb{R}_{r+1}^{n+1}$, $\dim W_2 = \dim W_1 = r+1$, $U^\perp = (W_1 \oplus W_2)^\perp$ and U does not contain any isotropic vector. The decomposition $W_1 \oplus W_2 \oplus U = \mathbb{R}_{r+1}^{n+1}$ is called a Witt decomposition (see [19, p. 160, Exercise 9]). Let Q be the quadratic form on $\mathbb{R}_{r+1}^{n+1}$ given in the standard basis by $Q(x, y) = \langle x, y \rangle_{\mathbb{R}_{r+1}^{n+1}}$. Clearly, there exists an orthonormal basis $\{e_1, \cdots, e_{n+1}\}$ of $\mathbb{R}_{r+1}^{n+1}$ with $Q(e_i, e_i) = -1$ for $i \in \{1, \cdots, r+1\}$, $Q(e_j, e_j) = 1$ for $j \in \{r+2, \cdots, n+1\}$ and such that $\{w_1, \cdots, w_{r+1}\}$, $\{w_{r+1}, \cdots, w_{2r+2}\}$, $\{w_{2r+3}, \cdots, w_{n+1}\}$ are bases of W_1, W_2 and U respectively, with

$$w_i = \frac{1}{2}(e_i - e_{i+r+1}), \quad w_{i+r+1} = \frac{1}{2}(e_i + e_{i+r+1}), \quad w_k = e_k,$$

for any $i \in \{1, \cdots, r+1\}$ and $k \in \{2r+3, \cdots, n+1\}$. Any $g \in G \subset SO_0(n-r, r+1)$ is a linear transformation on $\mathbb{R}_{r+1}^{n+1}$, which can be written with respect to the basis $\{w_1, \cdots, w_{n+1}\}$ in the form

$$g = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & D \end{pmatrix},$$

with $A, B \in \mathrm{GL}(r+1, \mathbb{R})$ and $D \in \mathrm{GL}(n-2r-1, \mathbb{R})$. By our assumption of $g \in SO_0(n-r, r+1)$, we easily get that $B = (A^{-1})^t$ and $D \in O(n-2r-1)$. By the connectedness of G, we naturally have that

$A \in \mathrm{GL}_+(r+1, \mathbb{R})$. The effectivity of G simply implies $n = 2r + 1$. If follows that G is a Lie subgroup of $U_0^\pi(r+1)$. Let

$$G_1 = \left\{ A \in \mathrm{GL}_+(r+1, \mathbb{R}) \; \Big| \; \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} \in G \right\}.$$

With respect to the basis $\{w_i\}$, the quadratic Q writes as $Q(z_1, \cdots, z_{2r+2}) = z_1 z_{r+2} + \cdots + z_{r+1} z_{2r+2}$. Hence, for any $(z_1, \cdots, z_{r+1}) \neq 0$, we obviously have $Q(z_1, \cdots, z_{r+1}, -\frac{z_1}{R}, \cdots, -\frac{z_{r+1}}{R}) = -1$ with $R = z_1^2 + \cdots + z_{r+1}^2$. It follows that G_1 acts effectively and transitively on $W_1 \setminus \{0\} = \mathbb{R}^{r+1} \setminus \{0\}$. Clearly, the Lie group $G_1 \subset \mathrm{GL}(r+1, \mathbb{R})$ acts irreducibly on $\mathbb{R}^{r+1}$ and, hence, by [10, Proposition 19.1] its Lie algebra $\mathfrak{g}_1$ is reductive and $\dim Z(\mathfrak{g}_1) \leq 1$. Thus, G is reductive and $\dim Z(\mathfrak{g}) \leq 1$. The non-semisimple Lie group G decomposes as $G = (G, G)(Z(G))^0$, where (G, G) is its semisimple part, and $(Z(G))^0$ is the connected component containing the identity of the center of G.

Like in § 3.2.1, we see that the associated compact form G^* acts transitively and effectively on a sphere S^{2r+2} and hence $G^* \in \{SU(r+1)U(1), Sp((r+1)/2)U(1)\}$. It follows that (i) the abelian part $(Z(G))^0$ of the reductive non-semisimple Lie group G is either $U(1)$ or $U_0^\pi(1)$ and (ii) the only possibilities for the semisimple part (G, G), with $(G, G) \subset SU^\pi(r+1)$ are: $(G, G) = Sp^\pi((r+1)/2)$ or $(G, G) = SU^\pi(r+1)$.

If $(Z(G))^0 = U(1)$, then there exists a complex structure $I \in U(1) \subset G$ on $\mathbb{R}^{2r+2}$. Moreover, there exists a para-complex structure $J \in (G, G) \subset G$ on $\mathbb{R}^{2r+2}$ because $(G, G) \in \{SU^\pi(r+1), Sp^\pi((r+1)/2)\}$. Thus $\{I, J, IJ\} \subset G$ is a para-quaternionic structure on $\mathbb{R}^{2r+2}$, and, in consequence, the only possibility for G is $G = Sp^\pi((r+1)/2)U(1)$, that is, the case (20) in Table 1.

When $(Z(G))^0 = U_0^\pi(1)$, we get 2 possibilities for G: $G = Sp^\pi((r+1)/2)U_0^\pi(1)$ or $G = SU^\pi(r+1)U_0^\pi(1)$, which correspond to the cases (18-19) in Table 1. □

Unlike in the real pseudo-hyperbolic case, the groups $SU(m-s, s+1)$, $Sp(m-s, s+1)$, $Sp^\pi(m+1)$, $SU^\pi(m+1)$, $Sp^\pi(m+1)$, $Sp(m-s, s+1)$ and $Sp^\pi(m+1)$, act only almost effectively on $\mathbb{C}H_s^m$, $\mathbb{C}H_{2s+1}^{2m+1}$, $\mathbb{C}H_m^{2m+1}$, $\mathbb{A}P^m$, $\mathbb{A}P^{2m+1}$, $\mathbb{H}H_s^m$ and $\mathbb{B}P^m$, respectively. In order to make these actions effective, one has to consider the action of the quotient of each group by its center (see [3, §7.12 Note on effectivity]). Let $Z_{m+1} = \{\exp(2\pi i k/(m+1) \mid k = 0, \cdots, m\}$. Note that

$$Z(SU(m-s, s+1)) = Z_{m+1}, \quad Z(Sp(m-s, s+1)) = Z_2,$$

$$Z(Sp^\pi(m+1)) = Z(Sp(m, \mathbb{R})) = Z_2,$$

$$Z(SU^\pi(m+1)) = Z(SL(m+1, \mathbb{R})) = \{x \in \mathbb{R} \mid x^{m+1} = 1\}.$$

Theorem 3.2. *Let G be a connected Lie group. One of the following holds:*

(1) *G is a closed subgroup of $SO_0(2n - 2r, 2r + 2)$ acting on $\mathbb{C}H_r^n$,*
(2) *G is a closed subgroup of $SO_0(n + 1, n + 1)$ acting on $\mathbb{A}P^n$,*
(3) *G is a closed subgroup of $SO_0(4m - 4s, 4s + 4)$ acting on $\mathbb{H}H_s^m$,*
(4) *G is a closed subgroup of $SO_0(2m + 2, 2m + 2)$ acting on $\mathbb{B}P^m$,*

and the action is effective and transitive if and only if G is contained in Table 2.

Table 2

No.	G	H	G/H
(1)	$SU(m-s, s+1)/Z_{m+1}$	$S(U(1)U(m-s, s))/Z_{m+1}$	$\mathbb{C}H_s^m$
(2)	$Sp(m-s, s+1)/Z_2$	$U(1)Sp(m-s, s)/Z_2$	$\mathbb{C}H_{2s+1}^{2m+1}$
(3)	$Sp^\pi(m+1)/Z_2$	$Sp^\pi(m)U(1)/Z_2$	$\mathbb{C}H_m^{2m+1}$
(4)	$SU^\pi(m+1)/Z_2, \quad$ if m is odd $SU^\pi(m+1), \qquad$ if m is even	$S(U^\pi(m)U^\pi(1))/Z_2, \quad$ if m is odd $S(U^\pi(m)U^\pi(1)), \qquad$ if m is even	$\mathbb{A}P^m$
(5)	$Sp^\pi(m+1)/Z_2$	$Sp^\pi(m)U_0^\pi(1)/Z_2$	$\mathbb{A}P^{2m+1}$
(6)	$Sp(m-s, s+1)/Z_2$	$Sp(m-s, s)Sp(1)/Z_2$	$\mathbb{H}H_s^m$
(7)	$Sp^\pi(m+1)/Z_2$	$Sp^\pi(m)Sp^\pi(1)/Z_2$	$\mathbb{B}P^m$

Proof. Clearly, if G is one of the groups in Table 2, then G acts effectively and transitively on the corresponding space. We shall prove that if G is a subgroup satisfying (1-4) and acting transitively and effectively, then it is contained in Table 2.

The transitivity of G on $\mathbb{C}H_r^n = H_{2r+1}^{2n+1}/U(1)$ implies that $GU(1)$ acts transitively on H_{2r+1}^{2n+1}. There exists a complex structure I on $\mathbb{R}^{2n+2}$ such that $I \in Z(GU(1))$ (e.g. take $I = i\mathrm{Id}_{2n+2} \in GU(1) \subset SO_0(2n - 2r, 2r + 2)$), and therefore, $GU(1) \subset U(m - s, s + 1)$. By the transitivity of $GU(1)$ on $H_{2r+1}^{2n+1} = SU(n - r, r + 1)U(1)/SU(n - r, r)U(1)$, we get

$$su(n - r, r + 1) + u(1) = su(n - r, r) + u(1) + \mathfrak{g} + u(1) = su(n - r, r) + u(1) + \mathfrak{g}.$$

It follows that G acts transitively on $H_{2r+1}^{2n+1} = SU(n-r, r+1)U(1)/SU(n-r, r)U(1)$. On the other hand, the effectivity of G on $\mathbb{C}H_s^m$ clearly implies that G acts also effectively on H_{2r+1}^{2n+1} and $G \cap U(1) = \{e\}$. Hence, by Table 1 of Theorem 3.1, $G \in \{SU(m - s, s + 1)/Z_{m+1}, Sp(m - s, s + 1)/Z_2, Sp^\pi(m + 1)/Z_2\}$.

We now repeat the argument above for the other cases. The transitivity of G on $\mathbb{A}P^n = H_n^{2n+1}/U_0^\pi(1)$, implies the transitivity of $GU_0^\pi(1)$ on H_n^{2n+1}. The existence of a para-complex structure I on $\mathbb{R}^{2n+2}$ such that $I \in Z(GU_0^\pi(1))$, implies that $GU_0^\pi(1)$ is a subgroup of $SU^\pi(n + 1)U_0^\pi(1)$. It follows that

$$sl(n + 1) + \mathbb{R} = sl(n) + \mathbb{R} + \mathfrak{g} + \mathbb{R} = sl(n) + \mathbb{R} + \mathfrak{g}.$$

Therefore, G acts transitivity on $H_n^{2n+1} = GL_+(n + 1, \mathbb{R})/GL_+(n, \mathbb{R})$. Obviously, the effectivity of G on $\mathbb{A}P^n$ implies the effectivity on H_n^{2n+1} and $G \cap U_0^\pi(1) = \{e\}$. Hence, by Table 1, G falls in the cases (4-5) of Table 2.

Analogously, we get that if G acts effectively and transitively on $\mathbb{H}H_s^m = H_{4s+3}^{4m+3}/Sp(1)$ or $\mathbb{B}P^m = H_{2m+1}^{4m+3}/Sp^\pi(1)$, then $G = Sp(m - s, s + 1)/Z_2$ or $G = Sp^\pi(m + 1)/Z_2$, respectively. $\square$

4. THE PROOF OF THE MAIN THEOREM

Proof of Theorem 1.1. Let (G, H) be a pair of Lie groups contained in Tables 1 or 2. We denote by $\mathfrak{g}, \mathfrak{h}$ their associated Lie algebras and by $\mathrm{ad} : \mathfrak{g} \to \mathrm{gl}(\mathfrak{g})$ the adjoint representation of $\mathfrak{g}$. When $\mathfrak{h}$ is not semisimple, then the isotropy representation $\chi = \mathrm{ad} : \mathfrak{h} \to \mathrm{gl}(\mathfrak{g})$ is completely reducible simply because the center $Z(\mathfrak{h}) \in \{U(1), U_0^\pi(1)\}$ acts by semisimple endomorphisms. When $\mathfrak{h}$ is semisimple, $ad : \mathfrak{h} \to \mathrm{gl}(\mathfrak{g})$ is always completely reducible (see [10, Theorem 6.3]). It follows that there exits a subspace $\mathfrak{m}$ in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $[\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}$. Such a homogeneous space G/H is called reductive.

Let $(\cdot, \cdot)$ be an $ad(\mathfrak{h})$-invariant symmetric non-degenerate bilinear form on $\mathfrak{m}$, associated to a G-invariant pseudo-Riemannian metric g on G/H. Let $\mathfrak{m} = \mathfrak{m}_+ \oplus \mathfrak{m}_-$ be an orthogonal decomposition of $\mathfrak{m}$ such that $(\cdot, \cdot)$ is positive definite on $\mathfrak{m}_+$ and negative definite on $\mathfrak{m}_-$. There exists a Cartan involution T of $\mathfrak{g}$ such that $\mathfrak{m}_+ \subset \mathfrak{g}_+$ and $\mathfrak{m}_- \subset \mathfrak{g}_-$, where

$$\mathfrak{g}_+ = \{X \in \mathfrak{g} \,|\, T(X) = X\}, \quad \mathfrak{g}_- = \{X \in \mathfrak{g} \,|\, T(X) = -X\}.$$

As in §3.2.1, changing the point where the isotropy is computed, we may assume that the isotropy Lie algebra $\mathfrak{h}$ is T-invariant. We have $T(\mathfrak{h}) = \mathfrak{h}$ and thus, $T(\mathfrak{m}) = \mathfrak{m}$. Letting $\mathfrak{h}_\pm = \mathfrak{g}_\pm \cap \mathfrak{h}$, we note that $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$.

Now, we define the compact forms $\mathfrak{g}^* = \mathfrak{g}_+ + i\mathfrak{g}_-$, $\mathfrak{h}^* = \mathfrak{h}_+ + i\mathfrak{h}_-$; let $\mathfrak{m}^* = \mathfrak{m}_+ + i\mathfrak{m}_-$, and take G^* and H^* to be the connected analytic Lie groups, with $\mathrm{Lie}(G^*) = \mathfrak{g}^*$ and $\mathrm{Lie}(H^*) = \mathfrak{h}^*$. Clearly, G^*/H^* is a compact homogeneous space, and the associated bilinear form $(\cdot, \cdot)^*$ on $\mathfrak{m}^*$ is positive definite and its associated G^*-invariant metric g^* is Riemannian (see [11] for the definition of $(\cdot, \cdot)^*$). Moreover, $\mathfrak{m}_+^*$ and $\mathfrak{m}_-^*$ are orthogonal to each others with respect to $(\cdot, \cdot)^*$. It means that $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, (\cdot, \cdot))$ is a T-dual to $(\mathfrak{g}^*, \mathfrak{h}^*, \mathfrak{m}^*, (\cdot, \cdot)^*)$ (see Kath [11, Definition 3.1]).

The compact dual triples $(G^*, H^*, G^*/H^*)$ of all triples $(G, H, G/H)$ of Tables 1 and 2, with a non-compact G, are listed in the next table.

Table 3

No.	G	H	G/H	G^*	H^*	G^*/H^*
(1)	$SO_0(n-r,r+1)$	$SO_0(n-r,r)$	H_r^n	$SO(n+1)$	$SO(n)$	S^n
(2)	G_2^*	$SU(2,1)$	H_2^6	G_2	$SU(3)$	S^6
(3)	G_2^*	$SL(3,\mathbb{R})$	H_3^6	G_2	$SU(3)$	S^6
(4)	$Spin_0(4,3)$	G_2^*	H_3^7	$Spin(7)$	G_2	S^7
(5)	$SU(m-s,s+1)$	$SU(m-s,s)$	H_{2s+1}^{2m+1}	$SU(m+1)$	$SU(m)$	S^{2m+1}
(6)	$U(m-s,s+1)$	$U(m-s,s)$	H_{2s+1}^{2m+1}	$U(m+1)$	$U(m)$	S^{2m+1}
(7)	$SU^\pi(m+1)$	$SU^\pi(m)$	H_m^{2m+1}	$SU(m+1)$	$SU(m)$	S^{2m+1}
(8)	$U_0^\pi(m+1)$	$U_0^\pi(m)$	H_m^{2m+1}	$U(m+1)$	$U(m)$	S^{2m+1}
(9)	$Spin_0(8,1)$	$Spin(7)$	H_7^{15}	$Spin(9)$	$Spin(7)$	S^{15}
(10)	$Spin_0(5,4)$	$Spin_0(4,3)$	H_7^{15}	$Spin(9)$	$Spin(7)$	S^{15}
(11)	$Sp(m-s,s+1)$	$Sp(m-s,s)$	H_{4s+3}^{4m+3}	$Sp(m+1)$	$Sp(m)$	S^{4m+3}
(12)	$Sp(m-s,s+1)U(1)$	$Sp(m-s,s)U(1)$	H_{4s+3}^{4m+3}	$Sp(m+1)U(1)$	$Sp(m)U(1)$	S^{4m+3}
(13)	$Sp(m-s,s+1)Sp(1)$	$Sp(m-s,s)Sp(1)$	H_{4s+3}^{4m+3}	$Sp(m+1)Sp(1)$	$Sp(m)Sp(1)$	S^{4m+3}
(14)	$Sp^\pi(m+1)$	$Sp^\pi(m)$	H_{2m+1}^{4m+3}	$Sp(m+1)$	$Sp(m)$	S^{4m+3}
(15)	$Sp^\pi(m+1)U(1)$	$Sp^\pi(m)U(1)$	H_{2m+1}^{4m+3}	$Sp(m+1)U(1)$	$Sp(m)U(1)$	S^{4m+3}
(16)	$Sp^\pi(m+1)U_0^\pi(1)$	$Sp^\pi(m)U_0^\pi(1)$	H_{2m+1}^{4m+3}	$Sp(m+1)U(1)$	$Sp(m)U(1)$	S^{4m+3}
(17)	$Sp^\pi(m+1)Sp^\pi(1)$	$Sp^\pi(m)Sp^\pi(1)$	H_{2m+1}^{4m+3}	$Sp(m+1)Sp(1)$	$Sp(m)Sp(1)$	S^{4m+3}
(18)	$SU(m-s,s+1)$	$S(U(m-s,s)U(1))$	$\mathbb{C}H_s^m$	$SU(m+1)$	$S(U(m)U(1))$	$\mathbb{C}P^m$
(19)	$Sp(m-s,s+1)$	$Sp(m-s,s)U(1)$	$\mathbb{C}H_{2s+1}^{2m+1}$	$Sp(m+1)$	$Sp(m)U(1)$	$\mathbb{C}P^{2m+1}$
(20)	$Sp^\pi(m+1)$	$Sp^\pi(m)U(1)$	$\mathbb{C}H_m^{2m+1}$	$Sp(m+1)$	$Sp(m)U(1)$	$\mathbb{C}P^{2m+1}$
(21)	$SU^\pi(m+1)$	$S(U^\pi(m)U^\pi(1))$	$\mathbb{A}P^m$	$SU(m+1)$	$S(U(m)U(1))$	$\mathbb{C}P^m$
(22)	$Sp^\pi(m+1)$	$Sp^\pi(m)U_0^\pi(1)$	$\mathbb{A}P^{2m+1}$	$Sp(m+1)$	$Sp(m)U(1)$	$\mathbb{C}P^{2m+1}$
(23)	$Sp(m-s,s+1)$	$Sp(m-s,s)Sp(1)$	$\mathbb{H}H_s^m$	$Sp(m+1)$	$Sp(m)Sp(1)$	$\mathbb{H}P^m$
(24)	$Sp^\pi(m+1)$	$Sp^\pi(m)Sp^\pi(1)$	$\mathbb{B}P^m$	$Sp(m+1)$	$Sp(m)Sp(1)$	$\mathbb{H}P^m$

By Kath [11, Corollary 4.1], the G-homogeneous Einstein pseudo-Riemannian metrics on G/H are in one-to-one correspondence to the G^*-homogeneous Einstein Riemannian metrics on G^*/H^*. Thus, by Ziller's classification of homogeneous Einstein Riemannian metrics on sphere and projective spaces (see Ziller [22]), we get the following:

(i) for the cases (1-8) of Table 3, the only G-homogeneous Einstein pseudo-Riemannian is the constant curvature metric,

(ii) for each of cases (9-11, 14) of Table 3, we have only two G-homogeneous Einstein pseudo-Riemannian metrics: the constant curvature one and the Einstein metric of the canonical variation,

(iii) for each of (19, 20, 22), we have only two G-homogeneous Einstein pseudo-Riemannian on G/H,

(iv) the cases (12-13) are special cases of (11), and the cases (15-17) are special cases of (14),

(v) for the cases (18, 21, 23, 24), we have only one G-homogeneous Einstein pseudo-Riemannian on G/H.

□

Remark 4.1. We recall from Ziller [22] that the homogeneous Einstein Riemannian metrics on S^{4n+3} (associated to the canonical variation of the Hopf fibration $S^{4n+3} \to \mathbb{C}P^n$) are normal homogeneous, but the homogeneous Einstein Riemannian metrics on S^{15} (associated to the Hopf fibration $S^{15} \to S^8$) and on $\mathbb{C}P^{2n+1}$ are not even naturally reductive. Since the notions of normal homogeneity and natural reductivity are preserved under duality, it follows that 2 homogeneous Einstein metrics on H_7^{15}, namely the $Sp(2,2)$ and $Sp^\pi(4)$-invariant metrics, and the Einstein metrics on H_{4s+3}^{4m+3} are normal homogeneous, but the non-canonical homogeneous Einstein metrics on $\mathbb{C}H_{2s+1}^{2m+1}$, $\mathbb{A}P^{2m+1}$ and the other 2 non-canonical Einstein metrics on H_7^{15} (the $Spin_0(5,4)$ and $Spin_0(8,1)$-invariant metrics) are not naturally reductive.

References

1. Gabriel Bădiţoiu, *Classification of Pseudo-Riemannian submersions with totally geodesic fibres from pseudo-hyperbolic spaces*, Proc. London Math. Soc. **105** (2012), 1315–1338, DOI:10.1112/plms/pds027, arXiv:1001.4490. MR 3004106

2. Marcel Berger, *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France **83** (1955), 279–330, http://www.numdam.org/item?id=BSMF_1955__83__279_0 . MR 0079806 (18,149a)

3. Arthur L. Besse, *Einstein manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, 2008, Reprint of the 1987 edition. MR 2371700 (2008k:53084)

4. Novica Blažić, *Paraquaternionic projective space and pseudo-Riemannian geometry*, Publ. Inst. Math. (Beograd) (N.S.) **60(74)** (1996), 101–107, www.emis.de/journals/PIMB/074/11.html. MR 1428897 (97k:53034)

5. Armand Borel, *Some remarks about Lie groups transitive on spheres and tori*, Bull. Amer. Math. Soc. **55** (1949), 580–587, DOI:10.1090/S0002-9904-1949-09251-0. MR 0029915 (10,680c)

6. Vicente Cortés, Christoph Mayer, Thomas Mohaupt, and Frank Saueressig, *Special geometry of Euclidean supersymmetry. I. Vector multiplets*, J. High Energy Phys. (2004), no. 3, 028, 73 pp. (electronic), DOI:10.1088/1126-6708/2004/03/028. MR 2061551 (2005c:53055)

7. Sadettin Erdem, *Paracomplex projective models and harmonic maps into them*, Beiträge Algebra Geom. **40** (1999), no. 2, 385–398, www.emis.de/journals/BAG/vol.40/no.2/10.html. MR 1720113 (2000h:53038)

8. P. M. Gadea and A. Montesinos Amilibia, *Spaces of constant para-holomorphic sectional curvature*, Pacific J. Math. **136** (1989), no. 1, 85–101, projecteuclid.org/euclid.pjm/1102650845. MR 971936 (90d:53043)

9. F. Reese Harvey, *Spinors and calibrations*, Perspectives in Mathematics, vol. 9, Academic Press Inc., Boston, MA, 1990. MR 1045637 (91e:53056)

10. James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York, 1978, Second printing, revised. MR 499562 (81b:17007)

11. I. Kath, *Pseudo-Riemannian T-duals of compact Riemannian homogeneous spaces*, Transform. Groups **5** (2000), no. 2, 157–179, DOI:10.1007/BF01236467. MR 1762116 (2001f:53103)

12. Matthias Krahe, *Para-pluriharmonic maps and twistor spaces*, Handbook of pseudo-Riemannian geometry and supersymmetry, IRMA Lect. Math. Theor. Phys., vol. 16, Eur. Math. Soc., Zürich, 2010, DOI:10.4171/079-1/15, pp. 497–557. MR 2681600 (2011f:53145)

13. Tadayuki Matsuzawa, *Einstein metrics and fibred Riemannian structures*, Kodai Math. J. **6** (1983), no. 3, 340–345, DOI:10.2996/kmj/1138036800. MR 717324 (85c:53075)

14. Deane Montgomery and Hans Samelson, *Transformation groups of spheres*, Ann. of Math. (2) **44** (1943), 454–470, www.jstor.org/stable/1968975. MR 0008817 (5,60b)

15. G. D. Mostow, *Some new decomposition theorems for semi-simple groups*, Mem. Amer. Math. Soc. **1955** (1955), no. 14, 31–54. MR 0069829 (16,1087g)

16. A. L. Oniščik, *Transitive compact transformation groups*, Eleven papers on topology and algebra, American Mathematical Society Translations. Series 2, vol. 55, American Mathematical Society, Providence, R.I., 1966, pp. 153–194. MR 0155935 (27 #5868)

17. A. L. Oniščik, *Decompositions of reductive Lie groups*, Math. USSR Sb. **9** (1969), 515–554, DOI:10.1070/SM1969v009n04ABEH001292. MR 0277660 (43 #3393)

18. Arkadi L. Onishchik, *Topology of transitive transformation groups*, Johann Ambrosius Barth Verlag GmbH, Leipzig, 1994, An official russian edition is available for online reading at www.rfbr.ru/rffi/ru/books/o_61720. MR 1266842 (95e:57058)

19. Arkady L. Onishchik and Rolf Sulanke, *Projective and Cayley-Klein geometries*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006. MR 2253811 (2007f:51002)

20. Joseph A. Wolf, *Homogeneous manifolds of constant curvature*, Comment. Math. Helv. **36** (1961), 112–147, DOI:10.1007/BF02566896. MR 0133085 (24 #A2921)

21. Joseph A. Wolf, *Isotropic manifolds of indefinite metric*, Comment. Math. Helv. **39** (1964), 21–64, DOI:10.1007/BF02566943. MR 0179744 (31 #3987)

22. W. Ziller, *Homogeneous Einstein metrics on spheres and projective spaces*, Math. Ann. **259** (1982), no. 3, 351–358, DOI:10.1007/BF01456947. MR 661203 (84h:53062)

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