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Semi-Riemannian submersions from real and complex pseudo-hyperbolic spaces

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Abstract

We classify the semi-Riemannian submersions from a pseudo-hyperbolic space onto a Riemannian manifold under the assumption that the fibres are connected and totally geodesic. Also we obtain the classification of the semi-Riemannian submersions from a complex pseudo-hyperbolic space onto a Riemannian manifold under the assumption that the fibres are complex, connected and totally geodesic submanifolds. © 2002 Published by Elsevier Science B.V.

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Introduction

The theory of Riemannian submersions was initiated by O'Neill [13] and Gray [8]. Presently, there is an extensive literature on the Riemannian submersions with different conditions imposed on the total space and on the fibres. A systematic exposition could be found in Besse's book [2]. Semi-Riemannian submersions were introduced by O'Neill in his book [14].

The class of harmonic Riemannian submersions, and in particular of those with totally geodesic fibres, is contained in the class of horizontally homothetic harmonic morphisms. For important results concerning the geometry of harmonic morphisms we refer to [1]. Wood constructs examples of harmonic morphisms from Riemannian submersions with totally geodesic fibres by horizontally

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conformal deformation of the metric. Recently, Fuglede studied harmonic morphisms between semi-Riemannian manifolds (see [7]). In this paper we solve the classification problem of the semi-Riemannian submersions with totally geodesic fibres from real and complex pseudo-hyperbolic spaces.

Escobales [5,6] and Ranjan [15] classified Riemannian submersions with totally geodesic fibres from a sphere S^n and from a complex projective space $\mathbb{C}P^n$. Magid [12] classified the semi-Riemannian submersions with totally geodesic fibres from an anti-de Sitter space onto a Riemannian manifold. In Section 2 we classify the semi-Riemannian submersions with totally geodesic fibres from a pseudo-hyperbolic space onto a Riemannian manifold. Also we obtain the classification of the semi-Riemannian submersions with connected, complex, totally geodesic fibres from a complex pseudo-hyperbolic space onto a Riemannian manifold.

1. Preliminaries and examples

Definition 1. Let (M,g) be an m-dimensional connected semi-Riemannian manifold of index s $(0 \leqslant s \leqslant m)$, let (B,g') be an n-dimensional connected semi-Riemannian manifold of index $s' \leqslant s$ $(0 \leqslant s' \leqslant n)$. A *semi-Riemannian submersion* (see [14]) is a smooth map $\pi : M \to B$ which is onto and satisfies the following three axioms:

(a) $\pi_*|_p$ is onto for all $p \in M$;
(b) the fibres $\pi^{-1}(b)$, $b \in B$ are semi-Riemannian submanifolds of M;
(c) π_* preserves scalar products of vectors normal to fibres.

We shall always assume that the dimension of the fibres $\dim M - \dim B$ is positive and the fibres are connected.

The tangent vectors to fibres are called vertical and those normal to fibres are called horizontal. We denote by $\mathcal{V}$ the vertical distribution and by $\mathcal{H}$ the horizontal distribution.

O'Neill [13] has characterized the geometry of a Riemannian submersion in terms of the tensor fields T, A defined by

$$A_E F = h\nabla_{hE} vF + v\nabla_{hE} hF, \qquad T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF,$$

for every E, F tangent vector fields to M. Here ∇ is the Levi-Civita connection of g, the symbols v and h are the orthogonal projections on $\mathcal{V}$ and $\mathcal{H}$, respectively. The letters U, V will always denote vertical vector fields, X, Y, Z horizontal vector fields. Notice that $T_U V$ is the second fundamental form of each fibre and $A_X Y$ is a natural *obstruction* to integrability of horizontal distribution $\mathcal{H}$. The tensor A is called O'Neill's integrability tensor. For basic properties of Riemannian submersions and examples see [2,8,13]. A vector field X on M is said to be *basic* if X is horizontal and π-related to a vector field X' on B. Notice that every vector field X' on B has a unique horizontal lifting X to M and X is basic. The following lemma is well known (see [13]).

Lemma 1.1. *We suppose X and Y are basic vector fields on M which are π-related to X' and Y'. Then*

(a) *$h[X,Y]$ is basic and π-related to $[X',Y']$;*
(b) *$h\nabla_X Y$ is basic and π-related to $\nabla'_{X'} Y'$, where ∇' is the Levi-Civita connection on B;*

The O'Neill's integrability tensor A has the following properties (see [13] or [2]).

Lemma 1.2. *Let X, Y be horizontal vector fields and E, F be vector fields on M. Then each of the following holds*:

(a) $A_X Y = -A_Y X$;
(b) $A_{hE} F = A_E F$;
(c) A_E *maps the horizontal subspace into the vertical one and the vertical subspace into the horizontal one*;
(d) $g(A_X E, F) = -g(E, A_X F)$;
(e) *If moreover X is basic then $A_X V = h\nabla_V X$ for every vertical vector field V*;
(f) $g((\nabla_Y A)_X E, F) = g(E, (\nabla_Y A)_X F)$.

Let $\hat{g}$ be the induced metric on fibre $\pi^{-1}(\pi(p))$, $p \in M$. We denote by R, R', $\widehat{R}$ the Riemann tensors of the metrics g, g', $\hat{g}$ respectively.

The following equations, usually called *O'Neill's equations*, characterize the geometry of a semi-Riemannian submersion (see [2,8,13]).

Proposition 1.3. *For every vertical vector fields U, V, W, W' and for every horizontal vector fields X, Y, Z, Z', we have the following formulae*:

(i) $R(U, V, W, W') = \widehat{R}(U, V, W, W') - g(T_U W, T_V W') + g(T_V W, T_U W')$,
(ii) $R(U, V, W, X) = g((\nabla_V T)_U W, X) - g((\nabla_U T)_V W, X)$,
(iii) $R(X, U, Y, V) = g((\nabla_X T)_U V, Y) - g(T_U X, T_V Y) + g((\nabla_U A)_X Y, V) + g(A_X U, A_Y V)$,
(iv) $R(U, V, X, Y) = g((\nabla_U A)_X Y, V) - g((\nabla_V A)_X Y, U) + g(A_X U, A_Y V) - g(A_X V, A_Y U) - g(T_U X, T_V Y) + g(T_V X, T_U Y)$,
(v) $R(X, Y, Z, U) = g((\nabla_Z A)_X Y, U) + g(A_X Y, T_U Z) - g(A_Y Z, T_U X) - g(A_Z X, T_U Y)$,
(vi) $R(X, Y, Z, Z') = R'(\pi_* X, \pi_* Y, \pi_* Z, \pi_* Z') - 2g(A_X Y, A_Z Z') + g(A_Y Z, A_X Z') - g(A_X Z, A_Y Z')$.

Using O'Neill's equations, we get the following lemma.

Lemma 1.4. *If $\pi : (M, g) \to (B, g')$ is a semi-Riemannian submersion with totally geodesic fibres then*:

(a) $R(U, V, U, V) = \widehat{R}(U, V, U, V)$;
(b) $R(X, U, X, U) = g(A_X U, A_X U)$;
(c) $R(X, Y, X, Y) = R'(\pi_* X, \pi_* Y, \pi_* X, \pi_* Y) - 3g(A_X Y, A_X Y)$.

We recall the definitions of real and complex pseudo-hyperbolic spaces (see [14] and [3]).

Definition 2. Let $\langle\cdot,\cdot\rangle$ be the symmetric bilinear form on $\mathbb{R}^{m+1}$ given by

$$\langle x, y\rangle = -\sum_{i=0}^{s} x_i y_i + \sum_{i=s+1}^{m} x_i y_i$$

for $x = (x_0, \ldots, x_m)$, $y = (y_0, \ldots, y_m) \in \mathbb{R}^{m+1}$. For $s > 0$ let $H_s^m = \{x \in \mathbb{R}^{m+1} \mid \langle x, x\rangle = -1\}$ be the semi-Riemannian submanifold of

$$\mathbb{R}^{m+1}_{s+1} = \left(\mathbb{R}^{m+1}, ds^2 = -dx^0 \otimes dx^0 - \cdots - dx^s \otimes dx^s + dx^{s+1} \otimes dx^{s+1} + \cdots + dx^m \otimes dx^m\right).$$

H^m_s is called the m-dimensional (*real*) *pseudo-hyperbolic space* of index s. We notice that H^m_s has constant sectional curvature -1 and the curvature tensor is given by

$$R(X,Y,X,Y) = -g(X,X)g(Y,Y) + g(X,Y)^2.$$

H^m_s can be written as homogeneous space, namely we have

$$H^m_s = SO(s+1, m-s)/SO(s, m-s),$$

$$H^{2m+1}_{2s+1} = SU(s+1, m-s)/SU(s, m-s),$$

$$H^{4m+3}_{4s+3} = Sp(s+1, m-s)/Sp(s, m-s)$$

(see [16]).

Definition 3. Let $(\cdot,\cdot)$ be the hermitian scalar product on $\mathbb{C}^{m+1}$ given by

$$(z,w) = -\sum_{i=0}^{s} z_i \bar{w}_i + \sum_{i=s+1}^{m} z_i \bar{w}_i$$

for $z = (z_0, \ldots, z_m)$, $w = (w_0, \ldots, w_m) \in \mathbb{C}^{m+1}$. Let M be the real hypersurface of $\mathbb{C}^{m+1}$ given by $M = \{z \in \mathbb{C}^{m+1} \mid (z,z) = -1\}$ and endowed with the induced metric of

$$\left(\mathbb{C}^{m+1}, ds^2 = -dz^0 \otimes d\bar{z}^0 - \cdots - dz^s \otimes d\bar{z}^s + dz^{s+1} \otimes d\bar{z}^{s+1} + \cdots + dz^m \otimes d\bar{z}^m\right).$$

The natural action of $S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ on $\mathbb{C}^{m+1}$ induces an action on M. Let $\mathbb{C}H^m_s = M/S^1$ endowed with the unique indefinite Kähler metric of index $2s$ such that the projection $M \to M/S^1$ becomes a semi-Riemannian submersion (see [3]). $\mathbb{C}H^m_s$ is called the *complex pseudo-hyperbolic space*. Notice that $\mathbb{C}H^m_s$ has constant holomorphic sectional curvature -4 and the curvature tensor is given by

$$R(X,Y,X,Y) = -g(X,X)g(Y,Y) + g(X,Y)^2 - 3g(I_0X,Y)^2,$$

where I_0 is the natural complex structure on $\mathbb{C}H^m_s$. $\mathbb{C}H^m_s$ is a homogeneous space, namely we have (see [16]) $\mathbb{C}H^m_s = SU(s+1, m-s)/S(U(1)U(s, m-s))$ and

$$\mathbb{C}H^{2m+1}_{2s+1} = Sp(s+1, m-s)/U(1)Sp(s, m-s).$$

We denote by $H^n(-4)$ the hyperbolic space with sectional curvature -4, by $\mathbb{H}H^n$ the quaternionic hyperbolic space of real dimension $4n$ with quaternionic sectional curvature -4.

Many explicit examples of semi-Riemannian submersions with totally geodesic fibres can be given following a standard construction (see [2] for Riemannian case). Let G be a Lie group and K, H two compact Lie subgroups of G with $K \subset H$. Let $\pi : G/K \to G/H$ be the associated bundle with fibre H/K to the H-principal bundle $p : G \to G/H$. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k} \subset \mathfrak{h}$ the corresponding Lie subalgebras of K and H. We choose an $Ad(H)$-invariant complement $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$, and an $Ad(K)$-invariant complement $\mathfrak{p}$ to $\mathfrak{k}$ in $\mathfrak{h}$. An $ad(H)$-invariant nondegenerate bilinear symmetric form on $\mathfrak{m}$ defines a G-invariant semi-Riemannian metric g' on G/H and an $ad(K)$-invariant nondegenerate bilinear symmetric form on $\mathfrak{p}$ defines a H-invariant semi-Riemannian metric $\hat{g}$ on H/K. The orthogonal direct sum for these nondegenerate bilinear symmetric forms on $\mathfrak{p} \oplus \mathfrak{m}$ defines a G-invariant semi-Riemannian metric g on G/K. The following theorem is proved in [2].

Theorem 1.5. *The map $\pi : (G/K, g) \to (G/H, g')$ is a semi-Riemannian submersion with totally geodesic fibres.*

Using this theorem we get the following examples.

Example 1. Let $G = SU(1,n)$, $H = S(U(1)U(n))$, $K = SU(n)$. We have the semi-Riemannian submersion

$$H_1^{2n+1} = SU(1,n)/SU(n) \to \mathbb{C}H^n = SU(1,n)/S\big(U(1)U(n)\big).$$

Example 2. Let $G = Sp(1,n)$, $H = Sp(1)Sp(n)$, $K = Sp(n)$. We get the semi-Riemannian submersion

$$H_3^{4n+3} = Sp(1,n)/Sp(n) \to \mathbb{H}H^n = Sp(1,n)/Sp(1)Sp(n).$$

Example 3. Let $G = Spin(1,8)$, $H = Spin(8)$, $K = Spin(7)$. We have the semi-Riemannian submersion

$$H_7^{15} = Spin(1,8)/Spin(7) \to H^8(-4) = Spin(1,8)/Spin(8).$$

Example 4. Let $G = Sp(1,n)$, $H = Sp(1)Sp(n)$, $K = U(1)Sp(n)$. We obtain the semi-Riemannian submersion

$$\mathbb{C}H_1^{2n+1} = Sp(1,n)/U(1)Sp(n) \to \mathbb{H}H^n = Sp(1,n)/Sp(1)Sp(n).$$

Definition 4. Two semi-Riemannian submersions $\pi, \pi' : (M, g) \to (B, g')$ are called *equivalent* if there is an isometry f of M which induces an isometry $\tilde{f}$ of B so that $\pi' \circ f = \tilde{f} \circ \pi$. In this case the pair $(f, \tilde{f})$ is called a *bundle isometry*.

We shall need the following theorem, which is the semi-Riemannian version of Theorem 2.2 in [5].

Theorem 1.6. *Let $\pi_1, \pi_2 : M \to B$ be semi-Riemannian submersions from a connected complete semi-Riemannian manifold onto a semi-Riemannian manifold. Assume the fibres of these submersions are connected and totally geodesic. Suppose f is an isometry of M which satisfies the following two properties at a given point $p \in M$:*

(1) *$f_{*p} : T_pM \to T_{f(p)}M$ maps $\mathcal{H}_{1p}$ onto $\mathcal{H}_{2f(p)}$, where $\mathcal{H}_i$ denotes the horizontal distribution of π_i, $i \in \{1, 2\}$;*
(2) *for every $E, F \in T_pM$, $f_* A_{1E} F = A_{2 f_* E} f_* F$, where A_i are the integrability tensors associated with π_i.*

Then f induces an isometry $\tilde{f}$ of B so that the pair $(f, \tilde{f})$ is a bundle isometry between π_1 and π_2. In particular, π_1 and π_2 are equivalent.

2. Semi-Riemannian submersions with totally geodesic fibres

Proposition 2.1. *If $\pi : H_s^m \to B^n$ is a semi-Riemannian submersion with totally geodesic fibres from an m-dimensional pseudo-hyperbolic space of index s onto an n-dimensional Riemannian manifold then $m = n + s$, the induced metrics on fibres are negative definite and B has negative sectional curvature.*

Proof. By Lemma 1.4(b), we get $g(A_X V, A_X V) = -g(X, X)g(V, V) \geqslant 0$ for every horizontal vector X and for every vertical vector V. Therefore $g(V, V) \leqslant 0$ for every vertical vector V. By Lemma 1.4(c), we have

$$R'(\pi_*X, \pi_*Y, \pi_*X, \pi_*Y) = -g'(\pi_*X, \pi_*X)g'(\pi_*Y, \pi_*Y) + g'(\pi_*X, \pi_*Y)^2 + 3g(A_X Y, A_X Y) < 0$$

for every linearly independent horizontal vectors X and Y. □

Proposition 2.2. *Let $\pi : (M_s^{n+s}, g) \to (B^n, g')$ be a semi-Riemannian submersion from an $(n + s)$-dimensional semi-Riemannian manifold of index $s \geqslant 1$ onto an n-dimensional Riemannian manifold. We suppose M is geodesically complete and simply connected. Then B is complete and simply connected. If moreover B has nonpositive curvature then the fibres are simply connected.*

Proof. Since M is geodesically complete, the base space B is complete.

Let $\tilde{g}$ be the Riemannian metric on M defined by

$$\tilde{g}(E, F) = g(hE, hF) - g(vE, vF)$$

for every E, F vector fields on M. Since $\tilde{g}$ is a horizontally complete Riemannian metric (this means that any maximal horizontal geodesic is defined on the entire real line) and B is a complete Riemannian manifold then $\mathcal{H}$ is an Ehresmann connection for π (see Theorem 1 in [17]). By Theorem 9.40 in [2], it follows $\pi : M \to B$ is a locally trivial fibration and we have an exact homotopy sequence

$$\cdots \to \pi_2(B) \to \pi_1(\textit{fibre}) \to \pi_1(M) \to \pi_1(B) \to 0.$$

Since M is simply connected, we have $\pi_1(B) = 0$.

If B has nonpositive curvature, then $\pi_2(B) = 0$ by theorem of Hadamard. It follows $\pi_1(\textit{fibre}) = 0$. □

Theorem 2.3. *If $\pi : H_s^m \to B^n$ is a semi-Riemannian submersion with totally geodesic fibres from a pseudo-hyperbolic space of index $s > 1$ onto a Riemannian manifold then B is a Riemannian symmetric space of rank one, noncompact and simply connected, any fibre is diffeomorphic to S^s and $s \in \{3, 7\}$.*

Proof. In order to prove that B is a locally symmetric space we need to check that $\nabla' R' \equiv 0$.

Let X_0', X', Y', Z' be vector fields on B and let X_0, X, Y, Z be the horizontal liftings of X_0', X', Y', Z' respectively. By definition of the covariant derivative we have

$$\begin{aligned}(\nabla'_{X_0'} R')(X', Y', Z') \\ = \nabla'_{X_0'} R'(X', Y')Z' - R'(\nabla'_{X_0'} X', Y')Z' - R'(X', \nabla'_{X_0'} Y')Z' - R'(X', Y')\nabla'_{X_0'} Z'. \qquad (2.1)\end{aligned}$$

In order to prove that the curvature tensor R' of the base space is parallel, we have to lift all vector fields in relation (2.1). By Lemma 1.1, the horizontal liftings of $\nabla'_{X_0'} X'$, $\nabla'_{X_0'} Y'$ and $\nabla'_{X_0'} Z'$ are $h\nabla_{X_0} X$, $h\nabla_{X_0} Y$ and $h\nabla_{X_0} Z$, respectively.

We denote by $R^h(X,Y)Z$ the horizontal lifting of $R'(X',Y')Z'$. The convention for Riemann tensor used here is $R(X,Y)=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}$. O'Neill's equation (vi) gives us the following relation

$$R^h(X,Y)Z=h\big(R(X,Y)Z\big)+2A_ZA_XY-A_XA_YZ-A_YA_ZX.$$

Using this relation we compute

$$\begin{aligned}
&(\nabla'_{X_0}R')(X',Y',Z')\\
&\quad=\pi_*\big[h\nabla_{X_0}\big(R^h(X,Y)Z\big)-R^h(h\nabla_{X_0}X,Y)Z-R^h(X,h\nabla_{X_0}Y)Z-R^h(X,Y)h\nabla_{X_0}Z\big]\\
&\quad=\pi_*\big[h\nabla_{X_0}h\big(R(X,Y)Z\big)-hR(h\nabla_{X_0}X,Y)Z-hR(X,h\nabla_{X_0}Y)Z-hR(X,Y)h\nabla_{X_0}Z\\
&\qquad+2(h\nabla_{X_0}A_ZA_XY-A_{h\nabla_{X_0}Z}A_XY-A_ZA_{h\nabla_{X_0}X}Y-A_ZA_Xh\nabla_{X_0}Y)\\
&\qquad-(h\nabla_{X_0}A_XA_YZ-A_{h\nabla_{X_0}X}A_YZ-A_XA_{h\nabla_{X_0}Y}Z-A_XA_Yh\nabla_{X_0}Z)\\
&\qquad-(h\nabla_{X_0}A_YA_ZX-A_{h\nabla_{X_0}Y}A_ZX-A_YA_{h\nabla_{X_0}Z}X-A_YA_Zh\nabla_{X_0}X)\big].
\end{aligned}\tag{2.2}$$

Since H_s^m has constant curvature, we have $R(X,Y,Z,U)=0$ for every vertical vector U and for every horizontal vector fields X,Y,Z. This implies $R(X,Y)Z$ is horizontal and $R(X,U)Y$, $R(U,X)Y$, $R(X,Y)U$ are vertical. Hence

$$\begin{aligned}
&\pi_*\big(\nabla_{X_0}h\big(R(X,Y)Z\big)-hR(h\nabla_{X_0}X,Y)Z-hR(X,h\nabla_{X_0}Y)Z-hR(X,Y)h\nabla_{X_0}Z\big)\\
&\quad=\pi_*\big(\nabla_{X_0}R(X,Y)Z\big)-\pi_*\big(R(\nabla_{X_0}X,Y)Z-R(v\nabla_{X_0}X,Y)Z\big)\\
&\qquad-\pi_*\big(R(X,Y)\nabla_{X_0}Z-R(X,Y)v\nabla_{X_0}Z\big)\\
&\quad=\pi_*\big[(\nabla_{X_0}R)(X,Y,Z)\big].
\end{aligned}$$

Since H_s^m has constant curvature, we get $(\nabla_{X_0}R)(X,Y,Z)=0$. So the sum of the first four terms in relation (2.2) is zero.

We have

$$\begin{aligned}
&h\nabla_{X_0}A_ZA_XY-A_{h\nabla_{X_0}Z}A_XY-A_ZA_{h\nabla_{X_0}X}Y-A_ZA_Xh\nabla_{X_0}Y\\
&\quad=h\big((\nabla_{X_0}A)_Z(A_XY)\big)-A_Z\big(v(\nabla_{X_0}A)_XY\big).
\end{aligned}$$

For the case of totally geodesic fibres, O'Neill's equation (v) becomes

$$R(X,Y,Z,U)=g\big((\nabla_ZA)_XY,U\big).$$

By Lemma 1.2(f) and by the hypothesis of constant curvature total space we get

$$g\big((\nabla_ZA)_XU,Y\big)=g\big((\nabla_ZA)_XY,U\big)=0$$

for every horizontal vector fields X,Y,Z and for every vertical vector field U. It follows $h(\nabla_ZA)_XU=0$ and $v(\nabla_ZA)_XY=0$ for every horizontal vector fields X,Y,Z and for every vertical vector field U. Therefore $h((\nabla_{X_0}A)_Z(A_XY))=0$ and $v((\nabla_{X_0}A)_XY)=0$. This implies

$$h\nabla_{X_0}A_ZA_XY-A_{h\nabla_{X_0}Z}A_XY-A_ZA_{h\nabla_{X_0}X}Y-A_ZA_Xh\nabla_{X_0}Y=0.$$

By circular permutations of (X,Y,Z) in the last relation we get

$$h\nabla_{X_0}A_XA_YZ-A_{h\nabla_{X_0}X}A_YZ-A_XA_{h\nabla_{X_0}Y}Z-A_XA_Yh\nabla_{X_0}Z=0,$$

$$h\nabla_{X_0}A_YA_ZX-A_{h\nabla_{X_0}Y}A_ZX-A_YA_{h\nabla_{X_0}Z}X-A_YA_Zh\nabla_{X_0}X=0.$$

So the sum of all terms in relation (2.2) is zero.

We proved that $(\nabla'_{X'_0} R')(X', Y', Z') = 0$ for every vector fields X'_0, X', Y', Z', so B is a locally symmetric space. By Proposition 2.2, B is simply connected and complete. Therefore B is a Riemannian symmetric space. By Proposition 2.1, B has negative sectional curvature. Hence B is a noncompact Riemannian symmetric space of rank one.

Let $b \in B$. Since $\pi^{-1}(b)$ is a totally geodesic submanifold of a geodesically complete manifold, $\pi^{-1}(b)$ is itself geodesically complete. Since $R'(X', Y', X', Y') \leqslant 0$ for every X', Y' tangent vectors to B, we have $\pi_1(\mathit{fibre}) = 0$, by Proposition 2.2. Since $(\pi^{-1}(b), \hat{g})$ is a complete, simply connected semi-Riemannian manifold of dimension r and of index r and with constant sectional curvature -1, it follows $(\pi^{-1}(b), \hat{g})$ is isometric to H^s_s (see Proposition 23 from p. 227 in [14]). Hence any fibre is diffeomorphic to S^s.

We shall prove below that the tangent bundle of any fibre is trivial. From a well known result of Adams it follows that $s \in \{1, 3, 7\}$.

Lemma 2.4. *The tangent bundle of any fibre is trivial.*

Proof. Since $g(A_X V, A_X V) = -g(X, X)g(V, V)$, we have that $A_X : \mathcal{V} \to \mathcal{H}$, $V \mapsto A_X V$ is an injective map and $\dim \mathcal{V} \leqslant \dim \mathcal{H}$, if $g(X, X) \neq 0$.

For any horizontal vector field X, we denote by $A^*_X : \mathcal{H} \to \mathcal{V}$ the map given by $A^*_X(Y) = A_X Y$. By O'Neill's equation (iv), we have $g(A_X V, A_X W) = -g(X, X)g(V, W)$ for every vertical vector fields V and W. Hence, by Lemma 1.2(d), we get $A^*_X A_X V = g(X, X)V$ for every vertical vector field V. If $g(X, X) \neq 0$ anywhere then A^*_X is surjective and hence $\dim \mathcal{V} = \dim \mathcal{H} - \dim \ker A^*_X$. By Lemma 1.2(d), we have $A_X X = 0$. This implies $\dim \ker A^*_X \geqslant 1$.

Let $b \in B$ and $x \in T_b B$ with $g(x, x) = 1$. We denote by X the horizontal lifting along the fibre $\pi^{-1}(b)$ of the vector x. Let p an arbitrary point in $\pi^{-1}(b)$ and let $\{X(p), y_1, \ldots, y_l\}$ be an orthonormal basis of the vector space $\ker A^*_{X(p)}$. Since π_{*p} sends isometrically $\mathcal{H}_p$ into $T_b B$ we have $\{\pi_* X(p), \pi_* y_1, \ldots, \pi_* y_l\}$ is a linearly independent system which can be completed to a basis of $T_b B$ with a system of vectors $\{x_{l+1}, \ldots, x_{n-1}\}$. Let $X, X_1, X_2, \ldots, X_{n-1}$ be the horizontal liftings along the fibre $\pi^{-1}(b)$ of $x = \pi_* X(p), \pi_* y_1, \ldots, \pi_* y_l, x_{l+1}, \ldots, x_{n-1}$, respectively.

By Lemma 1.4, we have for every $q \in \pi^{-1}(b)$ and for every $i \in \{1, \ldots, l\}$

$$
\begin{aligned}
&3g\big(A_{X(q)} X_i(q), A_{X(q)} X_i(q)\big) \\
&\quad = R'\big(\pi_* X(q), \pi_* X_i(q), \pi_* X(q), \pi_* X_i(q)\big) - R\big(X(q), X_i(q), X(q), X_i(q)\big) \\
&\quad = R'(x, \pi_* y_i, x, \pi_* y_i) + g\big(X(q), X(q)\big) g\big(X_i(q), X_i(q)\big) - g\big(X(q), X_i(q)\big)^2 \\
&\quad = R'(x, \pi_* y_i, x, \pi_* y_i) + g'\big(\pi_* X(q), \pi_* X(q)\big) g'\big(\pi_* X_i(q), \pi_* X_i(q)\big) - g'\big(\pi_* X(q), \pi_* X_i(q)\big)^2 \\
&\quad = 3g\big(A_{X(p)} X_i(p), A_{X(p)} X_i(p)\big) \\
&\quad = 0.
\end{aligned}
$$

Since the induced metrics on fibre $\pi^{-1}(b)$ are negative definite, we get $A_{X(q)} X_i(q) = 0$.

By Lemma 1.2(a), we have $A_{X(q)} X(q) = 0$. We proved that $\{X(q), X_1(q), \ldots, X_l(q)\} \subset \ker A^*_{X(q)}$. Since π_{*q} sends isometrically $\mathcal{H}_q$ into $T_b B$, we get $\{X(q), X_1(q), \ldots, X_l(q)\}$ is a basis of the vector space $\ker A^*_{X(q)}$ for every point $q \in \pi^{-1}(b)$.

Let $V_{l+1} = A_X^* X_{l+1}, \dots, V_{n-1} = A_X^* X_{n-1}$ be tangent vector fields to the fibre $\pi^{-1}(b)$. We denote by Q_q the vector subspace of $\mathcal{H}_q$ spanned by $\{X_{l+1}(q), X_{l+2}(q), \dots, X_{n-1}(q)\}$. Let $\tilde{g}$ be the Riemannian metric on $\pi^{-1}(b)$ given by $\tilde{g}(V, W) = -g(V, W)$ for every V, W vector fields tangent to $\pi^{-1}(b)$. Since $\dim \mathcal{V}_q = \dim Q_q$ and $g(A_X V, A_X V) = \tilde{g}(V, V)$, we get $A_{X(q)} : (\mathcal{V}_q, \tilde{g}) \to (Q_q, g)$ is an isometry for every $q \in \pi^{-1}(b)$.

So $\{V_{l+1}, \dots, V_{n-1}\}$ is a global frame for the tangent bundle of $\pi^{-1}(b)$. It follows the tangent bundle of the fibre $\pi^{-1}(b)$ is trivial. $\square$

This ends the proof of Theorem 2.3. $\square$

By the classification of the Riemannian symmetric spaces of rank one of noncompact type, we have B is isometric to one of the following spaces:

(1) $H^n(c)$ real hyperbolic space with constant sectional curvature c;
(2) $\mathbb{C}H^k(c)$ complex hyperbolic space with holomorphic sectional curvature c;
(3) $\mathbb{H}H^k(c)$ quaternionic hyperbolic space with quaternionic sectional curvature c;
(4) $\mathbb{C}aH^2(c)$ Cayley hyperbolic plane with Cayley sectional curvature c.

This will give us more information about the relation between the dimension of fibres and the geometry of base space.

Proposition 2.5. *Let $\pi : H_s^{n+s} \to B^n$ be a semi-Riemannian submersion with totally geodesic fibres.*

(a) *If $s = 3$ then $n = 4k$ and B^n is isometric to $\mathbb{H}H^k$.*
(b) *If $s = 7$ then we have one of the following situations*:
  (i) *$n = 8$ and B^n is isometric to $H^8(-4)$; or*
  (ii) *$n = 16$ and B^n is isometric to $\mathbb{C}aH^2$.*

Proof. Let Y, Z be two linear independent horizontal vectors and let $Y' = \pi_* Y$, $Z' = \pi_* Z$. By Proposition 2.1, the metric induced on fibres are negative definite. This implies $g(A_Z Y, A_Z Y) \leqslant 0$. By Lemma 1.4, we get

$$K'(Z', Y') = \frac{R'(Z', Y', Z', Y')}{g'(Z', Z')g'(Y', Y') - g'(Z', Y')^2} = -1 + \frac{3g(A_Z Y, A_Z Y)}{g(Z, Z)g(Y, Y) - g(Z, Y)^2} \leqslant -1.$$

By Schwartz inequality applied to the positive definite scalar product induced on $\mathcal{H}$, we have

$$-g(A_Z Y, A_Z Y) = g(A_Z A_Z Y, Y) \leqslant \sqrt{g(A_Z A_Z Y, A_Z A_Z Y)}\sqrt{g(Y, Y)}.$$

By Lemma 1.4, we get

$$-g(A_Z Y, A_Z Y) \leqslant \sqrt{-g(A_Z Y, A_Z Y)g(Z, Z)}\sqrt{g(Y, Y)}.$$

Thus $-g(A_Z Y, A_Z Y) \leqslant g(Z, Z)g(Y, Y)$. Therefore

$$K'(Z', Y') = -1 + \frac{3g(A_Z Y, A_Z Y)}{g(Z, Z)g(Y, Y)} \geqslant -4$$

for every orthogonal vectors Z' and Y'.

We proved that $-4 \leqslant K' \leqslant -1$.

We shall prove that if the base space B has constant curvature c then $c = -4$. It is sufficient to see that for any point $b \in B$ there is a 2-plane $\alpha \in T_b B$ such that $K(\alpha) = -4$. We choose $\alpha = \{\pi_* Z, \pi_* A_Z V\}$ where Z is a horizontal vector and V is a vertical vector. By Lemma 1.4, we have

$$R'(\pi_* Z, \pi_* A_Z V, \pi_* Z, \pi_* A_Z V) = R(Z, A_Z V, Z, A_Z V) + 3g\big(A_Z(A_Z V), A_Z(A_Z V)\big). \tag{2.3}$$

We notice that Z and $A_Z V$ are orthogonal, because, by Lemma 1.2, we have

$$g(Z, A_Z V) = -g(A_Z Z, V) = 0.$$

By Lemma 1.4, we have

$$g(A_X U, A_X U) = -g(X, X)g(U, U)$$

for every horizontal vector X and for every vertical vector U. By Lemma 1.2(d), we get $g(A_X A_X U, U) = g(X, X)g(U, U)$. Hence, by polarization, we find $A_X A_X U = g(X, X)U$ for every horizontal vector X and for every vertical vector U. Therefore relation (2.3) becomes

$$\begin{aligned} R'(\pi_* Z, \pi_* A_Z V, \pi_* Z, \pi_* A_Z V) &= -g(Z, Z)g(A_Z V, A_Z V) + 3g(Z, Z)^2 g(V, V) \\ &= 4g(Z, Z)^2 g(V, V) \\ &= -4\big(g'(\pi_* Z, \pi_* Z)g'(\pi_* A_Z V, \pi_* A_Z V) - g'(\pi_* Z, \pi_* A_Z V)^2\big). \end{aligned}$$

Then $K'(\pi_* Z, \pi_* A_Z V) = -4$. Therefore if the base space B has constant curvature c then $c = -4$.

Let X be a horizontal vector field. By Lemma 1.4, $Y \in \ker A_X^*$ if and only if

$$R'(\pi_* X, \pi_* Y, \pi_* X, \pi_* Y) = -g'(\pi_* X, \pi_* X)g'(\pi_* Y, \pi_* Y) + g'(\pi_* X, \pi_* Y)^2.$$

For every $X' \in T_{\pi(p)} B$, we denote by

$$\mathcal{L}_{X'} = \big\{Y' \in T_{\pi(p)} B \mid R'(X', Y', X', Y') = -g'(X', X')g'(Y', Y') + g'(X', Y')^2\big\}.$$

With this notation, $\pi_*(\ker A^*_{X(p)}) = \mathcal{L}_{\pi_* X(p)}$. Since π_* sends isometrically $\mathcal{H}_p$ into $T_{\pi(p)} B$, we have $\dim \mathcal{H} - \dim \mathcal{V} = \dim \ker A^*_{X(p)} = \dim \mathcal{L}_{\pi_* X(p)}$.

We compute $\dim \mathcal{L}_{X'}$ from the geometry of B. We have the following possibilities for B:

Case 1. $B = H^k(-4)$.

The curvature tensor of hyperbolic space $H^k(-4)$ is given by

$$R'(X', Y', X', Y') = -4\big(g'(X', X')g'(Y', Y') - g'(X', Y')^2\big).$$

We have $\mathcal{L}_{X'} = \{\lambda X' \mid \lambda \in \mathbb{R}\}$. Hence $\dim \mathcal{L}_{X'} = 1$. It follows $\dim \mathcal{H} = \dim \mathcal{V} + 1$.

If $s = 3$ then B^n is isometric to $H^4(-4)$, which falls in the case (a), since $H^4(-4)$ is isometric to $\mathbb{H}H^1$.

If $s = 7$ then $\dim \mathcal{H} = 8$ and this is the case b(ii).

Case 2. $B = \mathbb{C}H^k$.

Let I_0 be the natural complex structure on $\mathbb{C}H^k$. The curvature tensor of complex hyperbolic space $\mathbb{C}H^k$ with $-4 \leqslant K' \leqslant -1$ is given by

$$R'(X', Y', X', Y') = -\big(g'(X', X')g'(Y', Y') - g'(X', Y')^2 + 3g'(I_0 X', Y')^2\big).$$

We get $\mathcal{L}_{X'} = \{I_0 X'\}^{\perp}$. So $\dim \mathcal{L}_{X'} = 2k - 1 = \dim \mathcal{H} - 1$. It follows $\dim \mathcal{V} = 1$.

Case 3. $B = \mathbb{H}H^k$.

Let $\{I_0, J_0, K_0\}$ be local almost complex structures which rise to the quaternionic structure on $\mathbb{H}H^k$. The curvature tensor of the quaternionic hyperbolic space $\mathbb{H}H^k$ with $-4 \leqslant K' \leqslant -1$ (see [9]) is given by

$$R'(X', Y', X', Y') = -g'(X', X')g'(Y', Y') + g'(X', Y')^2 \\ - 3g'(I_0X', Y')^2 - 3g'(J_0X', Y')^2 - 3g'(K_0X', Y')^2.$$

It follows that $Y' \in \mathcal{L}_{X'}$ if and only if $g'(I_0X', Y') = g'(J_0X', Y') = g'(K_0X', Y') = 0$. Therefore $\mathcal{L}_{X'} = \{I_0X', J_0X', K_0X'\}^{\perp}$. Hence $\dim \mathcal{L}_{X'} = 4k - 3 = \dim \mathcal{H} - 3$. We get $\dim \mathcal{V} = 3$.

Case 4. $B = \mathbb{C}aH^2$.

Let $\{I_0, J_0, K_0, M_0, M_0I_0, M_0J_0, M_0K_0\}$ be local almost complex structures which rise to the Cayley structure on $\mathbb{C}aH^2$ Cayley hyperbolic plane. The curvature tensor of the Cayley plane $\mathbb{C}aH^2$ with $-4 \leqslant K' \leqslant -1$ (see [4]) is given by

$$R'(X', Y', X', Y') = -g'(X', X')g'(Y', Y') + g'(X', Y')^2 - 3g'(I_0X', Y')^2 \\ - 3g'(J_0X', Y')^2 - 3g'(K_0X', Y')^2 - 3g'(M_0I_0X', Y')^2 \\ - 3g'(M_0J_0X', Y')^2 - 3g'(M_0K_0X', Y')^2.$$

We get

$$\mathcal{L}_{X'} = \{I_0X', J_0X', K_0X', M_0X', M_0I_0X', M_0J_0X', M_0K_0X'\}^{\perp}.$$

So $\dim \mathcal{L}_{X'} = \dim \mathcal{H} - 7$. It follows $\dim \mathcal{V} = 7$. □

Summarizing all of the above, we obtain our main classification result.

Main Theorem 2.6. *Let $\pi : H^m_s \to B$ be a semi-Riemannian submersion with totally geodesic fibres from a pseudo-hyperbolic space onto a Riemannian manifold. Then the semi-Riemannian submersion π is equivalent to one of the following canonical semi-Riemannian submersions, given by Examples* (1)–(3)

(a) $H^{2k+1}_1 \to \mathbb{C}H^k$,
(b) $H^{4k+3}_3 \to \mathbb{H}H^k$,
(c) $H^{15}_7 \to H^8(-4)$.

Proof. The index of the pseudo-hyperbolic space cannot be $s = 0$. Indeed, by Lemma 1.4, for $s = 0$, we get $0 \leqslant g(A_XV, A_XV) = -g(X, X)g(V, V) \leqslant 0$ for every horizontal vector X and for every vertical vector V. But this is not possible.

By [12], any semi-Riemannian submersion with totally geodesic fibres, from a pseudo-hyperbolic space of index 1 onto a Riemannian manifold is equivalent to the canonical semi-Riemannian submersion $H^{2k+1}_1 \to \mathbb{C}H^k$.

It remains to study the case $s > 1$. By Theorem 2.3 and Proposition 2.5, any semi-Riemannian submersion with totally geodesic fibres from a pseudo-hyperbolic space of index $s > 1$ onto a Riemannian manifold is one of the following types:

(1) $H^{4k+3}_3 \to \mathbb{H}H^k$, or

(2) $H_7^{15} \to H^8(-4)$, or
(3) $H_7^{23} \to \mathbb{C}aH^2$.

In order to prove that any two semi-Riemannian submersions in one of the categories (1) or (2) are equivalent we shall modify Ranjan's argument (see [15]) to our situation. In the category (3), we shall prove there are no such semi-Riemannian submersions with totally geodesic fibres.

First, we shall prove the uniqueness in the case $H_3^{4k+3} \to \mathbb{H}H^k$. Let $p \in H_3^{4k+3}$ and let $\mathcal{U}: \mathcal{V}_p \to End(\mathcal{H}_p)$ the map given by $\mathcal{U}(v)(x) = A_x v$ for every $v \in \mathcal{V}_p$ and for every $x \in \mathcal{H}_p$. We denote $\mathcal{U}(v)$ by A^v. It is trivial to see that A^v is skew-symmetric (i.e., $g(A^v x, y) = -g(x, A^v y)$). The O'Neill's equation $g(A_x v, A_x v) = -g(x,x)g(v,v)$ becomes $g(A^v x, A^v x) = -g(x,x)g(v,v)$. This implies $g(A^v A^v x, x) = g(x,x)g(v,v)$. Hence, by polarization in x, we have $g(A^v A^v x, y) = g(x,y)g(v,v)$ for every $y \in \mathcal{H}_p$. So $A^v A^v x = g(v,v)x$. Again by polarization we get $A^v A^w + A^w A^v = 2g(v,w)Id$. Let $\tilde{g}$ be the Riemannian metric given by $\tilde{g}(v,w) = -g(v,w)$ for every $v, w \in \mathcal{V}_p$. It follows $A^v A^w + A^w A^v = -2\tilde{g}(v,w)Id_{\mathcal{V}_p}$. This is the condition which allows us to extend $\mathcal{U}$ to a representation of the Clifford algebra $Cl(\mathcal{V}_p, \tilde{g}_p)$ of $\mathcal{V}_p$. We also denote by $\mathcal{U}$ the extension of $\mathcal{U}$. Since $\dim \mathcal{V}_p = 3$ and $\tilde{g}_p$ is positive definite, $Cl(\mathcal{V}_p, \tilde{g}_p)$ has at most two types of irreducible representations. We notice that $\mathcal{H}_p$ is a $Cl(\mathcal{V}_p, \tilde{g}_p)$-module which splits in simple modules of dimension 4. The next step is to show that any two such simple modules in decomposition of $\mathcal{H}_p$ are equivalent. Let $\{v_1, v_2, v_3\}$ be an orthonormal basis of $(\mathcal{V}_p, \tilde{g}_p)$. Since the affiliation of a simple $Cl(\mathcal{V}_p, \tilde{g}_p)$-module to one of the two possible types is decided by the action of $v_1 v_2 v_3$, it is sufficient to check that $A^{v_1} A^{v_2} A^{v_3} = Id_{\mathcal{V}_p}$.

Consider the function $x \mapsto g(A^{v_1} A^{v_2} A^{v_3} x, x)$ defined on the unit sphere in $\mathcal{H}_p$. We have

$$g\big(A^{v_1} A^{v_2} A^{v_3} x, x\big) = -g\big(A^{v_2} A^{v_3} x, A^{v_1} x\big) = g(A_x A_{A_x v_3} v_2, v_1).$$

A straightforward computation shows that $A_x A_{A_x v_3} v_2$ is orthogonal to v_2 and v_3. Hence $A_x A_{A_x v_3} v_2$ is a multiple of v_1.

By polarization of the relation $A_x A_x v = g(x,x)v$, we get $A_x A_y + A_y A_x = 2g(x,y)Id$ for every horizontal vectors x and y. In particular, we have

$$A_x A_{A_x v_3} v_2 = -A_{A_x v_3} A_x v_2 + 2g(x, A_x v_3) v_2 = -A_{A_x v_3} A_x v_2.$$

Let S be the vector subspace of $\mathcal{H}_p$ spanned by $\{x, A_x v_1, A_x v_2, A_x v_3\}$. By Lemma 1.4, we get $K'(\pi_* x, \pi_* A_x v_i) = -4$ for all $i \in \{1,2,3\}$. By geometry of $\mathbb{H}H^n$, there exists a unique totally geodesic hyperbolic line $\mathbb{H}H^1$ passing through $\pi(p)$ such that $T_{\pi(p)}\mathbb{H}H^1 = \pi_* S$. Notice that for every orthonormal vectors $y, z \in T_{\pi(p)}\mathbb{H}H^1$, $K'(y,z) = -4$. In particular we have $K'(\pi_* A_x v_2, \pi_* A_x v_3) = -4$. Hence $g(A_{A_x v_3} A_x v_2, A_{A_x v_3} A_x v_2) = -1$. It follows that $A_x A_{A_x v_3} v_2 = \pm v_1$. Hence $g(A^{v_1} A^{v_2} A^{v_3} x, x) = \pm 1$ for all unit vectors x. Since the function $x \mapsto g(A^{v_1} A^{v_2} A^{v_3} x, x)$ defined on the unit sphere in $\mathcal{H}_p$ is continuous, we get either

(i) $g(A^{v_1} A^{v_2} A^{v_3} x, x) = 1$ for any unit horizontal vector x, or
(ii) $g(A^{v_1} A^{v_2} A^{v_3} x, x) = -1$ for any unit horizontal vector x.

We may assume the case (i) holds.

If the case (ii) is happen, we replace the orthonormal basis $\{v_1, v_2, v_3\}$ of $(\mathcal{V}_p, \tilde{g}_p)$ with the orthonormal basis $\{v_1, v_2, -v_3\}$. So for this new basis we are in the case (i).

Since $A^{v_1}A^{v_2}A^{v_3}$ is an isometry, we have

$$g\left(A^{v_1}A^{v_2}A^{v_3}x, A^{v_1}A^{v_2}A^{v_3}x\right)g(x,x) = g(x,x)^2 = 1 = g\left(A^{v_1}A^{v_2}A^{v_3}x, x\right)^2$$

for all unit horizontal vectors x.

So the Schwartz inequality for the scalar product $g|_{\mathcal{H}_p}$

$$g\left(A^{v_1}A^{v_2}A^{v_3}x, x\right)^2 \leqslant g\left(A^{v_1}A^{v_2}A^{v_3}x, A^{v_1}A^{v_2}A^{v_3}x\right)g(x,x)$$

becomes equality. It follows that $A^{v_1}A^{v_2}A^{v_3}x = \lambda x$ for some λ. Because $A^{v_1}A^{v_2}A^{v_3}$ is an isometry and we assumed the case (i), it follows $\lambda = 1$. We proved that $A^{v_1}A^{v_2}A^{v_3}x = x$ for all unit horizontal vectors x. Obviously, $A^{v_1}A^{v_2}A^{v_3}x = x$ for all $x \in \mathcal{H}_p$.

Let $\pi' : H_3^{4k+3} \to \mathbb{H}H^k$ be another semi-Riemannian submersion with totally geodesic fibres. For an arbitrary chosen point $q \in H_3^{4k+3}$, we consider horizontal and vertical subspaces $\mathcal{H}'_q$ and $\mathcal{V}'_q$. Let $\{v'_1, v'_2, v'_3\}$ be an orthonormal basis in $\mathcal{V}'_q$ such that $v'_1v'_2v'_3$ acts on $\mathcal{H}'_q$ as Id. Let $L_1 : \mathcal{V}'_q \to \mathcal{V}_p$ be the isometry given by $L_1(v'_i) = v_i$ for all $i \in \{1, 2, 3\}$ and let $Cl(L_1) : Cl(\mathcal{V}'_q) \to Cl(\mathcal{V}_p)$ be the extension of L_1 to the Clifford algebras. The composition $\mathcal{U} \circ Cl(L_1) : Cl(\mathcal{V}'_q) \to End(\mathcal{H}_p)$ makes $\mathcal{H}_p$ to be a $Cl(\mathcal{V}'_q)$-module of dimension $4k$. Let $\mathcal{H}_p = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_k$ and $\mathcal{H}'_q = \mathcal{H}'_1 \oplus \cdots \oplus \mathcal{H}'_k$ be the decomposition of $\mathcal{H}_p$ and $\mathcal{H}'_q$ in simple $Cl(\mathcal{V}'_q)$-modules, respectively. For each i there is $f_i : \mathcal{H}'_i \to \mathcal{H}_i$ an equivalence of $Cl(\mathcal{V}'_q)$-modules, which after a rescaling by a constant number is an isometry which preserves the O'Neill's integrability tensors. Taking the direct sum of all these isometries, we obtain an isometry $L_2 : \mathcal{H}'_q \to \mathcal{H}_p$ which preserves the O'Neill's integrability tensors. Therefore $L = L_1 \oplus L_2 : T_qH_3^{4k+3} \to T_pH_3^{4k+3}$ is an isometry which maps $\mathcal{H}'_q$ onto $\mathcal{H}_p$ and A' onto A. Since H_3^{4k+3} is a simply connected complete symmetric space, there is an isometry $f : H_3^{4k+3} \to H_3^{4k+3}$ such that $f(q) = p$ and $f_{*q} = L$ (see Corollary 2.3.14 in [16]). Therefore, by Theorem 1.6, we get π and π' are equivalent.

Now, we shall prove that any two semi-Riemannian submersions $\pi, \pi' : H_7^{15} \to H^8(-4)$ with totally geodesic fibres are equivalent. The proof is analogous to the case (1), but it is easier.

Let $p, q \in H_7^{15}$ and let $\mathcal{H}_p$, $\mathcal{V}_p$ be the horizontal and vertical subspaces in $T_pH_7^{15}$ for π, let $\mathcal{H}'_q$, $\mathcal{V}'_q$ be the horizontal and vertical subspaces in $T_qH_7^{15}$ for π'. Let $\{v_1, \ldots, v_7\}$ be an orthonormal basis of $(\mathcal{V}_p, \tilde{g}_p)$ and $\{v'_1, \ldots, v'_7\}$ be an orthonormal basis of $(\mathcal{V}'_q, \tilde{g}_q)$ such that $A^{v_1}A^{v_2}\ldots A^{v_7} = Id$ and $A^{v'_1}A^{v'_2}\ldots A^{v'_7} = Id$. Since $\dim \mathcal{V}_p = 7$, the irreducible $Cl(\mathcal{V}_p, \tilde{g}_p)$-modules are 8-dimensional. Since $\dim \mathcal{H}_p = 8$, we get $\mathcal{H}_p$ is simple. Because $A^{v_1}A^{v_2}\ldots A^{v_7} = Id$ and $A^{v'_1}A^{v'_2}\ldots A^{v'_7} = Id$ we get $\mathcal{H}'_q$ and $\mathcal{H}_p$ are $Cl(\mathcal{V}'_q)$-modules equivalent. Analogously to the case (1), we can construct an isometry $L = L_1 \oplus L_2 : T_qH_7^{15} \to T_pH_7^{15}$, which map $\mathcal{H}'_q$ onto $\mathcal{H}_p$ and A' onto A. This produces an isometry $f : H_7^{15} \to H_7^{15}$ such that $f(q) = p$ and $f_{*q} = L$ (see Corollary 2.3.14 in [16]). Again by Theorem 1.6, we get π and π' are equivalent.

Now, we prove that there are no $\pi : H_7^{23} \to \mathbb{C}aH^2$ semi-Riemannian submersions with totally geodesic fibres. The proof is analogous to that of Ranjan (see Proposition 5.1 in [15]).

$\mathcal{H}_p$ becomes a $Cl(\mathcal{V}_p)$-module by considering the extension of the map $\mathcal{U} : \mathcal{V}_p \to \mathrm{End}(\mathcal{H}_p)$, $\mathcal{U}(V)(X) = A_XV$ to the Clifford algebra $Cl(\mathcal{V}_p)$. Here $Cl(\mathcal{V}_p, \tilde{g}_p)$ denotes the Clifford algebra of $(\mathcal{V}_p, \tilde{g}_p)$, $\tilde{g}(U, V) = -g(U, V)$ for every $U, V \in \mathcal{V}_p$. Since $\tilde{g}_p$ is positive definite, we have $Cl(\mathcal{V}_p) \simeq \mathbb{R}(8) \oplus \mathbb{R}(8)$. Hence, $\mathcal{H}_p$ splits into two 8-dimensional irreducible $Cl(\mathcal{V}_p)$-modules. Since the induced metrics on fibres are negative definite we get $\pi^{-1}(\mathbb{C}aH^1)$ is totally geodesic in H_7^{23} and isometric to H_7^{15}, by Theorem 2.5 in [6]. Here $\mathbb{C}aH^1$ denotes the Cayley hyperbolic line through π_*X; we choose S be the horizontal space of the restricted submersion $\tilde{\pi}: H_7^{15} \to \mathbb{C}aH^1 = H^8(-4)$. So for every $X \in \mathcal{H}_p$, $g(X, X) \neq 0$ we find an irreducible $Cl(\mathcal{V}_p)$-submodule S of $\mathcal{H}_p$ passing through X. Since $\dim \mathcal{V}_p \geqslant 4$, we get a contradiction. □

Escobales [6] classified Riemannian submersions from complex projective spaces under the assumption that the fibres are connected, complex, totally geodesic submanifolds. Using the Main Theorem 2.6, we obtain a classification of semi-Riemannian submersions from a complex pseudo-hyperbolic space onto a Riemannian manifold under the assumption that the fibres are connected, complex, totally geodesic submanifolds.

Proposition 2.7. *If $\pi: \mathbb{C}H_s^m \to B^n$ is a semi-Riemannian submersion with complex, connected, totally geodesic fibres then $2m = n + 2s$, the induced metrics on fibres are negative definite and the fibres are diffeomorphic to $\mathbb{C}P^s$.*

Proof. We denote by J the natural almost complex structure on $\mathbb{C}H_s^m$. By Lemma 1.4, we have

$$\text{(a)} \quad \widehat{R}(U, V, U, V) = R(U, V, U, V) = -\big(g(U, U)g(V, V) - g(U, V)^2 + 3g(U, JV)^2\big).$$

Hence the fibres have constant holomorphic curvature -4.

$$\text{(b)} \quad g(A_X U, A_X U) = -\big(g(U, U)g(X, X) + 3g(X, JU)^2\big) = -g(U, U)g(X, X),$$

since the fibres are complex submanifolds. We obtain $g(U, U) \leqslant 0$ for every vertical vector field U.

$$\text{(c)} \quad \begin{aligned} R'(\pi_* X, \pi_* Y, \pi_* X, \pi_* Y) &= R(X, Y, X, Y) + 3g(A_X Y, A_X Y) \\ &= -\big(g(X, X)g(Y, Y) - g(X, Y)^2 + 3g(X, JY)^2\big) + 3g(A_X Y, A_X Y) \leqslant 0, \end{aligned}$$

since the induced metrics on fibres are negative definite. By Proposition 2.2, it follows that the fibres are simply connected. Since the fibres are complete, simply connected, complex manifolds with constant holomorphic curvature -4, we have that the fibres are isometric to $\mathbb{C}H_s^s$. □

Theorem 2.8. *If $\pi: \mathbb{C}H_s^m \to B$ is a semi-Riemannian submersion with connected, complex, totally geodesic fibres from a complex pseudo-hyperbolic space, then π is, up to equivalence, the canonical semi-Riemannian submersion given by Example* 4

$$\mathbb{C}H_1^{2k+1} \to \mathbb{H}H^k.$$

Proof. Let $\theta: H_{2s+1}^{2m+1} \to \mathbb{C}H_s^m$ be the canonical semi-Riemannian submersion with totally geodesic fibres given in the Definition 3 (see also [3] or [10]). We have $\tilde{\pi} = \pi \circ \theta: H_{2s+1}^{2m+1} \to B$ is a semi-Riemannian submersion with totally geodesic fibres, by Theorem 2.5 in [6]. Since the dimension of fibres of $\tilde{\pi}$ is greater than or equal to 2, we get, by Main Theorem 2.6, the following possible situations:

(i) $m = 2k + 1$, $2s + 1 = 3$ and B is isometric to $\mathbb{H}H^k$ or
(ii) $m = 7$, $2s + 1 = 7$ and B is isometric to $H^8(-4)$.

First, we shall prove that any two semi-Riemannian submersions $\pi, \pi': \mathbb{C}H_1^{2k+1} \to \mathbb{H}H^k$ with connected, complex, totally geodesic fibres are equivalent.

By proof of Proposition 2.7, we have $g(A_X U, A_X U) = -g(U,U)g(X,X)$. Let $p,q \in \mathbb{C}H_1^{2k+1}$. By proof of the main theorem, this implies $A^v A^w + A^w A^v = -2\tilde{g}(v,w)Id$. The extension of $\mathcal{U}:\mathcal{V}_p \to End(\mathcal{H}_p)$ constructed in proof of the main theorem, to the Clifford algebra $Cl(\mathcal{V}_p, \tilde{g}_p)$ makes $\mathcal{H}_p$ a $Cl(\mathcal{V}_p, \tilde{g}_p)$-module which splits in k irreducible modules of dimension 4. By classification of irreducible representation for case $\dim \mathcal{V}_p = 2$ and $\tilde{g}_p$ positive definite, we have any two such irreducible $Cl(\mathcal{V}_p, \tilde{g}_p)$-modules are equivalent. Like in proof of the main theorem, we may construct an isometry $L = L_1 \oplus L_2 : T_q\mathbb{C}H_1^{2k+1} \to T_p\mathbb{C}H_1^{2k+1}$, which maps $\mathcal{H}'_q$ onto $\mathcal{H}_p$ and A' onto A. This produces an isometry $f:\mathbb{C}H_1^{2k+1} \to \mathbb{C}H_1^{2k+1}$ with $f(q) = p$ and $f_{*q} = L$ (see Corollary 2.3.14 in [16]). Again by Theorem 1.6, we get π and π' are equivalent.

For the case (ii) we shall obtain that there are no $\pi:\mathbb{C}H_3^7 \to H^8(-4)$ semi-Riemannian submersions with complex, connected, totally geodesic fibres.

Proposition 2.9. *There are no $\pi:\mathbb{C}H_3^7 \to H^8(-4)$ semi-Riemannian submersions with connected, complex, totally geodesic fibres.*

Proof. The proof is based on Ranjan's argument (see proof of main theorem in [15]). Here, we show how to modify Ranjan's argument to our different situation.

Suppose there is $\pi:\mathbb{C}H_3^7 \to H^8(-4)$ a semi-Riemannian submersion with complex, connected, totally geodesic fibres. By Main Theorem 2.6, $\tilde{\pi} = \pi \circ \theta : H_7^{15} \to H^8(-4)$ is equivalent to the canonical semi-Riemannian submersion $Spin(1,8)/Spin(7) \to Spin(1,8)/Spin(8)$ given by Example 3.

Let $\sigma : Spin(1,8) \to SO(8,8)$ be the spin representation of $Spin(1,8)$. $Spin(1,8)$ acts on $H^8(-4)$ via double covering map $Spin(1,8) \to SO(1,8)$ and transitively on $H_7^{15} \subset \mathbb{R}_8^{16}$. We denote by $Cl^0(\mathbb{R}_1^9)$ the even component of Clifford algebra $Cl(\mathbb{R}_1^9)$. Notice that $Cl^0(\mathbb{R}_1^9) \cong M(16,\mathbb{R})$, $Cl(\mathbb{R}_1^9) \cong M(16,\mathbb{R}) \oplus M(16,\mathbb{R})$ and the volume element ω in $Cl(\mathbb{R}_1^9)$ satisfies $\omega^2 = 1$ (see [11]).

For any $b \in H^8(-4)$, let G_b be the isotropy group of b in $Spin(1,8)$. If we restrict $\sigma|_{G_b}$ then $\sigma|_{G_b}$ breaks $\mathbb{R}_8^{16}$ into two $\frac{1}{2}$-spin representations. We will denote them by $\mathbb{R}_\pm^8$. Hence $\mathbb{R}_+^8 \cap H_7^{15} = \tilde{\pi}^{-1}(b)$. Let $b^\perp = \{x \in \mathbb{R}_1^9 \mid \langle x, b\rangle = 0\}$. We have $Cl(b^\perp) \cap Spin(1,8) = G_b$, $\dim b^\perp = 8$ and the following diagram is commutative

$$\begin{array}{ccc} G_b & \longrightarrow & Cl^0(b^\perp) \\ \downarrow & & \downarrow \\ Spin(1,8) & \longrightarrow & Cl^0(\mathbb{R}_1^9), \end{array}$$

where all arrows are standard inclusions. Let $\{e_1,\dots,e_8\}$ be an orientated basis of $b^\perp$. Then $z' = e_1 \dots e_8$ lies in the centre of $Cl^0(b^\perp)$ and z' acts by Id on $\mathbb{R}_+^8$ and $-Id$ on $\mathbb{R}_-^8$. We have $Cl(\sigma)(z') = \pm 1$ on $\mathbb{R}_\pm^8$.

Since $\mathbb{R}_+^8 \cap H_7^{15} = \tilde{\pi}^{-1}(b)$, $\mathbb{R}_+^8$ is invariant under J and so is $\mathbb{R}_-^8$. Here J denotes the natural complex structure on $\mathbb{R}^{16} = \mathbb{C}^8$. Hence $Cl(\sigma)(z')$ commutes with J. Let $z \in Cl(\mathbb{R}_1^9)$ be the generator of the center of $Cl(\mathbb{R}_1^9)$. We have either $z = e_1e_2\dots e_8 b$ or $zb = -e_1e_2\dots e_8 = -z'$. Therefore $Cl(\sigma)(zb)$ commutes with J for every $b \in H^8(-4)$ and hence for every $b \in \mathbb{R}^9$.

Consider the linear map $\alpha : \mathbb{R}^9 \to M(16,\mathbb{R})$ given by $b \mapsto Cl(\sigma)(zb)$. It has the following properties:

(i) It factors through $M(8,\mathbb{C}) \subset M(16,\mathbb{R})$;
(ii) $[Cl(\sigma)(zb)]^2 = Cl(\sigma)((zb)^2) = Cl(\sigma)(-|b|^2) = -|b|^2 Id$.

Hence α extends to a homomorphism $Cl(\alpha): Cl(\mathbb{R}^9_1) \to M(8, \mathbb{C})$. But $Cl(\mathbb{R}^9_1) \cong M(16, \mathbb{R}) \oplus M(16, \mathbb{R})$ (see [11]). So the above homomorphism is impossible to exist. We get the required contradiction. $\square$

This ends the proof of Theorem 2.8. $\square$

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References

[1] P. Baird, J.C. Wood, Harmonic morphisms between Riemannian manifolds, London Math. Soc. Monogr. (N.S.), Oxford Univ. Press (to appear).
[2] A.L. Besse, Einstein Manifolds, Springer, Berlin, 1987.
[3] M. Barros, A. Romero, Indefinite Kähler manifolds, Math. Ann. 261 (1982) 55–62.
[4] P. Coulton, J. Glazebrook, Submanifolds of the Cayley projective plane with bounded second fundamental form, Geom. Dedicata 33 (1990) 265–275.
[5] R. Escobales, Riemannian submersions with totally geodesic fibers, J. Differential Geom. 10 (1975) 253–276.
[6] R. Escobales, Riemannian submersions from complex projective spaces, J. Differential Geom. 13 (1978) 93–107.
[7] B. Fuglede, Harmonic morphisms between semi-Riemannian manifolds, Ann. Acad. Sci. Fenn. Math. 21 (1996) 31–50.
[8] A. Gray, Pseudo-Riemannian almost product manifolds and submersions, J. Math. Mech. 16 (1967) 715–737.
[9] S. Ishihara, Quaternionic Kähler manifolds, J. Differential Geom. 9 (1974) 483–500.
[10] J.K. Kwon, Y.J. Suh, On sectional and Ricci curvatures of semi-Riemannian submersions, Kodai Math. J. 20 (1997) 53–66.
[11] H.B. Lawson, M.-L. Michelsohn, Spin Geometry, Princeton Univ. Press, Princeton, 1989.
[12] M.A. Magid, Submersions from anti-de Sitter space with totally geodesic fibres, J. Differential Geom. 16 (1981) 323–331.
[13] B. O'Neill, The fundamental equations of a submersion, Michigan Math. J. 13 (1966) 459–469.
[14] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, London, 1983.
[15] A. Ranjan, Riemannian submersions of spheres with totally geodesics fibers, Osaka J. Math. 22 (1985) 243–260.
[16] J. Wolf, Spaces of Constant Curvature, McGraw-Hill, New York, 1967.
[17] N.I. Zhukova, Submersions with an Ehresmann connection, Izvestiya VUZ Matematika 32 (1988) 25–33.