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Spectral geometry of Riemannian Legendre foliations

by
Gabriel Băditoiu, Stere Ianuş* and Anna Maria Pastore

Abstract

We obtain geometric characterizations of isospectral minimal Riemannian Legendre foliations on compact Sasakian manifolds of constant φ-sectional curvature.

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1 Preliminaries

Let $\mathcal{F}$ be a Riemannian foliation on an m-dimensional compact Riemannian manifold (M, g). We denote by L and $Q = L^{\perp}$ the tangent and normal bundles of $\mathcal{F}$, and that gives the decomposition of the tangent bundle $TM = L \oplus L^{\perp}$. Let Δ_g be the Laplace operator associated to g and let ∇ be the Bott connection of the normal bundle $Q = TM/L$ (see [16, pp. 20-21]). The Jacobi operator $\mathcal{J}_\nabla$ of $\mathcal{F}$, defined by $\mathcal{J}_\nabla s = (d_\nabla^* d_\nabla - \rho_\nabla)s$ for any s section of the normal bundle, is a second order elliptic operator (see [15]). The compactness of M implies that the spectra of Δ_g and $\mathcal{J}_\nabla$ are discrete. Using Gilkey's theory ([8, 14]), one can write their associated asymptotic expansions:

$$Tr\, e^{-t\Delta_g} = \sum_{i=1}^{\infty} e^{-t\lambda_i} \underset{t\searrow 0}{\widetilde{\quad}} (4\pi t)^{-\frac{m}{2}} \sum_{s=0}^{\infty} t^s a_s(\Delta_g)$$

$$Tr\, e^{-t\mathcal{J}_\nabla} = \sum_{i=1}^{\infty} e^{-t\mu_i} \underset{t\searrow 0}{\widetilde{\quad}} (4\pi t)^{-\frac{m}{2}} \sum_{s=0}^{\infty} t^s b_s(\mathcal{J}_\nabla)$$

where

$$a_s(\Delta_g) = \int_M a_s(x, \Delta_g) dv_g$$

*Stere Ianuş passed away on April 8th, 2010.

$$b_s(\mathcal{J}_\nabla) = \int_M b_s(X, \mathcal{J}_\nabla) dv_g$$

are invariants of Δ_g and $\mathcal{J}_\nabla$ depending only on their corresponding discrete spectra

$$Spec(M, g) = \{0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_i \leq \ldots \uparrow \infty\}$$

$$Spec(\mathcal{F}, \mathcal{J}_\nabla) = \{\mu_1 \leq \mu_2 \leq \ldots \leq \mu_i \leq \ldots \uparrow \infty\}$$

We restrict our attention to the first coefficients a_s and b_s for $s \in \{0, 1, 2\}$ which encodes certain properties of the spectral geometry of $(M, \mathcal{F})$. We recall the following theorem from [8, 11].

Theorem 1.1. *Let $\mathcal{F}$ be a Riemannian foliation of codimension $q \geq 2$ on a compact Riemannian manifold (M, g). Then*

$$\begin{aligned} a_0(\Delta_g) &= a_0 = Vol_g(M) \\ a_1(\Delta_g) &= a_1 = \tfrac{1}{6}\int_M \tau dv_g \\ a_2(\Delta_g) &= a_2 = \tfrac{1}{360}\int_M (2\|R\|^2 - 2\|\rho\|^2 + 5\tau^2) dv_g \end{aligned} \tag{1}$$

$$\begin{aligned} b_0(\mathcal{J}_\nabla) &= b_0 = qVol_g(M) \\ b_1(\mathcal{J}_\nabla) &= b_1 = qa_1 + \int_M \tau_\nabla dv_g \\ b_2(\mathcal{J}_\nabla) &= b_2 = qa_2 + \tfrac{1}{12}\int_M (2\tau\tau_\nabla + 6\|\rho_\nabla\|^2 - \|R_\nabla\|^2) dv_g\,, \end{aligned} \tag{2}$$

where R, ρ are the Riemann and the Ricci tensor fields, τ is the scalar curvature of g, and $R_\nabla, \rho_\nabla, \tau_\nabla$ are those associated to the Bott connection ∇ of the transverse bundle $Q = TM/L$.

The following theorem, due to Nishikawa, Tondeur and Vanhecke [11], is fundamental for the spectral geometry of a Riemannian foliation.

Theorem 1.2. *Let (M, g) and (M_0, g_0) be two compact Riemannian manifolds endowed with Riemannian foliations $\mathcal{F}$ and $\mathcal{F}_0$ of codimensions q and q_0, respectively. If $\mathcal{F}$ and $\mathcal{F}_0$ are isospectral, that is*

$$Spec(M, g) = Spec(M_0, g_0), \qquad Spec(\mathcal{F}, \mathcal{J}_\nabla) = Spec(\mathcal{F}_0, \mathcal{J}_{\nabla_0}),$$

then the following hold:

i) $\dim M = \dim M_0\,, \quad \mathrm{Vol}(M) = \mathrm{Vol}(M_0)\,, \quad q = q_0,$

ii) $\int_M \tau dv_g = \int_{M_0} \tau_0 dv_{g_0}\,, \quad \int_M \tau_\nabla dv_g = \int_{M_0} \tau_{\nabla_0} dv_{g_0},$

iii) $\int_M (2\|R\|^2 - 2\|\rho\|^2 + 5\tau^2) dv_g = \int_{M_0} (2\|R_0\|^2 - 2\|\rho_0\|^2 + 5\tau_0^2) dv_{g_0},$

iv) $\int_M (2\tau\tau_\nabla + 6\|\rho_\nabla\|^2 - \|R_\nabla\|^2) dv_g = \int_{M_0} (2\tau_0\tau_{\nabla_0} + 6\|\rho_{\nabla_0}\|^2 - \|R_{\nabla_0}\|^2) dv_{g_0}.$

We recall that in the one-codimension case the isospectral Riemannian foliations are completely determined by the spectrum of Δ_g and for this reason we shall assume throughout the paper that the codimension $q \geq 2$.

2 Spectral Invariants of a Riemannian Legendre foliation

In this section we compute the spectral invariants a_s, b_s, for $s \in \{0, 1, 2\}$ of a Riemannian Legendre foliation with minimal leaves on a Sasakian manifold M of constant φ-sectional curvature and then we obtain certain geometric properties of two such isospectral Riemannian foliations. First, we recall the notion of Riemannian Legendre foliation.

Definition 2.1. Let M be a $(2n+1)$-dimensional compact manifold endowed with a Sasakian structure (φ, ξ, η, g) and let $\mathcal{D} = Ker\,\eta = Im\,\varphi$ be the $2n$-dimensional distribution on M orthogonal to the 1-dimensional distribution generated by ξ. A Riemannian foliation $\mathcal{L}$ on M is said to be a **Riemannian Legendre foliation** if the leaves are n-dimensional and $L_x \subset \mathcal{D}_x$, for each $x \in M$. Note that $\varphi(L) \subset L^{\perp}$.

Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold of constant φ-sectional curvature c and of dimension $2n+1 \geq 5$. We recall that φ is an endomorphism of tangent bundle, ξ is a vector field on M, η is the 1-form dual to ξ with respect to g, satisfying:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \varphi(\eta(X)) = 0,$$
$$g(X,Y) = g(\varphi(X), \varphi(Y)) + \eta(X)\eta(Y),$$
$$\nabla^M_X \xi = -\varphi(X), \quad (\nabla^M_X \varphi)(Y) = g(X,Y)\xi - \eta(Y)X,$$

for any vector fields X, Y. By [3], its curvature tensor is given by

$$\begin{aligned} R(X,Y)Z = \;& \tfrac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ & + \tfrac{c-1}{4}\{g(Z,\varphi Y)\varphi X - g(Z,\varphi X)\varphi Y \\ & + 2g(X,\varphi Y)\varphi Z - g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi \\ & - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y\}\,, \end{aligned} \tag{3}$$

and its Ricci tensor and scalar curvature satisfy

$$\rho(X,Y) = \frac{n(c+3)+c-1}{2} g(X,Y) - \frac{(n+1)(c-1)}{2}\eta(X)\eta(Y)\,, \tag{4}$$

$$\tau = \frac{n}{2}(2n+1)(c+3) + \frac{n}{2}(c-1)\,. \tag{5}$$

Let $\mathcal{L}$ be a Riemannian Legendre foliation on M and $(e_i, \varphi e_i, \xi)$, $i \in \{1, \ldots, n\}$ be a local orthonormal basis of TM adapted to the foliation $\mathcal{L}$, which means that $(e_1, \ldots, e_n)$ is a local basis of L and $(\varphi e_1, \ldots \varphi e_n)$ is a basis of $\varphi(L)$. The curvature tensor writes as

$$\begin{aligned} & R(e_i, \xi, e_k, \xi) = R(\varphi e_i, \xi, \varphi e_k, \xi) = \delta_{ik}, \\ & R(e_i, e_j, e_k, e_m) = R(\varphi e_i, \varphi e_j, \varphi e_k, \varphi e_m) = \tfrac{c+3}{4}(\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}), \\ & R(e_i, e_j, \varphi e_k, \varphi e_m) = R(\varphi e_i, \varphi e_j, e_k, e_m) = \tfrac{c-1}{4}(\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}), \\ & R(e_i, \varphi e_j, e_k, \varphi e_m) = \tfrac{c+3}{4}\delta_{ik}\delta_{jm} + \tfrac{c-1}{4}\delta_{im}\delta_{jk} + \tfrac{c-1}{2}\delta_{ij}\delta_{km}, \end{aligned} \tag{6}$$

the other expressions being equal to zero. The Ricci tensor is given by

$$\begin{aligned}
&\rho(e_i,\xi)=\rho(\varphi e_i,\xi)=\rho(e_i,\varphi e_j)=0,\\
&\rho(\xi,\xi)=2n,\\
&\rho(e_i,e_j)=\rho(\varphi e_i,\varphi e_j)=\tfrac{n(c+3)+c-1}{2}\delta_{ij}.
\end{aligned} \tag{7}$$

The square of the Hilbert-Schmidt norm of R, defined to be

$$\|R\|^2=\sum_{a,b,c,d} g(R(e_a,e_b)e_c,e_d)g(R(e_a,e_b)e_c,e_d), \tag{8}$$

in any orthonormal basis, writes, in the fixed adapted basis, as

$$\begin{aligned}
\|R\|^2 &= 2[(\tfrac{c+3}{4})^2+(\tfrac{c-1}{4})^2]\textstyle\sum_{i,j,k,m}(\delta_{ik}\delta_{jm}-\delta_{im}\delta_{jk})^2+8\sum_{ij}\delta_{ij}\\
&\quad +4\textstyle\sum_{i,j,k,m}(\tfrac{c+3}{4}\delta_{ik}\delta_{jm}+\tfrac{c-1}{4}\delta_{im}\delta_{jk}+\tfrac{c-1}{2}\delta_{ij}\delta_{km})^2\\
&= [\tfrac{(c+3)^2}{8}+\tfrac{(c-1)^2}{8}](2n^2-2n)+8n+4\{\tfrac{(c+3)^2}{16}n^2+\tfrac{(c-1)^2}{16}n^2\\
&\quad +\tfrac{(c-1)^2}{4}n^2+2\tfrac{c+3}{4}(\tfrac{c-1}{4}n+\tfrac{c-1}{2}n)+\tfrac{(c-1)^2}{4}n\}\\
&= \tfrac{[(c-1)^2+(c+3)^2]n(n-1)}{4}+\tfrac{(c+3)^2}{4}n^2+\tfrac{(c-1)^2}{4}(5n^2+4n)+\tfrac{3(c+3)(c-1)n}{2}+8n\\
&= \tfrac{(c+3)^2n(2n-1)}{4}+\tfrac{(c-1)^2n(6n+3)}{4}+\tfrac{3(c+3)(c-1)n}{2}+8n.
\end{aligned} \tag{9}$$

Furthermore, for the norm of the Ricci tensor, computing

$$\|\rho\|^2=\sum_{a,b}\rho(e_a,e_b)\rho(e_a,e_b),$$

in the adapted basis, one has

$$\|\rho\|^2=2\left(\frac{n(c+3)+c-1}{2}\right)^2 n+4n^2\,. \tag{10}$$

Summarizing the above computations, by Theorem 1.1 we get the following proposition.

Proposition 2.2. *If $\mathcal{L}$ is a Riemannian Legendre foliation on a compact $(2n+1)$-dimensional*

Sasakian manifold of constant φ-sectional curvature c, then the spectral invariants satisfy:

$$\begin{aligned}
a_0(\Delta_g) &= a_0 = Vol_g(M) && (11)\\
a_1(\Delta_g) &= a_1 = \frac{n((2n+1)(c+3)+c-1)}{12} Vol_g(M) && (12)\\
a_2(\Delta_g) &= a_2 = \frac{n}{1440}\Big(64-32n+(c+3)^2(-2+9n+16n^2+20n^3) && (13)\\
&\quad +(c+3)(c-1)(12+2n+20n^2)+(c-1)^2(2+17n)\Big)Vol_g(M)\\
b_0(\mathcal{J}_\nabla) &= b_0 = (n+1)Vol_g(M) && (14)\\
b_1(\mathcal{J}_\nabla) &= b_1 = (n+1)a_1 + \int_M \tau_\nabla dv_g && (15)\\
b_2(\mathcal{J}_\nabla) &= b_2 = (n+1)a_2 + \frac{1}{12}\int_M (2\tau\tau_\nabla + 6\|\rho_\nabla\|^2 - \|R_\nabla\|^2)dv_g\,. && (16)
\end{aligned}$$

Since the foliation $\mathcal{L}$ is assumed to be Riemannian, there exists, locally, a Riemannian submersion whose vertical and horizontal distributions are L and $L^\perp = \varphi(L)\oplus[\xi]$, respectively. Let A and T be the O'Neill tensors (see [7], [16, p. 49]).

Proposition 2.3. *Let $\mathcal{L}$ be a Riemannian Legendre minimal foliation on a Sasakian manifold with constant φ-sectional curvature c. Then*

a) $\tau_\nabla = 3\|A\|^2 + \frac{n}{4}((c+3)(n-1)+8)$;

b) $\|A\|^2 = \|T\|^2 + n(c+1)$.

Proof: Let $X, Y \in \Gamma(\varphi(L)\oplus[\xi])$. From the theory of Riemannian submersions [7] we have

$$K(X,Y) = K_\nabla(X,Y) - 3\|A_XY\|^2, \tag{17}$$

where ∇ is the connection associated to $TM/L \simeq \varphi(L)\oplus[\xi]$, which in the case of a Riemannian foliation coincides with the connection induced by the Levi-Civita connection of M on the horizontal distribution.

Fixing an adapted local orthonormal basis of the foliation, by [15], we see the transverse scalar curvature can be written as

$$\begin{aligned}
\tau_\nabla &= \sum_{i\neq j=1}^{n} K(\varphi e_i,\varphi e_j) + 2\sum_{i=1}^{n} K(\varphi e_i,\xi) + 3\|A\|^2\\
&= \sum_{i\neq j=1}^{n} R(\varphi e_i,\varphi e_j,\varphi e_i,\varphi e_j) + 2\sum_{i=1}^{n} R(\varphi e_i,\xi,\varphi e_i,\xi) + 3\|A\|^2 \qquad (18)\\
&= \tfrac{c+3}{4}n(n-1) + 2n + 3\|A\|^2\,,
\end{aligned}$$

which implies a).

Denoting by τ^{mixed} the mixed scalar curvature, defined by

$$\tau^{mixed} = \sum_{i=1}^{n}\sum_{j=n+1}^{2n} R(e_i,f_j,e_i,f_j) + \sum_{i=1}^{n} R(e_i,\xi,e_i,\xi)\,, \tag{19}$$

where $f_{n+i} = \varphi e_i$, we obtain:

$$\tau^{mixed} = \sum_{i,j} c\delta_{ij} + n = (c+1)n\,. \tag{20}$$

We denote by H the mean curvature of the leaves. We recall Ranjan's formula ([13])

$$\tau^{mixed} = div(H) + \|H\|^2 + \|A\|^2 - \|T\|^2\,. \tag{21}$$

Specializing to the case $H = 0$, the relations (21) and (20) simply imply b). □

Theorem 2.4. *Let $\mathcal{L}$ and $\mathcal{L}_0$ be two Riemannian Legendre minimal foliations on compact Sasakian manifolds $(M, \varphi, \xi, \eta, g)$ and $(M_0, \varphi_0, \xi_0, \eta_0, g_0)$. If (M, g) and (M, g_0) have constant φ, and φ_0-sectional curvature c and c_0, and if $\mathcal{L}$ and $\mathcal{L}_0$ are isospectral, then*

a) $\dim M = \dim M_0$, $Vol(M) = Vol(M_0)$, $c = c_0$,

b) $\int_M \|A\|^2 dv_g = \int_{M_0} \|A_0\|^2 dv_{g_0}$, *and*

c) $\int_M \|T\|^2 dv_g = \int_{M_0} \|T_0\|^2 dv_{g_0}$.

Proof: By Theorem 1.2i) and ii), we see that $n = n_0$ and $\mathrm{Vol}(g) = \mathrm{Vol}(g_0)$ and

$$\int_M \tau\, dv_g = \int_{M_0} \tau_0\, dv_{g_0}, \text{ and } \int_M \tau_\nabla dv_g = \int_{M_0} \tau_{0\,\nabla_0} dv_{g_0}.$$

Therefore by (5), we get $c = c_0$, which by Proposition 2.3a) simply implies a). Now, by Proposition 2.3b) we get b) . □

Corollary 2.5. *Under the hypotheses of Theorem 2.4, the following statements hold:*

(a) If $\varphi(L) \oplus [\xi]$ is integrable, then so is $\varphi(L_0) \oplus [\xi]$.

(b) If $\mathcal{L}$ is totally geodesic, then so is $\mathcal{L}_0$.

Proof: It is sufficient to observe that $A = 0 \Rightarrow A_0 = 0$ and that $T = 0 \Rightarrow T_0 = 0$. □

3 The invariants b_1 and b_2

We shall explicitly compute b_1, b_2 of Proposition 2.2. By Theorem 1.1 and Proposition 2.3 we get

$$b_1(\mathcal{J}_\nabla) = \frac{n}{12}[(c+3)(2n^2+6n-2) + 3(n+1)(c-1) + 2n] + 3\int_M \|A\|^2 dv_g\,. \tag{22}$$

Now, we proceed to the computation of each term involved in (16) of b_2. Let (X_i, U_j) be an orthonormal basis adapted to the foliation with X_i horizontal and U_j vertical. We introduce the notations from [2]

$$(A_X, A_Y) = \sum_i g(A_X X_i, A_Y X_i) = \sum_j g(A_X U_j, A_Y U_j)\,, \tag{23}$$

$$(TX, TY) = \sum_j g(T_{U_j}X, T_{U_J}Y). \tag{24}$$

Since $\mathcal{L}$ is assumed to be minimal, by Proposition 9.36 in [2], we have

$$\rho_\nabla(X,Y) = \rho(X,Y) + 2(A_X, A_Y) + (TX, TY), \tag{25}$$

Setting $f_i = \varphi e_i$, for $i \in \{1, \dots, n\}$, and $f_{n+1} = \xi$, we have

$$\|\rho_\nabla\|^2 = \sum_{i,j=1}^{n} \rho_\nabla(f_i, f_j)^2 + \rho_\nabla(\xi, \xi)^2 \tag{26}$$

and we obtain

$$\begin{aligned}\|\rho_\nabla\|^2 &= \sum_{i,j=1}^{n+1} \rho(f_i, f_j)^2 + 2\sum_{i,j=1}^{n+1} \rho(f_i, f_j)[2(A_{f_i}, A_{f_j}) + (Tf_i, Tf_j)] \\ &+ \sum_{i,j=1}^{n+1} [2(A_{f_i}, A_{f_j}) + (Tf_i, Tf_j)]^2\,.\end{aligned} \tag{27}$$

We easily see that

$$T_{e_i}\xi = 0, \quad T_{e_i}\varphi e_j = \varphi(T_{e_i}e_j), \quad A_{\varphi e_i}\varphi e_j = \varphi(A_{\varphi e_i}e_j), \quad A_\xi \varphi e_i = e_i,$$

$$2(A_\xi, A_\xi) + (T\xi, T\xi) = 2\sum_{i=1}^{n} (A_\xi \varphi e_i, A_\xi \varphi e_i) = 2n.$$

We also know from (7) that

$$\rho(\xi, \xi) = 2n$$

$$\sum_{i,j=1}^{n} \rho(f_i, f_j)^2 = \left[\frac{n(c+3) + c - 1}{2}\right]^2 n.$$

Setting

$$l = \sum_{i,j=1}^{n+1} \rho(f_i, f_j)^2,$$

we obtain:

$$\begin{aligned}\|\rho_\nabla\|^2 &= l + 2\textstyle\sum_{i,j} \frac{n(c+3)+c-1}{2}\delta_{ij}\left[2(A_{f_i}, A_{f_j}) + (Tf_i, Tf_j)\right] \\ &\quad +2(2n)^2 + \sum_{i,j=1}^{n+1}\left[2(A_{f_i}, A_{f_j}) + (Tf_i, Tf_j)\right]^2 \\ &= (l+8n^2) + (n(c+3)+c-1)\left[2\|A\|^2 + \|T\|^2\right] \\ &\quad + \sum_{i,j=1}^{n+1}\left[2(A_{f_i}, A_{f_j}) + (Tf_i, Tf_j)\right]^2 .\end{aligned} \tag{28}$$

Proposition 3.1. *If $\mathcal{L}$ is a Riemannian foliation on (M,g), then*

$$\sum_{i=1}^{n} R(X, e_i, Y, e_i) = \frac{1}{2}(g(\nabla^M_Y H, X) + g(\nabla^M_X H, Y)) + (A_X, A_Y) - (TX, TY), \tag{29}$$

where ∇^M is the Levi-Civita connection of (M,g), H is the mean curvature of the leaves, $(e_1, \ldots, e_n)$ is a local basis of the vertical distribution L, and X, Y are horizontal.

Proof: From the theory of Riemannian submersions, for any horizontal vectors X, Y and vertical vector U, we have

$$R(X,U,Y,U) \quad = \quad g((\nabla^M_X T)_U U, Y) - g(T_U X, T_U Y) + g((\nabla^M_U A)_X Y, U) + g(A_X U, A_Y U).$$

Therefore

$$\begin{aligned}\sum_{i=1}^{n} R(X, e_i, Y, e_i) &= \sum_{i=1}^{n}(g((\nabla^M_X T)_{e_i} e_i, Y) - g(T_{e_i} X, T_{e_i} Y) \\ &\quad + g((\nabla^M_{e_i} A)_X Y, e_i) + g(A_X e_i, A_Y e_i)).\end{aligned}$$

The covariant derivative of T satisfies

$$\begin{aligned}\sum_i g((\nabla^M_X T)_{e_i} e_i, Y) &= g(\nabla^M_X H, Y) - \sum_i g(T_{v(\nabla^M_X e_i)} e_i, Y) - \sum_i g(T_{e_i} v(\nabla^M_X e_i), Y) \\ &= g(\nabla^M_X H, Y) - 2\sum_i g(T_{e_i} v(\nabla^M_X e_i), Y)\,,\end{aligned}$$

since $T_U W = T_W U$ and $\sum_i T_{e_i} e_i = H$.
Setting

$$v(\nabla^M_X e_i) = \sum_j h_{ij} e_j$$

we note that

$$h_{ij} = g(v(\nabla^M_X e_i), e_j) = X(g(e_i, e_j)) - g(e_i, \nabla^M_X e_j) = -g(e_i, v(\nabla^M_X e_j)) = -h_{ji};$$

$$\sum_{i,j} g(T_{e_i} h_{ij} e_j, Y) = \sum_{i,j} h_{ij} g(T_{e_i} e_j, Y) = 0;$$

$$\sum_i g((\nabla^M_{e_i} A)_X Y, e_i) = \frac{1}{2}\left(g(\nabla^M_Y H, X) - g(\nabla^M_X H, Y)\right).$$

Therefore

$$\begin{aligned} \textstyle\sum_i R(X, e_i, Y, e_i) &= g(\nabla^M_X H, Y) - (TX, TY) + (A_X, A_Y) \\ &+ \tfrac{1}{2}(g(\nabla^M_Y H, X) - g(\nabla^M_X H, Y)), \end{aligned} \tag{30}$$

and (29) follows. ∎

From (30), one can obtain the following proposition.

Proposition 3.2. *If $\mathcal{L}$ is a Riemannian foliation with minimal leaves then*

$$\sum_{k=1}^{n} R(f_i, e_k, f_j, e_k) = (A_{f_i}, A_{f_j}) - (Tf_i, Tf_j). \tag{31}$$

Now, we consider a Riemannian Legendre foliation with minimal leaves on a Sasakian manifold M of constant φ-sectional curvature c and we fix a local orthonormal basis (e_i, f_i, f_{n+1}) adapted to the foliation, that is $f_{n+1} = \xi$ and $f_i = \varphi e_i$ for any $i \in \{1, \ldots, n\}$ and $\{e_1, \ldots, e_n\}$ is a local basis of a leaf L.

Setting

$$S(f_i, f_j) = \sum_{k=1}^{n} R(f_i, e_k, f_j, e_k)$$

and using relations (6), we obtain that

$$S(\varphi e_i, \xi) = \sum_{k=1}^{n} R(\varphi e_i, e_k, \xi, e_k) = 0,$$

$$S(\xi, \xi) = n, \quad S(f_i, f_j) = S(\varphi e_i, \varphi e_j) = d\delta_{ij},$$

where $d = \frac{c+3}{4} n + \frac{3(c-1)}{4}$.

By Proposition 2.3b) and equations (28) and (31), it follows:

$$\begin{aligned} \|\rho_\nabla\|^2 &= (l + 8n^2) + [n(c+3) + c - 1][3\|A\|^2 - c(n+1)] \\ &+ \sum_{i,j=1}^{n+1} [3(A_{f_i}, A_{f_j}) - S(f_i, f_j)]^2. \end{aligned} \tag{32}$$

Denoting by E the last sum of the previous relation, we have:

$$\begin{aligned} E &= \sum_{i=1}^{n}[3(A_{\varphi e_i}, A_{\varphi e_i}) - d]^2 + [2(A_\xi, A_\xi) + (T\xi, T\xi)]^2 + 2\sum_{i<j}^{n+1}[3(A_{f_i}, A_{f_j})]^2 \\ &= 9\sum_{i=1}^{n}(A_{\varphi e_i}, A_{\varphi e_i})^2 + d^2 n - 6d\sum_{i=1}^{n}(A_{\varphi e_i}, A_{\varphi e_i}) + 4n^2 + 18\sum_{i<j}^{n+1}(A_{f_i}, A_{f_j})^2. \end{aligned} \tag{33}$$

Summarizing, we conclude:

Proposition 3.3. *Under the hypothesis of Proposition 2.2, the following holds*

$$\begin{aligned} \|\rho_\nabla\|^2 &= 9\sum_{i=1}^{n}(A_{\varphi e_i}, A_{\varphi e_i})^2 + 18\sum_{i<j}^{n+1}(A_{f_i}, A_{f_j})^2 \\ &\quad + nd(d+6) - 6d\|A\|^2 + 16n^2 \\ &\quad + (n(c+3)+c-1)\left(3\|A\|^2 - c(n+1)\right) \\ &\quad + n\left(\frac{n(c+3)+c-1}{2}\right)^2, \end{aligned} \tag{34}$$

where $d = \frac{c+3}{4}n + \frac{3(c-1)}{4}$.

To compute $\|R_\nabla\|^2$, we notice that

$$\begin{aligned} R_\nabla(f_i, f_j, f_k, f_l) &= R(f_i, f_j, f_k, f_l) + 2g(A_{f_i}f_j, A_{f_k}f_l) \\ &\quad - g(A_{f_j}f_k, A_{f_i}f_l) - g(A_{f_k}f_i, A_{f_j}f_l)\,, \end{aligned}$$

we consider the following tensor of type $(0,4)$ associated to the horizontal distribution

$$V(X, Y, Z, Z') = 2g(A_X Y, A_Z Z') - g(A_Y Z, A_X Z') - g(A_Z X, A_Y Z')$$

and we set

$$\|V\|^2 = \sum_{i,j,k,l=1}^{n+1} V(f_i, f_j, f_k, f_l)^2.$$

We can write:

$$\begin{aligned} \|R_\nabla\|^2 &= \sum_{i,j,k,l=1}^{n+1}(R_\nabla(f_i, f_j, f_k, f_l))^2 \\ &= \sum_{i,j,k,l=1}^{n+1}(R(f_i, f_j, f_k, f_l))^2 + 2\sum_{i,j,k,l=1}^{n+1} R(f_i, f_j, f_k, f_l)\{2g(A_{f_i}f_j, A_{f_k}f_l) \\ &\quad - g(A_{f_j}f_k, A_{f_i}f_l) - g(A_{f_k}f_i, A_{f_j}f_l)\} + \|V\|^2\,. \end{aligned}$$

Then

$$
\begin{aligned}
l' &= \sum_{i,j,k,l=1}^{n+1} R(f_i, f_j, f_k, f_l)^2 \qquad (35)\\
&= \sum_{i,j,k,l=1}^{n} R(\varphi e_i, \varphi e_j, \varphi e_k, \varphi e_l)^2 + 4\sum_{i,k=1}^{n} R(\varphi e_i, \xi, \varphi e_k, \xi)^2\\
&= (\frac{c+3}{4})^2 \sum_{i,j,k,l=1}^{n} (\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl})^2 + 4\sum_{i,k} \delta_{ik}^2\\
&= \frac{(c+3)^2}{16}[2n^2 - 2\sum_{i,j}\delta_{ij}] + 4n = \frac{(c+3)^2(n-1)n}{8} + 4n\,,
\end{aligned}
$$

and thus,

$$
\begin{aligned}
\|R_\nabla\|^2 &= l' + 2\sum_{i,j,k,l=1}^{n} \frac{c+3}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})[2g(A_{\varphi e_i}\varphi e_j, A_{\varphi e_k}\varphi e_l)\\
&\quad -g(A_{\varphi e_j}\varphi e_k, A_{\varphi e_i}\varphi e_l) - g(A_{\varphi e_k}\varphi e_i, A_{\varphi e_j}\varphi e_l)]\\
&\quad +8\sum_{i,k} R(\varphi e_i, \xi, \varphi e_k, \xi)[2g(A_{\varphi e_i}\xi, A_{\varphi e_k}\xi) - g(A_\xi\varphi e_k, A_\xi\varphi e_i)]\\
&= l' + \|V\|^2 + 2\frac{c+3}{4}\sum_{i,j}[2g(A_{\varphi e_i}\varphi e_j, A_{\varphi e_i}\varphi e_j)\\
&\quad -g(A_{\varphi e_j}\varphi e_i, A_{\varphi e_i}\varphi e_j) - g(A_{\varphi e_i}\varphi e_i, A_{\varphi e_j}\varphi e_j)]\\
&\quad -2\frac{c+3}{4}\sum_{i,j}[2g(A_{\varphi e_i}\varphi e_j, A_{\varphi e_j}\varphi e_i)\\
&\quad -g(A_{\varphi e_j}\varphi e_j, A_{\varphi e_i}\varphi e_i) - g(A_{\varphi e_j}\varphi e_i, A_{\varphi e_j}\varphi e_i)] + 24n\\
&= l' + 24n - 3n(c+3) + 3(c+3)\|A\|^2 + \|V\|^2\,.
\end{aligned}
$$

Therefore

$$
\|R_\nabla\|^2 = \frac{(c+3)^2(n-1)n}{8} - 3n(c+3) + 28n + 3(c+3)\|A\|^2 + \|V\|^2\,. \qquad (36)
$$

On the other hand, we get

$$
\begin{aligned}
\textstyle\sum_j V(f_i, f_j, f_k, f_j) &= \textstyle\sum_j [2g(A_{f_i}f_j, A_{f_k}f_j) - g(A_{f_j}f_k, A_{f_i}f_j) - g(A_{f_k}f_i, A_{f_j}f_j)]\\
&= 3\textstyle\sum_j g(A_{f_i}f_j, A_{f_k}f_j) = 3(A_{f_i}, A_{f_k})\,,
\end{aligned}
$$

and thus

$$
(C_{24}V)(f_i, f_k) = 3(A_{f_i}, A_{f_k})\,, \qquad (37)
$$

where C_{24} denotes the contraction of the tensor with respect to the indices 2 and 4. The Hilbert-Schmidt norm of the $(0,2)$ tensor $C_{24}V$ along the horizontal distribution satisfies

$$\|C_{24}V\|^2 \quad = \quad 9\sum_{i=1}^{n}(A_{\varphi e_i}, A_{\varphi e_i})^2 + 18\sum_{i<j}^{n+1}(A_{f_i}, A_{f_j})^2 + 9n^2. \tag{38}$$

Summarizing, by Proposition 2.2, we obtain that

$$\begin{aligned} b_2 \quad &= \quad (n+1)a_2 + \frac{1}{12}\int_M 2\tau\tau_\nabla + 6\|\rho_\nabla\|^2 - \|R_\nabla\|^2 dv_g \qquad (39)\\ &= \quad (n+1)a_2 + \frac{1}{12}\int_M 6\|(C_{24}V)\|^2 - \|V\|^2 dv_g \\ &\quad + \frac{((c+3)(-3+12n+6n^2)+(c-1)(-9+3n))}{12}\int_M \|A\|^2 dv_g \\ &\quad + \frac{\mathrm{Vol_g(M)}}{12}\Big(42n^2 - 28n + (c+3)n(3+11n+4n^2) \\ &\quad + \frac{1}{8}(c+3)^2 n(-11-15n+13n^2+4n^3) \\ &\quad + (c-1)n(27+2n) + \frac{1}{8}(c-1)^2(-36+3n) \\ &\quad + \frac{1}{4}(c+3)(c-1)(-6-24n+2n^2+n^3)\Big). \end{aligned}$$

By (39) and Theorems 1.1, 1.2, 2.4, we now get our main result.

Theorem 3.4. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ and $(M_0^{2n_0+1}, \varphi_0, \xi_0, \eta_0, g_0)$ be compact isospectral Sasakian manifolds with constant φ-sectional curvature c and constant φ_0-sectional curvature c_0 respectively. If $\mathcal{L}$ and $\mathcal{L}_0$ are Riemannian minimal Legendre foliations on M and M_0 such that $Spec(\mathcal{L}, \mathcal{J}_\nabla) = Spec(\mathcal{L}_0, \mathcal{J}_{\nabla_0})$, then*

1) $\dim M = \dim M_0\,, \quad Vol(M) = Vol(M_0)\,, \quad c = c_0,$

2) $\int_M \|A\|^2 dv_g = \int_{M_0} \|A_0\|^2 dv_{g_0}\,, \quad \int_M \|T\|^2 dv_g = \int_{M_0} \|T_0\|^2 dv_{g_0},$

3) $\int_M [6\|C_{24}V\|^2 - \|V\|^2] dv_g = \int_{M_0} [6\|C_{24}V_0\|^2 - \|V_0\|^2] dv_{g_0}.$

4 Concluding remarks

Let $\mathcal{L}$ be a Riemannian Legendre foliation with totally geodesic leaves on (M,g) and assume that M has the constant curvature $c=1$ and that $2n+1 = \dim M$. In this particular case, we would like to point that the condition 3) of Theorem 3.4 is implied by 1). Indeed, for any Riemannian totally geodesic foliation $\mathcal{L}$ on a constant curvature space M with $\dim Q = \dim L + 1$, one can see that

$$g(A_Y W, A_Y W) = g(Y,Y)g(W,W), \text{ for any } W \in L \text{ and for any } Y \in Q,$$

$A_X : L \to Q^{\perp X} = \{Y \in Q \mid g(Y, X) = 0\}$ is a bijection for any unit vector X, and R_∇ has the constant curvature 4 (see the argument of [1, Prop. 4.6]). Thus, we simply have

$$\begin{aligned}(A_{\varphi e_i}, A_{\varphi e_i}) &= \sum_{k=1}^{n} g(\varphi e_i, \varphi e_i) g(e_k, e_k) = n, \\ (A_{f_i}, A_{f_j}) &= \sum_{k=1}^{n+1} g(f_i, f_j) g(e_k, e_k) = 0, \quad \text{for any } i < j,\end{aligned}$$

and therefore, by (38), $\|C_{24}V\|^2 = 18n^2$. We easily see that

$$\begin{aligned}\|V\|^2 &= \sum_{i,j,k,l=1}^{n+1} V(f_i, f_j, f_k, f_l)^2 = \sum_{i,j,k,l=1}^{n+1} (R_\nabla(f_i, f_j, f_k, f_l) - R(f_i, f_j, f_k, f_l))^2 \\ &= \sum_{i,j,k,l=1}^{n+1} [(4-1)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})]^2 = 18n^2 + 18n.\end{aligned}$$

This concludes that

$$\int_M [6\|C_{24}V\|^2 - \|V\|^2] dv_g = (90n^2 - 18n) Vol(M). \tag{40}$$

Example 1. Let $(S^3, \varphi, \xi, \eta, g)$ be the standard contact metric structure on S^3, which we now recall. Let $\mathbb{H} = \{x_1 + ix_2 + jx_3 + kx_4 \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be the algebra of quaternion numbers, where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$. The set of unit quaternions is identified with S^3. Let (I, J, K) be the quaternionic structure on $\mathbb{H}$ given for $I, J, K : \mathbb{H} \to \mathbb{H}$, by $I(h) = ih$, $J(h) = jh$, $K(h) = kh$ for any $h \in \mathbb{H}$. Let N be the unit outer normal vector field on S^3 and let g be the Riemann metric with constant curvature $c = 1$. We set $\xi = -IN$, η the dual form of ξ; $\varphi(Z)$ the projection of $I(Z)$ onto tangent space of S^3, for any vector field Z of S^3. Note that (φ, ξ, η, g) is the standard contact metric on S^3 and the its φ-sectional curvature is $c = 1$.

Let (x_1, x_2, x_3, x_4) be the Cartesian coordinate system on $\mathbb{R}^4 = \mathbb{H}$. It is easy to see that

$$\xi = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4}.$$

Setting $W = -JN$ and $Y = -KN$, we have

$$W = x_3 \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4};$$

$$Y = x_4 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_4}.$$

One can easily compute the Lie brackets between these vectors: $[W, \xi] = -2Y$, $[Y, \xi] = 2W$, $[W, Y] = 2\xi$ (see [10]). The distributions $L = \mathrm{span}\{W\}$ and $L' = \mathrm{span}\{Y\}$ define two non-degenerate Legendre foliations (see [10, Example 7.1]). Since $[W, Y] = 2\xi$ and $\varphi(W) = Y$, by [10, Lemma 6.6], L and L' are Riemannian Legendre foliations on the Sasakian space form S^3

and both of them are totally geodesic. By theorem 3.4, any Riemannian minimal Legendre foliations on a compact Sasakian space form M_0 with φ_0-sectional curvature c_0, isospectral to the foliation $\mathcal{L}$ (defined above) on the standard Sasakian space form $(S^3, \varphi, \xi, \eta)$ is totally geodesic, $c_0 = 1$, $\dim M_0 = 3$, and

$$\int_M [6\|C_{24}V\|^2 - \|V\|^2] dv_g = 72\mathrm{Vol}(M_0) = 72\mathrm{Vol}(M),$$

$$\int_M \|A\|^2 dv_g = 2Vol(M).$$

It is well known that a typical example of a Sasakian space form is a $\mathcal{D}$-homothetic deformation of the standard contact metric structure of an odd-dimensional sphere S^{2n+1}, which we now recall (see [4, Example 7.4.1]). For a contact metric structure (φ, ξ, η, g), one defines the $\mathcal{D}$-homothetic deformation $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ by

$$\bar{\varphi} = \varphi, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\eta} = a\eta, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta,$$

where a is a positive constant (see [4, p. 114]). By [4, Theorem 7.15], a compact simply connected Sasakian space form with φ-sectional curvature $c > -3$ is a $\mathcal{D}$-homothetic deformation of the standard contact metric structure on S^{2n+1} and $c = \frac{4}{a} - 3$ (for some $a > 0$). Since $\mathrm{Ker}\,\bar{\eta} = \mathrm{Ker}\,\eta$, the problem of finding Riemannian Legendre foliations on such a compact Sasakian space form (with $c > -3$) reduces to the one on the standard sphere S^{2n+1} (i.e. c=1).

Case $n = 1$. One can apply a $\mathcal{D}$-homothetic deformation to Example 1 to obtain an example for any $c > -3$.

Case $n = 2$. From [9], there are no Riemannian foliations with two-dimensional leaves on a standard sphere, which in particular means that there are no Riemannian Legendre foliations on S^5.

Case $n = 3$. By [9, Theorem 5.3], we get, in particular, that any Riemannian foliation with 3-dimensional leaves on S^7 is given uniquely (up to equivalence) by a direct sum of irreducible unitary representations of $SU(2)$, namely $\rho_1 \oplus \rho_1$, or by ρ_3, where ρ_k is the action of $SU(2)$ on the set of complex homogeneous polynomials in two variables of degree k.
Note that $\rho_1 \oplus \rho_1$ corresponds to the Hopf fibration $S^7 \to S^4$ (see [9]), which is not a Legendre foliation. In fact, no leaf of $\rho_1 \oplus \rho_1$ is Legendrian, simply because L, the tangent distribution of the leaves, is generated by $-\xi = IN$, JN, KN, where (I, J, K) is the standard quaternionic structure on $\mathbb{H}^2 = \mathbb{R}^8$ and N is the unit outer vector field to S^7 (see [6, p. 265]).
In [12, p. 365], Ohnita constructed a unique minimal Legendrian orbit on S^7 under the action of ρ_3, which means that only one leaf of the Riemannian foliation given by ρ_3 on S^7 is both minimal and Legendrian. This concludes that ρ_3 does not provide a Riemannian Legendre foliation with minimal leaves.

Finally, another typical example of a Riemannian Legendre foliation with totally geodesic leaves is given by the tangent sphere bundle $\pi : T_1P \to P$ of a Riemannian manifold (P, h). Assume that T_1P is endowed with the standard contact metric structure.

If $\dim P > 2$, then T_1P is never a Sasakian space form (see [5]). Note that if P has constant curvature, then T_1P admits a non-Sasakian contact metric structure of constant φ-sectional curvature c^2 if and only if $c = 2 \pm \sqrt{5}$ (see [4, Theorem 9.9]).

If $\dim P = 2$ and if T_1P is a Sasakian space form, then P has constant curvature $c = 1$ (see [4, Theorem 9.3]). Note that $T_1S^2 \simeq \mathbb{R}P^3$ (see [4, p. 142]) and the Riemannian Legendre foliation on the universal cover of T_1S^2 is equivalent to Example 1.

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"Simion Stoilow" Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700
Bucharest, Romania
E-mail: Gabriel.Baditoiu@imar.ro

Department of Mathematics, University of Bari "Aldo Moro", Italy
E-mail: pastore@dm.uniba.it