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Lax Pair Equations and Connes-Kreimer Renormalization

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Abstract: We find a Lax pair equation corresponding to the Connes-Kreimer Birkhoff factorization of the character group of a Hopf algebra. This flow preserves the locality of counterterms. In particular, we obtain a flow for the character given by Feynman rules, and relate this flow to the Renormalization Group Flow.

1. Introduction

In the theory of integrable systems, many classical mechanical systems are described by a Lax pair equation associated to a coadjoint orbit of a semisimple Lie group, for example via the Adler-Kostant-Symes theorem [1]. Solutions are given by a Birkhoff factorization on the group, and in some cases this technique extends to loop group formulations of physically interesting systems such as the Toda lattice [9,13]. By the work of Connes-Kreimer [2], there is a Birkhoff factorization of characters on general Hopf algebras, in particular on the Kreimer Hopf algebra of 1PI Feynman diagrams. In this paper, we reverse the usual procedure in integrable systems: we construct a Lax pair equation $\frac{dL}{dt} = [L, M]$ on the Lie algebra of infinitesimal characters of the Hopf algebra whose solution is given precisely by the Connes-Kreimer Birkhoff factorization (Theorem 5.9). The Lax pair equation is nontrivial in the sense that it is not an infinitesimal inner automorphism. The main technical issue, that the Lie algebra of infinitesimal characters is not semisimple, is overcome by passing to the double Lie algebra with the simplest possible Lie algebra structure. In particular, the Lax pair equation induces a flow for the character given by Feynman rules in dimensional regularization. This flow has the physical significance that it preserves locality, the independence of the character's counterterm from the mass parameter. The flow also induces Lax pair flows for the β-functions of characters.

In Sects. 1–4, we introduce a method to produce a Lax pair on any Lie algebra from equations of motion on the double Lie algebra. In Sect. 5, we apply this method to the particular case of the Lie algebra of infinitesimal characters of a Hopf algebra, and prove

Theorem 5.9. Although we focus on the minimal subtraction scheme for simplicity, the main results hold in any renormalization scheme.

In Sects. 6–8, we discuss physical implications of the Lax pair flow. These implications are of two types: results in Sect. 7 which say that local characters remain local under the flow, and results in Sect. 8 which compare the Lax pair flow to the Renormalization Group Flow (RGF) usually considered in quantum field theory.

As discussed in the beginning of Sect. 6, the RGF is a flow on the group $G_{\mathbb{C}}$ of scalar valued (i.e renormalized) characters on the Hopf algebra of Feynman diagrams, while the Lax pair flow is on the Lie algebra $\mathfrak{g}_{\mathcal{A}}$ of infinitesimal characters with values in Laurent series. There are several ways to compare the RGF to the Lax pair flow, all of which involve some identifications of the different spaces for the flows.

Working at the group level, we can consider an unrenormalized character φ (e.g. before dimensional regularization and minimal subtraction) as an element of the corresponding group $G_{\mathcal{A}}$. Manchon's bijection [12] $\tilde{R} : G_{\mathcal{A}} \to \mathfrak{g}_{\mathcal{A}}$ transfers φ to an infinitesimal character $\tilde{R}(\varphi)$; this bijection has better behavior than the logarithm map. This infinitesimal character has a Lax pair flow, which we can then transfer back to $G_{\mathcal{A}}$ to obtain a flow φ_t. Finally, we can compare the flow of the renormalized scalar valued characters $(\varphi_t)_+(0)$ in Connes-Kreimer notation to the RGF of $\varphi_+(0)$. However, even in simple examples these two flows are not the same.

Working at the infinitesimal level, we consider the β-function of a renormalized character, since the β-function is essentially the infinitesimal generator of the RGF. The β-function of a character is an element of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of scalar valued characters, so in Sect. 6 we extend the β-function to a β-character in $\mathfrak{g}_{\mathcal{A}}$. (This extension has previously appeared in the literature in a different context.) This material is used in the main results in Sects. 7–8.

In preparation for the study of the RGF, in Sect. 7 we discuss "how physical" the Lax pair flow is. We first show that the Lax pair flow is trivial on primitives in the Hopf algebra. We then prove the main result (Theorem 7.3) that local characters for the minimal subtraction scheme remain local under the Lax pair flow, so the flow stays inside the set of physically plausible characters. We also discuss the dependence of this result on the renormalization scheme, and identify characters for which the Lax pair flow is trivial.

From the results in Sect. 7, it seems unlikely that one can directly identify the RGF with a Lax pair flow, so in Sect. 8 we track how the RGF changes under the Lax pair flow. For example, even if the Lax pair flow is nontrivial for a given initial physically plausible character, one might hope that the RGF is unchanged. In Sect. 8, we give a criterion (Corollary 8.4) for when the RGF is fixed under the Lax pair flow. We show that the β-character and the β-function satisfy Lax pair equations, and briefly discuss the complete integrability of Lax pair flows of characters and β-functions.

We would like to thank Dirk Kreimer for suggesting we investigate the connection between the Connes-Kreimer factorization and integrable systems, Dominique Manchon for helpful conversations, and the referee for valuable suggestions.

2. The Double Lie Algebra and its Associated Lie Group

There is a well known method to associate a Lax pair equation to a Casimir element on the dual $\mathfrak{g}^*$ of a semisimple Lie algebra $\mathfrak{g}$ [13]. The semisimplicity is used to produce an Ad-invariant, symmetric, non-degenerate bilinear form on $\mathfrak{g}$, allowing an identification of $\mathfrak{g}$ with $\mathfrak{g}^*$. For a general Lie algebra $\mathfrak{g}$, there may be no such bilinear form. To produce a Lax pair, we need to extend $\mathfrak{g}$ to a larger Lie algebra with the desired bilinear form. We do this by constructing a Lie bialgebra structure on $\mathfrak{g}$, whose definition we now recall (see e.g. [10]).

Definition 2.1. *A Lie bialgebra is a Lie algebra* $(\mathfrak{g}, [\cdot, \cdot])$ *with a linear map* $\gamma : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ *such that*

(a) ${}^t\gamma : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ *defines a Lie bracket on* $\mathfrak{g}^*$*,*
(b) γ *is a* 1*-cocycle of* $\mathfrak{g}$*, i.e.*

$$\mathrm{ad}_x^{(2)}(\gamma(y)) - \mathrm{ad}_y^{(2)}(\gamma(x)) - \gamma([x, y]) = 0,$$

where $\mathrm{ad}_x^{(2)} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ *is given by* $\mathrm{ad}_x^{(2)}(y \otimes z) = \mathrm{ad}_x(y) \otimes z + y \otimes \mathrm{ad}_x(z) = [x, y] \otimes z + y \otimes [x, z]$.

A Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot], \gamma)$ *induces an Lie algebra structure on the* **double Lie algebra** $\mathfrak{g} \oplus \mathfrak{g}^*$ *by*

$$[X, Y]_{\mathfrak{g}\oplus\mathfrak{g}^*} = [X, Y],$$
$$[X^*, Y^*]_{\mathfrak{g}\oplus\mathfrak{g}^*} = {}^t\gamma(X \otimes Y),$$
$$[X, Y^*] = \mathrm{ad}_X^*(Y^*),$$

for $X, Y \in \mathfrak{g}$ *and* $X^*, Y^* \in \mathfrak{g}^*$*, where* ad^* *is the coadjoint representation given by* $\mathrm{ad}_X^*(Y^*)(Z) = -Y^*(\mathrm{ad}_X(Z))$ *for* $Z \in \mathfrak{g}$.

Since it is difficult to construct explicitly the Lie group associated to the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$, we will choose the trivial Lie bialgebra given by the cocycle $\gamma = 0$ and denote by $\delta = \mathfrak{g} \oplus \mathfrak{g}^*$ the associated Lie algebra. Let $\{Y_i, i = 1, \ldots, l\}$ be a basis of $\mathfrak{g}$, with dual basis $\{Y_i^*\}$. The Lie bracket $[\cdot, \cdot]_\delta$ on δ is given by

$$[Y_i, Y_j]_\delta = [Y_i, Y_j],\ [Y_i^*, Y_j^*]_\delta = 0,\ [Y_i, Y_j^*]_\delta = -\sum_k c_{ik}^j Y_k^*,$$

where the c_{ik}^j are the structure constants: $[Y_i, Y_j] = \sum_k c_{ij}^k Y_k$. The Lie group naturally associated to δ is given by the following proposition.

Proposition 2.2. *Let* G *be the simply connected Lie group with Lie algebra* $\mathfrak{g}$ *and let* $\theta : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ *be the coadjoint representation* $\theta(g, X) = \mathrm{Ad}_G^*(g)(X)$*. Then the Lie algebra of the semi-direct product* $\tilde{G} = G \ltimes_\theta \mathfrak{g}^*$ *is the double Lie algebra* δ.

Proof. The Lie group law on the semi-direct product $\tilde{G}$ is given by

$$(g, X) \cdot (g', X') = (gg', X + \theta(g, X')).$$

Let $\tilde{\mathfrak{g}}$ be the Lie algebra of $\tilde{G}$. Then the bracket on $\tilde{\mathfrak{g}}$ is given by

$$[X, Y^*]_{\tilde{\mathfrak{g}}} = d\theta(X, Y^*),\quad [X, Y]_{\tilde{\mathfrak{g}}} = [X, Y],\quad [X^*, Y^*]_{\tilde{\mathfrak{g}}} = 0,$$

for left-invariant vector fields X, Y of G and $X^*, Y^* \in \mathfrak{g}^*$. We have $d\theta(X, Y^*) = d\mathrm{Ad}_G^*(X)(Y^*) = [X, Y^*]_\delta$, since $d\mathrm{Ad}_G = \mathrm{ad}_{\mathfrak{g}}$. $\square$

The main point of this construction is existence of a good bilinear form on the double.

Lemma 2.3. *The natural pairing* $\langle\cdot,\cdot\rangle : \delta\otimes\delta \to \mathbb{C}$ *given by*

$$\langle (a,b^*),(c,d^*)\rangle = d^*(a)+b^*(c), \quad a,c\in\mathfrak{g}, \quad b^*,d^*\in\mathfrak{g}^*,$$

is an Ad-*invariant symmetric non-degenerate bilinear form on the Lie algebra* δ.

Proof. By [10], this bilinear form is ad-invariant. Since $\tilde{G}$ is simply connected, the Ad-invariance follows. As an explicit example, we have

$$\mathrm{Ad}_{\tilde{G}}((g,0))(Y_i,0) = (\mathrm{Ad}_G(g)(Y_i),0), \quad \text{and} \quad \mathrm{Ad}_{\tilde{G}}((g,0))(0,Y_j^*) = (0,\mathrm{Ad}_G^*(g)(Y_j^*)),$$

from which the invariance under $\mathrm{Ad}_{\tilde{\mathrm{G}}}(g,0)$ follows. □

3. The Loop Algebra of a Lie Algebra

Following [1], we consider the loop algebra

$$L\delta = \left\{ L(\lambda) = \sum_{j=M}^{N} \lambda^j L_j \mid M,N\in\mathbb{Z}, L_j\in\delta \right\}.$$

The natural Lie bracket on $L\delta$ is given by

$$\left[\sum \lambda^i L_i, \sum \lambda^j L_j'\right] = \sum_k \lambda^k \sum_{i+j=k} [L_i, L_j'].$$

Set

$$L\delta_+ = \left\{ L(\lambda) = \sum_{j=0}^{N} \lambda^j L_j \mid N\in\mathbb{Z}^+\cup\{0\}, L_j\in\delta \right\},$$
$$L\delta_- = \left\{ L(\lambda) = \sum_{j=-M}^{-1} \lambda^j L_j \mid M\in\mathbb{Z}^+, L_j\in\delta \right\}.$$

Let $P_+ : L\delta \to L\delta_+$ and $P_- : L\delta \to L\delta_-$ be the natural projections and set $R = P_+ - P_-$.

The natural pairing $\langle\cdot,\cdot\rangle$ on δ yields an Ad-invariant, symmetric, non-degenerate pairing on $L\delta$ by setting

$$\left\langle \sum_{i=M}^{N} \lambda^i L_i, \sum_{j=M'}^{N'} \lambda^j L_j' \right\rangle = \sum_{i+j=-1} \langle L_i, L_j'\rangle.$$

For our choice of basis $\{Y_i\}$ of $\mathfrak{g}$, we get an isomorphism

$$I : L(\delta^*) \to L\delta \tag{3.1}$$

with

$$I\left(\sum L_i^j Y_j \lambda^i\right) = \sum L_i^j Y_j^* \lambda^{-1-i}.$$

We will need the following lemmas.

Lemma 3.1 [1]. *We have the following natural identifications*:

$$L\delta_+ = L(\delta^*)_- \quad \text{and} \quad L\delta_- = L(\delta^*)_+.$$

Lemma 3.2 [13, Lem. 4.1]. *Let φ be an* Ad-*invariant polynomial on δ. Then*

$$\varphi_{m,n}[L(\lambda)] = \mathrm{Res}_{\lambda=0}(\lambda^{-n}\varphi(\lambda^m L(\lambda)))$$

is an Ad-*invariant polynomial on $L\delta$ for $m, n \in \mathbb{Z}$.*

As a double Lie algebra, δ has an Ad-invariant polynomial, the quadratic polynomial

$$\psi(Y) = \langle Y, Y\rangle$$

associated to the natural pairing. Let $Y_{l+i} = Y_i^*$ for $i \in \{1, \dots, l = \dim(\mathfrak{g})\}$, so elements of $L\delta$ can be written $L(\lambda) = \sum_{j=1}^{2l}\sum_{i=-M}^{N} L_i^j Y_j \lambda^i$. Then the Ad-invariant polynomials

$$\psi_{m,n}(L(\lambda)) = \mathrm{Res}_{\lambda=0}(\lambda^{-n}\psi(\lambda^m L(\lambda))), \tag{3.2}$$

defined as in Lemma 3.2 are given by

$$\psi_{m,n}(L(\lambda)) = 2\sum_{j=1}^{l}\sum_{i+k-n+2m=-1} L_i^j L_k^{j+l}. \tag{3.3}$$

Note that powers of ψ are also Ad-invariant polynomials on δ, so

$$\psi_{m,n}^k(L(\lambda)) = \mathrm{Res}_{\lambda=0}(\lambda^{-n}\psi^k(\lambda^m L(\lambda))) \tag{3.4}$$

are Ad-invariant polynomials on $L\delta$. It would be interesting to classify all Ad-invariant polynomials on $L\delta$ in general.

4. The Lax Pair Equation

Let P_+, P_- be endomorphisms of a Lie algebra $\mathfrak{h}$ and set $R = P_+ - P_-$. Assume that

$$[X, Y]_R = [P_+X, P_+Y] - [P_-X, P_-Y]$$

is a Lie bracket on $\mathfrak{h}$. From [13, Theorem 2.1], the equations of motion induced by a Casimir (i.e. Ad-invariant) function φ on $\mathfrak{h}^*$ are given by

$$\frac{dL}{dt} = -\mathrm{ad}_{\mathfrak{h}}^* M \cdot L, \tag{4.1}$$

for $L \in \mathfrak{h}^*$, where $M = \frac{1}{2}R(d\varphi(L)) \in \mathfrak{h}$.

Now we take $\mathfrak{h} = (L\delta)^* = L(\delta^*)$, with δ a finite dimensional Lie algebra and with the understanding that $(L\delta)^*$ is the graded dual with respect to the standard $\mathbb{Z}$-grading on $L\delta$. Let $P_\pm$ be the projections of $L\delta^*$ onto $L\delta^*_\pm$. After identifying $L\delta^* = L\delta$ and $\mathrm{ad}^* = -\mathrm{ad}$ via the map I in (3.1), the equations of motion (4.1) can be written in Lax pair form

$$\frac{dL}{dt} = [M, L], \tag{4.2}$$

where $M = \frac{1}{2}R(I(d\varphi(L(\lambda)))) \in L\delta$, and φ is a Casimir function on $L\delta^* = L\delta$ [13, Th. 2.1]. Finding a solution for (4.2) reduces to the Riemann-Hilbert (or Birkhoff) factorization problem. The following theorem is a corollary of [1, Th. 4.37] [13, Th. 2.2].

Theorem 4.1. *Let φ be a Casimir function on $L\delta$ and set $X = I(d\varphi(L(\lambda))) \in L\delta$, for $L(\lambda) = L(0)(\lambda) \in L\delta$. Let $g_{\pm}(t)$ be the smooth curves in $L\tilde{G}$ which solve the factorization problem*

$$\exp(-tX) = g_{-}(t)^{-1} g_{+}(t),$$

with $g_{\pm}(0) = e$, and with $g_{+}(t) = g_{+}(t)(\lambda)$ holomorphic in $\lambda \in \mathbb{C}$ and $g_{-}(t)$ a polynomial in $1/\lambda$ with no constant term. Let $M = \frac{1}{2} R(I(d\varphi(L(\lambda)))) \in L\delta$. Then the integral curve $L(t)$ of the Lax pair equation

$$\frac{dL}{dt} = [L, M]$$

is given by

$$L(t) = \mathrm{Ad}_{L\tilde{G}} g_{\pm}(t) \cdot L(0). \tag{4.3}$$

This Lax pair equation projects to a Lax pair equation on the loop algebra of the original Lie algebra $\mathfrak{g}$. Let π_1 be either the projection of $\tilde{G}$ onto G or its differential from δ onto $\mathfrak{g}$. This extends to a projection of $L\delta$ onto $L\mathfrak{g}$. The projection of (4.2) onto $L\mathfrak{g}$ is

$$\frac{d(\pi_1(L(t)))}{dt} = [\pi_1(L), \pi_1(M)], \tag{4.4}$$

since $\pi_1 = d\pi_1$ commutes with the bracket. Thus the equations of motion (4.2) induce a Lax pair equation on $L\mathfrak{g}$, although this is not the equations of motion for a Casimir on $L\mathfrak{g}$.

Theorem 4.2. *The Lax pair equation of Theorem 4.1 projects to a Lax pair equation on $L\mathfrak{g}$.*

Remark 4.3. The content of this theorem is that a Lax pair equation on the Lie algebra of a semi-direct product $G \ltimes G'$ evolves on an adjoint orbit, and the projection onto $\mathfrak{g}$ evolves on an adjoint orbit and is still in Lax pair form. Lax pair equations often appear as equations of motion for some Hamiltonian, but the projection may not be the equations of motion for any function on the smaller Lie algebra. We thank B. Khesin for this observation.

When $\psi_{m,n}$ is the Casimir function on $L\delta$ given by (3.2), X can be written nicely in terms of $L(\lambda)$.

Proposition 4.4. *Let $X = I(d\psi_{m,n}(L(\lambda)))$. Then*

$$X = 2\lambda^{-n+2m} L(\lambda). \tag{4.5}$$

Proof. Write $L(\lambda) = \sum_{i,j} L_i^j \lambda^i Y_j$. By formula (3.3), we have

$$\frac{\partial \psi_{m,n}}{\partial L_p^t} = \begin{cases} 2L_{n-1-2m-p}^{t+l}, & \text{if } t \le l \\ 2L_{n-1-2m-p}^{t-l}, & \text{if } t > l. \end{cases} \tag{4.6}$$

Therefore

$$\begin{aligned}
X &= I(d\psi_{m,n}(L(\lambda))) = \sum_{p,t} \frac{\partial \psi_{m,n}}{\partial L_p^t} \lambda^{-1-p} Y_t^* \\
&= 2\lambda^{-n+2m} \sum_p \left(\sum_{t=1}^{l} L_{n-1-2m-p}^{t+l} Y_{t+l} \lambda^{n-1-2m-p} \right. \\
&\quad \left. + \sum_{t=l+1}^{2l} L_{n-1-2m-p}^{t-l} Y_{t-l} \lambda^{n-1-2m-p} \right) \\
&= 2\lambda^{-n+2m} L(\lambda).
\end{aligned} \tag{4.7}$$

□

5. The Main Theorem for Hopf Algebras

In this section we give formulas for the Birkhoff decomposition of a loop in the Lie group of characters of a Hopf algebra and produce the Lax pair equations associated to the Birkhoff decomposition. We present two approaches, both motivated by the Connes-Kreimer Hopf algebra of 1PI Feynman graphs. First, in analogy to truncating Feynman integral calculations at a certain loop level, we truncate a (possibly infinitely generated) Hopf algebra to a finitely generated Hopf algebra, and solve Lax pair equations on the finite dimensional piece (Theorem 5.4). We also discuss the compatibility of solutions related to different truncations. Second, we solve a Lax pair equation associated to Ad-covariant maps on the full Hopf algebra (Theorem 5.9). These results are proven for the minimal subtraction scheme, but apply to other renormalization schemes.

In Sect. 5.1, we introduce notation and prove a Birkhoff decomposition for the loop group associated to a doubled Lie algebra. In Sect. 5.2, we introduce the truncation process and prove Theorem 5.4. In Sect. 5.3, we treat the Feynman rules character and prove Theorem 5.9. In Sect. 5.4, we note that the methods of this section apply to any renormalization scheme given by a linear projection satisfying the Rota-Baxter equation.

5.1. Birkhoff decompositions for doubled Lie algebras. Let $\mathcal{H} = (\mathcal{H}, 1, \mu, \Delta, \varepsilon, S)$ be a graded connected Hopf algebra over $\mathbb{C}$. Let $\mathcal{A}$ be a unital commutative algebra with unit $1_{\mathcal{A}}$. Unless stated otherwise, $\mathcal{A}$ will be the algebra of Laurent series; the only other occurrence in this paper is $\mathcal{A} = \mathbb{C}$.

Definition 5.1. *The* **character group** $G_{\mathcal{A}}$ *of the Hopf algebra* $\mathcal{H}$ *is the set of algebra morphisms* $\phi : \mathcal{H} \to \mathcal{A}$ *with* $\phi(1) = 1_{\mathcal{A}}$. *The group law is given by the convolution product*

$$(\psi_1 \star \psi_2)(h) = \langle \psi_1 \otimes \psi_2, \Delta h \rangle;$$

the unit element is ε.

Definition 5.2. *An* $\mathcal{A}$*-valued* **infinitesimal character** *of a Hopf algebra* $\mathcal{H}$ *is a* $\mathbb{C}$*-linear map* $Z : \mathcal{H} \to \mathcal{A}$ *satisfying*

$$\langle Z, hk \rangle = \langle Z, h \rangle \varepsilon(k) + \varepsilon(h) \langle Z, k \rangle.$$

The set of infinitesimal characters is denoted by $\mathfrak{g}_{\mathcal{A}}$ *and is endowed with a Lie algebra bracket:*

$$[Z, Z'] = Z \star Z' - Z' \star Z, \quad \text{for } Z,\ Z' \in \mathfrak{g}_{\mathcal{A}},$$

where $\langle Z \star Z', h\rangle = \langle Z \otimes Z', \Delta(h)\rangle$. *Notice that* $Z(1) = 0$.

For a finitely generated Hopf algebra, $G_{\mathbb{C}}$ is a Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and for any Hopf algebra and any $\mathcal{A}$, the same is true at least formally.

We recall that $\delta = \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}^*$ is the double of $\mathfrak{g}_{\mathbb{C}}$ and the $\mathfrak{g}_{\mathbb{C}}^*$ is the graded dual of $\mathfrak{g}_{\mathbb{C}}$. We consider the algebra $\Omega\delta = \delta \otimes \mathcal{A}$ of formal Laurent series with values in δ

$$\Omega\delta = \left\{ L(\lambda) = \sum_{j=-N}^{\infty} \lambda^j L_j \mid L_j \in \delta, N \in \mathbb{Z} \right\}.$$

The natural Lie bracket on $\Omega\delta$ is

$$\left[\sum \lambda^i L_i, \sum \lambda^j L'_j\right] = \sum_k \lambda^k \sum_{i+j=k} [L_i, L'_j].$$

Set

$$\Omega\delta_+ = \left\{ L(\lambda) = \sum_{j=0}^{\infty} \lambda^j L_j \mid L_j \in \delta \right\},$$

$$\Omega\delta_- = \left\{ L(\lambda) = \sum_{j=-N}^{-1} \lambda^j L_j \mid L_j \in \delta, N \in \mathbb{Z}^+ \right\}.$$

Recall that for any Lie group K, a loop $L(\lambda)$ with values in K has a Birkhoff decomposition if $L(\lambda) = L(\lambda)_-^{-1} L(\lambda)_+$ with $L(\lambda)_-^{-1}$ holomorphic in $\lambda^{-1} \in \mathbb{P}^1 - \{0\}$ and $L(\lambda)_+$ holomorphic in $\lambda \in \mathbb{P}^1 - \{\infty\}$. In the next lemma, $\tilde{G}$ refers to $G \ltimes_\theta \mathfrak{g}^*$ as in Prop. 2.2.

We prove the existence of a Birkhoff decomposition for any element $(g, \alpha) \in \Omega\tilde{G}$.

Theorem 5.3. *Every* $(g, \alpha) \in \Omega\tilde{G} = G_{\mathcal{A}} \ltimes_{Ad^*_{G_{\mathcal{A}}}} \mathfrak{g}^*_{\mathcal{A}}$ *has a Birkhoff decomposition* $(g, \alpha) = (g_-, \alpha_-)^{-1}(g_+, \alpha_+)$ *with* (g_+, α_+) *holomorphic in* λ *and* (g_-, α_-) *a polynomial in* λ^{-1} *without constant term.*

Proof. We recall that $(g_1, \alpha_1)(g_2, \alpha_2) = (g_1 g_2, \alpha_1 + \mathrm{Ad}^*(g_1)(\alpha_2))$. Thus $(g, \alpha) = (g_-, \alpha_-)^{-1}(g_+, \alpha_+)$ if and only if $g = g_-^{-1} g_+$ and $\alpha = \mathrm{Ad}^*(g_-^{-1})(-\alpha_- + \alpha_+)$. Let $g = g_-^{-1} g_+$ be the Birkhoff decomposition of g in $G_{\mathcal{A}}$ given in [2,6,12]. Set $\alpha_+ = P_+(\mathrm{Ad}^*(g_-)(\alpha))$ and $\alpha_- = -P_-(\mathrm{Ad}^*(g_-)(\alpha))$, where P_+ and P_- are the holomorphic and pole part, respectively. Then for this choice of α_+ and α_-, we have $(g, \alpha) = (g_-, \alpha_-)^{-1}(g_+, \alpha_+)$. Note that the Birkhoff decomposition is unique. □

5.2. Lax pair equations for the truncated Lie algebra of infinitesimal characters. For a finitely generated Hopf algebra, we can apply Theorems 4.1, 4.2 to produce a Lax pair equation on $L\delta$ and on the loop space of infinitesimal characters $L\mathfrak{g}$. However, the common Hopf algebras of 1PI Feynman diagrams and rooted trees are not finitely generated.

As we now explain, we can truncate the Hopf algebra to a finitely generated Hopf algebra, and use the Birkhoff decomposition to solve a Lax pair equation on the infinitesimal character group of the truncation. A graded Hopf algebra $\mathcal{H} = \oplus_{n\in\mathbb{N}}\mathcal{H}_n$ is said to be of **finite type** if each homogeneous component $\mathcal{H}_n$ is a finite dimensional vector space. Let $\mathcal{B} = \{T_i\}_{i\in\mathbb{N}}$ be a minimal set of homogeneous generators of the Hopf algebra H such that $\deg(T_i) \le \deg(T_j)$ if $i < j$ and such that $T_0 = 1$. For $i > 0$, we define the $\mathbb{C}$-valued infinitesimal character Z_i on generators by $Z_i(T_j) = \delta_{ij}$. The Lie algebra of infinitesimal characters $\mathfrak{g}$ is a graded Lie algebra generated by $\{Z_i\}_{i>0}$. Let $\mathfrak{g}^{(k)}$ be the vector space generated by $\{Z_i \mid \deg(T_i) \le k\}$. We define $\deg(Z_i) = \deg(T_i)$ and set

$$[Z_i, Z_j]_{\mathfrak{g}^{(k)}} = \begin{cases} [Z_i, Z_j] & \text{if } \deg(Z_i) + \deg(Z_j) \le k \\ 0 & \text{if } \deg(Z_i) + \deg(Z_j) > k \end{cases}$$

We identify $\varphi \in G_{\mathbb{C}}$ with $\{\varphi(T_i)\} \in \mathbb{C}^{\mathbb{N}}$ and on $\mathbb{C}^{\mathbb{N}}$ we set a group law given by $\{\varphi_1(T_i)\} \oplus \{\varphi_2(T_i)\} = \{(\varphi_1 \star \varphi_2)(T_i)\}$. $G^{(k)} = \{\{\varphi(T_i)\}_{\{i \mid \deg(T_i)\le k\}} \mid \varphi \in G_{\mathbb{C}}\}$ is a finite dimensional Lie subgroup of $G_{\mathbb{C}} = (\mathbb{C}^{\mathbb{N}}, \oplus)$ and the Lie algebra of $G^{(k)}$ is $\mathfrak{g}^{(k)}$. There is no loss of information under this identification, as $\varphi(T_iT_j) = \varphi(T_i)\varphi(T_j)$.

Let $\delta^{(k)}$ be the double Lie algebra of $\mathfrak{g}^{(k)}$ and let $\tilde{G}^{(k)}$ be the simply connected Lie group with $\mathrm{Lie}(\tilde{G}^{(k)}) = \delta^{(k)}$ as in Proposition 2.2. The following theorem is a restatement of Theorem 4.1 in our new setup.

Theorem 5.4. *Let $\mathcal{H} = \oplus_n\mathcal{H}_n$ be a graded connected Hopf algebra of finite type, and let $\psi : L\delta^{(k)} \to \mathbb{C}$ be a Casimir function (e.g. $\psi(L) = \psi_{m,n}(L(\lambda)) = \mathrm{Res}_{\lambda=0}$ $(\lambda^m\psi(\lambda^nL(\lambda)))$ with $\psi : \delta^{(k)} \times \delta^{(k)} \to \mathbb{C}$ the natural paring of $\delta^{(k)}$). Set $X = I(d\psi(L_0))$ for $L_0 \in L\delta^{(k)}$. Then the solution in $L\delta^{(k)}$ of*

$$\frac{dL}{dt} = [L, M]_{L\delta^{(k)}}, \quad M = \frac{1}{2}R(I(d\psi(L))) \tag{5.1}$$

with initial condition $L(0) = L_0$ is given by

$$L(t) = \mathrm{Ad}_{L\tilde{G}^{(k)}}\, g_{\pm}(t) \cdot L_0, \tag{5.2}$$

where $\exp(-tX)$ *has the Connes-Kreimer Birkhoff factorization*
$\exp(-tX) = g_-(t)^{-1}g_+(t)$.

Remark 5.5. (i) If $L_0 \in L\delta$, there exists $k \in \mathbb{N}$ such that $L_0 \in L\delta^{(k)}$. Indeed $L_0 \in L\delta$ is generated over $\mathbb{C}[\lambda, \lambda^{-1}]$ by a finite number of $\{Z_i\}$, and we can choose $k \ge \max\{\deg(Z_i)\}$.

(ii) While the Hopf algebra of rooted trees and the Connes-Kreimer Hopf algebra of 1PI Feynman diagrams satisfy the hypothesis of Theorem 5.4, the Feynman rules character does not lie in $L\tilde{G}$, as explained below.

In the next sections, we will investigate the relationship between the Lax pair flow $L(t)$ and the Renormalization Group Equation. In preparation, we project from $L\delta^{(k)}$ to $L\mathfrak{g}^{(k)}$ via π_1 as in Sect. 4.

Corollary 5.6. *Let ψ be a Casimir function on $L\delta^{(k)}$. Set $L_0 \in L\mathfrak{g}^{(k)} \subset L\delta^{(k)}$, $X = \pi_1(I(d\psi(L_0)))$. Then the solution of the following equation in $L\mathfrak{g}^{(k)}$*

$$\frac{dL}{dt} = [L, M_1]_{L\mathfrak{g}^{(k)}}, \quad M_1 = \pi_1\left(\frac{1}{2}R(I(d\psi(L)))\right) \tag{5.3}$$

with initial condition $L(0) = L_0$ is given by

$$L(t) = \mathrm{Ad}_{LG^{(k)}} g_\pm(t) \cdot L_0, \tag{5.4}$$

where $\exp(-tX)$ has the Connes-Kreimer Birkhoff factorization in $L\mathfrak{g}^{(k)}$

$$\exp(-tX) = g_-(t)^{-1} g_+(t).$$

Remark 5.7. (i) For Feynman graphs, this truncation corresponds to halting calculations after a certain loop level. From our point of view, this truncation is somewhat crude. $\mathfrak{g}^{(k)}$ is not a subalgebra of $\mathfrak{g}$, and if $k < \ell$, $\mathfrak{g}^{(k)}$ is not a subalgebra of $\mathfrak{g}^{(\ell)}$. Although the Casimirs $\psi_{m,n}$ and the exponential map restrict well from $\mathfrak{g}$ to $\mathfrak{g}^{(k)}$, the Birkhoff decomposition $\exp(-tX)$ of $X \in L\mathfrak{g}^{(k)}$ is very different from the Birkhoff decompositions in $L\mathfrak{g}$, $L\mathfrak{g}^{(\ell)}$. In fact, if $g \in G^{(k)}$ has Birkhoff decomposition $g = g_-^{-1} g_+$ in G, there does not seem to be $f(k) \in \mathbb{N}$ such that $g_\pm \in G^{(f(k))}$.

(ii) It would interesting to know, especially for the Hopf algebras of Feynman graphs or rooted trees, whether there exists a larger connected graded Hopf algebra $\mathcal{H}'$ containing $\mathcal{H}$ such that the associated infinitesimal Lie algebra $\mathrm{Lie}(G'_{\mathbb{C}})$ is the double δ. This would provide a Lax pair equation associated to an equation of motion on the infinitesimal Lie algebra of $\mathcal{H}'$. The most natural candidate, the Drinfeld double $\mathcal{D}(\mathcal{H})$ of $\mathcal{H}$, does not work since the dimension of the Lie algebra associated to $\mathcal{D}(\mathcal{H})$ is larger than the dimension of δ.

5.3. Lax pair equations in the general case. In [2], Connes and Kreimer give a Birkhoff decomposition for the character group of the Hopf algebra of 1PI graphs, and in particular for the Feynman rules character $\varphi(\lambda)$ given by minimal subtraction and dimensional regularization. The truncation process treated above does not handle the Feynman rules character, as the regularized toy model character defined in [5,7,11] of the Hopf algebra of integer decorated rooted trees and the Feynman rules character are not polynomials in λ, λ^{-1}, but Laurent series in λ. Thus Corollary 5.6 does not apply, as in our notation $\log(\varphi(\lambda)) \in \Omega\mathfrak{g}\backslash L\mathfrak{g}$. This and Remark 5.7(i) force us to consider a direct approach in $\Omega\mathfrak{g}$ as in the next theorem. However, we cannot expect that the Lax pair equation is associated to any Hamiltonian equation, and we replace Casimirs with Ad-covariant functions.

Definition 5.8 [14]. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. A map $f : \mathfrak{g} \to \mathfrak{g}$ is* Ad*-covariant if* $\mathrm{Ad}(g)(f(L)) = f(\mathrm{Ad}(g)(L))$ *for all $g \in G$, $L \in \mathfrak{g}$.*

Theorem 5.9. *Let $\mathcal{H}$ be a connected graded commutative Hopf algebra with $\mathfrak{g}_{\mathcal{A}}$ the associated Lie algebra of infinitesimal characters with values in Laurent series. Let $f : \mathfrak{g}_{\mathcal{A}} \to \mathfrak{g}_{\mathcal{A}}$ be an* Ad*-covariant map. Let $L_0 \in \mathfrak{g}_{\mathcal{A}}$ satisfy $[f(L_0), L_0] = 0$. Set $X = f(L_0)$. Then the solution of*

$$\frac{dL}{dt} = [L, M], \quad M = \frac{1}{2}R(f(L)) \tag{5.5}$$

with initial condition $L(0) = L_0$ *is given by*

$$L(t) = \mathrm{Ad}_G g_\pm(t) \cdot L_0, \tag{5.6}$$

where $\exp(-tX)$ *has the Connes-Kreimer Birkhoff factorization* $\exp(-tX) = g_-(t)^{-1} g_+(t)$.

Proof. The proof is similar to [13, Theorem 2.2]. First notice that

$$\begin{aligned}\frac{d}{dt}\left(\mathrm{Ad}(g_-(t)^{-1}g_+(t)) \cdot L_0\right) &= \frac{d}{dt}(\exp(-tX)L_0 \exp(tX)) \\ &= -\exp(-tX)XL_0 \exp(tX) + \exp(-tX)L_0X \exp(tX) \\ &= -\exp(-tX)[X, L_0] \exp(tX) = 0,\end{aligned}$$

which implies $\mathrm{Ad}(g_-(t)^{-1}g_+(t)) \cdot L_0 = L_0$ and $\mathrm{Ad}(g_-(t)) \cdot L_0 = \mathrm{Ad}(g_+(t)) \cdot L_0$. Set $L(t) = \mathrm{Ad}(g_\pm(t)) \cdot L_0 = g_\pm(t)L_0 g_\pm(t)^{-1}$. As usual,

$$\frac{dL}{dt} = \left[\frac{dg_\pm(t)}{dt} g_\pm(t)^{-1}, L(t)\right],$$

so

$$\frac{dL}{dt} = \frac{1}{2}\left[\frac{dg_+(t)}{dt} g_+(t)^{-1} + \frac{dg_-(t)}{dt} g_-(t)^{-1}, L(t)\right].$$

The Birkhoff factorization $g_+(t) = g_-(t) \exp(-tX)$ gives

$$\frac{dg_+(t)}{dt} = \frac{dg_-(t)}{dt} \exp(-tX) + g_-(t)(-X) \exp(-tX),$$

and so

$$\frac{dg_+(t)}{dt} g_+(t)^{-1} = \frac{dg_-(t)}{dt} g_-(t)^{-1} + g_-(t)(-X)g_-(t)^{-1}.$$

Thus

$$\begin{aligned}2M &= R(f(L(t))) = R(f(\mathrm{Ad}(g_-(t)) \cdot L_0)) = R(\mathrm{Ad}(g_-(t)) \cdot f(L_0)) \\ &= R(\mathrm{Ad}(g_-(t)) \cdot X)) = -R(\frac{dg_+(t)}{dt} g_+(t)^{-1}) + R(\frac{dg_-(t)}{dt} g_-(t)^{-1}) \\ &= -\frac{dg_+(t)}{dt} g_+(t)^{-1} - \frac{dg_-(t)}{dt} g_-(t)^{-1}.\end{aligned}$$

Here we use $\left(\frac{dg_\pm(t)}{dt} g_\pm(t)^{-1}\right)(x) \in \mathcal{A}_\pm$ for $x \in \mathcal{H}$. Thus $\frac{dL}{dt} = [L, M]$. □

Remark 5.10. In a particular case of Theorem 5.9, we get a Hamiltonian system. First, since the proof of Theorem 5.9 depends on the splitting $\mathfrak{g}_\mathcal{A} = \mathfrak{g}_{\mathcal{A}_-} \oplus \mathfrak{g}_{\mathcal{A}_+}$ of the Lie algebra and the Birkhoff decomposition of the Lie group, and since this splitting and Birkhoff decomposition exist (see Theorem 5.3) on $L\delta^k$, Theorem 5.9 extends to $L\delta^k$. Let $\psi : L\delta^k \to \mathbb{C}$ be a Casimir function on $L\delta^k$. For $f(L) = (\nabla\psi)(L)$ the gradient of ψ, f is an Ad-invariant function on $L\delta^k$. Since ψ is a Casimir function, $[L, (\nabla\psi)(L)] = 0$ for all L (see [1,14]), so in particular $[f(L_0), L_0] = 0$. Thus, the Hamiltonian system (5.5) satisfies (5.6). Since $I(d\psi) = \nabla\psi$, Theorem 5.4 is a particular case of the $L\delta^k$ version of Theorem 5.9. It is natural to ask if the Hamiltonian system (5.1) is completely integrable; this is discussed briefly in Sect. 8.

If $f : \mathfrak{g}_{\mathcal{A}} \to \mathfrak{g}_{\mathcal{A}}$ is given by $f(L) = 2\lambda^{-n+2m}L$, then f is Ad-covariant and $[f(L_0), L_0] = [2\lambda^{-n+2m}L_0, L_0] = 0$.

Corollary 5.11. *Let $\mathcal{H}$ be a connected graded commutative Hopf algebra with $\mathfrak{g}_{\mathcal{A}}$ the Lie algebra of infinitesimal characters with values in Laurent series. Pick $L_0 \in \mathfrak{g}_{\mathcal{A}}$ and set $X = 2\lambda^{-n+2m}L_0$. Then the solution of*

$$\frac{dL}{dt} = [L, M], \quad M = R(\lambda^{-n+2m}L) \tag{5.7}$$

with initial condition $L(0) = L_0$ is given by

$$L(t) = \mathrm{Ad}_{G_{\mathcal{A}}} g_{\pm}(t) \cdot L_0, \tag{5.8}$$

where $\exp(-tX)$ *has the Connes-Kreimer Birkhoff factorization* $\exp(-tX) = g_-(t)^{-1}g_+(t)$.

Remark 5.12. Let φ be the Feynman rules character. We can find the Birkhoff factorization of φ itself within this framework by adjusting the initial condition. Namely, set $L_0(\lambda) = \frac{1}{2}\lambda^{n-2m}\exp^{-1}(\varphi(\lambda))$. Then $\exp(X) = \varphi$ by Prop. 4.4, so the solution of (5.7) involves the Birkhoff factorization $\varphi = g_-(-1)^{-1}g_+(-1)$. Namely, we have

$$L(-1) = \frac{\lambda^{n-2m}}{2}\mathrm{Ad}_{G_{\mathcal{A}}} g_{\pm}(-1)\exp^{-1}(\varphi).$$

5.4. Other renormalization schemes. Although the renormalization scheme considered so far is the minimal subtraction scheme, the results on Lax pair equations are valid for any suitably defined renormalization scheme. Roughly speaking, different renormalization schemes correspond to different splittings of $\mathcal{A}$, as we now briefly explain. However, for simplicity in most sections we will treat only the minimal subtraction scheme.

Let $\mathcal{A}$ be the algebra of Laurent series. Let $\pi : \mathcal{A} \to \mathcal{A}$ be a Rota-Baxter map [6], which by definition is a linear map satisfying the Rota-Baxter equation:

$$\pi(ab) + \pi(a)\pi(b) = \pi(a\pi(b)) + \pi(\pi(a)b)$$

for $a, b \in \mathcal{A}$. Let $R : \mathfrak{g}_{\mathcal{A}} \to \mathfrak{g}_{\mathcal{A}}$ given by $R(X) = X - 2\pi \circ X$ for any infinitesimal character $X : \mathcal{H} \to \mathcal{A}$. By [6, Prop. 2.6], R satisfies the modified classical Yang-Baxter equation (mCYBE)

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = -[X, Y],$$

which implies that R is an R-operator, i.e. the bracket $[\cdot, \cdot]_R$ is a Lie bracket (cf. [13]). If additionally we assume that the Rota-Baxter map is a projection, $\pi^2 = \pi$, then $\mathcal{A}$ splits into a direct sum of two subalgebras $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ with $\mathcal{A}_- = \mathrm{Im}(\pi)$, and the Birkhoff decomposition is unique. As a consequence, the results of this section extend to any Rota-Baxter projection π. For the minimal subtraction scheme, π is just the projection of a Laurent series onto its pole part.

In summary, in Sect. 5 we have presented methods for applying the Connes-Kreimer renormalization theory to produce Lax pair equations for both truncated and full character algebras. This theory both encodes the traditional Bogoliubov-Parasiuk-Hepp-Zimmermann procedure and emphasizes the pro-unipotent complex group of characters associated to the commutative Hopf algebra of Feynman graphs (see [4, Ch. 1, Sect. 6]). This pro-unipotent group is by definition a projective limit of finite dimensional unipotent Lie groups, and has an associated pro-nilpotent Lie algebra, a projective limit of nilpotent Lie algebras. In our case, the finite dimensional nilpotent Lie algebras are the double of the infinitesimal characters of the truncated Hopf algebras, and the corresponding unipotent groups are the exponentials of the nilpotent algebras.

6. The Connes-Kreimer β-Function and the β-Character

6.1. Overview of Sects. 6–8. The next three sections are devoted to studying "how physical" the Lax pair flow is. This vague question can be approached in at least two ways: (i) given a property of a physically plausible character, we ask if this property is preserved under the Lax pair flow (see Sect. 7); (ii) we compare the Lax pair flow to the renormalization group flow (RGF) which is fundamental in quantum field theory (see Sect. 8).

Before addressing either topic, we have to recall where various flows live. A character for us is an element of $G_{\mathcal{A}}$, the homomorphisms from the Hopf algebra of Feynman diagrams to the algebra $\mathcal{A}$ of Laurent series (although many of our results hold in more generality). A renormalized character, such as Feynman rules given by dimensional regularization and minimal subtraction, is an element of $G_{\mathbb{C}}$, the homomorphisms from the Hopf algebra to $\mathbb{C}$. The RGF is a flow on $G_{\mathbb{C}}$. In contrast, the Lax pair flow is on the Lie algebra $\mathfrak{g}_{\mathcal{A}}$ of infinitesimal characters, which is formally the Lie algebra of $G_{\mathcal{A}}$.

As an example of topic (i), in Sect. 7 we ask if the physically necessary property of locality (see Def. 6.2) of an unrenormalized character $\varphi \in G_{\mathcal{A}}$ is preserved under the Lax pair flow. To make sense of this question, we must choose a bijection $\alpha : G_{\mathcal{A}} \to \mathfrak{g}_{\mathcal{A}}$, take $\alpha(\varphi) \in \mathfrak{g}_{\mathcal{A}}$, let $\alpha(\varphi)$ flow to $L(t)$ under the Lax pair flow, and set $\varphi_t = \alpha^{-1}(L(t))$. The question, "If φ is local, is φ_t local," is now well defined but depends on the choice of α and a choice of Casimir function for the Lax pair flow. A natural choice of α is the logarithm, the inverse of the exponential map, but it turns out that another bijection due to Manchon has much better behavior.

As an example of (ii), we can ask to what extent the RGF is related to the Lax pair flow. Since these flows live on different spaces, there are several ways to interpret this question, and each interpretation involves a choice of identification. For example, we can ask if the family of renormalized scalar characters $[\alpha^{-1}(L(t))]_+(\lambda = 0) \in G_{\mathbb{C}}$ ever coincides with the RGF of a character $\varphi \in G_{\mathcal{A}}$. The answer to this is negative for the bijections mentioned above.

Since the RGF does not equal the Lax pair flow under these identifications, it is better to ask how the RGF is affected by the Lax pair flow. In particular, we consider the β-function $\beta = \beta_\varphi \in \mathfrak{g}_{\mathbb{C}}$, the infinitesimal generator of the RGF. We can extend β to an element $\tilde{\beta} \in \mathfrak{g}_{\mathcal{A}}$, and ask for the behavior of $\tilde{\beta}$ under the Lax pair flow. We show in Sect. 8 that $\tilde{\beta}$ and hence β satisfy a Lax pair flow. This allows us to give a fixed point equation (Corollary 8.4) which has a solution iff the RGF of (suitably identified) φ_t coincide.

In summary, we will see in Sect. 7 that the physically important property of locality is preserved under the Lax pair flow. After preliminary work in Sect. 6 on the β-character, we will see in Sect. 8 that the RGF of a family of characters φ_t given by a Lax pair flow is itself controlled by a Lax pair flow.

6.2. The β-character. As mentioned above, we extend the (scalar) β-function $\beta_\varphi \in \mathfrak{g}_{\mathbb{C}}$ of a local character φ to an infinitesimal character $\tilde{\beta}_\varphi \in \mathfrak{g}_{\mathcal{A}}$ (Lemma 6.6). This "β-character" has already appeared in the literature: $\tilde{\beta}_\varphi = \lambda \tilde{R}(\varphi)$, in the language of [12] explained below (Lemma 6.7).

To define the β-character, we recall material from [3,7,12]. Throughout this section, $\mathcal{A}$ denotes the algebra of Laurent series.

Let $\mathcal{H} = \bigoplus_n \mathcal{H}_n$ be a connected graded Hopf algebra. Let Y be the biderivation on $\mathcal{H}$ given on homogeneous elements by

$$Y : \mathcal{H}_n \to \mathcal{H}_n, \quad Y(x) = nx \quad \text{for } x \in \mathcal{H}_n.$$

Definition 6.1 [12]. *We define the bijection $\tilde{R} : G_{\mathcal{A}} \to \mathfrak{g}_{\mathcal{A}}$ by*

$$\tilde{R}(\varphi) = \varphi^{-1} \star (\varphi \circ Y).$$

We now define an action of $\mathbb{C}$ on $G_{\mathcal{A}}$. For $s \in \mathbb{C}$ and $\varphi \in G_{\mathcal{A}}$ we define $\varphi^s(x)$ on an homogeneous element $x \in \mathcal{H}$ by

$$\varphi^s(x)(\lambda) = e^{s\lambda|x|}\varphi(x)(\lambda), \tag{6.1}$$

for any $\lambda \in \mathbb{C}$, where $|x|$ is the degree of x.

Definition 6.2. *Let*

$$G_{\mathcal{A}}^{\Phi} = \{\varphi \in G_{\mathcal{A}} \mid \frac{d}{ds}(\varphi^s)_- = 0\}, \tag{6.2}$$

be the set of characters with the negative part of the Birkhoff decomposition independent of s. Elements of $G_{\mathcal{A}}^{\Phi}$ are called **local characters**.

The dimensional regularized Feynman rule character φ is local. Referring to [3,7], the physical meaning of locality is that the counterterm φ_- does not depend on the mass parameter μ: $\frac{\partial \varphi_-}{\partial \mu} = 0$, and this in turn reflects the locality of the Lagrangian.

Proposition 6.3. ([3,12,7]) *Let $\varphi \in G_{\mathcal{A}}^{\Phi}$. Then the limit*

$$F_\varphi(s) = \lim_{\lambda \to 0} \varphi^{-1}(\lambda) \star \varphi^s(\lambda)$$

exists and is a one-parameter subgroup in $G_{\mathcal{A}} \cap G_{\mathbb{C}}$ of scalar valued characters of $\mathcal{H}$.

Notice that $(\varphi^{-1}(\lambda) \star \varphi^s(\lambda))(\Gamma) \in \mathcal{A}_+$ as

$$\varphi^{-1}(\lambda) \star \varphi^s(\lambda) = \varphi_+^{-1} \star \varphi_- \star (\varphi^s)_-^{-1} \star (\varphi^s)_+ = \varphi_+^{-1} \star (\varphi^s)_+.$$

Definition 6.4. *For $\varphi \in G_{\mathcal{A}}^{\Phi}$, the β-function of φ is defined to be $\beta_\varphi = -(\mathrm{Res}(\varphi_-)) \circ Y$.*

We have [3]

$$\beta_\varphi = \frac{d}{ds}\Big|_{s=0} F_{\varphi_-^{-1}}(s),$$

where $F_{\varphi_-^{-1}}$, the one-parameter subgroup associated to φ_-^{-1}, also belongs to $G_{\mathcal{A}}^{\Phi}$.

To relate the β-function $\beta_\varphi \in \mathfrak{g}_{\mathbb{C}}$ to our Lax pair equations, which live on $\mathfrak{g}_{\mathcal{A}}$, we can either consider $\mathfrak{g}_{\mathbb{C}}$ as a subset of $\mathfrak{g}_{\mathcal{A}}$, or we can extend β_φ to an element of $\mathfrak{g}_{\mathcal{A}}$. Since $\mathfrak{g}_{\mathbb{C}}$ is not preserved under the Lax pair flow, we take the second approach.

Definition 6.5. *For $\varphi \in G^{\Phi}_{\mathcal{A}}$, $x \in H$, set*

$$\tilde{\beta}_{\varphi}(x)(\lambda) = \frac{d}{ds}\Big|_{s=0}(\varphi^{-1} \star \varphi^{s})(x)(\lambda).$$

The following lemma establishes that $\tilde{\beta}$ is an infinitesimal character.

Lemma 6.6. *Let $\varphi \in G^{\Phi}_{\mathcal{A}}$.*

(i) *$\tilde{\beta}_{\varphi}$ is an infinitesimal character in $\mathfrak{g}_{\mathcal{A}}$.*
(ii) *$\tilde{\beta}_{\varphi}$ is holomorphic (i.e. $\tilde{\beta}_{\varphi}(x) \in \mathcal{A}_{+}$ for any x).*

Proof. (i) For two homogeneous elements $x, y \in \mathcal{H}$, we have:

$$\varphi^{s}(xy) = e^{s|xy|\lambda}\varphi(xy) = e^{s|x|\lambda}\varphi(x)e^{s|y|\lambda}\varphi(y) = \varphi^{s}(x)\varphi^{s}(y).$$

Therefore $\varphi^{-1} \star \varphi^{s} \in G_{\mathcal{A}}$. Since $\varphi^{-1} \star \varphi^{0} = e$ we get

$$\frac{d}{ds}\Big|_{s=0}\varphi^{-1} \star \varphi^{s} \in \mathfrak{g}_{\mathcal{A}}.$$

(ii) Since $\frac{d}{ds}(\varphi^{s})_{-} = 0$, we get

$$\tilde{\beta}_{\varphi} = \frac{d}{ds}\Big|_{s=0}\Big((\varphi_{+})^{-1} \star \varphi_{-} \star ((\varphi^{s})_{-})^{-1} \star (\varphi^{s})_{+}\Big) = (\varphi_{+})^{-1} \star \frac{d}{ds}\Big|_{s=0}(\varphi^{s})_{+}.$$

Then

$$\tilde{\beta}_{\varphi}(x) = (\varphi_{+})^{-1}(x')\frac{d}{ds}\Big|_{s=0}(\varphi^{s})_{+}(x'') = (\varphi_{+})(S(x'))\frac{d}{ds}\Big|_{s=0}(\varphi^{s})_{+}(x'').$$

Therefore $\tilde{\beta}_{\varphi}(x) \in \mathcal{A}_{+}$. $\square$

Lemma 6.7. *If $\varphi \in G^{\Phi}_{\mathcal{A}}$ then*

(i) *$\tilde{\beta}_{\varphi} = \lambda\tilde{R}(\varphi)$,*
(ii) *$\beta_{\varphi} = \mathrm{Ad}(\varphi_{+}(0))(\tilde{\beta}_{\varphi}\big|_{\lambda=0})$,*
(iii) *$\tilde{\beta}_{\varphi_{-}}(\lambda)$ is independent of λ and satisfies $\tilde{\beta}_{\varphi_{-}}(x)(\lambda = 0) = -\beta_{\varphi}(x)$.*

Proof. (i) For $\Delta(x) = x' \otimes x''$, we have

$$\begin{aligned}\tilde{\beta}_{\varphi}(x)(\lambda) &= \frac{d}{ds}\Big|_{s=0}(\varphi^{-1} \star \varphi^{s})(x)(\lambda) = \varphi^{-1}(x')\frac{d}{ds}\Big|_{s=0}(\varphi^{s})(x'')\\ &= \varphi^{-1}(x')\lambda \cdot \deg(x'')\varphi(x'') = \lambda\varphi^{-1}(x')\varphi \circ Y(x'') = \lambda(\varphi^{-1} \star (\varphi \circ Y))(x)\\ &= \lambda\tilde{R}(\varphi)(x).\end{aligned}$$

(ii) The cocycle property of $\tilde{R}$ [7], $\tilde{R}(\phi_1 \star \phi_2) = \tilde{R}(\phi_2) + \phi_2^{-1} \star \tilde{R}(\phi_1) \star \phi_2$, implies that

$$\lambda\tilde{R}(\varphi) = \lambda\tilde{R}(\varphi_{-}^{-1} \star \varphi_{+}) = \lambda\tilde{R}(\varphi_{+}) + \varphi_{+}^{-1} \star \lambda\tilde{R}(\varphi_{-}^{-1}) \star \varphi_{+}. \tag{6.3}$$

Since $\tilde{R}(\varphi_{+}) = \varphi_{+}^{-1} \star (\varphi_{+} \circ Y)$ is always holomorphic and since $\lambda\tilde{R}(\varphi_{-}^{-1}) = \mathrm{Res}(\varphi_{-}^{-1}) \circ Y = -\mathrm{Res}(\varphi_{-}) \circ Y = \beta$ by [12, Th. IV.4.4], when we evaluate (6.3) at $\lambda = 0$ we get $\tilde{\beta}(\varphi)\big|_{\lambda=0} = \mathrm{Ad}(\varphi_{+}^{-1}(0))\beta$.

(iii) The Birkhoff decomposition of $\varphi_- = (\varphi_-)_-^{-1} \star (\varphi_-)_+$ is given by $(\varphi_-)_- = \varphi_-^{-1}$ and $(\varphi_-)_+ = \varepsilon$. By definition, $\beta_{\varphi_-} = -\mathrm{Res}((\varphi_-)_-) \circ Y = -\mathrm{Res}(\varphi_-^{-1}) \circ Y = \mathrm{Res}(\varphi_-) \circ Y = -\beta_\varphi$. Applying (ii) to φ_-, we get

$$-\beta_\varphi = \beta_{\varphi_-} = \mathrm{Ad}(\varepsilon\big|_{\lambda=0})(\tilde{\beta}_{\varphi_-}\big|_{\lambda=0}) = \tilde{\beta}_{\varphi_-}\big|_{\lambda=0}.$$

By part (i), $\tilde{\beta}_{\varphi_-} = \lambda \tilde{R}(\varphi_-)$, which by [12, Th. IV.4.4] belongs to $\mathfrak{g}_{\mathbb{C}}$, i.e. it does not depend on λ. □

These results will be used in Sect. 8.

7. The Lax Pair Flow and Locality in Minimal Subtraction and Other Renormalization Schemes

The Lax pair flow lives on the Lie algebra $\mathfrak{g}_{\mathcal{A}}$ of infinitesimal characters. As described in the Introduction, the bijection $\tilde{R}^{-1} : \mathfrak{g}_{\mathcal{A}} \to G_{\mathcal{A}}$ of [12] transfers the Lax pair flow to the Lie group. The main result in this section is that using $\tilde{R}^{-1}$, local characters remain local under the Lax pair flow (Theorem 7.3). This would not be true if we used the exponential map from $\mathfrak{g}_{\mathcal{A}}$ to $G_{\mathcal{A}}$. We also discuss to what extent these results are independent of the choice of renormalization scheme.

This section is organized as follows. In Sect. 7.1, we prove that the Lax pair flow is trivial on primitive elements in the Hopf algebra, and prove the locality of characters under the Lax pair flow. In Sect. 7.2, we describe how the results of this section carry over for renormalization schemes other than minimal subtraction, and identify certain characters which are fixed points of the Lax pair flow.

7.1. The Lax pair flow: the role of primitives, pole order, and locality. In this subsection, we use the minimal subtraction scheme for renormalization.

We first show that the Lax pair flow is trivial on primitive elements.

Proposition 7.1. *If x is a primitive element of $\mathcal{H}$, then the solution $L(t)(x)$ of the Lax pair flow does not depend on t.*

Proof. We first compute the adjoint representation in terms of the Hopf algebra coproduct. We use Sweedler's notation for the reduced coproduct $\tilde{\Delta}(x) = x' \otimes x''$, where $\tilde{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$. Notice that $\deg(x') + \deg(x'') = \deg(x)$ and $1 \leq \deg(x'),\ \deg(x'') < \deg(x)$. For $x \neq 1$ and $\tilde{\Delta}(x') = (x')' \otimes (x')''$, we have

$$\begin{aligned}
((\varphi_1\varphi_2)\varphi_1^{-1})(x) &= \langle (\varphi_1\varphi_2) \otimes \varphi_1^{-1}, x \otimes 1 + 1 \otimes x + x' \otimes x'' \rangle \\
&= (\varphi_1\varphi_2)(x) + \varphi_1^{-1}(x) + (\varphi_1\varphi_2)(x')\varphi_1^{-1}(x'') \\
&= \varphi_1(x) + \varphi_2(x) + \varphi_1(x')\varphi_2(x'') + \varphi_1^{-1}(x) + \big(\varphi_1(x') + \varphi_2(x') \\
&\quad + \varphi_1((x')')\varphi_2((x')'')\big)\,\varphi_1^{-1}(x'') \\
&= \varphi_2(x) + \varphi_1(x')\varphi_2(x'') + \Big(\varphi_1(x) + \varphi_1^{-1}(x) + \varphi_1(x')\varphi_1^{-1}(x'')\Big) \\
&\quad + \varphi_2(x')\varphi_1^{-1}(x'') + \varphi_1((x')')\varphi_2((x')'')\varphi_1^{-1}(x'').
\end{aligned}$$

Differentiating with respect to φ_2 and setting $L = \dot{\varphi}_2$ gives the adjoint representation:

$$\begin{aligned} \mathrm{Ad}(\varphi_1)(L)(x) = {} & L(x) + \varphi_1(x')L(x'') + L(x')\varphi_1(Sx'') \\ & + \varphi_1((x')')L((x')'')\varphi_1(Sx''), \end{aligned} \tag{7.1}$$

where S is the antipode of the Hopf algebra.

For a primitive element x, the reduced coproduct vanishes, $\tilde{\Delta}(x) = 0$, thus by relation (7.1), $L(t)(x) = (\mathrm{Ad}(g_\pm(t))(L_0))\,(x) = L_0(x)$. □

Thus everything of interest in the Lax pair flow occurs off the primitives.

The following lemma will be used in the proof of the locality of the Lax pair flow.

Lemma 7.2. *(i) If the initial condition $L_0 \in \mathfrak{g}_\mathcal{A}$ is holomorphic in λ, then the solution $L(t) = \mathrm{Ad}(g_+(t))L_0$ of the Lax pair equation is holomorphic in λ.*
(ii) If $L_0 \in \mathfrak{g}_\mathcal{A}$ has a pole of order n, then $L(t) = \mathrm{Ad}(g_+(t))L_0$ has a pole of order at most n.

Proof. By (7.1), we have

$$\begin{aligned} \mathrm{Ad}(g_+(t))(L_0)(x) = {} & L_0(x) + g_+(t)(x')L_0(x'') + L_0(x')g_+(t)(Sx'') \\ & + g_+(t)((x')')L_0((x')'')g_+(t)(Sx''). \end{aligned} \tag{7.2}$$

Notice that $g_+(t)(x)$ is holomorphic for $x \in H$. If L_0 is holomorphic, then every term of the right hand side of (7.2) is holomorphic, so $\mathrm{Ad}(g_+(t))(L_0)$ is holomorphic. Since multiplication with a holomorphic series cannot increase the pole order, $L(t)$ cannot have a pole order greater than the pole order of L_0. □

We now show that local characters remain local under the Lax pair flow.

Theorem 7.3. *For a local character $\varphi \in G^\Phi_\mathcal{A}$, let $L(t)$ be the solution of the Lax pair equation (5.5) for any* Ad*-covariant function f, with the initial condition $L_0 = \tilde{R}(\varphi)$. Let φ_t be the flow given by*

$$\varphi_t = \tilde{R}^{-1}(L(t)).$$

Then φ_t is a local character for all t.

Proof. Recall from [12, Th. IV.4.1] that $\lambda\tilde{R} : G_\mathcal{A} \to \mathfrak{g}_\mathcal{A}$ restricts to a bijection from $G^\Phi_\mathcal{A}$ to $\mathfrak{g}_{\mathcal{A}+}$, where $\mathfrak{g}_{\mathcal{A}+}$ is the set of infinitesimal characters on $\mathcal{H}$ with values in $\mathcal{A}_+ = \mathbb{C}[[\lambda]]$. This can be rephrased to $\tilde{R}$ is a bijection between $G^\Phi_\mathcal{A}$ and the set of Laurent series with the pole order at most one. In particular, $L_0 = \tilde{R}(\varphi)$ has a pole of order at most one. By Lemma 7.2, $L(t)$ has the pole order at most one, which implies that $\varphi_t = \tilde{R}^{-1}(L(t))$ is a local character. □

Corollary 7.4. *If x is a primitive element of $\mathcal{H}$, then $\varphi_t(x) = \varphi(x)$ and $\beta_{\varphi_t}(x) = \beta_\varphi(x)$ for all t.*

Proof. By its definition, $\varphi_t = \tilde{R}^{-1}(L(t))$. By Proposition 7.1, for x primitive we have $L(t)(x) = L_0(x)$ for all t, therefore $\varphi_t(x) = \varphi(x)$ for all t. Thus

$$\beta_{\varphi_t}(x) = (-\mathrm{Res}(\varphi_t)_- \circ Y)\,(x) = -|x|\mathrm{Res}\,((\varphi_t)_-(x)) = -|x|\mathrm{Res}\,(\varphi_-(x)) = \beta_\varphi(x).$$

□

7.2. Lax pair flows for arbitrary renormalization schemes. In this subsection, we prove that Theorem 7.3 on the locality of the Lax pair flow holds for an arbitrary renormalization scheme, i.e. for any Rota-Baxter projection $\pi : \mathcal{A} \to \mathcal{A}$ as in Sect. 5.4. Recall that for such projections the Birkhoff decomposition is uniquely determined and that Theorem 5.9 holds.

We first need to establish two lemmas. The next lemma proves that the key result [12, Th. IV.4.1] holds for any Rota-Baxter projection. Manchon's proof seems to work only for the minimal subtraction scheme, so we rework the proof.

Lemma 7.5. *The map $\lambda\tilde{R} : G_{\mathcal{A}} \to \mathfrak{g}_{\mathcal{A}}$ restricts to a bijection from $G^{\Phi}_{\mathcal{A}}$ to $\mathfrak{g}_{\mathcal{A}+}$, where $\mathfrak{g}_{\mathcal{A}+}$ is the set infinitesimal characters on $\mathcal{H}$ with values in $\mathcal{A}_+$*

Proof. The definition of $\tilde{R} : G_{\mathcal{A}} \to \mathfrak{g}_{\mathcal{A}}$ is independent of the renormalization scheme and by [12], $\tilde{R}$ is bijective (see also the scattering map in [3]).

It is sufficient to show that $\lambda\tilde{R}(G^{\Phi}_{\mathcal{A}}) = \mathfrak{g}_{\mathcal{A}+}$. First, we point out that the proofs of Lemma 6.7(i), i.e. $\lambda\tilde{R}(\varphi)(\varphi) = \tilde{\beta}_{\varphi}$, and of Lemma 6.6, i.e. $\tilde{\beta}_{\varphi} \in \mathfrak{g}_{\mathcal{A}_+}$ remain identical for any Rota-Baxter projection. This implies that if $\varphi \in G^{\Phi}_{\mathcal{A}}$, then $\lambda\tilde{R}(\varphi) \in \mathfrak{g}_{\mathcal{A}_+}$.

Let $\lambda\tilde{R}(\varphi) \in \mathfrak{g}_{\mathcal{A}_+}$, with $\varphi \in G_{\mathcal{A}}$. We prove the converse of Lemma 6.6(ii).

$$\lambda\tilde{R}(\varphi) = \tilde{\beta}_{\varphi} = (\varphi_+)^{-1} \star \varphi_- \star \left.\frac{d}{ds}\right|_{s=0} ((\varphi^s)_-)^{-1} \star \varphi_+ + (\varphi_+)^{-1} \star \left.\frac{d}{ds}\right|_{s=0} (\varphi^s)_+$$

which implies

$$\varphi_- \star \frac{d}{ds}\Big|_{s=0} \left(((\varphi^s)_-)^{-1}\right) = \varphi_+ \star \left(\lambda\tilde{R}(\varphi) - (\varphi_+)^{-1} \star \frac{d}{ds}\Big|_{s=0}(\varphi^s)_+\right) \star (\varphi_+)^{-1}$$
$$\in \mathfrak{g}_{\mathcal{A}_+} \cap \mathfrak{g}_{\mathcal{A}_-} = \{0\}.$$

Thus $(\varphi^s)_-$ does not depend on s, so $\varphi \in G^{\Phi}_{\mathcal{A}}$. □

The following lemma is due to Manchon [12, Lem. IV.4.3], and its proof extends to any renormalization scheme (i.e. Rota-Baxter projection).

Lemma 7.6. *Let $\varphi \in G^{\Phi}_{\mathcal{A}}$.*

(1) Then $(\varphi_-)^{-1} \in G^{\Phi}_{\mathcal{A}}$.
(2) If $h \in G_{\mathcal{A}+}$, then $\varphi \star h \in G^{\Phi}_{\mathcal{A}}$.

We can now show that locality of characters is preserved under the Lax pair flow via the $\tilde{R}$ identification of any renormalization scheme.

Theorem 7.7. *For a local character $\varphi \in G^{\Phi}_{\mathcal{A}}$, let $L(t)$ be the solution of the Lax pair equation (5.5) for any* Ad*-covariant function f, with the initial condition $L_0 = \tilde{R}(\varphi)$. Let φ_t be the flow given by*

$$\varphi_t = \tilde{R}^{-1}(L(t)).$$

Then φ_t is a local character for all t.

Proof. By [7],

$$\tilde{R}(\varphi \star \xi) = \tilde{R}(\xi) + \xi^{*-1} \star \tilde{R}(\varphi) \star \xi.$$

Taking $\xi = g_+(t)^{-1}$ and multiplying by λ, we get

$$\lambda \tilde{R}(\varphi \star g_+(t)^{-1}) = \lambda \tilde{R}(g_+(t)^{-1}) + g_+(t) \star \lambda \tilde{R}(\varphi) \star g_+(t)^{-1}.$$

Since $\varphi \in G^{\Phi}_{\mathcal{A}}$ and $g(t)_+^{-1} \in G^{\Phi}_{\mathcal{A}+}$, by Lemma 7.6, $\varphi \star g(t)_+^{*-1} \in G^{\Phi}_{\mathcal{A}}$. Thus $\lambda \tilde{R}(\varphi \star g_+(t)^{-1}) \in \mathfrak{g}_{\mathcal{A}_+}$. Since $g(t)_+^{-1} \in G_{\mathcal{A}+}$, by definition of $\tilde{R}$ we get $\lambda \tilde{R}(g(t)_+^{-1}) \in \mathfrak{g}_{\mathcal{A}_+}$. It follows that

$$\begin{aligned} \varphi_t &= \tilde{R}^{-1}(\mathrm{Ad}(g_+(t))L_0) = (\lambda \tilde{R})^{-1}(\lambda \mathrm{Ad}(g_+(t))\tilde{R}(\varphi)) \\ &= (\lambda \tilde{R})^{-1}(g_+(t) \star \lambda \tilde{R}(\varphi) \star g_+(t)^{-1}) \in G^{\Phi}_{\mathcal{A}}. \end{aligned} \tag{7.3}$$

□

We can also show that for certain initial conditions, the flow φ_t is constant.

Proposition 7.8. *If $\varphi \in G^{\Phi}_{\mathcal{A}}$ and $\varphi_+ = \varepsilon$ (i.e. φ has only a pole part), then the flow φ_t of Theorem 7.7 for the* Ad*-covariant function $f(L) = \lambda^{-n+2m}L$ has $\varphi_t = \varphi$ for all t.*

Proof. If we show that either $g_\pm(t) = \varepsilon$, then

$$\varphi_t = \tilde{R}^{-1}\left(g_\pm(t) \star \tilde{R}(\varphi) \star g_\pm(t)^{-1}\right) = \tilde{R}^{-1}\left(\varepsilon \star \tilde{R}(\varphi) \star \varepsilon^{-1}\right) = \varphi.$$

$g_\pm(t)$ are given by the Birkhoff decomposition of

$$g(t) = \exp(-2t\lambda^{-n+2m}L_0) = \sum_{k=0}^{\infty} \frac{(-2t\lambda^{-n+2m-1})^k(\lambda L_0)^k}{k!}, \tag{7.4}$$

where $\lambda L_0 = \lambda \tilde{R}(\varphi) \in \mathfrak{g}_{\mathbb{C}}$ [12, Th. IV.4.4]. If $-n + 2m - 1 \geq 0$, then $g(t)(x) \in \mathcal{A}_+$ for any x, which implies $g_-(t) = \varepsilon$. Similarly, if $-n + 2m - 1 < 0$, then $g(t)(x) \in \mathcal{A}_-$ for any x, which implies $g_+(t) = \varepsilon$. Notice that the right hand side of (7.4) is a finite sum, namely up to $k = \deg(x)$ when evaluated on $x \in \mathcal{H}$. □

Starting with a local character, we can produce examples of the previous theorem.

Corollary 7.9. *If $\varphi \in G^{\Phi}_{\mathcal{A}}$, then the flow φ_t associated to the* Ad*-covariant function $f(L) = \lambda^{-n+2m}L$ has $((\varphi_-)^{-1})_t = \varphi$ for all t.*

Proof. Since $\varphi \in G^{\Phi}_{\mathcal{A}}$, by Lemma 7.6, $(\varphi_-)^{-1} \in G^{\Phi}_{\mathcal{A}}$. By the previous proposition, it follows that $((\varphi_-)^{-1})_t = \varphi$. □

Remark 7.10. Lemma 6.7(ii) and the results in the next section cannot be extended to an arbitrary renormalization scheme (with the same proofs). This comes from the following simple fact. If R is a Rota-Baxter map, then $\mathrm{Id} - R$ is also a Rota-Baxter map. In particular, when R is the minimal subtraction scheme, $\mathrm{Id} - R$, the projection to the holomorphic part of the Laurent series, is also a Rota-Baxter projection and thus if we renormalize with respect to $\mathrm{Id} - R$, then $\varphi_+^{Id-R}(x) = -\varphi_-^R(x)$ for any primitive element x. Therefore $\varphi_+^{Id-R}(\lambda = 0)$ might not be defined.

8. The β-Function, the Renormalization Group Flow, and the Lax Pair Flow

It is natural to ask if the Lax pair flow can be identified with the more usual Renormalization Group Flow (RGF). As mentioned above, the RGF $(\varphi^t)_+(\lambda = 0)$ lives in the Lie group of characters $G_{\mathbb{C}}$, while the Lax pair flow $L(t)$ lives in the Lie algebra $\mathfrak{g}_{\mathcal{A}}$. To match these flows, we can transfer the Lax pair flow to the Lie group level using either of the maps $\tilde{R}^{-1}$ and exp, namely by defining

$$\varphi_t = \tilde{R}^{-1}(L(t)) \quad \text{and} \quad \chi_t = \exp(L(t)) \tag{8.1}$$

and then setting $\lambda = 0$. However, it is easy to show that even on a commutative, graded connected Hopf algebra $\mathcal{H}$, $\varphi_t \neq \varphi^t$ and $\chi_t \neq \varphi^t$, where $\mathcal{A}$ is the algebra of Laurent series.

In Sect. 8.1, we give a criterion (Corollary 8.4) under which the RGF is independent of the Lax flow parameter t. Strictly speaking, we compare RGFs translated back to the identity in the character group in order to make the comparison. In Sect. 8.2, we show that both the β-function and the β-character satisfy Lax pair equations (Proposition 8.5, Lemma 8.6, Theorem 8.7). Finally, in Sect. 8.3 we make some preliminary remarks on the complete integrability of the Lax pair flows for characters and for β-functions.

Throughout this section we work with the minimal subtraction renormalization scheme.

8.1. Relations between the Renormalization Group Flow and the Lax pair flow. Local characters satisfy the abstract Renormalized Group Equation [8], which we now recall. For a local character $\varphi \in G^{\Phi}_{\mathcal{A}}$ with φ^s given by (6.1), the renormalized characters are defined by $\varphi_{\text{ren}}(s) = (\varphi^s)_+(\lambda = 0)$.

Theorem 8.1. *For $\varphi \in G^{\Phi}_{\mathcal{A}}$, the renormalized characters $\varphi_{\text{ren}}(s)$ satisfy the abstract Renormalized Group Equation:*

$$\frac{\partial}{\partial s}\varphi_{\text{ren}}(s) = \beta_\varphi \star \varphi_{\text{ren}}(s).$$

Here our parameter s corresponds to e^s in [8].

The abstract RGE of a local character φ can be written as $(d/ds)(\varphi_{ren}(s)) \star \varphi_{ren}(s)^{-1} = \beta_\varphi$, thus the renormalized group flow φ_{ren} is in fact the integral flow associated to the beta function and in consequence

$$\varphi_{ren}(s) = \exp(s\beta_\varphi)\varphi_{ren}(0).$$

We now give an expression for the β-function of a character under the Lax pair flow from Theorem 7.3. This will be put into Lax pair form in Proposition 8.5.

Proposition 8.2. *In the setup of Theorem 5.9, we have $\beta_{\varphi_t} = Ad(A(t))(\beta_\varphi)$ with $A(t)$ given by*

$$A(t) = (\varphi_t)_+(0)\; g_+(t)(0)\; \varphi_+^{-1}(0) \in G_{\mathbb{C}},$$

and where $g_+(t)$ is given as in Theorem 5.9.

Proof. By Lemma 6.7, we get $\tilde{\beta}_{\varphi_t} = \lambda\tilde{R}(\varphi_t) = \lambda L(t)$, which by Theorem 5.9 becomes $\tilde{\beta}_{\varphi_t} = \lambda Ad(g_+(t))L_0 = g_+(t)(\lambda L_0)g_+(t)^{-1} = g_+(t)(\tilde{\beta}_\varphi)g_+(t)^{-1}$. Since $\tilde{\beta}_\varphi$ and $g_+(t)$

are holomorphic, evaluating at $\lambda = 0$, we get that $\tilde{\beta}_{\varphi_t}(0) = Ad(g_+(t)(0))(\tilde{\beta}_\varphi)$, which by Lemma 6.7 implies that $\beta_{\varphi_t} = Ad\left((\varphi_t)_+(0)g_+(t)(0)\varphi_+^{-1}(0)\right)(\beta_\varphi)$. $\square$

Exponentiating the formula of $s\beta_{\varphi_t}$ given in the previous proposition, a straightforward computation gives the RGF of the local character φ_t in terms of the RGF of the character φ:

$$(\varphi_t)_{ren}(s) = (\varphi_t)_+(0) \star Ad(g_+(t)(0))\left(\varphi_+(0)^{-1}\varphi_{ren}(s)\right).$$

We can now relate the RGF $(\varphi_t)_{ren}(s)$ to other flows in this paper.

Assume as usual that φ is a local character. To compare the flows $(\varphi_t)_{ren}(s)$ and $\varphi_{ren}(s)$, we introduce the translated RGF $\Omega_{\varphi_t}(s)$ of $(\varphi_t)_{ren}(s)$ by

$$\Omega_{\varphi_t}(s) = ((\varphi_t)_+(0))^{-1}(\varphi_t)_{ren}(s).$$

A natural question is to find the values of t where the translated RG flows $\Omega_{\varphi_t}(s)$, $\Omega_\varphi(s)$ coincide. Notice that $\Omega_{\varphi_t}(s)$, $\Omega_\varphi(s)$ coincide at s = 0. We set

$$\tilde{\beta}_0(t) = \tilde{\beta}_{\varphi_t}\Big|_{\lambda=0}.$$

By its definition, φ_t satisfies $L(t) = \tilde{R}(\varphi_t)$, so by Lemma 6.7,

$$\tilde{\beta}_0(t) = \lambda\tilde{R}(\varphi_t)\Big|_{\lambda=0} = \lambda L(t)\Big|_{\lambda=0} = \mathrm{Res}(L(t)).$$

Lemma 8.3. *If φ is a local character, then $\Omega_{\varphi_t}(s) = \exp(s\tilde{\beta}_0(t))$.*

Proof. We consider the Taylor expansion of $\Omega_{\varphi_t}(s)$ at $s = 0$:

$$\Omega_{\varphi_t}(s) = \sum_{k\geq 0}\frac{d^k\Omega_{\varphi_t}(0)}{ds^k}\frac{s^k}{k!}.$$

By the abstract RGE Theorem 8.1 and Lemma 6.7, we have

$$\frac{d^k\Omega_{\varphi_t}(s)}{ds^k}\Big|_{s=0} = ((\varphi_t)_+(0))^{-1}\star\beta_{\varphi_t}^{*k}\star(\varphi_t)_{\mathrm{ren}}(s)\Big|_{s=0} = \left(Ad(((\varphi_t)_+(0))^{-1})(\beta_{\varphi_t})\right)^{*k}.$$

Therefore $\Omega_{\varphi_t}(s) = \exp(s\tilde{\beta}_0(t))$. $\square$

Recall that $L(t) = \mathrm{Ad}(g_+(t))(\tilde{R}(\varphi))$ and that, by its definition, $\tilde{R}(\varphi_t) = L(t)$. Thus

$$\tilde{\beta}_{\varphi_t} = \lambda\tilde{R}(\varphi_t) = \mathrm{Ad}(g_+(t))(\lambda\tilde{R}(\varphi)) = \mathrm{Ad}(g_+(t))(\tilde{\beta}_\varphi),$$

which evaluated at $\lambda = 0$ gives

$$\tilde{\beta}_0(t) = \mathrm{Ad}(g_+(t)(0))(\tilde{\beta}_0(0))).$$

This implies the following equation for Ω_{φ_t}.

Corollary 8.4. *For fixed t, the translated RGF $\Omega_{\varphi_t}(s)$ equals $\Omega_\varphi(s)$ iff*

$$\mathrm{Ad}(g_+(t)(0))\cdot\tilde{\beta}_0(0) = \tilde{\beta}_0(0). \tag{8.2}$$

By Theorem 5.9,

$$g_+(t) = (\exp(-tf(L_0)))_+ = \left(\exp\left(-tf\left(\frac{\tilde{\beta}_\varphi}{\lambda}\right)\right)\right)_+$$

depends only on the initial character φ and the choice of an Ad-covariant function f. Thus we can consider (8.2) as a fixed point equation for the t flow of φ. This equation is satisfied for a cocommutative Hopf algebra (for which the adjoint representation is trivial), in the setup of Proposition 7.8, and in Theorem 8.8 below. However, for the Hopf algebra of integer decorated rooted trees and the regularized toy model character defined in [5,7,11], the only value of t for which $\Omega_{\varphi_t} = \Omega_\varphi$ is $t = 0$.

8.2. Lax pair equations for the β-function. We now show that the β-functions and β-characters of φ_t also satisfy a Lax pair flow.

Proposition 8.5. *In the setup of Theorem 5.9, we have*

$$\frac{d\beta_{\varphi_t}}{dt} = \left[\frac{d((\varphi_t)_+(0))}{dt}((\varphi_t)_+(0))^{-1} + \mathrm{Ad}((\varphi_t)_+(0))(M_+(0)), \beta_{\varphi_t}\right],$$

where M comes from the Lax pair equation $dL/dt = [L, M]$, and M_+ is the projection of M into $\mathfrak{g}_{\mathcal{A}+}$.

Proof. In the notation of Proposition 8.2, $\beta_{\varphi_t} = \mathrm{Ad}(A(t))(\beta_\varphi)$, which as usual implies that

$$\frac{d\beta_{\varphi_t}}{dt} = \left[\frac{dA(t)}{dt}A(t)^{-1}, \beta_{\varphi_t}\right].$$

We have

$$\begin{aligned}\frac{dA(t)}{dt}A(t)^{-1} &= \frac{d((\varphi_t)_+(0))}{dt}((\varphi_t)_+(0))^{-1}\\ &\quad+(\varphi_t)_+(0)\ \frac{dg_+(t)(0)}{dt}\ (g_+(t)(0))^{-1}\ ((\varphi_t)_+(0))^{-1}.\end{aligned}$$

By proof of Theorem 5.9, we have

$$\frac{d}{dt}\,(g_+(t)(0))\,(g_+(t)(0))^{-1}) = M_+(0).$$

□

Since $\tilde{\beta}_{\varphi_t}$ and its value at $\lambda = 0$ are simpler and carry better geometric properties, we now restrict our attention to them.

Lemma 8.6. *For a local character $\varphi \in G^\Phi_{\mathcal{A}}$, let φ_t be the flow from Theorem 7.3. Then*

$$\frac{d\tilde{\beta}_{\varphi_t}}{dt} = [\tilde{\beta}_{\varphi_t}, M]. \tag{8.3}$$

Proof. By Lemma 6.7, we get $\tilde{\beta}_{\varphi_t} = \lambda \tilde{R}(\varphi_t) = \lambda L(t)$. Then

$$\frac{d\tilde{\beta}_{\varphi_t}}{dt} = \frac{d(\lambda \tilde{R}(\varphi_t))}{dt} = \frac{\lambda dL(t)}{dt} = \lambda[L(t), M] = [\tilde{\beta}_{\varphi_t}, M].$$

□

Let the Taylor expansion of $\tilde{\beta}_{\varphi_t}$ be

$$\tilde{\beta}_{\varphi_t} = \sum_{k=0}^{\infty} \tilde{\beta}_k(t)\lambda^k.$$

In the setup of Corollary 5.11, we get a Lax pair equation for $\tilde{\beta}_0(t) = \tilde{\beta}_{\varphi_t}\big|_{\lambda=0}$.

Theorem 8.7. *For a local character $\varphi \in G_{\mathcal{A}}^{\Phi}$, let $L(t)$ be the Lax pair flow of Corollary 5.11 with initial condition $L_0 = \tilde{R}(\varphi)$. Let $\varphi_t = \tilde{R}^{-1}(L(t))$. Then*

(i) *For $-n + 2m \geq 1$, $\varphi_t = \varphi$ and hence $\beta_{\varphi_t} = \beta_\varphi$ and $\tilde{\beta}_0(t) = \tilde{\beta}_0(0)$ for all t.*
(ii) *For $-n + 2m \leq 0$, $\beta_{\varphi_t} \in \mathfrak{g}_{\mathbb{C}}$ satisfies*

$$\frac{d\tilde{\beta}_0(t)}{dt} = 2[\tilde{\beta}_0(t), \tilde{\beta}_{n-2m+1}(t)]. \tag{8.4}$$

Proof. By Theorem 7.3, φ_t are local characters, so by [12, Th. IV.4.], $\tilde{\beta}_{\varphi_t} = \lambda L(t) = \lambda \tilde{R}(\varphi_t)$ is holomorphic.

(i) If $-n + 2m \geq 1$, then $\lambda^{-n+2m} L(t)$ is holomorphic, which implies

$$M = R(\lambda^{-n+2m} L(t)) = \lambda^{-n+2m} L(t) = \lambda^{-n+2m-1} \tilde{\beta}_{\varphi_t}.$$

$L(t)$ satisfies the Lax pair equation

$$\frac{dL}{dt} = [L, M] = [L, \lambda^{-n+2m} L] = \lambda^{-n+2m}[L, L] = 0.$$

Thus $L(t) = L_0$ for all t, which gives $\varphi_t = \tilde{R}^{-1}(L(t)) = \tilde{R}^{-1}(L_0) = \varphi$ for all t.

(ii) For $-n + 2m \leq 0$, we have

$$\begin{aligned} M &= R(\lambda^{-n+2m} L(t)) = \lambda^{-n+2m} L(t) - 2P_-(\lambda^{-n+2m} L(t)) \\ &= \lambda^{-n+2m-1} \tilde{\beta}_{\varphi_t} - 2P_-(\lambda^{-n+2m-1} \tilde{\beta}_{\varphi_t}). \end{aligned}$$

Equation (8.3) becomes

$$\begin{aligned} \frac{d\tilde{\beta}_{\varphi_t}}{dt} &= -2[\tilde{\beta}_{\varphi_t}, P_-(\lambda^{-n+2m-1} \tilde{\beta}_{\varphi_t})] \\ &= -2[P_+(\lambda^{-n+2m-1} \tilde{\beta}_{\varphi_t}), \lambda^{n-2m+1} P_-(\lambda^{-n+2m-1} \tilde{\beta}_{\varphi_t})]. \end{aligned}$$

Expand $\tilde{\beta}_{\varphi_t}$ as

$$\tilde{\beta}_{\varphi_t} = \sum_{k=0}^{\infty} \tilde{\beta}_k(t)\lambda^k.$$

Then

$$\frac{d\tilde{\beta}_{\varphi_t}}{dt} = -2\left[\sum_{k=n-2m+1}^{\infty} \tilde{\beta}_k(t)\lambda^{k-n+2m-1}, \sum_{j=0}^{n-2m} \tilde{\beta}_j(t)\lambda^j\right],$$

and evaluating at $\lambda = 0$ gives

$$\frac{d\tilde{\beta}_0(t)}{dt} = -2[\tilde{\beta}_{n-2m+1}(t), \tilde{\beta}_0(t)] = 2[\tilde{\beta}_0(t), \tilde{\beta}_{n-2m+1}(t)]. \tag{8.5}$$

□

In the setup of Corollary 5.11, Proposition 8.5 can be restated as follows.

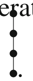
Corollary 8.8. *Let $\mathcal{H}$ be a connected graded commutative Hopf algebra with $\mathfrak{g}_{\mathcal{A}}$ the Lie algebra of infinitesimal characters with values in Laurent series. Pick $L_0 \in \mathfrak{g}_{\mathcal{A}}$ and set $X = 2\lambda^{-n+2m}L_0$. Then*

$$\frac{d\beta_{\varphi_t}}{dt} = \left[\beta_{\varphi_t}, -\frac{d((\varphi_t)_+(0))}{dt}((\varphi_t)_+(0))^{-1} + 2\mathrm{Ad}((\varphi_t)_+(0))\left(\tilde{\beta}_{n-2m+1}(t)\right)\right]. \tag{8.6}$$

Proof. A gauge transformation $X \to X' = \xi X \xi^{-1}$ changes a Lax pair equation of the form $(d/dt)X = [Y, X]$ into $(d/dt)X' = [Y', X']$ with $Y' = \xi Y \xi^{-1} + (d\xi/dt)\xi^{-1}$. Taking $X = \tilde{\beta}_0(t)$, $Y = -2\tilde{\beta}_{n-2m+1}(t)$, $\xi = (\varphi_t)_+(0)$ and using Lemma 6.7(ii), (8.4) becomes (8.6). □

8.3. Complete integrability for the flows of infinitesimal characters. We end with a brief discussion of the complete integrability of our Lax pair equations. We first discuss the complete integrability of the flow of infinitesimal characters $L(t)$ using spectral curve techniques as in [13] for a specific truncated Hopf algebra. We then give an example of a truncated Hopf algebra for which complete integrability can be shown by classical techniques. These results will be discussed more completely elsewhere.

8.3.1. Spectral curve techniques Let $\mathcal{H}^2$ be the Hopf subalgebra generated by the trees

$t_0 = 1_{\mathcal{T}}$, $t_1 = \bullet$, $t_2 =$ , $t_3 =$ , $t_4 =$ , $t_5 =$ .

For $T \in \{t_1, \ldots, t_5\}$, let Z_T be the corresponding infinitesimal character. The Lie algebra $\mathfrak{g}_2$ of scalar valued infinitesimal characters of $\mathcal{H}^2$ is generated by $Z_{t_1}, \ldots, Z_{t_5}$. Let G_1 be the scalar valued character group of $\mathcal{H}^2$, and let G_0 be the semi-direct product $G_1 \rtimes \mathbb{C}$ given by

$$(g, t) \cdot (g', t') = (g \cdot \theta_t(g'), t + t'),$$

where $\theta_t(g)(T) = e^{t\deg(T)}g(T)$ for homogenous T. Define a new variable Z_0 with $[Z_0, Z_{t_i}] = \deg(t_i)Z_{t_i}$, so formally $Z_0 = \frac{d}{d\theta}$. The Lie algebra $\mathfrak{g}_0$ of G_0 is generated by $Z_0, Z_{t_1}, \ldots, Z_{t_5}$. Let $\delta = \mathfrak{g}_0 \oplus \mathfrak{g}_0^*$ be the double Lie algebra associated to an arbitrary Lie bialgebra structure. The conditions a) and b) in Definition 2.1 of a Lie bialgebra can be written in a basis as a system of quadratic equations. We can solve this system explicitly, e.g. via Mathematica. It turns out that there are 43 families of Lie bialgebra structures γ on $\mathfrak{g}_0$. In more detail, the system of quadratic equations involves 90 variables. Mathematica gives 1 solution with 82 linear relations (and so 8 degrees of freedom), 7 solutions with 83 linear relations, 16 solutions with 84 linear relations, 13 solutions with 85 linear relations, 5 solutions with 86 linear relations, and 1 solution with 87 linear relations.

To any Lax equation of matrices with a spectral parameter, one can associate a spectral curve and study its algebro-geometric properties (see [13]). In our case, we consider the adjoint representation $\mathrm{ad} : \delta \to \mathfrak{gl}(\delta)$ and the induced adjoint representation of the loop algebra. Applying $\mathrm{ad} : L\delta \to \mathfrak{gl}(L\delta)$ to the Lax pair equation (5.1) of Theorem 5.4, for the Hopf algebra $\mathcal{H}^2$ we get a Lax pair equation in $\mathfrak{gl}(L\delta)$,

$$\frac{\mathrm{ad}(L)}{dt} = [\mathrm{ad}(L), \mathrm{ad}(M)]. \tag{8.7}$$

The spectral curve of (8.7) is given by the characteristic equation of $\mathrm{ad}(L(\lambda))$: $\Gamma_0 = \{(\lambda, \nu) \in \mathbb{C} - \{0\} \times \mathbb{C} \mid \det(\mathrm{ad}(L(\lambda)) - \nu \mathrm{Id}) = 0\}$.

The theory of the spectral curve and its Jacobian usually assumes that the spectral curve is irreducible. For all 43 families of Lie bialgebra structures that gives δ, the spectral curve itself is the union of degree one curves. Thus each irreducible component has a trivial Jacobian, and the spectral curve theory breaks down. We do not know if spectral curve techniques work for more complicated truncated Hopf algebras.

8.3.2. A completely integrable Lax pair equation We give an example of a completely integrable system associated to the Lax pair equation of Theorem 5.4. Let $\mathcal{H}^3$ be the Hopf subalgebra generated by the trees

$t_0 = 1_{\mathcal{T}}$, $t_1 =$ , $t_2 =$ , $t_4 =$ ,

let $\mathfrak{g}_3$ be the Lie algebra of infinitesimal characters, and let $\delta = \mathfrak{g}_3 \oplus \mathfrak{g}_3^*$ be the double Lie algebra (associated to the trivial Lie bialgebra structure). Let $L_0 = l_{-2}\lambda^{-2} + l_{-1}\lambda^{-1} + l_0 \in L\delta$. By Lemma 7.2, $L(t) = Ad(g_+(t))(L_0)$ has a pole of order at most two. By (7.1) and arguing as in Lemma 7.2, we conclude that $L(t) = Ad(g_-(t))(L_0)$ has no terms $c_i\lambda^i$, with $i \geq 1$ in its Laurent expansion. Thus $L(t) \in L_{-2,0}\delta = \{\sum_{k=-2}^{0} L^k\lambda^k,\ L^k \in \delta\}$.

Let $\{Y_1, Y_2, Y_3, Y_1^*, Y_2^*, Y_3^*\}$ be a basis of δ and write $x = \sum x_\alpha^i \lambda^i Y_\alpha + \sum x_{\alpha*}^i \lambda^i Y_\alpha^* \in L_{-2,0}\delta$. The truncated Poisson bracket $\{\cdot, \cdot\}_R$ is given by

$$\{x_a^i, x_b^j\}_R = \varepsilon_{i,j} C_{a,b}^c x_{c*}^{i+j},$$

with (i) $\varepsilon_{i,j} = 1$ if $i, j \geq 0$, $\varepsilon_{i,j} = -1$ if $i, j \leq -1$, $\varepsilon_{i,j} = 0$ otherwise; (ii) $[E_a, E_b] = C_{a,b}^c E_c$; (iii) $x_{c*}^{i+j} = 0$ if $i + j > 0$ or if $i + j < -2$. The rank of this Poisson bracket is four. Since the dimension of $L_{-2,0}\delta$ is 18, we need a set of $18 - (4/2) = 16$ linearly independent functions in involution to get a completely integrable system [1].

For $(i, k) \in \{(-2, k) \mid 1 \leq k \leq 3\} \cup \{(-1, 3), (0, 3)\}$ and $(i, r) \in \{(-2, r) \mid 1 \leq r \leq 3\} \cup \{(i, r) \mid -1 \leq i \leq 0, 1 \leq r \leq 2\}$, in the coordinates $\{x_k^i, x_{k*}^i\}_{\{-2 \leq i \leq 0,\ 1 \leq k \leq 3\}}$ set

$$H_k^i(x) = \frac{(x_k^i)^2}{2}, \quad H_{r+3}^i(x) = \frac{(x_{r*}^i)^2}{2},$$
$$H_1^0(x) = \frac{(x_1^0)^2}{2} + \frac{(x_2^0)^2}{2}, \quad H_6^0(x) = \frac{(x_1^0)^2}{2} + \frac{(x_2^0)^2}{2} + \frac{(x_{3*}^0)^2}{2},$$
$$H_1^{-1}(x) = x_1^{-1}x_1^{-2} + x_2^{-1}x_2^{-2}, \quad H_6^{-1}(x) = x_1^{-1}x_1^{-2} + x_2^{-1}x_2^{-2} + x_{3*}^{-1}x_{3*}^{-2}.$$

The set $\mathcal{S} = \{H_j^{-2},\ 1 \leq j \leq 6\} \cup \{H_1^0,\ H_1^{-1}\} \cup \{H_j^i,\ -1 \leq i \leq 0,\ 3 \leq j \leq 6\}$ is a set of sixteen linearly independent functions in involution. If ψ from Theorem 5.4 is a nonconstant Casimir function with respect to the truncated Poisson bracket $\{\cdot, \cdot\}$

on $L_{-2,0}\delta$, then $\mathcal{S} \cup \{\psi\}$ is in involution with respect to the truncated Poisson bracket $\{\cdot, \cdot\}_R$. There then exists a function $F \in \mathcal{S}$ such that $\{\psi\} \cup \mathcal{S} \setminus \{F\}$ is linearly independent (in the sense that their differentials are linearly independent on a dense open subset of $L_{-2,0}\delta$). Therefore Eq. (5.1) of Theorem 5.4 is completely integrable with respect to the truncated Poisson structure $\{\cdot, \cdot\}_R$ on $L_{-2,0}\delta$.

Remark 8.9. We can apply these techniques to the Lax pair flow for β-characters. For certain Hopf subalgebras, the equation

$$\frac{d\tilde{\beta}_0(t)}{dt} = 2[\tilde{\beta}_0(t), \tilde{\beta}_{n-2m+1}(t)] \tag{8.8}$$

from Theorem 8.7 is a Hamiltonian system. In some cases, (8.8) is an integrable system on the corresponding double Lie algebra δ.

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