APPROXIMATIONS FOR GIBBS STATES OF ARBITRARY HÖLDER POTENTIALS ON HYPERBOLIC FOLDED SETS

Eugen Mihailescu

Institute of Mathematics "Simion Stoilow" of the Romanian Academy
P.O Box 1-764, RO 014700
Bucharest, Romania

Abstract. We give approximations for the Gibbs states of arbitrary Hölder potentials ϕ, with the help of weighted sums of atomic measures on preimage sets, in the case of smooth *non-invertible* maps hyperbolic on *folded basic sets* Λ. The endomorphism may have also stable directions on Λ and is non-expanding in general. Folding of the phase space means that we do not have a foliation structure for the local unstable manifolds (instead they depend on the whole past and may intersect each other both inside and outside Λ). We consider here *simultaneously all* n-preimages in Λ of a point, instead of the usual way of taking only the consecutive preimages from some given prehistory. We thus obtain the weighted distribution of consecutive preimage sets, with respect to various equilibrium measures on the saddle-type folded set Λ. In particular we obtain the distribution of preimage sets on Λ, with respect to the measure of maximal entropy. Our result is not a direct application of Birkhoff Ergodic Theorem on the inverse limit $\hat{\Lambda}$, since the set of prehistories of a point is uncountable in general, and the speed of convergence may vary for different prehistories in $\hat{\Lambda}$. For hyperbolic toral endomorphisms, we obtain the distribution of the consecutive preimage sets towards an *inverse SRB measure*, for Lebesgue-almost all points.

1. **Introduction.** Gibbs states (equilibrium measures) of Hölder potentials for smooth maps appear naturally in chaotic dynamics and statistical physics and initially were studied in the invertible setting ([2], [15], [16], [3], etc).

The setting in which we work here is however *different* from both the hyperbolic diffeomorphism case (see [2]), as well as from the expanding case (see [13]). We assume that $f : M \rightarrow M$ is a smooth map (endomorphism), not necessarily a diffeomorphism, and that Λ is a basic set for f so that f is hyperbolic on Λ but not necessarily expanding. For such chaotic *non-expanding endomorphisms* there are no Markov partitions in general, so it is not possible to code the system using Markov partitions like in the diffeomorphism case. Also, the presence of stable directions on Λ makes the local inverse iterates of small balls to grow exponentially (up to a certain order).

For instance in [1] it was introduced a family of horseshoes with self intersections and it was proved that there are open sets of parameters which give non-injectivity of the map on its respective basic set. In the non-invertible case, if f is hyperbolic on Λ, we do not have a foliation structure for the local unstable manifolds; the local

---

unstable manifolds depend now on the whole prehistories (see [14]). This folding of the phase space is a major difference from the diffeomorphism case, since we are forced to work on $\hat{\Lambda}$ which is **not** a manifold. The local unstable manifolds may intersect each other and through any given point there may pass infinitely many local unstable manifolds. Moreover the number of f-preimages belonging to Λ *may vary* from point to point, so the map is not necessarily constant-to-1; hence the set Λ is **not** necessarily totally invariant.

The local unstable manifolds depend Holder on their respective prehistories in the canonical metric on $\hat{\Lambda}$ ([8]). The existence of several n-preimages in Λ for any point $x \in \Lambda$ means that we can have n-preimages $y \in f^{-n}(x) \cap \Lambda$ of x where the consecutive sum $S_n\phi(y) := \phi(y) + \ldots + \phi(f^{n-1}y)$ is *well behaved*, but also other n-preimages $z \in f^{-n}(x) \cap \Lambda$ where $S_n\phi(z)$ is *badly behaved*. In the case of smooth non-invertible expanding maps the problem of finding the weighted distributions of preimages was solved by Ruelle in [13]; in that situation it was important that the local inverse iterates contract uniformly on small balls.

In our present *non-invertible non-expanding setting*, we will describe the weighted distributions of preimages by studying the intersections between different tubular neighbourhoods of the (many) different local unstable manifolds. The **main results** of the paper are in Theorem 6 and its Corollaries:

Theorem 6. *Let $f : M \to M$ be a smooth map (say $\mathcal{C}^2$) on a smooth Riemannian manifold M, so that f is hyperbolic and finite-to-one on a basic set Λ; assume also that the critical set $\mathcal{C}_f$ of f does not intersect Λ. Let also ϕ a Holder continous potential on Λ and μ_ϕ be the equilibrium measure of ϕ on Λ. Then $\forall g \in \mathcal{C}(\Lambda, \mathbb{R})$,*

$$\int_\Lambda | < \frac{1}{n} \sum_{y \in f^{-n}(x) \cap \Lambda} \frac{e^{S_n\phi(y)}}{\sum_{z \in f^{-n}(x) \cap \Lambda} e^{S_n\phi(z)}} \cdot \sum_{i=0}^{n-1} \delta_{f^i y} - \mu_\phi, g > | d\mu_\phi(x) \underset{n \to \infty}{\longrightarrow} 0$$

In Corollary 7 we obtain an approximation for the equilibrium measure μ_ϕ, i.e the weak-$\star$ convergence of a sequence of weighted atomic probabilities of the above type towards μ_ϕ.

Corollary 7. *In the same setting as in Theorem 6, for any Holder potential ϕ with equilibrium measure μ_ϕ, it follows that there exists a subset $E \subset \Lambda$, with $\mu_\phi(E) = 1$ and an infinite subsequence $(n_k)_k$ such that for any $x \in E$ we have the weak-$\star$ convergence of measures:*

$$\frac{1}{n_k} \sum_{y \in f^{-n_k}(x) \cap \Lambda} \frac{e^{S_{n_k}\phi(y)}}{\sum_{z \in f^{-n_k}(x) \cap \Lambda} e^{S_{n_k}\phi(z)}} \cdot \sum_{i=0}^{n_k-1} \delta_{f^i y} \underset{k \to \infty}{\longrightarrow} \mu_\phi$$

In particular, if μ_0 is the measure of maximal entropy, it follows that for μ_0-almost all points $x \in \Lambda$, $\frac{1}{n_k} \sum_{y \in f^{-n_k}(x) \cap \Lambda} \frac{\sum_{i=0}^{n_k-1} \delta_{f^i y}}{\text{Card}(f^{-n_k}(x) \cap \Lambda)} \underset{k \to \infty}{\longrightarrow} \mu_0$, for a subsequence $(n_k)_k$.

Remark that, since μ_ϕ is positive on any open set (as any open set contains some small Bowen ball and one can apply Proposition 1), there exists a dense set in Λ of points x for which we have the above weak convergence of weighted atomic measures generated by x, towards μ_ϕ. Therefore in a physical non-reversible system, if we know the past trajectories of such a generic point x up to a high order n, then we can approximate the Gibbs state μ_ϕ as above.

We notice that the basic set Λ considered above is not necessarily an attractor, nor a repellor; thus the usual SRB (Sinai, Ruelle, Bowen), as well as the inverse SRB measure (when there is an inverse function f^{-1}) may not exist. One may compare this result with the (forward) SRB measure in the case of diffeomorphisms (see for example [2], [15], [16] or endomorphisms ([12], [11]); whenever μ_ϕ is equivalent to the Lebesgue measure, like in the case of toral endomorphisms and $\phi \equiv 0$, we obtain an *inverse SRB result*. Our setting and methods are however different due to the lack of an inverse function, the fact that unstable manifolds depend on whole prehistories (not just base points), and also to the fact that the number of preimages is not necessarily constant on Λ. We will apply combinatorial arguments in order to estimate the measure μ_ϕ on the intersections between different tubular unstable sets; and we will estimate carefully the equilibrium measure on the different parts of the consecutive preimage sets in Λ, by separating the prehistories from the point of view of the convergence properties of certain weighted sums of Dirac measures along them.

In our Theorem we average over **all n-preimages** of points, so we do not consider only one prehistory at a time, like in the usual Birkhoff Ergodic Theorem for $\hat{f}^{-1}$ on the inverse limit $\hat{\Lambda}$. This **simultaneous** consideration of all n-preimages is what makes the proof difficult. It cannot be obtained just by applying Birkhoff Ergodic Theorem to different prehistories since the speeds of convergence may be different over the *uncountable* collection of prehistories.

Among the **examples** of smooth endomorphisms on folded basic sets, let us mention the horseshoes with self-intersections from [1], the hyperbolic skew products with overlaps in their fibers from [10], or dynamical systems generating from certain non-reversible statistical physics models (see [15]).

We proved the existence of a very strong non-invertible character in the examples of [5], by using intersections of Cantor sets in fibers and estimates of the stable dimension by means of the thickness of such Cantor sets; in those examples, there exist infinitely many points each having uncountably many prehistories.

Examples may be obtained also from non-degenerate holomorphic maps on complex projective spaces (for instance [9], [6]).

An application of Theorem 6 will be in Corollary 8, where it will be applied to **Anosov endomorphisms**, in order to give the distribution of consecutive preimage sets, with respect to different equilibrium measures. A classical example of Anosov endomorphism is given by a toral endomorphism $f_A : \mathbb{T}^m \to \mathbb{T}^m, m \geq 2$, where f_A is the map induced on the m-dimensional torus by a matrix A with integer coefficients and $\det A \neq 0$; the map f_A is $|\det A|$-to-1 on $\mathbb{T}^m$ ([17]). If A has all its eigenvalues of absolute values different from 1, then f_A is a hyperbolic endomorphism and the above Theorem will apply. Since the equilibrium measure of any constant function is the Haar measure ([17]), we obtain the asymptotic distribution of the local inverse iterates toward an *inverse SRB measure* in this case. A generalization of this class of examples is given by smooth **perturbations of hyperbolic toral endomorphisms** on $\mathbb{T}^m, m \geq 2$. They will be again constant-to-1 and we can apply our main Theorem, to obtain the weighted distribution of preimages with respect to equilibrium measures of Holder potentials.

Remark: On algebraic-type manifolds, like the **infranilmanifolds**, the situation of Anosov endomorphisms is very different from that of Anosov diffeomorphisms. Indeed in the case of infranilmanifolds ([4], [18]), Franks and separately Manning showed that any Anosov diffeomorphism can be "linearized", i.e it is topologically

conjugate to some hyperbolic automorphism. Also Gromov ([4]) showed that if f is an expanding map on a compact manifold, then f is topologically conjugate to some expanding endomorphism on some infranilmanifold. However this is not the case for Anosov endomorphisms. As was proved in [18], if M is an infranilmanifold then there exists a $\mathcal{C}^1$ dense subset $\mathcal{U}$ in the set of "true" Anosov endomorphisms on M (i.e those endomorphisms which are not Anosov diffeomorphisms nor expanding maps), such that every $f \in \mathcal{U}$ is not shift equivalent (hence also not topologically equivalent) to any hyperbolic infranilmanifold endomorphism. In particular this applies to tori $\mathbb{T}^m, m \geq 2$, which are natural examples of infranilmanifolds. For such Anosov endomorphisms which are **neither diffeomorphisms nor expanding**, one cannot apply results similar to the ones from those two previous cases; however we can apply Theorem 6 to get the distribution of preimage sets with respect to equilibrium measures.

2. **Distributions of consecutive preimages on basic sets.** First let us establish some notations. The next definition is parallel to that of basic set from [3] (see also [6]).

Definition 1. Let $f : M \to M$ a smooth map (say $\mathcal{C}^2$) defined on the smooth manifold M. We will say that a compact f-invariant set Λ is a **basic set** for f if there exists a neighbourhood U of Λ such that $\Lambda = \underset{n \in \mathbb{Z}}{\cap} f^n(U)$, and if f is transitive on Λ. As we work here with non-invertible maps, such sets will also be called **folded basic sets**.

Definition 2. The **natural extension** (or **inverse limit**) of the dynamical system (f, Λ) is the dynamical system $(\hat{f}, \hat{\Lambda})$, where $\hat{\Lambda} := \{\hat{x} = (x, x_{-1}, x_{-2}, \ldots), f(x_{-i}) = x_{-i+1}, x_0 = x, x_{-i} \in \Lambda, i \geq 1\}$ and $\hat{f}(\hat{x}) := (f(x), x, x_{-1}, \ldots), \hat{x} \in \hat{\Lambda}$. It follows that $\hat{f}$ is a homeomorphism on $\hat{\Lambda}$. An element $\hat{x} = (x, x_{-1}, \ldots)$ of $\hat{\Lambda}$, starting with x, is called a **prehistory** of x (or *full prehistory* of x). The canonical projection $\pi : \hat{\Lambda} \to \Lambda$ is defined by $\pi(\hat{x}) = x, \hat{x} \in \hat{\Lambda}$. If $f(y) = x, y \in \Lambda$ we call y a **preimage** of x; if $f^n(z) = x, z \in \Lambda$, we call z an n-**preimage** of x (through f). A finite sequence $(x, x_{-1}, \ldots, x_{-n})$ will be called an n-*prehistory* of x.

Let us mention that if μ is an f-invariant probability measure on Λ, then there exists a unique probability $\hat{f}$-invariant measure $\hat{\mu}$ on $\hat{\Lambda}$ such that $\pi_*(\hat{\mu}) = \mu$. It can be seen that μ is ergodic if and only if $\hat{\mu}$ is ergodic on $\hat{\Lambda}$. Also the topological pressure of ϕ (denoted by $P_f(\phi)$ to emphasize dependence on f) is equal to the topological pressure of $\phi \circ \pi$, namely $P_{\hat{f}}(\phi \circ \pi)$; and μ is an equilibrium measure for $\phi : \Lambda \to \mathbb{R}$ if and only if $\hat{\mu}$ is an equilibrium measure for $\phi \circ \pi$.

The concept of hyperbolicity on Λ makes sense for non-invertible maps (i.e endomorphisms), but now the unstable tangent subspaces and the local unstable manifolds depend on whole prehistories, not only on the base points (for exp. [14]). In this hyperbolic setting we will denote by $E_x^s, W_r^s(x)$ the stable tangent subspace, respectively the local stable manifold at x (for $x \in \Lambda$); and by $E_{\hat{x}}^u, W_r^u(\hat{x})$ the unstable tangent subspace, respectively the local unstable manifold corresponding to the prehistory $\hat{x}$ (for $\hat{x} \in \hat{\Lambda}$). Also by $Df_s(x)$ we shall denote the stable derivative $Df_x|_{E_x^s}$ and by $Df_u(\hat{x})$ the unstable derivative $Df_x|_{E_{\hat{x}}^u}$. In [8] we studied the Holder dependence of local unstable manifolds with respect to the prehistories and proved a Bowen type formula, giving the unstable dimension as being the zero t^u of the pressure of the unstable potential $\Phi^u(\hat{x}) := -\log|Df_u(\hat{x})|, \hat{x} \in \hat{\Lambda}$. We proved

also that given a measurable partition of $\hat{\Lambda}$ subordinated to the unstable manifolds, the equilibrium measure of Φ^u has conditional measures which are geometric of exponent t^u.

If $f : M \to M$ is a smooth map which is hyperbolic on a basic set Λ, let $\hat{x} = (x, x_{-1}, \ldots, x_{-n}, \ldots) \in \hat{\Lambda}, n \geq 1, \varepsilon > 0$ small. Then we call an (n, ε)-**tubular unstable neighbourhood** (or tubular unstable set) the set

$$T_n(\hat{x}, \varepsilon) := \{y \in \Lambda, \exists \, y_{-n} \in \Lambda \cap f^{-n}(y), \text{s.t } d(f^i y_{-n}, f^i x_{-n}) < \varepsilon, i = 0, \ldots, n\}$$

The notion of tubular unstable set can be extended to those $y \in M$ which have an n-preimage $y_{-n} \in M$ with the above property, but since we will work in this paper only with measures supported on Λ, we preferred to give the definition restricted to Λ. It is important to keep in mind that tubular unstable sets corresponding to two different prehistories of the same point $x \in \Lambda$ *may not be the same*; still they intersect in a set containing x.

It is well-known that any f-invariant measure μ on Λ can be lifted to a unique $\hat{f}$-invariant measure $\hat{\mu}$ on $\hat{\Lambda}$ such that $\pi_*(\hat{\mu}) = \mu$. It will be important to see exactly how to calculate the measure $\hat{\mu}$ of an arbitrary closed set from $\hat{\Lambda}$, in terms of the μ-measures of sets in Λ.

Lemma 3. *Let $f : \Lambda \to \Lambda$ be a continuous map on a compact metric space Λ, and μ an f-invariant borelian probability measure on Λ. Let $\hat{\mu}$ be the unique $\hat{f}$-invariant probability measure on $\hat{\Lambda}$ with the property that $\pi_*(\hat{\mu}) = \mu$. Then for an arbitrary closed set $\hat{E} \subset \hat{\Lambda}$, we have that*

$$\hat{\mu}(\hat{E}) = \lim_n \mu(\{x_{-n}, \exists \hat{x} = (x, \ldots, x_{-n}, \ldots) \in \hat{E}\})$$

Proof. The arbitrary closed set $\hat{E}$ is not necessarily of the form $\pi^{-1}(E)$ for some E borelian set in Λ. Let us denote $\hat{E}_n := \hat{f}^{-n}\hat{E}, n \geq 1$; then $\hat{\mu}(\hat{E}_n) = \hat{\mu}(\hat{E})$ since $\hat{\mu}$ is $\hat{f}$-invariant. Let also $\hat{F}_n := \pi^{-1}(\pi(\hat{E}_n)), n \geq 1$. We will prove that

$$\hat{E} = \bigcap_{n \geq 0} \hat{f}^n(\hat{F}_n)$$

We have clearly $\hat{E} \subset \hat{f}^n(\hat{F}_n), n \geq 0$. Let now a prehistory $\hat{z} \in \bigcap_{n \geq 0} \hat{f}^n \hat{F}_n$; then if $\hat{z} = (z, z_{-1}, \ldots, z_{-n}, \ldots)$, we obtain that $z_{-n} \in \pi \hat{E}_n, \forall n \geq 0$, hence $\hat{z} \in \hat{E}$ since $\hat{E}$ is assumed closed. Thus we showed the equality $\hat{E} = \bigcap_{n \geq 0} \hat{f}^n(\hat{F}_n)$. Now let us notice that the above intersection is decreasing, since $\hat{f}^{n+1}\hat{F}_{n+1} \subset \hat{f}^n \hat{F}_n, n \geq 0$, since for a prehistory from $\hat{f}^{n+1}\hat{F}_{n+1}$ the $(n+1)$-th entry is from $\pi\hat{E}_{n+1}$ and the n-th entry is in $\pi\hat{E}_n$, whereas the $(n+1)$-entry of a prehistory from $\hat{f}^n\hat{F}_n$ can be **any** preimage of a point from $\pi\hat{E}_n$. Since the above intersection is decreasing, we get

$$\hat{\mu}(\hat{E}) = \lim_n \hat{\mu}(\hat{f}^n \hat{F}_n) = \lim_n \hat{\mu}(\hat{F}_n) =$$
$$= \lim_n \hat{\mu}(\pi^{-1}(\pi(\hat{E}_n))) = \lim_n \mu(\pi(\hat{E}_n)) = \lim_n \mu(\pi \circ \hat{f}^{-n}\hat{E})$$

We used that $\pi_*\hat{\mu} = \mu$ and that $\hat{\mu}$ is $\hat{f}$-invariant on $\hat{\Lambda}$. Therefore we obtain that $\hat{\mu}(\hat{E}) = \lim_n \mu(\{x_{-n}, \exists \hat{x} \in \hat{E}, \hat{x} = (x, \ldots, x_{-n}, \ldots)\})$. $\qquad \square$

For a basic set Λ for a smooth map f we will denote by $f_\Lambda^{-1}x, x \in \Lambda$ the set of f-preimages of x **which belong to** Λ. Similarly $f_\Lambda^{-n}x$ will denote the n-preimages of x belonging to Λ, i.e $f_\Lambda^{-n}x := f^{-n}x \cap \Lambda$. In general in this paper we will be

interested only in the preimages belonging to Λ. We will denote also by $S_n\phi(y)$ for a point $y \in \Lambda$, the **consecutive sum** $S_n\phi(y) := \phi(y) + \ldots + \phi(f^{n-1}y), n \geq 1$. Define then for $n \geq 1$ and $x \in \Lambda$, the probability measure:

$$\mu_n^x := \frac{1}{n} \sum_{y \in f_\Lambda^{-n}x} \frac{e^{S_n\phi(y)}}{\sum\limits_{z \in f_\Lambda^{-n}x} e^{S_n\phi(z)}} \cdot \sum_{i=0}^{n-1} \delta_{f^iy} \tag{1}$$

The $\mu_n^x, n \geq 1$ are probability measures and thus from the weak compactness of the unit ball in the space of measures, we obtain that any sequence of such measures (for $x \in \Lambda$ given) contains a convergent subsequence; the limit of such a sequence $(\mu_{n_k}^x)_{k \geq 1}$ is then an f-invariant probability measure on Λ. We will show that such limit measures have in fact an important thermodynamical property, namely they are equilibrium measures (i.e maximize in the Variational Principle, [17]).

We shall use in the sequel **equilibrium measures for endomorphisms**; the existence of these measures for non-invertible maps, and estimates on Bowen balls for these measures are similar to the corresponding properties for diffeomorphisms (see the proof of Proposition 4 below). We denote the Bowen ball $B_n(y, \varepsilon) := \{z \in \Lambda, d(f^iy, f^iz) < \varepsilon, i = 0, \ldots, n-1\}$, for $y \in \Lambda, n \geq 1, \varepsilon > 0$.

Proposition 4. *Let Λ be a hyperbolic basic set for a smooth endomorphism $f : M \to M$, and ϕ be a Holder continuous function on Λ. Then there exists a unique equilibrium measure μ_ϕ for ϕ on Λ which has the following properties:*

a) for any $\varepsilon > 0$ there exist positive constants $A_\varepsilon, B_\varepsilon$ and an integer $n_0 \geq 1$ such that for any $y \in \Lambda, n \geq n_0$,

$$A_\varepsilon e^{S_n\phi(y)-nP(\phi)} \leq \mu_\phi(B_n(y, \varepsilon)) \leq B_\varepsilon e^{S_n\phi(y)-nP(\phi)}$$

b) By working eventually with a finite iteration of f, we have

$$\mu_\phi = \lim_{n \to \infty} \frac{1}{P_\Lambda(f, \phi, n)} \sum_{x \in Fix(f^n) \cap \Lambda} e^{S_n\phi(x)}\delta_x,$$

where $P_\Lambda(f, \phi, n) := \sum\limits_{x \in Fix(f^n) \cap \Lambda} e^{S_n\phi(x)}, n \geq 1$.

Proof. a) We work in the natural extension $\hat{\Lambda}$ with the expansive homeomorphism $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$. The existence of a unique equilibrium measure for the Holder potential $\phi \circ \pi$ with respect to the expansive homeomorphism $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ follows from the standard theory for homeomorphisms on compact metric spaces (see for example [2], [3]); let us denote this equilibrium measure by $\hat{\mu}_\phi$. Then, given an $\hat{f}$-invariant probability measure $\hat{\mu}$ on $\hat{\Lambda}$, there exists a unique f-invariant measure μ on Λ such that $\pi_*\hat{\mu} = \mu$. If we take the measure $\hat{\mu}_\phi$ instead of $\hat{\mu}$, we will obtain a measure μ_ϕ. It is easy to show that $h_{\hat{\mu}}(\hat{f}) = h_\mu(f)$ and that $P_{\hat{f}}(\phi \circ \pi) = P_f(\phi), \forall \phi \in \mathcal{C}(\Lambda, \mathbb{R})$. Thus it follows that μ is an equilibrium measure for ϕ if and only if its unique $\hat{f}$-invariant lifting $\hat{\mu}$ is an equilibrium measure for $\phi \circ \pi$ on $\hat{\Lambda}$. Thus $\mu_\phi := \pi_*\hat{\mu}_\phi$ is the unique equilibrium measure for ϕ on Λ. Now we see that there exists a $k = k(\varepsilon) \geq 1$ such that $\hat{f}^k(\pi^{-1}B_n(y, \varepsilon)) \subset B_{n-k}(\hat{f}^k\hat{y}, 2\varepsilon) \subset \hat{\Lambda}$, for any $y \in \Lambda$. On the other hand for any $\hat{y} \in \hat{\Lambda}$, we have $\pi(B_n(\hat{y}, \varepsilon)) \subset B_n(y, \varepsilon)$. These two inclusions and the $\hat{f}$-invariance of $\hat{\mu}_\phi$, together with the estimates for the $\hat{\mu}_\phi$-measure of the Bowen balls in $\hat{\Lambda}$ (from [3]) imply that there exist positive constants $A_\varepsilon, B_\varepsilon$ (depending on $\varepsilon > 0$

and on ϕ) such that:

$$A_\varepsilon e^{S_n \phi(y) - nP(\phi)} \leq \mu_\phi(B_n(y, \varepsilon)) \leq B_\varepsilon e^{S_n(\phi)(y) - nP(\phi)}, \forall y \in \Lambda, n \geq 1$$

So the estimates for Bowen balls are true also for endomorphisms.

b) The iterate of f may be needed in order to have topological mixing (needed to guarantee specification, [3]). However without loss of generality we may assume that f is topologically mixing on Λ. If x is a periodic point for $f|_\Lambda$, say $f^m(x) = x$, then we obtain a periodic point for $\hat{f}$, namely the prehistory $\hat{x} = (x, f^{m-1}(x), \ldots, f(x), x, \ldots, f(x), x, \ldots) \in \hat{\Lambda}$. Conversely, if $\hat{x}$ is a periodic point for $\hat{f}$, then x is a periodic point (of the same period) for f. Similarly as for diffeomorphisms we prove that if f is hyperbolic then f satisfies specification. Then specification is used to show the convergence of the weighted sums of Dirac measures concentrated at periodic points towards μ_ϕ, in the same way as in [3].   □

Let us recall now a Lemma proved in [7], giving the relationship between the measures of different parts of the preimage of some set of positive measure; by approximating an arbitrary borelian set with sets having boundaries of μ_ϕ-measure zero from [7], we obtain:

Lemma 5. *In the above setting, let a Holder potential $\phi : \Lambda \to \mathbb{R}$ and its unique equilibrium measure μ_ϕ. Consider $\varepsilon > 0$, k disjoint Bowen balls $B_m(y_1, \varepsilon), \ldots, B_m(y_k, \varepsilon)$ and a borelian set $A \subset f^m B_m(y_1, \varepsilon) \cap \ldots \cap f^m B_m(y_k, \varepsilon)$ such that $\mu_\phi(A) > 0$; denote by $A_1 := f^{-m}A \cap B_m(y_1, \varepsilon), \ldots, A_k := f^{-m}A \cap B_m(y_k, \varepsilon)$. Then there exists a positive constant C_ε independent of $m, y_1, \ldots, y_m$ such that*

$$\frac{1}{C_\varepsilon} \mu_\phi(A_j) \cdot \frac{e^{S_m \phi(y_i)}}{e^{S_m \phi(y_j)}} \leq \mu_\phi(A_i) \leq C_\varepsilon \mu_\phi(A_j) \cdot \frac{e^{S_m \phi(y_i)}}{e^{S_m \phi(y_j)}}, i, j = 1, \ldots, m$$

Theorem 6. *Let $f : M \to M$ be a smooth (say $\mathcal{C}^2$) map on a Riemannian manifold M, which is hyperbolic and finite-to-one on a basic set Λ so that $\mathcal{C}_f \cap \Lambda = \emptyset$. Assume that ϕ is a Holder continuous potential on Λ and that μ_ϕ is the equilibrium measure of ϕ on Λ. Then with the notation from (1),*

$$\int_\Lambda | < \mu_n^x - \mu_\phi, g > |d\mu_\phi(x) \underset{n \to \infty}{\to} 0, \forall g \in \mathcal{C}(\Lambda, \mathbb{R})$$

Proof. We make the convention that all the preimages that we work with are in Λ. So we shall write $f^{-n}x$ for $f_\Lambda^{-n}x, n \geq 1, x \in \Lambda$.

If ϕ is a Holder continuous function on Λ if follows from Proposition 4 that there exists a unique equilibrium measure μ_ϕ for ϕ, and μ_ϕ is the push-forward of the equilibrium measure $\hat{\mu}_\phi$ of $\phi \circ \pi$ on $\hat{\Lambda}$. For simplicity of notation, we shall denote the measure μ_ϕ by μ, with ϕ being fixed. This measure is ergodic as being an equilibrium measure.

Let us fix now a continuous test function $g : \Lambda \to \mathbb{R}$. From von Neumann's L^1 Ergodic Theorem applied to the homeomorphism $\hat{f}^{-1} : \hat{\Lambda} \to \hat{\Lambda}$ and the potential $g \circ \pi$, we know that

$$\int_{\hat{\Lambda}} |\frac{1}{n} \sum_{i=0}^{n-1} g(x_{-i}) - \int_{\hat{\Lambda}} g \circ \pi \ d\hat{\mu}| \ d\hat{\mu}(\hat{x}) \underset{n \to \infty}{\to} 0, \tag{2}$$

where the prehistory $\hat{x} = (x, x_{-1}, x_{-2}, \ldots, x_{-i}, \ldots) \in \hat{\Lambda}$. Denote by

$$\Sigma_n(g, y) := \frac{\sum\limits_{i=0}^{n-1} g(f^i y)}{n} - \int_\Lambda g d\mu, y \in \Lambda, n \geq 2$$

Hence for an arbitrary small $\eta > 0$, we have from (2) that:

$$\hat{\mu}(\{\hat{x} = (x, x_{-1}, \ldots) \in \hat{\Lambda}, |\Sigma_n(g, x_{-n})| \geq \eta\}) \underset{n \to \infty}{\to} 0$$

So for any $\varepsilon' > 0, \varepsilon' = \varepsilon'(\eta) << \eta$, there exists $n(\eta) \geq 1$ so that if $n > n(\eta)$ then

$$\hat{\mu}(\hat{x}, |\Sigma_n(g, x_{-n})| \geq \eta) < \varepsilon'$$

But now $\{\hat{x} \in \hat{\Lambda}, |\Sigma_n(g, x_{-n})| \geq \eta\}$ is a closed set in $\hat{\Lambda}$, thus we can apply Lemma 3 to prove that:

$$\mu(x_{-n} \in \Lambda, |\Sigma_n(g, x_{-n})| \geq \eta) < \varepsilon', \tag{3}$$

if n is large enough (without loss of generality we can take $n > n(\eta)$).

Now let us consider a small $\varepsilon > 0$ with $\varepsilon < \varepsilon(\eta) << \eta$ such that $\omega(3\varepsilon) < \eta$, where $\omega(r)$ denotes in general the maximal oscillation of g on a ball of radius $r > 0$. Let us take also a maximal set of mutually disjoint n-Bowen balls $B_n(y, \varepsilon)$ in Λ; denote the set of such y by $\mathcal{F}_n$. Thus $\{B_n(y, \varepsilon), y \in \mathcal{F}_n\}$ is our maximal set. If $z \notin \mathcal{F}_n$, then from the definition, $B_n(z, \varepsilon)$ must intersect some Bowen ball $B_n(y, \varepsilon), y \in \mathcal{F}_n$. Thus $B_n(z, \varepsilon) \subset B_n(y, 3\varepsilon)$. Let us notice also that if $w \in B_n(z, 3\varepsilon)$ then

$$|\Sigma_n(g, w)| \leq |\Sigma_n(g, z)| + \omega(3\varepsilon) \tag{4}$$

In the sequel we will split different subsets of $\mathcal{F}_n$ in two disjoint subsets $\mathcal{R}_n, \mathcal{G}_n$, with $\mathcal{R}_n \subset \{x \in \Lambda, |\Sigma_n(g, x)| \geq 2\eta\}$ and $\mathcal{G}_n \subset \{x \in \Lambda, |\Sigma_n(g, x)| < 2\eta\}$. Intuitively $\mathcal{R}_n$ consists of the "bad" n-preimages (corresponding to g, η) and $\mathcal{G}_n$ are the "good" n-preimages.

Recall now that we denoted $f_\Lambda^{-n} x := \Lambda \cap f^{-n} x, n \geq 1, x \in \Lambda$, and that Λ is not necessarily totally invariant. Consider then

$$I_n(g, x) := \sum_{y \in f_\Lambda^{-n} x} \frac{e^{S_n \phi(y)}}{\sum\limits_{z \in f_\Lambda^{-n} x} e^{S_n \phi(z)}} \cdot |\Sigma_n(g, y)|, x \in \Lambda, n \geq 1$$

Denote also by

$$\Lambda(n, \eta) := \{x \in \Lambda, \text{s. t } x \text{ has at least one } n-\text{preimage } x_{-n} \text{ with } |\Sigma_n(g, x_{-n})| \geq 2\eta\}$$

The problem is that the unstable manifolds may depend on the whole prehistories, thus if we take $z, w \in f^{-n} x$, then $f^n B_n(w, \varepsilon)$ and $f^n B_n(z, \varepsilon)$ may be different unstable tubular sets; these unstable tubular sets intersect each other in a (possibly smaller) set containing x. By taking n-preimages for all points in $\mathcal{F}_n$ and then the corresponding tubular unstable sets as above, we shall obtain a collection of such tubular unstable sets which intersect each other in different smaller pieces, denoted generically by D. We have to estimate $\int_\Lambda I_n(g, x) d\mu(x)$, which is a sum of integrals on sets of type D. The problematic terms in this sum are those of type

$$\frac{e^{S_n \phi(y)}}{\sum\limits_{z \in f_\Lambda^{-n} x} e^{S_n \phi(z)}} \cdot |\Sigma_n(g, y)| \cdot \mu(D),$$

with $|\Sigma_n(g,y)| \geq 2\eta$ and such that, **at the same time**, there exist also good n-preimages for every point $x \in D$. In other words $x \in \Lambda(n,\eta)$, but x has also good n-preimages. The set of points where **all** n-preimages from Λ are bad, is easier to measure, by using Lemma 3. In the integral $\int_\Lambda I_n(g,x)d\mu(x)$ we have to deal both with the measures of subsets D, and with the n-preimages y of points $x \in D$, namely with the quantities $\Sigma_n(g,y)|$.

For the measures of subsets D we shall use the f-invariance of μ, namely $\mu(D) = \mu(f^{-n}(D))$. Then we will use the estimate on the measure of the set of bad preimages from Lemma 3, coupled with the existent control on the good preimages given by $|\Sigma_n(g,y)| < \eta$. For the rest of the preimages, the idea is to control the sum of the measures of these inverse iterates by the measure of the set of bad preimages.

First let us notice that if $x \in \Lambda$ and $y,z \in f^{-1}x \cap \Lambda, y \neq z$, then since the critical set $\mathcal{C}_f$ does not intersect Λ, it must exist a positive constant ε_0 so that $d(y,z) > \varepsilon_0$. Hence if similarly for some $n \geq 1$, we take two distinct n-preimages $y,z \in \Lambda$ of x ($f^n y = f^n z = x, y \neq z$), then we cannot have $y,z \in B_n(w,3\varepsilon)$ for any $w \in \Lambda$, if $\varepsilon << \varepsilon_0$. We know also that the Bowen balls $B_n(y,3\varepsilon)$ cover Λ when y ranges in $\mathcal{F}_n$, and $B_n(y_i,\varepsilon) \cap B_n(y_j,\varepsilon) = \emptyset$ for any two different points y_i, y_j from $\mathcal{F}_n$.

Let us take then a small $0 < \beta < 1$ such that $\eta < \beta$; to fix ideas we will consider $\beta = 3\eta$. If $x \in \Lambda$, denote by $R_n(x)$ the set of n-preimages $y \in \Lambda$ of x with $|\Sigma_n(g,y)| > 2\eta$ (in fact $R_n(x)$ depends on the η as well, but we do not record this here in order to simplify notation). Let us denote now

$$D_n(\beta,\eta) := \{x \in \Lambda(n,\eta), \frac{\sum\limits_{y \in R_n(x)} e^{S_n\phi(y)}}{\sum\limits_{y \in f^{-n}x} e^{S_n\phi(y)}} < \beta\} \tag{5}$$

Now if $||g|| := \sup\limits_{y \in \Lambda} |g(y)|$, it follows from definition that for any $y \in \Lambda$, we have $|\Sigma_n(g,y)| \leq 2||g||$; thus for a point $x \in D_n(\beta,\eta)$:

$$\sum_{y \in f_\Lambda^{-n}x} \frac{e^{S_n\phi(y)}}{\sum\limits_{z \in f_\Lambda^{-n}x} e^{S_n\phi(z)}} |\Sigma_n(g,y)| =$$

$$= \sum_{y \in R_n(x)} \frac{e^{S_n\phi(y)}}{\sum\limits_{z \in f_\Lambda^{-n}x} e^{S_n\phi(z)}} |\Sigma_n(g,y)| + \sum_{y \in f_\Lambda^{-n}x \backslash R_n(x)} \frac{e^{S_n\phi(y)}}{\sum\limits_{z \in f_\Lambda^{-n}x} e^{S_n\phi(z)}} |\Sigma_n(g,y)|$$

$$\leq 2||g|| \cdot \beta + \eta$$

Therefore

$$\int_{D_n(\beta,\eta)} I_n(g,x)d\mu(x) \leq 2||g||\beta + 2\eta \tag{6}$$

We will now restrict to the complement of $D_n(\beta,\eta)$ in Λ, so we work with points $x \in \Lambda$ for which $\frac{\sum\limits_{y \in R_n(x)} e^{S_n\phi(y)}}{\sum\limits_{y \in f^{-n}x} e^{S_n\phi(y)}} \geq \beta$. From Proposition 4 we know that for any $z \in \Lambda, n \geq 1, A_\varepsilon \cdot e^{S_n\phi(z)-nP(\phi)} \leq \mu(B_n(z,\varepsilon)) \leq B_\varepsilon \cdot e^{S_n\phi(z)-nP(\phi)}$. Therefore we have

$$\sum_{y \in f_\Lambda^{-n} x} \mu(B_n(y, \varepsilon)) \leq \frac{C_\varepsilon}{\beta} \cdot \sum_{z \in R_n(x)} \mu(B_n(z, \varepsilon)) \qquad (7)$$

Obviously from the definition of $\mathcal{F}_n$ we can place any n-preimage of x in a distinct Bowen ball of type $B_n(z, 3\varepsilon)$, for some $z \in \mathcal{F}_n$. From Proposition 4 it follows that the μ-measure of a Bowen ball of type $B_n(z, \varepsilon)$ is comparable to the μ-measure of the Bowen ball $B_n(z, 3\varepsilon)$ (by **comparable** we mean that their quotient is bounded below and above by a positive constant independent of z, n).

The idea is to split now the set $\Lambda \setminus D_n(\beta, \eta)$ into subsets A such that points from A have the same number of n-preimages. Let us assume that d is the maximum number of 1-preimages that a point may have in Λ. Then any point from Λ has at most d^n n-preimages in Λ. We will use next tubular unstable sets $T_n(\hat{x}, 3\varepsilon)$ obtained from different Bowen balls centered at the points of $\mathcal{F}_n$. For an integer $\ell \geq 2$ let us consider the sets of type

$$f^n(B_{i_1}) \cap \ldots \cap f^n(B_{i_\ell}) \setminus [\underset{\ell < |J| \leq d^n, J \subset \mathcal{F}_n}{\cup} \underset{j \in J}{\cap} f^n(B_j)],$$

where $B_j := B_n(y_j, 3\varepsilon)$ for some points $y_j \in \mathcal{F}_n$, and where we assume no repetitions among the respective Bowen balls. This means actually that we do not repeat a Bowen ball B_j in the above intersection; if y, z are different n-preimages of the same point x and ε is small enough, then we cannot have y, z in the same $B_n(\zeta, 3\varepsilon)$. Each point in such a set has exactly ℓ n-preimages, one in each of the Bowen balls $B_{i_1}, \ldots, B_{i_\ell}$.

Let us denote the collection of all such sets by $F(\ell, n, \varepsilon)$ and let $D \in F(\ell, n, \varepsilon)$. Consider the sets of type

$$D \setminus \underset{D' \in F(\ell, n, \varepsilon), D' \neq D}{\cup} D', D \in F(\ell, n, \varepsilon)$$

These sets are now mutually disjointed, borelian, and cover Λ. Their collection will be denoted by $\tilde{F}(\ell, n, \varepsilon)$.

We want to estimate the measure $\mu(\Lambda \setminus D_n(\beta, \eta))$. In order to do this, we will split $\Lambda \setminus D_n(\beta, \eta)$ into mutually disjoint borelian subsets, obtained by intersecting the sets of $\tilde{F}(\ell, n, \varepsilon)$ with $\Lambda \setminus D_n(\beta, \eta)$. Let us denote the collection of these intersections by $H(\ell, n, \varepsilon), \ell \geq 1$.

We will take an arbitrary subset $S \in H(\ell, n, \varepsilon)$, say $S \subset f^n(B_{i_1}) \cap \ldots \cap f^n(B_{i_\ell})$. From the f-invariance of μ, it follows that

$$\mu(S) = \mu(f^{-n} S) = \mu(f^{-n} S \cap B_{i_1}) + \ldots + \mu(f^{-n} S \cap B_{i_\ell})$$

We have that the Bowen balls $B_{i_s}, s = 1, \ldots, \ell$ are mutually disjoint as they contain different n-preimages of the same point (we know that for small ε one cannot have two different n-preimages of the same point, belonging to the same $B_n(y, 3\varepsilon), y \in \Lambda$). Let us denote by $S_n(i_1) := f^{-n} S \cap B_{i_1}, \ldots, S_n(i_k) := f^{-n} S \cap B_{i_k}$. Hence

$$\mu(S) = \mu(S_n(i_1)) + \ldots + \mu(S_n(i_k))$$

We assume that $\mu(S_n(i_1)) > 0$, otherwise we can take a different $S_n(i_j)$. But now from Lemma 5 and the fact that $S \subset \Lambda \setminus D_n(\beta, \eta)$, it follows that

$$\mu(S_n(i_1)) + \ldots + \mu(S_n(i_k)) \leq C_\varepsilon \frac{\mu(S_n(i_1))}{\mu(B_{i_1})} \left[\mu(B_{i_1}) + \ldots + \mu(B_{i_k}) \right] \leq$$

$$\leq C_\varepsilon \frac{\mu(S_n(i_1))}{\mu(B_{i_1})} \cdot \frac{1}{\beta} \sum_{B_j \in R_n(x), j \in \{i_1, \ldots, i_k\}} \mu(B_j) \tag{8}$$

Suppose that T is another disjoint set from some $H(p, n, \varepsilon)$, so that $T \subset f^n(B_{i_1}) \cap f^n(B_{j_2}) \cap \ldots \cap f^n(B_{j_p})$ and let $T_n(i_1), T_n(j_2), \ldots, T_n(j_p)$ be the corresponding parts of $f^{-n}T$ belonging respectively to $B_{i_1}, B_{j_2}, \ldots, B_{j_p}$.

So the sets $S_n(i_1), \ldots, S_n(i_\ell), T_n(i_1), T_n(j_2), \ldots, T_n(j_p)$ are mutually disjoint borelian subsets. We assumed that S and T have both n-preimages in B_{i_1} (for example). If they have n-preimages in completely different Bowen balls, then the situation will be simpler, since there will be no repetitions below in (9).

Let us estimate now $\mu(S) + \mu(T)$; for this we consider two points $x \in S$ and $y \in T$ and assume that $\{i_1, \ldots, i_\ell\} \cap \{i_1, j_2, \ldots, j_p\} \cap R_n(x) = \{l_1, \ldots, l_r\}$. Then from (8), we obtain:

$$\mu(S) + \mu(T) \leq$$

$$\leq \frac{C_\varepsilon}{\beta} \left[\frac{\mu(S_n(i_1))}{\mu(B_{i_1})} \cdot \sum_{B_j \in R_n(x), j \in \{i_1, \ldots, i_\ell\}} \mu(B_j) + \right.$$

$$\left. + \frac{\mu(T_n(t_1))}{\mu(B_{i_1})} \cdot \sum_{B_s \in R_n(y), s \in \{i_1, j_2, \ldots, j_p\}} \mu(B_s) \right]$$

$$\leq \frac{C_\varepsilon}{\beta \mu(B_{i_1})} \cdot \left[\mu(S_n(i_1)) + \mu(T_n(i_1)) \right] \cdot \left[\mu(B_{l_1}) + \ldots \mu(B_{l_r}) \right] +$$

$$+ C_\varepsilon \cdot \frac{\mu(S_n(i_1))}{\beta \mu(B_{i_1})} \cdot \Sigma(S, n) + C_\varepsilon \cdot \frac{\mu(T_n(i_1))}{\beta \mu(B_{i_1})} \cdot \Sigma(T, n), \tag{9}$$

where

$$\Sigma(S, n) := \sum_{j \in R_n(x) \cap \{i_1, \ldots, i_\ell\} \setminus \{l_1, \ldots, l_r\}} \mu(B_j) \text{ and}$$

$$\Sigma(T, n) := \sum_{j' \in \{i_1, j_2, \ldots, j_p\} \cap R_n(x) \setminus \{l_1, \ldots, l_r\}} \mu(B_{j'})$$

But recall that the subsets $S_n(i_1)$ and $T_n(i_1)$ are disjoint inside B_{i_1}, hence $\mu(S_n(i_1)) + \mu(T_n(i_1)) \leq \mu(B_{i_1})$. Since $\{l_1, \ldots, l_r\} = \{i_1, \ldots, i_l\} \cap \{i_1, j_2, \ldots, j_p\} \cap R_n(x)$, the sums $\Sigma(S, n)$ and $\Sigma(T, n)$ do not have common terms and do not have any term from the collection $\{B_{l_1}, \ldots, B_{l_r}\}$. Thus from (9) we obtain that

$$\mu(S) + \mu(T) \leq \frac{C_\varepsilon}{\beta} \left[\mu(B_{l_1}) + \ldots + \mu(B_{l_r}) + \Sigma(S, n) + \Sigma(T, n) \right]$$

From (4) we obtain that $B_i \subset \{y \in \Lambda, |\Sigma_n(g, y)| \geq \eta\}$. Also from the estimates of Proposition 1, we know that there exists a positive constant χ_ε such that $\mu(B_i) \leq \chi_\varepsilon \mu(B_n(y_i, \varepsilon)), i \geq 1$. Therefore, recalling also that the balls $B_n(y_i, \varepsilon), y_i \in \mathcal{F}_n$ are mutually disjoint we have that:

$$\sum_{1 \le k \le r} \mu(B_{l_k}) + \Sigma(S, n) + \Sigma(T, n) \le$$

$$\le \chi_\varepsilon \Big[\sum_{1 \le k \le r} \mu(B_n(y_{l_k}, \varepsilon)) + \sum_{j \in R_n(x) \cap \{i_1, \ldots, i_\ell\} \setminus \{l_1, \ldots, l_r\}} \mu(B_n(y_j, \varepsilon)) +$$

$$+ \sum_{j' \in R_n(x) \cap \{i_1, j_2, \ldots, j_p\} \setminus \{l_1, \ldots, l_r\}} \mu(B_n(y_{j'}, \varepsilon)) \Big] =$$

$$= \chi_\varepsilon \cdot \mu \big(\bigcup_{j \in R_n(x)} B_n(y_j, \varepsilon) \big) \le \chi_\varepsilon \cdot \mu(\{y, |\Sigma_n(g, y)| \ge \eta\})$$

But from (3) we can control the total measure of the set of bad n-preimages, which is thus smaller than ε'. Hence

$$\sum_{1 \le k \le r} \mu(B_{l_k}) + \Sigma(S, n) + \Sigma(T, n) \le \chi_\varepsilon \cdot \varepsilon' \tag{10}$$

This procedure can be used for any collection of mutually disjoint borelian subsets from the collections $H(\ell, n, \varepsilon), 1 \le \ell \le d^n$, not only for S, T. Indeed by using Lemma 5 and the disjointness of sets from $H(\ell, n, \varepsilon)$ (hence also the mutual disjointness of the sets of their n-preimages) we see that the weights associated in (9) to any measure $\mu(B_j)$ (where B_j corresponds to a bad n-preimage) never add up to more than 1.

Then similarly as in (10), by employing the control on the total measure of bad n-preimages from (3), we can conclude that for $n > n(\eta)$:

$$\mu(\Lambda \setminus D_n(\beta, \eta)) \le C_\varepsilon \cdot \frac{\varepsilon'}{\beta} \cdot \chi_\varepsilon = \tilde{C}_\varepsilon \cdot \frac{\varepsilon'}{\beta} \tag{11}$$

Therefore by using (6) and (11)

$$\int_\Lambda I_n(g, x) d\mu(x) \le 2\eta + \int_{\Lambda(\eta)} I_n(g, x) d\mu(x) \le$$

$$\le 2\eta + \int_{D_n(\beta, \eta)} I_n(g, x) d\mu(x) + \int_{\Lambda \setminus D_n(\beta, \eta)} I_n(g, x) d\mu(x) \le$$

$$\le 2\eta + 2||g||\beta + 2\eta + 2||g||\mu(\Lambda \setminus D_n(\beta, \eta)) \le$$

$$\le 4\eta + 2||g||\beta + 2||g||\tilde{C}_\varepsilon \frac{\varepsilon'}{\beta},$$

for $n > n(\eta)$. Recall however that we assumed before that $3\eta = \beta$. Assume that ε' is so small that $\tilde{C}_\varepsilon \cdot \frac{\varepsilon'}{3\eta} < \eta$. Then from the last displayed inequality, it follows that there exists a positive constant $C' = 4 + 8||g||$ so that:

$$\int_\Lambda I_n(g, x) d\mu(x) \le C' \cdot \eta, \text{ for } n > n(\eta)$$

This shows in conclusion that

$$\int_\Lambda I_n(g, x) d\mu(x) \underset{n \to \infty}{\to} 0, \forall g \in \mathcal{C}(\Lambda, \mathbb{R}).$$

$\square$

Hence we proved the convergence in integral (with respect to $d\mu_n(x)$) of the measures μ_n^x from (1), towards the equilibrium measure μ_ϕ of ϕ, in the hyperbolic non-invertible case.

Corollary 7. *In the same setting as in Theorem 6, for any Holder potential ϕ, it follows that there exists a subset $E \subset \Lambda$, with $\mu_\phi(E) = 1$ and an infinite subsequence $(n_k)_k$ such that for any $z \in E$ we have the weak convergence of measures*

$$\mu_{n_k}^z \underset{k \to \infty}{\to} \mu_\phi$$

In particular, if μ_0 is the measure of maximal entropy, it follows that for μ_0-almost all points $x \in \Lambda$, $\frac{1}{n_k} \sum_{y \in f^{-n_k}(x) \cap \Lambda} \frac{\sum_{i=0}^{n_k-1} \delta_{f^i y}}{Card(f^{-n_k}(x) \cap \Lambda)} \underset{k \to \infty}{\longrightarrow} \mu_0$, for a subsequence $(n_k)_k$.

Proof. Let us fix $g \in \mathcal{C}(\Lambda, \mathbb{R})$. From the convergence in μ_ϕ-measure of the sequence of functions $z \to \mu_n^z(g), n \geq 1$ obtained from Theorem 6, it follows that there exists a borelian set $E(g)$ with $\mu_\phi(E(g)) = 1$ and a subsequence $(n_p)_p$ so that $\mu_{n_p}^z(g) \underset{p}{\to} \mu_\phi(g), z \in E(g)$.

Let us consider now a sequence of functions $(g_m)_m$ dense in $\mathcal{C}(\Lambda, \mathbb{R})$. By applying a diagonal sequence procedure we shall obtain then a subsequence $(n_k)_k$ so that $\mu_{n_k}^z(g) \underset{k}{\to} \mu_\phi(g), \forall z \in \underset{m}{\cap} E(g_m)$. We notice also that $\mu_\phi(\underset{m}{\cap} E(g_m)) = 1$, since $\mu_\phi(E(g_m)) = 1, m \geq 1$. But since any continuous function g can be approximated in the uniform norm by the functions g_m, it will follow that $\mu_{n_k}^z(g) \underset{k}{\to} \mu_\phi(g), \forall z \in E := \underset{m}{\cap} E(g_m)$. Therefore we obtain that $\mu_{n_k}^z \underset{k}{\to} \mu_\phi, z \in E$, i.e we have weak convergence of the measures $\mu_{n_k}^z, k \geq 1$, on a set of z's having full μ_ϕ-measure in Λ. $\qquad \square$

The Theorem applies to **Anosov endomorphisms** in particular.

Corollary 8. *Assume that $f : M \to M$ is an Anosov endomorphism without critical points on a Riemannian manifold. Let also ϕ a Holder continuous potential on M and μ_ϕ the equilibrium measure of ϕ. Then*

$$\int_M | < \frac{1}{n} \sum_{y \in f^{-n}(x) \cap \Lambda} \frac{e^{S_n \phi(y)}}{\sum_{z \in f^{-n}(x) \cap \Lambda} e^{S_n \phi(z)}} \cdot \sum_{i=0}^{n-1} \delta_{f^i y} - \mu_\phi, g > | d\mu_\phi(x) \underset{n \to \infty}{\to} 0,$$

for any function $g \in \mathcal{C}(M, \mathbb{R})$.

We can compare these results to the usual SRB measure for the endomorphism f, defined as a measure μ having the property that for any measurable partition η of $\hat{M}$ subordinate to the lifts of the local unstable manifolds, and for $\hat{\mu}$ almost all $\hat{x} \in \hat{M}$, the projection of the conditional measure of $\hat{\mu}$, namely $\pi_*(\hat{\mu}_{\hat{x}}^\eta)$ is absolutely continuous with respect to the induced Lebesgue measure on $W_{\hat{x}}^u$ ([12]). In [12] it is shown that μ satisfies the SRB property for the Anosov endomorphism f if and only if $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x} \underset{n}{\to} \mu$ for Lebesgue almost every $x \in M$.

Corollary 9. *Let $f : M \to M$ be an Anosov endomorphism, $\phi : \Lambda \to \mathbb{R}$ a Holder potential and assume that the equilibrium measure μ_ϕ is absolutely continuous with respect to the Lebesgue measure on M. Then the measure μ_ϕ with this property is unique, it is an SRB measure and it also satisfies an inverse SRB condition in the sense that there exists a set E of full Lebesgue measure in M and a sequence $(n_k)_k$ such that $\mu_{n_k}^z \underset{k}{\to} \mu_\phi, z \in E$.*

Proof. The proof follows immediately from Theorem 6, Corollary 8 and from the results of [12] and [11]. The potential ϕ for which μ_ϕ is SRB, can be taken in fact to be the unstable potential. □

A classical example of Anosov endomorphism without critical points is a **toral hyperbolic endomorphism** $f_A : \mathbb{T}^m \to \mathbb{T}^m$, associated to an $m \times m$ integer valued matrix A, all of whose eigenvalues λ_i have absolute values different from 1. Each point from $\mathbb{T}^m$ has exactly $|\det A|$ f_A-preimages in $\mathbb{T}^m$. If we consider the potential $\phi \equiv 0$, then the equilibrium measure of ϕ is the Haar measure ω on $\mathbb{T}^m$ which is also the measure of maximal entropy (its entropy is equal to $\sum_{\lambda_i, |\lambda_i| > 1} \log |\lambda_i|$, where each eigenvalue is taken with its multiplicity). In [17] it was proved the asymptotic distribution of periodic points towards ω. Here we prove the convergence towards ω (which is also the unique measure of maximal entropy) of the measures $\{\mu_n^x\}_n$ which correspond to the potential $\phi \equiv 0$; this convergence happens for ω-almost all points $x \in \mathbb{T}^m$. We thus obtain the existence of an *inverse SRB measure* in this case.

Moreover Theorem 6 applies also to smooth (say $\mathcal{C}^2$) perturbations $f_{A,\varepsilon}$, of hyperbolic toral endomorphisms f_A. Indeed they will also be hyperbolic on the m-dimensional torus $\mathbb{T}^m$ and the basic set considered is the whole $\mathbb{T}^m$. Also the non-invertible map $f_{A,\varepsilon}$ remains $|\det(A)|$-to-1 on $\mathbb{T}^m$. We thus obtain the weighted distribution (with respect to a Holder potential ϕ on $\mathbb{T}^m$) of preimage sets of $f_{A,\varepsilon}$, with respect to the equilibrium measure μ_ϕ of ϕ, for **perturbations of hyperbolic toral endomorphisms**.

Anosov endomorphisms on **infranilmanifolds** represent a generalization of toral linear endomorphisms (see the Remark at the end of Section 1). Let us notice that our Theorem 6 applies to Anosov endomorphisms on infranilmanifolds which are not topologically conjugate to Anosov diffeomorphisms nor to expanding maps. Thus, besides Theorem 6, one cannot apply any of the previously known results for the distributions of preimages from the case of diffeomorphisms ([2]), or expanding endomorphisms ([13]).

Theorem 6 applies also to **hyperbolic basic sets of saddle type** for endomorphisms which *are not necessarily Anosov*, like the class of examples from [10], namely skew products with overlaps in their fibers

$$F : X \times V \to X \times V, F(x, y) = (f(x), h(x, y)),$$

where $f : X \to X$ is an expanding map on a compact metric space, while $h(x, \cdot) : V \to V$ (denoted also by h_x) is a contraction on an open convex set $V \subset \mathbb{R}^m$; h_x is assumed to depend continuously on $x \in X$. The basic set is in this case given by

$$\Lambda := \bigcup_{x \in X} \bigcap_{n=0}^{\infty} \bigcup_{z \in f^{-n}x} h_z^n(\bar{V}),$$

where $h_z^n := h_{f^{n-1}z} \circ \ldots \circ h_z, n \geq 1, z \in X$. In [10] we studied the conditional measures of equilibrium states induced on fibers and their relation to the stable dimension of fibers. So from Theorem 6 we obtain the weighted distributions of preimages of the non-invertible map F over Λ, with respect to equilibrium measures of Holder potentials.

We collect the above remarks in the following:

Corollary 10. *a) The conclusions of Corollary 8 hold in particular for toral hyperbolic endomorphisms and for smooth perturbations of these.*

b) The conclusions of Theorem 6 hold for the basic sets of hyperbolic skew products with overlaps in their fibers from [10]*, as well as for the attractors of the noninvertible horseshoes from* [1]*.*

REFERENCES

[1] H. G. Bothe, *Shift spaces and attractors in noninvertible horseshoes*, Fundamenta Math., **152** (1997), 267–289.

[2] R. Bowen, "Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms," Lecture Notes in Mathematics, **470**, Springer-Verlag, Berlin, 1975.

[3] A. Katok and B. Hasselblatt, "Introduction to the Modern Theory of Dynamical Systems," With a supplementary chapter by A. Katok and L. Mendoza, Encyclopedia of Mathematics and its Applications, **54**, Cambridge Univ. Press, Cambridge, 1995.

[4] R. Mane, "Ergodic Theory and Differentiable Dynamics," Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **8**, Springer-Verlag, Berlin, 1987.

[5] E. Mihailescu, *Unstable directions and fractal dimensions for skew products with overlaps in fibers*, Math. Zeitschrift, 2011, DOI: 10.1007/s00209-010-0761-y.

[6] E. Mihailescu, *Metric properties of some fractal sets and applications of inverse pressure*, Math. Proceed. Cambridge Phil. Soc., **148** (2010), 553–572.

[7] E. Mihailescu, *On a class of stable conditional measures*, Ergodic Th. and Dynam. Syst., **31** (2011), 1499–1515.

[8] E. Mihailescu, *Unstable manifolds and Hölder structures associated with noninvertible maps*, Discrete and Cont. Dynam. Syst., **14** (2006), 419–446.

[9] E. Mihailescu, *The set K^- for hyperbolic non-invertible maps*, Ergodic Th. and Dynam. Syst., **22** (2002), 873–887.

[10] E. Mihailescu and M. Urbański, *Transversal families of hyperbolic skew-products*, Discrete and Cont. Dynam. Syst., **21** (2008), 907–928.

[11] M. Qian and Z. S. Zhang, *Ergodic theory for Axiom A endomorphisms*, Ergodic Th. and Dyn. Syst., **15** (1995), 161–174.

[12] M. Qian and Z. Shu, *SRB measures and Pesin's entropy formula for endomorphisms*, Trans. Amer. Math. Soc., **354** (2002), 1453–1471.

[13] D. Ruelle, *The thermodynamic formalism for expanding maps*, Commun. Math. Physics, **125** (1989), 239–262.

[14] D. Ruelle, "Elements of Differentiable Dynamics and Bifurcation Theory," Academic Press, Inc., Boston, MA, 1989.

[15] D. Ruelle, *Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics*, J. Statist. Phys., **95** (1999), 393–468.

[16] Y. Sinaĭ, *Gibbs measures in ergodic theory*, Uspehi Mat. Nauk, **27** (1972), 21–64.

[17] P. Walters, "An Introduction to Ergodic Theory," Graduate Texts in Mathematics, **79**, Springer-Verlag, New York-Berlin, 1982.

[18] M. Zhang, "Anosov Endomorphisms: Shift Equivalence and Shift-Equivalence Classes," PH.D Thesis, Peking Univ., 1989.

E-mail address: Eugen.Mihailescu@imar.ro