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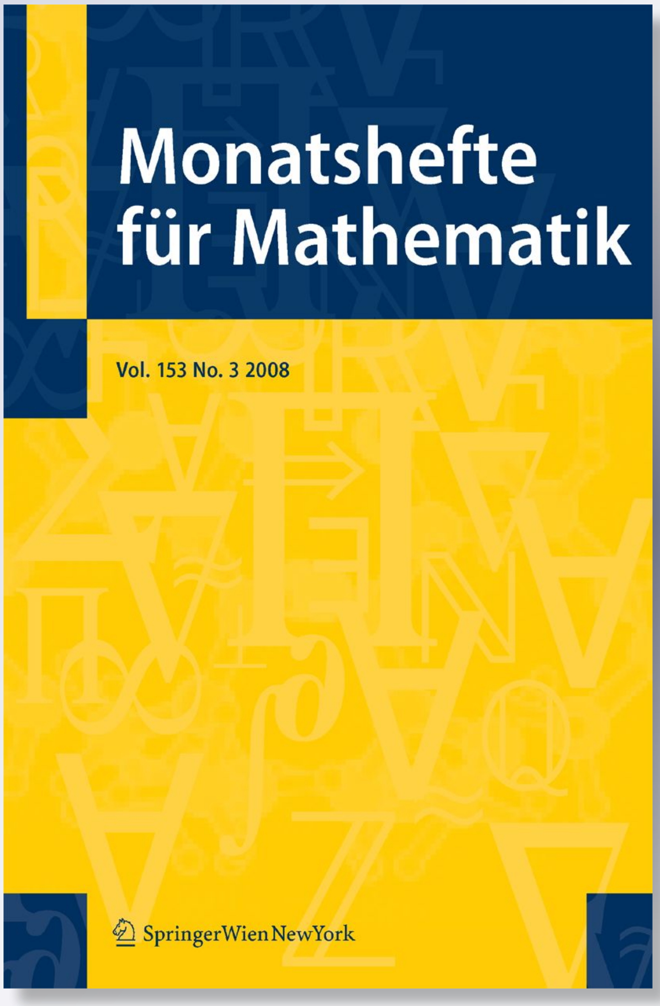

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On some coding and mixing properties for a class of chaotic systems

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Abstract We study certain ergodic properties of equilibrium measures of hyperbolic non-invertible maps f on basic sets with overlaps Λ. We prove that if the equilibrium measure μ_ϕ of a Holder potential ϕ, is 1-sided Bernoulli, then f is expanding from the point of view of a pointwise section dimension of μ_ϕ. If the measure of maximal entropy μ_0 is 1-sided Bernoulli, then f is shown to be distance expanding on Λ; and if μ_ϕ is 1-sided Bernoulli for f expanding, then μ_ϕ must be the measure of maximal entropy. These properties are very different from the case of hyperbolic diffeomorphisms. Another result is about the non 1-sided Bernoullicity for certain equilibrium measures for hyperbolic toral endomorphisms. We also prove the non-existence of generating Rokhlin partitions for measure-preserving endomorphisms in several cases, among which the case of hyperbolic non-expanding toral endomorphisms with Haar measure. Nevertheless the system (Λ, f, μ_ϕ) is shown to have always exponential decay of correlations on Holder observables and to be mixing of any order.

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1 Introduction and outline of main results

We investigate some ergodic properties of equilibrium measures on folded basic sets, i.e on locally maximal invariant sets for non-reversible smooth dynamical systems. Such systems appear naturally in statistical mechanics or in fractal theory. One central property in ergodic theory is the 1-sided (2-sided) Bernoullicity, or lack of it, i.e the possibility to code the measure-preserving system with a shift on a space of sequences. In a sense, 1-sided Bernoulli shifts represent the most chaotic and unpredictable non-reversible systems (see [15]). Parry and Walters showed in [18] that measurable endomorphisms of Lebesgue spaces behave **very differently** than automorphisms. Indeed for automorphisms Ornstein proved a famous result, namely that two invertible Bernoulli shifts on Lebesgue spaces are isomorphic if and only if they have the same measure theoretic entropy (see eg. [15]). However as Parry and Walters showed in [18] for measure-preserving *endomorphisms* $f : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$, the entropy alone $h_\mu(f)$ *does not* determine the conjugacy class. So the problem of coding for endomorphisms of Lebesgue spaces (in particular for 1-sided Bernoulli shifts) is subtle and there are no exhaustive classifications.

Hyperbolic diffeomorphisms on basic sets have Markov partitions (see [2]), and these are fundamental in establishing a coding to a 2-sided Bernoulli shift, of the diffeomorphism with an equilibrium measure of a Holder potential [2,3]; however such Markov partitions lack in general for endomorphisms. Endomorphisms on Lebesgue spaces present important differences from the automorphism/diffeomorphism case (for example [4,5,9–12,16,18,22,28], etc.) In [9] Mane proved that some iterate f^m of a rational map f is 1-sided Bernoulli with respect to the measure of maximal entropy on the Julia set of f.

In this paper we consider the significantly different case of equilibrium measures for smooth noninvertible maps (referred to also as *endomorphisms*) which are hyperbolic on basic sets with overlaps Λ; in general the map may have both stable and unstable directions on Λ. Here the local unstable manifolds do not form necessarily a foliation (unlike for hyperbolic diffeomorphisms), as they depend on the whole past. There are many examples of interesting and/or unexpected dynamical behaviour for endomorphisms, for instance: examples from statistical mechanics (see [22]); horseshoes with overlaps [1]; hyperbolic toral endomorphisms (see [8,27]), and endomorphisms on infranilmanifolds [8]; strange attractors and strange repellers with overlaps [11,12,24]; holomorphic maps in one complex variable and measures on their Julia sets [9]; holomorphic maps in higher dimension, hyperbolic on certain sets [12]; skew product endomorphisms with overlaps in fibers, having Cantor sets of points in fibers with infinitely many prehistories, as in [10]; parameterized families of skew products, satisfying a transversality condition [14], etc.

We denote by $\mathcal{B}(\Lambda)$ the σ-algebra of borelian sets on Λ; all our measures are borelian. In Theorem 1 we will show that, if the system (Λ, f, μ_ϕ) is 1-sided Bernoulli, with f a hyperbolic endomorphism and μ_ϕ the equilibrium measure of a Holder continuous potential ϕ, then f must be "expanding" on Λ from the point of view of μ_ϕ. In the proof of Theorem 1 we will use the notion of *folding entropy* introduced by Ruelle in [22]. Then in Theorem 2 we show that if the hyperbolic endomorphism f is 1-sided Bernoulli with respect to the measure of **maximal entropy** μ_0, then f must in fact be **(distance)-expanding** in the usual sense on Λ. And that, if f is expanding on Λ and if the equilibrium measure μ_ϕ is 1-sided Bernoulli, then μ_ϕ must be the measure of **maximal entropy** μ_0. Thus there exists a strong relation between 1-sided Bernoullicity, the distance expanding property and the measure of maximal entropy on Λ. In particular from Corollary 1 it will follow that **no** hyperbolic non-expanding toral endomorphism can be 1-sided Bernoulli with respect to the Lebesgue (Haar) measure.

In Theorem 3, we study hyperbolic **toral** endomorphisms and families of Holder potentials ϕ whose respective equilibrium measures μ_ϕ are **not 1-sided Bernoulli**. To do this we will employ commuting automorphisms in the case when the Jacobian-generated σ-algebra $\beta_{\mu_\phi}(f)$ is equal to $\mathcal{B}(\Lambda)$ (see [18,28]). The lack of 1-sided Bernoullicity above is in clear contrast to the case of hyperbolic toral *automorphisms* (for these see [6]); and in contrast with a class of 1-sided Bernoulli toral discontinuous skew-products given in [16].

In Theorem 4 we prove the **mixing** of arbitrary orders for equilibrium measures of Holder potentials for hyperbolic endomorphisms on folded basic sets. We obtain also Exponential Decay of Correlations on Holder observables.

We give then several classes of **examples** of hyperbolic saddle-type endomorphisms with equilibrium measures for which we check 1-sided Bernoullicity or lack of it.

Finally in Corollary 2 we prove the non-existence of generating **Rokhlin partitions** for certain endomorphisms with equilibrium measures. In particular an arbitrary hyperbolic non-expanding toral endomorphism with Haar measure does not have a generating Rokhlin partition.

2 Coding and mixing on folded basic sets

We will work with smooth (say $\mathcal{C}^2$), non-invertible maps $f : M \to M$ defined on a smooth Riemannian manifold M. A *locally maximal* set Λ is an invariant compact set which has a neighbourhood $U \subset M$ with $\Lambda = \cap_{n\in\mathbb{Z}} f^n(U)$. By *basic set* for f we mean here a locally maximal set Λ such that f is topologically mixing on Λ. As the map f is non-invertible on Λ, we will sometimes say that Λ is a *folded basic set* (or a *basic set with overlaps*, or *folded fractal*).

Our endomorphisms will be assumed *hyperbolic* on basic sets; the definition of hyperbolicity for *endomorphisms* (see [19,24]) is different than the one for diffeomorphisms and involves the various prehistories of points $x \in \Lambda$ with respect to f, namely sequences $\hat{x} = (x, x_{-1}, x_{-2}, \dots)$ consisting of consecutive preimages, i. e $f(x_{-i}) = x_{-i+1}, i \geq 1$. We need therefore the *inverse limit* (or *natural extension*) $\hat{\Lambda} := \{\hat{x}, \hat{x} = (x, x_{-1}, x_{-2}, \dots), x_{-i} \in \Lambda, i \geq 0$, s. t $\hat{x}$ is a prehistory of $x \in \Lambda\}$; this is a compact metric space with the canonical metric, $d(\hat{x}, \hat{y}) := \sum_{i\geq 0} \frac{d(x_{-i}, y_{-i})}{2^i}, \hat{x}, \hat{y} \in \hat{\Lambda}$. Notice that the canonical projection $\pi : \hat{\Lambda} \to \Lambda, \pi(\hat{x}) = x$, is Lipschitz continuous in the above metric. We have also the *shift homeomorphism* $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}, \hat{f}(\hat{x}) = (f(x), x, x_{-1}, \dots)$, $\hat{x} \in \hat{\Lambda}$.

Definition 1 Let Λ be a basic set for the smooth endomorphism $f : M \to M$. Then we say that f is **hyperbolic** on Λ if there exists a splitting of the tangent bundle over $\hat{\Lambda}$, $T_{\hat{\Lambda}}M = \{(\hat{x}, v), \hat{x} \in \hat{\Lambda}, v \in T_x M\}$ into a direct sum $T_{\hat{x}}M = E^s_x \oplus E^u_{\hat{x}}$ such that $Df_x(E^s_x) \subset E^s_{f(x)}$, $Df_x(E^u_{\hat{x}}) \subset E^u_{\hat{f}\hat{x}}$, $\hat{x} \in \hat{\Lambda}$ and Df contracts uniformly on E^s_x and Df expands uniformly on $E^u_{\hat{x}}$.

Associated to each prehistory we have local unstable manifolds $W^u_r(\hat{x})$ and local stable manifolds $W^s_r(x)$ (the local stable manifolds depend only on the base point). Since the unstable tangent spaces $E^u_{\hat{x}}$ depend on the whole past, there may exist many unstable manifolds going through the same point; this changes the dynamics on Λ, as compared with diffeomorphisms (see for instance [10,11,13,19,24], etc).

We adopt in this paper the above definition for basic set (where $f|_\Lambda$ is assumed topologically mixing), which is somewhat more restrictive than the usual one requiring that f be only topologically forward transitive on Λ. However for hyperbolic locally maximal sets this is not crucial. Indeed, if f were only transitive on Λ, then every point in Λ is nonwandering; hence by the same proof as in Corollary 6.4.19 from [7], it follows that: if f is hyperbolic on Λ, then its periodic points are dense in Λ. Thus as in the Spectral Decomposition Theorem ([7,24], etc.), there exists a finite partition of Λ, $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_k$ s.t for each $i = 1, \ldots, k$ there is a positive integer m_i s.t the iterate f^{m_i} invariates and is topologically mixing on Λ_i.

Now by *n-preimage* of $x \in \Lambda$ we consider any point $y \in f^{-n}(x) \cap \Lambda$; a word of caution is in place here: the set Λ is *not necessarily totally f-invariant*, so there may exist points $z \in M \setminus \Lambda$ such that $f^n(z) = x \in \Lambda$. However we work *only with the restriction of f to* Λ and will consider *only* those preimages remaining in Λ.

Since we work with a hyperbolic endomorphism f on Λ, we can lift it to the shift homeomorphism $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$. One can notice quickly that $\hat{f}$ is *expansive*. Indeed let us take $\varepsilon > 0$ small enough and assume that $\hat{x}, \hat{y} \in \hat{\Lambda}$ such that $d(\hat{f}^n\hat{x}, \hat{f}^n\hat{y}) \le \varepsilon, n \in \mathbb{Z}$. Then we would have $d(f^n x, f^n y) \le \varepsilon, n \ge 0$, thus $y \in W^s_\varepsilon(x)$ and $d(x_{-n}, y_{-n}) \le \varepsilon, n \ge 0$, so $y \in W^u_\varepsilon(\hat{x})$. But from [7, p. 272], one obtains that any hyperbolic locally maximal set has *local product structure*; hence from above it follows that $y = x$, and similarly $y_{-n} = x_{-n}, n \ge 0$. Thus $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ is expansive.

In the sequel we shall use also the *specification property* for homeomorphisms as defined in [7, p. 578]. The proof of the Specification Theorem 18.3.9 from [7] can be repeated for endomorphisms to show that if f is hyperbolic on the basic set Λ, then $f|_\Lambda$ has the specification property. From this we see easily that $\hat{f}$ has the specification property on $\hat{\Lambda}$ too; this follows since for a given specification $\hat{S} = \{\hat{x}^1 = (x^1, x^1_{-1}, \ldots), \ldots, \hat{x}^k = (x^k, x^k_{-1}, \ldots)\}$ in $\hat{\Lambda}$ we can apply the specification property of $f|_\Lambda$ to a specification S in Λ, formed with iterates of certain preimages $x^1_{-m}, \ldots, x^k_{-m}$ for $m > 0$ large enough.

So from the discussion above, we know that $\hat{f}$ is an expansive homeomorphism with the specification property on the inverse limit space $\hat{\Lambda}$.

Let now an f-invariant probability measure μ on the invariant set Λ. We always consider the compact set Λ endowed with the σ-algebra of its *borelian subsets*, denoted by $\mathcal{B}(\Lambda)$. All measures considered are borelian and probabilistic.

Consider a real valued Holder continuous potential ϕ on Λ. Then from [2] or [7, p. 635], there exists a unique equilibrium measure $\hat{\mu}_{\phi\circ\pi}$ on $\hat{\Lambda}$ for the Holder potential $\phi \circ \pi$, where $\pi : \hat{\Lambda} \to \Lambda$ is the canonical projection $\pi(\hat{x}) = x$. But any $\hat{f}$-invariant measure $\hat{\mu}$ on $\hat{\Lambda}$ has a unique push forward $\mu = \pi_*(\hat{\mu})$ and viceversa (see [25, p. 118]); also topological pressure is preserved by the canonical projection. So we obtain a unique equilibrium measure μ_ϕ on Λ for the non-invertible map f, and $\mu_\phi = \pi_* \hat{\mu}_{\phi \circ \pi}$.

By using the canonical metric on $\hat{\Lambda}$, we form the Bowen balls $\hat{B}_n(\hat{x}, \varepsilon) := \{\hat{y} \in \hat{\Lambda}, d(\hat{f}^i \hat{y}, \hat{f}^i \hat{x}) < \varepsilon, i = 0, \dots, n-1\}$. Then as in [7, p. 630], we can estimate $\hat{\mu}_{\phi\circ\pi}$ on these Bowen balls. But there exists a positive constant T depending on f such that $B_n(x, \frac{\varepsilon}{T}) \subset \pi(\hat{B}_n(\hat{x}, \varepsilon)) \subset B_n(x, \varepsilon), \hat{x} \in \hat{\Lambda}$; and $\pi^{-1}\pi(\hat{B}_n(\hat{x}, \varepsilon))$ is contained in a finite union of balls of type $\hat{B}_n(\hat{x}^i, T\varepsilon)$ for some prehistories $\hat{x}^i$ of x. On the other hand we have $P(\phi) = P(\phi \circ \pi)$. Hence from the estimates on $\hat{B}_n(\hat{x}, \varepsilon)$ obtained in [7, p. 630], and since $\mu_\phi = \pi_* \hat{\mu}_{\phi\circ\pi}$, we conclude that for any $\varepsilon > 0$ there are constants $A_\varepsilon, B_\varepsilon > 0$ s.t:

$$A_\varepsilon e^{S_n\phi(x) - nP(\phi)} \le \mu_\phi(B_n(x, \varepsilon)) \le B_\varepsilon e^{S_n\phi(x) - nP(\phi)}, x \in \Lambda, n > 0, \tag{1}$$

where $P(\phi)$ is the topological pressure of ϕ, $B_n(x, \varepsilon) := \{y \in \Lambda, d(f^i x, f^i y) \le \varepsilon, i = 0, \dots, n-1\}$ is a Bowen ball and $S_n\phi(x) := \phi(x) + \cdots + \phi(f^{n-1}x)$. Inspired by (1), we give the following:

Definition 2 Two quantities $Q_1(n, x), Q_2(n, x)$ depending on the variables $n > 1, x \in \Lambda$, are said to be *comparable*, i.e $Q_1(n, x) \approx Q_2(n, x)$, if there exist positive constants A, B such that $A \cdot Q_1(n, x) \le Q_2(n, x) \le B \cdot Q_1(n, x)$ for all n, x.

Let us now denote by $\Sigma_d^+ := \{1, \dots, d\}^{\mathbb{Z}^+}$ the space of sequences ω of $1, \dots, d$, indexed by the nonnegative integers. On Σ_d^+ we consider the shift $\sigma_d : \Sigma_d^+ \to \Sigma_d^+$; also for a probability vector $p = (p_1, \dots, p_d)$ we define the σ_d-invariant product measure ν_p, with the initial probabilities $\nu_p(\{\omega, \omega_0 = i\}) = p_i, i = 1, \dots, d$. The triple $(\Sigma_d^+, \sigma_d, \nu_p)$ is called a (model) *1-sided Bernoulli shift*. By extension we call *1-sided Bernoulli shift* any triple (X, f, μ), with μ f-invariant, which is measure-theoretically isomorphic to $(\Sigma_d^+, \sigma_d, \nu_p)$, for some $d \ge 1$ and $p = (p_1, \dots, p_d)$ a probabilistic vector.

In the sequel we will use the important notions of *Jacobian of an invariant measure* introduced by Parry in [17], and that of *index* of a countable-to-one endomorphism of Lebesgue spaces (see [18]). In short, the **Jacobian** of the f-invariant probability measure μ on the Lebesgue space (X, f, μ) is the Radon-Nikodym derivative of $\mu \circ f$ with respect to μ. If (X, f, μ) is a measure-preserving system (with some σ-algebra $\mathcal{B}$), and if ϵ is the point partition, one can form the fiber partition $\xi = f^{-1}\epsilon$ which is a measurable partition if f is countable-to-1 on (X, μ); let also $\pi : X \to X/\xi$ be the canonical projection. This partition induces a *factor space* $(X/\xi, g, \nu)$, where an arbitrary point z of X/ξ is a fiber $f^{-1}(x), x \in X, g(z) := \pi(x), z \in X/\xi$ and $\nu(E) := \mu(\pi^{-1}(E))$, E measurable in X/ξ. Now from the Rokhlin theory of measurable partitions (see [17,21], etc.), ξ induces a family of *conditional measures* on the fibers of f, $\{\mu_z\}_{z \in X/\xi}$ such that $\mu(A) = \int_{X/\xi} \mu_z(A \cap z) d\nu(z)$, for A measurable in X. This family of conditional measures is unique modulo ν. Notice that μ_z is a probability measure on the (at most countable) fiber $z = f^{-1}x$; its support supp μ_z is a subset of $f^{-1}x$. Then the **index** of (X, f, μ) is the measurable function

$$ind_\mu(f)(x) := \text{card}(\text{supp } \mu_z), z = f^{-1}x, \text{ for } \mu - \text{a.e } x \in X$$

For an f-invariant probability measure μ on Λ, let $\lambda_1(x) < \cdots < \lambda_{S(x)}(x) < 0$ be the negative Lyapunov exponents of μ with respect to f, which are defined for μ-a.e $x \in \Lambda$; let also the i-th partial stable manifold $W_i^s(x) := \{y \in M, \limsup_{n\to\infty} \frac{1}{n} \log d(f^n x, f^n y) \leq \lambda_i(x)\}, 1 \leq i \leq S(x)$. It is clear that the (usual) stable manifold of x, namely $W^s(x)$ is actually $W^s_{S(x)}(x)$. We also denote for $r > 0$ small, by $W^s_{i,r}$ the i-th partial stable manifold of radius r. In our case since we work with uniformly hyperbolic maps, r can be chosen independent of x.

One can find a measurable partition ξ of Λ, subordinate to the partial stable manifolds W_i^s (see for instance [26]) and can define the i-th **pointwise stable dimension** of μ, or the **dimension of μ on W_i^s-manifolds** as

$$\delta_i^s(\mu, x, \xi) := \liminf_{r\to 0} \frac{\log \mu_x^\xi(B^i(x,r))}{\log r},$$

where $\{\mu_x^\xi\}_x$ is the system of conditional measures of μ associated to the partition ξ and $B^i(x, r)$ is the ball of radius r centered at x inside W_i^s. It can be shown that $\delta_i^s(\mu, x, \xi)$ does not depend on ξ and it is constant along orbits.

Moreover we have $\delta_i^s(\mu, x, \xi) = \limsup_{r\to 0} \frac{\log \mu_x^\xi(B^i(x,r))}{\log r}$.

So if μ is ergodic, then the **pointwise i-th stable dimension of** μ, denoted by $\delta_i^s(\mu)$, is defined by $\delta_i^s(\mu) = \delta_i^s(\mu, x, \xi)$, μ-a.a $x \in \Lambda$, and $1 \leq i \leq S(x) = S$.

We show now that if the triple (Λ, f, μ_ϕ) is coded by a 1-sided Bernoulli shift, then f must be expanding on Λ from a certain measure-theoretical point of view. This is in **contrast** with the hyperbolic diffeomorphism case, where **all** equilibrium measures of Holder potentials can be coded with 2-sided Bernoulli shifts.

In general for a measurable partition ξ of Λ denote by $\xi(x)$ the unique (modulo μ) set of ξ which contains x. For a measurable partition ξ subordinated to the stable manifolds W_S^s, we can define the **stable dimension of** μ on $\xi(x)$ as:

$$HD^s(\mu, x) := HD(\mu_x^\xi) = \inf\{HD(Z), Z \subset \xi(x), \mu_x^\xi(Z) = 1\}, \mu - \text{a.e } x \in \Lambda$$

We remind the definition of *expanding map* from [7, p. 71]; the metric considered on Λ is the one induced from the Riemannian metric on M.

Theorem 1 *Let f be a smooth hyperbolic endomorphism on a connected basic set Λ; let also ϕ be a Holder continuous potential on Λ and μ_ϕ the unique equilibrium measure of ϕ. Then, if the measure-preserving system (Λ, f, μ_ϕ) is 1-sided Bernoulli, it follows that either f is distance-expanding on Λ, or the stable dimension of μ_ϕ is zero, i.e $HD^s(\mu_\phi, x) = 0$ for μ_ϕ-a.e $x \in \Lambda$.*

Proof Let us assume that (Λ, f, μ_ϕ) is 1-sided Bernoulli, i.e isomorphic to $(\Sigma_d^+, \sigma_d, \nu_p)$ for some $d > 1$ and probability vector p. Now the equilibrium measure of a Holder potential μ_ϕ is supported everywhere, since the μ_ϕ-measure of any ball is positive, from estimate (1). Thus, as the index function is preserved by isomorphisms (see [18]) and since any point from Λ has finitely many preimages, it follows that the fiber $f^{-1}(x)$ must contain d points for μ_ϕ-almost all $x \in \Lambda$. Also since we have an isomorphism with a 1-sided Bernoulli shift, we know from [17] that the Jacobian $J_{\mu_\phi}(f)$ of μ_ϕ, must be equal a.e with the Jacobian of the product measure ν_p.

Let us consider now a measurable partition ξ of Λ subordinated to the local stable manifolds W^s; by $\xi(x)$ we shall denote the set of ξ that contains x. We recall that $W^s_{S,r} = W^s_r$ notationally.

Since f is uniformly hyperbolic on Λ and thus the local stable/unstable manifolds have a fixed positive radius, it follows that we may take the partition ξ to be with borelian subsets of the stable manifolds which contain a smaller stable set of fixed radius, i. e there exist $r_0, r_1 > 0$ s.t $W^s_{r_1}(x) \subset \xi(x) \subset W^s_{r_0}(x)$, μ_ϕ-a.a $x \in \Lambda$. To this measurable partition ξ, we can associate (uniquely) a family of conditional measures of μ_ϕ; a generic element of this family is denoted by $\mu^\xi_{\phi,x}$ and it is a probability measure on the subset $\xi(x)$ of ξ (containing the point x).

We want to show now that for μ_ϕ-almost all points $x \in \Lambda$ we have that the conditional measure $\mu^\xi_{\phi,x}$ gives positive measure to any non-empty open subset in the local stable manifold $\xi(x)$. First we notice that if A is the intersection of a Bowen ball $B_m(y, \varepsilon)$ with a neighbourhood of the local unstable manifold $W^u_\varepsilon(\hat{\zeta})$, $\hat{\zeta} \in \hat{\Lambda}$, then the measure μ^ξ_ϕ induced on the factor space Λ/ξ has the property that:

$$\mu^\xi_\phi(A/\xi) = \mu_\phi(B_m(y, \varepsilon))$$

But we know from the definition of conditional measures that

$$\mu_\phi(A) = \int\limits_{A/\xi} \mu^\xi_{\phi,x}(A \cap \xi(x)) d\mu^\xi_\phi(\xi(x)),$$

where $\xi(x)$ are the leaves of the measurable partition ξ which intersect A (in the factor space Λ/ξ these leaves are identified with points). But $\mu_\phi(A) > 0$, since A is an open set in Λ (thus contains some Bowen ball); also $\mu^\xi_\phi(A/\xi) = \mu_\phi(B_m(y, \varepsilon)) > 0$. Thus from the essential uniqueness of the conditional measures, and since the sets of type A as above form a basis for open sets, we obtain that for μ^ξ_ϕ-almost all partition leaves $\xi(x) \in \Lambda$, $\mu^\xi_{\phi,x}(V) > 0$, for V a neighbourhood of z and $z \in \xi(x)$. This implies that

$$\text{supp}\mu^\xi_{\phi,x} = \xi(x) \cap \Lambda, \mu_\phi - a.e$$

We will now use Theorem 1.1 of [26] translated to our case, for the ergodic equilibrium measure μ_ϕ. In this case the Lyapunov exponents are all constant a.e and will be denoted simply by λ_i. Denote also by

$$\gamma_1 := \delta_1^s(\mu_\phi), \gamma_2 := \delta_2^s(\mu_\phi) - \delta_1^s(\mu_\phi), \ldots, \gamma_S := \delta_S^s(\mu_\phi) - \gamma_{S-1}$$

Recall now the notion of *folding entropy* $F_\mu(f)$ of an arbitrary f-invariant probability measure μ (see [22]), which is defined as the conditional entropy

$$F_\mu(f) := H_\mu(\epsilon | f^{-1}\epsilon),$$

where ϵ is the partition of M into single points.

We can consider thus the folding entropy $F_{\mu_\phi}(f)$ of an equilibrium measure μ_ϕ. From [17,22] it follows that the folding entropy $F_{\mu_\phi}(f)$ is equal to the integral of the logarithm of the Jacobian of μ_ϕ, i. e

$$F_{\mu_\phi}(f) = \int_\Lambda \log J_{\mu_\phi}(f) d\mu_\phi$$

And from [26] we have that:

$$h_{\mu_\phi}(f) = F_{\mu_\phi}(f) - \sum_{1 \le i \le S} \lambda_i \gamma_i(\mu_\phi), \tag{2}$$

Since (Λ, f, μ_ϕ) is isomorphic to $(\Sigma_m^+, \sigma_m, \nu_p)$ and since the Jacobian is preserved by isomorphisms of Lebesgue spaces (see [17]), it follows that

$$F_{\mu_\phi}(f) = \int_\Lambda \log J_{\mu_\phi}(f) d\mu_\phi = \int_{\Sigma_m^+} \log J_{\nu_p}(\sigma_m) d\nu_p = h_{\nu_p}(\sigma_m) = h_{\mu_\phi}(f)$$

Thus from (2) we obtain $\sum_{1 \le i \le S} \lambda_i \gamma_i(\mu_\phi) = 0$. But since we have a uniformly hyperbolic system, either f is distance-expanding on Λ (i. e it does not have stable directions), or $\lambda_i < 0, 1 \le i \le S$ and $\gamma_i(\mu_\phi) = \delta_i^s(\mu_\phi) = 0, 1 \le i \le S$.

Thus for a measurable partition ξ subordinated to the stable manifolds $W^s = W_S^s$,

$$\delta_S^s = \limsup_{r \to 0} \frac{\log \mu_{\phi,x}^\xi(B(y,r))}{\log r} = 0, \text{ for } \mu_\phi - \text{a.e } x, \text{ and } \mu_{\phi,x}^\xi - \text{a.e } y \in \xi(x)$$

So there exists a set $E \subset \Lambda$ with $\mu_\phi(E) = 1$ so that for any small $\beta > 0$, there exists $r(y, \beta) > 0$, $y \in E$ such that

$$\mu_{\phi,x}^\xi(B(y,r)) > r^\beta, 0 < r < r(y,\beta), y \in E \cap \xi(x), \tag{3}$$

for μ_ϕ-a.e $x \in \Lambda$. From the definition of conditional measures (see [17,21]), we deduce that if $\mu_\phi(E) = 1$ then for almost all x, $\mu_{\phi,x}^\xi(E \cap \xi(x)) = 1$. So for almost all leaves $\xi(x)$ of ξ, $\mu_{\phi,x}^\xi$-almost all points $y \in \xi(x)$ satisfy (3).

Now using the Vitali Covering Theorem, we can cover a set $E' \subset E \cap \xi(x)$ having $\mu_{\phi,x}^\xi(E') = 1$, with mutually disjoint balls $B(y, \rho(y))$ where $\rho(y) < r(y, \beta)$. Thus we obtain a cover with a family of mutually disjoint balls $B(y, \rho(y))$, $y \in F \subset E \cap \xi(x)$ and

$$1 \geq \sum_{y \in F} \mu^{\xi}_{\phi,x}(B(y, \rho(y))) \geq \sum_{y \in F} \rho(y)^{\beta}$$

Hence $HD(E') \leq \beta$ for μ_ϕ-almost all $x \in \Lambda$. But $\beta > 0$ is arbitrarily small; hence recalling also that $\mu^{\xi}_{\phi,x}(E \cap \xi(x)) = \mu^{\xi}_{\phi,x}(E') = 1$ we obtain

$$HD^s(\mu_\phi, x) = 0, \ \mu_\phi - \text{a.e } x \in \Lambda$$

□

For a system endowed with the measure of maximal entropy, we can say more:

Theorem 2 (a) *Let f be a smooth endomorphism on a Riemannian manifold M such that f is hyperbolic on the basic set Λ and the critical set $\mathcal{C}_f$ does not intersect Λ. Then if the system (Λ, f, μ_0) given by the measure of maximal entropy μ_0 is 1-sided Bernoulli, it follows that f is expanding on Λ.*
(b) *Assume f is an expanding endomorphism on Λ. If μ_ϕ is the equilibrium measure of the Holder potential ϕ and if (Λ, f, μ_ϕ) is 1-sided Bernoulli, then $\mu_\phi = \mu_0$, where μ_0 is the unique measure of maximal entropy for f on Λ.*

Proof (a) In the sequel we work with the restriction of f to Λ, $f|_\Lambda : \Lambda \to \Lambda$. From (1) and Definition 2 it follows that, for $\varepsilon > 0$ small enough,

$$\mu_0(B_n(x, \varepsilon)) \approx \frac{1}{e^{nh_{top}(f)}}, n > 0, x \in \Lambda,$$

and the comparability constants do not depend on n, x.

Assume that (Λ, f, μ_0) is isomorphic to $(\Sigma^+_d, \sigma_d, \nu_p)$ for a certain probability vector $p = (p_1, \ldots, p_d)$. Hence since the measure-theoretic entropy is preserved by isomorphisms (see [18]), it follows that

$$h_{\mu_0}(f) = h_{top}(f) = h_{\nu_p}(\sigma_d) \leq \log d \tag{4}$$

Also we know that the index is preserved by isomorphisms (see [18,28]), thus f is at least d-to-1 on $\Lambda \mu_0$-a.e.

Let us now consider a Rokhlin partition of (Λ, f, μ_0) with the sets $A_1, \ldots, A_d$ (see for example [17]); we have that $f|_{A_i} : A_i \to \Lambda$ is bijective (modulo μ_0) for any $i = 1, \ldots, d$. Denote $G := \{x \in \Lambda, |f^{-1}(x) \cap \Lambda| \geq d\}$. From above, we know that $\mu_0(G) = 1$. Let now $G_1 := f(G \cap A_1) \cap \ldots \cap f(G \cap A_d)$; this can be viewed also as the set of points x having at least d preimages in Λ, and such that each of its preimages has at least d preimages in turn. Notice now that since $\mu_0 \circ f$ is absolutely continuous with respect to μ_0 (see [17]), we obtain $\mu_0(f(G \cap A_i)) = \mu_0(f(A_i)) = 1, i = 1, \ldots, d$. Therefore $\mu_0(G_1) = 1$. In general define inductively

$$G_j := f(G_{j-1} \cap A_1) \cap \ldots \cap f(G_{j-1} \cap A_d), j \geq 2$$

Thus all points in G_j have at least $d^{j+1} f^{j+1}$-preimages in Λ, and by induction and a similar argument as above, we have $\mu_0(G_j) = 1, j \geq 1$. Also it is clear that $G_j \subset G_{j-1}, j \geq 1$ (mod μ_0), where $G_0 := G$.

But for any given $x \in \Lambda$, the set $f^{-n}(x) \cap \Lambda$ is an (n, ε)-separated set for $\varepsilon > 0$ small enough, since $\mathcal{C}_f \cap \Lambda = \emptyset$; so if $x \in G_n$, then there exist at least $d^n f^n$-preimages of x in Λ for $n > 2$. This implies that

$$h_{top}(f|_\Lambda) \geq \log d$$

This implies that $h_{\nu_p}(\sigma_d) = h_{\mu_0}(f) = \log d$, hence ν_p is the measure of maximal entropy on Σ_d^+. Therefore the probability vector p is equal to $(\frac{1}{d}, \ldots, \frac{1}{d})$. Hence

$$J_{\nu_p}(\sigma_d) = d, \nu_p - a.e$$

But the Jacobians are preserved by measure-theoretic isomorphisms, hence

$$J_{\mu_0}(f) = d, \mu_0 - \text{a.e, and } J_{\mu_0}(f^n) = d^n, n > 0, \mu_0 - \text{a.e}$$

Thus from the properties of Jacobians from [17], we obtain that

$$\mu_0(f^n(B_n(x, \varepsilon))) = \int\limits_{B_n(x,\varepsilon)} J_{\mu_0}(f^n) d\mu_0 = d^n \cdot \mu_0(B_n(x, \varepsilon)) \approx \frac{d^n}{e^{nh_{top}(f)}} = 1,$$

where the comparability constants do not depend on n, x.

This means that for $r > 0$ sufficiently small, the intersection $W_r^s(x) \cap \Lambda$ is equal to $\{x\}$, for $x \in \Lambda$. Hence f can be considered to be expanding on Λ since on Λ there are no points y close to x and forward-asymptotic to x, for any $x \in \Lambda$.

(b) Since f is assumed expanding on Λ now, we have from [23] or [8] that the equilibrium measure μ_ϕ is the weak limit of the sequence of measures

$$\mu_n^x := \sum_{y \in f^{-n}(x) \cap \Lambda} \frac{\delta_y \cdot e^{S_n \phi(y)}}{e^{nP(\phi)}}, n > 1,$$

i.e $\mu_n^x \to_{n \to \infty} \mu_\phi$ for any $x \in \Lambda$. This implies easily that the Jacobian of μ_ϕ in the expanding case is

$$J_{\mu_\phi}(f)(x) = e^{-\phi(x) + P(\phi)}, \tag{5}$$

for μ_ϕ-almost all $x \in \Lambda$.

On the other hand, the probability vector $p = (p_1, \ldots, p_d)$ gives the 1-sided Bernoulli measure ν_p on Σ_d^+, and we have the invariance of the Jacobians by the measure theoretic isomorphism. So $J_{\mu_\phi}(f) = J_{\nu_p}(\sigma_d)$ and $J_{\mu_\phi}(f)$ must take the values $\frac{1}{p_1}, \ldots, \frac{1}{p_d}$ respectively, on the sets of a measurable partition of Λ. But we showed in (5) that $J_{\mu_\phi}(f)$ is in fact equal μ_ϕ-a. e with the continuous function $e^{-\phi + P(\phi)}$. Since

μ_ϕ gives positive measure to open sets we obtain then that all the values $p_1, \dots, p_d$ must be equal, i.e $p_1 = \dots = p_d = \frac{1}{d}$. Also it follows that the continuous function ϕ must be constant a.e. Hence $\mu_\phi = \mu_0$, where μ_0 is the measure of maximal entropy. □

From the above Theorem we obtain immediately the following:

Corollary 1 *Let f_A be a hyperbolic endomorphism of the torus $\mathbb{T}^m$ $(m \geq 2)$, given by the integer valued matrix A. Assume that A has both eigenvalues of absolute value larger than* 1 *and eigenvalues of absolute value strictly less than* 1. *Then the measure-preserving system* $(\mathbb{T}^m, f_A, m)$ *is not* 1*-sided Bernoulli, where m is the Lebesgue (Haar) measure.*

We study now **other equilibrium measures** μ_ϕ for hyperbolic toral endomorphisms.

Theorem 3 *Consider a hyperbolic non-expanding toral endomorphism $f_A : \mathbb{T}^m \to \mathbb{T}^m$ associated to the integer valued matrix A. Assume $|det(A)| = 2$, let $\alpha \neq (0, \dots, 0)$ be a fixed point of f_A, and let ϕ be a periodic Holder continuous function of period α on $\mathbb{T}^m$. Then $(\mathbb{T}^m, f_A, \mu_\phi)$ is not isomorphic to $(\Sigma_2^+, \sigma_2, \nu_p)$, for $p = (p_1, p_2)$, $p_1 \neq \frac{1}{2}$.*

Proof First remark that the number of f_A-preimages of any point in $\mathbb{T}^m$ is constant and equal to $|\det(A)|$. So f_A is 2-to-1 on $\mathbb{T}^m$, then the only 1-sided Bernoulli shifts which *could* possibly be isomorphic to $(\mathbb{T}^m, f_A, \mu_\phi)$ live on (Σ_2^+, σ_2). Assume then that $(\mathbb{T}^m, f_A, \mu_\phi)$ is isomorphic to $(\Sigma_2^+, \sigma_2, \nu_{(p_1,p_2)})$ with $p_1 \neq \frac{1}{2}$.

Since A is hyperbolic, 1 is not an eigenvalue for A, so $A - I$ is invertible. Now remark that for an integer-valued matrix A, there exist exactly $|\det(A - I)|$ isolated fixed points for f_A on $\mathbb{T}^m$. Since in our case $A - I$ is invertible, we have that $\det(A - I) \neq 0$, so there exist isolated fixed points for f_A.

Let α be such a fixed point for f_A in $\mathbb{T}^m$. Denote by $T_\alpha(x) := x - \alpha = (x_1 - \alpha_1, \dots, x_m - \alpha_m)$, $x \in \mathbb{T}^m$. It can be seen easily that T_α is well defined and that it is a bijection on $\mathbb{T}^m$. Also since α is fixed point for f_A, T_α commutes with f_A, i.e

$$T_\alpha \circ f_A = f_A \circ T_\alpha \tag{6}$$

We want to show now that T_α preserves the measure μ_ϕ if ϕ is periodic of period α. For this recall how the equilibrium measure μ_ϕ was constructed: μ_ϕ is the weak limit of a sequence of probability measures of type

$$\mu_n := \sum_{y \in \mathrm{Fix}(f_A^n) \cap \Lambda} \frac{e^{S_n\phi(y)} \delta_y}{\sum_{y \in \mathrm{Fix}(f_A^n) \cap \Lambda} e^{S_n\phi(y)}}$$

Now if B is a borelian set in Λ with $\mu_\phi(\partial B) = \mu_\phi(\partial T_\alpha(B)) = 0$, then we know that $\mu_n(B) \to \mu_\phi(B)$. Now $\mu_n(B) = \sum_{y \in \mathrm{Fix}(f_A^n) \cap B} \frac{e^{S_n\phi(y)} \delta_y}{\sum_{y \in \mathrm{Fix}(f_A^n) \cap \Lambda} e^{S_n\phi(y)}}$ and $\mu_n(T_\alpha(B)) = \sum_{y \in \mathrm{Fix}(f_A^n) \cap T_\alpha(B)} \frac{e^{S_n\phi(y)} \delta_y}{\sum_{y \in \mathrm{Fix}(f_A^n) \cap \Lambda} e^{S_n\phi(y)}}$. But $y \in \mathrm{Fix}(f_A^n) \cap B$ if and only if $T_\alpha(y) \in \text{Fix}(f_A^n) \cap T_\alpha(B)$, since f_A is linear and α is a fixed point for f_A; at the same time notice that $S_n\phi(y) = S_n\phi(y-\alpha), n \geq 1$ since ϕ was chosen to be periodic of period α. Therefore we obtain that $\mu_n(B) = \mu_n(T_\alpha(B)), n \geq 1$ and thus for the type of sets B considered above, we have $\mu_\phi(B) = \mu_\phi(T_\alpha(B))$. But the sets B considered above form a sufficient family of borelians, hence

$$\mu_\phi(B) = \mu_\phi(T_\alpha(B)),$$

for any borelian set B in Λ.

Hence we proved that the nontrivial automorphism T_α preserves the measure μ_ϕ and commutes with f_A. Let now $\beta_{\mu_\phi}(f_A)$ be the smallest σ-algebra contained in $\mathcal{B}(\mathbb{T}^m)$ with respect to which the Jacobian $J_{\mu_\phi}(f_A)$ is measurable and s.t $f_A^{-1}\beta_{\mu_\phi}(f_A) \subset \beta_{\mu_\phi}(f_A)$. The fact that the system $(\mathbb{T}^m, f_A, \mu_\phi)$ was assumed measure-theoretically isomorphic to $(\Sigma_2^+, \sigma_2, \nu_{(p_1,p_2)})$ with $p_1 \neq \frac{1}{2}$ implies that:

$$\beta_{\mu_\phi}(f_A) = \mathcal{B}(\mathbb{T}^m)$$

(see [28]). Notice that if p_1 were $\frac{1}{2}$, then the last statement would not hold. Now if $\beta_{\mu_\phi}(f_A)$ is equal to the σ-algebra of borelians on $\mathbb{T}^m$ and if we have a nontrivial automorphism T_α commuting with f_A and preserving μ_ϕ, we can apply [4] Theorem 2.21 (see also [18,28]) in order to get a contradiction. In conclusion we obtain that $(\mathbb{T}^m, f_A, \mu_\phi)$ is **not** a 1-sided $\{p_1, p_2\}$ Bernoulli shift with $p_1 \neq \frac{1}{2}$. □

We now prove mixing of any order (see [20] for definition) and Exponential Decay of Correlations (see [2,3] for definitions) in general, for the triple (Λ, f, μ_ϕ).

Theorem 4 *Let f be a smooth endomorphism on M, hyperbolic on a basic set Λ and let ϕ be a Holder continuous potential defined on Λ; let μ_ϕ be the unique equilibrium measure of ϕ. Then:*

(a) *the measure-preserving system (Λ, f, μ_ϕ) is mixing of any order.*
(b) *the measure μ_ϕ has Exponential Decay of Correlations on Holder observables.*

Proof (a) By assumption the map f is uniformly hyperbolic on Λ, so as in [7, p. 272], we obtain that f has local product structure on Λ, and similarly $\hat{f}$ has local product structure on $\hat{\Lambda}$ with local stable sets (defined for some $\delta > 0$ small enough):

$$V_{\hat{x}}^- := \{\hat{y} \in \hat{\Lambda}, d(\hat{f}^n\hat{y}, \hat{f}^n\hat{x}) < \delta, n \geq 0\},$$

and local unstable sets

$$V_{\hat{x}}^+ := \{\hat{y} \in \hat{\Lambda}, d(\hat{f}^{-n}\hat{y}, \hat{f}^{-n}\hat{x}) < \delta, n \geq 0\}, \hat{x} \in \hat{\Lambda}$$

This implies that $(\hat{\Lambda}, \hat{f})$ has a Smale space structure, as defined in [25].

Now since the potential ϕ on Λ is Holder continuous and as $\pi : \hat{\Lambda} \to \Lambda$ is Lipschitz continuous, it follows that $\hat{\phi} := \phi \circ \pi : \hat{\Lambda} \to \mathbb{R}$ is Holder continuous; so to the unique equilibrium measure μ_ϕ of ϕ it corresponds the unique equilibrium measure

$\mu_{\hat{\phi}}$ of $\hat{\phi}$ on $\hat{\Lambda}$ s.t $\mu_\phi = \pi_* \mu_{\hat{\phi}}$. We have that $P_f(\phi) = P_{\hat{f}}(\hat{\phi})$ and $h_{\mu_\phi}(f) = h_{\mu_{\hat{\phi}}}(\hat{f})$. Also $\int_\Lambda \phi d\mu_\phi = \int_{\hat{\Lambda}} \phi \circ \pi d\mu_{\hat{\phi}}$.

Now we assumed that f is topologically mixing on Λ, which implies easily that $\hat{f}$ is topologically mixing on $\hat{\Lambda}$ (this is standard proof by considering certain pre-images of large order). But from [25] Corollary 7.10 d) we have then, that $(\hat{\Lambda}, \hat{f}, \mu_{\hat{\phi}})$ is isomorphic to a Bernoulli automorphism. Hence as Bernoulli automorphisms are Kolmogorov (by [8, p. 161], it follows that $(\hat{\Lambda}, \hat{f}, \mu_{\hat{\phi}})$ is mixing of any order. Thus (Λ, f, μ_ϕ) is mixing of any order (see [20]).

(b) We have Exponential Decay of Correlations on Holder observables, for the inverse limit $(\hat{\Lambda}, \hat{f}, \hat{\mu}_\phi)$ since from a), this is a Bernoulli automorphism (see [2,3]).

Then due to the bijective correspondence between f-invariant probabilities on Λ and $\hat{f}$-invariant probabilities on $\hat{\Lambda}$, and by the invariance of measure-theoretic entropies and integrals discussed above, we obtain Exponential Decay of Correlations on Holder observables for the system (Λ, f, μ_ϕ) as well. □

More Examples: Theorems 2 and 4 apply also to the examples of hyperbolic skew-product endomorphisms constructed in [10] and in [14].

For the nonlinear skew products with overlaps in fibers $f_\alpha(x, y) = (g(x), h_\alpha(x, y))$ and their basic sets Λ_α from [10], we showed that there exist Cantor sets in fibers, such that every point in such a set has uncountably many prehistories in $\hat{\Lambda}$. We also proved in Corollary 2 of [10] that the stable dimension in that case is non-zero, at any point of Λ_α, by using properties of the thickness of the intersection of Cantor sets. In fact if α is small enough, we proved that this stable dimension is close to 1. Thus the examples of [10] are non-invertible, hyperbolic and *non-expanding* on Λ_α, since the stable dimension is strictly positive. Hence we can apply Theorem 2 to prove that the system with the measure of maximal entropy (Λ, f, μ_0) is **not 1-sided Bernoulli**. More generally for the equilibrium measure μ_ϕ of an arbitrary Holder potential ϕ, we know from Theorem 1 that the system (Λ, f, μ_ϕ) is not 1-sided Bernoulli, as long as the stable dimension of μ_ϕ is non-zero a.e.

Also, for the family of parameterized hyperbolic skew products F_λ satisfying the transversality condition from [14] we proved for almost all parameters λ, a Bowen-type formula (on the natural extension) for the stable dimension of the respective basic set Λ_λ. One such example is

$$F_\lambda(x, y) = (f(x), \lambda_i + \Phi_i(x, y, \lambda)), x \in X_i, i = 1, \ldots, d,$$

where f and X_i are given by an iterated function system, and λ_i are real parameters.

Another example from [14] with transversality condition and defined on an open set $W \subset \mathbb{C}^2$, is:

$$F_\lambda(z, w) = \left(z^2 + c, h(z) + \frac{1}{5}w^2 + \lambda z^2\right),$$

where $|c|$ is small enough, h is a Lipschitz function satisfying a growth condition and $|\lambda| < \frac{1}{6}$ is a complex parameter. But since these examples satisfy the transversality condition, we can find the stable dimension as the zero of the pressure function of a certain potential, on the natural extension $\hat{\Lambda}$; but since $h_{top}(f|_\Lambda) > 0$, we obtain that the stable dimension is positive, hence the function F_λ is not expanding on Λ. So the system with the measure of maximal entropy $(\Lambda_\lambda, F_\lambda, \mu_{0,\lambda})$ is **not 1-sided Bernoulli**.

Also from Theorem 4 we have Exponential Decay of Correlations on Holder observables and mixing of any order, for all equilibrium measures of Holder potentials for the above examples of [10,14]. □

An important notion related to the coding problem for endomorphisms on Lebesgue spaces is that of *Rokhlin partition*. Let ϵ be the point partition on the Lebesgue space (X, f, μ), where μ is an f-invariant probability measure defined on the σ-algebra $\mathcal{B}$ on X. We denote by $\mathcal{P}_1 = \{E_1, \ldots, E_{m-1}\}$ a partition of X into measurable subsets so that $f|_{E_i}$ is a bijection a.e between E_i and X, $i = 0, \ldots, m-1$. Such a partition exists and it is called a **Rokhlin partition** (see [5,17,20]). Clearly it is not uniquely defined.

In general, given a Rokhlin partition $\mathcal{P}_1$, define the measurable partition

$$\mathcal{P} := \bigvee_{i \geq 1} T^{-i}\mathcal{P}_1$$

The measurable partition $\mathcal{P}_1$ is called **a 1-sided generator** for (X, f, μ) if the smallest sub-σ-algebra of $\mathcal{B}(\Lambda)$ containing $\mathcal{P}$ and complete with respect to μ, is equal modulo μ to the borelian σ-algebra $\mathcal{B}(\Lambda)$. In this case we will say also that $\mathcal{P}_1$ is a *generating partition*.

Corollary 2 (a) *Let an endomorphism f hyperbolic and non-expanding on a basic set Λ. Then there exists no generating Rokhlin partition $\mathcal{P}_1$ of (Λ, f, μ_0) s.t J_{μ_0} is piecewise constant a.e on the sets of $\mathcal{P}_1$ (where μ_0 is the measure of maximal entropy).*

Also if f is expanding on Λ but $\mu_\phi \neq \mu_0$, then there is no generating Rokhlin partition $\mathcal{P}_1$ of (Λ, f, μ_ϕ) s.t $J_{\mu_\phi}(f)$ piecewise constant a.e on the sets of $\mathcal{P}_1$.

(b) *A hyperbolic non-expanding toral endomorphism $f_A : \mathbb{T}^m \to \mathbb{T}^m$, $m \geq 2$, does not have generating Rokhlin partitions with respect to the Lebesgue measure.*

Proof (a) If there exists a generating Rokhlin partition $\mathcal{P}_1$ for (Λ, f, μ_ϕ) s.t the Jacobian J_{μ_ϕ} is constant μ_ϕ-a.e on the sets of the partition $\mathcal{P}_1$, then from Proposition 3.7 of [5] it follows that (Λ, f, μ_ϕ) is isomorphic to a 1-sided Bernoulli shift. But this gives then a contradiction with Theorem 2, since we assumed that f is non-expanding on Λ.

Hence no Rokhlin partition can be a generator for hyperbolic non-expanding endomorphisms as above equipped with the measure of maximal entropy.

Same conclusion holds if f is expanding on Λ, but μ_ϕ is not the measure of maximal entropy.

(b) This follows immediately from (a) for $\phi \equiv 0$, since the Jacobian of the Lebesgue measure μ_0 with respect to f_A, is constant and equal to $|\det(A)|$ a.e. Thus if there were generating Rokhlin partitions then from [5] it would follow that the system were 1-sided Bernoulli with respect to the Lebesgue measure, which is also the measure of maximal entropy.

Thus we obtain a contradiction with respect to Corollary 1. □

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