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Equilibrium measures on saddle sets of holomorphic maps on $\mathbb{P}^2$

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Abstract We consider the case of hyperbolic basic sets Λ of saddle type for holomorphic maps $f : \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$. We study equilibrium measures μ_ϕ associated to a class of Hölder potentials ϕ on Λ, and find the measures μ_ϕ of iterates of arbitrary Bowen balls. Estimates for the pointwise dimension δ_{μ_ϕ} of μ_ϕ that involve Lyapunov exponents and a correction term are found, and also a formula for the Hausdorff dimension of μ_ϕ in the case when the preimage counting function is constant on Λ. For terminal/minimal saddle sets we prove that an invariant measure ν obtained as a wedge product of two positive closed currents, is in fact the measure of maximal entropy for the *restriction* $f|_\Lambda$. This allows then to obtain formulas for the measure ν of arbitrary balls, and to give a formula for the pointwise dimension and the Hausdorff dimension of ν.

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1 Introduction

The dynamics of holomorphic endomorphisms in higher dimensions presents many interesting geometric and ergodic aspects based on the interplay of complex dynamics, hyperbolic smooth dynamics and ergodic theory (see [10,12,13], etc.) In this paper we study the problem of holomorphic endomorphisms of $\mathbb{P}^2\mathbb{C}$ which are hyperbolic on basic sets of saddle type Λ (see [12] and [25] for hyperbolicity in the non-invertible case). An arbitrary holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$ is given by three homogeneous polynomials $[P_0 : P_1 : P_2]$ each of them having the same degree d. We will say that d is the *degree* (or the *algebraic degree*) of f. For such maps $f : \mathbb{P}^2 \to \mathbb{P}^2$ Fornaess and Sibony have defined a positive closed current $T = \lim_n \frac{(f^n)^*\omega}{d^n}$ which can be written locally in $\mathbb{C}^3\setminus\{0\}$ as dd^cG where G is the Green function associated to f. This allows them to define a probability measure $\mu = T \wedge T$ which is f-invariant and mixing (see [13]); it was shown in [3] that μ is the unique measure of maximal entropy $\log d^2$ for f.

In the case when f is hyperbolic on a basic set Λ one may study also the measure of maximal entropy of the *restriction* of f on Λ. This measure has different properties than μ; for instance if Λ is a saddle set then it has both negative and positive Lyapunov exponents.

Given a compact f-invariant set Λ one forms the *natural extension* (or *inverse limit*) $\hat{\Lambda} := \{(x, x_{-1}, x_{-2}, \ldots)\}, f(x_{-i}) = x_{-i+1}, x_{-i} \in \Lambda, i \geq 1\}$. The natural extension is a compact metric space with the canonical metric (see [12,25], etc.) On the natural extension $\hat{\Lambda}$ there exists a shift homeomorphism $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ defined by $\hat{f}(\hat{x}) = (f(x), x, x_{-1}, \ldots), \hat{x} \in \hat{\Lambda}$. We denote the canonical projection by $\pi : \hat{\Lambda} \to \Lambda, \pi(\hat{x}) = x, \hat{x} \in \hat{\Lambda}$.

Hyperbolicity for endomorphisms is defined as a continuous splitting of the tangent bundle over $\hat{\Lambda}$ into stable and unstable directions (see [12,25]); the stable directions depend only on base points, but unstable directions depend nevertheless on whole prehistories $\hat{x} \in \hat{\Lambda}$ (i.e. past trajectories) and not only on x. Hyperbolic maps on basic sets were also studied for instance in [7,19,22], etc. If f is hyperbolic on Λ then we have local stable manifolds $W^s_r(x)$ and local unstable manifolds $W^u_r(\hat{x})$ where $\hat{x} \in \hat{\Lambda}$. Also notice that Λ is not necessarily totally invariant. Thus for $x \in \Lambda$ we may have some f-preimages of x in Λ and others outside Λ. Moreover the number of f-preimages of x that remain in Λ may vary with x. So the endomorphism case is subtle and very different from the case of diffeomorphisms.

Now given an arbitrary probability measure μ on a compact metric space X one can define the *lower pointwise dimension* and the *upper pointwise dimension* at $x \in X$ respectively by:

$$\underline{\delta}_\mu(x) := \liminf_{\rho\to 0} \frac{\log \mu(B(x,\rho))}{\log \rho}, \quad \text{and} \quad \bar{\delta}_\mu(x) := \limsup_{\rho\to 0} \frac{\log \mu(B(x,\rho))}{\log \rho}$$

In case they coincide, we call the common value $\delta_\mu(x)$ the *pointwise dimension of* μ at $x \in X$ (see [24]). Also one can define the *Hausdorff dimension* of μ by:

$$HD(\mu) := \inf\{HD(Z), Z \text{ borelian set with } \mu(X\setminus Z) = 0\}$$

In [31] Young proved that for a hyperbolic measure μ (i.e. without zero Lyapunov exponents) invariated by a smooth diffeomorphism f of a surface, we have μ-a.e the formula

$$\delta_\mu = h_\mu \left(\frac{1}{\chi_u(\mu)} - \frac{1}{\chi_s(\mu)} \right),$$

where $\chi_s(\mu)$, $\chi_u(\mu)$ are the negative, respectively positive Lyapunov exponents of μ.

For analytic endomorphisms f on the Riemann sphere $\mathbb{P}^1\mathbb{C}$, Manning proved in [16] that if f is hyperbolic on its Julia set $J(f)$ and has no critical points in $J(f)$, then for any ergodic f-invariant probability measure μ on $J(f)$ the Hausdorff dimension of μ is given by:

$$HD(\mu) = \frac{h_\mu}{\chi(\mu)},$$

where $\chi(\mu)$ is the (only) Lyapunov exponent of μ. This formula was later extended by Mane (see [15]) to the case of all rational maps (i.e. not only hyperbolic) and invariant ergodic probabilities with positive Lyapunov exponent.

However the situation for higher dimensional endomorphisms and their invariant measures is different (see also [11]). Barreira Pesin and Schmeling [1] proved a product property for invariant hyperbolic measures under smooth diffeomorphisms, which allows to compute the pointwise dimension as the sum of stable and unstable pointwise dimensions. In the case of polynomial endomorphisms Binder and DeMarco gave in [2] estimates for the Hausdorff dimension of the measure of maximal entropy $\mu = T \wedge T$ which involve Lyapunov exponents of μ. In [8] Dinh and Dupont extended those estimates to the case of meromorphic endomorphisms of $\mathbb{P}^k$. De Thelin studied in [6] invariant measures (in general non-ergodic) with both negative and positive Lyapunov exponents. And in [9] Dupont obtained in the case of holomorphic endomorphisms on $\mathbb{P}^k$ a lower bound for the lower pointwise dimension of an ergodic f-invariant measure with positive Lyapunov exponents.

Our case here is different from those above, in that we study the measure of maximal entropy of the *restriction* of the endomorphism f to a saddle basic set Λ, and not the measure of maximal entropy on the whole of $\mathbb{P}^2$. In fact we will consider more generally equilibrium (Gibbs) measures for Hölder potentials ϕ on Λ, such that ϕ satisfies an inequality relating the number of f-preimages remaining in Λ and the topological pressure of ϕ.

In the case when Λ is a *terminal* saddle basic set, i.e. when the iterates of f form a normal family on $W^u(\hat{\Lambda})\setminus\Lambda$, Diller and Jonsson have introduced a measure $\nu_i = \sigma^u \wedge T$ which is f-invariant and supported on Λ. For the case of a *minimal* saddle basic set Λ for an s-hyperbolic map on $\mathbb{P}^2$, Fornaess and Sibony introduced in [12] a probability measure $\nu = T \wedge \sigma$ as a wedge product of positive closed currents; this measure is also f-invariant and mixing. Examples of terminal sets can be obtained by perturbations of already known examples (see [7,12]).

Our **main results** are:

First we study **equilibrium measures of Hölder potentials** on a basic set Λ. If the smooth endomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ is hyperbolic on Λ and if ϕ is a Hölder potential on Λ, there exists a unique equilibrium measure μ_ϕ for ϕ, i.e. a measure which maximizes in the Variational Principle (see [14,30], etc.) $P(\phi) = \sup\{h_\mu + \int \phi d\mu, \mu f - \text{invariant}\}$. Equilibrium measures for stable potentials were also used for instance in [19] to show that the stable dimension cannot be 2 on a basic saddle set Λ in $\mathbb{P}^2$.

In the sequel we shall use Hölder continuous potentials ϕ which satisfy the following inequality

$$\phi + \log d' < P(\phi) \quad \text{on } \Lambda, \tag{1}$$

where d' is an upper bound on the number of f-preimages in Λ of an arbitrary point. Notice that d' may even be 1, when the restriction to Λ is a homeomorphism (as in the examples from [22]). The inequality (1) will help in estimating the measure μ_ϕ of arbitrary iterates of Bowen balls. For instance if $\phi \equiv 0$ it says that $h_{top}(f|_\Lambda) > \log d'$, which in a sense is the more interesting case. In Corollary 4 we give a case when $\log d' = h_{top}(f|_\Lambda)$, a condition which implies that f is expanding on Λ (see also [17]). In the same Corollary 4 and in Examples we discuss also other cases for the upper bound d'.

In Theorem 1 we will give precise estimates of the measure μ_ϕ of an **arbitrary iterate** of a Bowen ball. This will help us obtain estimates and in some cases even exact formulas for the **pointwise dimension** and the **Hausdorff dimension** of μ_ϕ. In particular we prove that the measure μ_ϕ is exact dimensional in those cases. In Corollary 1 we give estimates for δ_{μ_ϕ} in the case when the number of preimages remaining in Λ is not constant.

In particular Theorem 1 applies to negative potentials of type $t \log |Df_s(x)|$, and in Corollary 2 we obtain the pointwise dimension of the equilibrium measure μ_s of the stable potential $\delta^s \log |Df_s(x)|$, where δ^s is the Hausdorff dimension of the intersection between a stable manifold and Λ.

We prove in Theorem 2 that for a terminal set Λ the measure ν_i from above, is in fact the **measure of maximal entropy** μ_0 of the restriction $f|_\Lambda$; and if Λ is minimal and c-hyperbolic for the Axiom A holomorphic map f, then the measure ν from above is equal to μ_0 as well.

In Corollaries 3, 4 we estimate, and in certain cases give formulas for the pointwise dimension of the measures ν_i, ν on terminal, respectively minimal saddle sets; in Corollary 4 we give complete formulas for the pointwise dimension of ν, for all the possible **minimal c-hyperbolic** saddle sets of a map of degree 2.

In the end we will give also **examples** of holomorphic maps and equilibrium measures of Hölder potentials on terminal saddle sets, for which the (upper/lower) pointwise dimension can be estimated/computed. Some of these examples are obtained as perturbations of polynomial maps and we can see that the bound d' on the number of preimages can vary by changing the parameters; for instance in some cases $d' = 1$, in other cases $d' = 2$, etc. See also the classification from [17], of perturbations of $(z, w) \to (z^2 + c, w^2)$ in terms of 1-sided Bernoullicity and of their preimage counting function behavior. Other examples of equilibrium measures on terminal sets for holomorphic maps will be obtained by Ueda's method.

2 Equilibrium measures of Hölder potentials. Green currents. Transversal measures

In the sequel we consider a holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$ of degree d; this means that f is given as $[f_0 : f_1 : f_2]$, where f_0, f_1, f_2 are homogeneous polynomials in coordinates (z_0, z_1, z_2), of common degree d. We know also that the topological entropy of f on $\mathbb{P}^2$ is equal to $\log d^2$ (see [13]).

We will work on a *basic set* Λ, i.e. an f-invariant compact set Λ for which there exists a neighbourhood U such that $\cap_{n\in\mathbb{Z}} f^n(U) = \Lambda$ and $f|_\Lambda$ is topologically transitive. If the non-invertible map f is hyperbolic on the basic set Λ, then from the Spectral Decomposition Theorem (see [14,25]) Λ can be written as the union of finitely many mutually disjoint subsets Λ_i s.t there exists a positive integer m with $f^m(\Lambda_i) = \Lambda_i, i$ and $f^m|_{\Lambda_i}$ topologically mixing.

Notice that we only have forward invariance of Λ, but **not** total invariance; this means that $f(\Lambda) = \Lambda$ but an arbitrary point $x \in \Lambda$ may have in general also f-preimages *outside* Λ.

On a basic set Λ for f, let us now consider a continuous potential $\phi : \Lambda \to \mathbb{R}$. Then from the Variational Principle, we know that the topological pressure satisfies $P(\phi) = \sup\{h_\mu + \int \phi d\mu, \mu$ f-invariant probability measure on $\Lambda\}$. If ϕ is Hölder continuous on Λ and f is hyperbolic on Λ, then there exists a *unique* measure μ_ϕ which attains the supremum in the Variational Principle, and it is called the **equilibrium measure**, or the **Gibbs state** of ϕ (see [14] for the diffeomorphism case and [20] for the endomorphism case). It follows from above that $h_{\mu_\phi} + \int \phi d\mu_\phi = P(\phi)$.

Denote by $B_n(x, \varepsilon) := \{y \in \Lambda, d(f^i y, f^i x) < \varepsilon, 0 \le i \le n-1\}$ a *Bowen ball*, i.e. the set of points which ε-follow the orbit of order n of x.

Recall now that there exists a unique correspondence between f-invariant measures m on Λ and $\hat{f}$-invariant measures $\hat{m}$ on $\hat{\Lambda}$ and that $\pi_*\hat{m} = m$. Then by working with the homeomorphism $\hat{f}$ on $\hat{\Lambda}$ and then projecting, it was proved in [20] that the equilibrium measure μ_ϕ satisfies the following estimates on Bowen balls:

$$\frac{1}{C} e^{S_n\phi(x) - nP(\phi)} \le \mu_\phi(B_n(x, \varepsilon)) \le C e^{S_n\phi(x) - nP(\phi)}, \tag{2}$$

for every $n > 0$, where $S_n\phi(x) := \phi(x) + \cdots + \phi(f^{n-1}(x))$ is the consecutive sum and C is a positive constant independent of x, n.

Definition 1 Given a basic set Λ for the map f, denote by $d(x) := Card\{f^{-1}(x) \cap \Lambda\}, x \in \Lambda$ and call it the **preimage counting function** on Λ.

If Λ is a *connected* basic set such that the critical set does not intersect Λ, i.e. $C_f \cap \Lambda = \emptyset$ and if there exists a neighbourhood U of Λ with $f^{-1}(\Lambda) \cap U = \Lambda$, then the preimage counting function $d(\cdot)$ is constant on Λ (see [19]). Notice also that the preimage counting function is **not necessarily preserved** when taking perturbations; see Example 1 at the end of paper, where by perturbing a 2-to-1 basic set we may obtain a basic set on which the restriction is 1-to-1.

Now let us recall the following Lemma proved in [18] which relates the measures of various subsets of Bowen balls, subsets which iterate to the same image:

Lemma 1 *Let f be an endomorphism, hyperbolic on a basic set Λ; consider also a Holder continuous potential ϕ on Λ and μ_ϕ be the unique equilibrium measure of ϕ. Let a small $\varepsilon > 0$, two disjoint Bowen balls $B_k(y_1, \varepsilon)$, $B_m(y_2, \varepsilon)$ and a borelian set $A \subset f^k(B_k(y_1, \varepsilon)) \cap f^m(B_m(y_2, \varepsilon))$, s.t $\mu_\phi(A) > 0$; denote by $A_1 := f^{-k}A \cap B_k(y_1, \varepsilon)$, $A_2 := f^{-m}A \cap B_m(y_2, \varepsilon)$ and assume that $\mu_\phi(\partial A_1) = \mu_\phi(\partial A_2) = 0$. Then there exists a positive constant C_ε independent of k, m, y_1, y_2 such that*

$$\frac{1}{C_\varepsilon}\mu_\phi(A_2) \cdot \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} \cdot e^{(m-k)P(\phi)} \leq \mu_\phi(A_1) \leq C_\varepsilon \mu_\phi(A_2) \cdot \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} \cdot e^{(m-k)P(\phi)}$$

We shall say that two functions $Q_1(n, x)$ and $Q_2(n, x)$ are **comparable** if there exists a positive constant C such that $\frac{1}{C}Q_1(n, x) \leq Q_2(n, x) \leq CQ_1(n, x)$ for all $n > 0$ and x; this will be denoted by $Q_1 \approx Q_2$. The constant C is sometimes called a *comparability constant*.

The next Definition is similar to that of s-hyperbolicity (see [12]), but it refers only to a fixed basic set, not to the whole nonwandering set.

Definition 2 Let Λ be a basic set for a map f on a manifold M. We say that f is **c-hyperbolic** on Λ if f is hyperbolic on Λ, there exists a neighbourhood U of Λ with $f^{-1}(\Lambda) \cap U = \Lambda$ and if the critical set C_f of f does not intersect Λ.

For a holomorphic endomorphism f on $\mathbb{P}^2$, let us remind now some properties of the associated positive closed **Green current** T (see [13] for more details). First there exists a continuous plurisubharmonic function G on $\mathbb{C}^3\setminus\{0\}$ called the Green function of f, satisfying $G(F(z)) = d \cdot G(z)$ where $F : \mathbb{C}^3\setminus\{0\} \to \mathbb{C}^3\setminus\{0\}$ is the lift of f relative to the canonical projection $\pi_2 : \mathbb{C}^3 \to \mathbb{P}^2$. We have $G \in \mathcal{P}_1$, where $\mathcal{P}_1$ is the cone of plurisubharmonic functions u on $\mathbb{C}^3\setminus\{0\}$ satisfying the homogeneity condition $u(\lambda z) = \log|\lambda| + u(z)$, $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^3\setminus\{0\}$. Recall also that

$$\pi_2^* T = dd^c G,$$

and that the Green measure $\mu = T \wedge T$ is mixing.

In [12] Fornaess and Sibony studied also *s-hyperbolic* holomorphic maps on $\mathbb{P}^2$ and *minimal* saddle basic sets, for the ordering $\Lambda_i \succ \Lambda_j$ if $W^u(\hat{\Lambda}_i) \cap W^s(\Lambda_j) \neq \emptyset$. A related notion introduced in [7] is that of a *terminal* set in the case of a holomorphic map f on $\mathbb{P}^2$. Here f is not assumed to have Axiom A and the condition refers only to Λ itself. A saddle set Λ is called *terminal* if for any $\hat{x} \in \hat{\Lambda}$, the iterates of f restricted to $W^u_{loc}(\hat{x})\setminus\Lambda$ form a normal family. Notice that if f is Axiom A and if Λ is minimal, then for any $\hat{x} \in \hat{\Lambda}$ the global unstable set $W^u(\hat{x})$ does not intersect any global stable set of any other basic set, thus $W^u(\hat{\Lambda})\setminus\Lambda$ is contained in the union of basins of attraction of attracting cycles; hence in this case minimal sets are also terminal. Examples of minimal sets for holomorphic maps on $\mathbb{P}^2$ were given in [12], and examples of terminal sets in [7]. Various types of hyperbolicity for holomorphic maps, and the associated sets of points whose prehistories do not always converge to the support of the corresponding Green measure were studied also in [21].

In [12], Fornaess and Sibony constructed positive closed currents σ on minimal sets for s-hyperbolic maps, by using forward iterates of unstable disks (or equivalently of disks which are transverse to local stable directions). If D is an unstable disk then

$$\frac{f_\star^n([D])}{d^n} \to \sigma \cdot \int D \wedge T$$

Then using the positive closed (1, 1) current σ, they constructed an invariant measure ν on Λ as

$$\nu = \sigma \wedge T$$

Let us remind now the properties of the **transversal measures** $\hat{\mu}_x^s$ associated to a hyperbolic structure on Λ; they are built in the same fashion as in [27] (see also [28]), but on the natural extension $\hat{\Lambda}$. The key to that proof is the existence for diffeomorphisms of a Markov partition; in our endomorphism case, we have instead a Markov partition on the inverse limit $\hat{\Lambda}$ (see [26]). Moreover the inverse limit $\hat{\Lambda}$ has local product structure, in fact it is a Smale space (see [14,26]).

One obtains then a system of transversal measures $\hat{\mu}_x^s$ on $\hat{W}_{loc}^s(x)$, where we denote by $\hat{W}_{loc}^s(x)$ and $\hat{W}_{loc}^u(\hat{x})$ the lifts to $\hat{\Lambda}$ of the local stable intersection $W_{loc}^s(x) \cap \Lambda$, respectively of the local unstable intersection $W_{loc}^u(\hat{x}) \cap \Lambda$. More precisely $\hat{W}_{loc}^s(x) := \pi^{-1}(W_{loc}^s(x) \cap \Lambda)$ and $\hat{W}_{loc}^u(\hat{x}) := \pi^{-1}(W_{loc}^u(\hat{x}) \cap \Lambda)$, $\hat{x} \in \hat{\Lambda}$.

Let us assume without loss of generality that all the stable and unstable local manifolds we work with, are of size r for some $r > 0$ small enough. Then the measures $\hat{\mu}_x^s$ satisfy the following properties:

(i) if $\chi_{x,y}^s : \hat{W}_r^s(x) \to \hat{W}_r^s(y)$ is the holonomy map given by $\chi_{x,y}^s(\hat{\xi}) = \hat{W}_r^u(\hat{\xi}) \cap \hat{W}_r^s(y)$, then $\hat{\mu}_x^s(A) = \hat{\mu}_y^s(\chi_{x,y}^s(A))$ for any borelian set A.

(ii) $\hat{f}_\star \hat{\mu}_x^s = e^{h_{top}(f|_\Lambda)} \hat{\mu}_{f(x)}^s|_{\hat{f}(\hat{W}_r^s(x))}$

(iii) $\mathrm{supp}\hat{\mu}_x^s = \hat{W}^s(x)$.

In fact from [27] and [28] applied to our case on $\hat{\Lambda}$, it follows that there exist also unstable transversal measures, denoted by $\hat{\mu}_{\hat{x}}^u$ on $\hat{W}_r^u(\hat{x})$, $\hat{x} \in \hat{\Lambda}$ with similar properties. And moreover the measure of maximal entropy on $\hat{\Lambda}$ denoted by $\hat{\mu}_0$, can be written as the product of transversal stable measures $\hat{\mu}_y^s$ with transversal unstable measures $\hat{\mu}_{\hat{x}}$ i.e.

$$\hat{\mu}_0(\phi) = \int_{\hat{W}_r^s(x)} \left(\int_{\hat{W}_r^u(\hat{y})} \phi \, d\hat{\mu}_{\hat{y}}^u \right) d\hat{\mu}_x^s(\hat{y}), \tag{3}$$

for any function ϕ defined on a neighbourhood of $\hat{x} \in \hat{\Lambda}$.

Transversal measures associated to stable/unstable foliations are subject to a unicity result by Bowen and Marcus [5], which can be applied on the natural extension $\hat{\Lambda}$.

Recall now that in [7] Diller and Jonsson introduced a positive current σ^u by using transversal measures (see also the diffeomorphism case in [27,28]); namely in a neighbourhood of $x \in \Lambda$,

$$\langle \sigma^u, \chi \rangle = \int_{\hat{W}^s_{loc}} \left(\int_{W^u_{loc}(\hat{y})} \chi \right) d\hat{\mu}^s_x(\hat{y}),$$

where $\hat{\mu}^s_x$ are transversal measures on $\hat{W}^s_{loc}(x) := \pi^{-1}(W^s_{loc}(x)$. Here we use a different notation for these measures, in order to emphasize that they are supported on lifts of local stable manifolds. If Λ is terminal, then they defined an invariant probability measure on Λ,

$$\nu_i = \sigma^u \wedge T$$

We use the notation ν_i in order to emphasize the way the current σ^u was constructed with the help of the inverse limit.

In general from the Spectral Decomposition Theorem a basic set Λ can be written as a union $\Lambda_1 \cup \cdots \cup \Lambda_m$ of mutually disjoint compact subsets, and there exist positive integers $n_1, \ldots, n_m$ s.t f^{n_j} invariates Λ_j and f^{n_j} is topologically mixing on Λ_j (see [14]). Thus it is natural to assume that f is topologically mixing on Λ.

In Theorem 2 we will prove that the measures ν, ν_i defined above are both equal to the measure of maximal entropy of $f|_\Lambda$ if Λ is (topologically) mixing.

3 The measure μ_ϕ of an iterate of a Bowen ball, and pointwise dimensions. Geometric description of the measure of maximal entropy on saddle sets

Define first the set

$$B(n, k, z, \varepsilon) := f^n(B_{n+k}(z, \varepsilon)), \quad z \in \Lambda, n > 0, k > 0$$

This set is an iterate of a Bowen ball and when n and k vary, we can adjust the sides of $B(n, k, z, \varepsilon)$ arbitrarily; in particular we can make it have (almost) equal sides in the stable and unstable directions. The idea of adjusting in order to obtain "round balls" used in estimating the pointwise dimension, was also employed in [31]. For more generality we state the next Theorem in the real setting for a function which is conformal on both stable and unstable manifolds, although our main application in this article will be to holomorphic maps on $\mathbb{P}^2$.

Theorem 1 *Let $f : M \to M$ be a $\mathcal{C}^2$ map on a Riemannian manifold M and $\Lambda \subset M$ be a basic set such that f is c-hyperbolic on Λ, f is conformal on both stable and unstable local manifolds over Λ and the preimage counting function is constant and equal to d' on Λ. Consider also a Hölder continuous potential ϕ on Λ which satisfies $\phi(x) + \log d' < P(\phi)$, $\forall x \in \Lambda$, and let μ_ϕ its equilibrium measure. Then for any $z \in \Lambda$, positive integers k, n and small $\varepsilon > 0$ we have the following formula, with comparability constants independent of z, k, n:*

$$\mu_\phi(B(n, k, z, \varepsilon)) \approx \frac{e^{S_{n+k}\phi(z)}}{(d')^k}$$

Moreover the pointwise dimension of μ_ϕ exists μ_ϕ-a.e, is denoted by δ_{μ_ϕ}, and we have:

$$\delta_{\mu_\phi} = HD(\mu_\phi) = h_{\mu_\phi}\left(\frac{1}{\chi_u(\mu_\phi)} - \frac{1}{\chi_s(\mu_\phi)}\right) + \log d' \cdot \frac{1}{\chi_s(\mu_\phi)}$$

Proof Recall that $S_n\phi(y)$ is defined as the consecutive sum $\phi(y) + \phi(f(y)) + \cdots + \phi(f^{n-1}(y))$, $y \in \Lambda$. Let us take a point $z \in \Lambda$, a positive integer n and $x := f^n(z)$. By definition $B(n, k, z, \varepsilon) = f^n(B_{n+k}(z, \varepsilon))$. Now we assumed that the preimage counting function $d(\cdot)$ is constant on Λ and equal to d'. Thus every point y from Λ has exactly d' f-preimages remaining in Λ.

Next we assumed $\phi + \log d' < P(\phi)$; so if we define the real-valued function $\bar\phi := \phi - P(\phi) + \log d'$, then $P(\bar\phi) = P(\phi) - P(\phi) + \log d' = \log d'$ and also $\bar\phi < 0$ on Λ. Then since ϕ and $\bar\phi$ are cohomologous, they have the same equilibrium measure μ_ϕ. Therefore we will assume in the sequel that $\phi < 0$ on Λ and $P(\phi) = \log d'$.

Consider now prehistories in $\hat\Lambda$ of an arbitrary point $y \in B(n, k, z, \varepsilon) \subset \Lambda$; for such a prehistory $\hat y = (y, y_{-1}, y_{-2}, \ldots)$ let us denote by $n(\hat y)$ the smallest positive integer m satisfying $S_m\phi(y_{-m}) \le S_n\phi(z)$; since $\phi < 0$ on Λ, it is clear that such an m must exist for any prehistory $\hat y \in \hat\Lambda$. So $S_{n(\hat y)-1}\phi(y_{-n(\hat y)+1}) > S_n\phi(z)$ while $S_{n(\hat y)}\phi(y_{-n(\hat y)}) \le S_n\phi(z)$. Call such a finite prehistory $(y, y_{-1}, \ldots, y_{-n(\hat y)})$ a *maximal prehistory*.

We intend to get an estimate of the measure $\mu_\phi(B(n, k, z, \varepsilon)$ from the f-invariance of μ_ϕ and the comparison estimates between the different pieces of its preimage set using Lemma 1. We will write $B(n, k, z, \varepsilon)$ as a union of subsets E which are contained in forward iterates of Bowen balls; the question is how these iterates intersect and what is the relation between various components of preimage sets of different orders.

In fact since we know that $B(n, k, z, \varepsilon) = f^n(B_{n+k}(z, \varepsilon))$, it means that every point in $B(n, k, z, \varepsilon)$ has an f^n-preimage in $B_{n+k}(z, \varepsilon)$. But in estimating $\mu_\phi(B(n, k, z, \varepsilon))$ we have to consider *all* f^n-preimages in Λ of points from $B(n, k, z, \varepsilon)$, in order to use the f-invariance of μ_ϕ; so we will compare various f^m-preimages of subsets of $B(n, k, z, \varepsilon)$ with the corresponding f^n-preimages from $B_{n+k}(z, \varepsilon)$. Our standard for comparison of all these preimages of various orders of points in $B(n, k, z, \varepsilon)$, will be those preimages belonging to $B_{n+k}(z, \varepsilon)$.

Take an arbitrary point $y \in B(n, k, z, \varepsilon)$ and a prehistory $\hat y \in \hat\Lambda$ of y; then $y \in f^n(B_{n+k}(z, \varepsilon)) \cap f^{n(\hat y)}(B_{n(\hat y)}(y_{-n(\hat y)}, \varepsilon))$. Let us take all the prehistories $\hat y$ of y in $\hat\Lambda$; along each such prehistory we go until reaching the preimage of order $n(\hat y)$. It is clear that there exists only a finite collection $\mathcal{P}(y)$ of such maximal prehistories of y, since $\phi < 0$ on the compact set Λ and since we cannot continue to add indefinitely values of ϕ on consecutive preimages until reaching the value $S_n\phi(z)$. Denote by $E(y)$ the intersection of the iterates $f^{n(\hat y)}(B_{n(\hat y)}(y_{-n(\hat y)}, \varepsilon))$ over all the prehistories $\hat y$ of y in $\hat\Lambda$ (i.e. in fact over the finite prehistories of $\mathcal{P}(y)$).

We shall cover the set $B(n, k, z, \varepsilon)$ with mutually disjoint subsets of various sets of type $E(y)$, $y \in B(n, k, z, \varepsilon)$. Actually we can cover $B(n, k, z, \varepsilon)$ with a collection $\mathcal{F}$ of sets F, each such F belonging to $E(y)$ for some $y \in B(n, k, z, \varepsilon)$. Now if $F \subset E(y)$, denote by

$$F(\hat{y}) := B(y_{-n(\hat{y})}, \varepsilon) \cap f^{-n(\hat{y})}(F)$$

From the definition of $E(y)$ we know that $F = f^{n(\hat{y})}(F(\hat{y}))$, where we recall that $n(\hat{y})$ was defined above with respect to $S_n\phi(z)$. Recall also that for $y \in \Lambda$, the set of "maximal" prehistories of type $(y, y_{-1}, \dots, y_{-n(\hat{y})})$ for $\hat{y} \in \hat{\Lambda}$ prehistory of y, is denoted by $\mathcal{P}(y)$.

Take now two prehistories $\hat{y}, \hat{y}'$ of y belonging to $\hat{\Lambda}$ and go along these prehistories until we reach $n(\hat{y})$ and $n(\hat{y}')$ respectively. We want to compare the measure μ_ϕ on the preimages of F along these maximal prehistories by using Lemma 1.

$$\begin{aligned} \frac{1}{C}\mu_\phi(F(\hat{y}))\frac{e^{S_{n(\hat{y}')}\phi(y'_{-n(\hat{y}')})}}{e^{S_{n(\hat{y})}\phi(y_{-n(\hat{y})})}} \cdot e^{(n(\hat{y})-n(\hat{y}'))P(\phi)} &\le \mu_\phi(F(\hat{y}')) \\ \le C\mu_\phi(F(\hat{y}))\frac{e^{S_{n(\hat{y}')}\phi(y'_{-n(\hat{y}')})}}{e^{S_{n(\hat{y})}\phi(y_{-n(\hat{y})})}} \cdot e^{(n(\hat{y})-n(\hat{y}'))P(\phi)}, & \end{aligned} \tag{4}$$

where C is a positive constant independent of $y, \hat{y}, \hat{y}', x$.

On the other hand from the definition of $n(\hat{y}), n(\hat{y}')$ we know that $|S_{n(\hat{y})}\phi(y_{-n(\hat{y})}) - S_n\phi(z)| \le M$ and $|S_{n(\hat{y}')}\phi(y'_{-n(\hat{y}')}) - S_n\phi(z)| \le M$, for some positive constant M. Therefore since $P(\phi) = \log d'$ we obtain (by taking perhaps a larger C) that:

$$\frac{1}{C}\mu_\phi(F(\hat{y}))e^{(n(\hat{y})-n(\hat{y}'))\log d'} \le \mu_\phi(F(\hat{y}')) \le C\mu_\phi(F(\hat{y}))e^{(n(\hat{y})-n(\hat{y}'))\log d'} \tag{5}$$

We add now the measures of various preimages $F(\hat{y})$ of F, over all finite "maximal" prehistories from $\mathcal{P}(y)$, in order to obtain the measure $\mu_\phi(F)$, where recall that $F \subset E(y)$. We take into consideration the fact that any point in Λ has $(d')^m$ f^m-preimages in Λ, $m > 0$. Thus if $n(\hat{y})$ is say the largest maximal order associated to any prehistory from $\mathcal{P}(y)$, and if $\hat{y}'$ is another prehistory of y with $n(\hat{y}') = n(\hat{y}) - 1$, then from (5) it follows that

$$\mu_\phi(F(\hat{y})) \approx \mu_\phi(F(\hat{y}')) \cdot \frac{1}{d'},$$

where the comparability constant is C above (i.e. a positive universal constant). Now if we add the measures $\mu_\phi(F(\hat{y}))$ over all prehistories which coincide with $\hat{y}'$ up to order $n(\hat{y}) - 1$, we obtain $\mu_\phi(F(\hat{y}'))$.

Similarly we order the integers $n(\hat{y}), \hat{y} \in \mathcal{P}(y)$ in decreasing order and then add successively the measures of preimages $F(\hat{y})$ using (5) and the fact that each point has exactly d' f-preimages in Λ. Thus if we compare the measures of $F(\hat{y})$ with the measure of the f^n-preimage $F(f^n z, \dots, z)$ of F in $B_{n+k}(z, \varepsilon)$, we obtain that

$$\sum_{\hat{y} \in \mathcal{P}(y)} \mu_\phi(F(\hat{y})) \approx \mu_\phi(F(f^n z, f^{n-1}z, \dots, z)) \cdot (d')^n$$

But now the sets $F \in \mathcal{F}$ were chosen mutually disjoint modulo μ_ϕ, hence their preimages will be mutually disjoint too (recall that $C_f \cap \Lambda = \emptyset$); thus by adding over $F \in \mathcal{F}$

$$\sum_{F\in\mathcal{F}}\sum_{\hat{y}\in\mathcal{P}(y),F\subset E(y)}\mu_\phi(F(\hat{y}))\approx\mu_\phi(B_{n+k}(z,\varepsilon))\cdot(d')^n$$

Thus from the f-invariance of μ_ϕ and by adding as above all the measures of preimages along maximal prehistories, we obtain the following formula for the measure μ_ϕ of an arbitrary "rectangle" $B(n,k,z,\varepsilon)$ with sides in the stable and in the unstable directions and centered on Λ:

$$\mu_\phi(B(n,k,z,\varepsilon))\approx\mu_\phi(B_{n+k}(z,\varepsilon))\cdot(d')^n\approx\frac{e^{S_{n+k}\phi(z)}}{(d')^k},\tag{6}$$

where the comparability constants are independent of n,k,z,x. Then the above formula helps us obtain the measure μ_ϕ of an *arbitrary ball* centered on Λ.

As an application we obtain the pointwise dimension of such an equilibrium measure μ_ϕ. We know from definition that $B(n,k,z,\varepsilon)=f^n(B_{n+k}(z,\varepsilon))$. Since f is holomorphic on $\mathbb{P}^2$, it follows from conformality on the local stable/unstable manifolds that the set $B(n,k,z,\varepsilon)$ has a side comparable to $\varepsilon|Df_s^n(z)|$ in the stable direction, and a side comparable to $\varepsilon|Df_u^n(z)||Df_u^{n+k}(z)|^{-1}=\varepsilon|Df_u^k(x)|^{-1}$ in the unstable direction.

Now for some n,k the set $B(n,k,z,\varepsilon)$ becomes "round", i.e. the stable side and the unstable side become comparable with a fixed comparability constant. So if $x=f^n(z)$, we want

$$\rho_n:=\varepsilon|Df_s^n(z)|\approx\varepsilon|Df_u^k(x)|^{-1}$$

In general for a continuous function $\psi:\Lambda\to\mathbb{R}$, an f-invariant ergodic probability measure μ on Λ and $\tau>0$, let us define the following set of well-behaved points with respect to μ:

$$G_n(\psi,\mu,\tau):=\left\{y\in\Lambda,|\frac{1}{n}S_n\psi(y)-\int\psi d\mu|<\tau\right\},\quad n>0$$

Then from Birkhoff Ergodic Theorem we have $\mu(G_n(\psi,\mu,\tau))\to 1$ when $n\to\infty$; so for every $\tau'>0$ there is $n(\tau')>0$ such that $\mu(G_n(\psi,\mu,\mu))>1-\tau'$ for $n>n(\tau')$.

We apply this to our case for the ergodic measure μ_ϕ and the functions $\log|Df_s|$ and $\log|Df_u|$ which are continuous and bounded on Λ (as f has no critical points in Λ). If $z\in G_n(\log|Df_s|,\mu_\phi,\tau)$ and $x=f^n(z)\in G_k(\log|Df_u|,\mu_\phi,\tau)$, then

$$\left|\frac{1}{n}S_n\log|Df_s|(z)-\int\log|Df_s|\,d\mu_\phi\right|<\tau\quad\text{and}\quad\left|\frac{1}{k}S_k\log|Df_u|(x)-\int\log|Df_u|\,d\mu_\phi\right|<\tau$$

The question is how large is the set of such $z's$. From above it follows that $\mu_\phi(f^{-n}(G_k(\log|Df_u|,\mu_\phi,\tau))=\mu_\phi(G_k(\log|Df_u|,\mu_\phi,\tau))>1-\tau'$ and $\mu_\phi(G_n(\log|Df_s|,\mu_\phi,\tau))>1-\tau'$, for $n>n(\tau')$. Thus for n large enough:

$$\mu_\phi(G_n(\log|Df_s|,\mu_\phi,\tau)\cap f^{-n}G_k(\log|Df_u|,\mu_\phi,\tau))>1-2\tau'\tag{7}$$

We now come back to the problem of the pointwise dimension of μ_ϕ. It is clear from above that if $B(n, k, z, \varepsilon)$ has comparable sides (i.e. it is “round”), then k must depend on n, so denote it by $k(z, n)$. Let us consider also

$$\chi_s(\mu_\phi) := \int \log |Df_s| \, d\mu_\phi, \; \chi_u(\mu_\phi) := \int \log |Df_u| \, d\mu_\phi,$$

the *Lyapunov exponents* of the ergodic measure μ_ϕ in the stable, respectively unstable directions; they will be denoted also by χ_s, χ_u for simplicity. As μ_ϕ is ergodic (see [30]) we have that μ_ϕ-a.e, $\frac{1}{n} S_n \log |Df_s| \to_{n\to\infty} \chi_s$ and $\frac{1}{k} S_k \log |Df_u| \to_{k\to\infty} \chi_u$. Thus if $|Df_s^n(z)| \approx |Df_u^{k(z,n)}(x)|^{-1}$, it follows that

$$\frac{-\log C}{n} - \frac{S_{k(z,n)} \log |Df_u|(x)}{k(z,n)} \cdot \frac{k(z,n)}{n} \le \frac{S_n \log |Df_s|(z)}{n}$$
$$\le \frac{\log C}{n} - \frac{S_{k(z,n)} \log |Df_u|(x)}{k(z,n)} \cdot \frac{k(z,n)}{n}$$

Thus since $n \to \infty$, we have $\frac{k(z,n)}{n} \to_{n\to\infty} \frac{-\chi_s}{\chi_u}$. But then, using also (6) one sees that if $B(n, k, z, \varepsilon)$ is a “round”ball, i.e. with sides of comparable size $\rho_n = \varepsilon |Df_s^n(z)|$ then for μ_ϕ-a.e $z \in \Lambda$

$$\frac{\log \mu_\phi(B(n, k(z,n), z, \varepsilon))}{\log \rho_n} \underset{n\to\infty}{\to} \frac{\int \phi d\mu_\phi - \frac{\chi_s}{\chi_u} \cdot \int \phi d\mu_\phi + \frac{\chi_s}{\chi_u} \cdot \log d'}{\chi_s}$$

Therefore the pointwise dimension of μ_ϕ is well-defined and for μ_ϕ-almost all $z \in \Lambda$, it is given by:

$$\delta_{\mu_\phi}(z) = \int \phi \, d\mu_\phi \cdot \left(\frac{1}{\chi_s} - \frac{1}{\chi_u} \right) + \frac{\log d'}{\chi_u}$$

But μ_ϕ is the equilibrium measure for ϕ and we assumed $P(\phi) = \log d'$, so $P(\phi) = \log d' = h_{\mu_\phi} + \int \phi d\mu_\phi$. Hence from above for μ_ϕ-a.e $z \in \Lambda$ we obtain

$$\delta_{\mu_\phi}(z) = h_{\mu_\phi} \left(\frac{1}{\chi_u} - \frac{1}{\chi_s} \right) + \log d' \cdot \frac{1}{\chi_s},$$

In conclusion the measure μ_ϕ is exact dimensional on Λ and it satisfies the above formula.

The fact that the Hausdorff dimension of μ_ϕ takes the same value as δ_{μ_ϕ} follows from a criterion of Young (see [31]), since the pointwise dimension is constant μ_ϕ-a.e. □

Even if the preimage counting function $d(\cdot)$ is not constant on Λ, still we obtain bounds for the measure of iterates of Bowen balls, and estimates for the lower pointwise dimension:

Corollary 1 *In the setting of Theorem 1 assume the preimage counting function satisfies $d(x) \le d'$ for μ_ϕ-a.e $x \in \Lambda$ and that $\phi(x) + \log d' < P(\phi)$ for all $x \in \Lambda$; then for μ_ϕ-a.e $x \in \Lambda$*

$$\underline{\delta}_{\mu_\phi}(x) \ge h_{\mu_\phi}\left(\frac{1}{\chi_u(\mu_\phi)} - \frac{1}{\chi_s(\mu_\phi)}\right) + \log d' \cdot \frac{1}{\chi_s(\mu_\phi)}$$

Proof Clear by applying (6).

An application of Theorem 1 will be to equilibrium states of potentials of type $t\Phi^s$ on folded fractals Λ, where $\Phi^s(x) := \log|Df_s(x)|$, $x \in \Lambda$. In our noninvertible case, it is somewhat similar to the product structure formula for pointwise dimension from [1].

Corollary 2 *In the same setting as in Theorem 1, denote by μ_s the equilibrium measure of $\delta^s\Phi^s$, where $\delta^s(x) := HD(W_r^s(x) \cap \Lambda)$ is assumed to be positive for some $x \in \Lambda$. Then $\delta^s = \delta^s(x)$ does not depend on x and for μ_s-a.e $x \in \Lambda$,*

$$\delta_{\mu_s}(x) = \frac{h_{\mu_s}}{\chi_u(\mu_s)} + \delta^s$$

Proof From [17] and references therein it follows that if f is conformal on local stable manifolds over Λ and d'-to-1 on Λ, then the stable dimension δ^s does not depend on x, and that it is equal to the unique zero of the pressure function $t \to P(t\Phi^s - \log d')$. Thus $P(\delta^s\Phi^s) = \log d'$ and if $\delta^s > 0$, it follows that the condition

$$\delta^s\Phi^s + \log d' < P(\delta^s\Phi^s)$$

is satisfied. Now $P(\delta^s\Phi^s) = h_{\mu_s} + \delta^s \int \Phi^s d\mu_s = h_{\mu_s} + \delta^s\chi_s(\mu_s) = \log d'$. Hence from Theorem 1 we obtain μ_s-a.e:

$$\begin{aligned}\delta_{\mu_s}(x) &= h_{\mu_s}\left(\frac{1}{\chi_u(\mu_s)} - \frac{1}{\chi_s(\mu_s)}\right) + \frac{\log d'}{\chi_s(\mu_s)} \\ &= \frac{h_{\mu_s}}{\chi_u(\mu_s)} + \frac{\delta^s\chi_s(\mu_s)}{\chi_s(\mu_s)} = \frac{h_{\mu_s}}{\chi_u(\mu_s)} + \delta^s\end{aligned}$$

□

Next we shall return to the holomorphic case and prove that the measure of maximal entropy of the restriction of f to terminal sets can be described geometrically as a wedge product of positive currents, by using a unicity result of Bowen and Marcus with respect to local holonomy maps.

Theorem 2 (*a*) *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic map of degree d and Λ be a terminal mixing saddle set. Then ν_i is equal to the measure of maximal entropy μ_0 on Λ.*

(*b*) *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be an Axiom A holomorphic map of degree d, which is c-hyperbolic on the mixing minimal saddle set Λ. Then $\nu_i = \nu = \mu_0$, where μ_0 is the maximal entropy measure of $f|_\Lambda$.*

Proof (a) Recall from Sect. 2 the construction of the current σ^u; then in [7] the measure ν_i is defined as the wedge product $\sigma^u \wedge T$, where the positive closed current σ^u is constructed with the help of the stable transversal measures $\hat{\mu}^s_x, x \in \Lambda$. Recall also that $\pi|_{\hat{W}^u_r(\hat{x})} : \hat{W}^u_r(\hat{x}) \to W^u_r(\hat{x})$ is a bijection (see [20]), so any function ϕ on $\hat{W}^u_r(\hat{x})$ determines uniquely a function denoted again with ϕ on $W^u_r(\hat{x})$. Then from the measure ν_i we can form a system of measures on the lifts of local unstable manifolds $\hat{W}^u_r(\hat{x}), \hat{x} \in \hat{\Lambda}$ in the following way:

$$\hat{\nu}^u_{\hat{x}}(\phi) = \int_{W^u_r(\hat{x})} \phi T|_{W^u_r(\hat{x})}$$

We assumed that f is mixing on Λ; in fact (topological) mixing of f on Λ is equivalent to mixing of $\hat{f}$ on $\hat{\Lambda}$. Define stable holonomy maps between lifts to $\hat{\Lambda}$ of local unstable manifolds, namely

$$\chi^u_{\hat{x},\hat{y}} : \hat{W}^u_r(\hat{x}) \to \hat{W}^u_r(\hat{y}),\ \chi^u_{\hat{x},\hat{y}}(\hat{\xi}) := \hat{W}^u_r(\hat{y}) \cap \hat{W}^s_r(\hat{\xi}),\ \hat{\xi} \in \hat{W}^u_r(\hat{x})$$

We wish to prove that the measures $\hat{\nu}^u_{\hat{x}}$ are transversal and invariant with respect to stable holonomy maps in the Smale space structure of $\hat{\Lambda}$, in the sense of Bowen and Marcus [5]. From the way the local unstable manifolds were constructed as determined by prehistories, it follows that there is a bijection between $W^u_r(\hat{x}) \cap \Lambda$ and its lift $\hat{W}^u_r(\hat{x})$ (see also [20]). Given a borelian set $\hat{A} \subset \hat{W}^u_r(\hat{x})$, there exists a unique borelian set $A \subset W^u_r(\hat{x}) \cap \Lambda$ such that π is a bijection between $\hat{A}$ and A. From the definition of $\hat{\nu}^u_{\hat{x}}$, we know that

$$\hat{\nu}^u_{\hat{x}}(\hat{A}) = \int_{A \cap W^u_r(\hat{x}) \cap \Lambda} dd^c G|_{W^u_r(\hat{x})}$$

Denote now the unstable intersection $W^u_r(\hat{x}) \cap \Lambda$ by $Z(\hat{x})$ for $\hat{x} \in \hat{\Lambda}$. Consider points x, y in a subset of Λ belonging to an open set $V \in \mathbb{P}^2$ so there exists a holomorphic inverse $s : V \to \mathbb{C}^3 \setminus \{0\}$ of π_2. Then for r small we can identify $Z(\hat{x}), Z(\hat{y})$ with their respective lifts to $\mathbb{C}^3 \setminus \{0\}$ for any prehistories $\hat{x}, \hat{y} \in \hat{\Lambda}$. Since there are no critical points of f in Λ and since we work on Λ, it follows that $Z(\hat{x})$ can be split into mutually disjoint subsets on which f^n is injective, i.e. $Z(\hat{x}) = \cup_i Z_{i,n}(\hat{x})$, $f^n|_{Z_{i,n}(\hat{x})} : Z_{i,n}(\hat{x}) \to Z^n_i(\hat{x})$ is bijective, and moreover $Z^n_i(\hat{x}), i$ are mutually disjoint. It follows that $f^n(Z(\hat{x})) = \cup_i Z^n_i(\hat{x})$. Now if $Z(\hat{x})$ is contained in V, then $f^n(Z(\hat{x}))$ may not be contained in V; but, if $f^n(Z(\hat{x}))$ is contained say in $V_1 \cup V_2$ where V_1, V_2 are open sets in $\mathbb{P}^2$ as above, with respective local inverses s_1, s_2 of π_2, and if $V_1 \cap V_2 \neq \emptyset$, then there exists a holomorphic function ρ on $V_1 \cap V_2$ so that $s_1 = \rho s_2$ on $V_1 \cap V_2$. So

$$dd^c(G \circ s_1) = dd^c(G(\rho s_2)) = dd^c \log|\rho| + dd^c(G \circ s_2) = dd^c(G \circ s_2)$$

This implies that working with $dd^c G$ on $\mathbb{C}^3 \setminus \{0\}$ is the same as working on $\mathbb{P}^2$.

Now $G \circ F = d \cdot G$ and $f^n : Z_{i,n}(\hat{x}) \to Z_i^n(\hat{x})$ is bijective hence $\int_{Z_i^n(\hat{x})} dd^c G = d^n \int_{Z_{i,n}(\hat{x})} dd^c G$. Thus by adding over all the indices i we obtain:

$$\int\limits_{f^n(Z(\hat{x}))} dd^c G = d^n \int\limits_{Z(\hat{x})} dd^c G \tag{8}$$

Now let $x, y \in \Lambda$ closer than $r/2$ and iterate $Z(\hat{x})$ and $Z(\hat{y})$ for some prehistories $\hat{x}, \hat{y} \in \hat{\Lambda}$. We also take as above, the subsets $Z_{i,n}(\hat{y})$ such that $f^n : Z_{i,n}(\hat{y}) \to Z_i^n(\hat{y})$ is a bijection, $Z_i^n(\hat{y}), i$ are mutually disjoint and $f^n(Z(\hat{y}) = \cup Z_i^n(\hat{y})$. If $Z_{i,n}(\hat{x}), Z_{i,n}(\hat{y})$ has diameter small enough, then it follows that $Z_i^n(\hat{x}), Z_i^n(\hat{y})$ both have diameter bounded above by r and they are very close to each other, in fact $d(Z_i^n(\hat{x}), Z_i^n(\hat{y})) \to 0$ for each i, when $n \to \infty$. This follows as in the Laminated Distortion Lemma (see [19]) since the distances between iterates of points on stable manifolds decrease exponentially, and the unstable derivative $|Df_u|$ is Hölder continuous.

Now if ψ is a smooth test function equal to 1 on a fixed neighbourhood of Z_i^n we have $\int_{Z_i^n(\hat{x})} dd^c G = \int_{Z_i^n(\hat{x})} \psi dd^c G = \int_{Z_i^n(\hat{x})} G dd^c \psi$ hence since $dd^c \psi$ is continuous and $Z_i^n(\hat{x})$ and $Z_i^n(\hat{y})$ are close, we obtain similar to [12] that for n large enough

$$\left| \int\limits_{f^n(A) \cap Z_i^n(\hat{x})} dd^c G - \int\limits_{f^n(\chi^u_{\hat{x},\hat{y}}(A)) \cap Z_i^n(\hat{y})} dd^c G \right| \le \varepsilon m_2(Z_i^n(\hat{x})),$$

where m_2 is the Lebesgue measure on $\mathbb{P}^2$. Now we add these inequalities over i and use the fact proved in Proposition 5.3 of [12] that $m_2(f^n(Z(\hat{x}))) \le C d^n, n > 0$. Hence by dividing with d^n, using (8), and letting $n \to \infty$ we obtain $\int_{A \cap Z(\hat{x})} dd^c G = \int_{\chi^u_{\hat{x},\hat{y}}(A) \cap Z(\hat{y})} dd^c G$. We lift then to the natural extension, keeping in mind that there exists a homeomorphism between $Z(\hat{x})$ and $\hat{W}^u_r(\hat{x})$. Hence on $\hat{\Lambda}$ we have:

$$\hat{\nu}^u_{\hat{x}}(\hat{A}) = \hat{\nu}_{\hat{y}}(\chi^u_{\hat{x},\hat{y}}(\hat{A})), \hat{A} \text{ borelian set in} \hat{W}^u_r(\hat{x})$$

The above equality can be extended next to general borelian sets contained in global unstable sets $\hat{W}^u(\hat{x}) = \cup_{n \ge 0} \hat{f}^n(\hat{W}^u_r(\hat{x})), \hat{x} \in \hat{\Lambda}$. Thus by a theorem of Bowen and Marcus (see main result of [5]), extended to the mixing homeomorphism $\hat{f}$ on $\hat{\Lambda}$, it follows that there exists a positive constant γ such that

$$\hat{\nu}^u_{\hat{x}} = \gamma \cdot \hat{\mu}^u_{0,\hat{x}},$$

for any $\hat{x} \in \hat{\Lambda}$, where $\hat{\mu}^u_{0,\hat{x}}$ are the transversal measures given by the measure of maximal entropy $\hat{\mu}_0$ on $\hat{\Lambda}$ (as in [27,28]); see also (3). In fact if μ_0 is the unique measure of maximal entropy on Λ and if $\hat{\mu}_0$ is the unique measure of maximal entropy on $\hat{\Lambda}$, then

$$\mu_0 = \pi_* \hat{\mu}_0 \quad \text{and} \quad h_{\mu_0} = h_{top}(f|_\Lambda) = h_{top}(\hat{f}|_{\hat{\Lambda}}) = h_{\hat{\mu}_0}$$

The measure ν_i is constructed with the transversal stable measures $\hat{\mu}^s_x$ (which we denote also by $\hat{\mu}^s_{0,x}$). Now from [26] we know that any f-invariant measure μ on Λ can be lifted uniquely to an $\hat{f}$-invariant measure $\hat{\mu}$ on $\hat{\Lambda}$ such that $\pi_* \hat{\mu} = \mu$. In our case we denote by $\hat{\nu}_i$ this unique lift of ν_i to $\hat{\Lambda}$. Since both $\hat{\nu}_i$ and $\hat{\mu}_0$ are ergodic probabilities on $\hat{\Lambda}$, it follows that $\gamma = 1$ and that

$$\hat{\mu}_0 = \hat{\nu}_i, \quad \text{hence} \quad \mu_0 = \nu_i$$

(b) Let us assume now that f has Axiom A, that Λ is a minimal basic set (i.e. the unstable set of Λ does not intersect the stable set of any other basic set Λ') and that f is c-hyperbolic on Λ. Then from Sect. 2 there exists a positive closed $(1, 1)$ current σ supported on the global unstable set $W^u(\hat{\Lambda})$ such that if D is a local disk transverse to the stable direction, then $\frac{f^n_*([D])}{d^n} \to (\int [D] \wedge T)\sigma$. Without loss of generality assume that the disk D is chosen such that $\int [D] \wedge T = 1$; and also that T has no mass on the boundary ∂D of D.

We have from [12] that on a neighbourhood Δ of a point $x \in \Lambda$ there exists a measure λ on the space of holomorphic maps from a local unstable disk Δ_1 to a local stable disk Δ_2 such that

$$\sigma = \int [W^u_r(\hat{y})] d\lambda(g_{\hat{y}}),$$

where $W^u_r(\hat{y})$ are local unstable manifolds intersecting Δ, $[W^u_r(\hat{y})]$ are the respective currents of integration and $g_{\hat{y}} : \Delta_1 \to \Delta_2$ is an arbitrary holomorphic map whose graph is $W^u_r(\hat{y})$. Then $\nu = \sigma \wedge T$ is supported only on Λ; hence we can define measures $\hat{\nu}^s_x$ on $\hat{W}^s_r(x)$ by

$$\hat{\nu}^s_x(\hat{A}) = \lambda(\{g_{\hat{y}}, \hat{y} \in \hat{A}\})$$

Thus from the way the function $g_{\hat{y}}$ was defined, namely as a function whose graph is $W^u_r(\hat{y})$, it follows that these measures are invariant to the local holonomy map between $\hat{W}^s_r(x)$ and $\hat{W}^s_r(y)$ for x, y close. Also by covering with small flow boxes it follows we can extend this property globally. Therefore from [5] we obtain that

$$\hat{\nu}^s_x = \gamma \cdot \hat{\mu}^s_{0,x},$$

where the constant $\gamma > 0$ does not depend on $x \in \Lambda$. Now ν was defined as integration of T on local unstable manifolds followed by integration with respect to transversal measures; by using a) we obtain that $\hat{\nu} = \hat{\mu}_0$, and thus $\nu = \mu_0$. Hence on minimal saddle basic sets the measure ν is equal to the measure ν_i and both are equal to the measure of maximal entropy μ_0 on Λ. □

Now that we know that the measure ν_i is equal to the measure of maximal entropy μ_0, and to ν when f has Axiom A and Λ is minimal, we find its pointwise dimension.

Corollary 3 (*a*) *Let* Λ *be a mixing terminal saddle set for a holomorphic map* $f : \mathbb{P}^2 \to \mathbb{P}^2$ *of degree d, s.t* Λ *does not intersect the critical set* C_f *of* f*. If each point in* Λ *has at most* d' f*-preimages in* Λ *and if* $d' < d$*, then for* μ_ϕ*-a.e z,*

$$\underline{\delta}_{\nu_i}(z) \geq \log d \cdot \left(\frac{1}{\chi_u(\nu_i)} - \frac{1}{\chi_s(\nu_i)} \right) + \log d' \cdot \frac{1}{\chi_s(\nu_i)}$$

(*b*) *If* Λ *is a mixing terminal saddle set for a holomorphic map* f *on* $\mathbb{P}^2$ *of degree d, if* $C_f \cap \Lambda = \emptyset$ *and if the preimage counting function is constant equal to* d' *on* Λ *for* $d' \leq d$*, then we have:*

$$\delta_{\nu_i} = HD(\nu_i) = \log d \cdot \left(\frac{1}{\int \log |Df_u| \, d\nu_i} - \frac{1}{\int \log |Df_s| \, d\nu_i} \right) + \log d' \cdot \frac{1}{\int \log |Df_s| \, d\nu_i},$$

(*c*) *Let* $f : \mathbb{P}^2 \to \mathbb{P}^2$ *be a holomorphic Axiom A map of degree d, which is c-hyperbolic on a connected minimal saddle set* Λ*. Then the preimage counting function is constant on* Λ*, with value denoted* d'*, and*

$$\delta_\nu = HD(\nu) = \log d \cdot \left(\frac{1}{\chi_u(\nu)} - \frac{1}{\chi_s(\nu)} \right) + \log d' \cdot \frac{1}{\chi_s(\nu)}$$

Proof (a) We use the result of [7] that the topological entropy of $f|_\Lambda$ is equal to $\log d$ if Λ is a terminal saddle basic set. From Theorem 2 we have that $\nu_i = \mu_0$, the measure of maximal entropy on Λ; hence $h_{\mu_0} = h_{top}(f|_\Lambda)$. Then the inequality follows from Corollary 1 in case the number of preimages in Λ is bounded above by d'.

(b) If Λ is terminal, then from Theorem 2 we know that ν_i is equal to μ_0 the measure of maximal entropy of $f|_\Lambda$; also since the topological entropy of $f|_\Lambda$ is $\log d$, it follows that $h_{\nu_i} = \log d$.

If the preimage counting function is constant and equal to d', then in case $d' < d$, we have that condition $\phi + \log d' < P(\phi)$ is satisfied on Λ for $\phi \equiv 0$. So one can apply Theorem 1 in order to obtain the pointwise dimension of ν_i; and since δ_{ν_i} is constant, then $HD(\nu_i) = \delta_{\nu_i}$.

There remains only the case $d' = d$. In this case every point in Λ has d f-preimages in Λ and $h_{top}(f|_\Lambda) = \log d$ (from [7]). Thus the unique zero of the function $t \to P(t \log |Df_s| - \log d)$ is equal to 0; so from [17] the function $f|_\Lambda$ is expanding. In this expanding case we have that $B_n(z, \varepsilon)$ is itself a round ball of μ_0-measure comparable to $\frac{1}{d^n}$ [from the estimates of equilibrium measures on Bowen balls in (2)]. At the same time the radius of this ball is comparable to $\varepsilon |Df_u^n(z)|^{-1}$. Hence the lower and upper pointwise dimensions of ν_i coincide and the pointwise dimension and Hausdorff dimension of ν_i are both equal to

$$\delta_{\mu_0} = HD(\mu_0) = \frac{\log d}{\chi_u(\mu_0)}$$

(c) In this case if Λ is connected and f is c-hyperbolic on Λ, it follows from [19] that the preimage counting function is constant on Λ. Also Λ cannot be written as a disjoint union of compact sets, so it is mixing for an iterate of f. So we can apply Theorem 1 for the measure ν which, according to Theorem 2 is equal to μ_0, and thus has entropy $\log d$.

□

Remark If f is a smooth endomorphism hyperbolic on a basic set Λ, then by taking a smooth perturbation g of f, it follows that g has also a basic set Λ_g on which it is hyperbolic (see [25]). Also if Λ is **connected** then $\hat{\Lambda}$ is connected, so from the conjugacy of $\hat{f}|_{\hat{\Lambda}}$ to $\hat{g}|_{\hat{\Lambda}_g}$, $\hat{\Lambda}_g$ is connected and thus Λ_g is connected too.

However the dynamics of perturbations g may be very **different**; for instance perturbations of toral endomorphisms are not conjugated necessarily to the original maps, and may even have infinitely many unstable manifolds through a given point. See also the examples of perturbations of polynomial maps from [22]. □

For minimal c-hyperbolic sets of maps of degree 2 we can determine the possible values of the pointwise dimension of ν; recall that the preimage counting function is constant if Λ is connected.

Corollary 4 *Let f be an Axiom A holomorphic map on $\mathbb{P}^2$ of degree 2, which is c-hyperbolic on a connected minimal saddle set Λ and let ν be the measure of maximal entropy of $f|_\Lambda$. Then we have exactly one of the following two possibilities:*

(*1*) *the preimage counting function of f is equal to 1 on Λ; then $f|_\Lambda$ is a homeomorphism and*

$$\delta_\nu = \log 2 \cdot \left(\frac{1}{\int \log |Df_u| \, d\nu} - \frac{1}{\int \log |Df_s| \, d\nu} \right)$$

(*2*) *or, the preimage counting function of f is equal to 2 on Λ; then $f|_\Lambda$ is expanding and*

$$\delta_\nu = \log 2 \cdot \frac{1}{\int \log |Df_u| \, d\nu}$$

Proof If f has Axiom A and Λ is minimal then Λ is terminal, thus from [7] it follows that $h_{top}(f|_\Lambda) = \log 2$. Now if Λ is connected and if there exists a neighbourhood U of Λ with $f^{-1}(\Lambda) \cap U = \Lambda$ then the preimage counting function is constant on Λ. If the preimage counting function is equal to d' on Λ it follows that $d' \le 2$, otherwise from Misiurewicz–Przytycki Theorem (see [14]) we would have $h_{top}(f|_\Lambda) \ge \log d' > \log 2$ which is impossible, as we saw above.

So we either have $d' = 1$ or $d' = 2$. In the first case we can apply Theorem 1 for the potential $\phi \equiv 0$ since $\log d' < P(0) = \log d$ on Λ. In this case $f|_\Lambda$ is a homeomorphism (like for instance the family of polynomial perturbations constructed in [22]).

In the second case if $d' = 2$, the stable dimension $\delta^s := HD(W_r^s(z) \cap \Lambda)$ is equal to the unique zero of the pressure function $t \to P(t \log |Df_s| - \log 2)$ (see [19] and references therein); but since $h_{top}(f|_\Lambda) = \log 2$, the zero of this pressure function is indeed equal to 0. Then $\delta^s = 0$ and we can apply the result of [17] saying that in this case, f must be expanding on Λ.

From Theorem 2 the measure ν is equal to μ_0, i.e. the measure of maximal entropy. Therefore from Theorem 1 the pointwise dimension of ν is

$$\delta_\nu = \log 2 \cdot \frac{1}{\int \log |Df_u| \, d\nu}$$

□

Examples and remarks

(1) First notice that the map $f_0(z, w) = (z^2 + c, w^2)$ is 2-to-1 and expanding on $\Lambda_0 = \{p_0(c)\} \times S^1$, where $p_0(c)$ is the fixed attracting point of $z \to z^2 + c$ for $|c|$ is small enough. In [22] there was constructed a class of **perturbations** of f_0 which are **homeomorphic** on their respective basic sets. This shows in particular that the preimage counting function is **not** necessarily preserved by perturbations. These are maps of type

$$f_\varepsilon(z, w) = (z^2 + c + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2, w^2),$$

for $b \neq 0$, $|c|$ small and $0 < \varepsilon < \varepsilon(a, b, c, d, e)$. Then f_ε has a basic set Λ_ε (close to Λ_0), on which it is hyperbolic and has a homeomorphic restriction.

For $0 < \varepsilon < \varepsilon(a, b, c, d, e)$ it also follows that f_ε is c-hyperbolic on Λ_ε since there are no critical points of f_ε in Λ_ε and since there exists a neighbourhood U of Λ_ε such that $f_\varepsilon^{-1}(\Lambda_\varepsilon) \cap U = \Lambda_\varepsilon$. Indeed for ε fixed if there were no such neighbourhood, then for any neighbourhood V of Λ_ε there would exist a point $y \in V \backslash \Lambda_\varepsilon$ with $f_\varepsilon(y) \in \Lambda_\varepsilon$. Thus for any $n > 0$ there would exist points $y_n \in B(\Lambda_\varepsilon, \frac{1}{n})$ with $f_\varepsilon(y_n) = x_n \in \Lambda_\varepsilon$, and since Λ is compact we can assume $x_n \to x \in \Lambda_\varepsilon$. Since there are no critical points of f_ε in Λ_ε, it follows that there exists a positive distance η_0 s. t $d(y_n, z_n) > \eta_0$, where z_n is another preimage of x_n belonging to Λ_ε (such z_n must exist since $f_\varepsilon(\Lambda_\varepsilon) = \Lambda_\varepsilon$). Then without loss of generality we can assume that $y_n \to y$ so $y \in \Lambda$ since $d(y_n, \Lambda) < \frac{1}{n}$. But perhaps after passing to a subsequence, $z_n \to z \in \Lambda$; then from above $d(y, z) > \eta_0/2$. But this is a contradiction since f_ε is homeomorphic on Λ_ε. Hence there must exist a neighbourhood U of Λ_ε satisfying

$$f_\varepsilon^{-1}(\Lambda_\varepsilon) \cap U = \Lambda_\varepsilon$$

Also notice that if we fix $a, b, c, d, e, \varepsilon$ as above and perturb now f_ε, we obtain another map g which has a saddle basic set Λ_g on which g is hyperbolic and homeomorphic. This example shows that the bound d' on the number of preimages remaining in Λ, can be very different among perturbations. For endomorphisms which are hyperbolic on basic sets Λ, there exists a connection between the maximality of the number of preimages, 1-sided Bernoullicity of the measure of maximal entropy and the expanding property on Λ (see [17]).

(2) Examples of **terminal sets** can be obtained by perturbations of known examples f and Λ; if $W^u(\hat{\Lambda})\setminus\Lambda$ is contained in the union of basins of finitely many attracting cycles of f, then any small perturbation g has a saddle basic set Λ_g close to Λ, and $W^u(\hat{\Lambda}_g)\setminus\Lambda_g$ is also contained in the union of basins of attraction of g; hence Λ_g is terminal too.

Also the topological entropy of restrictions is preserved by perturbations, i.e. $h_{top}(f|_\Lambda) = h_{top}(g|_{\Lambda_g})$. Thus by perturbing the known examples (like products of hyperbolic rational maps, or Ueda type examples of [12], see [12]), we obtain more examples of terminal sets. As noticed before if Λ is connected, then the basic set Λ_g is connected too. And if f is mixing on Λ then $\hat{f}$ is mixing on $\hat{\Lambda}$; hence from the conjugacy on inverse limits, we obtain that g is mixing on Λ_g as well.

We can take for instance examples constructed by Ueda's method (see [29]); if $\Phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ is the Segre map $\Phi([z_0 : z_1], [w_0 : w_1]) = [z_0w_0 : z_1w_1 : z_0w_1 + z_1w_0]$, and $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map then there exists $F : \mathbb{P}^2 \to \mathbb{P}^2$ holomorphic of the same degree as f, so that

$$\Phi(f, f) = F \circ \Phi$$

If f is hyperbolic on its Julia set $J(f)$ (i.e. expanding), then F is hyperbolic on basic sets of type

$$\Lambda = \Phi(\{\text{periodic sink of } f\} \times J(f))$$

The saddle set Λ is terminal and topologically mixing for F. Let us consider now also a holomorphic **perturbation** G of F with a corresponding basic set Λ_G, which is close to Λ. From above it follows that Λ_G is terminal and mixing saddle set for G.

Consider also Hölder potentials ϕ on Λ_G satisfying inequality (1) with respect to G; for instance, in the setting of Corollary 3 we can take ϕ sufficiently small in $\mathcal{C}^0$-norm s.t (1) is still satisfied. Now for each such ϕ we have an equilibrium measure μ_ϕ on Λ_G. Then it follows that one can apply Theorem 1, Corollary 1 and Corollary 3 in order to obtain the values of μ_ϕ on iterates of Bowen balls in Λ_G, and also in order to estimate the (upper/lower) pointwise dimensions of μ_ϕ.

In particular we obtain information about the (upper/lower) pointwise dimensions for the measure $\mu_{0,G}$ of maximal entropy of the restriction $G|_{\Lambda_G}$.

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