Math. Ann. 314, 39–52 (1999)

Mathematische Annalen

Periodic points for actions of tori in Stein manifolds

Eugen Mihailescu

Mathematics Department, University of Michigan, Ann Arbor, MI 48109, USA
(e-mail: mihailes@math.lsa.umich.edu)

Received: 20 February 1998

Mathematics Subject Classification (1991): 32M05, 32S20, 32A27, 58F08

0. Introduction

In one complex variable dynamics, Sullivan's theorem ([6]) gives a complete classification of the Fatou components that can appear for a rational map f.

Consequently we can have only periodic components U and U can be one of the following: 1. U attracting basin; 2. U parabolic domain; 3. U Siegel disk; 4. U Herman ring.

Cases 3. and 4. are called *rotation domains*. In these cases, the rational map f is conjugated on U to an irrational rotation, hence by taking all the iterates of f and their limits we obtain an S^1-action on U which has at most one fixed (periodic) point in U.

The goal of this paper is to give conditions when the generalization of the one variable situation is true. In particular we consider actions of tori on Stein manifolds and study their periodic (fixed) points.

Outline of the paper:

The main results of the paper are contained in Theorems 2.1, 3.3, 3.10 and Proposition 3.7.

In Sect. 1 we introduce notation and basic definitions; also some examples of Siegel domains are reviewed.

In Sect. 2 we will prove that the number of periodic points (of all periods) for a large class of actions of tori on Stein manifolds is finite.

Then, Sect. 3 will give topological conditions on the manifold guaranteeing the existence of at most one fixed point. In particular, this will become true when the second cohomology group with integer coefficients vanishes.

Finally, Sect. 4 introduces an example of an S^1-action on a connected and Kobayashi hyperbolic Stein manifold having exactly m fixed points, $m \geq 2$.

The author would like to thank John-Erik Fornæss for many useful discussions. I am also grateful to the referee for many careful comments about this paper.

1. Basic facts and examples

In the following we will consider Ω a Stein manifold, which is also supposed to be Kobayashi hyperbolic. If Ω is hyperbolic, a theorem of Kobayashi ([14]) states that its group of holomorphic automorphisms, $Aut(\Omega)$, is a Lie group. The following lemma follows from the structure theorem of commutative Lie groups (for example Onishchik [16]); for the convenience of the reader, a proof of the lemma may be found in Ueda [18].

Lemma 1.1 *Let G be a Lie group and assume there exists an element f in G such that the subgroup generated by f is infinite cyclic and there exists a subsequence $(f^{j_n})_n$ converging to the identity element. Then the closure of $(f^n)_n$ is compact and it contains a torus T^s, for some $s > 0$.*

Hence by taking $G = Aut(\Omega)$ with the compact-open topology, and $f \in Aut(\Omega)$ so that the hypothesis of the lemma is satisfied, we will obtain a $\boldsymbol{T^s}$-action on Ω.

For future reference, let us write down that in the notation of Ueda, H is the closure of $(f^j)_j$ in G and H_0 is the connected component of H containing the identity element.

If $s = \dim_{\mathbb{C}} \Omega$, then the action is described by a theorem of Barrett-Bedford-Dadok [3]:

Theorem 1.2 ([3]) *In the above assumptions (so $G = Aut(\Omega)$), if $s = dim_{\mathbb{C}} \Omega$, there exists $\tilde{\Phi} : \Omega \to U$, a biholomorphism of Ω to a Reinhardt domain U, and an integer $l > 0$ so that $\tilde{\Phi} \circ f^l = R \circ \tilde{\Phi}$, with $R(z_1, ..., z_s) = (e^{i\alpha_1} z_1, ..., e^{i\alpha_s} z_s)$.*

Consequently we may have at most one fixed point of f in Ω in this case. However there is no classification as the one above for general actions of tori on Stein manifolds.

A natural example of a Stein manifold with a torus action is provided by a Siegel domain for a holomorphic map on a projective space.

In the following definition the notion of degree of a holomorphic mapping $f : \mathbb{P}^n \to \mathbb{P}^n$ will be in the sense of Fornaess–Sibony ([9, 11]), i.e as degree of a homogeneous polynomial lifting for f,$F : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{C}^{n+1} \setminus 0$.

Definition: Let $f : \mathbb{P}^n \to \mathbb{P}^n$ a holomorphic map of degree d ≥ 2.

$\mathcal{F}$ = $\{$z $\in \mathbb{P}^n$, z has a neighbourhood V s.t $(f^j)_j$ forms a normal family in V$\}$ is its Fatou set.

For properties of Fatou sets, [9-10-11] are good references. A connected component U of $\mathcal{F}$ is called a Siegel domain if there exists a subsequence $(f^{j_\nu})_\nu$ converging uniformly on compacts to id_U. This notion has been introduced by Fornæss and Sibony ([8, 9]) as a generalization of the Siegel disks and Herman rings from one variable.

Directly from the definition it is clear that any Siegel domain is periodic, i.e there is $m > 0$ s.t $f^m(U) = U$. By a theorem of Ueda ([19]), any such Siegel domain is Stein and Kobayashi hyperbolic. It follows easily that $f : U \to U$ is a biholomorphism (we may assume WLOG that $m = 1$, hence that $f(U) = U$) and, from degree considerations, also that there is no $q > 0$ s.t $f^q/_U = id_U$.

We notice also that, since U is Kobayashi hyperbolic, $G = Aut(\Omega)$ is a Lie group and we are in the conditions of Lemma 1.1.

Remark: Examples of Siegel domains in $\mathbb{P}^2$ can be constructed by the method of Ueda ([20]), starting from rational functions on $\mathbb{P}^1$ that have Siegel disks or Herman rings.

The main idea is to double cover $\mathbb{P}^2$ with $\mathbb{P}^1 \times \mathbb{P}^1$ by means of a projection π such that the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ is the branch locus and $\pi([\xi : \eta], [\xi\prime : \eta\prime]) = \pi([\xi\prime : \eta\prime], [\xi : \eta])$. Then, if $f : \mathbb{P}^1 \to \mathbb{P}^1$ is rational, there will exist a holomorphic mapping $F : \mathbb{P}^2 \to \mathbb{P}^2$ s.t $F \circ \pi = \pi \circ \hat{f}$, where $\hat{f} = (f, f)$.

Hence if U is a Siegel disk for f, then since f has no periodic points of period two in U (other than the fixed point), the associated map F has only one fixed point in π $(U \times U)$. Similarly, if U is a Herman ring for f, we obtain that $\pi(U \times U)$ is a Siegel domain for F with no fixed points.

Taking the iterates of F on U and their limits in $Aut(\Omega)$ will provide us with an example of an action on a Siegel domain which has no fixed points.

2. Periodic points

Theorem 2.1 *Let Ω be a Stein manifold of dimension n and assume that Ω is Kobayashi hyperbolic. Assume also that we are in the conditions from the hypothesis of Lemma 1.1 with $G = Aut(\Omega)$ and that the holomorphic automorphism f satisfies the following condition:*

($\star$)

for all $m > 0$, m integer, there exist only at most a finite number of solutions *of the equation $f^m(x) = x$ (we call these solutions periodic points of period m).*

Then the number of periodic points of f of all periods, belonging to Ω is finite.

Proof As in the proof of Lemma 1.1 (see Ueda [18]), if H_0 is the connected component of $(f^j)_j$ in $Aut(\Omega)$ containing the identity, then $\exists p \in \mathbb{Z}$ such that H_0 is the closure of $(f^{pj})_j$ and there is a Lie group isomorphism $\Phi : H_0 \to T^s$, for some $s > 0$.

Then $\Phi(f^p) = (e^{i\alpha_1}z_1, ..., e^{i\alpha_s}z_s)$.

Assume that one of the $\alpha_1, ..., \alpha_s$ is rational, for instance $\alpha_1 = \frac{m_1}{n_1}$.

Then $\Phi(f^{pm}) = (e^{2\pi i m\alpha_1}, ..., e^{2\pi i m\alpha_s})$ and obviously $e^{2\pi i m\alpha_1}$ takes only a finite number of values. Therefore, $(f^{pm})_m$ would not be dense in H_0, contradiction. So all α_j's are irrational.

Suppose now that we can find an infinite number of periodic points in Ω. Because of the condition $(\star)$, we should have a sequence of integers $m_j \to \infty$ and points p_j s.t p_j is a periodic point of period m_j, for all j.

Let $j > 0$ be fixed and consider m arbitrary s.t $f^m \in H_0$. Since all α_k's are irrational, it is clear that the closed subgroup generated by f^{pm_j} is still dense in H_0, for all j. Consequently there is a sequence k_q s.t $f^{pm_jk_q} \to f^m$ in H_0, when $q \to \infty$.

So $f^{pm_jk_q}(p_j) = \underbrace{f^{m_j} \circ ... \circ f^{m_j}}_{p\cdot k_q}(p_j) = p_j \to f^m(p_j)$, hence p_j is a periodic point of period m. But j was chosen arbitrarily, so all points p_j are periodic of period m, which would contradict $(\star)$. $\diamond$

Remark Let $f : \mathbb{P}^n \to \mathbb{P}^n$ a holomorphic mapping of degree $d \geq 2$ and Ω a Siegel domain of f. Now Ω is Stein and Kobayashi hyperbolic and f satisfies condition $(\star)$, (Fornæss–Sibony [9]).

Hence we get that there exists at most a finite number of periodic points of f (of all periods) belonging to Ω.

3. Fixed points for holomorphic S^1-actions in Stein manifolds

In this paragraph we shall specialize our results to the case of Stein manifolds of dimension 2. We consider $f \in Aut(\Omega)$, a holomorphic automorphism of Ω, s.t the closure of $(f^n)_n$ in $Aut(\Omega)$ (with the compact-open topology) is isomorphic as a Lie group to S^1.

Assume that 0 is a fixed point of f in Ω and also that f satisfies $(\star)$..

Ω is not supposed for the moment to be Kobayashi hyperbolic. First let us state a well-known local linearization result (see also [1] and [14]).

Lemma 3.1 *If Ω is a complex manifold of dimension 2 and $f \in Aut(\Omega)$ is generating an S^1-action as above, then there will exist a local biholomorphism ϕ from a neighbourhood U of 0 in Ω to a neighbourhood W of 0 in $\mathbb{C}^2$, so that*

$$\phi \circ f = A \cdot \phi \text{ on } U\,,$$

where $A := Df(0)$.
Note: we denote by 0 both the point in Ω and the point in $\mathbb{C}^2$; which one is meant will become clear from the context.

Knowing that f generates an S^1 action we can say more about the matrix A from the previous Lemma.

Lemma 3.2 *In the same setting as in Lemma 3.1, there will exist $q_1, q_2 \in \mathbb{Z}$ and λ with $|\lambda| = 1$ and λ is not a root of unity, such that $A = \begin{pmatrix} \lambda^{q_1} & 0 \\ 0 & \lambda^{q_2} \end{pmatrix}$.*

The proof of this lemma is relatively easy and we will not give it here (see also [7]). Among matrices that have the above form, we can differentiate between those for which q_1, q_2 have the same sign and the ones with q_1, q_2 of opposite signs. This classification does not depend on which λ we choose above and it will prove to be essential in the study of S^1 -actions.

Case 1
Let us consider first the case when q_1, q_2 have the same sign.WLOG we may suppose that q_1, q_2 are both positive, otherwise we just take f^{-1} instead of f and clearly f^{-1} would have the same fixed points as f. Although versions of the next theorem are more or less known ([7]), we will give nevertheless a brief proof for future reference and notation.

Theorem 3.3 *Let Ω be a Stein manifold of dimension 2 with a holomorphic S^1 -action generated by an automorphism f as above.*

If the eigenvalues of the derivative of f at 0 look like $\lambda_1 = \lambda^{q_1}, \lambda_2 = \lambda^{q_2}$, $q_1, q_2 \in \mathbb{Z}$ and λ not a root of unity, and if q_1, q_2 have the same sign, then the action will have exactly one fixed point in Ω.

Proof Assume as above that 0 is a fixed point for f. First there is a neighbourhood U of 0 s.t we can continue the orbits of points in U to analytic disks in U going through 0.

Suppose now there exists another fixed point $p \in \Omega$. Since Ω is Stein, we find g holomorphic on Ω s.t $g(0) = 0, g(p) = 1$.

Define $\tilde{g}(z) = \int_{S^1} g(\theta \cdot z) d\mu(\theta)$, where $\theta \cdot z$ denotes the action of θ on z and μ is the normalized Haar measure on S^1; $\tilde{g}$ is well defined and holomorphic on Ω and this is independent of the relative signs of q_1, q_2. Also, $\tilde{g}$ must be constant on orbits and since both 0 and p are fixed points, $\tilde{g}(0) = 0, \tilde{g}(p) = 1$.

Near 0 the orbits of the action are boundaries of analytic disks passing through 0, therefore taking the restriction of $\tilde{g}$ to one of these analytic disks, we get that $\tilde{g}$ is constant in the interior. But since $\tilde{g}(0) = 0$, it follows that $\tilde{g} \equiv 0$ near 0, hence in the entire Ω. On the other hand $\tilde{g}(p) = 1$, which leads to a contradiction. $\diamond$

Case 2

The case when q_1, q_2 have different signs is more complicated, mainly because we do not have analytic disks passing through 0 and whose boundaries are orbits of the action. In this situation the orbits can only be "complexified" up to analytic annuli in Ω. In what follows we will also need the condition Ω Kobayashi hyperbolic.

In the sequel we will obtain a foliation of Ω with 1-dimensional complex varieties. Let us summarize how to obtain this foliation.

As in the proof of Theorem 3.3, we can consider the average over orbits of a fixed arbitrary non-constant holomorphic function $g \in O(\Omega)$ (assuming such a function g exists s.t $\tilde{g}$ is not constant).

Notation: *If $f \in Aut(\Omega)$ generates an S^1-action and if $g \in O(\Omega)$, then define its averaged function $\tilde{g}(z) = \int_{S^1} g(\theta \cdot z) d\mu(\theta)$, where μ is the normalized Haar measure on S^1 and $\theta \cdot z$ denotes the action of θ on z through the S^1-action generated by f.*

Like in the proof of Theorem 3.3, one can easily check that $\tilde{g}$ is holomorphic on Ω and it is constant on the orbits of the S^1-action.

If we take the level sets $L_C(\tilde{g})$ of this averaged function, we will obtain complex 1-dimensional varieties in Ω. Since $\tilde{g}$ is constant on orbits one can see that the action preserves the sets $L_C(\tilde{g})$.

Also, if $z_0 \in \Omega$ and $O(z_0)$ denotes its orbit, $\exists C$ complex number, s.t $O(z_0) \subset L_C(\tilde{g})$. Then, we will consider the global irreducible component of $L_0(\tilde{g})$ containing the local "z-axis" near 0.

One can show that this global irreducible component is an analytic submanifold of dimension 1, invariant to the action of f, playing the role of a "global z-axis" in Ω. This will be used in the end for the topological condition from Theorem 3.9

Let us make this submanifold construction more precise in the next lemmas and in Proposition 3.7.

Firstly one has to make sure there is a non-constant averaged function $\tilde{g}$ as above.

Lemma 3.4 *In the notation from 3.4, if Ω is a Stein manifold, there exists a holomorphic function $g \in O(\Omega)$ such that $g(0) = 0$, but $\tilde{g} \not\equiv 0$.*

Proof First we will recall briefly the definition of an Oka–Weil domain ([12])

Definition: An Oka–Weil domain U in a holomorphic variety V is an open subset of V with the property that there are finitely many global holomorphic functions $f_1, ..., f_N \in O(V)$ s.t the restrictions of these functions to U are the coordinate functions of a biholomorphic mapping between U and a holomorphic subvariety of an open polydisk in $\mathbb{C}^N$. $\diamond$

Hence, since Ω is a Stein manifold, we can find a neighbourhood of 0, denoted by U, which is an Oka–Weil domain.

We can also assume that U is invariant to the action and that f is linearizable in U. Suppose now that for all $g \in O(\Omega)$ with $g(0) = 0$, we have $\tilde{g} \equiv 0$. Consider the function $h \in O(U)$ which in the local linearizing coordinates (z, w) has the formula $(z, w) \to z^{|q_2|} \cdot w^{q_1}$, if $q_1 > 0, q_2 < 0$. Then, since U is an Oka–Weil domain one can approximate h uniformly on compacts of U by global holomorphic functions g_n (Gunning [12]).

By taking $g_n - g_n(0)$, we can suppose that $g_n(0) = 0$.

Then $\tilde{g_n}(z, w) = \int_{S^1} g_n(\lambda^{q_1} \cdot z, \frac{1}{\lambda^{|q_2|}} \cdot w) d\mu(\lambda)$, for $(z, w) \in U$, and clearly we get $\tilde{g_n} \to \tilde{h}$ on U. But if $\tilde{g_n} \equiv 0, \forall n$, then also $\tilde{h} \equiv 0$. On the other hand by the formula for $\tilde{h}$, $\tilde{h}(z, w) = \int_{S^1} \lambda^{q_1|q_2|} \cdot z^{|q_2|} \cdot \frac{1}{\lambda^{|q_2| \cdot q_1}} \cdot w^{q_1} d\mu(\lambda) = z^{|q_2|} \cdot w^{q_1} \not\equiv 0$, contradiction. Hence we proved that there is a function $g \in O(\Omega)$, $g(0) = 0$ s.t $\tilde{g} \not\equiv 0$. $\diamond$

In the following we will only consider functions g of the type referred in Lemma 3.5

Lemma 3.5 *In the above setting, the analytic set $L_0(\tilde{g})$ has only two local irreducible components near 0.*

Proof Since $\tilde{g}(0) = 0$, but $\tilde{g} \not\equiv 0$, we can decompose $L_0(\tilde{g})$ near 0 as a union of a finite number of irreducible analytic sets: $L_0(\tilde{g}) \cap U = S_1 \cup ... \cup S_l$.

Also, $\forall k$, $f^k(S_1) \subset S_1 \cup \ldots \cup S_l$. But $f^k(S_1)$ is an irreducible analytic set, so $f^k(S_1) \subset \{S_1, .., S_l\}$, hence since f is a biholomorphism of U, there will exist k_1 s.t $f^{k_1}(S_1) = S_1$. Similarly we can prove that $\exists k$ s.t $f^k(S_i) = S_i, \forall i = 1..l$.

Because the problem is local, f can be assumed to be equal to $A = \begin{pmatrix} \lambda^{q_1} & 0 \\ 0 & \lambda^{q_2} \end{pmatrix}$, q_1, q_2 of different signs, for example $q_1 > 0$, $q_2 < 0$.

Since f^k has the same form as f, we may assume WLOG that $f(S_i) = S_i, i = 1..l$. The question is reduced now to studying for which germs of analytic sets S_i, at 0, we get invariance by f. An easy exercise in one complex variable will prove that the only analytic germs at 0, which are invariated by the above matrix are the coordinate axes. This implies that $l = 2$, so $L_0(\tilde{g})$ as only two irreducible local components near 0. $\diamond$

In the sequel let us fix a holomorphic function g_0 on Ω s.t $g_0(0) = 0$ and $\tilde{g_0} \not\equiv 0$ in Ω. Denote by S the global irreducible component of $L_0(\tilde{g_0})$

which contains the local "z-axis" near 0. Here and in the sequel, by local z-axis near 0 in Ω we mean the preimage of the z-axis near $0 \in \mathbb{C}^2$ by the local biholomorphism ϕ of Lemma 3.1.

Remark In fact, since any two irreducible analytic subsets of Ω of dimension 1 coinciding in an open set are actually identical globally (this follows from the identity principle), one can see that the set S does not depend on the defining function $\tilde{g_0}$.

We will show that this set S plays the role of a "global z-axis" passing through 0 in Ω.

Proposition 3.6 *Let g_0 be one of the functions whose existence is proved in Lemma 3.5, i.e so that $g_0 \in O(\Omega)$, $g_0(0) = 0$, but the averaged function $\tilde{g_0} \not\equiv 0$. If S is the global irreducible component of $\tilde{g_0}^{-1}(0)$ containing the preimage of the local z-axis near $0 \in \mathbb{C}^2$ by the biholomorphism ϕ of Lemma 3.1, then $Sing(S) = \emptyset$, so S is a complex submanifold of dimension 1 in Ω.*

Proof $Sing(S)$ is an analytic set of dimension 0, so it is a discrete subset in Ω, without accumulation points in the interior. We will show first that 0 is not a singular point of S; the proof in general will follow then easily from this.

Lemma 3.7 *In the above notation, the point 0 (fixed by the action) is not a singular point for S.*

Proof Assume 0 is a singular point of S; this will be shown to give a contradiction. $\exists$ a small neighbourhood in which S is the union of the two local coordinate axes(one can actually identify the z- and w-axes near 0 in $\mathbb{C}^2$ with their preimages in Ω). We can also assume that in this neighbourhood, 0 is the only singular point of S. Let now $\tilde{S} = S \setminus Sing(S)$; $\tilde{S}$ is a Riemann surface and it is hyperbolic (since it is contained in Ω which was supposed to be Kobayashi hyperbolic). Therefore $f|\tilde{S} : \tilde{S} \to \tilde{S}$ is a biholomorphism.

According to Milnor's classification theorem ([15]), any holomorphic map f of a hyperbolic Riemann surface $\tilde{S}$ falls into one of the following four mutually disjoint cases:

1. f has a unique attracting fixed point in $\tilde{S}$, or
2. every orbit diverges to infinity w.r.t the Kobayashi distance $d_{\tilde{S}}$, or
3. f is an automorphism of finite order, or
4. $\tilde{S}$ is isomorphic to the unit disk Δ, the punctured unit disk $\Delta^\star$ or an annulus, and f corresponds to an irrational rotation.

In our case, since f has a sequence of iterates converging to identity, we cannot have 1. or 2. Also case 3. cannot appear because of the local form of f. Here and in the rest of the paper we are still in the case when the exponents

q_1, q_2 of the derivative of f at 0 have different signs. Then f is isomorphic to $\Delta, \Delta^\star$ or an annulus by means of a map β, and f is conjugated to an irrational rotation.

So $\beta : \tilde{S} \to \Delta, \Delta^\star$, or an annulus *A*, is a biholomorphism and let an invariant neighbourhood U of 0 in which f can be linearized and s.t according to the previous lemma 3.6, $\tilde{S} \cap U$ is the union of the two local coordinate axes, minus the point 0. Denote the two connected components of $\tilde{S} \cap U$ by S_1, S_2; obviously $f(S_1) = S_1, f(S_2) = S_2$. Consider now $\beta(S_1)$ and $\beta(S_2)$ open connected subsets in one of $\Delta, \Delta^\star$ or an annulus, in which f corresponds to an irrational rotation.

If we take for example $\beta(S_1) \cap$ segment on the positive axis belonging to $\Delta, \Delta^\star$ or an annulus centered at 0, we will obtain an open subset of this segment. By rotating this open subset, the original set $\beta(S_1)$ will be reobtained; hence this open subset of the interval must be connected. But the only connected open subsets of a real interval are real open subintervals. Hence $\beta(S_1)$ must be actually a set of the form $\{0 \leq |z| < r\}$, or of the form $\{0 < |z| < r$, for some $r < 1$. The same can be said about $\beta(S_2)$.

Since $S_1 \cap S_2 = \emptyset \Longrightarrow \beta(S_1) \cap \beta(S_2) = \emptyset$.

According to the previous discussion, $\beta(S_1)$ and $\beta(S_2)$ are both annuli or disks or punctured disks centered at 0 in $\mathbb{C}$. Hence if they are disjoint, they cannot be both punctured disks.

On the other hand S_1, S_2 were defined as the local coordinate axes minus 0, hence they should be both biholomorphic to punctured disks.

But a disk, a punctured disk and an annulus cannot be biholomorphic to each other, therefore a contradiction. This shows that our assumption was wrong, so 0 is a regular point of S. $\diamond$

Returning to the proof of Proposition 3.7, it is clear that $f(Sing(S)) = Sing(S)$; let now $z_0 \in Sing(S)$ and consider its orbit $O(z_0)$. Since $O(z_0) = \overline{(f^n(z_0))_n}$ is compact and $f^n(z_0) \in Sing(S), \forall n$, it follows that $O(z_0)$ is a finite set of points. WLOG one can assume that z_0 is a fixed point for f.

If $\mathrm{D}f(z_0)$ has eigenvalues of the type $\lambda^{q_1}, \lambda^{q_2}$, with q_1, q_2 of opposite signs, then $f|S$ is conjugated to an irrational rotation and we get a contradiction as above. Hence $Sing(S) = \emptyset$. $\diamond$

Remark The above proposition has an interest in itself. Indeed, consider S_1 the global irreducible component of $L_0(\tilde{g_0})$ containing the local “w-axis” near 0. Since S cannot contain both the “z-axis” and the “w-axis” near 0(Lemma 3.6), it follows that $S_1 \neq S$. Also $f(S_1) = S_1$.

In fact the following result is true:

Corollary 3.8 $S \cap S_1 = \{0\}$.

This can be proved in a similar way, by looking at the orbit of a point from $S \cap S_1$ and at the action of f on S.

Remark (continued) S and S_1 are complex submanifolds of dimension 1 in Ω, they are invariated by the action and they extend the local "coordinate axes" near 0. One can interpret them as "global axes of coordinates" in Ω. Next we want to prove that the action, under certain topological restrictions, has only one fixed point in Ω. First we need to introduce some notions and facts from multidimensional residue theory. Assume for the moment that X is a complex manifold of dimension n and S is an analytic submanifold of codimension 1 in X.

Definition ([2]) Let ω a closed regular form on $X \setminus S$.

ω is said to have a polar singularity of first order if, $\forall a \in S$, if $\{s = 0\}$ is a minimal defining equation for S, $s \cdot \omega$ can be locally extended as a regular form across S. $\diamond$

Suppose now that ω is a form like in the definition. Then by a theorem of Leray ([2]), if $a \in S$, there exist U_a a neighbourhood of a, and regular forms ψ and θ defined in U_a, s.t

$$(1) \quad \omega = \frac{ds}{s} \wedge \psi + \theta\,,$$

where $S \cap U_a = \{s = 0\}$, grad $s \neq 0$.

Moreover the form $\psi|_{S\cap U_a}$ can be extended globally to the entire S and it is closed. This restricted global form on S is called the *residue form of* ω.

Also if ω is holomorphic in $X \setminus S$, then the form $\psi|_S$ is holomorphic on S. The holomorphic forms on $X \setminus S$ which have polar singularities of first order play a special role in the case when $dim_X = 2$ and X is Stein. Denote by $\Omega^p(X, S)$ the sheaf of germs of p-forms χ that are holomorphic in $X \setminus S$ and s.t χ and $d\chi$ have polar singularities of first order on S.

In 1986, G.Raby ([17]) proved the following theorem relating the forms in $\Omega^p(X, S)$ to $H^1(X \setminus S, \mathbb{C})$:

Theorem: If X is a Stein manifold of dimension 2 and S is a submanifold of X of dimension 1, then

$$H^1(X \setminus S, \mathbb{C}) = H^1(\Gamma(X, \Omega^\bullet(X, S)))$$

$\diamond$

So each cohomology class from $H^1(X \setminus S, \mathbb{C})$ contains a closed holomorphic form on $X \setminus S$ with polar singularities of first order on S.

Theorem 3.9 *Let Ω be a Stein and Kobayashi hyperbolic manifold of complex dimension 2 with an S^1-action as above (i.e generated by an automorphism f satisfying ($\star$), $f(0) = 0$ and with exponents q_1, q_2 with opposite* *signs). Assume the restriction morphism $\rho : H^1(\Omega) \to H^1(\Omega \setminus S)$ is not surjective, where S is the global analytic set of Proposition 3.7. Then the S^1-action has exactly one fixed point in Ω, namely 0.*

Proof By hypothesis, there exists a regular 1- form ω on $\Omega \setminus S$ which is not cohomologous to the restriction of a regular form on Ω. From Raby's Theorem, we can find $\tilde{\omega}$ holomorphic form on $\Omega \setminus S$, with polar singularities of first order on S, s.t $\tilde{\omega} \sim \omega$. Then the Leray expression (1) gives that locally

$$\tilde{\omega} = \frac{ds}{s} \wedge \Phi + \psi\,,$$

where s is chosen to be irreducible.

Also, $\Phi|_S$ is a holomomorphic function on S which, as a differential form is closed, so Φ is holomorphic near S and it is a constant C when restricted to S. Let a point Q = $(z_0, w_0), w_0 \neq 0$ close to 0. We can also assume that in the neighbourhood U, $s(z, w) = w$. It is easily seen that the orbit of Q, denoted now by $\gamma(Q)$, is homologous in $U \setminus S$ to a circle of the form $(z_0, \lambda \cdot w_0), \lambda \in S^1$.

Then by Leray's Residue Theorem ([2]),

$$\frac{1}{2\pi i}\int_{\gamma(Q)} \tilde{\omega} = \frac{1}{2\pi i}\int_{\{(z_0,\lambda w_0),\lambda\in S^1\}} \tilde{\omega} = \Phi(z_0, 0) = C\,,$$

since Φ was constant on S.

Now, assume we would have another fixed point for the action, denoted by P. From the description of S given in the proof of Proposition 3.7, $P \notin S$. Since S is an analytic set in Ω, $\Omega \setminus S$ is connected; join Q to P by a simple path $\eta|[0, 1]$ contained in $\Omega \setminus S$.

"Rotating" η will give us that $\gamma(Q)$ and P are homologous in $\Omega \setminus S$.

Let us make this more precise. Indeed if we take the orbits of all points of η by the S^1-action (we identify the parametrization of η with its image), we will get a homology between the orbit of $\eta(0)$ and that of $\eta(1)$

But the orbit of Q is $\gamma(Q)$ and the orbit of P is P itself since this point was assumed to be fixed for the S^1-action. Also, because $f(\Omega \setminus S) = \Omega \setminus S$ and f is the biholomorphism of Ω generating our S^1- action, none of the orbits of points from $\eta([0, 1])$ will intersect S.

This proves that $\gamma(Q)$ and P are homologous as paths in $\Omega \setminus S$.

Nevertheless, $\tilde{\omega}$ is a closed form on $\Omega \setminus S$ and $\gamma(Q)$ is homologous to P in $\Omega \setminus S$.

Thus, $\frac{1}{2\pi i}\int_{\gamma(Q)} \tilde{\omega} = \frac{1}{2\pi i}\int_{\gamma(P)} \tilde{\omega} = 0 \Longrightarrow C = 0$.

Therefore Φ is holomorphic on a neighbourhood of S and vanishes on S.

So, if s is a local irreducible defining function for S, s will divide Φ.

Hence $\tilde{\omega} = \frac{ds}{s} \cdot \Phi + \Psi$ has no singularities on S, which means that it can be extended across S. So $\tilde{\omega}$ is the restriction to $\Omega \setminus S$ of a differential form on Ω, hence contradiction. In conclusion 0 is the only fixed point of the action. $\diamond$

Corollary 3.10 *Any holomorphic S^1-action with a discrete set of fixed points, on a 2-dimensional Stein manifold Ω which is Kobayashi hyperbolic and has $H^2(\Omega, \mathbb{Z}) = 0$, has in fact at most one fixed point.*

Proof We will use the fact that, if Ω is Stein and $H^2(\Omega, \mathbb{Z}) = 0$, then the second Cousin problem can always be solved. By taking the local minimal defining functions for the “global z-axis” S, and constants equal to 1 outside S, we obtain a multiplicative Cousin data. Solving it will provide us with a global defining function, g_0 for S.

Now we can take the 1-form $\omega = \frac{dg_0}{g_0}$, which is closed on $\Omega \setminus S$ and has polar singularities of the first order on S.

Hence the hypothesis of Theorem 3.10 is fulfilled and there is at most one fixed point of the action in Ω. $\diamond$

There seems to exist also a more general relationship between the second cohomology group $H^2(\Omega, \mathbb{Z})$ and the maximum number of fixed points in Ω. This question will be investigated in a future article.

4. Some examples

Let Ω a 2-dimensional Stein and Kobayashi hyperbolic manifold as before.We also have an S^1-action on Ω by holomorphic automorphisms which have discrete sets of fixed points. A natural question would be to generalize the existence of only at most one fixed point to the case when the eigenvalues of the derivative A are of the form $\lambda^{q_1}, \lambda^{q_2}$, with q_1, q_2 of opposite signs. The following example shows this is not possible unless suplementary conditions are imposed.

Example: Let X = connected component of the set $\{(z, w, t) \in \mathbb{C}^3, z^m - 1 + wt = 0, |z| < 3, |w| < 3, |t| < 3\}$ containing the point (1,0,0), where $m \geq 2$ is a positive integer.

The S^1-action on X is given by $\lambda \cdot (z, w, t) \to (z, \lambda w, \frac{1}{\lambda} t), \lambda \in S^1$.

First we notice that X is a manifold of dimension 2, i.e it has no singular points as a subvariety in $\mathbb{C}^3$. Indeed $d(z^m - 1 + wt)(z, w, t) = (mz^{m-1}, t, w) = (0, 0, 0)$ iff $(z, w, t) = (0, 0, 0)$. But $(0, 0, 0) \notin$ X; also it is easy to see that the action is well -defined in the sense that it preserves the analytic set, the respective connected component, and $|\lambda w| = |w|, |\frac{1}{\lambda} t| = |t|, \forall \lambda \in S^1$. Next X is Stein and, since it is bounded, it is also Kobayashi hyperbolic.

We have that $(\epsilon_m, 0, 0) \in$ X,where ϵ_m is an m-root of unity ; indeed we can join $(\epsilon_m, 0, 0)$ to $(\epsilon_m, 0, 1)$ in X by a path $u \to (\epsilon_m, 0, u), u \in [0, 1]$,and then $(\epsilon_m, 0, 1)$ to (0,1,1) by a path $u \to ((1-u)\epsilon_m, 1-(1-u)^m, 1), u \in [0, 1]$, which is contained in X as well.

We can do the same thing for a path going from (1,0,0) to (1,0,1) and then on to $(0, 1, 1)$. In fact the m points of the form $(\epsilon_m, 0, 0)$ are the only fixed points of this action on X. Indeed, if $\lambda \cdot (z, w, t) = (z, w, t), \forall \lambda \in S^1 \Longrightarrow w = t = 0 \Longrightarrow z$ is an m-root of unity, if $(z, w, t) \in$ X.

In conclusion we obtained a Stein, Kobayashi hyperbolic manifold of dimension 2 with an S^1-action having exactly m fixed points. ◇

By a suitable modification of the above we can give an example showing that Corollary 3.11 does not cover all cases of Theorem 3.10.

Let X = connected component of $\{(z, w, t) \in \mathbb{C}^3, z + z^2 - 1 + wt = 0, |z| < 1, |w| < 1, |t| < 1\}$ containing the point $(\frac{-1+\sqrt{5}}{2}, 0, 0)$, with the action $\lambda \cdot (z, w, t) \to (z, \lambda w, \frac{1}{\lambda} t)$.

As before, X is Stein, Kobayashi hyperbolic manifold and it can be checked that the action has only one fixed point.

We will show that the "global z-axis S" is defined as the set of zeros of a global holomorphic function, but $H^2(X, \mathbb{Z}) \neq 0$. So there are cases when $\rho : H^1(X) \to H^1(X \setminus S)$ is not surjective and $H^2(X, \mathbb{Z}) \neq 0$.

The set S is given as $\{(z, w, t) \in X, t = 0\}$, so $\frac{dt}{t} \in H^1(X \setminus S)$, and $\frac{dt}{t} \notin \rho(H^1(X))$. We will prove first that the homology group $H_2(X \setminus \{P\}, \mathbb{Z})$ has a nonzero free part, where $P = (\frac{-1+\sqrt{5}}{2}, 0, 0)$.

Then, since the real dimension of X is 4, if $H_2(X \setminus \{P\}, \mathbb{Z})$ has nonzero free part, the same is true for $H_2(X, \mathbb{Z})$.

Now, if $(z, w, t) \in X \setminus \{P\}$, then $w \neq 0, t \neq 0$, so can write $t = \frac{1-z-z^2}{w}$. This means that $X \setminus \{P\} \cong D :=$ respective connected component of $\{(z, w) \in \mathbb{C}^2, |z| < 1, |w| < 1, z \neq \frac{-1+\sqrt{5}}{2}, |\frac{1-z-z^2}{w}| < 1\}$.

Denote by D_1 the first projection of D as a subset of $\Delta \times \Delta$. Now $\frac{-1+\sqrt{5}}{2} \notin D_1$, but $\exists V = V(\frac{-1+\sqrt{5}}{2}) \subset \Delta$ s.t $\forall z \in V \setminus \{\frac{-1+\sqrt{5}}{2}\}, |1 - z - z^2| < |w|$ if $|w| > \varepsilon_0$, for some $\varepsilon_0 > 0$. So D_1 is open, connected, and contains $V \setminus \{\frac{-1+\sqrt{5}}{2}\}$.

From the definition of the set D, the second projection of D on the unit disk, D_2, is contained in $\Delta^\star$. Also, since $|\lambda \cdot w| = |w|, \forall \lambda \in S^1$,we get that D_2 is invariant to rotations. So D_2 is open, connected, invariant to rotations and contained in $\Delta^\star \Longrightarrow D_2$ is an annulus or a punctured disk.

Take next the product of a small closed loop around $\frac{-1+\sqrt{5}}{2}$ in D_1 with a larger circle around 0 in D_2; we can choose the loops so that to obtain a 2-simplex in D. By using Kunneth formula (Bredon [5]) it is easy to see that this 2-simplex will span a non-zero free subgroup in $H_2(D, \mathbb{Z})$.

Hence $H_2(X, \mathbb{Z})$ will have a non-zero free part. From the Universal Coefficient Theorem ([5]), there exists an epimorphism: $H^2(X, \mathbb{Z}) \to Hom(H_2(X, \mathbb{Z}), \mathbb{Z}) \to 0$.

Since $H_2(X, \mathbb{Z})$ has a non-zero free subgroup, $Hom(H_2(X, \mathbb{Z}), \mathbb{Z}) \neq 0 \Longrightarrow H^2(X, \mathbb{Z}) \neq 0$.

References

1. M. Abate: Iteration theory of holomorphic maps on taut manifolds, Mediteranean Press, 1989
2. L.A. Aizenberg, A.K. Tsikh, A.P. Yuzhakov: Multidimensional residues and applications, in Encyclopedia of Mathematical Sciences, vol 8, Springer-Verlag 1994
3. D. Barrett, E. Bedford, J. Dadok: $\mathbf{T}^n$-actions on holomorphically separable complex manifolds, Math. Zeitschrift **202**, 65–82 (1989)
4. E. Bedford, J. Smillie: Polynomial diffeomorphisms of $\mathbb{C}^2$, Journal of the AMS, vol. 4, 4, October 1991
5. G.E. Bredon: Topology and geometry, Springer-Verlag 1995
6. L. Carleson, T. Gamelin: Complex dynamics, Springer-Verlag 1993
7. H.Cartan: Les fonctions de deux variables complexes et le probleme de la representation analytique, J. Math. Pures Appl. (9) **1931**, 1–114
8. J.E. Fornæss, N. Sibony: Classification of recurrent domains for some holomorphic maps, Math. Annalen, **301**, no 4, 813–820, 1995
9. J.E. Fornæss, N. Sibony: Complex dynamics in higher dimensions, Asterisque 1994, no 222 5, 201–231
10. J.E. Fornæss, N. Sibony: Complex dynamics in higher dimensions, part 2, Modern methods in complex analysis, Princeton, NJ 1992, 135–182
11. J.E. Fornæss, N. Sibony: Complex dynamics in higher dimensions, in Complex Potential Theory (P.M Gauthier and G.Sabidussi eds.), Kluwer Academic Publishers, Dordrecht 1994
12. R. Gunning: Introduction to holomorphic functions of several variables, vol 3, Homological Theory, Wadsworth and Brooks / Cole Math. Series 1990
13. M. Herman: Recent results and some open questions on Siegel's linearization theorem of germs of complex analytic diffeomorphisms of $\mathbb{C}^n$ near a fixed point, eighth International Congress on Math.Physics (Marseille 1986), 138–184, World Sci. Publishing, Singapore 1987
14. S. Kobayashi: Hyperbolic manifolds and holomorphic mappings, Marcel–Dekker Inc., New - York 1970
15. J. Milnor: Dynamics in one complex variable, preprint Stony Brook
16. A.L. Onishchik (Ed.): Lie groups and Lie algebras, Encyclopaedia of Mathematical Sciences vol. 20, Springer-Verlag 1993
17. G. Raby: Parametrix, cohomologie et formes meromorphes, in Seminaire d'Analyse P.Lelong, P.Dolbeault, H.Skoda 1985/1986 Lecture Notes 1295 Springer-Verlag 1987
18. T. Ueda: Critical orbits of holomorphic maps on projective spaces, to appear in the Journal of Geometric Analysis
19. T. Ueda: Fatou sets in complex dynamics on projective spaces, J. Math. Soc. Japan, **46**, 545–555, 1994
20. T. Ueda: Complex dynamical systems on projective spaces, in Proceedings of the RIMS Conference, World Sci. Publishing, Singapore, 1992