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Upper Estimates for Stable Dimensions on Fractal Sets with Variable Numbers of Foldings

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For a hyperbolic map f on a saddle-type fractal Λ with self-intersections, the number of f-preimages of a point x in Λ may depend on x. This makes estimates of the stable dimensions more difficult than for diffeomorphisms or for maps which are constant-to-one. We employ the thermodynamic formalism in order to derive estimates for the stable Hausdorff dimension function δ^s on Λ, in the case when f is conformal on local stable manifolds. These estimates are in terms of a continuous function ω on Λ, which bounds the preimage counting function from below. As a corollary, we obtain that, if δ^s attains its maximal possible value in Λ, then the stable dimension is constant throughout Λ, and the preimage counting function is constant on at least an open dense subset of Λ. In particular, this shows that, if at some point in Λ the stable dimension is equal to the analogue of the similarity dimension in the stable direction at that point, then f behaves very much like a homeomorphism on Λ. Finally, we also obtain results about the stable upper box dimension for this class of fractals. We end the paper with a discussion of two explicit examples.

1 Introduction and Statement of Results

In this paper, we investigate fractal basic sets Λ of saddle type that are invariant under a noninvertible $\mathcal{C}^2$-endomorphism f of a Riemann manifold M into itself; these fractals

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are *basic sets of* f in the sense that Λ is compact, f-invariant, $f|_\Lambda$ is topologically transitive and there exists a neighbourhood U of Λ satisfying $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$. The fact that f is noninvertible produces complicated overlaps and foldings within Λ, which influence the Hausdorff dimension of the sections through Λ; also the number of overlaps does not necessarily have to be constant.

We will always assume that f is hyperbolic on Λ in the sense of Ruelle [23], that is, for each backward orbit $\hat{x} = (x, x_{-1}, x_{-2}, \ldots)$ of x in Λ, where $f(x_{-1}) = x$ and $f(x_{-(i+1)}) = x_{-i} \in \Lambda$ for all $i \in \mathbb{N}$, there exists a continuous splitting of the tangent bundle over the space $\hat{\Lambda}$ of all backward orbits of elements of Λ, called the *natural extension* (or inverse limit) of the tuple (Λ, f), into stable spaces E^s_x and unstable spaces $E^u_{\hat{x}}$. It is well known that $\hat{\Lambda}$ is a compact metric space and that the lift $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ of f to $\hat{\Lambda}$, given by $\hat{f}(\hat{x}) := (f(x), x, x_{-1}, x_{-2}, \ldots)$, is a homeomorphism. Note that natural extensions play an important role in the study of the dynamics of endomorphisms (see for instance [11, 23]). As in the diffeomorphism case, for a hyperbolic endomorphism f on Λ, there exist local stable manifolds $W^s_r(x)$ and local unstable manifolds $W^u_r(\hat{x})$, for each $x \in \Lambda$ and $\hat{x} \in \hat{\Lambda}$. Note that there may be infinitely many local unstable manifolds through a given point in Λ and, unlike in the diffeomorphism case, these do not necessarily give rise to a foliation.

We will consider in the sequel, the *stable dimension at the point* $x \in \Lambda$, which is defined by

$$\delta^s(x) := \dim_H(W^s_r(x) \cap \Lambda),$$

where $\dim_H$ refers to the Hausdorff dimension. To give estimates for the stable dimension is much more delicate than for the unstable dimension $\delta^u(\hat{x}) := \dim_H(W^u_r(\hat{x}) \cap \Lambda)$. In fact, in [11] it was shown that $\delta^u(\hat{x})$ is constant on $\hat{\Lambda}$ and that its value is given by the unique zero of the pressure function $P_{\hat{f}|_{\hat{\Lambda}}}(-t \log |Df_u|)$, where $|Df_u(x)|$ denotes the norm of the derivative of f restricted to $E^u_{\hat{x}}$. However, for the stable dimension we cannot expect that a similar formula holds in general (see [12, 13, 25], etc.)

Before we state our main result, let us point out that in this paper we consider a special type of hyperbolic endomorphisms which will be called *c-hyperbolic*. A map f is *c-hyperbolic* on Λ if it is hyperbolic as an endomorphism over Λ, if it is conformal on all local stable manifolds and if the set Λ does not contain any critical point of f.

Also, let us introduce the *preimage counting function* $\Delta : \Lambda \to \mathbb{N}$, which is given for each $x \in \Lambda$ by

$$\Delta(x) := \mathrm{Card}(f^{-1}(x) \cap \Lambda).$$

One immediately verifies that Δ is upper semi-continuous and bounded on Λ (see e.g. [15, Lemma 1]). Moreover, the *stable potential function* Φ^{s} on Λ is defined by $\Phi^{\mathrm{s}}(x) := \log|Df_s(x)|$, where $|Df_s(x)|$ denotes the norm of the derivative of f restricted to E^{s}_x. We are now in the position to state the main result of this paper.

Theorem 1. Let $f: M \to M$ be a $\mathcal{C}^2$-endomorphism which is c-hyperbolic on a basic set Λ of f and for which there exists a continuous function $\omega: \Lambda \to (0, \infty)$ such that $\Delta(x) \geq \omega(x)$, for all $x \in \Lambda$. Then, the following upper estimate is true for any point $x \in \Lambda$:

$$\delta^{\mathrm{s}}(x) \leq t_\omega,$$

where t_ω is the unique zero of the pressure function $t \mapsto P(t\Phi^{\mathrm{s}} - \log\omega)$, associated to the potential function $t\Phi^{\mathrm{s}} - \log\omega$ on Λ. □

Let us point out that one of the difficulties in proving this theorem is that the map f is not necessarily expanding and that its inverse branches do not necessarily contract small balls. In fact, some directions are even expanding in backward time. Another difficulty is that the number of preimages of a point that remain in Λ is not always constant over Λ.

The reader might like to recall that in their pioneering work Bowen [4] and Ruelle [22] employed the thermodynamic formalism in order to derive dimension formulae for rational maps. In fact, in the diffeomorphism case, it turned out that the stable and the unstable dimension can in general be computed both as the zero of the pressure function of the stable potential, respectively, the unstable potential (see [9]); for further applications of the thermodynamic formalism in dimension theory, we refer to [1, 18]. It is important to note that for an endomorphism f in higher dimension, a hyperbolic basic set is not necessarily totally invariant. This is of course significantly different from the case of Julia sets of rational maps in the complex 1D case.

Examples of perturbations of toral endomorphisms that are Anosov and the unstable manifolds of which depend on the whole prehistory were given in [19]. Another class of noninvertible hyperbolic maps with crossed invariant horseshoes was given by Bothe [2]. Also, Simon [26] gave another class of noninvertible endomorphisms, for which the Hausdorff dimension of the associated attractors can be computed with the help of a pressure formula just as in the invertible case.

Examples of nonlinear hyperbolic skew products having Cantor sets of overlaps in their fibers, were given in [13], where the strongly noninvertible character of these maps has been established, and where it was shown that these skew products are far from being constant-to-one. In [13], it was shown that for these dynamical systems, there exist Cantor sets in each of their fibers such that, through each point of these Cantor sets, there pass uncountably many different local unstable manifolds. Also we mention that, for this family of strongly noninvertible maps, one has information about the function $\Delta(\cdot)$; it was shown in [13] that on some subsets in the respective associated basic set Λ, we have $\Delta = 1$, while on other subsets we have $\Delta = 2$. Hence, we can use this in order to construct continuous functions ω as in Theorem 1, such that $\omega \not\equiv 1$ on Λ and so that $\Delta(x) \geq \omega(x),\ x \in \Lambda$.

Another class of c-hyperbolic endomorphisms can be found by considering hyperbolic basic sets of saddle type for holomorphic maps $f : \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$ on the 2-dimensional complex projective space [11]. Let us also mention that in [12], the stable dimension on basic sets was related to a notion of inverse pressure.

The paper continues by showing that an application of Theorem 1 gives rise to the following proposition. Here, we consider the situation in which δ^s attains a maximal value and show that in this case, δ^s has to be constant throughout Λ and that Δ has to be equal to its least value d on an open dense subset.

Proposition 1. If in addition to the assumptions in Theorem 1 we have that the minimal value of Δ on Λ is equal to d, and that there exists a point $x \in \Lambda$ at which δ^{s} is equal to the unique zero t_d of the pressure function $t \mapsto P(t\Phi^{\mathrm{s}} - \log d)$, then Δ is equal to d on an open dense subset of Λ, and $\delta^{\mathrm{s}}(y)$ is equal to t_d, for all $y \in \Lambda$. □

Note that the latter proposition can be applied in particular in the case when d is equal to 1 and there is no overlap. In this situation, the stable dimension is equal to the similarity dimension, and the proposition guarantees that there exists an open dense set of points in Λ, each of these points having precisely one f-preimage in Λ. Therefore, in this case the map behaves almost like a homeomorphism, when restricted to Λ. This particular situation is somewhat parallel to a result of Schief [24], although the setting and proofs are completely different. We summarize these results in the following corollary:

Corollary 1. Let $f : M \to M$ be a $\mathcal{C}^2$-endomorphism which is c-hyperbolic on a basic set Λ of f and for which there exists a point $x \in \Lambda$ such that $\delta^{\mathrm{s}}(x)$ is equal to the unique zero t_1 of the pressure function $t \mapsto P(t\Phi^{\mathrm{s}})$. Then there exists an open dense set of points in Λ, each of them having precisely one f-preimage in Λ. Moreover, we have that $\delta^{\mathrm{s}}(y) = t_1$, for all points $y \in \Lambda$. □

Also in Corollary 3 from Section 4, we will show how the above Corollary 1, can be applied to a class of translations of horseshoes with overlaps previously studied by Simon and Solomyak in [27].

Let us now remark that a combination of Theorem 1 with the main theorem in [15] gives rise to the following result.

Corollary 2. If in addition to the assumptions in Theorem 1 we have that the preimage counting function Δ is locally constant on Λ, then it follows that $\delta^s(x) = t_\omega$, for all $x \in \Lambda$. Here, t_ω is given as in Theorem 1. □

Finally, we consider the *stable upper box dimension* $\beta^s(x)$, which is given as the upper box dimension $\overline{\dim}_B(W^s_r(x) \cap \Lambda)$ of the set $W^s_r(x) \cap \Lambda$, for each $x \in \Lambda$. For a general discussion of upper box dimension for fractal sets, we refer to [10, 18].

We show that the stable upper box dimension function $\beta^s(\cdot)$ is constant throughout Λ, and that in the situation when Δ is bounded from below, then similarly as in Theorem 1, one derives an upper bound for its value. These results are summarized in the following proposition.

Proposition 2. Let $f: M \to M$ be a $\mathcal{C}^2$-endomorphism which is c-hyperbolic on a saddle basic set Λ of f. Then the following hold:

(a) If there exists a continuous function $\omega: \Lambda \to (0, \infty)$ such that $\Delta(x) \geq \omega(x)$, for all $x \in \Lambda$ and if t_ω is defined as in Theorem 1, then we have

$$\beta^s(y) \leq t_\omega \quad \text{for all } y \in \Lambda.$$

(b) The function β^s is constant on Λ. □

In particular, the above results apply for hyperbolic basic sets of saddle type for holomorphic maps $f: \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$. We will end the paper by giving two further explicit examples in which the above results can be applied. Our first example will be concerned with certain horseshoes with overlaps in $\mathbb{R}^3$ considered in [27]. The second example will be on basic sets for a family of hyperbolic skew products studied in [13].

We close Section 1 with some comments on how the results in our paper relate to previous work in this area.

In [6] (see also [16, 28]), Falconer studied self-affine fractals with overlaps obtained from finitely many linear contractions $T_i(x) = \lambda_i x,\ i = 1, \ldots, \ell$ in $\mathbb{R}$ satisfying $0 < |\lambda_i| < 1$ and $\sum_{1 \leq i \leq \ell} |\lambda_i| < 1$. He showed that the Hausdorff dimension of the invariant set of the family of translated contractions $\{T_i + a_i, : 1 \le i \le \ell\}$ is equal to s, for Lebesgue almost all $(a_1, \ldots, a_\ell) \in \mathbb{R} \times \cdots \times \mathbb{R}$; where s represents the *similarity dimension*, defined as the solution of the equation

$$\sum_{1 \le i \le \ell} |\lambda_i|^s = 1.$$

We remark that this result may be extended also to similarities on $\mathbb{R}^n$. However, the result fails if the condition $\sum_{1 \le i \le \ell} |\lambda_i| < 1$ is not satisfied, as observed by Edgar [5], who based his argument on a result by Przytycki and Urbański [20]. Indeed, if $T_1 = T_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \lambda \end{pmatrix}$ and if $|\lambda| > \frac{1}{2}$, then for Lebesgue almost every $a = (a_1, a_2) \in \mathbb{R}^2$ the attractor $\Lambda(a)$ of the system $\{T_1 + a_1, T_2 + a_2\}$ remains the same; and moreover if $1/\lambda$ is a Pisot number (i.e., an algebraic integer such that the absolute value of all its algebraic conjugates is < 1), then $\dim_H(\Lambda(a)) < 2 - (\log(1/\lambda))/\log 2$ (see e.g., [28]). This shows that fractals originating from overlapping constructions can have Hausdorff dimension *less* than their similarity dimension.

In [24], Schief considered self-similar fractal sets K and showed that if for the similarity dimension σ of K one has that the σ-dimensional Hausdorff measure $\mathcal{H}^\sigma(K)$ is positive, then K satisfies the strong open set condition, that is, the system behaves similarly to a homeomorphism on K. Note that this result is in the spirit of our results in this paper, although the setting and the ideas of our proofs differ significantly from the approach in [24]. More precisely, the assumptions in Proposition 1 are much weaker than the ones in [24]. Namely, in order to obtain the "almost injectivity" of the system associated with Λ, we only require that the stable dimension $\delta^s(x)$ is equal to the zero t_1 of the pressure function $t \to P(t\Phi^s)$, for some $x \in \Lambda$; we do not require that $\mathcal{H}^{t_1}(W_r^s(x) \cap \Lambda) > 0$. In our case here, t_1 is the analog of the similarity dimension in the stable direction, in the sense that it represents the dimension which one would obtain if the system were invertible. In particular, if there exists some $x \in \Lambda$ for which $\mathcal{H}^{t_1}(W_r^s(x) \cap \Lambda) > 0$ is positive, then we have that the stable dimension is everywhere equal to t_1 and that there exists an open dense set of points in Λ which precisely have one preimage in Λ.

Also, in [25] Schmeling and Troubetzkoy introduced a class of endomorphisms which are piecewise smooth and have hyperbolic attractors, and showed that the Young dimension formula holds if and only if the endomorphism is invertible SRB-almost everywhere. Moreover, in the papers [7, 14, 17], Hausdorff dimension on noninvertible hyperbolic attractors was studied, in the case of various types of endomorphisms satisfying transversality conditions.

Mihailescu and Urbański [15] studied c-hyperbolic maps on Λ, for which Δ is bounded from *above* by a continuous map η on Λ. In that paper the authors obtained a lower estimate for stable dimension, namely $\delta^s(x) \geq t_\eta$ for all $x \in \Lambda$, where t_η represents the unique zero of the pressure function $t \to P(t\Phi^s - \log \eta)$. Note that the proof for the upper estimate in this paper, is very different from the proof for the lower estimate in [15]. However, we can combine these two estimates, as done in Corollary 2, to obtain that if the preimage counting function Δ is locally constant on Λ, then the stable dimension is equal to t_Δ throughout Λ.

2 Proof of Theorem 1

For ease of exposition, let us first consider the situation in which ω is locally constant and takes on only two different positive integer values on Λ, namely d_1 on the set V_1 and d_2 on the set V_2. We then have that $V_1 \cup V_2 = \Lambda$ and that V_1 and V_2 are two disjoint compact subsets of Λ. Hence, there exists $\varepsilon_0 > 0$ such that the distance $d(V_1, V_2)$ between V_1 and V_2 is greater than ε_0. For $x \in \Lambda$ and $n \in \mathbb{N}$, let $B_n(x, \varepsilon) := \{y \in \Lambda : d(f^i(y), f^i(x)) < \varepsilon,\ 0 \leq i \leq n-1\}$ refer to the n-Bowen ball centered at x of radius $\varepsilon > 0$. Note that for $0 < \varepsilon < \varepsilon_0$ we have that if $y \in B_n(x, \varepsilon)$ then $f^i(y)$ and $f^i(x)$ both belong to either V_1 or V_2, for each $0 \leq i \leq n-1$. Recall that $\Phi^s(x) := \log |Df_s(x)|$, $x \in \Lambda$. Now, let $t > t_\omega$ be fixed. By definition of t_ω, we have that there exists $\beta > 0$ such that

$$P(t\Phi^s - \log \omega) < -\beta.$$

Hence, by choosing $\varepsilon > 0$ sufficiently small, there exists a constant $C > 0$ such that for each $n \in \mathbb{N}$ large enough, there exists a minimal (n, ε)-spanning set E_n for Λ such that

$$\sum_{z \in E_n} (\operatorname{diam} U_n(z))^t \cdot \frac{1}{\Delta(f(z)) \cdot \ldots \cdot \Delta(f^n(z))} < C\, \mathrm{e}^{-\beta n} < 1, \tag{1}$$

where we have set $U_n(z) := f^n(B_n(z, \varepsilon)) \cap W^s_r(x) \cap \Lambda$. Note that in here we have used the fact that the set $U_n(z)$ is the intersection of an unstable tubular neighbourhood with the fixed stable manifold $W^s_r(x)$. Also, we used that $|Df^n_s(z)|$ is uniformly comparable to $\operatorname{diam} U_n(z)$, which follows from the fact that f is conformal on local stable manifolds.

In the sequel, let us denote $W := W^s_r(x) \cap \Lambda$. Our aim is to show that $\dim_{\mathrm{H}}(W) \leq t$, for each $t > t_\omega$. The main idea of the proof is to extract suitable covers of W out of the large set of covers which are given by taking n-preimages, such that at each step a different sum will be minimized. Note that we say that a point y is a *k-preimage of x* if $f^k(y) = x$. Each such n-preimage will be included in a Bowen ball of type $B_n(z, \varepsilon)$, for some $z \in E_n$. This procedure is delicate, since at each step the number of preimages of points belonging to Λ varies. The idea is to consider the k iterates of n-preimages, then to subdivide Λ into various different subsets and finally, to find suitable covers of these subsets, which minimize certain sums determined by the kth level.

First, note that since Λ is covered by the set of Bowen balls $\{B_n(z,\varepsilon): z\in E_n\}$, it follows that $\{U_n(z): z\in E_n\}$ covers W. However, this cover is far too rich and we will have to extract a suitable subcover. Indeed, by using a well-known theorem by Besicovitch (see for e.g., [10]), there exists a subcover $\{5U_n(z): z\in \mathcal{G}(0)\}$ of W such that $\{U_n(z): z\in \mathcal{G}(0)\}$ consists of pairwise disjoint sets. Note that, since f is conformal on local stable manifolds, we can assume that the sets $U_n(z)$ are in fact balls, and we shall denote the radii of these balls by $r(n,z)$, respectively; also, we write $5U_n(z)$ to denote the ball of radius $5r(n,z)$ centered at the center of $U_n(z)$.

The next step is to "inflate" this cover, that is, to enlarge it to a "richer" cover of W. For this, we consider an $(n-1)$-preimage of w in Λ, which we denote by $w(n-1)$, for each point $w\in W$. Let us assume that $w(n-1)\in V_1$ and hence, that $w(n-1)$ has at least d_1 1-preimages in Λ. Now, since E_n is (n,ε)-spanning, for each point $\xi\in\Lambda$, there exists at least one point $y\in E_n$ such that $\xi\in B_n(y,\varepsilon)$. However, we cannot have two 1-preimages of some $w(n-1)$ belonging to different Bowen balls $B_n(y,\varepsilon)$ and $B_n(y',\varepsilon)$ such that y and y' are both in $\mathcal{G}(0)$. This is an immediate consequence of the fact that the collection $\{U_n(z): z\in\mathcal{G}(0)\}$ consists of pairwise disjoint sets.

Therefore, by way of successive eliminations, we can find d_1 pairwise disjoint families, denoted by $\mathcal{F}(1,d_1;1),\ldots,\mathcal{F}(1,d_1;d_1)$, such that $\{5U_n(z): z\in\mathcal{F}(1,d_2;i)\}$ is a cover of the set $\{w\in W: w(n-1)\in V_1\}$, for each $1\le i\le d_1$. Obviously, for $w(n-1)\in V_2$, we can proceed in a similar way, which then gives rise to d_2 mutually disjoint families $\mathcal{F}(1,d_2;1),\ldots,\mathcal{F}(1,d_2;d_2)$ for which we have that $\{5U_n(z): z\in\mathcal{F}(1,d_2;j)\}$ is a cover of $\{w\in W: w(n-1)\in V_2\}$, for each $1\le j\le d_2$. Note that, since $d(V_1,V_2)>0$, we have that $\mathcal{F}(1,d_1;i)\cap\mathcal{F}(1,d_2;j)=\emptyset$, for all i and j, and that by construction we have that the so obtained disjoint families are all contained in E_n. Next, we define the collection:

$$\mathcal{F}(1) := \bigcup_{i=1}^{2}\bigcup_{1\le j\le d_i}\mathcal{F}(1,d_i,j)$$

and let $\mathcal{G}(1,d_k)$ be determined, for $k\in\{1,2\}$, by the minimizing condition

$$\sum_{z\in\mathcal{G}(1,d_k)}\frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^2(z))\cdots\Delta(f^n(z))} = \min\left\{\sum_{z\in\mathcal{F}(1,d_k;i)}\frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^2(z))\cdots\Delta(f^n(z))} : i\in\{1,\ldots,d_k\}\right\}.$$

For $\mathcal{G}(1) := \mathcal{G}(1, d_1) \cup \mathcal{G}(1, d_2)$, we then obtain, by adding the sums over $\mathcal{G}(1, d_1)$ and $\mathcal{G}(1, d_2)$,

$$\sum_{z \in \mathcal{G}(1)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^2(z)) \cdots \Delta(f^n(z))} \leq \sum_{z \in \mathcal{F}(1)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z)) \cdots \Delta(f^n(z))}. \tag{2}$$

Note that here we have used the trivial fact that for each $x \in \Lambda$ we have that $\sum_{y \in \Lambda,\, f(y)=x} 1/\Delta(x) = 1$. Also, note that the sum over the family $\mathcal{G}(1)$ on the left-hand side of the inequality in (2) is smaller than the sum over the larger family $\mathcal{F}(1)$ on the right-hand side. However, and this is the crucial point, the summands on the right-hand side have one more factor in their denominator than the summands on the left-hand side.

Let us now bring the argument to its next level by enlarging the family $\mathcal{F}(1)$ as follows. Recall that for each $w \in W$ we have fixed an $(n-1)$-preimage $w(n-1) \in \Lambda$. We now define $w(n-2) := f(w(n-1))$ and consider not only $w(n-1)$ but also the other 1-preimages of $w(n-2)$ in Λ. Subsequently, we will then take the 1-preimages of these 1-preimages of $w(n-2)$ and obtain new covers of W. Indeed similarly as before, if $w(n-2) \in V_1$ then we can construct, by successive eliminations, pairwise disjoint families $\mathcal{F}(2, d_1; 1), \ldots, \mathcal{F}(2, d_1; d_1)$ by selecting the 1-preimages of the ith preimage of $w(n-2)$, for each $1 \leq i \leq d_1$. In fact one of these families is $\mathcal{F}(1)$. As in the first step, the sets $\{5U_n(z) : z \in \mathcal{F}(2, d_1; i)\}$ cover $\{w \in W : w(n-2) \in V_1\}$, for each i. Let us remark that the procedure of successive elimination works, since if we take for instance the family $\mathcal{F}(2, d_1; 1)$, then for an arbitrary $w \in W$ we cannot have two 1-preimages y and y' of $w(n-2)$ and 1-preimages ξ of y and ξ' of y' such that ξ and ξ' are both contained in either $B_n(z, \varepsilon)$ or $B_n(z', \varepsilon)$, for some $z, z' \in \mathcal{F}(2, d_1; 1)$. Indeed, since $f^2(B_n(z, \varepsilon)) \cap f^2(B_n(z', \varepsilon)) \neq \emptyset$, in this situation it would follow that $U_n(z) \cap U_n(z') \neq \emptyset$ and hence we would have a contradiction. This implies that there exist d_1 disjoint families $\mathcal{F}(2, d_1; i)$ corresponding to the d_1 1-preimages of $w(n-2) \in V_1$.

Clearly, we can proceed analogously in the case in which $w(n-2) \in V_2$, which then gives rise to pairwise disjoint families $\mathcal{F}(2, d_2; 1), \ldots, \mathcal{F}(2, d_2; d_2)$ for which $\{5U_n(z) : z \in \mathcal{F}(2, d_2; j)\}$ covers $\{w \in W : w(n-2) \in V_2\}$, for each j. Note that we cannot have repetitions of points from E_n when taking the union of the collections $\mathcal{F}(2, d_i; j)$ over all $i \in \{1, 2\}$ and $1 \leq j \leq d_i$. Indeed, if we would have two 1-preimages $y, y' \in \Lambda$ of some $w(n-2)$ and two 1-preimages $\xi, \xi' \in \Lambda$ of y, and y', respectively, so that $\xi \in B_n(z, \varepsilon)$ and $\xi' \in B_n(z', \varepsilon)$, for some $z, z' \in \mathcal{F}(2, d_1; i)$, then it would follow that $U_n(z) \cap U_n(z') \neq \emptyset$, which gives a contradiction. Moreover, by construction we have that $\mathcal{F}(2, d_1; i) \cap \mathcal{F}(2, d_2; j) = \emptyset$, for all i and j. This follows, since if $f^2(z) \in V_1$, for some $z \in \mathcal{F}(2, d_1; i)$, and if at the same time $f^2(z') \in V_2$, for some $z' \in \mathcal{F}(2, d_2; j)$, then it would follow that $V_1 \cap V_2 \neq \emptyset$ and hence, we would get a contradiction.

Now, as in the first step, for each $i \in \{1, 2\}$ and $1 \le j \le d_i$ there exists a family $\mathcal{G}(2, d_i, j)$ in $\{\mathcal{F}(2, d_k; \ell) : k \in \{1, 2\}, 1 \le \ell \le d_j\}$ satisfying

$$\sum_{z \in \mathcal{G}(2, d_i, j)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^2(z)) \cdots \Delta(f^n(z))} \le \sum_{z \in \mathcal{F}(2, d_i; j)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z))\Delta(f^2(z)) \cdots \Delta(f^n(z))}. \tag{3}$$

Among these so obtained families $\mathcal{G}(2, d_i; j)$ we now choose for each $i \in \{1, 2\}$ a particular family, which will be denoted by $\mathcal{G}(2, d_i)$, such that we have

$$\sum_{z \in \mathcal{G}(2, d_i)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^3(z)) \cdots \Delta(f^n(z))} = \min \left\{ \sum_{z \in \mathcal{G}(2, d_i; j)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^3(z)) \cdots \Delta(f^n(z))} : j \in \{1, \ldots, d_i\} \right\}. \tag{4}$$

Combining (3) and (4), we now obtain, for each $i \in \{1, 2\}$, that

$$\begin{aligned} \sum_{z \in \mathcal{G}(2, d_i)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^3(z)) \cdots \Delta(f^n(z))} &\le \sum_{z \in \bigcup_{1 \le j \le d_i} \mathcal{G}(2, d_i; j)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^2(z)) \ldots \Delta(f^n(z))} \\ &\le \sum_{z \in \bigcup_{1 \le j \le d_i} \mathcal{F}(2, d_i; j)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z))\Delta(f^2(z)) \ldots \Delta(f^n(z))}. \end{aligned}$$

Therefore, by defining the collections

$$\mathcal{F}(2) := \bigcup_{i \in \{1,2\}} \bigcup_{1 \le j \le d_i} \mathcal{F}(2, d_i; j) \quad \text{and} \quad \mathcal{G}(2) := \mathcal{G}(2, d_1) \cup \mathcal{G}(2, d_2),$$

we have now shown that

$$\sum_{z \in \mathcal{G}(2)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^3(z)) \cdots \Delta(f^n(z))} \le \sum_{z \in \mathcal{F}(2)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z)) \cdots \Delta(f^n(z))}. \tag{5}$$

Continuing the above procedure, assume that we have constructed a family $\mathcal{F}(k) \subset E_n$ and a subfamily $\mathcal{G}(k)$ so that the sets $(U_n(z))_{z \in \mathcal{G}(k)}$ 5-cover W (meaning that $W \subset \bigcup_{z \in \mathcal{G}(k)} 5U_n(z)$), and such that

$$\sum_{z \in \mathcal{G}(k)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^{k+1}(z)) \cdots \Delta(f^n(z))} \le \sum_{z \in \mathcal{F}(k)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z)) \cdots \Delta(f^n(z))}.$$

For each $w \in W$, we then take the kth iterate of $w(n-1)$ and denote it by $w(n-k-1)$; this is an $(n-k-1)$-preimage of w in Λ. Now, if $w(n-k-1) \in V_1$ then it has d_1 1-preimages in Λ and to each of these we can apply the same procedure from step k.

In this way, we obtain by successive eliminations d_1 mutually disjoint families $\mathcal{F}(k+1, d_1; i)$, $1 \le i \le d_1$ and inside each of these a subfamily $\mathcal{G}(k+1, d_1; i)$ such that

$$\sum_{z \in \mathcal{G}(k+1, d_1; i)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^{k+1}(z)) \cdots \Delta(f^n(z))} \le \sum_{z \in \mathcal{F}(k+1, d_1; i)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z)) \cdots \Delta(f^n(z))}.$$

The successive elimination procedure works, since we cannot have two different 1-preimages y and y' of $w(n-k-1)$ having $(n-k)$-preimages $\xi \in \Lambda$ and $\xi' \in \Lambda$, respectively, such that $\xi \in B_n(z, \varepsilon)$, $\xi' \in B_n(z', \varepsilon)$, for some $z, z' \in \mathcal{F}(k+1, d_1; i)$. Indeed, it would then follow that the family $\{U_n(z) : z \in \mathcal{F}(k+1, d_1; i)\}$ does not consist of pairwise disjoint sets, which clearly is a contradiction. Moreover, since $V_1 \cap V_2 = \emptyset$, we must have $\mathcal{F}(k+1, d_1; i) \cap \mathcal{F}(k+1, d_2; j) = \emptyset$. Hence, there is no repetition of elements, when we consider the union

$$\mathcal{F}(k+1) := \bigcup_{1 \le j \le d_1} \mathcal{F}(k+1, d_1; j) \cup \bigcup_{1 \le j \le d_2} \mathcal{F}(k+1, d_2; j)$$

Now among the collections $\mathcal{G}(k+1, d_1; i)$, for $1 \le i \le d_1$, let us consider the collection which gives rise to the smallest sum of type

$$\sum_{z \in \mathcal{G}(k+1, d_1; i)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^{k+2}(z)) \cdots \Delta(f^n(z))}.$$

Denote this minimizing collection by $\mathcal{G}(k+1, d_1)$. Similarly, we obtain the collection $\mathcal{G}(k+1, d_2)$. We now have that

$$\sum_{z \in \mathcal{G}(k+1, d_1)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^{k+2}(z)) \cdots \Delta(f^n(z))} \le \sum_{z \in \bigcup_{1 \le i \le d_1} \mathcal{G}(k+1, d_1; i)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^{k+1}(z)) \cdots \Delta(f^n(z))}$$

$$\le \sum_{z \in \bigcup_{1 \le i \le d_1} \mathcal{F}(k+1, d_1; i)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z)) \cdots \Delta(f^n(z))}. \qquad (6)$$

Of course, we can proceed similarly for $\mathcal{G}(k+1, d_2)$. With $\mathcal{G}(k+1) := \mathcal{G}(k+1, d_1) \cup \mathcal{G}(k+1, d_2)$, it follows from above that

$$\sum_{z \in \mathcal{G}(k+1)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^{k+2}(z)) \cdots \Delta(f^n(z))} \le \sum_{z \in \bigcup_{1 \le i \le d_1} \mathcal{F}(k+1)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z)) \cdots \Delta(f^n(z))}.$$

Therefore, we obtain by finite induction a union $\mathcal{F}(n)$ of families in E_n, as well as one particular family $\mathcal{G}(n)$ such that $\{5U_n(z) : z \in \mathcal{G}(n)\}$ covers the set W and has the property that

$$\sum_{z\in\mathcal{G}(n)} (\operatorname{diam} U_n(z))^t \leq \sum_{z\in\mathcal{F}(n)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z))\cdots\Delta(f^n(z))}.$$

By combining this with the observation in (1) at the start of the proof, we obtain that

$$\sum_{z\in\mathcal{G}(n)} (\operatorname{diam} U_n(z))^t < 1.$$

Since $\{5U_n(z) : z \in \mathcal{G}(n)\}$ is a covering of the set $W = W_r^s(x) \cap \Lambda$, we can now conclude that

$$\delta^s(x) \leq t < t_\omega.$$

In the more general case in which ω is a continuous function on Λ with the property that $\omega(x) \leq \Delta(x)$, for all $x \in \Lambda$, we proceed as follows. First note that by the continuity of the function ω, we have that there exists an increasing positive function ρ on $(0, \infty)$, such that $\rho(\varepsilon)$ decreases to zero for ε tending to zero from above, and such that for any y, z with $d(y, z) \leq \varepsilon$ we have

$$|\omega(y) - \omega(z)| \leq \rho(\varepsilon).$$

Since if $y \in B_n(z, \varepsilon)$ then $f^i(y) \in B(f^i z, \varepsilon)$, the latter implies that if $y \in B_n(z, \varepsilon)$ then $|\omega(f^i(y)) - \omega(f^i(z))| \leq \rho(\varepsilon)$. Hence, since $\Delta(x) \geq \omega(x)$ for all $x \in \Lambda$, it follows that for each $0 \leq i \leq n-1$ we have

$$\Delta(f^i(y)) \geq \omega(f^i(y)) \geq \omega(f^i(z)) - \rho(\varepsilon).$$

Now in order to proceed, let us define the ε-pressure function P_ε, for some arbitrary potential function ψ, by the formula

$$P_\varepsilon(\psi) := \liminf_{n\to\infty} \frac{1}{n} \log \inf \left\{ \sum_{x\in E} \exp\left(\sum_{k=0}^{n-1} \psi(f^k(x)) \right) : E \text{ is a } (n, \varepsilon)\text{-spanning set for } \Lambda \right\}$$

and let t_ε denote the unique zero of $P_\varepsilon(t\Phi^s - \log(\omega - \rho(\varepsilon)))$. Then let $t > t_\varepsilon$ be fixed and note that the above proof goes through in the same way if in the sums appearing there, we replace the function Δ by the function $\omega - \rho(\varepsilon)$. Indeed, this follows since, for all $0 \leq i \leq n-1$, we have that $\Delta(f^i y) \geq \omega(f^i(y)) \geq \omega(f^i(z)) - \rho(\varepsilon)$, for each $y \in B_n(z, \varepsilon)$ and for some arbitrary fixed element z contained in some minimal (n, ε)-spanning set E_n for Λ.

In this way, the above inductive procedure gives rise to a family $\mathcal{F}(n) \subset E_n$ and also to a particular family $\mathcal{G}(n)$, such that $\{5U_n(z) : z \in \mathcal{G}(n)\}$ covers the set W and such that

$$\sum_{z\in\mathcal{G}(n)} (\operatorname{diam} U_n(z))^t \le \sum_{z\in\mathcal{F}(n)} \frac{(\operatorname{diam} U_n(z))^t}{(\omega(f(z)) - \rho(\varepsilon))\cdots(\omega(f^n(z)) - \rho(\varepsilon))} < 1.$$

Now, for $\eta > 0$ sufficiently small and $0 < \varepsilon < \eta$, let $\tau_{\varepsilon,\eta}$ refer to the unique zero of the pressure function $P_\varepsilon(t\Phi^{\mathrm{s}} - \log(\omega - \rho(\eta)))$ and let τ_η denote the unique zero of the pressure function $P(t\Phi^{\mathrm{s}} - \log(\omega - \rho(\eta)))$. Since $\lim_{\varepsilon\to 0} P_\varepsilon(\psi) = P(\psi)$ for each continuous function ψ, it follows that $\lim_{\varepsilon\to 0} \tau_{\varepsilon,\eta} = \tau_\eta$.

On the other hand, let us note that for $0 < \varepsilon < \eta$, we have that $\rho(\varepsilon) < \rho(\eta)$ and therefore

$$t\Phi^{\mathrm{s}} - \log(\omega - \rho(\varepsilon)) \le t\Phi^{\mathrm{s}} - \log(\omega - \rho(\eta)).$$

This implies that $\tau_\varepsilon \le \tau_{\varepsilon,\eta}$. Now, consider some arbitrary fixed $t > \tau_\eta$. For $\varepsilon > 0$ sufficiently small, we then have that $t > \tau_{\varepsilon,\eta} \ge \tau_\varepsilon$. Hence from the above, we obtain that for t in this range and for n sufficiently large, there exists a cover $\{5U_n(z) : z \in \mathcal{G}(n)\}$ of W, such that we have the inequality

$$\sum_{z\in\mathcal{G}(n)} (\operatorname{diam} U_n(z))^t < 1.$$

This shows that $t \ge \dim_{\mathrm{H}}(W)$ and therefore, since $t > \tau_\eta$ was chosen to be arbitrary, it follows that $\tau_\eta \ge \dim_{\mathrm{H}}(W)$. Finally, observe that the continuity of the pressure function implies that $\lim_{\eta\to 0} \tau_\eta = t_\omega$, and this then allows to deduce the desired inequality

$$\dim_{\mathrm{H}}(W) \le t_\omega.$$

3 Proofs of Propositions 1 and 2

Proof of Proposition 1. Recall that here we assume that d is the minimal value of Δ on Λ. Then note that, since Δ is upper semi-continuous on Λ and takes on only integer values, it follows that if $\Delta(x) = d$ for some $x \in \Lambda$, then we have that the preimage counting function Δ must be equal to d on some open neighbourhood of x. This implies that the set

$$A := \{x \in \Lambda : \Delta(x) = d\}$$

has to be open in Λ. In order to show that A is dense in Λ, assume to the contrary, that there exists a nonempty open set $V \subset \Lambda$ such that $\Delta(x) \ge d + 1$, for all $x \in V$. In this situation, we can find a Lipschitz continuous function ψ on Λ such that $d \le \psi(x) \le \Delta(x)$, for all $x \in \Lambda$, and such that $\psi \equiv d+1$ on some open ball contained in V.

Now note that Theorem 1 implies that $t_\psi \ge \delta^s(x)$, for all $x \in \Lambda$. Also, since $\psi(x) \ge d$ for all $x \in \Lambda$, we have that $t_\psi \le t_d$. Therefore, if, for some $x \in \Lambda$, we have that $t_d = \delta^s(x)$, then it follows that

$$t_d = t_\psi = \delta^s(x).$$

Let us now consider the unique equilibrium measure μ_ψ for the Hölder continuous potential $t_d \Phi^s - \log \psi$ (the existence and the uniqueness of μ_ψ are guaranteed, since f is hyperbolic on Λ, see [3, 8, 11]). Also, since μ_ψ is an f-invariant probability measure for which the Variational Principle holds for the potential $t_d \Phi^s - \log d$, we have that

$$0 = P(t_d \Phi^s - \log d) = P(t_d \Phi^s - \log \psi) = h_{\mu_\psi} + \int_\Lambda (t_d \Phi^s - \log \psi)\, \mathrm{d}\mu_\psi \ge h_{\mu_\psi} + \int_\Lambda (t_d \Phi^s - \log d)\, \mathrm{d}\mu_\psi.$$

This shows that

$$\int_\Lambda \log \psi \, \mathrm{d}\mu_\psi \le \int_\Lambda \log d \, \mathrm{d}\mu_\psi.$$

However, recall that $\log \psi(y) > \log d$, for all y in some open ball contained in V. Moreover, since μ_ψ is an equilibrium measure, we have that μ_ψ is positive on Bowen balls [3, 21] and hence, it is positive on any open set in Λ. Clearly, this gives a contradiction and therefore, it follows that $\Delta \equiv d$ on a dense open set in Λ.

Now in order to show that, if $\Delta \equiv d$ on an open dense set then $\delta^s(y) = t_d$ for all $y \in \Lambda$, let us define the following set:

$$A_n := \{x \in \Lambda : x \text{ has precisely } d^n n\text{-preimages } y_i \text{ and } \Delta(f^j(y_i)) = d, \text{ for all } 0 \le j \le n \text{ and } 1 \le i \le d^n\}$$

The aim is to show that A_n is open and dense in Λ, for each $n \in \mathbb{N}$. For this, we first show that A_1 is open in Λ. By definition, we have that if $x \in A_1$ then $x \in A$ and hence, x has precisely d 1-preimages $x_1, \ldots, x_d \in A$. Now, let y be a point close to x. Since A is open, we can assume without loss of generality that $y \in A$ and hence, y has precisely d preimages $y_1, \ldots, y_d \in \Lambda$. Since d is the least value Δ can attain on Λ and since f has no critical points in Λ, we have that each of the y_i is close to one of the x_j. Since A is open and since the x_j are contained in A, it follows that $y_i \in A$, for all $1 \le i \le d$ provided y is close enough to x. This shows that $y \in A_1$ and hence it follows that A_1 is open in Λ.

In order to show that A_1 is dense in Λ, consider some open set V in Λ. Since A is dense in Λ, there exists some point $y \in A \cap V$, which must have precisely d 1-preimages $y_1, \dots, y_d \in \Lambda$. Now, let $B \subset A$ be a small ball centered at y. For each $1 \le i \le d$, choose a sufficiently small ball B_i centered at y_i such that the resulting family of balls is pairwise disjoint and such that f is injective on B_i and on $B \subset f(B_i)$. The aim is to show that $B \cap f(B_i \cap A)$ is open and dense in B. Indeed, if $z \in B \cap f(B_i \cap A)$, then z has a 1-preimage $z_i \in B_i \cap A$. Now, if z' is close enough to z, then z' belongs to A and hence z' has a 1-preimage $z'_i \in B_i$ which lies close to z_i. Since $z_i \in A$ and since A is open, it follows that $z'_i \in A$. This implies then that $B \cap f(B_i \cap A)$ is open in B. Also, if there were a nonempty open set $B' \subset B$ such that $B' \cap f(B_i \cap A) = \emptyset$, then $B_i \cap f^{-1}(B')$ would be open and nonempty. Clearly, this contradicts the fact that A dense in Λ. This shows that $B \cap f(B_i \cap A)$ must be open and dense in B, for all $1 \le i \le d$. Since a finite intersection of open and dense subsets is again open and dense, it now follows that A_1 has to be open and dense in Λ.

Clearly, the same methods as in the previous argument can be used to prove by way of induction that A_n is open and dense in Λ, for each $n \in \mathbb{N}$. Therefore, we now have that, for each $n \in \mathbb{N}$, there exists an open dense set A_n such that for every $y \in A_n$ there exist exactly d^n n-preimages $y_1, \dots, y_{d^n} \in \Lambda$ of y such that $\Delta(f^i y_j) = d$, for each $0 \le i \le n$ and $1 \le j \le d^n$. This shows that in the proofs of Theorem 1 and the main theorem of [15] one can work exclusively with points from $\bigcup_{n \in \mathbb{N}} A_n$. Indeed, since A_n is open and dense in Λ, it follows that for every $z \in E_n$ we can take a point z' sufficiently close to z such that $f^n(z') \in A_n$; thus, we obtain a set E'_n with the same number of elements as E_n, which is again (n, ε)-spanning and which can be used in the condition on the pressure, in order to obtain good covers of $W^s_r(x) \cap \Lambda$. Then all the iterates up to order n of any $z' \in E'_n$ will have exactly d 1-preimages in Λ and thus we obtain $\delta^s(x) = t_d$, for all points $x \in \Lambda$. ■

Proof of Proposition 2. (a) In the sequel, let $x \in \Lambda$ be fixed and put $W := W^s_r(x) \cap \Lambda$. As in the proof of Theorem 1, for each $\varepsilon > 0$ sufficiently small, there exists $n_0 \in \mathbb{N}$ and a minimal (n_0, ε)-spanning set E_{n_0} for Λ such that for each $t > t_\omega$ sufficiently large we have, for some fixed $\beta > 0$,

$$\sum_{z \in E_{n_0}} \frac{|Df_s^{n_0}(z)|^t}{\omega(f(z) \cdots \omega(f^{n_0} z)} < e^{-\beta n_0} < \frac{1}{2}. \tag{7}$$

Let us assume $E_{n_0} =: \{e_1, \dots, e_{m_0}\}$. As before, define $U_n(z) := f^n(B_n(z, \varepsilon)) \cap W^s_r(x)$, for $n \in \mathbb{N}$ and $z \in \Lambda$. The aim is to construct a covering of W which consists of sets of comparable diameter. For this, let $\{|Df_s^{n_0}(z)| : z \in E_{n_0}\} =: \{\delta_1, \dots, \delta_{m_0}\}$ and then define for

$n \in \mathbb{N}$ the value $\chi(n)$ by

$$\chi(n) := \inf \left\{ \prod_{i=1}^{n} \delta_{j_i} : 1 \le j_i \le m_0 \right\}.$$

Now, for each $w \in \Lambda$ and for each nn_0-preimage $w(-nn_0) \in \Lambda$ of w, we have that $f^{jn_0}(w(-nn_0)) \in B_{n_0}(z_j, \varepsilon)$, for each $0 \le j \le n-1$. From this, we deduce that $|Df_s^{nn_0}(w(-nn_0))| \ge \chi(n)$. Next observe that in general, given any full prehistory $\hat{w} = (w, w_{-1}, \ldots) \in \hat{\Lambda}$ of some element $w \in \Lambda$, there exists $k(\hat{w}, n) \in \mathbb{N}$ such that $|Df_s^{k(\hat{w},n)n_0}(w_{-k(\hat{w},n)n_0})|$ is comparable to $\chi(n)$, that is,

$$C_0^{-1} \cdot \chi(n) < |Df_s^{k(\hat{w},n)n_0}(w_{-k(\hat{w},n)n_0})| < C_0 \cdot \chi(n),$$

where we have put $C_0 := \sup_{z \in \Lambda} \cdot |Df_s^{n_0}(z)|$. This shows that for $w \in W$ we have that the diameter $\operatorname{diam} U_{k(\hat{w},n)n_0}(w_{-k(\hat{w},n)n_0})$ is comparable to $\chi(n)$, where the comparability constant does depend neither on w nor on n. Hence, the sets $U_{k(\hat{w},n)n_0}(w_{-k(\hat{w},n)n_0})$ provide a covering of W and their diameters are all of size comparable to $\chi(n)$. For later use, let us remark that one can choose a point $z_{k(\hat{w},n)}(\hat{w}) \in E_{n_0}$ such that $w_{-n_0 k(\hat{w},n)} \in B_{n_0}(z_{k(\hat{w},n)}(\hat{w}), \varepsilon)$ and similarly, points $z_{k(\hat{w},n)-j}(\hat{w}) \in E_{n_0}$ such that $f^{n_0 j}(w_{-n_0 k(\hat{w},n)}) \in B_{n_0}(z_{k(\hat{w},n)-j}(\hat{w}), \varepsilon)$, for each $1 \le j < k(\hat{w}, n)$. Then recalling that $E_{n_0} =: \{e_1, \ldots, e_{m_0}\}$, the inequality in (7) reads:

$$\sum_{i=1}^{m_0} \frac{\delta_i^t}{\omega(f(e_i)) \cdots \omega(f^{n_0}(e_i))} < \frac{1}{2}.$$

By raising both sides of this inequality to the power $p \in \mathbb{N}$ and then summing over p, we obtain

$$\sum_{p \in \mathbb{N}} \left(\sum_{i=1}^{m_0} \frac{\delta_i^t}{\omega(f(e_i)) \cdots \omega(f^{n_0}(e_i))} \right)^p$$
$$= \sum_{p \in \mathbb{N}} \sum_{(i_1,\ldots,i_p) \in \{1,\ldots,m_0\}^p} \frac{\delta_{i_1}^t \cdots \delta_{i_p}^t}{(\omega(f(e_{i_1})) \cdots \omega(f^{n_0}(e_{i_1}))) \cdot \ldots \cdot (\omega(f(e_{i_p})) \cdots \omega(f^{n_0}(e_{i_p})))} < 1. \tag{8}$$

Let us now again consider some point $w \in \Lambda$ and its full prehistory $\hat{w} = (w, w_{-1}, \ldots) \in \hat{\Lambda}$. By the above, we then have that the orbit of $w_{-k(\hat{w},n)n_0}$ under the map $f^{k(\hat{w},n)n_0}$ is shadowed by the consecutive linking of the n_0-orbits of $k(\hat{w}, n)$ points from E_{n_0}. Then, the summand of the corresponding sum, associated with this orbit, is of the form

$$\frac{(\operatorname{diam} U_{k(\hat{w},n)n_0}(w_{-k(\hat{w},n)n_0}))^t}{(\omega(z_{k(\hat{w},n)}(\hat{w})) \ldots \omega(f^{n_0}(z_{k(\hat{w},n)}(\hat{w})))) \cdots (\omega(z_1(\hat{w})) \cdots \omega(f^{n_0}(z_1(\hat{w}))))}. \tag{9}$$

We can now use the procedure of successive partial minimization and elimination, which we used in the proof of Theorem 1, and this then leads to a covering of $W^s_r(x) \cap \Lambda$ consisting of sets of diameter comparable to $\chi(n)$. Indeed, as in the proof of Theorem 1, here we use the fact that the denominators of the terms in (8) are products of evaluations of ω along the forward orbit of the preimages.

In this fashion, we obtain a sum with terms as in (9), which is smaller than or equal to the sum in (8). To this sum, we can apply the repeated partial minimization procedure as in the proof of Theorem 1, in order to extract a subcover $\mathcal{V}$ such that in the associated sum the denominators are successively eliminated; therefore, we arrive at the inequality

$$\sum_{U \in \mathcal{V}} (\operatorname{diam} U)^t < 1.$$

From this, it clearly follows then, that

$$\beta^s(y) \leq t_\omega \quad \text{for all } y \in W^s_r(x) \cap \Lambda.$$

(b) The aim is to show that the stable upper box-counting dimension β^s is constant on Λ. For this, note that since f is transitive on Λ, there must exist a point $x \in \Lambda$ the set of preimages of which is dense in Λ. Therefore, if $y \in \Lambda$ is some fixed arbitrary point and if $\varepsilon > 0$, then there exists some n-preimage x_{-n} of x such that $d(x_{-n}, y) = \varepsilon$, for certain $n \in \mathbb{N}$.

Then the local product structure on Λ (see [8]) implies that, if for some $z \in \Lambda$ the local unstable manifold $W^u_r(\hat{z})$ intersects $W^s_r(y)$, then it also intersects $W^s_r(x_{-n})$ at a unique point contained in Λ. Likewise, any local unstable manifold, which intersects $W^s_r(x_{-n})$, will also intersect $W^s_r(y)$ in a point from Λ. Note that if $W^s_r(y) \cap \Lambda$ is covered by balls $U \in \mathcal{U}$ of radius $\varepsilon > 0$, then the set $W^s_r(x_{-n}) \cap \Lambda$ is covered by the same number of balls of radius at most $C'\varepsilon$, for some fixed constant $C' > 0$. This follows, since the intersection $W^s_r(x_{-n}) \cap \bigcup_{\hat{z} \in \hat{\Lambda}, z \in U} W^u_r(\hat{z})$ is contained in a ball of radius $C'\varepsilon$; indeed $d(x_{-n}, y) = \varepsilon$ and the inclination of local unstable manifolds with respect to $W^s_r(y)$ is bounded from below, a consequence of the uniform hyperbolicity of f on Λ.

Also, if we cover $W^s_r(x_{-n}) \cap \Lambda$ with balls of radius ε, then we can consider all the local unstable manifolds through the points of each of these balls, in order to obtain balls of radius at most $C'\varepsilon$ that are contained in $W^s_r(y)$. However, by setting $\varepsilon' := \varepsilon |Df_s(x_{-n})|^n$ for $\varepsilon > 0$ sufficiently small, we have that every covering of $W^s_r(x) \cap \Lambda$ by balls of radius ε' determines a covering of $W^s_r(x_{-n}) \cap \Lambda$ by balls of radius ε; and vice versa. Therefore, we obtain that

$$\beta^s(y) = \beta^s(x) \quad \text{for all } y \in \Lambda,$$

and consequently it follows that, the stable upper box dimension function is constant on the fractal Λ. ■

Remark. Let us assume for a moment that the following condition is satisfied: if D is the maximum possible value of Δ on Λ, then, for each $1 \le i \le D-1$, the sets $\Lambda_i := \{x \in \Lambda : \Delta(x) \le i\}$ have their respective closure contained in Λ_{i+1}. Note that, by the upper semi-continuity of Δ on Λ, we have that the set $\Lambda_D := \{x \in \Lambda : \Delta(x) = D\}$ is closed in Λ. Also, the upper semi-continuity of Δ implies that Λ_i is open in Λ, for each $1 \le i \le D-2$. Owing to our assumption here, it is possible to fix some neighbourhood $\Lambda_i(\varepsilon)$ of $\bar{\Lambda}_i$ such that $\Lambda_i(\varepsilon) \subset \Lambda_{i+1}$, for each $1 \le i \le D-2$. Also, let us fix some neighbourhood $\Lambda_{D-1}(\varepsilon)$ of the closure of Λ_{D-1}. Then define $K_0 := \Lambda_D \setminus \Lambda_{D-1}(\varepsilon)$, $K_1 := \bar{\Lambda}_{D-1} \setminus \Lambda_{D-2}(\varepsilon)$, $K_2 := \bar{\Lambda}_{D-2} \setminus \Lambda_{D-3}(\varepsilon), \ldots, K_{D-1} := \bar{\Lambda}_1$ and note that the family $\{K_j : 0 \le j < D\}$ consists of pairwise disjoint compact sets.

Using the above disjoint subsets, we infer that there exists a continuous function ψ on Λ such that $\psi(x) = D$ for all $x \in K_0$, $D-1 \le \psi(x) \le D$ for $x \in \Lambda_{D-1}(\varepsilon) \setminus \bar{\Lambda}_{D-1}$, $\psi(x) = D-1$ for $x \in K_1$, and $D-2 \le \psi(x) \le D-1$ for $x \in \Lambda_{D-2}(\varepsilon) \setminus \bar{\Lambda}_{D-2}$, which can be continued until we reach Λ_1. By construction, we then have that $\Delta(x) \ge \psi(x)$, for all $x \in \Lambda$. By applying Theorem 1, it follows that $\delta^s(x) \le t_{\psi_\varepsilon}$, for all $x \in \Lambda$ and $\varepsilon > 0$. Also, by choosing $\varepsilon \ge \varepsilon'$ appropriately, we can assume that $\Lambda_i(\varepsilon') \subset \Lambda_i(\varepsilon)$. Therefore, we have for each $x \in \Lambda$ that $\psi_\varepsilon(x)$ is increasing, for ε tending to zero. This implies that there exists t_* such that t_{ψ_ε} tends to t_* when ε tending to zero. Therefore, $\delta^s(x) \le t_*$, for every point $x \in \Lambda$. □

4 Two Examples

Example 1. We assume that the reader is familiar with the type of horseshoes introduced by Simon and Solomyak [27]. They considered horseshoes with overlaps in $\mathbb{R}^3$, which are given by a $\mathcal{C}^{1+\epsilon}$-transformation f, defined by

$$f(x, y, z) := (\gamma(x, z), \eta(y, z), \psi(z)) \quad \text{for all } (x, y, z) \in [0, 1] \times [0, 1] \times \mathcal{I},$$

where $\mathcal{I} := \bigcup_{i=1}^m \in I_i$ denotes the union of m compact pairwise disjoint intervals $I_1, \ldots, I_m \subset (0, 1)$; we also assume that $m \ge 3$, that $\lambda_1 < |\gamma'_x|, |\eta'_y| < \lambda_2$ for some $0 < \lambda_1 < \lambda_2 < \frac{1}{2}$, that $|\psi'| > 1$ on $\mathcal{I}$, and that $\psi(I_i) = [0, 1]$, for all $i = 1, \ldots, m$. The basic set Λ of f is defined as before, namely $\Lambda := \bigcap_{n\in\mathbb{Z}} f^n([0,1]^3)$. Let us now consider the following smooth perturbations f_τ of f:

$$f_\tau(x, y, z) := (\gamma(x, z) + \tau_{i,1}, \eta(y, z) + \tau_{i,2}, \psi(z)) \quad \text{for all } (x, y, z) \in [0,1] \times [0,1] \times I_i, 1 \le i \le m. \tag{10}$$

We will say that the parameter $\tau := (\tau_{1,1}, \tau_{1,2}, \dots, \tau_{m,1}, \tau_{m,2})$ is *f-admissible* if we have the condition:

$$f_\tau\left(\bigcup_{1\le i\le m} [0,1]^2 \times I_i\right) \subset (0,1)^2 \times [0,1].$$

It can be checked that the set of f-admissible parameters τ, is a nonempty open subset of $\mathbb{R}^{2m}$. Also, due to the expansion in the z-direction as well as the contractions with respect to the (x, y)-coordinates, one can show that f_τ is hyperbolic on the basic set Λ_τ associated with f_τ.

As in [27], one then verifies that for Lebesgue almost every f-admissible τ, we have that the stable dimension of Λ_τ is given by the maximum of the zeros s_1, and s_2, respectively, of certain pressure functions of $\log|\gamma'_x|$, and $\log|\eta'_y|$, respectively, on the symbolic space Σ_m. Let us now assume that on $[0,1] \times \mathcal{I}$, we have

$$|\gamma'_x| = |\eta'_y| \equiv 1/m.$$

Then, from the proof of Theorem 1(i) of [27] and the fact that in this case both zeros s_1 and s_2 are equal to 1, it follows that the stable dimension of Λ_τ is equal to 1, for Lebesgue-almost every f-admissible τ.

However, in the above case we have that the zero $t_{1,\tau}$ of the pressure function $t \mapsto P_{f_\tau|_{\Lambda_\tau}}(t\Phi^s_\tau)$ for the stable potential function Φ^s_τ, is also equal to 1. This follows since $\Phi^s_\tau \equiv -\log m$, and since the topological entropy of $f_\tau|_{\Lambda_\tau}$ is equal to $\log m$. This latter fact about entropy holds since the spanning sets of $f_\tau|_{\Lambda_\tau}$ are determined only by the dynamics of ψ in the z coordinate; however, this dynamics in the z-direction is conjugated to the shift σ_m on Σ_m (since ψ expands I_i onto the whole interval $[0,1]$ for each $i = 1, \dots, m$) and, as is well known, $h(\sigma_m) = \log m$.

This shows then that $t_{1,\tau} = 1$. Also note that if $|\gamma'_x| = |\eta'_y| \equiv 1/m$ on $[0,1] \times \mathcal{I}$, then f_τ is c-conformal. Therefore, since $\delta^s = t_{1,\tau}$, we can now apply Corollary 1, which then gives that almost every horseshoe f_τ has an open dense set of points in its associated basic set Λ_τ such that each of these points has precisely one f_τ-preimage in Λ_τ. In conclusion, for the above choice of auxiliary functions γ and η, we have now shown that Lebesgue-almost every translation f_τ has a similar behavior as that of a homeomorphism, on its associated basic set Λ_τ. We summarize this result in the following:

Corollary 3. Let $(f_\tau)_\tau$ denote the family of horseshoes with overlaps given in (10), and assume that on $[0, 1] \times \mathcal{I}$, we have $|\gamma'_x| = |\eta'_y| \equiv 1/m$. Also, let $\Lambda_\tau := \bigcap_{n\in\mathbb{Z}} f_\tau^n([0, 1]^3)$ denote the associated basic set of f_τ. Then for Lebesgue-almost every f-admissible parameter τ, there exists an open dense set A_τ in Λ_τ, such that every $x \in A_\tau$ has precisely one f_τ-preimage in Λ_τ. □

Example 2. In [13], the first author gave an example of a family of nonlinear hyperbolic skew products for which the preimage counting function is not constant on their associated basic sets. Let us first briefly recall the construction of this family. For $\alpha \in (0, 1)$, let $I_1^\alpha, I_2^\alpha \subset I := [0, 1]$ be two intervals such that $I_1^\alpha \subset [\frac{1}{2} - \epsilon(\alpha), \frac{1}{2} + \epsilon(\alpha)]$ and $I_2^\alpha \subset [1 - \alpha - \epsilon(\alpha), 1 - \alpha + \epsilon(\alpha)]$, for some $0 < \epsilon(\alpha) < \alpha^2$ sufficiently small. Let $g : I_1^\alpha \cup I_2^\alpha \to I$ be a strictly increasing smooth function with the property that $g(I_1^\alpha) = g(I_2^\alpha) = I$. Also, assume that there exists a large number $\beta > 0$ such that $\beta^2 > g'(x) > \beta$, for each $x \in I_1^\alpha \cup I_2^\alpha$. Then there exist intervals $I_{11}^\alpha, I_{12}^\alpha \subset I_1^\alpha$ and $I_{21}^\alpha, I_{22}^\alpha \subset I_2^\alpha$ such that $g(I_{11}^\alpha) = g(I_{21}^\alpha) = I_1^\alpha$ and $g(I_{12}^\alpha) = g(I_{22}^\alpha) = I_2^\alpha$. For

$$J^\alpha := I_{11}^\alpha \cup I_{12}^\alpha \cup I_{21}^\alpha \cup I_{22}^\alpha, \quad \text{and} \quad J_*^\alpha := \{x \in J^\alpha : g^i(x) \in J^\alpha \text{ for all } i \geq 0\},$$

one can then define the skew-product $f_\alpha : J_*^\alpha \times I \to J_*^\alpha \times I$ by the formula

$$f_\alpha(x, y) := (g(x), h_\alpha(x, y)) \quad \text{where } h_\alpha(x, y) := \begin{cases} \psi_{1,\alpha}(x) + s_{1,\alpha} y, & x \in I_{11}^\alpha, \\ \psi_{2,\alpha}(x) + s_{2,\alpha} y, & x \in I_{21}^\alpha, \\ \psi_{3,\alpha}(x) - s_{3,\alpha} y, & x \in I_{12}^\alpha, \\ s_{4,\alpha} y, & x \in I_{22}^\alpha, \end{cases} \tag{11}$$

where $s_{1,\alpha}, \ldots, s_{4,\alpha} \in (\frac{1}{2} - \varepsilon_0, \frac{1}{2} + \varepsilon_0)$ denote some arbitrary fixed numbers close to $\frac{1}{2}$ and $\psi_{1,\alpha}, \psi_{2,\alpha}, \psi_{3,\alpha} : I \to \mathbb{R}$ are $\mathcal{C}^2$-functions which are ε_0-close (with respect to the $\mathcal{C}^1$-metric) to the linear functions given by $x \mapsto x$, $x \mapsto 1 - x$ and $x \mapsto 1$, respectively. Let us also use the following shorter notation:

$$h_{x,\alpha}(y) := h_\alpha(x, y) \quad \text{for any } (x, y) \text{ for which this is well-defined.}$$

By defining $h_{z,\alpha}^n := h_{f^n(z),\alpha} \circ \ldots \circ h_{z,\alpha}$ for each $n \geq 0$, the basic set Λ_α of the above system is given by

$$\Lambda_\alpha = \bigcup_{x\in J_*^\alpha} \bigcap_{n\geq 0} \bigcup_{z\in g^{-n}(x)\cap J_*^\alpha} h_{z,\alpha}^n(I).$$

In [13], it was shown that for α small enough, the map f_α is a hyperbolic endomorphism on Λ_α and that there exist two infinite sets $A_\alpha, B_\alpha \subset \Lambda_\alpha$, which are both not dense in Λ_α, such that on A_α the preimage counting function Δ is constant equal to 1, whereas for points in B_α we have that Δ is constant equal to 2. Since Δ is upper semi-continuous on Λ_α, it follows that A_α is open in Λ_α and that B_α is closed in Λ_α.

We now want to obtain an estimate of the stable dimension $\delta^s(\cdot)$ by using Theorem 1. There exists a nonconstant continuous function ω such that $1 \le \omega(x) \le 2$ for all $x \in \Lambda_\alpha$, and $\omega(x) = 1$ for all $x \in A_\alpha$. Hence, we have $\omega(x) \le \Delta(x),\ x \in \Lambda_\alpha$. The function ω may be viewed as an "approximation" of the function $\chi_{B_\alpha} + 1$. Then, an application of Theorem 1 gives the following estimate:

$$\delta^s(y) \le t_\omega \quad \text{for all } y \in \Lambda_\alpha$$

Similarly, as before consider an increasing sequence of nonconstant continuous functions ω_m such that $\omega_m \equiv 1$ on A_α, and $1 \le \omega_m \le 2$ on Λ_α. For each member of this sequence, we can now argue as before. This leads to improvements of the upper bounds for $\delta^s(y)$. Namely with t_{ω_m} denoting the unique zero of the pressure function associated with ω_m, we have that the positive numbers t_{ω_m} are decreasing when $m \to \infty$, and

$$\delta^s(y) \le t_{\omega_m} \quad \text{for all } y \in \Lambda_\alpha,\ m \in \mathbb{N}.$$

In particular, by applying Proposition 2, we also obtain that the stable upper box dimension β^s is constant on Λ_α and bounded above by t_{ω_m}, for each $m \in \mathbb{N}$. □

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