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On a class of stable conditional measures

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Abstract. The dynamics of endomorphisms (smooth non-invertible maps) presents many differences from that of diffeomorphisms or that of expanding maps; most methods from those cases do not work if the map has a basic set of saddle type with self-intersections. In this paper we study the conditional measures of a certain class of equilibrium measures, corresponding to a measurable partition subordinated to local stable manifolds. We show that these conditional measures are geometric probabilities on the local stable manifolds, thus answering in particular the questions related to the stable pointwise Hausdorff and box dimensions. These stable conditional measures are shown to be absolutely continuous if and only if the respective basic set is a non-invertible repeller. We find also invariant measures of maximal stable dimension, on folded basic sets. Examples are given, too, for such non-reversible systems.

1. *Background and outline of the paper*

In this paper we will study non-invertible smooth (say $\mathcal{C}^2$) maps on a Riemannian manifold M, called *endomorphisms*, which are uniformly hyperbolic on a basic set Λ. Here by a *basic set* for an endomorphism $f : M \to M$, we understand a compact topologically transitive set Λ, which has a neighbourhood U such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$.

Considering non-invertible transformations makes sense from the point of view of applications, since the evolution of a non-reversible physical system is usually given by a time-dependent differential equation $dx(t)/dt = F(x(t))$ whose solution, the flow $(f^t)_t$, may not consist necessarily of diffeomorphisms. However, if we look at the ergodic (qualitative) properties of the associated flow (equilibrium measures, Lyapunov exponents, conditional measures associated to measurable partitions), we may replace it with a discrete non-invertible dynamical system [**5**].

The theory of hyperbolic diffeomorphisms (Axiom A) has been studied by many authors (see for example [**4, 5, 7, 19**] and the references therein); also, the theory of expanding maps was studied extensively (see for instance [**18**]), and the fact that the local inverse iterates are contracting on small balls was crucial in that case.

However, the theory of smooth non-invertible maps which have saddle basic sets is significantly different from the two above-mentioned cases. Most methods of proof from diffeomorphisms or expanding maps do not work here due to the complicated *overlappings and foldings* that the endomorphism may have in the basic set Λ. The unstable manifolds depend in general on the *choice of a sequence* of consecutive preimages, not only on the initial point (as in the case of diffeomorphisms). So, the unstable manifolds do not form a foliation; instead, they may intersect each other both inside and outside Λ. Moreover, the local inverse iterates do not contract necessarily on small balls; instead, they will grow exponentially (at least for some time) in the stable direction. Also, an arbitrary basic set Λ is not necessarily totally invariant for f, and there do not always exist Markov partitions on Λ. We mention also that endomorphisms on Lebesgue spaces behave differently than invertible transformations even from the point of view of classifications in ergodic theory; see [**15**].

We will work in the following with a hyperbolic endomorphism f on a basic set Λ; such a set is also called a *folded basic set* (or a basic set with *self-intersections*). By an *n-preimage* of a point x we mean a point y such that $f^n(y) = x$. By *prehistory* of x we understand a sequence of consecutive preimages of x, belonging to Λ, and denoted by $\hat{x} = (x, x_{-1}, x_{-2}, \ldots)$, where $f(x_{-n}) = x_{-n+1}$, $n > 0$, with $x_0 = x$. And, by an *inverse limit* of (f, Λ) we mean the space of all such prehistories, denoted by $\hat{\Lambda}$. For more about these aspects, see [**11, 17**]. By the definition of a basic set Λ, we assume that f is *topologically transitive* on Λ as an endomorphism, i.e. that there exists a point in Λ whose iterates are dense in Λ.

Hyperbolicity is defined for endomorphisms (see [**19**]) similarly as for diffeomorphisms, with the crucial difference that now the unstable spaces (and thus the local unstable manifolds) depend on whole prehistories; so we have the stable tangent spaces E^s_x, $x \in \Lambda$, the unstable tangent spaces $E^u_{\hat{x}}$, $\hat{x} \in \hat{\Lambda}$, the *local stable manifolds* $W^s_r(x)$, $x \in \Lambda$ and the *local unstable manifolds* $W^u_r(\hat{x})$, $\hat{x} \in \hat{\Lambda}$. As there may be (infinitely) many unstable manifolds going through a point, we do not have here a well-defined holonomy map between stable manifolds, by contrast to the diffeomorphism case. For more details on endomorphisms, see [**9, 11, 13, 19**], etc.

Definition 1. Consider a smooth (say $\mathcal{C}^2$) non-invertible map f which is hyperbolic on the basic set Λ, such that the critical set of f does not intersect Λ. Define the *stable potential* of f as $\Phi^s(y) := \log |Df_s(y)|$, $y \in \Lambda$. By *stable dimension* (at a point $x \in \Lambda$) we understand the Hausdorff dimension $\delta^s(x) := HD(W^s_r(x) \cap \Lambda)$. Denote also by $\mathcal{C}_f$ the set of *critical points* of f.

We will say that f is *c-hyperbolic* on Λ if f is hyperbolic on Λ, $\mathcal{C}_f \cap \Lambda = \emptyset$ and f is conformal on the local stable manifolds over Λ.

The relations between thermodynamic formalism and the dynamics of diffeomorphisms or expanding maps form a rich field (see for instance [**1, 4, 5, 8, 18**], etc). And, in [**12–14**], we studied some aspects of the thermodynamic formalism for non-invertible smooth maps.

Examples of hyperbolic endomorphisms are numerous, for instance hyperbolic solenoids and horseshoes with self-intersections [**3**], polynomial maps in higher dimension hyperbolic on certain basic sets, skew products with overlaps in their fibers [**14**], hyperbolic toral endomorphisms or perturbations of these, etc.

In this non-invertible setting, a special importance is presented by *constant-to-1 endomorphisms*. For such endomorphisms, we study the family of conditional measures of a certain equilibrium measure, a family associated to a measurable partition subordinated to local stable manifolds.

If a *topological condition* is satisfied, namely if the number of preimages remaining in Λ is constant along Λ, we showed in [**13**] the following.

THEOREM. (Independence of stable dimension) *If the endomorphism f is c-hyperbolic on the basic set Λ (see Definition 1) and if the number of f-preimages of any point from Λ, remaining in Λ, is constant and equal to d, then the stable dimension $\delta^s(x)$ is equal to the unique zero t^s_d of the pressure function $t \to P(t\Phi^s - \log d)$, for any $x \in \Lambda$. The common value of the stable dimension along Λ will be denoted by δ^s.*

In fact, if f is *open* on Λ, we proved (see [**13**] and [**12**, Proposition 1]) the following proposition.

PROPOSITION. [**12, 13**] *Let us consider an endomorphism $f : M \to M$ which has a basic set Λ, disjoint from the critical set of f. Assume that Λ is connected and $f|_\Lambda : \Lambda \to \Lambda$ is open. Then, the cardinality of the set $f^{-1}(x) \cap \Lambda$ is constant, when x ranges in Λ.*

Hence, the *openness* of f on the folded basic set Λ is very much related to the condition that the number of preimages of a point, remaining in Λ, is *constant*. Examples of hyperbolic open endomorphisms on saddle sets are given at the end of the paper.

Definition 2. Let us consider an endomorphism f c-hyperbolic on the basic set Λ, such that the number of f-preimages of any point from Λ, remaining in Λ, is constant and equal to d. Then, we call the equilibrium measure of $\delta^s \cdot \Phi^s$ the *stable equilibrium measure* of f on Λ, and denote it by μ_s.

We notice that, since the stable foliation is Lipschitz continuous for endomorphisms (see [**13**]), the potential $\delta^s \cdot \Phi^s$ is Holder continuous; thus, it can be shown by lifting the measure to the inverse limit $\hat{\Lambda}$ that there exists a unique equilibrium measure μ_s of $\delta^s \cdot \Phi^s$ (we can apply the results for homeomorphisms from [**7**] on the inverse limit $\hat{\Lambda}$, in order to get the uniqueness).

We will show in Theorem 1 that if the number of f-preimages in Λ is constant, then the *conditional measures* of μ_s associated to a measurable partition subordinated to the local stable manifolds are *geometric probabilities* of exponent δ^s. This will answer then in Corollary 1 the question of the pointwise Hausdorff dimension and the pointwise box dimension of the equilibrium measure μ_s on local stable manifolds (see for instance [**1**] for definitions). In the constant-to-1 non-invertible case, we show in particular in Corollary 2 that these stable conditional measures are measures of *maximal dimension* (in the sense of [**2**]) on the intersections of local stable manifolds with the folded basic set Λ.

Our approach will be different both from the case of diffeomorphisms and from that of expanding maps. In Proposition 1 (which is the main ingredient for the proof of Theorem 1), we compare the equilibrium measure on various different components of the preimage set of a small 'cylinder' around an unstable manifold. We will have to carefully estimate the equilibrium measure μ_s on the different pieces of the iterates of Bowen balls, in order to get good estimates for the cylinders around local unstable manifolds, $B(W^u_r(\hat{x}), \varepsilon)$. This will be done by a process of *disintegrating* the measure on the various components of the preimages of borelian sets, and then by successive re-combinations. Thus, we will re-obtain the measure μ_s on an arbitrary open set, and then will use the essential uniqueness of the family of conditional measures of μ_s; for background on conditional measures associated to measurable partitions on Lebesgue spaces, see [**17**].

In Corollary 3 we prove that the conditional measures of μ_s on the local stable manifolds over Λ are *absolutely continuous* if and only if the stable dimension is equal to the real dimension of the stable tangent space dim E^s_x, and we show that this is equivalent to Λ being a folded repeller.

We will also give at the end examples of hyperbolic constant-to-1 *folded basic sets* for which Theorem 1 and its corollaries do apply. In particular, we provide examples of folded repellers obtained for perturbation endomorphisms, which are not Anosov and for which we prove the absolute continuity of the stable conditional measures on their (nonlinear) stable manifolds.

2. *Main proofs and applications*

For our first result we assume only that f is a smooth endomorphism which is hyperbolic on a basic set Λ. We will give a comparison between the values of an arbitrary equilibrium measure μ_ϕ (corresponding to a Holder continuous potential ϕ on Λ) on the different pieces/components of the preimages of a borelian set; this will be useful when we will estimate, later on, the measure μ_s on certain sets.

By a *Bowen ball* $B_n(x, \varepsilon)$ we understand the set $\{y \in \Lambda, d(f^i y, f^i x) < \varepsilon, i = 0, \ldots, n\}$, for $x \in \Lambda$ and $n > 1$. If ϕ is a continuous real function on Λ and m is a positive integer, we denote by $S_m\phi(y) := \phi(y) + \phi(f(y)) + \cdots + \phi(f^m(y))$ the consecutive sum of ϕ on the n-orbit of $y \in \Lambda$. And, by $P(\phi)$ we denote the *topological pressure* of the potential ϕ with respect to the function $f|_\Lambda$.

PROPOSITION 1. *Let f be an endomorphism, hyperbolic on a basic set Λ; consider also a Holder continuous potential ϕ on Λ and let μ_ϕ be the unique equilibrium measure of ϕ. Let us consider a small $\varepsilon > 0$, two disjoint Bowen balls $B_k(y_1, \varepsilon)$, $B_m(y_2, \varepsilon)$ and a borelian set $A \subset f^k(B_k(y_1, \varepsilon)) \cap f^m(B_m(y_2, \varepsilon))$, such that $\mu_\phi(A) > 0$; denote $A_1 := f^{-k}A \cap B_k(y_1, \varepsilon)$, $A_2 := f^{-m}A \cap B_m(y_2, \varepsilon)$ and assume that $\mu_\phi(\partial A_1) = \mu_\phi(\partial A_2) = 0$. Then, there exists a positive constant C_ε independent of k, m, y_1, y_2 such that*

$$\frac{1}{C_\varepsilon}\mu_\phi(A_2) \cdot \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} \cdot P(\phi)^{m-k} \le \mu_\phi(A_1) \le C_\varepsilon \mu_\phi(A_2) \cdot \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} \cdot P(\phi)^{m-k}.$$

Proof. Let us fix a Holder potential ϕ. We will denote the equilibrium measure μ_ϕ by μ to simplify notation. We will work with f restricted to Λ.

From construction, we have $f^k(A_1) = f^m(A_2)$; this holds since $f(\Lambda) = \Lambda$, as the definition of the basic set $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$ given at the beginning of §1 tells us (see also [**7**]).

Assume for example that $m \geq k$. Now, the equilibrium measure μ can be considered as the limit of the sequence of measures (see [**7**])

$$\tilde{\mu}_n := \frac{1}{P(f, \phi, n)} \cdot \sum_{x \in \text{Fix}(f^n)} e^{S_n\phi(x)} \delta_x,$$

where $P(f, \phi, n) := \sum_{x \in \text{Fix}(f^n)} e^{S_n\phi(x)}$, $n \geq 1$.

So, we have

$$\tilde{\mu}_n(A_1) = \frac{1}{P(f, \phi, n)} \cdot \sum_{x \in \text{Fix}(f^n) \cap A_1} e^{S_n\phi(x)}, \quad n \geq 1. \tag{1}$$

Let us consider now a periodic point $x \in \text{Fix}(f^n) \cap A_1$; by definition of A_1, it follows that $f^k(x) \in A$, so there exists a point $y \in A_2$ such that $f^m(y) = f^k(x)$. However, the point y does not have to be periodic.

Now, we will use the specification property [**4, 7**] on the hyperbolic compact locally maximal set Λ: if $\varepsilon > 0$ is fixed, then there exists a constant $M_\varepsilon > 0$ such that for all $n >> M_\varepsilon$ there exists a $z \in \text{Fix}(f^{n+m-k})$ such that z ε-shadows the $(n + m - k - M_\varepsilon)$-orbit of y.

Let now V be an arbitrary neighbourhood of the set A_2 such that $V \subset B_m(y_2, \varepsilon)$. Consider two points $x, \tilde{x} \in \text{Fix}(f^n) \cap A_1$ and assume that the same periodic point $z \in V \cap \text{Fix}(f^{n+m-k})$ corresponds to both x and $\tilde{x}$ by the above procedure. This means that the $(n - k - M_\varepsilon)$-orbit of $f^m z$ ε-shadows the $(n - k - M_\varepsilon)$-orbit of $f^k x$ and also the $(n - k - M_\varepsilon)$-orbit of $f^k \tilde{x}$. Hence, the $(n - M_\varepsilon - k)$-orbit of $f^k x$ 2ε-shadows the $(n - M_\varepsilon - k)$-orbit of $f^k \tilde{x}$. But, recall that we chose $x, \tilde{x} \in A_1 \subset B_k(y_1, \varepsilon)$; hence, $\tilde{x} \in B_{n-M_\varepsilon}(x, 2\varepsilon)$.

Now, we can split the set $B_{n-M_\varepsilon}(x, 2\varepsilon)$ into at most N_ε smaller Bowen balls of type $B_n(\zeta, 2\varepsilon)$. In each of these $(n, 2\varepsilon)$-Bowen balls $B_n(\zeta, 2\varepsilon)$ we may have at most one fixed point for f^n. This holds since fixed points for f^n are solutions to the equation $f^n \xi = \xi$ and, on tangent spaces, we have that $Df^n - \text{Id}$ is a linear map without eigenvalues of absolute value 1. Thus, if $d(f^i \xi, f^i \zeta) < 2\varepsilon$, $i = 0, \ldots, n$ and if ε is small enough, we can apply the inverse function theorem at each step. Therefore, there exists only one fixed point for f^n in each Bowen ball $B_n(\zeta, 2\varepsilon)$. Hence, there exist at most N_ε periodic points from $\text{Fix}(f^n) \cap \Lambda$ having the same periodic point $z \in V$ attached to them by the above procedure.

Let us notice also that, if $x, \tilde{x}$ have the same point $z \in V \cap \text{Fix}(f^{n+m-k})$ attached to them, then, as before, $\tilde{x} \in B_{n-M_\varepsilon}(x, 2\varepsilon)$. So, the distances between iterates are growing exponentially in the unstable direction, and decrease exponentially in the stable direction. Thus, we can use the Holder continuity of ϕ and a bounded distortion lemma to prove that

$$|S_n\phi(x) - S_n\phi(\tilde{x})| \leq \tilde{C}_\varepsilon,$$

for some positive constant $\tilde{C}_\varepsilon$ depending on ϕ (but independent of n, x). This can be used then in the estimate for $\tilde{\mu}_n(A_1)$, according to equation (1). We use the fact that if $z \in B_{n+m-k-M_\varepsilon}(y, \varepsilon)$, then $f^m(z) \in B_{n-M_\varepsilon-k}(f^m y, \varepsilon)$; also recall that $f^k x = f^m y$, so $f^m z \in B_{n-M_\varepsilon-k}(f^k x, \varepsilon)$. Then, from the Holder continuity of ϕ and the fact that $x \in A_1 \subset B_m(y_1, \varepsilon)$, it follows again by a bounded distortion lemma that there exists a constant $\tilde{C}_\varepsilon$ (denoted as before without loss of generality) satisfying

$$|S_{n+m-k}\phi(z) - S_n\phi(x)| \le |S_k\phi(y_1) - S_m\phi(y_2)| + \tilde{C}_\varepsilon, \tag{2}$$

for $n > n(\varepsilon, m)$.

But, from [**7**, Proposition 20.3.3] (which extends immediately to endomorphisms), we have that there exists a positive constant c_ε such that for sufficiently large n,

$$\frac{1}{c_\varepsilon} e^{nP(\phi)} \le P(f, \phi, n) \le c_\varepsilon e^{nP(\phi)},$$

where the expression $P(f, \phi, n)$ was defined immediately before equation (1). Hence, in our case, if $n > n(\varepsilon, m)$, we obtain

$$\frac{1}{c_\varepsilon} e^{(n+m-k)P(\phi)} \le P(f, \phi, n+m-k) \le c_\varepsilon e^{(n+m-k)P(\phi)}$$

and

$$\frac{1}{c_\varepsilon} e^{nP(\phi)} \le P(f, \phi, n) \le c_\varepsilon e^{nP(\phi)}. \tag{3}$$

Recall also that there are at most N_ε points $x \in \mathrm{Fix}(f^n)$ which have the same attached $z \in V \cap \mathrm{Fix}(f^n)$. Therefore, by using equations (1), (2) and (3), we can infer that there exists a constant $C_\varepsilon > 0$ such that for n large enough ($n > n(\varepsilon, m)$),

$$\tilde{\mu}_n(A_1) \le C_\varepsilon \tilde{\mu}_{n+m-k}(V) \cdot \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} \cdot P(\phi)^{m-k}, \tag{4}$$

where we recall that $A_1 \subset B_m(y_1, \varepsilon)$, $A_2 \subset B_m(y_2, \varepsilon)$. But, since $\partial A_1, \partial A_2$ have μ-measure zero, we obtain

$$\mu(A_1) \le C_\varepsilon \mu(V) \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} \cdot P(\phi)^{m-k}.$$

But, V has been chosen arbitrarily as a neighbourhood of A_2; hence,

$$\mu(A_1) \le C_\varepsilon \mu(A_2) \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} P(\phi)^{m-k}.$$

Similarly, we prove also the other inequality; hence, we are done. □

Let us recall a few notions about measurable partitions (see [**17**]). Let ζ be a partition of a Lebesgue space $(X, \mathcal{B}, \mu)$ with $\mathcal{B}$-measurable sets. Subsets of X that are unions of elements of ζ are called ζ-sets. For an arbitrary point $x \in X$ (modulo μ), we denote the unique set which contains x by $\zeta(x)$. By a *basis* for ζ we understand a countable collection $\{B_\alpha, \alpha \in A\}$ of measurable ζ-sets so that for any two elements $C, C' \in \zeta$, there exists some $\alpha \in A$ with $C \subset B_\alpha$, $C' \cap B_\alpha = \emptyset$ or vice versa, i.e. $C \cap B_\alpha = \emptyset$, $C' \subset B_\alpha$. A partition ζ is called *measurable* if it has a basis as above.

Now, we recall briefly the notion of a *family of conditional measures* associated to a measurable partition ζ. Assume that we have an endomorphism f on a compact set Λ, and consider a probability borelian measure μ on Λ which is f-invariant. If ζ is a measurable partition of $(\Lambda, \mathcal{B}, \mu)$, denote by $(\Lambda/\zeta, \mu_\zeta)$ the *factor space* of Λ relative to ζ. Then, we can attach an *essentially unique* collection of *conditional measures* $\{\mu_C\}_{C \in \zeta}$ satisfying two conditions (see [**17**]):

(i) (C, μ_C) is a Lebesgue space;
(ii) for any measurable set $B \subset \Lambda$, the set $B \cap C$ is measurable in C for μ_ζ-almost all points $C \in \Lambda/\zeta$, the function $C \to \mu_C(B \cap C)$ is measurable on Λ/ζ and $\mu(B) = \int_{\Lambda/\zeta} \mu_C(B \cap C)\, d\mu_\zeta(C)$.

Definition 3. If f is a hyperbolic map on a basic set Λ and if μ is an f-invariant borelian measure on Λ, then a measurable partition ζ of $(\Lambda, \mathcal{B}(\Lambda), \mu)$ is said to be *subordinated to the local stable manifolds* if for μ-almost everywhere $x \in \Lambda$, we have $\zeta(x) \subset W^s_{\text{loc}}(x)$, and $\zeta(x)$ contains an open neighbourhood of x in $W^s_{\text{loc}}(x)$ (with respect to the topology induced on the local stable manifold).

Let us fix an f-invariant borelian measure μ on Λ. Since we work with a uniformly hyperbolic endomorphism, we can *construct a measurable partition* ξ (with respect to μ) subordinated to the local stable manifolds, in the following way: first we know that there is a small $r_0 > 0$ such that for each $x \in \Lambda$ there exists a local stable manifold $W^s_{r_0}(x)$. Then, it is possible to take a countable partition $\mathcal{P}$ of Λ (modulo μ) with open sets, each having diameter less than r_0 and such that the boundary of each set from $\mathcal{P}$ has μ-measure zero (see for example [**7**]). Now, for every open set $U \in \mathcal{P}$, and $x \in U \subset \Lambda$, we consider the intersection between U and the unique local stable manifold going through x; denote this intersection by $\xi(x)$. It is clear that $\xi(x) = \xi(y)$ if and only if both x, y are in the same set $U \in \mathcal{P}$ and they are on the same local stable manifold $W^s_{r_0}(z)$ for some $z \in \Lambda$. Now, take the collection ξ of all the borelian sets $\xi(x)$, $x \in U$, $U \in \mathcal{P}$. We see easily that ξ is a partition of Λ (modulo sets of μ-measure zero) and that ξ is measurable, since $\mathcal{P}$ was assumed countable and, inside each member $U \in \mathcal{P}$, we can separate any two local stable manifolds with the help of a countable collection of ξ-sets (which are neighbourhoods of local stable manifolds). Therefore, we have concluded the construction of the measurable partition ξ which is subordinated to the local stable manifolds. Modulo a set of μ-measure zero, we have thus a partition with pieces of local stable manifolds, $\xi(x) \subset W^s_{r(y(x))}(y(x))$, $x \in \Lambda$. In fact, without loss of generality, we may assume that for each member $A \in \xi$ there exists some $x(A) \in \Lambda$ and $r(A) \in (0, r_0)$ so that $W^s_{r(A)/2}(x(A)) \cap \Lambda \subset A \subset W^s_{r(A)}(x(A)) \cap \Lambda$.

Remark 1. From the construction above it follows that, outside a set of μ-measure zero, the radius $r(A)$ can be taken to vary continuously, i.e. there exists a constant $\chi > 0$ such that for each x in a set of full μ-measure in Λ, there exists a neighbourhood $U(x)$ of x with $r(\xi(z))/r(\xi(z')) \le \chi$, $z, z' \in U(x)$.

Notation. In our uniformly hyperbolic setting, with the partition ξ constructed above, we denote the conditional measure μ_A by μ^s_A, for $W^s_{r(A)/2}(x(A)) \cap \Lambda \subset A \subset W^s_{r(A)}(x(A)) \cap \Lambda$, $A \in \xi$. We will also denote the set of centres $\{x(A), A \in \xi\}$ by S. In particular, if $\mu = \mu_s$, we denote the conditional measures by $\mu^s_{s,A}$ for $A \in \xi$, or by $\mu^s_{s,x}$ when $\xi(x) = A$ for μ_s-almost everywhere $x \in \Lambda$. Also, we shall denote the probability measure induced by μ_s on the factor space Λ/ξ by $(\mu_s)_\xi$.

Now, if f is a d-to-1 c-hyperbolic endomorphism on the basic set Λ, we showed in [**13**] that the stable dimension $\delta^s(x)$ at *any point* $x \in \Lambda$ is independent of x, and is equal to the unique zero of the pressure function $t \to P(t\Phi^s - \log d)$. Thus, we can talk in this case about the *stable dimension of* Λ and will denote it by δ^s.

THEOREM 1. *Let f be a smooth endomorphism on a Riemannian manifold M, and assume that f is c-hyperbolic on a basic set of saddle type Λ. Let us assume moreover that f is d-to-1 on Λ. Assume that $\Phi^s(y) := \log |Df_s(y)|$, $y \in \Lambda$, that δ^s is the stable dimension of Λ and that μ_s is the equilibrium measure of the potential $\delta^s \Phi^s$ on Λ. Then, the conditional measures of μ_s associated to the partition ξ, namely $\mu^s_{s,A}$, are geometric probabilities, i.e. for $(\mu_s)_\xi$-almost all points $\pi_\xi(A)$ of Λ/ξ (corresponding to sets $A \in \xi$), there exists a positive constant C_A such that*

$$C_A^{-1}\rho^{\delta^s} \le \mu^s_{s,A}(B(y,\rho)) \le C_A \rho^{\delta^s}, \quad y \in A \cap \Lambda, 0 < \rho < \frac{r(A)}{2}.$$

Proof. By using the partition ξ subordinated to local stable manifolds from above, we can associate conditional measures of μ_s, denoted by $\mu^s_{s,A}$, $A \in \xi$. We want to estimate the measure $\mu^s_{s,A}$ of a small arbitrary ball $B(y,\rho)$ centred at some $y \in A$, where $W^s_{r(A)/2}(x) \cap \Lambda \subset A \subset W^s_{r(A)}(x) \cap \Lambda$, $x = x(A)$.

Let us first consider an arbitrary set $f^n(B_n(z,\varepsilon))$, where we recall that $B_n(z,\varepsilon)$ denotes a Bowen ball, and where $\varepsilon > 0$ is arbitrary but small. This set (i.e. $f^n(B_n(z,\varepsilon))$) is actually a neighbourhood of the local unstable manifold $W^u_\varepsilon(\hat{f}^n z)$ corresponding to some prehistory $(f^n z, f^{n-1}z, \ldots, z, \ldots)$. We will estimate next the μ_s-measure of a cross section of a set $f^n(B_n(z,\varepsilon))$, i.e. an intersection of type

$$B(n,z;k,x;\varepsilon) := f^n(B_n(z,\varepsilon)) \cap B_k(x,\varepsilon),$$

for arbitrary $z, x \in \Lambda$ and positive integers n, k .

Now, let us estimate the μ_s-measure of $B(n,z;k,x,\varepsilon)$. Notice that $B(n,z;k,x;\varepsilon)$ is contained in $f^n(B_{n+k}(z,\varepsilon))$. Without loss of generality, we can assume that $z = x_{-n}$, i.e. that z itself is the unique n-preimage of x inside $B_n(z,\varepsilon)$; if not, then we can replace z by a point x_{-n} which is ε-shadowed by z up to order $n+k$, and thus the dynamical behaviour of z up to order $n+k$ will be the same as that of x_{-n}.

Let us denote the positive quantity $|Df^n_s(z)| \cdot \varepsilon$ by ρ. Since the endomorphism f is conformal on local stable manifolds, the diameter of the intersection $f^n(B_n(z,\varepsilon)) \cap W^s_r(f^n z)$ is equal to 2ρ.

Now, recall that we assumed without loss of generality that $f^n z = x$, and consider all the finite prehistories of the point x, in Λ. We will call then a ρ-*maximal prehistory* of x any finite prehistory $(x, x_{-1}, \ldots, x_{-p})$ so that $|Df^{p-1}_s(x_{-p+1})| \cdot \varepsilon \ge \rho$ but $|Df^p_s(x_{-p})| \cdot \varepsilon < \rho$. Clearly, given any prehistory $\hat{x} = (x, x_{-1}, \ldots)$ of x, there exists some positive integer $n(\hat{x},\rho)$ such that $(x, x_{-1}, \ldots, x_{-n(\hat{x},\rho)})$ is a ρ-maximal prehistory. Let us denote

$$\mathcal{N}(x,\rho) := \{n(\hat{x},\rho), \hat{x} \text{ prehistory of } x \text{ from } \Lambda\}.$$

We will consider now the various components of the p-preimages of $B(n,z;k,x;\varepsilon)$, when p ranges in $\mathcal{N}(x,\rho)$. We extend the stable diameter of $B(n,z;k,x;\varepsilon)$ in backward time until we reach a diameter of at most ε. As the maximum expansion in backward time is realized on the stable manifolds (local inverse iterates contract all the unstable directions), it follows that for any prehistory $\hat{x}$ of x, there exists a component of $f^{-n(\hat{x},\rho)}(B(n,z;k,x;\varepsilon))$ inside the Bowen ball $B_{n(\hat{x},\rho)}(x_{-n(\hat{x},\rho)},\varepsilon)$; denote this component by $A(\hat{x},\rho)$. We see that all these components $A(\hat{x},\rho)$ are mutually disjoint if $\varepsilon << \varepsilon_0$, where ε_0 is the local injectivity constant of f on Λ (recall that there are no critical points in Λ). Indeed, if the sets $A(\hat{x}, \rho)$ and $A(\hat{x}', \rho)$ would intersect for some prehistories $\hat{x} = (x, x_{-1}, \ldots)$, $\hat{x}' = (x, x'_{-1}, \ldots)$ of x then, since they are contained in Bowen balls, their forward iterates would be 2ε-close. But, then we get a contradiction since the prehistories $\hat{x}, \hat{x}'$ must contain different preimages $x_p, x - p'$ at some level p, and these different preimages must be at a distance of at least ε_0 from each other. Hence, either $A(\hat{x}, \rho) = A(\hat{x}', \rho)$ or $A(\hat{x}, \rho) \cap A(\hat{x}', \rho) = \emptyset$.

Now, we will use the f-invariance of the equilibrium measure μ_s in order to estimate the μ_s-measure of the set $B(n, z; k, x; \varepsilon)$. Recall that $f^n z = x$ and $\varepsilon|Df_s^n(z)| =: \rho$. Then, we have

$$\mu_s(B(n, z; k, x; \varepsilon)) = \sum_{\hat{x} \text{ prehistory of } x} \mu_s(A(\hat{x}, \rho)),$$

since we showed above that the sets $A(\hat{x}, \rho)$ either coincide or are disjoint.

Now, let us take two sets $A(\hat{x}, \rho)$, $A(\hat{x}', \rho)$, one of them with $n(\hat{x}, \rho) = p$ and the other with $n(\hat{x}', \rho) = p'$. We proved in [**13**] that for a d-to-1 c-hyperbolic endomorphism f on the basic set Λ, we have $\delta^s = t_d^s$, where t_d^s is the unique zero of the pressure function $t \to P(t\Phi^s - \log d)$. Therefore, we can use

$$P(\delta^s \Phi^s) = \log d. \tag{5}$$

Then, from the definition of $A(\hat{x}, \varepsilon)$ and by using Proposition 1 (since by taking n, z, k, x, ε appropriately, we can assume that the measure μ_s on the boundaries of $A(\hat{x}, \rho)$, $A(\hat{x}', \rho)$ is zero), we can compare the measure μ_s on two sets $A(\hat{x}, \rho)$, $A(\hat{x}', \rho)$ as follows:

$$\begin{aligned}\frac{1}{C_\varepsilon}\mu_s(A(\hat{x}', \rho))&\frac{|Df_s^p(x_{-p})|^{\delta^s}}{|Df_s^{p'}(x'_{-p'})|^{\delta^s}} \cdot d^{p'-p}\\ &\le \mu_s(A(\hat{x}, \rho)) \le C_\varepsilon \mu_s(A(\hat{x}', \rho))\frac{|Df_s^p(x_{-p})|^{\delta^s}}{|Df_s^{p'}(x'_{-p'})|^{\delta^s}} \cdot d^{p'-p}.\end{aligned} \tag{6}$$

In general, if for two variable quantities Q_1, Q_2, there exists a positive universal constant c such that $1/cQ_2 \le Q_1 \le cQ_2$, we say that Q_1, Q_2 are *comparable*, and will denote this by $Q_1 \approx Q_2$; the constant c is called the *comparability constant*.

But, from the definition of $n(\hat{x}, \rho)$, $n(\hat{x}', \rho)$ above (as being the length of the ρ-maximal prehistory along $\hat{x}$, respectively, $\hat{x}'$), and since $n(\hat{x}, \rho) = p$, $n(\hat{x}', \rho) = p'$, we obtain that $(x, x_{-1}, \ldots, x_{-p})$ and $(x', x'_{-1}, \ldots, x'_{-p'})$ are two ρ-maximal prehistories. So, there exists a constant $C > 0$ *independent* of $\hat{x}, \hat{x}'$ (for instance take $C = \sup_{y\in\Lambda}(1/|Df_s(y)|)$, as we assumed that f has no critical points in Λ), such that

$$\frac{1}{C}|Df_s^{p'}(x'_{-p'})| \le |Df_s^p(x_{-p})| \le C|Df_s^{p'}(x'_{-p'})|.$$

Therefore, from relation (6), we obtain

$$\frac{1}{C_\varepsilon}\mu_s(A(\hat{x}', \rho))d^{p'-p} \le \mu_s(A(\hat{x}, \rho)) \le C_\varepsilon \mu_s(A(\hat{x}', \rho))d^{p'-p}, \tag{7}$$

where we used the same constant C_ε as in equation (6), without loss of generality. Hence, the proof will now be reduced to a combinatorial argument about the different pieces/components of the preimages of various orders of $B(n, z; k, x; \varepsilon)$.

However, we assumed that every point from Λ has exactly d f-preimages inside Λ. We use equation (7) in order to compare the μ_s-measures of the different pieces $A(\hat{x}, \rho)$, which will then be added successively. Recall that one of these components $A(\hat{x}, \rho)$ is precisely $B_{n+k}(z, \varepsilon)$. The comparisons will always be made with respect to this component $B_{n+k}(z, \varepsilon)$. Let us order the integers from $\mathcal{N}(x, \rho)$ as

$$n_1 > n_2 > \cdots > n_T.$$

We shall add first the measures $\mu_s(A(\hat{x}, \rho))$ over all the sets corresponding to $\hat{x}$ with $n(\hat{x}, \rho) = n_1$, then over those prehistories with $n(\hat{x}, \rho) = n_2$, etc. And, we will use that any point from Λ has exactly d^m m-preimages belonging to Λ for any $m \geq 1$. Therefore, by such successive addition and by using equation (7), we obtain

$$\begin{aligned}\mu_s(B_{n+k}(z, \varepsilon)) \cdot d^n &\leq \mu_s(B(n, z; k, x; \varepsilon)) \\ &= \sum_{\hat{x} \text{ prehistory of } x} \mu_s(A(\hat{x}, \rho)) \leq \mu_s(B_{n+k}(z, \varepsilon)) \cdot d^n,\end{aligned}$$

with the positive constant C_ε independent of n, k, z, x.

We use now [**11**, Theorem 1] which gave estimates for equilibrium measures on Bowen balls, similar to those from the case of diffeomorphisms (see [**7**] for example); this was done by lifting to an equilibrium measure on $\hat{\Lambda}$. Hence, from the last displayed formula and equation (5), we obtain

$$\frac{1}{C_\varepsilon} \frac{|Df_s^{n+k}(z)|^{\delta^s}}{d^k} \leq \mu_s(B(n, z; k, x; \varepsilon)) \leq C_\varepsilon \frac{|Df_s^{n+k}(z)|^{\delta^s}}{d^k}. \tag{8}$$

Let us prove now that, if we vary z, x, k, n, then we can write any open (borelian) set in Λ as a union of mutually disjoint sets (modulo μ_s), of type $B(n, z; k, x; \varepsilon)$. Consider sets of type $B(n, z; k, x; \varepsilon) = f^n(B_n(z, \varepsilon)) \cap B_k(x, \varepsilon)$, with $f^n(z) = x$, and such that the stable side $\varepsilon|Df_s^n(z)|$ is comparable to the unstable side $\varepsilon|Df_u^k(x)|^{-1}$, i.e. more precisely such that

$$\frac{1}{\lambda_u}|Df_u^k(x)|^{-1} \leq |Df_s^n(z)| \leq \lambda_u |Df_u^k(x)|^{-1}, \tag{9}$$

where $\lambda_u := \sup_{y \in \Lambda} |Df_u(y)|$; such sets will be called *round*. Notice also that there exists a sufficiently large constant $M > 1$, independent of n, z, k, x, such that, if $r_n(z) := |Df_s^n(z)|/M$, then we have $B(x, r_n(z)) \subset B(n, z; k, x; \varepsilon) \subset B(x, M \cdot r_n(z))$. We see now from equation (9) and since $\mathcal{C}_f \cap \Lambda = \emptyset$ that if $B(n, z; k, x; \varepsilon)$ is round, then there exists a constant $C_1 > 0$ independent of n, z, x, k such that

$$C_1^{-1} k \leq n \leq C_1 k. \tag{10}$$

Now, consider some $\ell \in \mathbb{Z}$, for which there exists another round set $B(n + \ell, z'; k', x'; \varepsilon)$ with $f^{n+\ell}(z') = x' = x$ and stable side $\varepsilon|Df_s^{n+\ell}(z')|$ comparable with $\varepsilon|Df_s^n(z)|$, with a fixed comparability constant, namely

$$\inf_{y \in \Lambda}(|Df_s(y)|^2) \cdot |Df_s^n(z)| \leq |Df_s^{n+\ell}(z')| \leq \sup_{y \in \Lambda}(|Df_s(y)|^{-2}) \cdot |Df_s^n(z)|. \tag{11}$$

In fact, one sees from the uniform hyperbolicity of f, relation (9), (11) and $\mathcal{C}_f \cap \Lambda = \emptyset$ that $|k - k'|$ depends only on Df on Λ and that $|k - k'|$ is smaller than some universal constant $k_0 > 0$. Thus, by applying equations (8), (10) and (11), we obtain that there exists a constant $C_2 > 1$, independent of n, z, z', x, k, k', so that

$$C_2^{-1} \cdot \mu_s(B(n + \ell, z'; k', x'; \varepsilon)) \le \mu_s(B(n, z; k, x; \varepsilon)) \\ \le C_2 \cdot \mu_s(B(n + \ell, z'; k', x'; \varepsilon)). \tag{12}$$

In other words, μ_s is a doubling measure on Λ. Now, by varying n, we see that each point x from Λ is the centre of round sets $B(n, z; k, x; \varepsilon)$ having arbitrarily small diameters. Therefore, from equation (12), we can apply variants of the Vitali covering theorem (see [**6**, Theorems 2.8.7 or 2.8.17]), for the family of round sets $B(n, z; k, x; \varepsilon)$ which cover Λ finely with respect to μ_s; in these variants of the Vitali theorem, the covering sets are not necessarily balls. Therefore, we conclude that we can cover Λ, modulo μ_s, with a union of mutually disjoint sets $B(n, z; k, x; \varepsilon)$.

Now, let us study in more detail the conditions from the definition of conditional measures.

From the construction of the measurable partition ξ, we have that $W^s_{r(A)/2}(x) \cap \Lambda \subset A \subset W^s_{r(A)}(x) \cap \Lambda$, $x = x(A) \in S$ and the radii $r(A)$ vary continuously with A. So, from Remark 1, we can split an arbitrary set $U \in \mathcal{P}$, modulo μ_s, into a disjoint union of open sets V, each being a ξ-set, so there exists $r = r(V) > 0$ such that for all $A \in \xi$ intersecting V, we have $W^s_{r/2}(x(A)) \cap \Lambda \subset A \subset W^s_r(x(A)) \cap \Lambda$. Hence, locally, on a subset $V \subset U \in \mathcal{P}$, we can consider that ξ is, modulo a set of μ_s-measure zero, a foliation with local stable manifolds $W^s_r(x)$ of the same size $r = r(V)$. The intersections of these local stable manifolds with Λ are then identified with points in the factor space Λ/ξ.

We will work for the rest of the proof on an open set V as above, i.e. where the sets $A \in \xi$ can be assumed to be of type $W^s_r(x)$, of the same size $r = r(V)$. Take also $\varepsilon = r$.

Now, from the definition of the factor space Λ/ξ, the $(\mu_s)_\xi$-measure induced on the quotient space Λ/ξ is given by $(\mu_s)_\xi(E) = \mu_s(\pi_\xi^{-1}(E))$, where $\pi_\xi : \Lambda \to \Lambda/\xi$ is the canonical projection which collapses a set from ξ to a point. We notice that the projection $\pi_\xi(B(n, z; k, x; r))$ in Λ/ξ has $(\mu_s)_\xi$-measure equal to $\mu_s(B_k(x, r))$, since $\pi_\xi^{-1}(\pi_\xi(B(n, z; k, x; r))$ is $B_k(x, r)$. Now, since $P(\delta^s \Phi^s) = \log d$ (from relation (5)) and by using again the estimates of equilibrium states on Bowen balls, we obtain as in equation (8) that $\mu_s(B_k(x, r))$ is comparable to $|Df_s^k(x)|^{\delta^s}/d^k$ (with a comparability constant $c = c(V)$). Hence, from this argument, we obtain that

$$(\mu_s)_\xi(B_k(x, r)/\xi) = (\mu_s)_\xi(\pi_\xi(B(n, z; k, x; r)) = \mu_s(\pi_\xi^{-1}(\pi_\xi(B(n, z; k, x; r))) \\ = \mu_s(B_k(x, r)) \approx \frac{|Df_s^k(x)|^{\delta^s}}{d^k}, \tag{13}$$

with the comparability constant C_V. Now, by equation (8) and recalling that $f^n z = x$, and by taking $\rho := |Df_s^n(z)| r$, we obtain

$$\mu_s(B(n, z; k, x; r)) \approx \frac{|Df_s^k(x)|^{\delta^s}}{d^k} \cdot \rho^{\delta^s}, \tag{14}$$

where the comparability constant can be taken again as C_V (the size $r > 0$ is fixed for a fixed set V). So, from equations (13) and (14), we see immediately that

$$\frac{\mu_s(B(n, z; k, x; r))}{(\mu_s)_\xi(B_k(x, r)/\xi)} \approx \rho^{\delta^s}, \tag{15}$$

where $\rho = |Df_s^n(z)|r$. But, from the definition of conditional measures, we know that

$$\mu_s(B(n, z; k, x; r)) = \int_{B_k(x,r)/\xi} \mu_{s,A}^s(A \cap B(n, z; k, x; r))\, d(\mu_s)_\xi(\pi_\xi(A)). \tag{16}$$

Recall now that we showed above that any borelian set in Λ can be written, modulo μ_s, as a countable union of disjoint sets of type $B(n, z; k, x; r)$; and these sets form a basis for the open sets in V. Also, if we vary n, the radius $\rho = |Df_s^n(z)| \cdot r$ can be made arbitrarily small. Now, we have the essential uniqueness of the system of conditional measures associated to (μ_s, ξ) given in [**17**]. Consider some fixed arbitrary local unstable manifold $W_r^u(\hat{\zeta})$ which intersects any local stable manifold $A \subset V$ in some unique point $y = y_A$, from the local product structure of the basic hyperbolic set Λ; for instance, $\hat{\zeta}$ can be taken as a continuation of the finite prehistory $(f^n z, \dots, z)$, for the point z appearing in equation (16). Now, from equation (15), together with a Lebesgue type derivation theorem (see [**10**]) applied in formula (16) to the function $y_A \to \mu_{s,A}^s(B(y_A, \rho))$, $y_A \in \Lambda \cap W_r^u(\hat{\zeta})$, $y_A := \Lambda \cap W_r^u(\hat{\zeta}) \cap A$, we conclude that

$$\mu_{s,A}^s(B(y_A, \rho)) \approx \rho^{\delta_s},$$

for $(\mu_s)_\xi$-almost all points A in Λ/ξ. But, our $\rho := |Df_s^n(z)|r$ becomes arbitrarily small when $n \to \infty$; and, without loss of generality, by varying the unstable manifold $W_r^u(\hat{\zeta})$ (i.e. by varying z, n), we can take the point y arbitrarily inside A, since A is supposed to be the intersection of Λ with a local stable manifold. Thus, we obtain that $\mu_{s,A}^s$ satisfies a geometric probability condition with a constant C_V, i.e.

$$\frac{1}{C_V}\rho^{\delta^s} \le \mu_{s,A}^s(B(y, \rho)) \le C_V \rho^{\delta^s}, \quad y \in A, 0 < \rho < r/2,$$

for $(\mu_s)_\xi$-almost all $A \subset V$, $A \in \xi$. The comparability factor C_V is constant on V; in general, it can be taken locally constant on the complement in Λ of a set of μ_s-measure zero. The proof of the theorem is thus finished. □

Definition 4. Let f be a hyperbolic endomorphism on the folded basic set Λ, μ a borelian probability measure on Λ and ξ a measurable partition subordinated to local stable manifolds. Then, the conditional measure μ_A^s corresponding to $A \in \xi$ will be called *the stable conditional measure* of μ on A. When $\mu = \mu_s$, we denote this stable conditional measure by $\mu_{s,A}^s$.

Remark 2. We notice from the proof of Theorem 1 that, in fact, the stable conditional measures of μ_s do not depend on the measurable partition ξ constructed above, subordinated to local stable manifolds. Therefore, there exists a set $\Lambda(\mu_s)$ of full μ_s-measure inside Λ, such that for every $x \in \Lambda(\mu_s)$ there exists some small $r(x) > 0$ so that $W_{r(x)}^s(x)$ is contained in a set A from a measurable partition of type ξ (subordinated to local stable manifolds); then one can construct the stable conditional measure $\mu_{s,A}^s$. We denote this conditional measure also by $\mu_{s,x}^s$, $x \in \Lambda(\mu_s)$.

We recall now the notions of *lower*, respectively, *upper pointwise dimension* of a finite borelian measure μ on a compact space Λ (see for example [**1, 16**]). For $x \in \Lambda$, they are defined by

$$\underline{d}_\mu(x) := \liminf_{\rho\to 0} \frac{\log \mu(B(x,\rho))}{\log \rho} \quad \text{and} \quad \bar{d}_\mu(x) := \limsup_{\rho\to 0} \frac{\log \mu(B(x,\rho))}{\log \rho}.$$

If the lower pointwise dimension at x coincides with the upper pointwise dimension at x, we denote the common value by $d_\mu(x)$ and call it simply the *pointwise dimension* at x.

One can also define the *Hausdorff dimension, lower box dimension and upper box dimension* of μ respectively by

$$HD(\mu) := \inf\{HD(Z), \mu(\Lambda\setminus Z) = 0\},$$
$$\underline{\dim}_B(\mu) := \lim_{\delta\to 0} \inf\{\underline{\dim}_B(Z), \mu(\Lambda\setminus Z) \le \delta\},$$
$$\overline{\dim}_B(\mu) := \lim_{\delta\to 0} \inf\{\overline{\dim}_B(Z), \mu(\Lambda\setminus Z) \le \delta\}.$$

Assume now in general that f is a hyperbolic endomorphism on Λ and μ a probability measure on Λ, and let ξ be a measurable partition subordinated to local stable manifolds of f on Λ. We define then the *lower/upper stable pointwise dimension* of μ at y, for μ-almost everywhere $y \in \Lambda$, as the lower/upper pointwise dimension of the stable conditional measure μ^s_A at y, for $y \in A$, namely

$$\underline{d}^s_\mu(y) := \liminf_{\rho\to 0} \frac{\log \mu^s_A(B(y,\rho))}{\log \rho} \quad \text{and} \quad \bar{d}^s_\mu(y) := \limsup_{\rho\to 0} \frac{\log \mu^s_A(B(y,\rho))}{\log \rho}.$$

Similarly, we define the *stable Hausdorff dimension* of μ on $A \in \xi$, and the *stable lower/upper box dimension* of μ on A, respectively, as the quantities

$$HD^s(\mu, A) := HD(\mu^s_A), \quad \underline{\dim}^s_B(\mu, A) := \underline{\dim}_B(\mu^s_A),$$
$$\overline{\dim}^s_B(\mu, A) := \overline{\dim}_B(\mu^s_A), \quad A \in \xi.$$

When $\mu = \mu_s$, we denote $HD^s(\mu_s, x) := HD(\mu^s_{s,x})$, $\underline{\dim}^s_B(\mu_s, x) := \underline{\dim}_B(\mu^s_{s,x})$ and $\overline{\dim}^s_B(\mu_s, x) := \overline{\dim}_B(\mu^s_{s,x})$, for $x \in \Lambda(\mu_s)$.

Recall now the stable dimension δ^s from Definition 1 and the theorem of independence of the stable dimension given afterwards.

COROLLARY 1. *Let f be a c-hyperbolic, d-to-1 endomorphism on a basic set Λ, and μ_s be the equilibrium measure of the potential $\delta^s\Phi^s$. Then, the stable pointwise dimension of μ_s exists μ_s-almost everywhere on Λ and is equal to the stable dimension δ^s.*

Also, the stable Hausdorff dimension of μ_s, stable lower box dimension of μ_s and stable upper box dimension of μ_s are all equal to δ^s.

Proof. The proof follows from Theorem 1, since we proved that the stable conditional measures of the equilibrium measure μ_s are geometric probabilities.

For the second part of the corollary, we use [**1**, Theorem 2.1.6]. Indeed, since the stable conditional measures of μ_s are geometric probabilities of exponent δ^s, we conclude that the stable Hausdorff and lower/upper dimensions coincide, and are all equal to the stable dimension δ^s. □

Definition 5. We will say that a measure μ on Λ has *maximal stable dimension* on $A \in \xi$, $A \subset W^s_{r(x)}(x)$ if

$$HD^s(\mu, A) = \sup\{HD^s(\nu, A), \nu \text{ is an } f|_\Lambda\text{-invariant probability measure on } \Lambda\}.$$

This definition is similar to that of the measure of maximal dimension; see [**1, 2**], where measures of maximal dimension on hyperbolic sets of surface diffeomorphisms were studied. Our setting/methods for the maximal *stable* dimension in the non-invertible case are, however, different.

Now, since the stable Hausdorff dimension of any f-invariant probability measure ν on Λ is bounded above by $\delta^s := HD(W^s_r(x) \cap \Lambda)$, we see from Corollary 1 the following Corollary 2.

COROLLARY 2. *In the setting of Theorem 1, it follows that the stable equilibrium measure μ_s of f is of maximal stable dimension on $W^s_{r(x)}(x) \cap \Lambda$ among all f-invariant probability measures on Λ, for μ_s-almost everywhere $x \in \Lambda$. And, μ_s maximizes in a variational principle for stable dimension on Λ, i.e.*

$$\begin{aligned}\delta^s &= HD^s(\mu_s, x) = \sup\{HD^s(\nu, x), \nu \textit{ is an}\\ &\quad f|_\Lambda\textit{-invariant probability measure on } \Lambda\}, \quad \mu_s\textit{-a.e. } x\end{aligned}$$

We say now that the basic set Λ is a *repeller* (or *folded repeller*) if there exists a neighbourhood U of Λ such that $\bar{U} \subset f(U)$. And, that Λ is a *local repeller* if there are local stable manifolds of f contained inside Λ (see [**12**] for more on these notions in the case of endomorphisms).

COROLLARY 3. *Consider an open c-hyperbolic endomorphism f on a connected basic set Λ. Then, we have that the stable conditional measures $\mu^s_{s,x}$ of μ_s are absolutely continuous with respect to the induced Lebesgue measures on $W^s_{r(x)}(x)$, $x \in \Lambda(\mu_s)$ if and only if Λ is a non-invertible repeller.*

Proof. If f is open on a connected Λ, we saw in §1 that f is constant-to-1 on Λ.

The first part of the proof follows from Theorem 1 and from [**12**, Theorem 1]. Indeed, in [**12**] we showed that in the above setting, if none of the stable manifolds centred at x is contained in Λ, then δ^s is strictly less than the real dimension d_s of the manifold $W^s_{r(x)}(x)$ (the result in [**12**], given for the case when d_s is 2, can be generalized easily to other dimensions as long as the condition of conformality on stable manifolds is satisfied). Thus, in order to have absolute continuity of the stable conditional measures, we must have some local stable manifolds contained in Λ, equivalent to Λ being a local repeller (in the terminology of [**12**]). But, we proved in [**12**, Proposition 1] that when $f|_\Lambda : \Lambda \to \Lambda$ is open, then Λ is a local repeller if and only if Λ is a repeller.

The converse is clearly true since, if Λ is a repeller, then the local stable manifolds are contained inside Λ, and thus the stable dimension δ^s is equal to the dimension d_s of the manifold $W^s_{r(x)}(x)$.

Hence, from Theorem 1, it follows that the stable conditional measures of μ_s are geometric of exponent d_s; thus, they are absolutely continuous with respect to the respective induced Lebesgue measures. □

Let us give at the end some examples of c-hyperbolic endomorphisms which are constant-to-1 on basic sets, for which we will apply Theorem 1 and its corollaries.

Example 1. The first and simplest example is that of a product

$$f(z, w) = (f_1(z), f_2(w)), \quad (z, w) \in \mathbb{C}^2,$$

where f_1 has a fixed attracting point p and f_2 is expanding on a compact invariant set J. Then, the basic set that we consider is $\Lambda := \{p\} \times J$. For instance, take $f(z, w) = (z^2 + c, w^2)$, $c \neq 0$, $|c|$ small, on the basic set $\Lambda = \{p_c\} \times S^1$, where p_c denotes the unique fixed attracting point of $z \to z^2 + c$. The stable dimension here is equal to zero and the intersections of type $W^s_r(x) \cap \Lambda$ are singletons.

Example 2. We can take a hyperbolic toral endomorphism f_A on $\mathbb{T}^2$, where A is an integer-valued matrix with one eigenvalue of absolute value strictly less than 1, and another eigenvalue of absolute value strictly larger than 1. In this case we can take $\Lambda = \mathbb{T}^2$, and we have the stable dimension equal to 1. We see that f_A is $|\det(A)|$-to-1 on $\mathbb{T}^2$.

We may take also $f_{A,\varepsilon}$ a perturbation of f_A on $\mathbb{T}^2$. Then, again, $f_{A,\varepsilon}$ is $|\det(A)|$-to-1 on $\mathbb{T}^2$, and c-hyperbolic on $\mathbb{T}^2$. The stable dimension is equal to 1, but the stable potential Φ^s is not necessarily constant now. From Corollary 3, we see that the stable conditional measures of the equilibrium measure μ_s are absolutely continuous.

Example 3. We construct now examples of folded repellers which are not necessarily Anosov endomorphisms.

We remark first that if Λ is a repeller for an endomorphism f, with neighbourhood U so that $\bar{U} \subset f(U)$, then $f^{-1}(\Lambda) \cap U = \Lambda$. Therefore, if Λ is in addition connected, it follows easily that f is constant-to-1 on Λ. Let us show now that constant-to-1 repellers are stable under perturbations.

PROPOSITION 2. *Let Λ be a connected repeller for an endomorphism f so that f is hyperbolic on Λ, and consider a perturbation f_ε which is $\mathcal{C}^1$-close to f. Then, f_ε has a connected repeller Λ_ε close to Λ, and such that f_ε is hyperbolic on Λ_ε. Moreover, for any $x \in \Lambda_\varepsilon$, the number of f_ε-preimages of x belonging to Λ_ε is the same as the number of f-preimages in Λ of a point from Λ.*

Proof. Since Λ has a neighbourhood U so that $\bar{U} \subset f(U)$, it follows that for f_ε close enough to f, we will obtain $\bar{U} \subset f_\varepsilon(U)$. If f_ε is $\mathcal{C}^1$-close to f, then we can take the set $\Lambda_\varepsilon := \bigcap_{n\in\mathbb{Z}} f^n_\varepsilon(U)$, and it is quite well known that f_ε is hyperbolic on Λ_ε (see for example [**19**], etc).

We know that there exists a conjugating homeomorphism $H : \hat{\Lambda} \to \hat{\Lambda}_\varepsilon$ which commutes with $\hat{f}$ and $\hat{f}_\varepsilon$. The natural extension $\hat{\Lambda}$ is connected if and only if Λ is connected. Hence, $\hat{\Lambda}_\varepsilon$ is connected and so Λ_ε is also connected. Moreover, since $\bar{U} \subset f_\varepsilon(U)$, we obtain that Λ_ε is a connected repeller for f_ε.

Now, assume that $x \in \Lambda$ has d f-preimages in Λ. Then, if $C_f \cap \Lambda = \emptyset$ and if f_ε is $\mathcal{C}^1$-close enough to f, it follows that the local inverse branches of f_ε are close to the local inverse branches of f near Λ. Therefore, any point $y \in \Lambda_\varepsilon$ has exactly d f_ε-preimages in U, denoted by $y_1, \ldots, y_d$. Any of these f_ε-preimages from U has also an f_ε-preimage in U since $\bar{U} \subset f_\varepsilon(U)$, etc. Thus, $y_i \in \Lambda_\varepsilon = \bigcap_{n\in\mathbb{Z}} f_\varepsilon^n(U)$, $i = 1, \ldots, d$; hence, any point $y \in \Lambda_\varepsilon$ has exactly d f_ε-preimages belonging to the repeller Λ_ε. □

Let us now take the hyperbolic toral endomorphism f_A from Example 2, and the product $f(z, w) = (z^k, f_A(w))$, $(z, w) \in \mathbb{P}^1\mathbb{C} \times \mathbb{T}^2$, for some fixed $k \geq 2$. And, consider a $\mathcal{C}^1$-*perturbation* f_ε of f on $\mathbb{P}^1\mathbb{C} \times \mathbb{T}^2$. Since f is c-hyperbolic on its connected repeller $\Lambda := S^1 \times \mathbb{T}^2$, it follows from Proposition 2 that the perturbation f_ε also has a connected folded repeller Λ_ε, on which it is c-hyperbolic. Also, it follows from above that f_ε is constant-to-1 on Λ_ε, namely, it is $(k + |\det(A)|)$-to-1. The stable dimension $\delta^s(f_\varepsilon)$ of f_ε on Λ_ε is equal to 1 in this case.

We can form the stable potential of f_ε, namely $\Phi^s(f_\varepsilon)(z, w) := \log |D(f_\varepsilon)_s|(z, w)$, $(z, w) \in \Lambda_\varepsilon$, and the equilibrium measure $\mu_s(f_\varepsilon)$ of $\delta^s(f_\varepsilon) \cdot \Phi^s(f_\varepsilon)$, as in Theorem 1. Since the basic set Λ_ε is a repeller, we obtain from Corollary 3 that the stable conditional measures of $\mu_s(f_\varepsilon)$ are absolutely continuous on the local stable manifolds of f_ε (which in general are nonlinear submanifolds).

One actual example can be constructed by the above procedure, if we consider first the linear toral endomorphism $f_A(w) = (3w_1 + 2w_2, 2w_1 + 2w_2)$, $w = (w_1, w_2) \in \mathbb{R}^2/\mathbb{Z}^2$. The associated matrix A has one eigenvalue of absolute value less than 1 and the other eigenvalue larger than 1; hence, f_A is hyperbolic on $\mathbb{T}^2$. And, as above, we can take the product $f(z, w) = (z^k, f_A(w))$ for some $k \geq 2$.

Then, we consider the perturbation endomorphism

$$f_\varepsilon(z, w) := (z^k, 3w_1 + 2w_2 + \varepsilon \sin(2\pi(w_1 + 5w_2)), 2w_1 + 2w_2 + \varepsilon \cos(2\pi w_2) + \varepsilon \sin^2(\pi(w_1 - 2w_2))),$$

defined for $z \in \mathbb{P}^1\mathbb{C}$, $w \in \mathbb{T}^2$. We see that f_ε is well defined as an endomorphism on $\mathbb{P}^1\mathbb{C} \times \mathbb{T}^2$ and that it has a repeller Λ_ε close to $S^1 \times \mathbb{T}^2$, given by Proposition 2; namely, there exists a neighbourhood U of $S^1 \times \mathbb{T}^2$ so that

$$\Lambda_\varepsilon = \bigcap_{n\in\mathbb{Z}} f_\varepsilon^n(U).$$

Then, f_ε is c-hyperbolic on Λ_ε (see Definition 1) and it is $(k + 2)$-to-1 on Λ_ε. The stable potential $\Phi^s(f_\varepsilon)$ is not necessarily constant in this case.

We obtain as before that the stable conditional measures of $\mu_s(f_\varepsilon)$ are absolutely continuous, and that the stable pointwise dimension of $\mu_s(f_\varepsilon)$ is essentially equal to 1, on μ_s-almost all local stable manifolds over Λ_ε. □

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