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The set K^- for hyperbolic non-invertible maps

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Abstract. Any Axiom A holomorphic map on $\mathbb{P}^2\mathbb{C}$ has a natural ergodic measure μ. The set K^- is the set of points which have arbitrarily close neighbors with prehistories not convergent to supp μ.

This set is the analogue of the set of points with bounded backwards iterates from the case of Hénon mappings. For Hénon diffeomorphisms it was shown by Bedford and Smillie that K^- either has an empty interior or $\overset{\circ}{K}{}^- =$ union of the basins of finitely many repelling periodic points. For s-hyperbolic holomorphic endomorphisms of $\mathbb{P}^2$, we show here that the only possibility is $\overset{\circ}{K}{}^- = \emptyset$. This answers a question of Fornaess.

We also prove that $\overset{\circ}{K}{}^-$ is the union of finitely many repelling basins when the endomorphism is a perturbation of a Hénon mapping.

Several non-trivial examples of s-hyperbolic maps are discussed as well, some of them coming from perturbations of product maps and others from solenoids.

1. *Introduction and statements*

In [**BS**], Bedford and Smillie have studied the dynamics of the Hénon map on $\mathbb{C}^2$, $g(z, w) = (w, p(w) - az)$, where p is a monic polynomial of degree $d \geq 2$ and $a \neq 0$. This map defines a diffeomorphism of $\mathbb{C}^2$, of constant Jacobian.

By definition, $K^-(g) := \{(z, w) \in \mathbb{C}^2 : (g^{-n}(z, w))_n$ is bounded in $\mathbb{C}^2\}$ and $K(g) := \{(z, w) \in \mathbb{C}^2 : (g^{\pm n}(z, w))_n$ bounded in $\mathbb{C}^2\}$. It can be shown [**BS**] that $K^-(g) = W^u(K(g))$, i.e. $K^-(g) =$ the unstable set of $K(g) =$ union of global unstable sets of points from $K(g)$, and also that $K^-(g)$ is pseudoconvex.

In the case of diffeomorphisms, hyperbolicity over an invariant set is defined as a continuous splitting of the tangent bundle into stable and unstable bundles.

One then says that g is *hyperbolic if* g is hyperbolic on its Julia set $\mathcal{J}$. In this case, $W^u(\mathcal{J}) = \mathcal{J}^- \backslash \{s_1, \ldots, s_k\}$, where $\mathcal{J}^- = \partial K^-$ and $s_1, \ldots, s_k$ are the sinks (attracting periodic points) of g.

Bedford and Smillie [**BS**] also showed that if p is a saddle point of a hyperbolic polynomial diffeomorphism, then $\mathcal{J}^+ := \partial K^+$ is the closure of $W^s(p)$ and $\mathcal{J}^-$ is the closure of $W^u(p)$.

THEOREM. **[BS]** *If g is hyperbolic, then we have the following possibilities for $\overset{\circ}{K}{}^-$:*

(1) *if $|a| < 1$, then $\overset{\circ}{K}{}^- = \emptyset$;*

(2) *if $|a| = 1$, then $\overset{\circ}{K}{}^- = \overset{\circ}{K} = \emptyset$;*

(3) *if $|a| > 1$, then $\overset{\circ}{K}{}^- = \bigcup_{i=1}^m B(p_i)$, where $p_1, \ldots, p_m$ are repelling periodic points. Here $B(p_i) = \{z : f^{-n}(z) \to \mathcal{O}(p_i)$ orbit of $p_i\}$.*

Let us look now at the case of hyperbolic endomorphisms on $\mathbb{P}^2$. Since the map is not invertible anymore, the set K^- is not defined as in the case of diffeomorphisms. However, we can define it indirectly if f is holomorphic.

First, let us note that any holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$ can be lifted to $F : \mathbb{C}^3 \backslash \{0\} \to \mathbb{C}^3 \backslash \{0\}$, with $F = (F_1, F_2, F_3)$, where F_i are homogeneous polynomials, $i = \overline{1, 3}$.

The common degree d of F_1, F_2, F_3 is called the *degree of f*.

Definition 1. The non-wandering set $\Omega_f := \{x \in \mathbb{P}^2 : (\forall)$ neighborhood U_x of x, $\exists n \geq 1$ such that $f^n(U_x) \cap U_x \neq \emptyset\}$.

Ω_f is a compact set in $\mathbb{P}^2$ which will be denoted by Ω when no confusion arises.

Definition 2. The hyperbolic cover $\widehat{\Omega} := \{(x_n)_{n \leq 0} : x_n \in \Omega, (\forall) n$ and $f(x_n) = x_{n+1}, n \leq -1\}$.

$\widehat{\Omega}$ is a compact metric space with the metric $d(\hat{x}, \hat{y}) = \sum_{i \leq 0} d(x_i, y_i)/2^{|i|}$. $\widehat{\Omega}$ is also called the inverse limit of Ω.

Definition 3. f is hyperbolic on Ω if there exists a continuous splitting of $T_{\widehat{\Omega}}\mathbb{P}^2 := \{(\hat{x}, v) : v \in T_{x_0}\mathbb{P}^2, \hat{x} \in \widehat{\Omega}\}$, $T_{\hat{x}}\mathbb{P}^2 = E^s_{x_0} \oplus E^u_{\hat{x}}$, such that $Df(E^s_{x_0}) \subset E^s_{f x_0}$, $Df(E^u_{\hat{x}}) \subset E^u_{f\hat{x}}$ and $\exists \mathcal{C} > 0$, $\lambda > 1$ with $\|Df^n(x_0)(v)\| \leq \mathcal{C} \cdot \lambda^{-n} \cdot \|v\|$ for $v \in E^s_{x_0}$ and $\|Df^n(x_0)(v)\| \geq \mathcal{C} \cdot \lambda^n \cdot \|v\|$ for $v \in E^u_{\hat{x}}$. Given $x \in \Omega$, $\hat{x} \in \widehat{\Omega}$, $\hat{x} = (x_{-n})_{n \geq 0}$, $\varepsilon > 0$, we then define

$$W^s_\varepsilon(x) := \{y \in \mathbb{P}^2 : d(f^n y, f^n x) < \varepsilon, n \geq 0\}$$

$$W^u_\varepsilon(\hat{x}) := \{y \in \mathbb{P}^2 \text{ which has a prehistory } \hat{y} \text{ with } d(y_{-n}, x_{-n}) < \varepsilon, n \geq 0\}$$

$$W^s(x) = \{y : d(f^n y, f^n x) \to 0\}$$

$$W^u(\hat{x}) = \{y : \exists \hat{y} \text{ prehistory of } y \text{ with } d(y_{-n}, x_{-n}) \to 0\}.$$

If $\varepsilon > 0$ is small enough, the sets $W^s_\varepsilon(x)$, $W^u_\varepsilon(\hat{x})$ are complex manifolds.

If f is hyperbolic on its non-wandering set Ω, then $\Omega = S_0 \cup S_1 \cup S_2$, with S_i = set of points from Ω with unstable index i, $i = \overline{0, 2}$. If ω is the Kähler form on $\mathbb{P}^2$ (with $\int \omega \wedge \omega = 1$), Fornaess and Sibony **[FS2]** showed that $(f^n)^*\omega/d^n$ converge as currents to a positive closed current T whose support is equal to the Julia set of f.

From **[FS2]** it also follows that $\mu := T \wedge T$ is well-defined as a measure, it satisfies $f^*\mu = d^2\mu$ and is a measure of maximal entropy $\log d^2$.

THEOREM. **[FS2]** *If f holomorphic of degree d is hyperbolic on Ω, then:*

(i) *$\Omega_f \neq \mathcal{J}_f$, $\mathcal{J}_f \neq S_2$;*

(ii) *S_0 is the union of finitely many attracting periodic points;*

(iii) *if $S_\mu :=$ supp μ, then $S_\mu \subset S_2$, $f^{-1}(S_\mu) = S_\mu$.*

Definition 4. A holomorphic map f of degree d (the set of such maps is denoted by $\mathcal{H}_d$), is said to be *s-hyperbolic* if:

(i) f has Axiom A, i.e. f is hyperbolic on Ω and the periodic points of f, Per f, are dense in Ω;

(ii) $f^{-1}(S_2) = S_2$;

(iii) $\mathcal{C}_f \cap S_1 = \emptyset$, where $\mathcal{C}_f$ = critical set of f;

(iv) $\exists U$ neighborhood of S_1 such that $f^{-1}(S_1) \cap U = S_1$.

THEOREM. [**FS3**] *Given $f \in \mathcal{H}_d$ s-hyperbolic, $S_2 = S_\mu$ and the unstable set of S_2, $W^u(\widehat{S}_2)$ is open with locally pluripolar complement.*

Note. In [**FS3**], s-hyperbolicity is defined with slightly fewer restrictions; in (iii) they require an analytic set outside S_1, not necessarily equal to $\mathcal{C}_f$.

For our purposes, however, we prefer to work in the case $\mathcal{C}_f \cap S_1 = \emptyset$. Consider now a Hénon map on $\mathbb{C}^2$, $f(z, w) = (w, w^2 + az + c)$, $a \neq 0$ and $\tilde{f}[z : w : t] = [w^2 : w^2 + azt + ct^2 : t^2]$ its rational extension to $\mathbb{P}^2$.

It is proved in [**FS**] that $(\tilde{f}^n)^*\omega/2^n \to \tilde{\mu}^+$ and that $\tilde{\mu}^+ \wedge \tilde{\mu}^+ = \delta_{p_-}$, where $p_- = [1 : 0 : 0]$ is the only point of indeterminacy for $\tilde{f}$. $\tilde{\mu}^+$ is the analogue of the current T from the case when $f : \mathbb{P}^2 \to \mathbb{P}^2$ holomorphic.

p_- appears as the only 'repelling' point of f in the sense that $(\forall)U$ neighborhoods of p_- and $x \neq p_-$, $x \in U$, $\exists n_x \geq 1$ such that $\tilde{f}^{n_x} \notin U$, $n \geq n_x$. So for the Hénon map $f : \mathbb{C}^2 \to \mathbb{C}^2$, $K^- = \mathbb{C}^2 \backslash U^-$, with

$$U^- := \{z \in \mathbb{C}^2, (f^{-n}(z))_n \text{ converge to } p_- = \operatorname{supp} \delta_{p_-}\}.$$

However, $\delta_{p_-} = \tilde{\mu}^+ \wedge \tilde{\mu}^+$, so it makes sense to have the following definition.

Definition 5. Let $f \in \mathcal{H}_d$, a holomorphic endomorphism of $\mathbb{P}^2$. Then $U_f^- := \{z_0 \in \mathbb{P}^2$, $d(f^{-n}(z), \operatorname{supp} \mu) \to 0$, uniformly in a neighborhood of $z_0\}$.

Let $K_f^- := \mathbb{P}^2 \backslash U_f^-$. K_f^- will also be denoted simply by K^- when there is no danger of confusion. When f is s-hyperbolic, $K^- = \bigcup_{\hat{x} \in \widehat{S}_1} W^u(\hat{x}) \cup S_0$ and for small $\varepsilon > 0$, $\bigcup_{\hat{x} \in \widehat{S}_1} W^u_\varepsilon(\hat{x})$ contains a neighborhood of S_1 in $W^u(\widehat{S}_1)$ [**FS3**]. Also, the set K^- is connected. Consider now a hyperbolic Hénon mapping of $\mathbb{C}^2$ of the form $g(z, w) = (z^2 + aw, z)$, $|a| \gg 1$, and a perturbation of $f, f_\varepsilon(z, w) = (z^2 + aw, z + \varepsilon w^2)$. Let us define the following decomposition [**FS4**]:

$$\begin{aligned} R &= 10|a| + 3 \\ V &= \{(z, w), |z| \leq R, |w| \leq 8R\} \\ V^+ &= \{(z, w), |z| > R, |w| \leq 8|z|\} \\ V^- &= \{(z, w), |w| > 8R, 8|z| < |w|\}. \end{aligned}$$

PROPOSITION 1. *With the above notations, we have:*

(i) *$f_\varepsilon(V) \subset V \cup V^+$;*

(ii) *$f_\varepsilon(V^+) \subset V^+$ and f_ε multiplies the z-variable by at least three in V^+;*

(iii) *if $(z, w) \in V^-$, $|w| > 2/\varepsilon$, then $(z, w) \in$ basin of attraction of the line at infinity;*

(iv) *if $(z, w) \in V^-$, $|w| < 3/4\varepsilon$, then (z, w) is iterated forward until it lands in $V \cup V^+$;*

(v) *every point in V has at most one preimage in V.*

COROLLARY 1. *For a perturbation f_ε as above, the repelling periodic points are either in $\{(z, w) \in V^-, 3/4\varepsilon < |w| < 2/\varepsilon\}$ or in $\overset{\circ}{V}$.*

Proof. Indeed, if $(z, w) \in V^-, |w| > 2/\varepsilon$, then (z, w) is attracted to $(t = 0)$, so it cannot be periodic.

If $(z, w) \in V^-, |w| < 3/4\varepsilon$, then $\exists m$ such that $f^m(z, w) \in V \cup V^+ \Rightarrow (z, w)$ again cannot be periodic.

From Proposition 1, no periodic point belongs to V^+. Also $(t = 0)$ contains no repelling periodic point. So we can only have repelling periodic points in $\{(z, w) \in V^-, 3/4\varepsilon < |w| < 2/\varepsilon\}$ or in V. Now, if

$$|z| = R \Rightarrow |z^2 + aw| \geq R^2 - 8|a|R = R(R - 8|a|) > 3R,$$

hence no periodic point from V can have $|z| = R$.

Similarly no periodic point can have $|w| = 8R$. In conclusion, the repelling periodic points of f_ε belong either to $\{(z, w) \in V^-, 3/4\varepsilon < |w| < 2/\varepsilon\}$ or to $\overset{\circ}{V}$. □

PROPOSITION 2. *Let*

$$K_2 := \{(z, w) : f^n(z, w) \in \{(z', w'), |z'| < |w'|/8, 3/4\varepsilon < |w'| < 2/\varepsilon\}, (\forall)n\}.$$

Then f_ε is expanding on K_2 and $\text{supp}(\mu_\varepsilon) = K_2$ *(here μ_ε is the canonical measure of maximal entropy for f_ε).*

PROPOSITION 3. **[FS4]** *The non-wandering set Ω_ε of f_ε is the union of two disjoint closed sets $\Omega_1 := \Omega_\varepsilon \cap V$ and* $\text{supp}\,\mu_\varepsilon$. *The set Ω_ε does not intersect the critical set $\mathcal{C}$. Also, f_ε is bijective on Ω_1 and if $g(z, w) = (z^2 + aw, z)$ is a hyperbolic Hénon map, then f_ε is Axiom A on Ω_ε.*

COROLLARY 2. *If $f_\varepsilon(z, w) = (z^2 + aw, z + \varepsilon w^2), |a| \gg 1$, a perturbation of the hyperbolic Hénon map $g(z, w) = (z^2 + aw, z)$, then f_ε is with Axiom A on Ω_ε, but f_ε is not s-hyperbolic.*

Proof. The fact that f_ε is with Axiom A follows from Proposition 3. Because $|a| \gg 1$, g will have a finite number of repelling periodic points $\{p_1, \ldots, p_k\}$.

If $\varepsilon > 0$ small enough, then f_ε has repelling periodic points $p'_1, \ldots, p'_k$ close to $p_1, \ldots, p_k$, respectively, so $S_{2,\varepsilon} \supset \{p'_1, \ldots, p'_k\}$. However, from Proposition 2,

$$\text{supp}\,\mu_\varepsilon := S_{\mu_\varepsilon} \subset \{(z, w) \in V^-, 3/4\varepsilon < |w| < 2/\varepsilon\}.$$

Hence, if ε is small enough, $\{p'_1, \ldots, p'_k\} \cap S_{\mu_\varepsilon} = \emptyset \Rightarrow S_{2,\varepsilon} \neq S_{\mu_\varepsilon} \Rightarrow f_\varepsilon$ is not s-hyperbolic (according to the theorem from **[FS3]** cited above). □

In the case of small perturbations of hyperbolic Hénon mappings we will prove the following.

THEOREM 1. *For a perturbation $f_\varepsilon(z, w) = (z^2 + aw, z + \varepsilon w^2)$ of a hyperbolic Hénon map $g(z, w) = (z^2 + aw, z)$, we have that $\bigcup_{i=1}^k B_{p'_i} \subseteq \overset{\circ}{K}{}^-_\varepsilon \subset \overline{\bigcup_{i=1}^k B_{p'_i}}$. The line at infinity $(t = 0)$ is in ∂K^-_ε.*

This theorem, together with the next, will imply that there is a profound difference between perturbations of Hénon maps and s-hyperbolic maps, which translates in the structure of the set K^-.

THEOREM 2. *If f is an arbitrary s-hyperbolic holomorphic map on $\mathbb{P}^2$ (in the sense of Definition 4), then $\overset{\circ}{K}{}^- = \emptyset$.*

Proposition 4.3.2 of [**Mi**] gave a partial result in this direction, which is now improved by Theorem 2.

In [**Mi2**], using the concept of topological pressure we showed the following.

THEOREM. [**Mi2**] *Let f be an s-hyperbolic holomorphic endomorphism of $\mathbb{P}^2$ of degree d such that*

$$\sup_{\hat{y}\in\widehat{S}_1} \|Df|_{E^s_y}\| \cdot \|Df|_{E^u_{\hat{y}}}\|^{-1} < \inf_{x\in S_1} \|Df|_{E^s_x}\|^{\theta},$$

then both $HD(\widehat{W}^u(\widehat{S}_1))$ and $HD(K^-_f)$ are no greater than $2 + t^s/\theta$, with t^s being the only t such that the topological pressure $P(t\phi^s) = 0$.

In particular, if

$$\sup_{x\in S_1} |Df|_{E^s_x}| < \sqrt{\frac{1}{d}} \quad \textit{and} \quad \sup_{\hat{y}\in\widehat{S}_1} \|Df|_{E^s_{y_0}}\| \cdot \|Df|_{E^u_{\hat{y}}}\|^{-1} < \inf_{x\in S_1} \|Df|_{E^s_x}\|,$$

then $HD(K^-) < 4$.

So, if the conditions in the theorem hold, then, besides the conclusion $\overset{\circ}{K}{}^-_f = \emptyset$, we also have an estimate for $HD(K^-_f)$. We will end this section with some examples of Axiom A maps.

Example 1. Let $f[z : w : t] = [P(z,t) : Q(w,t) : t^d]$, P, Q hyperbolic on their Julia sets $\mathcal{J}_P, \mathcal{J}_Q$ (P, Q are taken to be polynomials of degree d in one variable). Then $S_2 = \mathcal{J}_P \times \mathcal{J}_Q$.

The basic sets of S_1 are, in $t = 1$, {periodic sinks of P} $\times \mathcal{J}_Q$ and $\mathcal{J}_P \times$ {periodic sinks of Q}; the basic set in $(t = 0)$ is the Julia set for $[P(z,0) : Q(w,0) : 0]$. Then f is clearly s-hyperbolic.

In this example the unstable manifolds for all prehistories in S_1 coincide. However, in general one would expect the unstable manifolds for different prehistories of a certain point to be distinct.

Example 2. If $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$, $\phi([z_0 : w_0], [z_1 : w_1]) = [z_0z_1 : w_0w_1 : z_0w_1 + w_0z_1]$, then ϕ gives a two to one cover of $\mathbb{P}^2$.

Let $f_0 : \mathbb{P}^1 \to \mathbb{P}^1$ rational hyperbolic map of degree d. Then $\exists f \in \mathcal{H}_d$ such that $\phi(f_0, f_0) = f \circ \phi$ and f is hyperbolic also. $S_2(f) = \phi(\mathcal{J}_0, \mathcal{J}_0)$ and the basic sets for $S_1(f)$ are ϕ (sink $\times \mathcal{J}_0$) so f is s-hyperbolic if $\mathcal{C}_f \cap S_1 = \emptyset$.

Example 3. The next example is a solenoid in $\mathbb{P}^2$, $f[z : w : t] = [z^2 : wt/10 + zt/2 + \varepsilon w^2 : t^2]$ (see also [**Ro**] for the original solenoid example in the real case).

The line at infinity ($t = 0$) is totally invariant and an attracting set. S_1 has three basic sets: a circle in ($t = 0$), a quasicircle in ($z = 0$) and a solenoid in $\{|z| = 1, |w| < 1 + \delta\}$, $\delta > 0$, small.

Now $S_2 \subset \{|z| = 1, t = 1\}$ and the eigenvalues r of the derivative $Df(z, w)$ are $2z$ and $1/10 + 2\varepsilon w$.

If $|z| = 1$, then $|2z| = 2$. If $|w| < 0.9/2\varepsilon$, then $|1/10+2\varepsilon w| < 1/10+2\varepsilon(0.9/2\varepsilon) = 1$ hence f not expanding on S_2, thus a contradiction.

On the other hand, if $|w| < 3/4\varepsilon \Rightarrow |w/10 + z/2 + \varepsilon w^2| \leq |w|(1/10 + \varepsilon|w|) + 0.5 < 0.85|w| + 0.5$.

Now, for small enough ε, $|w| > 4 \Rightarrow 0.5 < |w|/8 \Rightarrow |w_1| \leq 0.85|w| + |w|/8 = 0.975|w|$, where we used the notation $f(z, w) = (z_1, w_1)$. So, in this case $(z, w) \notin \Omega$. Therefore, if $(z, w) \in S_2$, then $|w| > 3/4\varepsilon$, $|z| = 1$.

Assume also that for $(z, w) \in S_2$, $|w| > 1.5/\varepsilon \Rightarrow |w/10 + z/2 + \varepsilon w^2| \geq \varepsilon|w|^2 - |w|/10 - |z|/2 > 1.3|w|$, so $(z, w) \notin \Omega$, a contradiction.

Hence, we have shown that $S_2 \subset \{(z, w) \in \mathbb{C}^2, |z| = 1, 3/4\varepsilon < |w| < 1.5/\varepsilon\}$.

Also, one can notice immediately that $S_2 \neq \emptyset$, because the point $p = (1, (0.9 + \sqrt{0.9^2 - 2\varepsilon})/2\varepsilon)$ is a repelling fixed point. The eigenvalues of the derivative of f at p are, in absolute value, two and $1 + \sqrt{0.9^2 - 2\varepsilon}$ and both of them are larger than one for small ε.

Now denote $\mathcal{A} := \{(z, w) \in \mathbb{C}^2, f^n(z, w) \in \{|z| = 1, 3/4\varepsilon \leq |w| \leq 1.5/\varepsilon\}, \forall n \geq 0\}$. It is obvious that $f(\mathcal{A}) = \mathcal{A}$. Also, if (z, w) is such that $f(z, w) \in \mathcal{A}$, then $|z| = 1$ and if $|w| < 3/4\varepsilon$, then $|w_1| < 0.975|w| < 3/4\varepsilon$, a contradiction. It was also shown before that if $|w| > 1.5/\varepsilon \Rightarrow |w_1| > 1.2|w|$, again a contradiction. In conclusion, $f^{-1}(\mathcal{A}) = \mathcal{A} \Rightarrow \mathcal{A}$ is totally invariant.

Our goal is to show that $\mathcal{A} = S_2$. First, we will prove that f is expanding on $\mathcal{A}$. $Df(z, w)(u, v) = (2zu, u/2 + (1/10 + 2\varepsilon w)v)$, and for $(z, w) \in \mathcal{A}$, $\|Df(z, w)(u, v)\| \geq 2|u|$.

Assume first that $2|u| > 1.1|v| \Rightarrow \|Df(z, w)(u, v)\| > 1.1|v|$. If, on the contrary, it happens that $2|u| \leq 1.1|v| \Rightarrow |u/2 + (1/10 + 2\varepsilon w)v| \geq |1/10 + 2\varepsilon w| \cdot |v| - |u|/2 \geq |1/10 + 2\varepsilon w| \cdot |v| - 1.1|v|/4 = |v|(|1/10 + 2\varepsilon w| - 1.1/4)$.

However, $|w| \geq 3/4\varepsilon \Rightarrow 2\varepsilon|w| > 1.5 \Rightarrow |1/10 + 2\varepsilon w| > 1.4$, hence $|u/2 + (1/10 + 2\varepsilon w)v| > 1.1|v|$ and $|2zu| > 1.1|u|$. This shows that f is expanding on $\mathcal{A}$.

So we have $S_\mu \subset S_2 \subset \mathcal{A}$ and $\mathcal{A}$ is a totally invariant set on which f is expanding. Hence, if we show that $S_2 = \mathcal{A}$, S_2 will also be totally invariant.

In order to do this, we will prove that $\mathcal{A} \subset \overline{\mathrm{Per}\, f}$, in other words that the repelling periodic points are dense in $\mathcal{A}$. This is proved by a classical argument, which is detailed in [**FS4**]. The only part which is different in our case is the fact that there is a neighborhood U of $\mathcal{A}$ so that $U \cap (C \cup f(C) \cup f^2(C) \cup \cdots) = \emptyset$, with C the critical set of f.

Indeed, since f expanding on $\mathcal{A}$ and $f^{-1}(\mathcal{A}) = \mathcal{A}$, it follows that $\exists U$, a neighborhood of $\mathcal{A}$ with $f^{-1}(U) \subset\subset U$. We can arrange to have $U \cap C = \emptyset$ because clearly $C \cap \mathcal{A} = \emptyset$. So, if $f^k(C) \cap U \neq \emptyset \Rightarrow C \cap U \neq \emptyset$, a contradiction.

The rest of the argument consists of showing that any limit of iterates of a given point is itself a limit of repelling periodic points and then that preimages of repelling periodic points can be approximated by periodic orbits like in Theorem 3.8 of [**FS4**]. In conclusion, since repelling periodic points are dense in $\mathcal{A}$, $\mathcal{A} = S_2$.

Let us notice that $S_1 = \{\text{circle in } t = 0\} \cup \{\text{quasicircle in } z = 0\} \cup \{\text{solenoid in } \{|z| = 1, |w| < \delta\}\}$. Hence, the set $A := \{z = 2t\}$ does not intersect S_1. Also, since f is injective in a neighborhood of the solenoid, it follows that $\exists V$ a neighborhood of S_1 such that $f^{-1}(S_1) \cap V = S_1$.

Recapping, we showed that f satisfies the following conditions:

(i) f is Axiom A;

(ii) $f^{-1}(S_2) = S_2$;

(iii) $\exists$ an algebraic variety A such that $A \cap S_1 = \emptyset$;

(iv) $\exists$ a neighborhood V of S_1 such that $f^{-1}(S_1) \cap V = S_1$.

Hence f is s-hyperbolic in the sense of [**FS3**].

Example 4. The last example is a perturbation of $(z^2 + c, w^2 + d)$, c, d small.

If c, d are small enough, then $P(z) := z^2 + c$ and $Q(w) := w^2 + d$ are both hyperbolic on their Julia sets and these Julia sets are quasicircles close to the unit circle.

Let $f_\varepsilon(z, w) := (P(z) + \varepsilon w + \varepsilon z, Q(w))$. The fixed points of P are $(1 + \sqrt{1 - 4c})/2$ and $(1 - \sqrt{1 - 4c})/2$. Denote by p_0 a fixed attracting point of P. f_ε will have a basic set close to $\{p_0\} \times J_Q$. Denote $K := \sup_{w \in J_Q} |w| + 2|p_0|$ and $\alpha := \sup_{(z,w)\in\Lambda_\varepsilon} |z - p_0|$ and assume the supremum in the definition of α is attained at z_0. Hence $\exists (z, w) \in \Lambda_\varepsilon$ such that $z_0 = z^2 + c + \varepsilon w + \varepsilon z$.

In this case, $z_0 - p_0 = z^2 + c - p_0^2 - c + \varepsilon w + \varepsilon z = (z - p_0)^2 + 2p_0(z - p_0) + \varepsilon w + \varepsilon z \Rightarrow \alpha \leq \alpha^2 + 2|p_0|\alpha + K\varepsilon$. Now α must be very small, so from the quadratic expression it follows that

$$0 < \alpha \leq \frac{1 - 2|p_0| - \sqrt{(1 - 2|p_0|)^2 - 4K\varepsilon}}{2}$$
$$= \frac{2K\varepsilon}{1 - 2|p_0| + \sqrt{(1 - 2|p_0|)^2 - 4K\varepsilon}} \leq K'\varepsilon,$$

with the constant K' close to $K/(1 - 2|p_0|) < 2$.

We will prove that f_ε is injective on Λ_ε. Indeed let us assume that $f_\varepsilon(z_{-1}, w_{-1}) = f_\varepsilon(z'_{-1}, w'_{-1})$. Then $P(z_{-1}) + \varepsilon w_{-1} + \varepsilon z_{-1} = P(z'_{-1}) + \varepsilon w'_{-1} + \varepsilon z'_{-1}$, therefore

$$(z_{-1} - z'_{-1})(z_{-1} + z'_{-1}) + \varepsilon(z_{-1} - z'_{-1}) = \varepsilon(w'_{-1} - w_{-1}).$$

Denote $K^\star := \sup |w_{-1} - w'_{-1}|$ when $w_{-1} \neq w'_{-1}$ are two points in J_Q with $Q(w_{-1}) = Q(w'_{-1})$; since ε is small, $K^\star$ is close to two.

So $\varepsilon K^\star \leq |(z_{-1} - z'_{-1})(z_{-1} + z'_{-1}) + \varepsilon(z_{-1} - z'_{-1})| \leq 5|p_0|K'\varepsilon + 2\varepsilon^2 K'$, which is a contradiction if c is chosen in the beginning such that $K^\star > 6|p_0|K'$. (p_0 depends on c and if c is small, then p_0 will be small too and at the same time, K, K', $K^\star$ are smaller than 1.5, 2 and 2.5, respectively.) In conclusion $w_{-1} = w'_{-1}$.

However, we assumed that $P(z_{-1}) + \varepsilon w_{-1} + \varepsilon z_{-1} = P(z'_{-1}) + \varepsilon w'_{-1} + \varepsilon z'_{-1}$, therefore

$$z_{-1}^2 - z'^2_{-1} = \varepsilon(z'_{-1} - z_{-1}) \Rightarrow |z_{-1} + z'_{-1}| = \varepsilon,$$

which is again a contradiction because for very small ε in comparison to p_0, z_{-1} and z'_{-1} are both close to p_0, hence $|z_{-1} + z'_{-1}|$ close to $2|p_0|$ which would give $2|p_0|$ close to ε, a contradiction. Therefore, on the basic set Λ_ε, f_ε is injective.

The other basic set in $(t = 1)$ is $\Lambda_{\varepsilon,2} := J_2 \times \{q_0\}$, with q_0 an attracting fixed point for Q and J_2 the Julia set of the rational function $z \to P(z) + \varepsilon q_0 + \varepsilon z$. However, from the one complex variable theory [**CG**], J_2 is totally invariant, therefore $f_\varepsilon^{-1}(\Lambda_{\varepsilon,2}) = \Lambda_{\varepsilon,2}$. This implies that condition (iv) of Definition 4 is satisfied.

The condition $f_\varepsilon^{-1}(S_2) = S_2$ is satisfied because the map $(z^2 + c, w^2 + d)$ is Axiom A without cycles; the fact that there exists a neighborhood W of S_2 such that $f^{-1}(W) \subset\subset W$ is invariant to small perturbations and because it holds for $(z^2 + c, w^2 + d)$, it will hold also for f_ε [**Mi2**]. Therefore f_ε is s-hyperbolic.

Finally, note that while for $f(z, w) = (P(z), Q(w))$, the unstable manifolds are easy to find, this is not the case for a general perturbation f_ε.

The results of the general theorem will still apply for maps for which the individual unstable manifolds are impossible to find analytically. *A priori*, a hyperbolic endomorphism will have an uncountable collection of unstable manifolds going through each of its saddle points, corresponding to the different prehistories of those points.

2. *Proofs of main results*

First we will make clear the definition of a basin of a repelling periodic point for an endomorphism.

LEMMA 1. *Assume $f : \mathbb{P}^2 \to \mathbb{P}^2$ holomorphic map which has a repelling periodic point at p; $\mathcal{O}(p) :=$ orbit of p. Denote $B_p := \{z \in \mathbb{P}^2 : z$ has a prehistory $(z_{-n})_{n\geq 0} : z_{-n} \longrightarrow_{n\to\infty} \mathcal{O}(p)\}$. Then B_p is open in $\mathbb{P}^2$.*

Proof. Without loss of generality, we can assume p is fixed. $\exists$, $V W$ neighborhoods of p such that $W \subset V$ and $f^{-1}(W) \cap V \subset\subset W$. Now, if $z \in B_p \Rightarrow \exists n_0$ with $z_{-n_0} \in W$, $z_{-n_0} \in f^{-n_0}(z)$. However, then $\exists U$ small neighborhood of z such that $(\forall) z' \in U$, $\exists z'_{-n_0} \in f^{-n_0}(z') \cap W$; then we use the fact that $f^{-1}(W) \cap V \subset\subset W$ to obtain that $U \subset B_p$. □

In the case of endomorphisms, B_p and $B_{p'}$ might intersect, even if $\mathcal{O}(p) \neq \mathcal{O}(p')$.

We now consider a Hénon map $g(z, w) = (z^2 + aw, z)$, $|a| \gg 1$ and a perturbation $f_\varepsilon(z, w) = (z^2 + aw, z + \varepsilon w^2)$ which extends as a holomorphic map on $\mathbb{P}^2$.

We have the partition V, V^+, V^- for $\mathbb{C}^2$ given in §1.

LEMMA 2. *$f_\varepsilon(V) \subset V \cup \mathcal{A}$, where $\mathcal{A} := \{(z, w) \in \mathbb{C}^2 : |z| > R, |w| \leq 8|z|, |z| > 4\varepsilon|w|^2\}$, if ε is small enough. The set $\mathcal{A}$ is invariant by f_ε. Hence $f_\varepsilon^n(V) \subset V \cup \mathcal{A}$, $(\forall) n \geq 0$.*

Proof. We already know from Proposition 1 that $f_\varepsilon(V) \subset V \cup V^+$. If $(z, w) \in V \Rightarrow |w| \leq 8R$, $|z| \leq R$ and if it happens that $f_\varepsilon(z, w) = (z_1, w_1) \in V^+ \Rightarrow |z_1| \geq 1/8|w_1| > 4\varepsilon|w_1|^2$ for small enough ε. Hence $f_\varepsilon(V) \subset V \cup \mathcal{A}$.

It remains to prove that $f_\varepsilon(\mathcal{A}) \subset \mathcal{A}$. If $(z, w) \in \mathcal{A}$, then

$$|z^2 + aw| \geq |z|^2 - |aw| \geq |z|^2 - 8|a||z|$$

and

$$|z + \varepsilon w^2|^2 \leq |z|^2 + \varepsilon^2|w|^4 + 2\varepsilon|z||w|^2.$$

So, the conclusion follows if we show that

$$|z|^2 - 8|a||z| > 4\varepsilon(|z|^2 + \varepsilon^2|w|^4 + 2\varepsilon|z||w|^2)$$
$$|z|^2(1-4\varepsilon) - 8|a||z| > 4\varepsilon^3|w|^4 + 8\varepsilon^2|z||w|^2. \tag{1}$$

However,

$$4\varepsilon|w|^2 < |z| \Rightarrow 16\varepsilon^2|w|^4 < |z|^2$$

and

$$8\varepsilon^2|z||w|^2 < 2\varepsilon|z|^2 \Rightarrow 4\varepsilon^3|w|^4 + 8\varepsilon^2|z||w|^2$$
$$< \frac{\varepsilon|z|^2}{4} + 2\varepsilon|z|^2 = \varepsilon|z|^2\left(\frac{1}{4}+2\right) = \frac{9}{4}\varepsilon|z|^2 < 0.1|z|^2,$$

for $\varepsilon < 0.1(4/9)$.

Now $|z|^2(1-4\varepsilon) - 8|a|\,|z| > 0.1|z|^2$, since $|z|(0.9-4\varepsilon) > 8|a|$ due to the fact that $|z| > R = 10|a| + 3$.

So this shows that inequality (1) is true $\Rightarrow f_\varepsilon(\mathcal{A}) \subset \mathcal{A}$. □

Proof of Theorem 1. It is obvious that, if $\varepsilon > 0$ is small enough, then f_ε will have repelling periodic points $p'_1, \ldots, p'_k$ close to the repelling periodic points of g ($p_1, \ldots, p_k$), and they do not belong to $S_{\mu,\varepsilon}$ (Corollary 1 and Proposition 2).

Also, if $\xi \in B_{p'_i}$, then $\exists(\xi_{-n})_{n\geq 0}$ prehistory of ξ, $\xi_{-n} \to \beta_i$ and the same is true for ξ' in a small neighborhood of ξ. So $\bigcup_{i=1}^{k} B_{p'_i} \subset \overset{\circ}{K}{}^-_\varepsilon$.

Now let $\mathcal{U}$ be a small open set $\subset K^-_\varepsilon$. From Proposition 1, $K^-_\varepsilon \subset V \cup V^+$. However, $f_\varepsilon(K^-_\varepsilon) = K^-_\varepsilon$. $\mathcal{U}$ is covered by $f_\varepsilon(K^-_\varepsilon)$. Now $f^{-1}(\mathcal{U}) \cap K^-_\varepsilon$ has a non-empty interior, because $f(f^{-1}(\mathcal{U}) \cap K^-_\varepsilon) = \mathcal{U}$. However, points in $f^{-1}(\mathcal{U}) \cap K^-_\varepsilon \cap V^+$ have smaller z-coordinates than their images in $\mathcal{U} \cap \mathcal{A}$. In general, $\mathcal{U}$ is covered by $f^n_\varepsilon(K^-_\varepsilon) \Rightarrow f^n(f^{-n}(\mathcal{U}) \cap K^-_\varepsilon) = \mathcal{U} \Rightarrow f^{-n}(\mathcal{U}) \cap K^-_\varepsilon$ has a non-empty interior.

However, if $\mathcal{U} \subset K^-_\varepsilon \cap \overset{\circ}{V}{}^+$, then $f^{-n}(\mathcal{U}) \cap K^-_\varepsilon$ is attracted towards V (since from Proposition 1, f_ε multiplies the z-coordinate of a point in V^+ by at least three).

Note also that if $|z| = R$, $(z, w) \in V$, then

$$f_\varepsilon(z, w) = (z^2 + aw, z + \varepsilon w^2) \Rightarrow |z^2 + aw| \geq R^2 - 8|a|R = R(R - 8|a|) > 3R,$$

so in fact $f^{-n}(\mathcal{U}) \cap K^-_\varepsilon$ would eventually be in V. Hence we would have a non-empty open set in $K^-_\varepsilon \cap V$. Without loss of generality, again denote this open set by $\mathcal{U}$; hence $\mathcal{U} \subset K^-_\varepsilon \cap V$, $\mathcal{U} \neq \emptyset$. Let us take $\xi_0 \in \mathcal{U} \subset K^-_\varepsilon \cap V$.

Suppose that for some $n_0 \geq 1$, $f^{-n_0}(\xi_0) \subset V^-$. In this case $\exists U^*$ small neighborhood of ξ_0 such that $U^* \subset \mathcal{U}$ and $(\forall)\xi \in U^*$, $f^{-n_0}(\xi) \in V^- \Rightarrow f^{-n}(\xi_0) \to S_\mu$, a contradiction to $\mathcal{U} \subset K^-_\varepsilon$. So for each $n_0 \geq 1$, $f^{-n_0}(\xi_0) \cap V \neq \emptyset$.

Now, from property (v) of Proposition 1, $|f^{-1}(\xi_0) \cap V| = 1$; there is only one point, denoted by ξ_{-1}, in $f^{-1}(\xi_0) \cap V$. If

$$z \in f^{-2}(\xi_0) \cap V \Rightarrow f^2(z) = \xi_0 \Rightarrow f(f(z)) = \xi_0 \Rightarrow f(z) \in f^{-1}(\xi_0) \Rightarrow f(z) = \xi_{-1} \in V$$

and, by the same property, there is only one z in $f^{-1}(\xi_{-1}) \cap V$.

We denote this by ξ_{-2}. Inductively, one shows that ξ_0 has a unique prehistory $(\xi_{-n})_{n\geq 0}$ contained in V. Therefore, on $\mathcal{U}$ we can define inverse iterates $(f_\varepsilon^{-n})_{n\geq 0}$ which take values in V. However, V is bounded $\Rightarrow (f_\varepsilon^{-n})_{n\geq 0}$ is a normal family. Assume that $\exists \xi \in \mathcal{U}$ such that $f_\varepsilon^{-n_i}(\xi) \to S_1, \varepsilon$. By eventually taking a subsequence, $f_\varepsilon^{-n_i} \to h$, h holomorphic on $\mathcal{U}$. So, it follows that for any tangent vector v at ξ, $(\|(Df_\varepsilon^{-n_i})_\xi(v)\|)_i$ is bounded. However, this is a contradiction to the existence of stable cones in the tangent bundle near S_1, therefore

$$f_\varepsilon^{-n}(\xi) \longrightarrow \bigcup_{i=1}^{k} \mathcal{O}(p_i').$$

So $\overset{\circ}{K}{}_\varepsilon^- \cap V \subset \bigcup_{i=1}^k B_{p_i'}$.

In general, if $\widetilde{\mathcal{U}}$ is a small open set in $\overset{\circ}{K}{}_\varepsilon^- \Rightarrow f^{-n}(\widetilde{\mathcal{U}}) \cap K_\varepsilon^- \subset V$ for large enough n and we prove that $\operatorname{int}(f^{-n}(\widetilde{\mathcal{U}}) \cap K_\varepsilon^-) \subset \bigcup_{i=1}^k B_{p_i'}$. Using that f is an open map it will now follow that $\overset{\circ}{K}{}_\varepsilon^- \subset \overline{\bigcup_{i=1}^k B_{p_i'}}$ (here we also use the fact that in our case K_ε^- does not have isolated points).

It remains to prove that $(t = 0) \subset \partial K_\varepsilon^- = K_\varepsilon^- \setminus \overset{\circ}{K}{}_\varepsilon^-$. First of all $(t = 0)$ is totally invariant and because $S_{\mu,\varepsilon} \cap (t = 0) = \emptyset$, it follows that $(t = 0) \subset K_\varepsilon^-$. Assume that $\exists$ a point $[z : w : 0] \in \overset{\circ}{K}{}_\varepsilon^-$ with $w \neq 0$. Since $K_\varepsilon^- \subset V \cup V^+$, we have $|w| \leq 8|z|$, hence $|z| \neq 0$.

However, $f_\varepsilon(K_\varepsilon^-) = K_\varepsilon^-$ and f_ε is open $\Rightarrow f_\varepsilon^n[z : w : 0] \in \overset{\circ}{K}{}_\varepsilon^-$, $n \geq 1$. However, $f_\varepsilon^n[z : w : 0] \to [1 : 0 : 0] \Rightarrow \exists$ a point $[1 : \tilde{w}_0 : 0]$, $\tilde{w}_0 \neq 0$ small, belonging to $\overset{\circ}{K}{}_\varepsilon^-$.

Consider a small neighborhood of $[1 : \tilde{w}_0 : 0]$ in K_ε^-; points in this neighborhood from $\mathbb{C}$ will be of the form $[1 : w_0 : t]$, $t \neq 0$ small and w_0 close to $\tilde{w}_0 \neq 0$.

So $[1 : w_0 : t] = [1/t : w_0/t : 1] =: [z_0 : w_0' : 1]$. Assume now that $(z, w) \in f_\varepsilon^{-1}(z_0, w_0')$

$$\begin{cases} z^2 + aw = z_0 \\ z + \varepsilon w^2 = w_0'. \end{cases} \tag{$*$}$$

Since $z_0 = 1/t$ is very large, $(z, w) \notin V$. Assume that $(z, w) \in \mathcal{A} \Rightarrow |z| > 4\varepsilon|w|^2$. From $(*)$ it will follow that

$$\tfrac{5}{4}|z| > |w_0'| > \tfrac{3}{4}|z|. \tag{2}$$

Also, $|z_0| = |z^2 + aw| \leq |z|^2 + 8|a|\,|z| < 1.1|z|^2$, because if $|z_0| \leq |z|^2 + 8|a||z| \Rightarrow |z|$ is large enough such that $8|a| < 0.1|z|$. So $|z_0| < 1.1|z|^2$.

Also, $|z^2 + aw| \geq |z|^2 - |a||w| \geq |z|^2 - 8|a||z| > 0.9|z|^2$ by the same argument as above. So

$$0.9|z|^2 < |z_0| < 1.1|z|^2 \Rightarrow \sqrt{\frac{|z_0|}{0.9}} > |z| > \sqrt{\frac{|z_0|}{1.1}}.$$

From (2) it follows $(5/(4 \cdot \sqrt{0.9}))\sqrt{|z_0|} > |w_0'| > (3/(4 \cdot \sqrt{1.1}))\sqrt{|z_0|}$. Writing that $w_0' = w_0/t$ and

$$z_0 = \frac{1}{t} \Rightarrow \frac{5}{4\sqrt{0.9}}\sqrt{\frac{1}{|t|}} > \left|\frac{w_0}{t}\right| > \frac{3}{4\sqrt{1.1}}\sqrt{\frac{1}{|t|}}, \quad 2\sqrt{|t|} > |w_0| > 0.6\sqrt{|t|}. \tag{3}$$

However, w_0 is arbitrarily close to $\tilde{w}_0 \neq 0$, which is fixed. So the above inequality (3) gives a contradiction because t is arbitrarily small (for example $|t| < |w_0|^2/4$ is enough). In this case, $f_\varepsilon^{-1}[1 : w_0 : t] \cap (\mathcal{A} \cup V) = \emptyset$. From Lemma 2, however, $f_\varepsilon^n(V) \subset \mathcal{A} \cup V$, $n \geq 1$,

$$f_\varepsilon^{-1}[1 : w_0 : t] \cap f_\varepsilon^n(V) = \emptyset \Rightarrow f_\varepsilon^{-n-1}([1 : w_0 : t]) \cap V = \emptyset, \quad n \geq 1.$$

Using that $f_\varepsilon^{-n}([1 : w_0 : t]) \cap V = \emptyset$ and the fact that, on V^+, f_ε multiplies the z-coordinate by at least three, hence $f_\varepsilon^{-n}[1 : w_0 : t]$ cannot accumulate in V^+, it follows that locally, $f_\varepsilon^{-n}[1 : w_0 : t] \longrightarrow_n S_\mu$. Hence $[1 : w_0 : t] \in U_\varepsilon^-$, a contradiction to $[1 : w_0 : t] \in K_\varepsilon^-$. So $(t = 0)\backslash[1 : 0 : 0] \subset \partial K_\varepsilon^-$.

Finally, let us assume $[1 : 0 : 0] \in \overset{\circ}{K}{}_\varepsilon^-$ and consider nearby points of the form $[1 : 0 : t]$, t small $[1 : 0 : t] = [1/t : 0 : 1]$ and if

$$(z, w) \in f^{-1}\left(\frac{1}{t}, 0\right) \Rightarrow \begin{cases} z^2 + aw = \dfrac{1}{t}, \\ z + \varepsilon w^2 = 0 \Rightarrow z = -\varepsilon w^2. \end{cases}$$

If t is small enough, $(z, w) \notin V$. If $(z, w) \in \mathcal{A} \Rightarrow |z| > 4\varepsilon|w|^2$ a contradiction to $z = -\varepsilon w^2$.

So

$$f_\varepsilon^{-1}[1 : 0 : t] \cap f_\varepsilon^n(V) = \emptyset \Rightarrow f_\varepsilon^{-n}[1 : 0 : t] \cap V = \emptyset, \quad n \geq 2.$$

However, $f_\varepsilon^{-n}[1 : 0 : t]$ cannot accumulate in V^+ either, which implies $f_\varepsilon^{-n}[1 : 0 : t] \to S_\mu$. If $f_\varepsilon^{-n_0}[1 : 0 : t] \subset V^-$, then $f_\varepsilon^{-n_0}(W) \subset V^-$ for a small neighborhood W of $[1 : 0 : t] \Rightarrow f_\varepsilon^{-n}(W) \subset V^-$, $(\forall)n \geq n_0 \Rightarrow W \subset U_\varepsilon^-$, a contradiction to the fact $[1 : 0 : t] \in K_\varepsilon^-$.

In conclusion, we obtain that the entire line at infinity $(t = 0)$ is in ∂K_ε^-. □

We are now ready to prove Theorem 2 for s-hyperbolic maps. The idea of the proof is to show that if $\overset{\circ}{K}{}^- \neq \emptyset$, then $(\forall)x \in \Lambda$, a basic set of S_1, there is a small complex disk contained in Λ, going through x. By Kontinuitätsatz, this would imply that the complement of Λ, $\mathbb{P}^2 \setminus \Lambda$ is a domain of holomorphy in $\mathbb{P}^2$. However, we know that because f is s-hyperbolic, $\mathcal{C}_f \cap S_1 = \emptyset \Rightarrow \mathcal{C}_f \subset \mathbb{P}^2\backslash\Lambda$ and we can use now a theorem of Takeuchi [**Ta**] saying that a domain of holomorphy in $\mathbb{P}^2$ cannot contain an analytic variety of dimension ≥ 1. This gives the contradiction, therefore $\overset{\circ}{K}{}^- = \emptyset$. This is detailed below.

Proof of Theorem 2. If f is s-hyperbolic, $f : \mathbb{P}^2 \to \mathbb{P}^2$, then a theorem of Fornaess and Sibony [**FS3**] states that $K^- = \bigcup_{\hat{x} \in \widehat{S}_1} W^u(\hat{x}) \cup S_0$.

So $K^- = W^u(\widehat{S}_1) \cup S_0$. If $\overset{\circ}{K}{}^- \neq \emptyset \Rightarrow \mathrm{int}(W^u(\widehat{S}_1)) \neq \emptyset$. However, $W^u(\widehat{S}_1) = \bigcup_{n \geq 0} f^n(W_\varepsilon^u(\widehat{S}_1))$, $\varepsilon > 0$. Then, from the Baire Category Theorem it would follow that for some $n \geq 1$, $\mathrm{int}(f^n(W_\varepsilon^u(\hat{S}_1))) \neq \emptyset \Rightarrow \mathrm{int}(W_\varepsilon^u(\hat{S}_1)) \neq \emptyset$ because f is open.

Denote by Λ a basic set of S_1 such that $W_\varepsilon^u(\hat{\Lambda})$ has an interior. Then let $x_0 \in \Lambda$ and U open, $U \subset K^-$, $U \subset B(x_0, \varepsilon)$. Denote by $F_x := (E_x^s)^\perp$ the orthogonal of the stable space at a point $x \in \Lambda$ and by $p_{x,\perp}$ the projection on $W_\varepsilon^s(x)$ along F_x. Assume also that $p_{x_0,\perp}(U) \supset B_s(x_0, \varepsilon)$, where $B_s(x_0, \varepsilon)$ is the ball of radius ε inside $W_{2\varepsilon}^s(x_0)$.

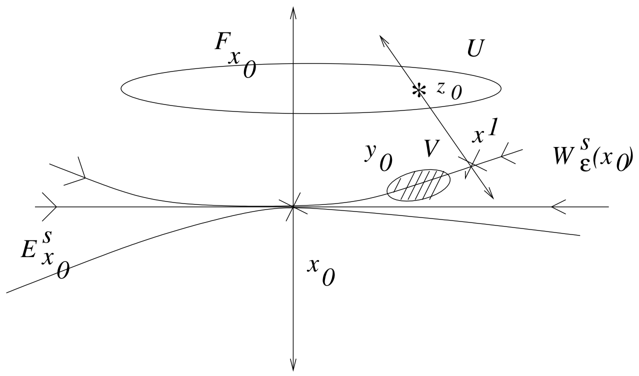

FIGURE 1.

Now take a small $V = B(y_0, \varepsilon')$ open inside $W^s_{2\varepsilon}(x_0)$ such that $d(x_0, y_0) < \varepsilon/100$, for example.

Our goal is to prove that $V \cap \Lambda \neq \emptyset$. In this way, it would follow that Λ is dense in $W^s_\varepsilon(x_0)$ hence $W^s_\varepsilon(x_0) \subset \Lambda$.

There exists a point $z_0 \in U$ such that $p_{x_0,\perp}(z_0) = y_0$ and since $z_0 \in W^u_\varepsilon(\hat{\Lambda})$ this implies $\exists$ an unstable local manifold going through z_0, which intersects $W^s_{2\varepsilon}(x_0)$ at a point x^1.

From the local product structure, $x^1 \in S_1 \Rightarrow x^1 \in \Lambda$. Let us assume that $x^1 \notin V$, otherwise we are finished. In general, the map $\hat{\Lambda} \ni \hat{x} \to W^u_\varepsilon(\hat{x})$ is continuous and $\hat{\Lambda}$ is compact, so the inclination of unstable manifolds w.r.t. stable spaces is always bounded away from zero and infinity.

So $\exists \alpha > 0$, fixed constant such that $d(x^1, y_0) < \alpha \cdot d(z_0, W^s_{2\varepsilon}(x_0))$. Without loss of generality, we assume that $\alpha \cdot d(z_0, W^s_{2\varepsilon}(x_0)) < d(x_0, y_0)$; if not, by taking preimages, this will eventually happen.

We consider now all the points in $f^{-1}(x^1) \cap S_1$, together with their stable sets. Denote by f_i^{-1} the branches of f^{-1} near x^1 (remember that $C_f \cap S_1 = \emptyset$)

$$d(y_{-1,i}, x^1_{-1,i}) \leq \frac{1}{\lambda_s} d(x^1, y_0), \quad \text{where } x^1_{-1,i} := f_i^{-1}(x^1), \, y_{-1,i} := f_i^{-1}(y_0)$$

with

$$0 < \lambda_s < \inf_{x \in \Lambda} \|Df|_{E^s_x}\| \quad (\text{since } \mathcal{C}_f \cap \Lambda = \emptyset).$$

Now, although not all $f_i^{-1}(U)$ are in K^-, $\exists i$ such that $\exists z^1_i \in W^u_\varepsilon(\hat{\Lambda})$ with $p_{x^1_{-1,i},\perp}(z^1_i) = y_{-1,i}$. Then an unstable local manifold containing z^1_i will intersect $W^s_{2\varepsilon}(x^1_{-1,i})$ in $x^2_i \in \Lambda$, and $d(x^2_i, y_{-1,i}) < \alpha\varepsilon/\lambda_u$, where $1 < \lambda_u < \inf_{\hat{x} \in \hat{\Lambda}} \|Df|_{E^u_{\hat{x}}}\|$.

We do this for all j, $z^1 \in W^u_\varepsilon(\widehat{S}_1)$ such that $p_{x^1_{-1,j,\perp}}(z^1) = y_{-1,j}$. If $U_{-1,i} := f_i^{-1}(U)$, then $p_{x^1_{-1,i}}(U_{-1,i}) \supset B(x^1_{-1,i}, \beta \cdot (1/\lambda_s)(\varepsilon - \beta' d(x_0, y_0)))$, where $\beta, \beta' > 0$ fixed constants determined by the maximum inclination of unstable vector spaces w.r.t. stable spaces. We repeat the above procedure, taking now $\{x^2_{-1,j}\}_j = \{f^{-1}(x^2_i)\}_i \cap \Lambda$, for all x^2_i's and $\{y_{-2,i}\}_i = f^{-2}(y_0) \cap \Lambda$.

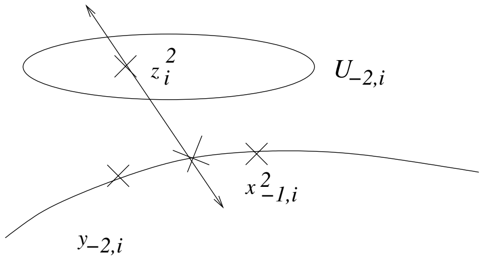

FIGURE 2.

If $U_{-2,i} = f_i^{-2}(U)$, then

$$d(U_{-2,i}, W^s_{x_i^2}) < \frac{\varepsilon}{\lambda_u^2} \cdot d(x_i^2, y_{-2,i}) < \alpha \frac{\varepsilon}{\lambda_u \lambda_s}$$

and

$$p_\perp(U_{-2,i}) \supset B\left(x^2_{-1,i}, \beta \cdot \frac{1}{\lambda_s}(d_1 - \beta' d(y_{-1,i}, x^1_{-1,i}))\right),$$

where d_1 is the diameter of $U_{-1,i}$, in the sense that d_1 is the radius of the projection of $U_{-1,i}$ along the respective unstable manifold.

We take now all the preimages of points $x^2_{-1,i}$ which are in Λ.

Over some of the $y_{-2,i}$'s there are points $z_i^2 \in K^-$ such that $p_{x^2_{-1,i,\perp}}(z_i^2) = y_{-2,i}$. This is true since $f^{-2}(U)$ is extending in the stable direction at the same rate as $f^{-2}(V)$.

We take all these points $z_i^2 \in K^-$ which exist over points in $f^{-2}(y_0)$. So, at the third step, $d(y_{-3,j}, x^3_{-1,j}) < \alpha\varepsilon/\lambda_u^2\lambda_s$, $d(U_{-3,j}, W^s_{2\varepsilon}(x^3_{-1,j})) < \varepsilon/\lambda_u^3$, and $p_\perp(U_{-3,j})$ covers $y_{-3,j}(\forall)j$. However, in this way we have considered all the points in $f^{-3}(y_0) \cap \Lambda$ and so $\exists j$ and $z_j^3 \in K^-$ with $p_\perp(z_j^3) = y_{-3,j}$ and clearly $z_j^3 \in W^u_{\varepsilon/\lambda_u^3}(\widehat{S}_1)$.

The procedure is repeated for large enough $n \geq 1$, inductively, hence $d(U_{-n,j}, W^s_{x^n_{-1,j}}) < \varepsilon/\lambda_u^n$ and

$$f^{-n}(V) \supseteq \bigcup_j B_s\left(y_{-n,j}, \frac{\operatorname{diam} V}{\lambda_s^n}\right)$$

(B_s denotes the ball in the stable local manifold).

The last inclusion will hold for all n for which $B_s(y_{-n,j}, \operatorname{diam} V/\lambda_s^n)$ makes sense, i.e. until the diameter of this stable ball is between $2\varepsilon/\lambda_s$ and 2ε.

Also, by definition $y_{-n,j} \in \Lambda$ and $\exists z_j^n \in K^-$ over some of the $y_{-n,j}$ (in the sense of orthogonal projection on the stable manifolds); also $z_j^n \in W^u_{\varepsilon/\lambda_u^n}(\widehat{S}_1)$. From the construction, x_j^{n+1} is the intersection between an unstable manifold going through z_j^n, and $W^s_{2\varepsilon}(x^n_{-1,j})$; z_j^n was chosen such that $p_\perp(z_j^n) = y_{-n,j}$. So $x_j^{n+1} \in \Lambda$ since S_1 has product structure and since $p_\perp(z_j^n) = y_{-n,j}$, we obtain that $d(x_j^{n+1}, y_{-n,j}) < \alpha\varepsilon/\lambda_u^n$. But this implies that eventually, for some n appropriate, $n = n(V)$,

$$x_j^{n+1} \in B_s\left(y_{-n,j}, \frac{1}{\lambda_s^n} \cdot \operatorname{diam} V\right) \cap S_1.$$

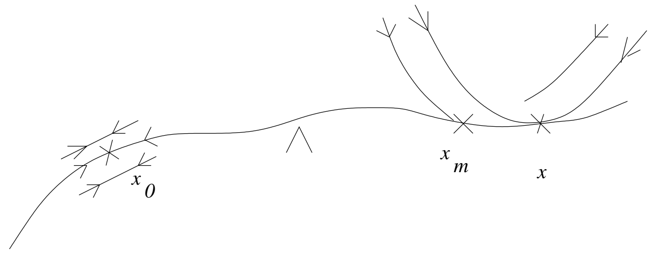

FIGURE 3.

However, $f^{-n}(V) \supseteq \bigcup_j B_s(y_{-n,j}, (\text{diam}\, V/\lambda_s^n))$. Hence we can find a point $\xi := x_j^{n+1} \in S_1 \cap f^{-n}(V)$, therefore

$$f^n(\xi) \in S_1 \cap V.$$

In conclusion, given V as an arbitrarily small open set in $W^s_\varepsilon(x_0)$ we can find in it a point from Λ, therefore

$$\Lambda \text{ is dense in } W^s_\varepsilon(x_0).$$

However, $\Lambda = \bar{\Lambda} \Rightarrow W^s_\varepsilon(x_0) \subset \Lambda$. The same argument will also show that $W^s_\varepsilon(x_0') \subset \Lambda$, $(\forall)x_0' \in B(x_0, \varepsilon/2)$. It only remains to prove that $W^s_\varepsilon(x) \subset \Lambda$, $(\forall)x \in \Lambda$.

Indeed, let $x \in \Lambda$ be arbitrary and $m \geq 1$. Let $B(x, 1/2^m)$; because $f|_\Lambda$ is mixing, $\exists N(m)$ such that $f^{N(m)}(B(x, 1/2^m)) \cap B(x_0, \varepsilon/2) \neq \emptyset$.

So $\exists x_m \in B(x, 1/2^m)$, such that $f^{N(m)}(x_m) \in B(x_0, \varepsilon/2)$; then $f^{N(m)}(W^s_\varepsilon(x_m)) \subseteq W^s_\varepsilon(f^{N(m)}(x_m)) \subset \Lambda$.

Now we know that f is s-hyperbolic which implies $\exists \varepsilon_0 > 0$ such that $\varepsilon_0 > 4\varepsilon$ and $f^{-1}(\Lambda) \cap B(\Lambda, \varepsilon_0) \subset \Lambda$. Therefore, if $f^{N(m)}(W^s_\varepsilon(x_m)) \subset \Lambda$ it follows that $f^{N(m)-1}(W^s_\varepsilon(x_m)) \subset \Lambda$; inductively $W^s_\varepsilon(x_m) \subset \Lambda$. However, m was chosen arbitrarily, so if we take a sequence $x_m \to x$, then $(\forall)m$, $W^s_\varepsilon(x_m) \subset \Lambda$.

Because the local stable manifolds depend continuously on their base point, $W^s_\varepsilon(x_m) \to W^s_\varepsilon(x)$ and since $W^s_\varepsilon(x_m) \subset \Lambda, \forall m, \Rightarrow W^s_\varepsilon(x) \subset \Lambda$. Hence, through each point x of Λ, $\exists$ a small complex disk, $W^s_\varepsilon(x)$ contained in Λ and then, by Kontinuitätsatz, $\mathbb{P}^2 \setminus \Lambda$ is a domain of holomorphy. Now a theorem of Takeuchi [**Ta**] says that a domain of holomorphy in $\mathbb{P}^n$, $n \geq 2$, cannot contain a compact complex variety of positive dimension.

In this way we will obtain a contradiction; indeed by the definition of s-hyperbolic maps, $\mathcal{C}_f \cap \Lambda = \emptyset$, hence the complex variety $\mathcal{C}_f$ is a subset of $\mathbb{P}^2 \setminus \Lambda$, which was shown to be a domain of holomorphy, therefore a contradiction with Takeuchi's result.

So we arrive at the conclusion of the theorem, $\overset{\circ}{K}{}^{-} = \emptyset$. $\square$

It would be an interesting open problem also to study the change in the structure of $K^-_{f_t}$, where $t \to f_t$ is a path connecting a small perturbation of a Hénon map with an s-hyperbolic endomorphism.

Another problem would be to see whether the dimension estimates in the Theorem from [**Mi2**], stated before the examples, are still true if one drops the additional assumptions on the stable derivative and to find out what makes the equality between $HD(W^s_\varepsilon(x) \cap \Lambda)$ and t^s break down for s-hyperbolic endomorphisms.

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