Habilitation Thesis

Smooth Ergodic Theory and Dimension in Dynamical Systems

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Contents:

Abstract

In the last 30 years, one of the most successfull developments in the study of smooth dynamical systems has been the use of thermodynamical formalism for problems in fractal dimension and ergodic theory.

Most of the research directions in this thesis are about smooth ergodic theory and thermodynamic formalism, with applications to dimension theory in hyperbolic dynamics, conformal and complex dynamics, and statistical properties of various types of measure-preserving systems.

This thesis presents my research achievements, after obtaining my PH.D in Mathematics at the University of Michigan (USA) in May 1999, under the supervision of Prof. J.E Fornaess.

After finishing the PH.D, I held academic positions for various durations, at several prestigious universities and institutes abroad; among these universities/institutes: Texas A&M Univ, USA, 1999-2001; Institut des Hautes Études Scientifiques, Paris, November-December 2013 and Aug-Sept 2014; Univ. of Bremen, Germany, March-May 2012; Max Planck Institut, Bonn, May 2011; Univ. of North Texas, February-March 2007; Instituto de Matematica Pura e Aplicada (IMPA), Brazil, February 2008; Erwin Schrödinger Math Institute, Vienna, Austria, June 2008; Scuola Normale Superiore, Pisa, Italy, May 2004; Univ. of North Texas, USA, 2002-2003, etc.

This thesis contains some of the results I published after my PH.D, in over 34 research papers, most of them in well-known ISI journals, such as Math. Annalen (2013, 1999), International Math Res Notices (2013), Ergodic Theory Dynam Syst (2011, 2011, 2002), Discrete and Cont Dynam Syst (2012, 2008, 2006, 2001), Math Zeitschrift (2011), Bulletin London Math Soc (2010), Commun Contemp Math (2004), J Stat Physics (2013, 2011, 2010), Mathematical Proceed Cambridge (2010), Monatshefte Math (2012), Canadian J Math (2008), Proceed American Math Soc (2013, 2011), Nonlinear Analysis (2010), Houston J Math (2005), etc. Together with M. Urbanski (U. North Texas), we also published a section of the book on "Fractal Geometry and Dynamics in Pure and Applied Mathematics", from the American Mathematical Society series Contemporary Mathematics. I also had invited articles in Discrete Cont. Dynam. Syst., Oberwolfach Reports, Rév. Roum. Math. Pures Appl., and in Ann. Univ. Bucharest.

The summary of the chapters of the thesis is the following.

In Chapter 2 we will give some of the theory of hyperbolicity for endomorphisms. In the case of non-invertible maps, the dynamical behaviour is very different than in the case of diffeomorphisms and there appear new phenomena for which one has to create new methods. For endomorphisms, the local unstable manifolds do not form a foliation, there may be uncountably many through a point (see [72]), and they may intersect both inside and outside the fractal Λ.

We will explain some of the new examples of endomorphisms with strong non-invertible behaviour, and the consequences of hyperbolicity in their case. A family of such examples, which present uncountably many unstable manifolds through certain points, and for which the stable dimension also behaves differently than in the diffeomorphism case, was given in [41]. In that paper we found a Newhouse-type phenomenon for intersections of Cantor sets in fibers, which translates into

a class of skew products which are far from being homeomorphism on the respective basic sets, but also far from being constant-to-1. Our class of examples is significant since it gives endomorphisms on actual fractals (non-Anosov), and involves a completely different technique to obtain examples with many unstable manifolds through a given point, than the toral examples given by Przytycki in [72]. We present also the case of holomorphic endomorphisms on $\mathbb{P}^2\mathbb{C}$, and of complex disks given by stable/unstable manifolds; a particular case is when the map is s-hyperbolic.

In Chapter 3, we will present several results about thermodynamical formalism and statistical properties for invariant measures on folded fractals. In this theory, we employed frequently equilibrium (Gibbs) measures of Hölder continuous potentials, which were shown to have good properties on iterates of Bowen balls. The Sinai-Ruelle-Bowen (or SRB) measures were introduced in the ground-breaking works of Sinai, Ruelle and Bowen and in the case of hyperbolic attractors represent the physically observable measures that give the asymptotic distribution of forward trajectories of Lebesgue-almost all points in some neighbourhood of the attractor. We introduced in [46] the new notion of inverse SRB measures for non-invertible dynamics, and studied it in the framework of equilibrium measures. Inverse SRB measures are not simply SRB measures for inverse function, since in this case the functions are not invertible. Instead they give asymptotic distributions of prehistories, and satisfy also a modified Pesin entropy formula.

We explain also some of our results about Jacobians of equilibrium measures, and about mixing of any order and decay of correlations for equilibrium measures. We will explain also some results which solve the problem of negative entropy production for measure-preserving endomorphisms on Lebesgue spaces, and which are related to the work of Ruelle.

In Chapter 4 we will present some results related to applications of thermodynamic formalism in dimension theory for non-invertible mappings.

We show that the problem of stable dimension is very different than in the diffeomorphism case, and give perturbation examples with a strange dynamical behaviour, obtained in [60]. We also give applications of a new notion of inverse pressure introduced by Mihailescu and Urbanski. Together with Urbanski and Stratmann, we solved the difficult dimension problem (see related problems of Verjovsky-Wu, Manning-McCluskey) for the stable intersections, in relation to the preimage counting function ([57], from [52]). In these papers, careful combinatorial study, together with smooth ergodic theory methods are employed in order to prove that if the stable dimension takes its minimal, resp. maximal value as zero of the pressure, then the number of preimages remaining in Λ is maximal, resp. minimal; the methods of proof are however completely different.

We then give applications of a notion of transversality for parametrized families of hyperbolic skew products; our result is related to some papers of Peres, Simon, Solomyak, etc. We also give a detailed description of the structure of the global unstable set in the holomorphic endomorphism, pertaining to a problem of Fornaess and Sibony.

In Chapter 5 we give several unexpected relations between ergodic theory of invariant measures and dimension theory on folded fractals. We give results about the coding and mixing properties for measure-preserving endomorphism. This problem is very subtle, and as Parry and Walters showed, it is profoundly different than in the case of automorphisms.

In [37] we solved a problem of Dajani and Hawkins in the setting of hyperbolic endomorphisms which are non-expanding on basic sets, showing the *non-existence of generating Rokhlin partitions*.

We found a surprising geometric flattening phenomenon in [43] that proves, in short, that if the stable dimension at a point is zero, then the whole fractal is contained in a global unstable manifold (if it is connected). Moreover answering some questions of Parry, Bruin and Hawkins, etc., we showed that there exists a connection between 1-sided Bernoullicity and expanding property on Λ.

We also showed that the stable conditional measures of certain equilibrium measures are geometric measures, and proved in [42] that these conditional measures maximize in a Variational Principle for the stable dimension; this answers a similar problem to one solved by Barreira in the diffeomorphism case.

Then we give results about the pointwise dimensions of equilibrium measures in the case of holomorphic endomorphisms on complex projective spaces, from a paper by Fornaess and Mihailescu, [21]. We also obtained a result parallel to a result by Briend and Duval about the Green measure, and proved that the measure of maximal entropy for the restriction to a terminal set, is in fact a wedge product of two positive closed currents. This shows that the measure of maximal entropy on a basic saddle set (which is singular with respect to Green measure), has a geometric description.

In Chapter 6 we give several results about dimension theory for iterated function systems (IFS). In general, IFS were studied under separation conditions, such as the Open Set Condition (Hutchinson, Falconer, Moran, etc). However if the Open Set Condition does not hold, then the dimension theory is much harder and there are few results in the literature.

In [56], Mihailescu and Urbanski attacked this problem using thermodynamic formalism and a so-called preimage counting function, together with an analysis of the equilibrium measures on limit sets of finite IFS with overlaps. We also proved that if the dimension of the limit set is minimal possible as zero of the corresponding pressure function, then the system is as far from Open Set Condition as possible. This analysis was later extended to the case of IFS with countably many generators, when new phenomena intervene, like the fact that the limit set is no longer compact, the conditions at the boundary at infinity, etc.

We then give in [51] a construction of a family of non-stationary infinitely generated Moran fractals, which are determined by asymptotic frequencies. For these fractals, we applied a form of thermodynamic formalism. We applied this construction in order to relate the dimension theory for these non-stationary Moran fractals, to the ergodic properties of f-expansions. In particular these relations are found for digits in m-expansions, β-expansions, Bolyai-Rényi expansions, continued fraction expansions, etc.

Rezumat

In ultimii 30 ani, una dintre cele mai importante directii in studiul sistemelor dinamice diferentiabile, a fost utilizarea formalismului termodinamic in probleme de dimensiuni fractale si teorie ergodica.

Majoritatea directiilor de cercetare din aceasta teza trateaza teorie ergodica diferentiabila si formalism termodinamic cu aplicatii la teoria dimensiunii in dinamica hiperbolica, dinamica conforma si complexa si proprietati statistice ale diferitelor tipuri de endomorfisme care invariaza masura.

Teza prezinta realizarile mele stiintifice dupa obtinerea doctoratului in matematica la University of Michigan, SUA in 1999, avand drept conducator de doctorat pe Prof. John Erik Fornaess.

Dupa terminarea doctoratului, am obtinut prin concurs posturi academice, la mai multe universitati si institute prestigioase din strainatate: Texas A&M Univ, USA, 1999-2001; Institut des Hautes Études Scientifiques, Paris, November-December 2013 and Aug-Sept 2014; Univ. of Bremen, Germany, March-May 2012; Max Planck Institut, Bonn, May 2011; Univ. of North Texas, February-March 2007; Instituto de Matematica Pura e Aplicada (IMPA), Brazil, February 2008; Erwin Schrödinger Math Institute, Vienna, Austria, June 2008; Scuola Normale Superiore, Pisa, Italy, May 2004; Univ. of North Texas, USA, 2002-2003, etc.

Teza contine multe dintre rezultatele publicate dupa doctorat, in peste 34 articole stiintifice, majoritatea in reviste ISI de prestigiu, precum Math. Annalen (2013, 1999), International Math Res Notices (2013), Ergodic Theory Dynam Syst (2011, 2011, 2002), Discrete and Cont Dynam Syst (2012, 2008, 2006, 2001), Math Zeitschrift (2011), Bulletin London Math Soc (2010), Commun Contemp Math (2004), J Stat Physics (2013, 2011, 2010), Mathematical Proceed Cambridge (2010), Monatshefte Math (2012), Canadian J Math (2008), Proceed American Math Soc (2013, 2011), Nonlinear Analysis (2010), Houston J Math (2005), etc. Impreuna cu M. Urbanski (U. North Texas), am publicat deasemenea o sectiune a cartii "Fractal Geometry and Dynamics in Pure and Applied Mathematics", din seria American Mathematical Society, Contemporary Mathematics. Am avut deasemenea articole invitate in Discrete Cont. Dynam. Syst., Oberwolfach Reports, Rév. Roum. Math. Pures Appl., and in Ann. Univ. Bucharest.

Iata mai jos sumarul capitolelor aceste teze de abilitare.

In Capitolul 2, vom prezenta unele notiuni de teoria hiperbolicitatii pentru endomorfisme. In cazul aplicatiilor ne-inversabile, comportamentul dinamic este foarte diferit decat in cazul difeomorfismelor si apar fenomene noi pentru care trebuie create noi metode. Pentru endomorfisme, varietatile locale instabile nu formeaza o laminare, pot exista nenumarabil de multe varietati printr-un punct (cf. [72]), si ele se pot intersecta atat in multimea Λ, cat si in afara ei.

Vom explica unele dintre exemplele de endomorfisme cu un comportament puternic ne-inversabil, si consecintele hiperbolicitatii in cazul lor. O familie de astfel de exemple, care prezinta nenumarabil de multe varietatie instabile prin puncte ale fractalului invariant, si pentru care dimensiunea stabila se comporta deasemenea diferit decat in cazul difeomorfismelor, a fost data in [41]. In acel articol am gasit un nou fenomen de tip Newhouse pentru intersectiile de multimi Cantor in fibre, care se traduce intr-o clasa de produse-skew care sunt departe de homeomorfisme pe multimile bazice respective, cat si departe de a fi constante-la-1. Clasa noastra de exemple este seminificativa, deoarece da endomorfisme pe fractali si nu endomorfisme Anosov, si deci implica tehnici noi de a obtine exemple cu nenumarabil de multe varietati instabile printr-un punct, decat exemplele torale date de Przytycki in [72]. Vom prezenta deasemenea cazul endomorfismelor olomorfe pe $\mathbb{P}^2\mathbb{C}$, si al discurilor complexe date de varietatile locale stabile/instabile; un caz particular este cel al aplicatiilor s-hiperbolice.

In Capitolul 3, vom prezenta rezultate despre formalismul termodinamic si proprietatile statistice pentru masuri invariante pe fractali cu suprapuneri.

In aceasta teorie, am utilizat frecvent masurile de echilibru (Gibbs) ale potentialilor Hölder continui, despre care s-a aratat ca au proprietati bune pe iteratele bilelor Bowen. Masurile Sinai-Ruelle-Bowen (SRB) au fost introduse in lucrarile fundamentale ale lui Sinai, Ruelle si Bowen, si in cazul atractorilor hiperbolici reprezinta masurile observabile fizic, care dau distributia asimptotica a traiectoriilor pentru Lebesgue-aproape toate punctele dintr-o vecinatate a atractorului.

Am introdus in [46] a notiune noua de masura SRB inversa pentru sisteme dinamice ne-inversabile, pe care am studiat-o ca o masura de echilibru. Masurile SRB inverse nu sunt masuri SRB ale functiilor inverse, deoarece in acest caz functiile nu sunt inversabile. In loc, aceste masuri dau distributia asimptotica a preistoriilor, si satisfac deasemenea o fomula Pesin modificata.

Explicam deasemenea unele dintre rezultatele noastre despre Jacobieni ai masurilor de echilibru, si despre mixing-ul de orice ordin si despre descresterea exponentiala a corelatiilor pentru masuri de echilibru. Vom prezenta deasemenea unele rezultate care rezolva problema productiei de entropie negativa pentru endomorfisme de spatii Lebesgue, si care sunt legate de cercetarile lui Ruelle.

In capitolul 4 vom prezenta unele rezultate legate de aplicatiile formalismului termodinamic in teoria dimensiunii pentru aplicatiile ne-inversabile.

Aratam ca problema dimensiunii stabile este foarte diferita de cea din cazul difeomorfismelor, si dam unele exemple de perturbatii care au un comportament dinamic neasteptat, exemple obtinute in [60]. Vom da deasemenea aplicatii ale uneir noi notiuni de presiune inversa, introdusa de Mihailescu si Urbanski. Impreuna cu Urbanski si Stratmann, am rezolvat dificila problema a dimensiunii (raspunzand unor intrebari ale lui Verjovsky-Wu, Manning-McCluskey, etc.), pentru dimesiunea stabila, in relatie cu functia de numarare a preimaginilor ([57], [52]). In aceste articole, am aplicat metode de combinatorica si de teorie ergodica diferentiabila, pentru a arata ca daca dimesiunea stabila atinge valoarea minima, resp. maxima, ca si zero al presiunii, atunci numarul de preimagini care raman in Λ este maxim, resp. minim; metodele de demonstratie insa difera profund.

Vom da apoi aplicatii ale unei notiuni de transversalitate pentru familii parametrizate de produse skew hiperbolice; rezultatele noastre sunt legate de unele articole ale lui Peres, Simon, Solomyak. Vom prezenta si o descriere detaliata a structurii multimii globale instabile in cazul olomorf, ceea ce raspunde unei probleme a lui Fornaess si Sibony.

In capitolul 5, vom da mai multe legaturi neasteptate intre teoria ergodica a masurilor invariante si teoria dimensiunii pe fractali cu suprapuneri. Vom include rezultate despre codarea si mixing-ul pentru endomorfisme care pastreaza masura. Aceasta problema este foarte subtila, si cum Parry si Walters au aratat, este profund diferita de cea in cazul automorfismelor.

In [37] am rezolvat o problema a lui Dajani si Hawkins, in cadrul endomorfismelor hiperbolice care nu sunt expanding pe multimi bazice, aratand *non-existenta partitiilor Rokhlin generatoare*.

Am gasit un fenomen de aplatizare geometrica surprinzator, in [43] care demonstreaza ca, daca dimensiunea stabila este zero intr-un punct, atunci intregul fractal este continut intr-o varietate instabila (daca este conex).

In plus, raspunzand unor intrebari ale lui Parry, Bruin, Hawkins, etc., am aratat ca exista o legatura intre Bernoullicitatea unilaterala si proprietatea de expansiune pe Λ.

Am aratat deasemenea ca masurile conditionale stabile ale anumitor masuri de echilibru, sunt masuri geometrice, si am demonstrat ca ca aceste masuri conditionale maximizeaza intr-un Principiu Variational pentru dimensiunea stabila; aceasta raspunde unei intrebari simuilare celei rezolvate de Barreira in cazul difeomorfismelor.

Apoi vom da rezultate despre dimensiunea punctuala a masurilor de echilibru in cazul endomorfismelor olomorfe pe spatii complexe proiective, obtinute de Fornaess si E. Mihailescu in [21]. Am obtionut deasemenea un rezultat paralel cu cel al lui Briend si Duval asyupra masurii Green, dar in cazul nostru foarte diferit, asupra masuri de entropie maximala a restrictiei la o multuime bazica de tip saddle. Aceasta masura am aratat ca este produsul wedge a doi curenti pozitivi inchisi. Aceasta arata ca masura de entropie maximala pe o multime de tip saddle are o descriere geometrica.

In Capitolul 6, vom da rezultate despre teoria dimensiunii pentru sisteme iterative de functii (IFS). In general, IFS au fost studiate sub conditii de separare, de exp. Conditia Multimii Deschise (OSC) de catre Hutchinson, Falconer, etc.

In [56], Mihailescu si Urbanski au atacat aceasta problema folosind formalismul termodinamic si o functie de numarare a preimaginilor, impreuna cu o analiza a masurilor de echilibru pe multimile limita ale IFS finite cu suprapuneri. Am aratat deasemenea ca daca dimensiunea multimii limita este minimala ca si zero al functiei de presiune corespunzatoare, atunci sistemul este cat de departe posibil de a avea Conditia Multimii Deschise. Aceasta analiza a fost extinsa mai tarziu la cazul IFS cu o multime numarabila de generatori, atunci cabnd apar noi fenomene, de exp. multimea limita nu mai este neaparat compacta, apar unele conditii la frontiera de la infinit, etc.

Am dat apoi in [51] constructia unei familii de fractali Moran non-stationari infinit generati, care sunt determinati de frecventele asimptotice. Pentru acesti fractali am aplicat o forma de formalism termodinamic. Am aplicat apoi aceasta constructie cu scopul de a lega teoria dimensiunii pentru acesti fractali Moran non-stationari, de proprietatile ergodice ale dezvoltarilor in f-serii. In particular am gasit aceste relatii pentru comportamentul numerelor care apar in dezvoltarile in m-serie, in dezvoltarile β, Bolyai-Rényi, fractiile continue, etc.

1 Introduction and overview of the thesis.

This thesis starts with an introduction to dynamical systems and smooth ergodic theory and especially to thermodynamic formalism. Thermodynamic formalism and dynamics are subjects of high current interest, as can be seen from the Fields medal that have been awarded recently in these fields (Smirnov, Yoccoz, McMullen, Lindenstrauss, etc.) Hyperbolicity has proved to be an important tool in dealing with long time behaviour of dynamical systems since the pioneering works of Smale ([91], Bowen [5], Ruelle [84], Sinai [90], etc.

I obtained my PH.D at University of Michigan, whose mathematics department is constantly ranked among the 10 best in the USA. My PH.D was under the supervision of Prof. John Erik Fornaess, one of the top mathematicians in the world, and one of the founders of higher dimensional complex dynamics.

After the PH.D, I worked or collaborated with several well-known foreign mathematicians, among them: M. Urbański (Univ. North Texas, USA), J.E Fornaess (Univ.Michigan, USA), B. Stratmann (Univ. of Bremen, Germany), M. Abate (Univ. Pisa, Italy), F. Przytycki (IMPAN, Poland), M. Denker (Göttingen Univ., Germany), M. Roychowdhury (Univ.Texas Pan-American, USA), etc.

After my visiting professorships at Texas A&M Univ. and at Univ. North Texas, USA, I decided to return to Bucharest (at IMAR). Hopefully, the Romanian academic system will improve and it will be possible to develop an active scientific life here too, including for example funds for research visits by and to foreign collaborators, stable predictable research grants, less bureaucracy, organizing of conferences in Romania and abroad, etc.

After the PH.D, I held academic positions at several prestigious universities and institutes abroad, Institut des Hautes Études Scientifiques, Paris, November-December 2013 and Aug-Sept. 2014; Texas A&M Univ, USA, 1999-2001; Univ. of North Texas, USA, 2002-2003; Univ. of Bremen, Germany, March-May 2012; Univ. of North Texas, February-March 2007; Max Planck Institut, Bonn, May 2011; Instituto de Matematica Pura e Aplicada (IMPA), Brazil, February 2008; Erwin Schrödinger Math Institute, Vienna, Austria, June 2008; Scuola Normale Superiore, Pisa, Italy, May 2004, etc.

I have written, alone or in collaboration, over 34 papers, most of them in well-known ISI journals such as Math. Annalen (2013, 1999), International Math Res. Notices (2013), Ergodic Th. and Dynam. Syst. (2011, 2011, 2002), Math. Zeitschrift (2011), Discrete and Cont. Dyn. Syst (2012, 2012, 2008, 2006, 2001), J. Statistical Physics (2013, 2011, 2010), Bull. London Math. Soc. (2010), Monatsh. Math (2012), Canadian J Math. (2008), Commun. Contemp. Math. (2004), etc., which are detailed in the attached CV.

Also, I have been director of a 4-person research grant PCE-Idei from CNCSIS Romania in the period 2009-2011, and a member in other grants. Also I participated in an NSF grant (director M. Urbanski) at Univ. North Texas, USA. I have been invited speaker at many high-level international conferences and seminars, among them Luminy, Oberwolfach, Bremen, Bedlewo, New York, Denton, Rio de Janeiro, Göttingen, Warsaw, Centro de Giorgi-Pisa, College Station, IHP-Paris, Bloomington, Ann Arbor, Orsay, Sydney, Alba Iulia, San Antonio, Washington DC, Seattle, etc. In 2007, I received the Simion Stoilow research prize of the Romanian Academy. Most of the results I obtained in the above mentioned papers are presented in this thesis.

Even since the papers and books of Eckmann-Ruelle [18], Ruelle [82], Przytycki [72], etc. it has been observed that the dynamics of hyperbolic *endomorphisms* is very different from that of diffeomorphisms and that many interesting new phenomena are generated by the non-invertibility and the non-existence of an inverse. As a matter of fact, it was shown that certain physical systems behave non-invertibly (for instance [18], [11], etc.)

In Chapter 2 we will review the definition of hyperbolicity for non-invertible systems, and will give several examples of non-invertible dynamics. Among these are the non-invertible horseshoes of Bothe [4], or those studied by Simon [88], etc. Then we shall give a class of examples found originally in [41], of hyperbolic skew products with overlaps in their fibers, which show a very strong non-invertible character also from the point of view of the stable dimension; these examples are however not constant-to-1 on their respective saddle basic sets.

Another large class of non-invertible maps is given by holomorphic endomorphisms on the complex projective space $\mathbb{P}^2\mathbb{C}$. Such examples were studied by Fornaess and Sibony [22], [23], etc. The behaviour of these maps on saddle sets is subtle and the methods intertwine ergodic theory, the theory of positive closed currents, and pointwise dimensions for invariant measures (as in [22], [23], [21], [48], etc.)

In Chapter 3 we will study some notions from the thermodynamic formalism, such as topological pressure, measure-theoretic and topological entropy. And also notions that take into considerations all the inverse branches of the map, such as the *inverse entropy* introduced by Mihailescu and Urbanski in [61]. A very important notion is that of *equilibrium measures* or *Gibbs state*), see [5], [27], [96], etc. A particular case of equilibrium measure is the measure of maximal entropy for a dynamical system.

Also, for hyperbolic attractors one is interested from a physical point of view, in the Sinai-Ruelle-Bowen measures (or *SRB measures*). These are measures that can be "seen", in the sen se that they give the asymptotic distribution of orbits for Lebesgue-almost all points in some neighbourhood of the hyperbolic attractor. In [46] E. Mihailescu introduced also a notion of *inverse SRB measure* for a saddle type non-invertible repellor. This notion is subtle, since the repellor Λ is not invertible, thus the map does not necessarily have an inverse on Λ and we cannot simply consider the SRB measure for the inverse. This inverse SRB measure was shown to be in fact an equilibrium measure and it satisfies a modified inverse Pesin entropy formula.

Then we study various statistical properties of equilibrium measures of Hölder potentials on folded fractals, such as exponential decay of correlations, mixing of any order, exactness, and 1-sided or 2-sided Bernoullicity. The notion of *entropy production* for probability measures was introduced in mathematics by Ruelle (see [80], [79]); it generalizes and makes precise some notions from statistical physics. In [53], Mihailescu and Urbanski studied entropy production for equilibrium measures on folded fractals, especially for inverse SRB measures on folded repellors and showed that the entropy production is negative in many instances.

In Chapter 4 we will use thermodynamic formalism to dimension estimates for hyperbolic endomorphisms, especially towards estimating the Hausdorff and the upper box dimension, of the sections through the fractal Λ, with local stable and unstable manifolds. We showed for instance in [47] that the behaviour of stable sections is completely different than that of unstable sections, in the case of non-invertible systems. The estimates will be done with zeros of various pressure functions.

Another method used to do estimates, will be with zeros of the inverse pressure functions; this method is designed especially for the stable dimension in the endomorphism case. In the paper [57], Mihailescu and Urbanski found significant relations between stable dimension and preimage counting function; this integer-valued function $d(x)$, is given by the number of f-preimages of x which still remain in the saddle basic set Λ, and in general $d(\cdot)$ is not continuous. In particular we showed that if there exists a point in Λ where the stable dimension is minimal, then at every point in Λ the stable dimension is minimal and the function f is constant-to-1 on Λ.

Also another class of results was obtained in [59], where we studied the *transversality condition* for parametrized families of hyperbolic non-invertible skew products. For families with transversality condition we proved that for Lebesgue-almost all parameters, we have Bowen-type formulas for the stable dimension, and we also gave several classes of examples, namely coming from iterated function systems and from higher dimensional complex dynamics.

In Chapter 5 we shall review several results of the author, as well as other authors, concerning the relationship between geometric properties of folded fractal sets, and ergodic properties of certain invariant measures. We shall investigate for instance the *family of conditional measures*, associated to the equilibrium measure of a stable potential and to a partition supported on local stable manifolds. In [42] we showed that these conditional measures are geometric, in the sense that they behave on small balls of radius r comparable to r^δ, for any $r > 0$ (for certain fixed δ).

Also in [43] we proved that in certain cases, if the stable dimension is zero at some point, then f is expanding on the saddle basic set Λ. Moreover we showed in [43] a surprising *geometric flattening phenomenon*, in the sense that if the stable dimension is zero at some point of Λ, then Λ is contained in a finite union of unstable manifolds.

Moreover answering some questions of Parry, Bruin and Hawkins, etc., we showed that there exists a connection between 1-sided Bernoullicity and expanding property on Λ, namely if the measure of maximal entropy is 1-sided Bernoulli, then f is expanding on Λ; we also proved that certain equilibrium measures cannot be 1-sided Bernoulli, and also studied mixing of any order for equilibrium measures (see [43], [37], etc.)

Then we will consider the problem of the pointwise dimension for equilibrium measures on folded saddle fractals for holomorphic endomorphisms $f : \mathbb{P}^2 \to \mathbb{P}^2$. This problem was solved by Fornaess and Mihailescu in [21]. For hyperbolic measures for diffeormorphisms this problem was solved by Barreira, Pesin and Schmeling in [2]. However the problem for endomorphisms is very different, and the various foldings of the basic set Λ usually produce new phenomena. Thus new methods and ideas must be devised to deal with this non-invertible case.

In the same paper [21], Fornaess and Mihailescu also showed that the measure of maximal entropy on Λ can be written as a wedge product of two positive closed currents, thus giving a geometric interpretation for the measure of maximal entropy of the restriction to Λ. This measure of maximal entropy is singular with respect to the Green measure ν_G which was shown by Briend-Duval [8] to be the measure of maximal entropy of f on the whole space $\mathbb{P}^2$.

In Chapter 6 we will investigate an important class of fractals sets, namely limit sets of finite or infinite conformal iterated function systems. First we will present results obtained by Mihailescu and Urbanski in [56], about the relations between the dimension of the limit set Λ of a finite conformal iterated function system with overlaps, and the preimage counting function.

These results open up a new type of approach towards estimating the Hausdorff and other types of dimension (lower/upper box dimension, etc.), for the limit set of IFS with overlaps. So far, transversality conditions were used for parametrized families of IFS; under these transversality conditions, Lebesgue almost all members of the family behave like IFS which satisfy the Open Set Condition, which allows Bowen type formulas for the dimension (for eg. Solomyak, [92]).

By contrast, we consider a fixed system of contractions, and we use the preimage counting function in [56], in order to find good estimates for the dimension. In general the preimage counting function $d(\cdot)$ is discontinuous on the fractal limit set Λ. However, in the case when $d(\cdot)$ is constant we find also exact formula for the dimension of Λ. Moreover we show that, if the Hausdorff dimension of Λ is equal to the minimal possible zero of the pressure function, corresponding to the maximal number D of preimages, then the number of preimages in Λ of any point of Λ is D.

Then in [55], Mihailescu and Urbanski studied the problem of the dimension of the limit set of an *infinite* conformal iterated function system with overlaps. Our paper is the first one in this new direction, as far as we know. The infinite case is very different from the finite case, and many new phenomena appear (for eg. [36]). First of all the limit set Λ is not compact anymore, also the infimum of the contraction rates could be zero. In the infinite case, the boundary at infinity $S(\infty)$ plays an important role.

Then we will remind the notion of f-expansion for real numbers; and examples of expansions like the m-ary expansion with $m \geq 2$ integer, the β-expansion, the Lüroth expansion, the continued fraction expansion, etc.

We will give some interesting connections found in [51], between classes of fractal sets constructed by a modified Moran method, and the behaviour of the digits of numbers in f-expansions, for certain functions f. In this way we obtain information about the size of the sets of real numbers with certain conditions on their digit expansions.

2 Smooth dynamical systems. Examples of non-invertible dynamics.

2.1 Hyperbolicity for smooth endomorphisms

Let us consider in the sequel a smooth (suppose $\mathcal{C}^2$) Riemannian manifold M, and a smooth ($\mathcal{C}^2$) map $f : M \to M$. We want to study the iterates of f and the invariant objects (sets, measures, etc.) with respect to this iteration. We will say that a transformation $f : \Lambda \to \Lambda$ is *topologically transitive* on a compact f-invariant set Λ if for any open sets (relatively to the induced topology) U, V in Λ there exists $n \geq 0$ such that $U \cap f^n(V) \neq \emptyset$.

Definition 2.1.1. *By* ***basic set*** *for a smooth endomorphism $f : M \to M$, we mean, as in [27], a compact f-invariant topologically transitive set Λ, such that there exists a neighbourhood U of Λ with $\Lambda = \bigcap\limits_{n \in \mathbb{Z}} f^n(U)$.*

Then, given a compact set Λ we have a useful construction, namely that of the *natural extension* (or **inverse limit**),

$$\hat{\Lambda} := \{\hat{x} = (x, x_{-1}, x_{-2}, \ldots), f(x_{-i}) = x_{-i+1}, x_{-i} \in \Lambda, i \geq 1\}$$

The infinite sequences of consecutive preimages of type $\hat{x}$ are called *prehistories* of x, for $x \in \Lambda$. On the inverse limit $\hat{\Lambda}$, we have a metris space structure, given by the canonical metric $d(\hat{x}, \hat{y}) = \sum\limits_{i \geq 0} \frac{d(x_{-i}, y_{-i})}{2^i}$, $\hat{x}, \hat{y} \in \hat{\Lambda}$.

Let us define also the *shift homeomorphism*

$$\hat{f} : \hat{\Lambda} \to \hat{\Lambda}, \hat{f}(\hat{x}) = (f(x), x, x_{-1}, \ldots)$$

Thus even if f itself is not invertible on Λ, we lift it to a homeomorphism $\hat{f}$ on the inverse limit $\hat{\Lambda}$. Denote also by $\pi : \hat{\Lambda} \to \Lambda$ the canonical projection on the first coordinate.

Hyperbolicity for endomorphisms is defined similarly as for diffeomorphisms, however there is an important difference from the diffeomorphism case, namely the unstable spaces/unstable manifolds depend now on whole prehistories; see Ruelle [82], Przytycki [72], [47], etc. We have thus a continous splitting of the tangent bundle over $\hat{\Lambda}$, $T_{\hat{\Lambda}} M := \{(\hat{x}, v), v \in T_x M, \hat{x} \in \hat{\Lambda}\}$, into contracting and expanding directions that depend on the whole backward history, i.e

$$T_{\hat{x}} M = E^s_x \oplus E^u_{\hat{x}}, \ \hat{x} \in \hat{\Lambda},$$

with $Df_x(E^s_x) \subset E^s_{f(x)}, Df_x(E^u_{\hat{x}}) \subset E^u_{\hat{f}\hat{x}}$, and Df is uniformly contracting on E^s_x and expanding on $E^u_{\hat{x}}$. The space E^s_x is called the *stable tangent space* at x, and $E^u_{\hat{x}}$ is called the *unstable tangent space* corresponding to the prehistory $\hat{x}$, for all $\hat{x} \in \hat{\Lambda}$.

Then for some $r > 0$, one can define the stable, and unstable local manifolds respectively as $W^s_r(x) := \{y \in M, d(f^i x, f^i y) < r, i \geq 0\}$ and $W^u_r(\hat{x}) := \{y \in M, y \text{ has a prehistory } \hat{y}, \text{with } d(y_{-i}, x_{-i}) < r, i \geq 0\}$, $\hat{x} \in \hat{\Lambda}$ (see [72], [82]).

In the case of a smooth endomorphism f which is uniformly hyperbolic on the basic set Λ, we shall denote by $\Phi^s(x) := \log |Df_s(x)|, x \in \Lambda$ the stable potential of f, and by $\Phi^u(\hat{x}) := -\log |Df_u(\hat{x})|, \hat{x} \in \hat{\Lambda}$ the unstable potential, where again we denoted by $Df_s(x), Df_u(\hat{x})$, the restrictions of Df to the stable space E^s_x, respectively to unstable space $E^u_{\hat{x}}, \hat{x} \in \hat{\Lambda}$. Also denote by:

$$\delta^s(x) := HD(W^s_r(x) \cap \Lambda), \text{ and by } \delta^u(\hat{x}) := HD(W^u_r(\hat{x}) \cap \Lambda),$$

the **stable dimension** at a point $x \in \Lambda$, respectively the **unstable dimension** at $\hat{x} \in \hat{\Lambda}$ (see also [47], [45]).

In the hyperbolic endomorphism case notice that the local unstable manifolds do not form a foliation (like in the diffeomorphism setting), but instead they can intersect inside and outside of the fractal set Λ. Another difficulty is also that in general we do not have Markov partitions on basic sets, like in the diffeomorphism case. Hence the problem of coding the system symbolically (for example using Bernoulli shifts) is much harder for endomorphisms, or even impossible. In addition, we may also have sudden drops in the fractal dimensions, caused by self-intersections in the basic set; the stable dimensions do not depend continuously on parameters (unlike for diffeomorphisms); see [60] for a discussion and examples of polynomial maps which become homeomorphisms when restricted to some invariant sets. Hence several subtle methods must be devised to overcome all these problems in the non-invertible situation.

2.2 Examples of non-invertible systems

We will give now several examples of endomorphisms f on basic sets Λ, and in some cases we shall also discuss their behaviour. In [18] Eckmann and Ruelle studied relations between attractors, SRB measures, dimension, Lyapunov exponents and entropy, and gave also interesting applications or relations to statistical physics (chaotic dynamics, turbulence theory, etc.). In the same paper [18], there appears a non-injective example in the plane, due to Ushiki et al., in which the computer picture of the attractor displays folded drapes, and hence non-invertible behaviour.

In [20], Falconer constructed a family of piecewise linear maps, which were proved to be homeomorphisms on their respective basic sets for Lebesgue-almost all parameters; thus in that case, one can write the Hausdorff dimension of the attractor as the solution of a Bowen equation. This kind of behaviour appears in general when there is a **transversality type condition** present for the respective parametrized family (as in [70], [93], or [59]).

In [59], Mihailescu and Urbanski gave actually examples of families of skew products having iterated function systems in the base, and also several examples of families of skew products from higher dimensional complex dynamics; these families satisfy the transversality condition. For these examples we proved that it is possible to find the stable dimension (i.e the Hausdorff dimension of the intersection between the basic set Λ and the local stable manifolds), as the zero of the pressure of a certain stable potential on the inverse limit $\hat{\Lambda}$. Let us recall these examples of skew products below:

Fix an expanding repeller $X \subset \mathbb{R}^p$ for some smooth map g, a bounded set $V \subset \mathbb{R}^q$, let also $d \geq 1$ and an open set $W \subset \mathbb{R}^d$. Consider a family $\Phi = \{\phi_x^\lambda : \bar{V} \to V\}_{(x,\lambda) \in W \times X}$ of maps from $\mathcal{C}^{1+\gamma}(\bar{V})$ satisfying the following conditions: (af) the absolute values of the derivatives of the contractions ϕ_x^λ are uniformly bounded away from 0 and 1. (bf) The map $(\lambda, x) \to \phi_x^\lambda \in C^{1+\gamma}(\bar{V})$ defined on $W \times X$ is continuous. (cf) (Transversality Condition)

$$\forall (x \in X)\, \forall (\lambda_0 \in W)\, \exists (\delta(x, \lambda_0) > 0)\, \exists (C_1 > 0)\, \forall (\hat{x}, \hat{y} \in p_0^{-1}(x))\, \forall (r > 0)$$
$$x_1 \neq y_1 \; l_d(\{\lambda \in B(\lambda_0, \delta(x, \lambda_0)) : ||\pi_\lambda(\hat{x}) - \pi_\lambda(\hat{y})|| \leq r\}) \leq C_1 r^q,$$

where l_d denotes the d-dimensional Lebesgue measure on $\mathbb{R}^d$ and $\pi_\lambda : \hat{X} \to \bar{V}$ is the canonical projection induced by the skew-product $F_\lambda : U \times \bar{V} : \mathbb{R}^p \times \bar{V}$ given by the formula

$$F_\lambda(x, y) = (f(x), \phi_x^\lambda(y)).$$

Any such family Φ is said to be *transversal* and the canonically induced family $\bar{\Phi} = \{F_\lambda\}_{\lambda \in W}$ is also called *transversal*.

The basic set here is $\Lambda := \underset{x \in X}{\cup} \overset{\infty}{\underset{n=0}{\cap}} \underset{z \in f^{-n}x}{\cup} h_z^n(\bar{V})$, where $h_z^n := h_{f^{n-1}z} \circ \ldots \circ h_z, n \geq 1, z \in X$. We studied then the conditional measures of equilibrium states induced on fibers and their relation to the stable dimension of fibers. We employed a transversality type condition in order to show that for Lebesgue almost all parameters, the stable dimension of the fibers is given by the zero of a Bowen type equation on $\hat{\Lambda}$. Several examples where these results can be applied were given in [59], among which some iterated function systems and examples from higher dimensional complex dynamics.

In [72] Przytycki studied a class of Anosov endomorphisms and gave examples of perturbations of hyperbolic toral endomorphisms such that, through any given point $x \in \mathbb{T}^m$ there pass uncountably many local unstable manifolds, which correspond to the different prehistories of x.

Another type of non-invertible dynamics is given by the family of hyperbolic horseshoes with overlaps, introduced by Bothe in [4]. Bothe proved in fact that the set of parameters for which the associated maps have such non-invertible horseshoes with overlaps, has non-empty interior.

Other examples of non-invertible behaviour were studied by Solomyak in [93], namely certain self-similar sets for families of iterated function systems with overlaps.

Also, in [56] Mihailescu and Urbanski investigated finite conformal iterated function systems with overlaps. These systems are notoriously difficult to study due to the lack of control on the foldings. So far in other papers such systems were studied only as part of a family of iterated function systems, and then by applying certain transversality type conditions which reduce almost all systems to IFS with open set condition. However in [56] we employed sophisticated combinatorial techniques and thermodynamic formlaism on inverse limits, in order to obtain relations between the preimage counting function and the dimension of the fractal limit sets.

We considered a finite system of conformal injective contraction maps $\phi_i, i \in I$ defined on a bounded set V in $\mathbb{R}^d$. Denote this system by $\mathcal{S}$. Then, according to well-known general results (for eg. [26]) there exists a unique compact fractal set Λ with the invariance property

$$\Lambda = \bigcup_{i \in I} \phi_i(\Lambda),$$

which can be obtained as the union of all intersections of iterates of maps ϕ_i:

$$\Lambda = \bigcup_{\omega \in I^\infty} \bigcap_{n \geq 0} \phi_{\omega|_n}(V),$$

where I^∞ is the space of all sequences indexed by positive integers, with elements in I, and for such a sequence ω, we denote by $\omega|_n = (\omega_1, \ldots, \omega_n)$, and where $\phi_{\omega|_n} = \phi_1 \circ \ldots \circ \phi_n$, for all $n \geq 1$.

We obtained in [56] that if the dimension of the limit set Λ of such a system is minimal, then every point in the limit set Λ has exactly the same number of preimages in Λ, which is maximal. This opens thus a whole new research direction about IFS with overlaps.

Moreover in [55] we studied the case of infinite conformal iterated function systems with overlaps. This case is significantly different from the finite case, although here too the preimage counting function plays an important role. However by contrast, the limit set of the system is not compact anymore, and the imfimum of the contraction rates of the functions in the system can be zero. We showed also that the boundary at infinity is important now.

Another large class of endomorphisms consists of holomorphic maps in one dimension or in several dimensions. The one-dimensional case (rational maps) has been studied more, and employed the first applications of thermodynamic formalism in order to calculate the Hausdorff dimension of Julia sets of rational hyperbolic maps (starting with Bowen [6], Ruelle [83]). In the higher dimensional case, we may have saddle type basic sets obtained as components of the non-wandering set, for Axiom A holomorphic maps on the projective space $\mathbb{P}^k, k \geq 2$; see [22]. As in the diffeomorphism case (for example Henon maps), the set K^- is very important; in the diffeomorphism case K^- is the set of points having bounded backward iterates, but in the holomorphic endomorphism case on $\mathbb{P}^k$, the set K^- is the complement of the set of points that have all of their preimages converging towards the support of the Green measure. Thus in a sense, K^- is the set of points with bounded backward iterates. In fact it can be shown in the s-hyperbolic endomorphism case, that K^- is the global unstable set $W^u(\hat{S}_1)$ union with the (finite) set of periodic attracting (see Fornaess-Sibony, [22]).

In [48] we proved that for s-hyperbolic holomorphic endomorphisms on $\mathbb{P}^k$, the set K^- has empty interior. Then in [45] we studied in greater detail the Hausdorff dimension of K^-, and the Hausdorff and upper box dimensions of the stable intersections, with the help of the inverse pressure (see also [61], [58]). Many differences appear in this case when compared to the case of holomorphic automorphisms on Stein manifolds ([50]).

In [94], Tsujii studied a class of dynamical systems generated by maps $T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, T(x,y) = (\ell x, \lambda y + g(x))$, where $\ell \geq 2$ is an integer, $0 < \lambda < 1$ and g is a $\mathcal{C}^2$-function on S^1. Then T is an Anosov endomorphism, which has thus a unique SRB measure μ. Let $\mathcal{D} \subset (0,1) \times C^2(S^1, \mathbb{R})$ be the set of pairs (λ, f) for which the SRB measure of the corresponding endomorphism T is absolutely continuous with respect to Lebesgue measure on $S^1 \times \mathbb{R}$; also let $\mathcal{D}^\circ$ the interior of $\mathcal{D}$ with respect to the product topology. In [94] Tsujii showed that there exist examples of endomorphisms as above, for which the SRB measure is totally singular. However he proved that "most" maps T have the following properties:

Theorem 2.2.1. *([94]) Let $\ell^{-1} < \lambda < 1$. There exists a finite collection of C^∞ functions $\phi_i : S^1 \to \mathbb{R}, i = 1, \dots, m$ s.t for any C^2 function $g \in C^2(S^1, \mathbb{R})$, the subset of $\mathbb{R}^m$*

$$\{(t_1, \dots, t_m) \in \mathbb{R}^m, (\lambda, g(x) + \sum_{i=1}^{m} t_i \phi_i(x)) \notin \mathcal{D}^\circ\}$$

is a null set with respect to the Lebesgue measure on $\mathbb{R}^m$. Consequently $\mathcal{D}$ contains an open and dense subset of $(\frac{1}{\ell}, 1) \times C^2(S^1, \mathbb{R})$.

2.3 Strong non-invertible hyperbolic behaviour

In the paper [41], we considered the dynamics of a family of hyperbolic skew products with overlaps in fibers f_α, which were shown to behave far from homeomorphisms and also far from being constant-to-1 maps. For these skew product maps, we showed that different prehistories of the same point may have different unstable tangent spaces associated to them, and we estimated also the angle between such unstable directions. Therefore the associated local unstable manifolds actually depend on whole prehistories, too. In order to present this class, let us fix first a small $\alpha \in (0,1)$; then take the subintervals $I_1^\alpha, I_2^\alpha \subset I = [0,1]$ so that I_1^α is contained in $[\frac{1}{2} - \epsilon(\alpha), \frac{1}{2} + \epsilon(\alpha)]$ and I_2^α is contained in $[1 - \alpha - \epsilon(\alpha), 1 - \alpha + \epsilon(\alpha)]$, for some small $\epsilon(\alpha) < \alpha^2$. Consider also a strictly increasing smooth map $g : I_1^\alpha \cup I_2^\alpha \to I$ such that $g(I_1^\alpha) = g(I_2^\alpha) = I$; assume there exists a large $\beta >> 1$ such that $\beta^2 > g'(x) > \beta >> 1, x \in I_1^\alpha \cup I_2^\alpha$. Thus there exist subintervals $I_{11}^\alpha, I_{12}^\alpha \subset I_1^\alpha, I_{21}^\alpha, I_{22}^\alpha \subset I_2^\alpha$ such that $g(I_{11}^\alpha) = g(I_{21}^\alpha) = I_1^\alpha$ and $g(I_{12}^\alpha) = g(I_{22}^\alpha) = I_2^\alpha$. Then let $J^\alpha := I_{11}^\alpha \cup I_{12}^\alpha \cup I_{21}^\alpha \cup I_{22}^\alpha$ and $J_*^\alpha := \{x \in J^\alpha, g^i(x) \in J^\alpha, i \geq 0\}$. Now define the skew-product endomorphism $f_\alpha : J_*^\alpha \times I \to J_*^\alpha \times I$,

$$f_\alpha(x,y) = (g(x), h_\alpha(x,y)), \text{ with } h_\alpha(x,y) = \begin{cases} \psi_{1,\alpha}(x) + s_{1,\alpha} y, \ x \in I_{11}^\alpha \\ \psi_{2,\alpha}(x) + s_{2,\alpha} y, \ x \in I_{21}^\alpha \\ \psi_{3,\alpha}(x) - s_{3,\alpha} y, \ x \in I_{12}^\alpha \\ s_{4,\alpha} y, \ x \in I_{22}^\alpha, \end{cases} \quad (1)$$

where for some small ε_0, we take $s_{1,\alpha}, s_{2,\alpha}, s_{3,\alpha}, s_{4,\alpha}$ to be positive numbers, ε_0-close to $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ respectively; and $\psi_{1,\alpha}(\cdot), \psi_{2,\alpha}(\cdot), \psi_{3,\alpha}(\cdot)$ are smooth (say $\mathcal{C}^2$) functions on I which are ε_0-close in the

$\mathcal{C}^1$-metric, to the affine functions $x \to x$, $x \to 1 - x$ and $x \to 1$, respectively. By $|g_1 - g_2|_{\mathcal{C}^1}$ let us denote the distance in the $\mathcal{C}^1(I)$-metric between two smooth functions on I, g_1 and g_2. Denote also the function $h_\alpha(x, \cdot) : I \to I$ by $h_{x,\alpha}(\cdot)$, for $x \in J_*^\alpha$. Now we shall define the fractal set

$$\Lambda(\alpha) := \bigcup_{x \in J_*^\alpha} \bigcap_{n \geq 0} \bigcup_{y \in g^{-n}x \cap J_*^\alpha} h_{y,\alpha}^n(I), \tag{2}$$

where $h_{y,\alpha}^n := h_{f^{n-1}y,\alpha} \circ \ldots \circ h_{y,\alpha}, n \geq 0$. For $x \in J_*^\alpha$ let also: $\Lambda_x(\alpha) := \bigcap\limits_{n \geq 0} \bigcup\limits_{y \in g^{-n}x \cap J_*^\alpha} h_{y,\alpha}^n(I)$, the **fiber** (or **slice**) of the fractal $\Lambda(\alpha)$ over x. Then we obtain the following:

Theorem 2.3.1. *([41]) There exists a function $\vartheta(\alpha) > 0$ defined for all positive small enough numbers α, with $\vartheta(\alpha) \underset{\alpha \to 0}{\to} 0$, such that if f_α is the map defined in (1) whose parameters satisfy:*

$$\max \left\{ |\psi_{1,\alpha}(x) - x|_{\mathcal{C}^1}, |\psi_{2,\alpha}(x) - 1 + x|_{\mathcal{C}^1}, |\psi_{3,\alpha}(x) - 1|_{\mathcal{C}^1}, |s_{i,\alpha} - \frac{1}{2}|, i = 1, \ldots, 4 \right\} < \vartheta(\alpha) \tag{3}$$

then we obtain:

a) For $x \in J_*^\alpha \cap I_1^\alpha$, there exists a Cantor set $F_x(\alpha) \subset \Lambda_x(\alpha)$, s. t every point of $F_x(\alpha)$ has two different f_α-preimages in $\Lambda(\alpha)$. And if $x \in J_*^\alpha \cap I_2^\alpha$, then there exists a Cantor set $F_x(\alpha) \subset \Lambda_x(\alpha)$ s. t every point of $F_x(\alpha)$ has two different f_α^2-preimages in $\Lambda(\alpha)$.

b) f_α is hyperbolic on $\Lambda(\alpha)$.

c) If $\hat{z}, \hat{z}' \in \hat{\Lambda}(\alpha)$ are two different prehistories of an arbitrary point $z \in \Lambda(\alpha)$, then $E_{\hat{z}}^u \neq E_{\hat{z}'}^u$.

Also we studied in [41] the unstable and the stable dimensions associated to the class of maps above on their respective invariant sets:

Theorem 2.3.2 ([41])**.** *For a small fixed $\alpha > 0$, let the function $f : \Lambda \to \Lambda$ defined in (1). Then the unstable dimension $\delta^u(\hat{z}) = t^u, \forall \hat{z} \in \hat{\Lambda}$, where t^u is the unique zero of the pressure function $t \to P_{\hat{f}|_{\hat{\Lambda}}}(t\Phi^u)$, and where $\Phi^u(\hat{y}) := -\log|Df_u(\hat{y})|, \hat{y} \in \hat{\Lambda}$. Consequently if $g'(x) > \beta(\alpha) >> 1$ on J_*, we have*

$$\delta^u(\hat{z}) < \frac{\log 2}{\log \frac{\beta(\alpha)}{2}}, \hat{z} \in \hat{\Lambda}$$

For the stable dimension function, we do not have exact formulas, but we can estimate it using the thickness of intersections of Cantor sets (see [64], [65] for background), in the fibers. Moreover this will show that the class of examples given above, is far from being a homeomorphism on its respective basic set, and also far from being constant-to-1 (in these two cases we have exact Bowen-type formulas for the stable dimension). The following two results were proved in [41].

Theorem 2.3.3 ([41])**.** *Let a sufficiently small $\alpha > 0$ and a function f defined as in (1), and assume that the parameters of f satisfy condition (3).*

a) Then the stable dimension $\delta^s(z) \leq t_s^- < 1$, for any point $z \in \Lambda$.

b) If $\eta(\cdot)$ is a continuous function on Λ such that $d(z) \leq \eta(z), z \in \Lambda$, it follows that $\delta^s(z) \geq t_\eta, z \in \Lambda$, where t_η is the unique zero of the function $t \to P(t\Phi^s - \log \eta)$.

Corollary 2.3.1 ([41]). *Let a small $\alpha > 0$ and a function f defined as in (1), s. t the parameters $s_i, \psi_j, i = 1, \dots, 4, j = 1, \dots, 3$ of f satisfy (3). Write Λ as the union $V_1 \cup V_2$, where V_1 is defined as the set of points having only one f-preimage inside Λ and V_2 is the set of points having exactly two f-preimages in Λ.*

a) Then $\delta^s(z) \in (\frac{\log 2}{\log(2+\frac{1}{\Delta(\alpha)})}, 1), z \in \Lambda$. So if α tends to 0, then the stable dimension at an arbitrary point of Λ may be made as close as we want to 1, but always strictly smaller than 1.

b) V_1 is an open uncountable set in Λ, and V_2 is a closed set in Λ.

c) Assume moreover that in the definition (1) of f, the contraction factors $s_i, i = 1, \dots, 4$ are all equal to $\frac{1}{2}$. Then V_2 is uncountable as well.

The above Corollary says thus that this family behaves, from the point of view of the stable dimension, very differently than a homeomorphism on Λ_α, and also from a 2-to-1 map on Λ_α, hence the preimage counting function is highly oscillating on the respective basic set.

2.4 Saddle sets for holomorphic maps on projective spaces

A particular case of endomorphisms is given by holomorphic maps on complex projective spaces $\mathbb{P}^k\mathbb{C}, k \geq 2$. In order to fix ideas, we shall assume that $k = 2$ and we study endomorphisms $f : \mathbb{P}^2 \to \mathbb{P}^2$. Any holomorphic endomorphism f on $\mathbb{P}^2$ can be written as

$$f([z_0 : z_1 : z_2]) = [P_0(z_0, z_1, z_2) : P_1(z_0, z_1, z_2) : P_2(z_0, z_1, z_2)], \ [z_0 : z_1 : z_2] \in \mathbb{P}^2,$$

where P_0, P_1, P_2 are homogeneous polynomials of the same degree $d \geq 2$, in z_0, z_1, z_2. In this case, the notion of basic set given before makes sense, and now if Λ is a basic set of saddle type, i.e f is hyperbolic with both contracting and expanding directions over Λ, we have that the local stable and unstable manifolds are complex disks.

A useful notion will be that of *s-hyperbolic map*, due to Fornaess and Sibony [22]. First we will give a general dynamical definition:

Definition 2.4.1. *Let a smooth ($\mathcal{C}^2$) map $f : M \to M$. Then the nonwandering set of f is the set*

$$\Omega(f) := \{x \in M, \forall U \, a \ neighbourhood \ of \ \ x, \exists n \geq 1, s.t f^n(U) \cap U \neq \emptyset\}$$

As can be seen from the definition, the important dynamical behaviour of f happens on $\Omega(f)$. If M is compact (for example in the case when $M = \mathbb{P}^2$), the non-wandering set is compact. Then, like in the case of any other compact subset, we can form the inverse limit $\hat{\Omega}(f)$.

Returning now to the case of a holomorphic endomorphism f on $\mathbb{P}^2$, if f is hyperbolic on $\Omega = \Omega(f)$, then we can write Ω as $S_0 \cup S_1 \cup S_2$, where S_i is the set of non-wandering points with unstable index $i \in \{0, 1, 2\}$.

Let us take now ω the Kähler form on $\mathbb{P}^2$; then Fornaess and Sibony showed that $(f^n)^*\omega/d^n$ converge as currents to a positive closed currenty T whose support is equal to the Julia set $J(f)$ of

f, defined as the complement of the set $F(f)$ of points x such that the iterates of f form a normal family on some neighbourhood of x; the set $F(f)$ is called the Fatou set of f.

Also one can define a measure $\mu = T \wedge T$, which satisfies $f^*\mu = d^2\mu$ and which is a measure of maximal entropy $\log d^2$ (we will come back to this later on). Then Fornaess and Sibony defined s-hyperbolic maps in [22] as holomorphic endomorphisms of $\mathbb{P}^2$ which are hyperbolic on $\Omega(f)$, their periodic points are dense in Ω (i.e f has Axiom A), and satisfy $f^{-1}(S_2) = S_2$, there exists an analytic variety C such that $C \cap S_1 = \emptyset$ and there exists a neighbourhood U of S_1 such that $f^{-1}(S_1) \cap U = S_1$.

Let us define also the set K^- which is the analogue of the set of points with bounded backward iterates from the Hénon map case.

Definition 2.4.2. *Consider the map $f : \mathbb{P}^2 \to \mathbb{P}^2$ which is holomorphic, and let $U^- := \{z \in \mathbb{P}^2, d(f^{-n}(z), supp\ \mu) \to 0,$ uniformly in a neighbourhood of $z\}$ and let $K^- := \mathbb{P}^2 \setminus U^-$.*

Theorem 2.4.1 ([22])**.** *If the holomorphic endomorphism f on $\mathbb{P}^2$ is s-hyperbolic, then:*

i) $K^- = W^u(\hat{S}_1) \cup S_0$

ii) $S_2 = supp\mu$

iii) the unstable set of S_2 is open with locally pluripolar complement.

In [48] we proved that there is a dichotomy between the dynamical behaviour of endomorphisms which are obtained by perturbations of Hénon maps, and s-hyperbolic maps. This difference was observed in the structure of the interior of the set K^-.

Theorem 2.4.2 ([48])**.** *For a perturbation $f_\varepsilon(z, w) = (z^2 + aw, z + \varepsilon w^2)$ of a hyperbolic Hénon map $g(z, w) = (z^2 + aw, z)$ we have that $intK^-$ contains the union of the repelling basins of finitely many repelling points.*

However for s-hyperbolic maps we have a completely different type of interior of the set K^-:

Theorem 2.4.3 ([48])**.** *If f is a s-hyperbolic holomorphic endomorphism on $\mathbb{P}^2$, then $intK^- = \emptyset$.*

There exist examples of Axiom A holomorphic endomorphisms (see [48]):

1) Consider $f([z : w : t]) = [P(z, t) : Q(w, t) : t^d]$, where P, Q are polynomials in one variable having degree d, and hyperbolic on their Julia sets J_P, J_Q. The expanding part of the non-wandering set is $S_2 = J_P \times J_Q$. The basic sets of S_1 are in $t = 1$, $J_P \times \{$periodic sinks of $Q\}$ and $\{$periodic sinks of $P\} \times J_Q$, respectively in $t = 0$, the Julia set of $[P(z, 0) : Q(w, 0) : 0]$. It is clear that f satisfies Axiom A.

2) Consider the map $\Phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ defined by $\Phi([z_0 : w_0], [z_1 : w_1]) = [z_0 z_1 : w_0 w_1 : z_0 w_1 + w_0 z_1]$. This gives a 2-to-1 cover of $\mathbb{P}^2$. Now let f_0 a rational map in one variable, having degree d; assume that f_0 is hyperbolic on its Julia set J_0. Then there exists a holomorphic map of degree d, $f : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\Phi(f_0, f_0) = f \circ \Phi$, and cf is hyperbolic too. Then $S_2(f) = \Phi(J_0, J_0)$ and the basic sets of S_1 are $\Phi(\text{sink} \times J_0)$.

3) Solenoids are important in differentiable dynamics (see for eg. [27]). We can construct also a solenoid in $\mathbb{P}^2$ as a perturbation, namely: $f([z : w : t]) = [z^2 : wt/10 + zt/2 + \varepsilon w^2 : t^2]$. Then it can be shown that f is s-hyperbolic.

3 Thermodynamic formalism and statistical properties of measures on folded fractals.

3.1 Entropy and topological pressure

Two of the most important notions in dynamical systems are those of entropy (topological, measure-theoretical) and topological pressure. Topological entropy is actually obtained as the pressure of the zero potential. These notions make connections with a functional analytic approach towards dynamical systems, and they have also important applications in dimension theory for invariant fractals (see for instance [6], [33], [71], [83], [99], etc.)

Let us first define the *topological entropy.* Consider a continuous map $f : X \to X$ on a compact metric space (X, d). By *Bowen ball* we will understand the set of points that remain within ε-distance of the iterates of a given point, up to an order. More precisely, a Bowen ball (see [5], [27], [96], etc.) is a set of type:

$$B_n(x, \varepsilon) := \{y \in X,\ d(f^i x, f^i y) < \varepsilon, i = 0, \dots n-1\}, \tag{4}$$

for $n \in \mathbb{N}, x \in X, \varepsilon > 0$. The Bowen balls are in fact balls in the distance $d_n(x, y) := \sup\{d(f^i x, f^i y), 0 \le i \le n-1\},\ x, y \in X$.

We will use now these Bowen balls in order to cover the space X as well as possible. We say that a set $E \subset X$ is (n, ε)*-spanning* if $X = \underset{x\in E}{\cup} B_n(x, \varepsilon)$. And the set F is called (n, ε)*-separated* if $d_n(x, y) > \varepsilon$ for all points $x, y \in F$. We will denote by $Sp(X, n, \varepsilon)$ the minimal number of (n, ε)-Bowen balls needed to cover X, thus the minimal cardinality of an (n, ε)-spanning set for X, and by $Sep(X, n, \varepsilon)$ the maximal cardinality of an (n, ε)-separated set in X. We do not write also f in the above notations, when things are clear from context. The following definition is then well-known:

Definition 3.1.1. *Let us define the topological entropy of f by* $h_{top}(f) = \lim\limits_{\varepsilon\to 0} \limsup\limits_{n\to\infty} \frac{1}{n} \log Sp(X, n, \varepsilon)$.

It can be shown that if f is expansive then there exists an $\varepsilon_0 > 0$ such that $h_{top}(f) = \limsup\limits_{n\to\infty} \frac{1}{n} \log Sp(X, n, \varepsilon_0)$ (for instance [96]).

The topological entropy gives a measure of the complexity of the behaviour of the iterates of f in the long run. The smaller the entropy, the more predictible is the system. For instance if f is contracting on X, then Bowen balls are determined only by the first iterate, and so the topological entropy of f is equal to 0.

Definition 3.1.2. *Let (X, d) a compact metric space and $f : X \to X$ a continuous map. Then a probability measure μ on X is said to be f-invariant if $\mu(f^{-1}(A)) = \mu(A)$, for any Borel set A.*

We will denote the space of all probability measures which are f-invariant, by $\mathcal{M}(X, f)$.

The definition of *measure-theoretic entropy* is more complicated and long, and we reffer to any classical text on ergodic theory/dynamical systems for this, for instance [27], [96], etc. Enough to say that for each f-invariant probability measure μ on X, there exists a number $h_\mu(f) \geq 0$ which describes the complexity of the behaviour of the trajectories of f modulo μ.

We will say that f is *expansive* if there exists a postive constant ε_0 such that if $x, y \in X$ and there exist prehistories $\hat{x}, \hat{y} \in \hat{X}$ with $d(x_{-i}, y_{-i}) < \varepsilon, i \geq 0$ and $d(f^i x, f^i) < \varepsilon, i \geq 0$, then $x = y$.

Theorem 3.1.1 ([96]). *If the map $f : X \to X$ is expansive, then the entropy map is upper semi-continuous, i.e if $\mu \in \mathcal{M}(X, f)$ and $\varepsilon > 0$, then there exists a neighbourhood U of μ in $\mathcal{M}(X, f)$ such that $\nu \in U$ implies that $h_\nu(f) < h_\mu(f) + \varepsilon$.*

We shall now define the *topological pressure*. Consider again a continuous map $f : X \to X$ on a compact metric space (X, d), and let also a continuous function $\phi : X \to \mathbb{R}$ (usually ϕ is chosen to be Hölder continuous). We shall also denote by

$$S_n\phi(x) := \phi(x) + \phi(f(x)) + \ldots + \phi(f^{n-1}(x)),$$

the n-th *consecutive sum* of ϕ on x, for any integer $n \geq 1$ and any $x \in X$.

Consider now the quantities

$$P_{sp}(f, \phi) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \inf\{\sum_{x \in E} e^{S_n\phi(x)},\ E \text{ an } (n, \varepsilon) - \text{spanning set for } X\}$$

$$P_{sep}(f, \phi) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup\{\sum_{x \in E} e^{S_n\phi(x)},\ E \text{ an } (n, \varepsilon) - \text{separated set for } X\}$$

Then it can be shown (for eg. [27], [96]) that $P_{sp}(f, \phi) = P_{sep}(f, \phi)$.

Definition 3.1.3. *The common value of $P_{sep}(f, \phi)$ and $P_{sp}(f, \phi)$ is called the topological pressure of the potential ϕ with respect to f; it is denoted by $P(f, \phi)$.*

From the definition above, it can be seen that $P(f, 0) = h_{top}(f)$.

Like with topological entropy, it can be shown that, if f is expansive, then there exists a constant $\delta_0 > 0$ such that the topological pressure $P(f, \phi)$ can be calculated only on (n, δ_0)-spanning sets (with $n \to \infty$), i.e only when $\varepsilon = \delta_0$, without having to take $\varepsilon \to 0$ (see [96]).

Let us give now several properties of the pressure functional, which will be used in the sequel. We denote the space of continuous real-valued functions on X by $\mathcal{C}(X, \mathbb{R})$. The following Theorem can be proved.

Theorem 3.1.2 ([96]). *Consider a continuous endomorphism on a compact metric space $f : X \to X$ and $\phi, \psi \in \mathcal{C}(X, \mathbb{R})$, and $\varepsilon > 0$. Then we have the following properties of topological pressure:*

1) $P(f, 0) = h_{top}(f)$

2) if $\phi \leq \psi$ on X, then $P(f, \phi) \leq P(f, \psi)$. Thus $h_{top}(f) + \inf \phi \leq P(f, \phi) \leq h_{top}(f) + \sup \phi$

3) $P(f, \cdot)$ is either finitely valued or consgtantly ∞

4) $|P(f,\phi) - P(f,\psi)| \leq ||\phi - \psi||$
5) if $h_{top}(f) < \infty$*, then* $P(f,\cdot)$ *is convex*
6) $P(f,\phi + c) = P(f,\phi) + c$*, for any constant* c
7) $P(f,\phi + \psi \circ f - \psi) = P(f,\phi)$
8) $P(f,\phi + \psi) \leq P(f,\phi) + P(f,\psi)$
9) $P(f,c\phi) \leq cP(f,\phi)$ *for any constant* $c \geq 1$*, and* $P(f,c\phi) \geq cP(f,\phi)$ *for any constant* $c \leq 1$
10) $|P(f,\phi)| \leq P(f,|\phi|)$.

Now, the properties of the pressure as a function of the transformation f are given in the next:

Theorem 3.1.3 ([96])**.** *Let* $f : X \to X$ *be a continuous map on a compact metric space* X*, and a potential* $\phi \in \mathcal{C}(X,\mathbb{R})$*. Then,*

1) for any $m \geq 1$*, we have* $P(f^m, S_m\phi) = mP(f,\phi)$*.*
2) if f *is a homeomorphism, then* $P(f^{-1},\phi) = P(f,\phi)$*.*
3) if Y *is a closed subset of* X *and* $f(Y) \subset Y$*, then* $P(f|_Y,\phi|_Y) \leq P(f,\phi)$*.*
4) if $f_i : X_i \to X_i$ *is a continuous map on a compact metric space* (X_i,d_i) *for* $i = 1,2$*, and if* $\Phi : X_1 \to X_2$ *is a surjective continuous map with* $\Phi \circ f_1 = f_2 \circ \Phi$*, then* $P(f_2,\phi) \leq P(f_1,\phi \circ \Phi)$*, for all* $\phi \in \mathcal{C}(X_2,\mathbb{R})$*. In particular this can be applied when* Φ *is a homeomorphism, thus pressure is a conjugacy invariant.*
5) Pressure behaves additively towards products of transformations on compact spaces, i.e if $f_i : X_i \to X_i, i = 1,2$ *are continuous transformations, and if* $\phi_i \in \mathcal{C}(X_i,\mathbb{R}), i = 1,2$*, then* $P(f_1 \times f_2, \phi_1 \times \phi_2) = P(f_1,\phi_1) + P(f_2,\phi_2)$*.*

3.2 Variational Principle and equilibrium measures

The Variational Principle was proved first for certain transformations by Ruelle and then for all transformations by Walters. It makes an important connection between pressure (and topological pressure), and measure-theoretic entropy. It corresponds also to certain phenomena in statistical physics ([84]).

Theorem 3.2.1 ([96])**.** *The Variational Principle for Topological Pressure*

Let a continuous map $f : X \to X$ *on a compact metric space* X*, and a potential* $\phi \in \mathcal{C}(X,\mathbb{R})$*. Then*

$$P(f,\phi) = \sup\{h_\mu(f) + \int \phi d\mu,\ \mu \in \mathcal{M}(X,f)\}$$

Since it is a very important notion, that will be used in the sequel, let us give now the definition of *ergodic measures* (see [34], [27], [96], etc.) We will say that the f-invariant measure μ is *ergodic* if the only borelian sets A for which $f^{-1}(A) = A$ are the ones with $\mu(A) = 1$ or $\mu(A) = 0$. We denote the ergodic f-invariant probability measures on X by $\mathcal{E}(X,f)$.

The following Corollary says that we can take in the supremum of the Variational Principle only ergodic measures, and that the only part that matters in calculating pressure is the non-wandering set. It can be proved using the decomposition of invariant measures in ergodic measures ([96]).

Corollary 3.2.1 ([96])**.** *Let $f : X \to X$ be a continuous map on a compact metric space X, and let $\phi \in \mathcal{C}(X, \mathbb{R})$. Then*

a) $P(f, \phi) = \sup\{h_\mu + \int \phi d\mu, \ \mu \in \mathcal{E}(X, f)\}$

b) $P(f, \phi) = P(f|_{\Omega(f)}, \phi|_{\Omega(f)})$

We give now the definition of *equilibrium measures* (or *Gibbs states*) which are important in ergodic theory and statistical physics, and have also applications in dimension theory of dynamical systems.

Definition 3.2.1. *Let the continuous map $f : X \to X$ on the compact metric space X, and $\phi \in \mathcal{C}(X, \mathbb{R})$. Then an f-invariant probability measure μ is called an equilibrium measure of ϕ if $P(f, \phi) = h_\mu(f) + \int \phi d\mu$. The set of equilibrium measures of ϕ is denoted by $\mathcal{M}_\phi(X, f)$.*

For certain potentials ϕ and certain maps f, the set of equilibrium measures of ϕ may be empty. The following Theorem gives some properties of equilibrium measures.

Theorem 3.2.2 ([96])**.** *Consider a continuous transformation $f : X \to X$ on a compact metric space, and let $\phi \in \mathcal{C}(X, \mathbb{R})$. Then*

a) $\mathcal{M}_\phi(X, f)$ is convex.

b) if $h_{top}(f) < \infty$, then the extreme points of $\mathcal{M}_\phi(X, f)$ are precisely the ergodic members of $\mathcal{M}_\phi(X, f)$.

c) if $h_{top}(f) < \infty$ and $\mathcal{M}_\phi(X, f) \neq \emptyset$, then $\mathcal{M}_\phi(X, f)$ contains an ergodic measure.

d) if the entropy map is upper semicontinuous then $\mathcal{M}_\phi(X, f)$ is compact and non-empty. In particular, if f is expansive on X, then every $\phi \in \mathcal{C}(X, \mathbb{R})$ has at least an equilibrium measure.

e) if $\phi, \psi \in \mathcal{C}(X, \mathbb{R})$, and if there exists a constant c such that $\phi - \psi - c$ belongs to the closure of the set $\{\chi \circ f - \chi, \ \chi \in \mathcal{C}(X, \mathbb{R})\}$, then ϕ and ψ have exactly the same set of equilibrium measures with respect to f.

By using tangent functionals, one can show that there is a dense set of potentials ϕ which have unique equilibrium measures.

Corollary 3.2.2 ([96])**.** *Consider the continuous transformation $f : X \to X$ on a compact metric space X and assume that the entropy map of f is upper semicontinous on $\mathcal{M}(X, f)$ (this happens for instance if f is expansive on X). Then there exists a dense set in $\mathcal{C}(X, \mathbb{R})$ such that each potential ϕ in this set has a unique equilibrium measure, i.e such that each $\mathcal{M}_\phi(X, f)$ has exactly one member.*

We will see now that the existence and uniqueness of equilibrium measures is guaranteed for hyperbolic maps f and Hölder continuous potentials ϕ. For this one needs the notion of specification property (see [5], [27]).

Theorem 3.2.3 ([5], [27]). *(Bowen) Let (X, d) be a compact metric space and $f : X \to X$ be an expansive homeomorphism with the specification property, and ϕ be a Hölder continuous potential on X. Then there exists exactly one equilibrium measure μ_ϕ of ϕ. It is mixing and we have*

$$\mu_\phi = \lim_{n\to\infty} \frac{1}{\sum\limits_{x\in Fix(f^n)} e^{S_n\phi(x)}} \sum_{x\in Fix(f^n)} e^{S_n\phi(x)}\delta_x \tag{5}$$

This Theorem can be extended to hyperbolic endomorphisms as follows:

Theorem 3.2.4 ([47]). *Consider a smooth map f on a Riemannian manifold M and Λ be a basic set so that f is hyperbolic on Λ. Let also a Hölder continuous potential ϕ on Λ. Then there exists a unique equilibrium measure μ_ϕ of ϕ on Λ, it is mixing and satisfies (5). Also for any $\varepsilon > 0$ there exist positive constants $A_\varepsilon, B_\varepsilon$ such that for every $x \in X$ and any $n \geq 1$, we have*

$$A_\varepsilon e^{S_n\phi(x)-nP(f,\phi)} \leq \mu_\phi(\overline{B_n(x,\varepsilon)}) \leq B_\varepsilon e^{S_n\phi(x)-nP(f,\phi)} \tag{6}$$

The proof of this Theorem requires to lift the measures to the inverse limit $\hat{\Lambda}$ and to compare with Bowen balls in $\hat{\Lambda}$, and also using the fact that hyperbolicity implies specification property (see [27]).

By using Livschitz Theorem ([27]) and the properties from the hyperbolic non-invertible case[47], we can prove also the following result that characterizes the set of potyentials which have a common equilibrium measure:

Theorem 3.2.5. *Consider a basic set Λ for the smooth endomorphism f and assume that f is hyperbolic on Λ (as an endomorphism). Let also ϕ, ψ be Hölder continous potentials on Λ such that the equilibrium measures μ_ϕ and μ_ψ coincide. Then there exists aconstant c and a Hölder continuous function χ on Λ such that $\psi(x) = \phi(x) + c + \chi(f(x)) - \chi(x), x \in \Lambda$.*

From the properties of the set $\mathcal{M}_\phi(X, f)$ above, it follows also that the converse of the last Theorem is true, which thus gives a complete characterization of the Hölder potentials that have a fixed equilibrium measure.

3.3 SRB and inverse SRB measures for endomorphisms

We will study now a special class of invariant measures on fractal attractors, which is important since it is physically observable, as it gives the asymptotic behaviour of trajectories of Lebesgue-almost all points in a neighborhood of the attractor. For hyperbolic attractors for diffeomorphisms, these inv ariant measures were first studied by Sinai, Ruelle and Bowen, and are thus called *Sinai-Ruelle-Bowen measures* (or *SRB measures*).

The SRB measures for Axiom A endomorphisms (smooth non-invertible maps), have been studied in [74]. In [32], Liu established a Pesin entropy formula in the case of an absolutely continuous invariant measure for an endomorphism. Also in [75], Qian and Zhu studied a notion of SRB measures in the non-uniform setting of invariant measures for smooth endomorphisms. They showed that a Pesin type entropy formula holds (see also the entropy formula for diffeomorphisms established earlier by Ledrappier and Young in [31]):

Theorem 3.3.1 (Pesin's entropy formula for endomorphisms, [75])**.** *Let $f : M \to M$ be a $\mathcal{C}^2$ endomorphism having an f-invariant probability borelian measure μ, so that* $\log |detDf| \in L^1(M, \mu)$. *Then the entropy formula*

$$h_\mu(f) = \int_M \sum_{\lambda_i > 0} \lambda_i(x) m_i(x) d\mu(x)$$

holds if and only if μ has the SRB property, i.e for every measurable partition η of the natural extension $\hat{M}$ subordinate to the unstable manifolds of (f, μ), we have that, for $\hat{\mu}$-a. e $\hat{x} \in \hat{M}$, $\pi(\hat{\mu}^\eta_{\hat{x}}) << m_{\hat{x}}$; here $m_{\hat{x}}$ is the Lebesgue measure induced on the local unstable manifold $W^u_{r(\hat{x})}(\hat{x})$, λ_i are the Lyapunov exponents of the measure μ and m_i their respective multiplicities.

Ruelle studied in [81] (see also [34]), the distribution of the preimages of points, for expanding maps; the main method was the use of Perron-Frobenius operators, and also the fact that the diameters of the images of small balls by branches of inverse iterates, decrease exponentially. However in our case of **non-invertible non-expanding maps**, this useful property does no longer hold.

In a series of papers, namely [38], [43], [46] we initiated a study of an analogue of the SRB measure for endomorphisms, but this time involving the various consecutive preimages of points. As noticed before, due to the non-invertibility of f, we cannot apply the case of the forward iterates, and the problem is difficult and subtle. New methods were developed, involving estimates of the equilibrium measures on pieces of neighbourhoods of unstable manifolds (corresponding to various prehistories), inverse pressure, non-Bernoullicity of some measures, combinatorial arguments, estimates of the lifts of measures on certain borelian sets from the natural extension, consideration of families of certain appropriate conditional measures, etc. A priori there may exist preimages of points from Λ which do not remain in Λ, as Λ is a fractal set, *not necessarily totally invariant.* First let us specify what we understand by repellor for an endomorphism:

Definition 3.3.1. *Let f be a smooth (say $\mathcal{C}^2$) endomorphism on a Riemannian manifold M and let Λ be a basic set for f. We say that Λ is a* ***repellor*** *for f if the critical set of f does not intersect Λ and if there exists a neighbourhood U of Λ such that $\bar{U} \subset f(U)$.*

We can prove the following result for the number of preimages remaining in the repellor:

Proposition 3.3.1. *In the setting of Definition 3.3.1, if Λ is a repellor for f, then $f^{-1}\Lambda \cap U = \Lambda$. If moreover Λ is assumed to be connected, the number of f-preimages that a point has in Λ is constant.*

Proof. Consider a point $x \in \Lambda$, and y be an f-preimage of x from U. Then $f^n y \in \Lambda, n \geq 1$. From Definition 3.3.1, since Λ is assumed to be a repellor, the point y has a preimage y_{-1} in U; then y_{-1}

has a preimage y_{-2} from U, and so on. Thus y has a full prehistory belonging to U and also its forward orbit belongs to U, hence $y \in \Lambda$ since Λ is a basic set. So $f^{-1}\Lambda \cap U = \Lambda$.

We prove now the second part of the statement. Let a point $x \in \Lambda$ and assume that it has d f-preimages in Λ, denoted $x_1, \dots, x_d$. Consider also another point $y \in \Lambda$ close to x. If y is close enough to x and since $\mathcal{C}_f \cap \Lambda = \emptyset$, it follows that y also has exactly d f-preimages in U, denoted by $y_1, \dots, y_d$. Since from the first part we know that $f^{-1}\Lambda \cap U = \Lambda$, we obtain then $y_1, \dots, y_d \in \Lambda$. In conclusion the number of f-preimages in Λ of a point is locally constant. If Λ is assumed to be connected, then the number of preimages belonging to Λ of any point from Λ, must be constant. $\square$

The importance of the fact that all points in Λ have a constant number of preimages remaining in Λ, is given by the following Theorem, proved in [46]:

Theorem 3.3.2. *([46]) Consider Λ to be a connected hyperbolic repellor for the smooth endomorphism $f : M \to M$; let us assume that the constant number of f-preimages belonging to Λ of any point from Λ is equal to d. Then $P(\Phi^s - \log d) = 0$.*

Then, using this we proved in [46] that in the case of a hyperbolic repellor *which is not necessarily expanding*, the distribution of consecutive preimages of Lebesgue almost all points in a neighborhood of the repellor is given by an **inverse SRB measure**; the important role of the inverse SRB measure is shown to be played here by the equilibrium measure of the stable potential. The methods of proof are however different from the diffeomorphism case, and involve a careful study of the types of behaviours of consecutive sums along various prehistories.

Theorem 3.3.3. *([46]) Let Λ be a connected hyperbolic repellor for a smooth endomorphism f : $M \to M$. There exists a neighbourhood V of Λ, $V \subset U$ such that if we denote by*

$$\mu_n^z := \frac{1}{n} \sum_{y \in f^{-n}z \cap U} \frac{1}{d(f(y)) \cdot \ldots \cdot d(f^n(y))} \sum_{i=1}^{n} \delta_{f^i y}, z \in V$$

where $d(y)$ is the number of f-preimages belonging to U of a point $y \in V$, then for any continuous function $g \in \mathcal{C}(U, \mathbb{R})$ we have

$$\int_V |\mu_n^z(g) - \mu_s(g)| dm(z) \underset{n\to\infty}{\to} 0,$$

where μ_s is the equilibrium measure of the stable potential $\Phi^s(x) := \log|det(Df_s(x))|, x \in \Lambda$.

Proof. We assume that U is the neighbourhood of Λ from Definition 3.3.1, i. e such that $\bar{U} \subset f(U)$. As we proved in Proposition 3.3.1, if Λ is a connected hyperbolic repellor, then any point from Λ has exactly d f-preimages belonging to Λ for some positive integer d. Moreover as was shown in the beginning of the proof of Theorem 3.3.2, there exists a neighbourhood V of Λ such that any point from V has d^n n-preimages in U, for $n \geq 1$.

If Λ is a hyperbolic repellor we have that all local stable manifolds must be contained in Λ. Indeed, otherwise there may exist small local stable manifolds which are not entirely contained in

Λ. Let $W^s_r(x), x \in \Lambda$ one such stable manifold, with a point $y \in W^s_r(x) \setminus \Lambda$; in this case since $y \in U$ (for small r) and since $\bar{U} \subset f(U)$, it follows that y has a full prehistory $\hat{y}$ in U, and from the fact that Λ is a basic set, we obtain that $y \in W^u_r(\hat{\xi})$ for some $\hat{\xi} \in \hat{\Lambda}$. But then $y = W^s_r(x) \cap W^u_r(\hat{\xi})$, hence $y \in \Lambda$ from the local product structure of Λ (since Λ is a basic set, see for example [27]); this gives a contradiction to our assumption. Hence there exists a small $r > 0$ such that all stable manifolds of size r are contained in Λ.

We shall denote by $\mathcal{C}(U)$ the space of real continuous functions on U. Let us fix now a Holder continuous function $g \in \mathcal{C}(U)$. We will apply the L^1 Birkhoff Ergodic Theorem ([34]) on $\hat{\Lambda}$ for the homeomorphism $\hat{f}^{-1}$, in order to obtain an estimate for the measure of the set of prehistories which are badly behaved. Similarly as in [27] or [**?**] we know that the stable distribution is Holder continuous, hence the stable potential on $\hat{\Lambda}$ is Holder too. This means that there exists a unique equilibrium measure for this potential on $\hat{\Lambda}$; so from the bijection between $\mathcal{M}(f)$ and $\mathcal{M}(\hat{f})$ it follows that there exists a unique equilibrium measure for Φ^s on Λ denoted by μ_s. This measure is ergodic and we can apply the L^1 Birkhoff Ergodic Theorem to the function $g \circ \pi$ on $\hat{\Lambda}$:

$$||\frac{1}{n}(g(x) + g \circ \pi(\hat{f}^{-1}(\hat{x})) + \ldots g \circ \pi(\hat{f}^{-n+1}(\hat{x})) - \int_\Lambda g \circ \pi d\hat{\mu}_s||_{L^1(\hat{\Lambda}, \hat{\mu}_s)} \underset{n\to\infty}{\to} 0 \qquad (7)$$

We make now the general observation that if $f : \Lambda \to \Lambda$ is a continuous map on a compact metric space Λ, μ is an f-invariant borelian probability measure on Λ and $\hat{\mu}$ is the unique $\hat{f}$-invariant probability measure on $\hat{\Lambda}$ with $\pi_*(\hat{\mu}) = \mu$, then for an arbitrary closed set $\hat{F} \subset \hat{\Lambda}$, we have that

$$\hat{\mu}(\hat{F}) = \lim_n \mu(\{x_{-n}, \exists \hat{x} = (x, \ldots, x_{-n}, \ldots) \in \hat{F}\}) \qquad (8)$$

Let us prove (8): first denote $\hat{F}_n := \hat{f}^{-n}\hat{F}, n \geq 1$; next notice that $\hat{\mu}(\hat{F}_n) = \hat{\mu}(\hat{F})$ since $\hat{\mu}$ is $\hat{f}$-invariant. Let also $\hat{G}_n := \pi^{-1}(\pi(\hat{F}_n)), n \geq 1$. We have $\hat{F} \subset \hat{f}^n(\hat{G}_n), n \geq 0$. Let now a prehistory $\hat{z} \in \bigcap_{n\geq 0} \hat{f}^n\hat{G}_n$; then if $\hat{z} = (z, z_{-1}, \ldots, z_{-n}, \ldots)$, we obtain that $z_{-n} \in \pi\hat{F}_n, \forall n \geq 0$, hence $\hat{z} \in \hat{F}$ since $\hat{F}$ is assumed closed. Thus we obtain $\hat{F} = \bigcap_{n\geq 0} \hat{f}^n(\hat{G}_n)$. Now the above intersection is decreasing, since $\hat{f}^{n+1}\hat{G}_{n+1} \subset \hat{f}^n\hat{G}_n, n \geq 0$. Since the above intersection is decreasing, we get that $\hat{\mu}(\hat{F}) = \lim_n \hat{\mu}(\hat{f}^n\hat{G}_n) = \lim_n \hat{\mu}(\hat{G}_n) = \lim_n \hat{\mu}(\pi^{-1}(\pi(\hat{F}_n))) = \lim_n \mu(\pi(\hat{F}_n)) = \lim_n \mu(\pi \circ \hat{f}^{-n}\hat{F})$, since $\hat{\mu}$ is $\hat{f}$-invariant. Therefore we obtain (8).

For a positive integer n, a continuous real function g defined on the neighbourhood U of Λ, and a point y so that $y, f(y), \ldots, f^{n-1}(y)$ are all in U, let us denote by

$$\Sigma_n(g, y) := \frac{g(y) + \ldots + g(f^{n-1}y)}{n} - \int g d\mu_s, n \geq 1, y \in \Lambda$$

Now from the convergence in $L^1(\hat{\Lambda}, \hat{\mu}_s)$ norm established in (7), it follows the convergence in $\hat{\mu}_s$-measure; i.e if we consider for a small $\eta > 0$ and an integer $n > 1$ the closed set:

$$\hat{F}_n(\eta) = \{\hat{x} = (x, x_{-1}, x_{-2}, \ldots) \in \hat{\Lambda}, |\Sigma_n(g, x_{-n})| \geq \eta\},$$

then we have the convergence

$$\hat{\mu}_s(\hat{F}_n(\eta)) \underset{n\to\infty}{\to} 0, \forall \eta > 0. \tag{9}$$

Thus from (9), (8) and the f-invariance of μ_s, we obtain that for any small $\eta > 0, \chi > 0$ there exists an integer $N(\eta, \chi) \geq 1$ so that:

$$\mu_s(x_{-n'} \in \Lambda \cap f^{-n'+n}(x_{-n}), |\Sigma_n(g, x_{-n})| \geq \eta) = \mu_s(x_{-n} \in \Lambda, |\Sigma_n(g, x_{-n})| \geq \eta) \ < \chi, \tag{10}$$

for $n' > n > N(\eta, \chi)$.

Let us consider now some small $\varepsilon > 0$. Recall that for $n \geq 1$ and $y \in \Lambda$, the Bowen ball $B_n(y, \varepsilon) := \{z \in M, d(f^i y, f^i z) < \varepsilon, i = 0, \ldots, n-1\}$. We shall prove that if $y \in \Lambda$ and $z \in B_n(y, \varepsilon)$ for n large enough, then the behaviour of $\Sigma_n(g, z)$ is similar to that of $\Sigma_n(g, y)$. More precisely, assume that $\eta > 0$ and that $y \in \Lambda$ satisfies $|\Sigma_n(g, y)| \geq \eta$. Then we will show that there exists $N(\eta) \geq 1$ so that

$$|\Sigma_n(g, z)| \geq \frac{\eta}{2}, \forall z \in B_n(y, \varepsilon), n > N(\eta) \tag{11}$$

Since g was assumed Holder, let us assume that it has a Holder exponent equal to α, i.e

$$|g(x) - g(y)| \leq C \cdot d(x, y)^\alpha, \forall x, y \in U,$$

where $d(x, y)$ is the Riemannian distance (from M) between x and y and $C > 0$ is a constant. The idea now is that, if $z \in B_n(y, \varepsilon)$, then for some time the iterates of z follow the iterates of y close to stable manifolds, and afterwards they follow the iterates of y closer and closer to unstable manifolds. We have in both cases an exponential growth of distances between iterates, and thus we can use the Holder continuity of g on U.

If $z \in B_n(y, \varepsilon), y \in \Lambda$ then we either have $z \in W^s_\varepsilon(y) \subset \Lambda$ or there exists a positive distance between z and the local stable manifold $W^s_\varepsilon(y)$. In the first case there exists some $\lambda_s \in (0, 1)$ such that $d(f^i z, f^i y) < \lambda_s^i \varepsilon, i = 0, \ldots, n-1$. This implies that, in the case when $z \in W^s_\varepsilon(y)$, for some $N_0 \geq 1$ we have:

$$|g(f^{N_0} y) + \ldots + g(f^{n-1} y) - g(f^{N_0} z) - \ldots - g(f^{n-1} z)| \leq \lambda_s^{\alpha N_0} \cdot C_0, \tag{12}$$

for some constant $C_0 > 0$ independent of n. If $z \in B_n(y, \varepsilon)$ but z is not necessarily on $W^s_\varepsilon(y)$, then the iterates of z will approach exponentially some local unstable manifolds at the corresponding iterates of y and their "projections" on these unstable manifolds increases exponentially, up to a maximum value less than ε (reached at level n). More precisely there exists some $N_0, N_1 \geq 1$ and some $\lambda \in (\lambda_s, 1)$ such that $d(f^i z, f^i y) \leq \lambda^i, i = N_0, \ldots, N_1 - 1$; notice that N_0, N_1, λ are independent of y, z, n. Now if the iterate $f^{N_1} z$ becomes much closer to $W^u_\varepsilon(f^{N_1} y)$ than to $W^s_\varepsilon(f^{N_1} y)$, it follows that all the higher order iterates will approach asymptotically the local unstable manifolds and $d(f^j y, f^j z)$ increases exponentially. We assume that N_1 has been taken such that for some $\lambda_u \in (\frac{1}{\inf_\Lambda |Df_u|}, 1)$, we have $d(f^j z, f^j y) \leq \lambda_u \cdot d(f^{j+1} z, f^{j+1} y), j = N_1, \ldots, n-2$. So the maximum such distance is $d(f^{n-1} y, f^{n-1} z)$ and we know that $d(f^{n-1} y, f^{n-1} z) < \varepsilon$ since $z \in B_n(y, \varepsilon)$. Hence

$$d(f^j z, f^j y) \leq \varepsilon \lambda_u^{n-j-1}, j = N_1, \ldots, n-1$$

Let us take now some $N_2 \geq 1$ such that $n - N_2 > N_1$; N_2 will be determined later. Thus from the Holder continuity of g on U we obtain (for some positive constant C) that:

$$\begin{aligned}
&|g(f^{N_0}z) + \ldots + g(f^{N_1-1}z) + g(f^{N_1}z) + \ldots + g(f^{n-N_2}z) + \ldots + g(f^{n-1}z) \\
&- g(f^{N_0}y) - \ldots g(f^{N_1-1}y) - g(f^{N_1}y) - \ldots - g(f^{n-N_2}y) - \ldots - g(f^{n-1}y)| \leq \\
&\leq C(\lambda^{\alpha N_0} + \lambda_u^{\alpha N_2}) + 2N_2||g||
\end{aligned} \tag{13}$$

Thus from (12) and (13) we obtain that, if $z \in B_n(y, \varepsilon)$ then:

$$|\Sigma_n(g, y) - \Sigma_n(g, z)| \leq \frac{1}{n}\left[2N_0||g|| + C(\lambda^{\alpha N_0} + \lambda_u^{\alpha N_2}) + 2N_2||g||\right] \tag{14}$$

From above, N_0, N_2 do not depend on n, y, z. Therefore we can choose some large $N(\eta)$ so that

$$\frac{1}{n}(2N_0||g|| + C(\lambda^{\alpha N_0} + \lambda_u^{\alpha N_2}) + 2N_2||g||) < \eta/2, \text{for } n > N(\eta)$$

This means that the relation from (11) holds. Let us denote now by:

$$I_n(g, x) := \frac{1}{d^n} \sum_{y \in f^{-n}(x) \cap U} |\Sigma_n(g, y)|, \tag{15}$$

for a continuous real function $g : U \to \mathbb{R}$, and $x \in V$. Recall that V is the neighbourhood of Λ, $\Lambda \subset V \subset U$, constructed in the proof of Theorem 3.3.3 so that every point $x \in V$ has d^n n-preimages in U for $n \geq 1$. For a fixed Holder continuous function g and a small $\eta > 0$, we will work with $n > N(\eta)$, where $N(\eta)$ was found above. From (14) and the discussion afterwards, we know that $|\Sigma_n(g, z) - \Sigma_n(g, y)| \leq \eta/2$ if $z \in B_n(y, \varepsilon)$ and $y \in \Lambda$.

Let us consider now an (n, ε)-separated set with maximal cardinality in Λ, denoted by $F_n(\varepsilon)$. It follows that any point $y \in V$ belongs to d^n tubular neighbourhoods, i.e $f^n(B_n(y_i, 3\varepsilon)), y_i \in F_n(\varepsilon)$ for $1 \leq i \leq d^n$. Let us denote as before $V_n(y_1, \ldots, y_{d^n}) := \bigcap\limits_{1 \leq i \leq d^n} f^n B_n(y_i, 3\varepsilon)$. Thus in the integral $\int_V I_n(g, x)dm(x)$, we can decompose V into the smaller pieces $V_n(y_1, \ldots, y_{d^n})$, for different choices of $y_1, \ldots, y_{d^n} \in F_n(\varepsilon)$.

We can use now relation (14) in order to replace in $\int_V I_n(g, x)dm(x)$, the term $|\Sigma_n(g, y)|$ with $|\Sigma_n(g, \zeta)|$, where $x \in V$ is arbitrary, $y \in f^{-n}x \cap U$ and $y \in B_n(\zeta, 3\varepsilon)$ for some $\zeta \in F_n(\varepsilon)$. Indeed let us fix some arbitrary small $\eta > 0$. Then we prove similarly as in (14) that if $n > N(\eta)$, then $|\Sigma_n(g, y)| \leq |\Sigma_n(g, \zeta)| + \eta/2$, if $y \in B_n(\zeta, 3\varepsilon)$ and $\zeta \in F_n(\varepsilon)$ ($N(\eta)$ can be assumed to be the same as in (14) without loss of generality).

So up to a small error of $\eta/2$ we can replace each of the terms $|\Sigma_n(g, y)|$ with the corresponding $|\Sigma_n(g, \zeta)|$. This implies that in the integral $\int_V I_n(g, x)dm(x)$, on each piece of type $V_n(y_1, \ldots, y_{d^n})$ in $f^n(B_n(y_j, 3\varepsilon))$ for $y_j \in F_n(\varepsilon)$, we integrate in fact $|\Sigma_n(g, y_j)|$, modulo an error of $\eta/2$. Then we will obtain that

$$\int_V I_n(g,x)dm(x) \le \frac{1}{d^n} \sum_{z_1,\dots,z_{d^n} \in F_n(\varepsilon)} \int_{V_n(z_1,\dots,z_{d^n})} \sum_{i=1}^n |\Sigma_n(g,z_i)|dm + \frac{\eta}{2} \cdot m(V)$$
$$\le \frac{1}{d^n} \sum_{z \in F_n(\varepsilon)} |\Sigma_n(g,z)| \cdot \sum_{z \in \{z_1,\dots,z_{d^n}\}} m(V_n(z_1,\dots,z_{d^n})) + m(V)\eta/2$$
$$\le \frac{1}{d^n} \sum_{z \in F_n(\varepsilon)} |\Sigma_n(g,z)| \cdot m(f^n B_n(z,3\varepsilon)) + m(V)\eta/2$$

So what we did is, we replaced $|\Sigma_n(g,y)|$ with $|\Sigma_n(g,z)|$ for all $y \in f^{-n}x \cap U$, where $y \in B_n(z,3\varepsilon), z \in F_n(\varepsilon)$, then we integrated the respective sums of $|\Sigma_n(g,z)|, z \in F_n(\varepsilon)$ on small pieces of tubular overlap $V_n(z_1,\dots,z_{d^n})$; lastly, we kept $|\Sigma_n(g,z)|$ fixed for an arbitrary $z \in F_n(\varepsilon)$ and added the measures of all intersections of $f^n B_n(z,3\varepsilon)$ with other tubular sets of type $f^n B_n(w,3\varepsilon), w \in F_n(\varepsilon)$. Thus by adding the measures of these overlaps, we recover $m(f^n B_n(z,3\varepsilon))$. In conclusion we obtain:

$$\int_V I_n(g,x)dm(x) \le C \cdot \sum_{y \in F_n(\varepsilon)} |\Sigma_n(g,y)| \cdot \frac{m(f^n(B_n(y,3\varepsilon))}{d^n} + \frac{\eta}{2} \cdot m(V) \tag{16}$$

But now recall that $m(f^n(B_n(y,3\varepsilon)))$ is comparable to $e^{S_n\Phi^s(y)}$, independently of $n, y \in \Lambda$. And from Theorem 3.3.2 we know that $P(\Phi^s) = \log d$. Hence from [46] it follows that, if μ_s denotes the unique equilibrium measure of Φ^s, then $\mu_s(B_n(y,\varepsilon/2))$ is comparable to $\frac{e^{S_n\Phi^s(y)}}{d^n}$, independently of n, y. Therefore combining with (16) we obtain that there exists a constant $C_1 > 0$ s.t:

$$\int_V I_n(g,x)dm(x) \le C_1 \left(\sum_{y \in F_n(\varepsilon)} |\Sigma_n(g,y)| \mu_s(B_n(y,\varepsilon/2)) + \eta \right), \tag{17}$$

for $n > N(\eta)$. We will split now the points of $F_n(\varepsilon)$ in two disjoint subsets denoted by $G_1(n,\varepsilon)$ and $G_2(n,\varepsilon)$, defined as follows:

$$G_1(n,\varepsilon) := \{y \in F_n(\varepsilon), |\Sigma_n(g,y)| < \eta\} \text{ and } G_2(n,\varepsilon) := \{z \in F_n(\varepsilon), |\Sigma_n(g,z)| \ge \eta\}$$

Recall that the Bowen balls $B_n(y,\varepsilon/2), y \in F_n(\varepsilon)$ are mutually disjointed since $F_n(\varepsilon)$ is (n,ε)-separated. Also if $y \in G_2(n,\varepsilon)$ and $z \in B_n(y,\varepsilon/2)$, we have $|\Sigma_n(g,z)| \ge \eta/2$ (from (11)); hence $B_n(y,\varepsilon/2) \cap \Lambda \subset \{z \in \Lambda, |\Sigma_n(g,z)| \ge \eta/2\}$. Consequently for a constant $C_\varepsilon > 0$,

$$\sum_{y \in F_n(\varepsilon)} |\Sigma_n(g,y)| \mu_s(B_n(y,\varepsilon/2)) = \sum_{y \in G_1(n,\varepsilon)} |\Sigma_n(g,y)| \mu_s(B_n(y,\varepsilon/2)) + \sum_{y \in G_2(n,\varepsilon)} |\Sigma_n(g,y)| \mu_s(B_n(y,\varepsilon/2)) \le$$
$$\le \eta \sum_{y \in G_1(n,\varepsilon)} \mu_s(B_n(y,\varepsilon/2)) + 2||g|| \mu_s(z \in \Lambda, |\Sigma_n(g,z)| \ge \frac{\eta}{2}) \cdot C_\varepsilon$$

But since the balls $B_n(y,\varepsilon/2), y \in F_n(\varepsilon)$ are mutually disjoint, we have $\sum\limits_{y \in G_1(n,\varepsilon)} \mu_s(B_n(y,\varepsilon/2)) \le 1$. Also $\mu_s(z \in \Lambda, |\Sigma_n(g,z)| \ge \eta/2) < \chi$ for $n > N(\eta/2,\chi)$, as follows from (10). Thus by using (31) we obtain for $n > \sup\{N(\eta), N(\eta, \chi)\}$

$$\int_V I_n(g,x)dm(x) \leq C_1(\eta + \eta + C_\varepsilon \cdot 2||g||\chi) = 2C_1(\eta + \chi \cdot C_\varepsilon ||g||)$$

Since $\eta, \chi > 0$ were taken arbitrarily, and by recalling the formula for $I_n(g,x)$ from (15) and the definition of μ_n^z, we obtain that:

$$\int_V |\mu_n^z(g) - \mu_s(g)|dm(z) \underset{n\to\infty}{\to} 0$$

Since Holder continuous functions g are dense in the uniform norm on $\mathcal{C}(U)$, we obtain the conclusion of the Theorem for all $g \in \mathcal{C}(U)$.

□

Corollary 3.3.1 ([46]). *In the same setting as in Theorem 3.3.3, it follows that there exists a borelian set $A \subset V$ with $m(V \setminus A) = 0$ and a subsequence $(n_k)_k$, such that for any point $z \in A$, we have the following weak convergence of measures on U*

$$\mu_{n_k}^z \underset{k\to\infty}{\to} \mu_s$$

Moreover we proved in [46] that a property like the one satisfied by usual SRB measures in regard to their conditional measures on unstable manifolds, is verified now by the inverse SRB measure, but on local stable manifolds:

Theorem 3.3.4. *([46]) Let Λ be a connected hyperbolic repellor for a smooth endomorphism $f : M \to M$ on a Riemannian manifold M; assume that f is d-to-1 on Λ. Then there exists a unique f-invariant probability measure μ^- on Λ satisfying an inverse Pesin entropy formula:*

$$h_{\mu^-}(f) = \log d - \int_\Lambda \sum_{i, \lambda_i(x) < 0} \lambda_i(x) m_i(x) d\mu^-(x)$$

In addition the measure μ^- has absolutely continuous conditional measures on local stable manifolds.

Connected hyperbolic repellors are very useful as examples since they preserve at perturbation the property of having a constant number of preimages of any point, remaining in the repellor (Proposition 3.3.1 above). Also, their hyperbolicity and connectedness are preserved by perturbations. Therefore one can construct examples like the one below, from [46]:

Example. Let us take $F : \mathbb{PC}^1 \times \mathbb{T}^2 \to \mathbb{PC}^1 \times \mathbb{T}^2$ given by:

$F([z_0 : z_1], (x,y)) = ([z_0^2 : z_1^2], f_A(x,y))$, where f_A is the toral endomorphism induced by the matrix $A = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$. Then F has a connected hyperbolic repellor $\Lambda := S^1 \times \mathbb{T}^2$. Consider the following perturbation of F, $F_\varepsilon : \mathbb{PC}^1 \times \mathbb{T}^2 \to \mathbb{PC}^1 \times \mathbb{T}^2$ given by:

$$F_\varepsilon([z_0 : z_1], (x,y)) = \big([z_0^2 + \varepsilon z_1^2 \cdot e^{2\pi i(2x+y)} : z_1^2], (2x + y + \varepsilon sin(2\pi(x+y)), 2x + 2y + \varepsilon cos^2(4\pi x))\big)$$

Then F_ε is well defined as a smooth endomorphism on $\mathbb{PC}^1 \times \mathbb{T}^2$ and it is a $\mathcal{C}^1$ perturbation of F. It follows from the discussion above that F_ε has a connected hyperbolic repellor Λ_ε (on which F_ε has both stable as well as unstable directions), and that Λ_ε is close to Λ. However Λ_ε is different from Λ, and it has a complicated structure with self-intersections; its projection on the second coordinate is $\mathbb{T}^2$. On Λ_ε we can apply Theorem 3.3.3 to get a physical measure μ_ε^- for the local inverse iterates of F_ε. This physical measure μ_ε^- is the equilibrium measure of the non-constant stable potential

$$\Phi_\varepsilon^s([z_0 : z_1], (x, y)) := \log |\det(DF_\varepsilon)_s([z_0 : z_1], (x, y))|, \text{for } ([z_0 : z_1], (x, y)) \in \Lambda_\varepsilon$$

$\square$

3.4 Statistical properties of equilibrium measures on folded fractals

The asymptotic distribution of preimages for expanding maps is given by equilibrium measures (see [81]). However, if the basic set Λ is of **saddle type**, the problem is very different and needs new methods for the proof. We lack the fact that the local inverse iterates act as contractions on small balls; in fact they are dilations in the stable directions in backward time, and this is changing completely the situation and the ideas of proof. In [38] we solved the above problem of the weighted preimage distribution with a Holder continuous weight ϕ, along a *general hyperbolic basic set* (i. e **not** necessarily a repellor, and **not** necessarily for an expanding map):

Theorem 3.4.1. *([38]) Let $f : M \to M$ be a smooth (say $\mathcal{C}^2$) map on a Riemannian manifold M, which is hyperbolic and finite-to-one on a basic set Λ so that $\mathcal{C}_f \cap \Lambda = \emptyset$. Assume that ϕ is a Holder continuous potential on Λ and that μ_ϕ is the equilibrium measure of ϕ on Λ. Then*

$$\int_\Lambda | < \frac{1}{n} \sum_{y \in f^{-n}x \cap \Lambda} \frac{e^{S_n\phi(y)}}{\sum\limits_{z \in f^{-n}x \cap \Lambda} e^{S_n\phi(z)}} \cdot \sum_{i=0}^{n-1} \delta_{f^i y} - \mu_\phi, g > | d\mu_\phi(x) \underset{n \to \infty}{\to} 0, \forall g \in \mathcal{C}(\Lambda, \mathbb{R})$$

The proof of this Theorem is difficult and is based on a careful study of the measure μ_ϕ of various pieces of Bowen balls, and of iterates of Bowen balls; one has to estimate the measure of the set of n-preimages y_{-n} behaving badly, i.e on which the consecutive averages $\frac{\phi(y)+\ldots+\phi(y_{-n})}{n}$ oscillate more than some ε from their median value $\int \phi d\mu_\phi$. As a Corollary, we obtained in [38] the following result giving the weak convergence of the above atomic measures along the same subsequence, for all points in a set of full μ_ϕ-measure, in the case of a basic set of saddle type Λ and a smooth non-invertible map f :

Corollary 3.4.1. *([38]) In the same setting as in Theorem 3.4.1, for any Holder potential ϕ, it follows that there exists a subset $E \subset \Lambda$, with $\mu_\phi(E) = 1$ and an infinite subsequence $(n_k)_k$ such that for any $z \in E$ we have the weak convergence of measures*

$$\mu_{n_k}^z \underset{k \to \infty}{\to} \mu_\phi$$

Corollary 3.4.2. *([38]) Assume that $f : M \to M$ is an Anosov endomorphism without critical points on a Riemannian manifold. Let also ϕ a Holder continuous potential on M and μ_ϕ the equilibrium measure of ϕ. Then*

$$\int_M | < \frac{1}{n} \sum_{y \in f^{-n}x} \frac{e^{S_n\phi(y)}}{\sum\limits_{z \in f^{-n}x} e^{S_n\phi(z)}} \cdot \sum_{i=0}^{n-1} \delta_{f^i y} - \mu_\phi, g > |d\mu_\phi(x) \underset{n\to\infty}{\to} 0, \forall g \in \mathcal{C}(\Lambda, \mathbb{R})$$

In particular, if μ_0 is the measure of maximal entropy, it follows that for μ_0-almost all points $x \in \Lambda$, $\frac{1}{n} \sum\limits_{y \in f^{-n}x} \frac{\sum\limits_{i=0}^{n-1} \delta_{f^i y}}{Card(f^{-n}x)} \underset{n\to\infty}{\to} \mu_0$

By applying some results from [32] and [75] we obtain sufficient conditions when the usual SRB measure is equal to our inverse SRB measure (see [46], [38]):

Corollary 3.4.3. *Let $f : M \to M$ be an Anosov endomorphism, $\phi : \Lambda \to \mathbb{R}$ a Holder potential and assume that the equilibrium measure μ_ϕ is absolutely continuous with respect to the Lebesgue measure on M. Then the measure μ_ϕ with this property is unique, it is an SRB measure and it also satisfies an inverse SRB condition in the sense that there exists a set E of full Lebesgue measure in M and a sequence $(n_k)_k$ such that $\mu^z_{n_k} \underset{k}{\to} \mu_\phi, z \in E$.*

One can give examples of Anosov endomorphisms so that each point has a constant number of preimages; take for instance the hyperbolic toral endomorphism $f_A : \mathbb{T}^m \to \mathbb{T}^m, m \geq 2$, given by an integer-valued matrix A whose eigenvalues λ_i all have absolute values different from 1. Each point of $\mathbb{T}^m$ has exactly $|\det A|$ preimages in $\mathbb{T}^m$. Then for any Hölder continuous potential ϕ on $\mathbb{T}^m$, one can apply the Corollary 5.1.5 in order to obtain the **weighted distribution of all n-preimages on** $\mathbb{T}^m$, asymptotically converging to the equilibrium measure μ_ϕ, when $n \to \infty$. In particular, if $\phi \equiv 0$, we obtain the distribution of the atomic measures supported on preimages of order smaller than n, towards the measure of maximal entropy (i. e towards the Lebesgue measure on $\mathbb{T}^m$). We notice also that Corollary 5.1.5 applies for Anosov endomorphisms on infranilmanifolds (see for example [34] for definitions). Moreover, Theorem 3.4.1 applies also to basic sets of saddle type which are not necessarily Anosov, like the examples from [59].

Two statistical properties of interest are the *exponential decay of correlations* and *mixing of any order*. We studied these properties for equilibrium measures of Hölder potentials on hyperbolic folded basic sets in [37]. First let us remind some definitions, from [5], [10], [77], [78], [96], etc.

Definition 3.4.1. *Given a transformation $f : M \to M$ we say that an f-invariant probability μ has* ***Exponential Decay of Correlations*** *on Hölder potentials, if there is some $\lambda \in (0, 1)$ such that for every integer $n \geq 1$,*

$$|\int \phi \cdot \psi \circ f^n d\mu - \int \phi d\mu \cdot \int \psi d\mu| \leq C(\phi, \psi)\lambda^n,$$

for any Hölder functions $\phi, \psi \in \mathcal{C}(M, \mathbb{R})$, where $C(\phi, \psi)$ depends only on the potentials ϕ, ψ.

Now for a Lebesgue space $(X, \mathcal{B}(X), \mu)$ and an endomorphism $f : X \to X$ preserving the probability measure μ, we say that $(X, f, \mathcal{B}(X), \mu)$ is an **exact endomorphism** ([77], [78], [34]) if

$$\bigcap_{n \geq 0} f^{-n}\mathcal{B}(X) = \mathcal{N},$$

where $\mathcal{N}$ is the σ-algebra containing only sets of μ-measure 0 or 1. Rokhlin [78], proved that $(X, f, \mathcal{B}(X), \mu)$ is exact if and only if for any measurable set of positive measure $A \subset X$, we have $\lim\limits_{n \to \infty} \mu(f^n A) = 1$.

Definition 3.4.2. *Consider m-tuples of positive integers $\Delta = (k_1, \ldots, k_m)$ and denote by $\ell(\Delta) := \inf |k_i - k_j|, 1 \leq i < j \leq m$. We say that the Lebesgue space $(X, f, \mathcal{B}(X), \mu)$ is **mixing of order m**, if for any arbitrary measurable sets $A_1, \ldots, A_m$ and for any sequences of m-tuples $\Delta^1 = (k_1^1, \ldots, k_m^1), \Delta^2 = (k_1^2, \ldots, k_m^2), \ldots$, with $\lim\limits_{n \to \infty} \ell(\Delta^n) = \infty$, we have*

$$\lim_{n \to \infty} \mu(\bigcap_{i=1}^{m} f^{-k_i^n} A_i) = \prod_{i=1}^{m} \mu(A_i)$$

If $m = 2$ we obtain the usual notion of mixing measure. An exact endomorphism is mixing of any order, as shown in [78]; however the converse is not true. For equilibrium measures on folded hyperbolic fractals we proved that:

Theorem 3.4.2 ([37])**.** *Let f be a smooth endomorphism on M, which is hyperbolic on the basic set Λ, and let ϕ be a Hölder continuous potential on Λ with its equilibrium measure μ_ϕ. Then:*
a) the measure-preserving system $(\Lambda, f|_\Lambda, \mu_\phi)$ is mixing of any order.
b) the probability measure μ_ϕ has Exponential Decay of Correlations on Hölder potentials.

Proof. a) The map f is uniformly hyperbolic on Λ, so as in [27], pg. 272 we obtain that f has local product structure on Λ, and thus $\hat{f}$ has local product structure on $\hat{\Lambda}$ with local stable sets (defined for some $\delta > 0$ small enough):

$$V_{\hat{x}}^{-} := \{\hat{y} \in \hat{\Lambda}, d(\hat{f}^n \hat{y}, \hat{f}^n \hat{x}) < \delta, n \geq 0\},$$

and local unstable sets

$$V_{\hat{x}}^{+} := \{\hat{y} \in \hat{\Lambda}, d(\hat{f}^{-n} \hat{y}, \hat{f}^{-n} \hat{x}) < \delta, n \geq 0\}, \hat{x} \in \hat{\Lambda}$$

Thus $(\hat{\Lambda}, \hat{f})$ has a Smale space structure, as defined in [84]. Now since the potential ϕ on Λ is Hölder continuous and as $\pi : \hat{\Lambda} \to \Lambda$ is Lipschitz continuous, it follows that $\hat{\phi} := \phi \circ \pi : \hat{\Lambda} \to \mathbb{R}$ is Hölder continuous; hence, to the equilibrium measure μ_ϕ of ϕ it corresponds the unique equilibrium measure $\mu_{\hat{\phi}}$ of $\hat{\phi}$ on $\hat{\Lambda}$ s.t $\mu_\phi = \pi_* \mu_{\hat{\phi}}$. We have then $P_f(\phi) = P_{\hat{f}}(\hat{\phi})$ and $h_{\mu_\phi}(f) = h_{\mu_{\hat{\phi}}}(\hat{f})$. Also $\int_\Lambda \phi d\mu_\phi = \int_{\hat{\Lambda}} \phi \circ \pi d\mu_{\hat{\phi}}$.

Now we assumed that f is topologically mixing on Λ (or if $f|_\Lambda$ is topologically transitive, then we can partition Λ into subsets on which some iterates of f are topologically mixing); this implies easily that $\hat{f}$ is topologically mixing on $\hat{\Lambda}$. Then from [84] Corollary 7.10 d) we have that $(\hat{\Lambda}, \hat{f}, \mu_{\hat{\phi}})$

is isomorphic to a Bernoulli automorphism. Hence as Bernoulli automorphisms are Kolmogorov (by [34], pg. 161), it follows that $(\hat{\Lambda}, \hat{f}, \mu_{\hat{\phi}})$ is mixing of any order. Therefore $(\Lambda, f|_\Lambda, \mu_\phi)$ is mixing of any order (see [78]).

b) We have Exponential Decay of Correlations on Hölder potentials, for the inverse limit transformation $(\hat{\Lambda}, \hat{f}, \hat{\mu}_\phi)$ since, from a) this is a Bernoulli automorphism (see [5], [10]).

Then due to the bijective correspondence between f-invariant probabilities on Λ and $\hat{f}$-invariant probabilities on $\hat{\Lambda}$, and from the invariance of measure-theoretic entropies, we obtain also the exponential decay of correlations on Hölder potentials for the endomorphism on Lebesgue spaces $(\Lambda, f|_\Lambda, \mu_\phi)$.

$\square$

3.5 Jacobians and asymptotic degrees with respect to equilibrium measures.

In the case when an endomorphism is not constant-to-one on a basic set Λ, there appears the question what do we understand by "degree" of the map on Λ. It turns out that in general one cannot define a good substitute for the degree, unless one considers it in the framework of invariant measures. Thus, it is better to define a notion of degree with respect to an invariant (especially equilibrium) measure. Also we know that usually the behaviour of a map on Λ is basically the same with that of its iterates; however it is better in general to consider iterates of higher and higher order, so that the trajectories become more uniformly spread within the set Λ (on average). In [54] we investigated the problems associated to such an asymptotic degree with respect to equilibrium measures, and also its relations to the Jacobians of such measures.

The *Jacobian* of an invariant measure μ with respect to an endomorphism f of a Lebesgue space X (see Parry, [67]) describes locally the ratio between $\mu(f(A))$ and $\mu(A)$, given that an arbitrary point in X may have several f-preimages and that, by invariance $\mu(f(A)) = \mu(f^{-1}(f(A)))$. Let $f : M \to M$ be a continuous endomorphism on the manifold M and μ an f-invariant probability on M; assume also that f is at most countable-to-one. Then as shown by Rohlin ([77], [67]), there exists a measurable partition $\xi = (A_0, A_1, \ldots)$ so that f is injective on each A_i, and the push-forward measure $((f|_{A_i})^{-1})_*\mu$ is absolutely continuous on A_i with respect to μ. The respective Radon-Nykodim derivative, will be called the **Jacobian** of μ with respect to f:

$$J_f(\mu)(x) = \frac{d\mu \circ (f|_{A_i})}{d\mu}(x),\ \mu - \text{a.e on } A_i, i \geq 0$$

Notice that from the f-invariance of μ, we have $J_f(\mu)(x) \geq 1, \mu - \text{a.e } x \in M$.

We proved the following result about the Jacobians of equilibrium measures, with respect to iterates of endomorphisms:

Proposition 3.5.1 ([54]). *Let f be a $\mathcal{C}^2$ hyperbolic endomorphism, restricted to a folded basic set Λ, which has no critical points in Λ; let also ϕ a Hölder continuous potential on Λ and let*

μ_ϕ the unique equilibrium measure of ϕ on Λ. Then there exists a comparability constant $C > 0$ independent of m, x s.t for $\mu_\phi - a.e$ $x \in \Lambda$ the Jacobian of μ_ϕ with respect to f^m satisfies:

$$C^{-1} \cdot \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}} \leq \ J_{f^m}(\mu_\phi)(x) \leq \ C \cdot \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}}, \tag{18}$$

Recall now (see [73]) that in the expanding case we have the following formula for pressure:

Theorem (Relation between preimage sets and pressure in the expanding case, [73]). *Let $f : X \to X$ a topologically transitive open distance expanding map, then for every Hölder continuous potential $\phi : X \to \mathbb{R}$ and every $x \in X$ we have the equality*

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{y \in f^{-n}(x)} e^{S_n\phi(y)} = P(\phi)$$

However for folded basic sets of saddle type this is no longer true, as we showed in [54]. First let us give a definition that will be used also in the next subsection; it gives the average entropy coming from the folding of the fractal under the endomorphism (for example if the map is a homeomorphism when restricted to Λ, then the folding entropy is equal to 0). For the next definition, see for instance [80], [79] or [32].

Definition 3.5.1. *Let $f : M \to M$ be an endomorphism and μ an f-invariant probability on M, then the **folding entropy** $F_f(\mu)$ of μ is the conditional entropy: $F_f(\mu) := H_\mu(\epsilon | f^{-1}\epsilon)$, where ϵ is the partition into single points.*

Theorem 3.5.1 (Relation between preimage sets and pressure in the saddle case, [54]). *Consider an endomorphism f which is hyperbolic on a saddle basic set Λ. Then for an arbitrary Hölder continuous potential ϕ on Λ and for its associated equilibrium measure μ_ϕ, we have that for μ_ϕ-a.e $x \in \Lambda$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{y \in f^{-n}(x)} e^{S_n\phi(y)} = P(\phi) + F_f(\mu_\phi) - h_{\mu_\phi}$$

Proof. From the properties of the Jacobian of μ_ϕ we know that it satisfies the Chain Rule, i.e $J_{f\circ g}(\mu_\phi)(x) = J_f(\mu_\phi)(g(x)) \cdot J_g(\mu_\phi)(x)$ for μ_ϕ-a.e $x \in \Lambda$. Hence μ_ϕ-a.e,

$$\log J_{f^m}(\mu_\phi)(x) = \log J_f(\mu_\phi)(x) + \ldots + \log J_f(\mu_\phi)(f^{m-1}(x))$$

This means that we can apply Birkhoff Ergodic Theorem and obtain that

$$\frac{\log J_{f^m}(\mu_\phi)}{m} \underset{m\to\infty}{\to} \int_\Lambda \log J_f(\mu_\phi) d\mu_\phi = F_f(\mu_\phi)$$

Apply now Proposition 3.5.1 to get μ_ϕ-a.e the convergence

$$\frac{\log \sum\limits_{y \in f^{-m}(f^m x)} e^{S_m\phi(y)} - \log e^{S_m\phi(x)}}{m} \underset{m\to\infty}{\to} F_f(\mu_\phi) \tag{19}$$

However from the Birkhoff Ergodic Theorem, $\frac{S_m\phi(x)}{m} \to \int \phi d\mu_\phi$ for μ_ϕ-a.e $x \in \Lambda$. Hence from (19) and from the fact that $P(\phi) = \int \phi d\mu_\phi + h_{\mu_\phi}$, we obtain:

$$\frac{\log \sum\limits_{y \in f^{-m}(f^m x)} e^{S_m\phi(y)}}{m} \underset{m\to\infty}{\to} F_f(\mu_\phi) + P(\phi) - h_{\mu_\phi}$$

□

We will give now a formula for the folding entropy of the equilibrium measure μ_ϕ in terms of an **"asymptotic logarithmic degree"** with respect to μ_ϕ. This will take into account at step n the n-preimages of points which behave well (are generic) with respect to μ_ϕ. To this end, for an f-invariant probability (borelian) measure μ on Λ let us define, for any small $\tau > 0$, $n > 0$ integer and $x \in \Lambda$ the set

$$G_n(x, \mu, \tau) := \{y \in f^{-n}(f^n x) \cap \Lambda, \text{ s.t } |\frac{S_n\phi(y)}{n} - \int \phi d\mu| < \tau\}, \tag{20}$$

Definition 3.5.2. *In the above setting, denote by $d_n(x, \mu, \tau) := CardG_n(x, \mu, \tau), x \in \Lambda, n > 0, \tau > 0$. The function $d_n(\cdot, \mu, \tau)$ is measurable, nonnegative and finite on Λ.*

We can now give the relation between the asymptotic degree and the folding entropy, proved in [54]:

Theorem 3.5.2 (Asymptotic logarithmic degree in terms of the folding entropy for μ_ϕ, [54])**.** *Let $f : M \to M$ be a $\mathcal{C}^2$ endomorphism and Λ a basic set for f so that f is hyperbolic on Λ and does not have critical points in Λ. Let also ϕ a Hölder continuous potential on Λ and μ_ϕ the equilibrium measure associated to ϕ. Then we have the following formula:*

$$F_f(\mu_\phi) = \lim_{\tau\to 0} \lim_{n\to\infty} \frac{1}{n} \int_\Lambda \log d_n(x, \mu_\phi, \tau) \; d\mu_\phi(x)$$

Therefore Theorem 3.5.2 allows us to define the *asymptotic logarithmic degree of* $f|_\Lambda$ (with respect to the measure of maximal entropy μ_0) by:

$$a_l(f, \Lambda) := \lim_n \frac{1}{n} \int_\Lambda \log d_n(x) d\mu_0(x) \tag{21}$$

The *asymptotic degree* of $f|_\Lambda$ is then defined as the number $d_\infty(f, \Lambda) := e^{a_l(f,\Lambda)}$.

As a matter of fact, for the examples in Section 2.3, we can even compute the asymptotic degree as being 2, even though the map itself is not constant-to-1 on Λ_α.

Corollary 3.5.1 ([54])**.** *For the above endomorphism f_α and its hyperbolic basic set Λ_α we have that f_α is not constant-to-one on Λ_α, and that the asymptotic degree satisfies:*

$$d_\infty(f_\alpha, \Lambda_\alpha) = 2$$

3.6 Entropy production for invariant measures on folded fractals.

An important notion in statistical physics is that of *entropy production* of a stationary state (see for instance [80], [79], etc.) Ruelle refined the notion from physics and defined a mathematical concept given in the next:

Definition 3.6.1. *Let $f : M \to M$ be a smooth endomorphism and μ be an f-invariant probability on M, then the* ***entropy production*** *of μ is defined by:*

$$e_f(\mu) := F_f(\mu) - \int \log |detDf(x)| d\mu(x),$$

where $F_f(\mu)$ is the folding entropy of μ.

Usually for physical purposes, it is useful to know whether a certain stationary state has positive, zero or negative entropy production, as these three cases correspond to different phenomena. Entropy production can be defined also for diffeomorphisms, but in that case without the folding entropy term.

Ruelle proved in [80] that a weak limit of a sum of iterates of μ has non-negative entropy production. The case when the entropy production is equal to zero is also related to the case when the measure is physically observable, namely when it is absolutely continuous with respect to the Lebesgue measure.

In [53] we studied the entropy production for the case of hyperbolic non-expanding endomorphisms, and for equilibrium measures for them, especially the case of inverse SRB measures (defined in Section 3.3). In the same paper we gave also examples of endomorphisms for which the entropy productions of the respective inverse SRB measures are negative.

In [53] we gave conditions for Anosov endomorphisms to have inverse SRB measures of negative entropy production, and also showed that a relatively large class of endomorphisms falls in this category. This can apply in particular to hyperbolic toral endomorphisms and their perturbations. We gave the statement however in ore generality, for hyperbolic non-expanding repellers.

Proposition 3.6.1 ([53]). *Let f be a $\mathcal{C}^2$ endomorphism on a connected Riemannian manifold M and let Λ be a hyperbolic saddle-type repeller for f such that f is d-to-1 on Λ, and f has no critical points in Λ. Consider an arbitrary small $\mathcal{C}^2$ perturbation g of f and let μ_g^- be the inverse SRB measure of g on the respective hyperbolic repeller Λ_g. Then:*

a) $e_g(\mu_g^-) \le 0$ and $F_g(\mu_g^-) = \log d$.

b) If f is an Anosov endomorphism on M, then there exists a neighbourhood V of f in $\mathcal{C}^2(M, M)$ and a set $W \subset V$ such that W is open and dense in the $\mathcal{C}^2$ topology in V and s.t for any $g \in W$ we have $e_g(\mu_g^-) < 0$.

We gave also concrete examples of hyperbolic repellers for which the respective inverse SRB measures have negative entropy productions. In a) below, we give an example of Anosov endomorphism (hyperbolic toral endomorphism), and in b) an example of a non-Anosov endomorphism (a perturbation of a saddle basic set), both of which have negative entropy production of their inverse SRb measures.

Corollary 3.6.1 ([53]). *a) Let the hyperbolic toral endomorphism on $\mathbb{T}^2$ given by $f(x,y) = (2x + 2y, 2x+3y)$ (mod 1) and its smooth perturbation*

$$g(x,y) = (2x + 2y + \varepsilon sin 2\pi y,\ 2x + 3y + 2\varepsilon sin 2\pi y)\ (mod\ 1)$$

Then the inverse SRB measure of g has negative entropy production; moreover the SRB measure of g has positive entropy production, i.e

$$e_g(\mu_g^-) < 0 \text{ and } e_g(\mu_g^+) > 0$$

b) Let now $f : \mathbb{PC}^1 \times \mathbb{T}^2 \to \mathbb{PC}^1 \times \mathbb{T}^2$ given by $f_g([z_0 : z_1], (x,y)) = ([z_0^k : z_1^k], g(x,y))$, where $k \geq 2$ is fixed, and g is a $\mathcal{C}^2$ perturbation of a hyperbolic toral endomorphism $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ without critical points. Then f_g has a connected hyperbolic repeller $\Lambda := S^1 \times \mathbb{T}^2$ in $\mathbb{P}^1\mathbb{C} \times \mathbb{T}^2$. Consider the smooth perturbation of f_g, $f_{\varepsilon,g} : \mathbb{PC}^1 \times \mathbb{T}^2 \to \mathbb{PC}^1 \times \mathbb{T}^2$,

$$f_{\varepsilon,g}([z_0 : z_1], (x,y)) := \left([z_0^k + \varepsilon z_1^k \cdot e^{2\pi i(2x+y)} : z_1^2], g(x,y)\right)$$

It follows from [46], that $f_{\varepsilon,g}$ has a connected saddle type repeller $\Lambda_{\varepsilon,g} := \bigcap_{n \leq 0} f_{\varepsilon,g}^n(V)$, for a neighbourhood V of Λ, and $\Lambda_{\varepsilon,g}$ is close to Λ; also $f_{\varepsilon,g}$ has an inverse SRB measure $\mu_{\varepsilon,g}^-$ on $\Lambda_{\varepsilon,g}$.

Then, the entropy production of the inverse SRB measure $\mu_{\varepsilon,g}^-$ on $\Lambda_{\varepsilon,g}$ is negative, i.e

$$e_{f_{\varepsilon,g}}(\mu_{\varepsilon,g}^-) < 0$$

Proof. We shall give here only the proof of b). From its construction, $\Lambda_{\varepsilon,g}$ is a connected repeller, hence it follows from Propositions 1 and 3 of [46] that the number of $f_{\varepsilon,g}$-preimages in $\Lambda_{\varepsilon,g}$ of a point from $\Lambda_{\varepsilon,g}$, is constant. Then we apply [46] to show that

$$h_{\mu_{\varepsilon,g}^-}(f_{\varepsilon,g}) = F_{f_{\varepsilon,g}}(\mu_{\varepsilon,g}^-) - \sum_{\lambda_i(\mu_{\varepsilon,g}^-) < 0} \lambda_i(\mu_{\varepsilon,g}^-),$$

where the Lyapunov exponents are repeated according to multiplicities. So we see that $e_{f_{\varepsilon,g}}(\mu_{\varepsilon,g}^-) \leq 0$. However we cannot have $e_{f_{\varepsilon,g}}(\mu_{\varepsilon,g}^-) = 0$ since otherwise it would follow that the inverse SRB measure $\mu_{\varepsilon,g}^-$ satisfies the equality in the (usual) Pesin formula. Then from a Volume Lemma (see [74]), this would imply that $\Lambda_{\varepsilon,g}$ is an attractor. However the basic set $\Lambda_{\varepsilon,g}$ is a hyperbolic repeller close to Λ, so it cannot have a neighbourhood U with $f_{\varepsilon,g}(U) \subset U$; hence the fractal $\Lambda_{\varepsilon,g}$ is not an attractor. Thus for any $\mathcal{C}^2$ perturbation g of f_A, the entropy production of the inverse SRB measure of the associated endomorphism $f_{\varepsilon,g}$ is negative, i.e $e_{f_{\varepsilon,g}}(\mu_{\varepsilon,g}^-) < 0$.

$\square$

4 Applications to dimension theory in hyperbolic dynamics on folded fractals.

4.1 Stable dimension and unstable dimension for hyperbolic endomorphisms

Thermodynamic formalism can be employed in order to obtain estimates for the Hausdorff (and upper box) dimension of various fractals obtained by iterative procedures. This was observed first in the papers by Bowen [6], and Ruelle [83]. Since then, thermodynamic formalism has proven to be a very useful tool and framework for difficult questions about dimension estimates (for instance [71], [1], [33], [73], [90], [89], [95], [98], [41], [45], [58], [57], [60], [59], etc.)

In the paper [83], Ruelle gave a formula for the Hausdorff dimension of the Julia set of a hyperbolic rational map f in one complex variable, as the zero of the pressure function $t \to P(-t \log |Df|)$. For hyperbolic diffeomorphisms on surfaces, Manning and McCluskey [33] showed that the Hausdorff dimension of the intersections between local stable manifolds and a hyperbolic basic set Λ is constant, and given as the zero of the pressure function $t \to P(t \log |Df_s|)$, and similarly for the dimension of the intersections between local unstable manifolds and Λ. This result has been found also in the case of Hénon automorphisms on $\mathbb{C}^2$ in [95].

However the situation for non-invertible hyperbolic maps is different and there appear new phenomena, due to the existence of several unstable manifolds going through the same point in Λ. We studied such phenomena and estimates in a series of papers, and showed that for the stable dimension (i.e the dimension of intersections between local stable manifolds and Λ), we do not have the same formulas as in the diffeomorphism case. As a matter of fact we proved that there exist strong connections between the preimage counting function on Λ and the stable dimension. This opens a new research direction, namely to investigate the thermodynamic formalism on folded fractals; we showed also that stable dimension is related to ergodic properties of equilibrium measures on Λ. Moreover we proved in [43] a surprising geometric phenomenon, showing that if the stable dimension is for example zero at some point in Λ, then Λ must be contained in a union of finitely many unstable manifolds and the measure of maximal entropy is 1-sided Bernoulli on it. The thermodynamic formalism and dimension estimates on folded fractals is related and answers to some questions of Ruelle, Simon, Solomyak, Fornaess, etc.

First let us fix some notation: consider f a smooth (say $\mathcal{C}^2$) endomorphism on a manifold M and Λ is a basic set for f, and assume that local stable manifolds and local unstable manifolds of size $r > 0$ do exist over Λ, for some small number $r > 0$. Let us denote throughout this thesis, by

$$\delta^s(x) := HD(W^s_r(x) \cap \Lambda),\ x \in \Lambda,$$

the *stable dimension* at x, where $HD(A)$ represents the Hausdorff dimension of a set A. And by

$$\delta^u(\hat{x}) := HD(W^u_r(\hat{x}) \cap \Lambda,\ \hat{x} \in \hat{\Lambda},$$

the *unstable dimension* corresponding to the prehistory $\hat{x} \in \hat{\Lambda}$. Also let us introduce the *stable potential* as the function

$$\Phi^s(x) := \log |Df_s(x)|, \ x \in \Lambda,$$

where $Df_s(x) := Df|_{E^s_x}, x \in \Lambda$. The *unstable potential* will be defined on the inverse limit $\hat{\Lambda}$ as

$$\Phi^u(\hat{x}) := -\log |Df_u(\hat{x})|, \ \hat{x} \in \hat{\Lambda},$$

where $Df_u(\hat{x}) = Df|_{E^u_{\hat{x}}}, \hat{x} \in \hat{\Lambda}$. We recall that in the non-invertible case, there may pass many (even uncountably many) local unstable manifolds through a point of Λ, and that unstable manifolds do not form a foliation; in fact they may intersect each other both inside and outside Λ.

Now we will present the situation for the unstable dimension; we showed in [47] that the unstable dimension is constant on $\hat{\Lambda}$, and that is given by a Bowen type equation for the unstable potential on $\hat{\Lambda}$. However as we will see later, no such formula is true in general for the stable dimension.

Theorem 4.1.1 ([47])**.** *Let as above a smooth map $f : M \to M$, which is hyperbolic on a basic set Λ and conformal on local unstable manifolds over Λ. Then the unstable dimension $\delta^u(\hat{x})$ is equal to the unique zero t^u of the pressure function $t \to P_{\hat{f}}(t\phi^u)$. In particular, the unstable dimension does not depend on $\hat{x}$.*

We start now to look into the problem of estimates for the stable dimension; in this case we do not usually have a Bowen type formula, due to the complicated intersections between local unstable manifolds and the existence of many unstable manifolds corresponding to the same point.

Theorem 4.1.2 ([60])**.** *Assume f is a smooth endomorphism as above, which is hyperbolic on a basic set Λ, and conformal on local unstable manifolds. Suppose that also that there are no critical points in Λ i.e $C_f \cap \Lambda = \emptyset$, and that $f|_\Lambda\colon\ \Lambda \to \Lambda$ has the property that each point $x \in \Lambda$ has at least d' preimages in Λ, $d' \le d$. Then $\delta^s(x) \le t^s_{d'}$, where $t^s_{d'}$ is the unique zero of the pressure function*

$$t \to P(t \log |Df_s| - \log d')$$

This estimate is independent of the point $x \in \Lambda$.

Proof. First of all, let us consider the function $t \to P(t \log |Df|_{E^s_y}| - \log d')$, which is well defined as $C_f \cap \Lambda = \emptyset$. Notice also that $P(t \log |Df|_{E^s_y}| - \log d') = P(t \log |Df|_{E^s_y}|) - \log d'$. It is strictly decreasing and at $t = 0$ takes the value $h(f|_\Lambda) - \log d' \ge 0$ and for t very large, it takes negative values. Hence it has exactly one zero denoted by $t^s_{d'}$, and $t^s_{d'} \ge 0$. Denote now $W := W^s_\delta(x) \cap \Lambda$ and by $\hat{W}$ its lift inside $\hat{\Lambda}$, i.e $\hat{W} := \pi^{-1}(W)$, where $\pi(\hat{x}) = x_0$ is the canonical projection from $\hat{\Lambda}$ to Λ. We know that $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ is a homeomorphism on the compact metric space $\hat{\Lambda}$.

Also $P(\phi) = P(\phi \circ \pi)$, for any continuous real function ϕ on Λ, i.e the topological pressure does not change by lifting the function to $\hat{\Lambda}$.

Let $\hat{E}$ be an $(n+1, \delta')$-separated set of maximal cardinality inside $\hat{f}^{-n}(\hat{W})$ with $\delta' << \delta$ to be determined in the course of the proof.

Since $t^s_{d'}$ is the unique zero of the pressure function, it follows that if we consider an arbitrary $t > t^s_{d'}$, then there exists $\beta < 0$ such that

$$P(t \log |Df|_{E^s_{\hat{y}}}| - \log d') < \beta < 0.$$

Thus if n is large enough,

$$\frac{1}{n+1} \log \sum_{\hat{z} \in \hat{E}} e^{S_{n+1}\Phi(\hat{z})} < \beta < 0,$$

where $\Phi(\hat{y}) := t \log |Df|_{E^s_{\hat{y}}}| - \log d'$. Hence,

$$\sum_{\hat{z} \in \hat{E}} |Df^n|_{E^s_{\hat{z}}}|^t < e^{(n+1)\beta} \cdot (d')^n.$$

But $\hat{E}$ has been taken as a separated set of maximal cardinality, hence it is also $(n+1, \delta')$ - spanning for the compact set $\hat{f}^{-n}(\hat{W})$ in the metric $d(\hat{\cdot}, \hat{\cdot})$ from $\hat{\Lambda}$. Thus the Bowen balls $B_{n+1}(\hat{z}, \delta') := \{\hat{w} \in \hat{\Lambda}, d(\hat{f}^k \hat{z}, \hat{f}^k \hat{w}) < \delta', k = 0, .., n\}, \hat{z} \in \hat{E}$, cover the entire set $\hat{f}^{-n}(\hat{W})$. From above, it follows that $\{\hat{f}^n(B_{n+1}(\hat{z}, \delta'))\}_{\hat{z} \in \hat{E}}$ cover the set $\hat{W}$, and for brevity, we will denote this collection of sets by $\{\hat{B}_j\}_{j \in J}$, where $\hat{E} = (\hat{z}^j)_{j \in J}$, J finite.

Consider now an arbitrary point y from W and $\hat{y}, \hat{y}'$ two prehistories of y in Λ which are different as n-prehistories, i.e there exists $0 < i \le n$ such that $y_{-i} \neq y'_{-i}$.

Can there be two such prehistories both in the same set $\hat{B}_j$? Assume $i \ge 0$ is the largest integer for which $y_{-i} = y'_{-i}$. Denote by l_0 the constant of injectivity of f near Λ, i.e, if z or z' belong to Λ and $f(z) = f(z')$ with $z \neq z'$, then $d(z, z') > l_0$ (here we use again that the critical set of f does not intersect Λ). Then for the prehistories $\hat{y}, \hat{y}'$ as above, $d(\hat{f}^{-i-1}(\hat{y}), \hat{f}^{-i-1}(\hat{y}')) > l_0 > 2\delta'$, if δ' is small enough. Thus $\hat{f}^{-n}(\hat{y})$ and $\hat{f}^{-n}(\hat{y}')$ cannot be in the same Bowen ball $B_{n+1}(\hat{\xi}, \delta'), \hat{\xi} \in \hat{E}$. So $\hat{y}, \hat{y}'$ cannot be in the same set $\hat{B}_j$, since $\hat{f}$ is a homeomorphism.

We take now the projections of $\hat{B}_j$ onto W, $B_j := \pi(\hat{B}_j) \cap W$. Let d_j denote the diameter of B_j and for all $j \in J$ take $\tilde{B}_j := B(x_j, d_j)$, where x_j is an arbitrary point in B_j. In general, by $MB(x, r)$ we shall denote the ball $B(x, Mr)$. From above, it can be seen that the multiplicity of the cover $\{\tilde{B}_j\}_j$ of W, is at least d'^n, if $0 < \delta' < \frac{l_0}{6}$. This holds because every point y from W has at least d'^n different n-prehistories in Λ and since every such n-prehistory of y is contained in a different set $\hat{B}_j$.

We want now to extract a subcover of W of multiplicity bounded by some universal constant C, coming from the following version of the Besicovitch Theorem :

Theorem (Guzman). *For each integer $n \ge 1$, there exists a positive integer $b(n)$, depending only on n, with the following property:*

Assume that A is an arbitrary bounded set of $\mathbb{R}^n$. Moreover suppose that we have some positive constant M, some function $r : A \to (0, \infty)$ and a system of sets $\{H(x)\}_{x \in A}$, satisfying the following two conditions:

(a) for every $x \in A$, $\bar{B}(x, r(x)) \subset H(x) \subset \bar{B}(x, Mr(x))$; (where $\bar{B}(y, s)$ represents in general the closed Euclidian ball of radius s centered at a point y);

(b) for each $x \in A$ and each $z \in H(x)$, the set $H(x)$ contains the convex hull of the set $\{z\} \cup \bar{B}(x, r(x))$.

Then out of $\{H(x)\}_{x \in A}$, we can extract a countable subcover of A of multiplicity bounded by $b(n)$. So one can select from among $\{H(x)\}_{x \in A}$ a sequence $\{H_k\}_k$ such that:

(i) A is covered by $\{H_k\}_k$;

(ii) no point of $\mathbb{R}^n$ is in more than $b(n)$ sets H_k.

Notice that if the sets $H(x)$ are convex, then condition (b) of the theorem above is immediately satisfied. Based on this theorem, we will prove the following covering theorem which will be useful in our context.

Theorem (Covering Theorem). *Let A be a bounded set of $\mathbb{R}^n$. Assume that A is covered by a family of balls $\{B(x_i, r_i)\}_{i \in I}$ centered at some points of A, where $r_i > 0$, for all $i \in I$. Then there exists a cover of A with balls $\{B(x_j, 2r_j)\}_{j \in J}$, where $J \subset I$ and the multiplicity of this cover is bounded by the universal constant $b(n)$.*

For each $x \in A$, choose one ball $B(x_i, r_i)$ containing x. Set $H(x) := B(x_i, 2r_i)$ and denote by $r(x)$ the radius r_i. Obviously the sets $\{H(x)\}_{x \in A}$ will cover A. They are also convex. For every $x \in A$, we have that $B(x, r(x)) \subset H(x) \subset B(x, 3r(x))$. Thus the assumptions of the previous theorem are satisfied and the proof is finished.

Coming back to the proof of Theorem 4.1.2, we apply the above Covering Theorem for $\tilde{B}_j$, $j \in J$. Hence, we obtain a subcover $2\tilde{B}_k$, where k belongs to a subset $K \subset J$ and such that the multiplicity of this subcover is bounded above by a universal constant $C > 0$ coming from the above Covering Theorem. But the multiplicity of the cover $2\tilde{B}_j$, $j \in J$ is larger than or equal to d'^n, if $\delta < l_0/6$. Hence if n is large enough, the remaining sets $\{\tilde{B}_j\}_{j \in J \setminus K}$, still cover W and we can extract again, out of them, a subcover of W of multiplicity bounded by C. Repeating the procedure and applying the Covering Theorem at each step, one can find at least $C^{-1} d'^n$ such subcovers, each with multiplicity bounded by C. But then, there exists a cover $\{2\tilde{B}_s\}_{s \in L}$ of W, $L \subset J$, corresponding to a subset $\hat{F}$ of $\hat{E}$ for which :

$$\sum_{\hat{z} \in \hat{F}} |Df|_{E^s_z}|^t \leq e^{n\beta} \cdot d'^n \cdot C(d')^{-n} = C \cdot e^{n\beta} \tag{22}$$

We make now the connection between the $\text{diam} 2\tilde{B}_s$ and $|Df^n|_{E^s_z}|$ using the bounded distortion property. First observe that in general, $B_j = \pi(\hat{f}^n B_{n+1}(\hat{y}, \delta)) = f^n(\pi(B_{n+1}(\hat{y}, \delta)))$. Also, if $\hat{\xi} \in B_{n+1}(\hat{y}, \delta)$, then $d(\hat{f}^k \hat{\xi}, \hat{f}^k \hat{y}) < \delta$, for $0 \leq k \leq n$, hence $d(f^k \xi, f^k y) < \delta$. This implies that $\xi \in W^s_\delta(y)$.

Hence, we can apply the property of bounded distortion on stable manifolds to conclude that, for a positive constant A,

$$\text{diam} 2\tilde{B}_j \leq A \cdot |Df^n|_{E^s_{x_j}}|.$$

Applying now inequality (22), and remembering that $\beta < 0$, we get that there exists a positive constant C' such that

$$\sum_{s \in L} (\text{diam} 2\tilde{B}_s)^t \leq C' \, e^{n\beta} \leq C'.$$

In conclusion, since t has been chosen arbitrarily larger than $t^s_{d'}$, we obtain $HD(W) \leq t^s_{d'}$. The proof of the last consequence of the statement is then straightforward if one uses the properties of the topological pressure from [96] and the fact that $P(\phi + c) = P(\phi) + c$, for any continuous real function ϕ and any constant c. Indeed, $0 = P(t^s_{d'} \log |Df|_{E^s_y}| - \log d') \leq P(t^s_{d'} \log \sup_{y \in \Lambda} |Df|_{E^s_y}| - \log d') = t^s_{d'} \log \sup_{y \in \Lambda} |Df|_{E^s_y}| - \log d' + h(f|_\Lambda)$ (since $t^s_{d'} \log \sup_{y \in \Lambda} |Df|_{E^s_y}|$ is in fact a constant, and $P(0) = h(f|_\Lambda)$). Thus,

$$t^s_{d'} \log \sup_{y \in \Lambda} |Df|_{E^s_y}| \geq \log d' - h(f|_\Lambda)$$

But $\log \sup_{y \in \Lambda} |Df|_{E^s_y}| < 0$, so we obtain the required inequality,

$$HD(W) \leq t^s_{d'} \leq \frac{h(f|_\Lambda) - \log d'}{|\log \sup_{y \in \Lambda} |Df|_{E^s_y}||}$$

□

Then in [58], by employing some combinatorial techniques and concatenations of prehistories, we proved that the stable dimension is in fact independent of the point in the case when f is constant-to-1 over Λ, and equal to the zero of a pressure function. This case happens for instance for s-hyperbolic maps ([22], [48]), as they are open when restricted to a minimal saddle basic set.

Theorem 4.1.3 ([58]). *Consider as above a smooth map f and a basic set of saddle type Λ which does not intersect the critical set $\mathcal{C}_f$, and such that f is conformal on local stable manifolds over Λ. Assume also that $f|_\Lambda : \Lambda \to \Lambda$ is open, in particular any point $x \in \Lambda$ has the same number d' of f-preimages in Λ. Then for any $x \in \Lambda$, we obtain that $\delta^s(x) = t^s_{d'}$, where $t^s_{d'}$ is the unique zero of the pressure function $t \to P(t\Phi^s - \log d')$.*

As examples we study a large class of maps obtained as perturbations of $(z^2 + c, w^2)$ $(0 \neq |c|$ small) and identify a set of elements of this class which are injective on their respective basic sets.

Theorem 4.1.4 ([60]). *Given the map $f_\varepsilon(z, w) = (z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2, w^2)$, there exist small positive constants $c(a, b, d, e)$ and $\varepsilon(a, b, c, d, e)$ such that, for $b \neq 0$, $0 \neq |c| < c(a, b, d, e)$ and $0 < \varepsilon < \varepsilon(a, b, c, d, e)$ we have that f_ε is injective on its basic set Λ_ε close to $\{p_0(c)\} \times S^1$ (where $p_0(c)$ is the attracting fixed point for $z^2 + c$).*

Proof. The existence of the basic set Λ_ε for the perturbations f_ε follows from the Stability Theorem for Endomorphisms ([72]). Assume $f_\varepsilon(z,w) = f_\varepsilon(z',w')$ for points $(z,w),(z',w') \in \Lambda_\varepsilon$, then

$$\begin{aligned} &z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2 = z'^2 + a\varepsilon z' + b\varepsilon w' + c + d\varepsilon z'w' + e\varepsilon w'^2 \\ &\Rightarrow (z^2 - z'^2) + \varepsilon[a(z-z') + b(w-w') + d(zw - z'w') + e(w^2 - w'^2)] = 0. \end{aligned}$$

Assume now that $w \neq w'$. Then $w' = -w$ since $w^2 = w'^2$

$$\Rightarrow (z - z')(z + z' + \varepsilon a) + \varepsilon[2bw + dw(z + z')] = 0.$$

$$\Rightarrow (z - z')(z + z' + \varepsilon a) = -\varepsilon[2bw + dw(z + z')]. \tag{23}$$

Denote by $p_0(c)$ the fixed attracting point for $z \to z^2 + c$, $0 \neq |c|$ small, and consider $\alpha := \sup\limits_{(z,w)\in\Lambda_\varepsilon} |z - p_0(c)|$; let (z_0, w_0) be a point where this supremum is attained on Λ_ε. Hence $\alpha = |z_0 - p_0(c)|$, and $p_0^2(c) + c = p_0(c)$. Now there exists $(z,w) \in \Lambda_\varepsilon$ such that

$$\begin{aligned} f_\varepsilon(z,w) = (z_0,w_0) &\Rightarrow z_0 = z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2 \\ &\Rightarrow z_0 - p_0(c) = z^2 - p_0^2(c) + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2 \\ &= (z - p_0(c))^2 + 2zp_0(c) - 2p_0^2(c) + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2 \Rightarrow \\ &\alpha \leq \alpha^2 + 2|p_0(c)|\alpha + K\varepsilon, \end{aligned}$$

with K, a positive constant depending on the parameters of the map. This last inequality implies $\alpha^2 + \alpha(2|p_0(c)| - 1) + K\varepsilon \geq 0$, and since $\alpha \ll 1$ (since Λ_ε is close to $\{p_0(c)\} \times S^1$) we obtain, with some constant $K' > 0$,

$$0 \leq \alpha \leq \frac{2K\varepsilon}{1 - 2|p_0(c)| + \sqrt{(1 - 2|p_0(c)|)^2 - 4K\varepsilon}} \leq K' \cdot \varepsilon \tag{24}$$

Notice that $|z - z'| = |z - p_0(c) + p_0(c) - z'| \leq |z - p_0(c)| + |z' - p_0(c)| \leq 2\alpha$, for z, z' first coordinates of some points from Λ_ε. Then, by taking absolute values in (23) and using that $|z - z'| \leq 2\alpha$, one obtains:

$$2\alpha(3|p_0(c)| + \varepsilon|a|) > \varepsilon|2bw + dw(z + z')|, \tag{25}$$

for $\varepsilon < \varepsilon(a,b,c,d,e)$, since z, z' are both ε-close to $p_0(c)$.

But for $(z,w) \in \Lambda_\varepsilon$, we know $|w| = 1$, and $|z + z'|$ is ε-close to $2|p_0(c)|$. Thus if $b \neq 0$, and $|c|$ is small enough in comparison to $|b|$, then $|p_0(c)|$ becomes so small that $|2bw + dw(z + z')| > |b| > 0$.

Hence from (24) and (25), we get $2K'\varepsilon(3|p_0(c)| + \varepsilon|a|) \geq \varepsilon|b| > 0$, which gives a contradiction if $|c| < c(a,b,d,e)$ for some convenient positive number $c(a,b,d,e) << 1$. (indeed, if $|c|$ is small with respect to $|b|$, then $|p_0(c)|$ is also small with respect to $|b|$, and we can take ε small, accordingly).

Hence, we proved that $w' = w$. Then, from $f_\varepsilon(z,w) = f_\varepsilon(z',w')$ it follows that $z^2 - z'^2 = -\varepsilon[a(z - z') + dw(z - z')]$. If $z \neq z'$, we would then get $z + z' = -\varepsilon(a + dw)$. But z, z' are both

ε -close to $p_0(c) \neq 0$, so if we choose $\varepsilon < \varepsilon(a, b, c, d, e)$ appropiately, then $|z + z'| > |p_0(c)| > 0$. However if $|a + dw| \neq 0$, we can take ε small enough such that $\varepsilon|a + dw| < |p_0(c)|$, which gives a contradiction. In case $|a + dw| = 0$, then we get a contradiction again since $p_0(c) \neq 0$.

Therefore we showed that $z = z', w = w'$, and consequently $f_\varepsilon|_{\Lambda_\varepsilon} : \Lambda_\varepsilon \to \Lambda_\varepsilon$ is an injective map. □

4.2 Inverse pressure and applications to dimension estimates

In order to study the dynamics of non-invertible systems, it is sometimes necessary to employ a different type of pressure, which takes into consideration all the possible backward trajectories of points, not only the forward ones. This kind of pressure encapsulates the behaviour of the preimages sets for iterates. It was introduced by Mihailescu and Urbanski in [61], and studied in [45] and [58]. In [45] we proved for instance that inverse pressure helps prove that the stable dimension over a basic set Λ is strictly less than the dimension of the stable tangent space, unless the fractal Λ is a repellor. This related then also to a problem of Fornaess-Sibony in higher dimensional complex dynamics on $\mathbb{P}^2$, showing that the Hausdorff dimension of the set K^- is strictly less than 4.

Let us recall the definition and properties of inverse pressure, which will be used later. We will be in the following setting:

Consider X a compact metric space, $f : X \to X$ is a continuous surjective map on X, and $Y \subseteq X$ is a subset of X. As f is surjective, for any point y of X, and any positive integer m, there exists $y_{-m} \in X$ such that $f^m(y_{-m}) = y$. By *prehistory of length* m (or m*-prehistory*, or *branch of length* m) of y, we will understand a collection of consecutive preimages of y, $C = (y, y_{-1}, ..., y_{-m})$, where $f(y_{-i}) = y_{-i+1}, i = 1, .., m, y_0 = y$. Given a prehistory C, we denote by $n(C)$ its length. Fix also a small number $\varepsilon > 0$, and denote by $\mathcal{C}_m$ the set of all m-prehistories of points from X. Then for such an m-prehistory C, let $X(C, \varepsilon)$ be the set of points ε*-shadowed* by C (in backward time) defined by: $X(C, \varepsilon) := \{z \in B(y_0, \varepsilon) : \exists z_{-1} \in f^{-1}(z) s.t.\ d(z_{-1}, y_{-1}) < \varepsilon, .., \exists z_{-m} \in f^{-1}(z_{-m+1}) s.t.\ d(z_{-m}, y_{-m}) < \varepsilon\}$. Given the m-prehistory $C = (y, y_{-1}, ..., y_{-m})$ and a real continuous function ϕ on X, we will define the *consecutive sum* of ϕ on C, $S_m^-\phi(C) = \phi(y) + \phi(y_{-1}) + ... + \phi(y_{-m})$. We may also use the usual notation $S_m^-\phi(y_{-m})$ instead of $S_m^-\phi(C)$.

Now we define the *inverse pressure* P^- by a procedure similar to that used in the case of Hausdorff outer measure. Let ϕ be an arbitrary continuous function, $\phi \in \mathcal{C}(X, \mathbb{R})$; let also λ a real number and N a positive integer. Let $\mathcal{C}_* := \underset{m\geq 0}{\cup} \mathcal{C}_m$. Then, a subset $\Gamma \subset \mathcal{C}_*$, ε*-covers* X if $X = \underset{C\in\Gamma}{\cup} X(C, \varepsilon)$. Define the following expression

$$M_f^-(\lambda, \phi, Y, N, \varepsilon) := \inf\{\sum_{C\in\Gamma} exp(-\lambda n(C) + S_{n(C)}^-\phi(C)), n(C) \geq N, \forall C \in \Gamma,$$
$$\text{and } \Gamma \subset \mathcal{C}_* \text{ s.t } Y \subset \underset{C\in\Gamma}{\cup} X(C, \varepsilon)\}$$

When the integer N increases, the set of collections Γ ε-covering X gets smaller, so the infimum increases in the previous expression. Hence $\lim\limits_{N\to\infty} M_f^-(\lambda,\phi,Y,N,\varepsilon)$ exists and will be denoted by $M_f^-(\lambda,\phi,Y,\varepsilon)$. Now, let $P_f^-(\phi,Y,\varepsilon) := \inf\{\lambda : M_f^-(\lambda,\phi,Y,\varepsilon)=0\}$. Consider two positive numbers $\varepsilon_1 < \varepsilon_2$ and compare $P_f^-(\phi,Y,\varepsilon_1)$ and $P_f^-(\phi,Y,\varepsilon_2)$. Given any prehistory C, we have $X(C,\varepsilon_1) \subset X(C,\varepsilon_2)$, so if $\Gamma \subset \mathcal{C}_*$ ε_1-covers Y, then Γ also ε_2-covers Y. Thus there are more candidates Γ in the expression of $M_f^-(\lambda,\phi,Y,N,\varepsilon_2)$ than in the expression of $M_f^-(\lambda,\phi,Y,N,\varepsilon_1)$. So for any N, $M_f^-(\lambda,\phi,Y,N,\varepsilon_2) \le M_f^-(\lambda,\phi,Y,N,\varepsilon_1)$. Hence $0 \le M_f^-(\lambda,\phi,Y,\varepsilon_2) \le M_f^-(\lambda,\phi,Y,\varepsilon_1)$, and then from definition, $P_f^-(\phi,Y,\varepsilon_2) \le P_f^-(\phi,Y,\varepsilon_1)$. Thus, when ε decreases to 0, $P_f^-(\phi,Y,\varepsilon)$ increases, so the limit $\lim\limits_{\varepsilon\to 0} P_f^-(\phi,Y,\varepsilon)$ does exist, and is denoted by $P_f^-(\phi,Y)$.

$P_f^-(\phi,Y)$ is called *the inverse pressure* (or *inverse upper pressure*) of ϕ on Y. $P_f^-(\phi,Y,\varepsilon)$ is called the *ε-inverse pressure* of ϕ on Y. This notion has been introduced in [61]. When the map f is clear from the context, we may drop the index f from the notations.

Also denote by $P_f^-(\phi), P_f^-(\phi,\varepsilon), M_f^-(\lambda,\phi,N,\varepsilon)$, etc, the respective quantities $P_f^-(\phi,X), P_f^-(\phi,X,\varepsilon)$, $M_f^-(\lambda,\phi,X,N,\varepsilon)$, etc., respectively. The following proposition provides some properties of P^-.

Proposition 4.2.1 ([61])**.** *Let $f : X \to X$ be a continuous surjective map on the compact metric space X, ε a positive number and $\phi \in \mathcal{C}(X,\mathbb{R})$. Then:*

i) If $Y_1 \subset Y_2 \subset X$, then $P_f^-(\phi,Y_1) \le P_f^-(\phi,Y_2)$ and $P_f^-(\phi,Y_1,\varepsilon) \le P_f^-(\phi,Y_2,\varepsilon)$.

ii) If $Y = \bigcup\limits_{j\in J} Y_j$ is a finite or countable union of subsets of X, then $P_f^-(\phi,Y,\varepsilon) = \sup\limits_{j\in J} P_f^-(\phi,Y_j,\varepsilon)$ and $P_f^-(\phi,Y) = \sup\limits_{j\in J} P_f^-(\phi,Y_j)$.

iii) If f is a homeomorphism on X, then $P_f^-(\phi) = P_f(\phi)$, where $P_f(\phi)$ denotes the usual (forward) topological pressure of ϕ with respect to the map f.

iv) $P_f^-(\phi,Y)$ is invariant to topological conjugacy, i.e if $f : X \to X, g : X' \to X'$ are continuous surjective maps and $\Psi : X \to X'$ is a homeomorphism such that $\Psi \circ f = g \circ \Psi$, then $P_f^-(\phi,Y) = P_g^-(\phi \circ \Psi^{-1}, \Psi(Y))$, for any subset $Y \subset X$.

Now, when f is surjective we proved that the limits P^- and $\tilde{P}^-$ coincide.

Theorem 4.2.1. *If $f : X \to X$ is surjective, then $P^-(\phi) = \tilde{P}^-(\phi)$, for any continuous function $\phi \in \mathcal{C}(X,\mathbb{R})$.*

In [58] we proved that the stable dimension can be bounded above always by the zero of the inverse pressure of the stable potential. This estimate is useful when the number of preimages that remain in Λ is not constant over Λ.

Theorem 4.2.2 ([58])**.** *Let a smooth endomorphism f, which is hyperbolic and conformal on local stable manifolds over a basic set Λ of saddle type, and such that the critical set of f does not intersect Λ. Then $\delta^s(x) \le t^s_-$, for any $x \in \Lambda$, where t^s_- is the unique zero of the inverse pressure function $t \to P^-(t\Phi^s)$.*

Proposition 4.2.2 ([61]). *In the above setting, let Λ be a basic set of saddle type, on which f is hyperbolic and conformal on local stable manifolds. If there exists a point $x \in \Lambda$ such that $\delta^s(x) \neq 0$, then it follows that Λ cannot be a finite graph (hence in particular Λ cannot be a Jordan curve).*

In the case of diffeomorphisms, a basic set Λ is called a repellor if there exists an open set U, $\Lambda \subset U$ such that $\bar{U} \subset f(U)$. However for non-invertible maps this condition alone, does not guarantee that local stable manifolds are contained inside Λ, due to the complicated structure of foldings, which may take a point from outside Λ, into a point from Λ. Besides being a repellor, one would need another condition like the openness of the map $f|_\Lambda$. In [45] we introduced thus a new notion of *local repellor*, which is similar to that of repellor, but refers to the non-invertible case.

Definition 4.2.1. *In the above setting, if the endomorphism f is hyperbolic on a basic set Λ, we say that Λ is alocal repellor if there exist local stable manifolds of f contained in Λ.*

We get that Λ is not a local repellor if it is not a repellor and if f is *preimage-transitive* on Λ, i.e if any point $y \in \Lambda$ has the set of all its preimages $\{z \in \Lambda, \exists n \geq 0,\ f^n(z) = y\}$ dense in Λ. Other cases when this happens is if $f|_\Lambda$ is open, or if f is s-hyperbolic on Λ (see [45]).

The inverse pressure can now be used in order to prove that the stable dimension on a fractal set Λ which is not a local repellor, is strictly less than 2 (in case the local stable manifolds have real dimension 2). Recall that P_ε^- is the inverse pressure with sets of diameter less or equal than ε. Denote by t_ε^s the unique zero of the pressure function $t \to P_\varepsilon^-(t\Phi^s)$, where as before $\Phi^s(x) = \log |Df_s(x),\ x \in \Lambda$.

Theorem 4.2.3 ([45]). *Consider a smooth endomorphism $f : M \to M$ on a Riemannian manifold of real dimension 4, which is hyperbolic on a basic set Λ, conformal on the local stable manifolds, which are assumed to have of real dimension 2, and such that the critical set $\mathcal{C}(f)$ does not intersect Λ. Assume also that Λ is not a local repellor. Then there exists a small $\varepsilon > 0$ such that for any point $x \in \Lambda$ we have*

$$\delta^s(x) \leq t^s(\varepsilon) < 2$$

Proof. We will denote by $W := W_r^s(x) \cap \Lambda$, for a fixed but arbitrary point $x \in \Lambda$, and by m_s the Lebesgue measure on a generic local stable manifold.

We know from [45] that $\delta^s(x) \leq t^s(\varepsilon)$, $\varepsilon > 0$ small, where $t^s(\varepsilon)$ is the unique zero of the function $t \to P^-(t\phi^s, \varepsilon)$, with $\phi^s(y) := \log |Df_s(y)|, y \in \Lambda$. Consider a fixed $\varepsilon > 0$ small enough (in particular $\varepsilon < \varepsilon_0$).

It remains to show that $P^-(2\phi^s, \varepsilon) < 0$, which will imply that $t^s(\varepsilon) < 2$. In order to do this, recall that $P^-(2\phi^s, \varepsilon)$ can be computed using $P_n^-(2\phi^s, \varepsilon)$. But from the Laminated Distortion Lemma, there exists a constant $\chi > 0$ such that, if $\omega \in M(C, \varepsilon), C = (y, y_{-1}, ..., y_{-n}) \in \mathcal{C}_n(\Lambda)$, and $(\omega, \omega_{-1}, ..., \omega_{-n})$ is the corresponding prehistory of ω which is ε-shadowed by C, then $\frac{1}{\chi}|Df_s^n(y_{-n})| \leq |Df_s^n(\omega_{-n})| \leq \chi |Df_s^n(y_{-n})|$. Thus $P_n^-(2\phi^s, \varepsilon) = \omega(\varepsilon) \cdot \inf\{\sum\limits_{C \in \Gamma} m_s(M(C, \varepsilon)), \Gamma \subset \mathcal{C}_n, \Gamma \varepsilon\text{-covering } \Lambda\}$, with $\omega(\varepsilon)$ some positive function of ε and $m_s(M(C,\varepsilon)) := m_s(W^s(y,\varepsilon) \cap M(C,\varepsilon))$. Then,

$$P^-(2\phi^s, \varepsilon) = \overline{\lim_{n\to\infty}} \frac{\log P_n^-(2\phi^s, \varepsilon)}{n}$$

We want now to find some $\upsilon \in (0,1)$ and a positive integer $N = N(\upsilon)$, such that for $n > N$, we have $P_{n+N}^-(2\phi^s, \varepsilon) \le \upsilon \cdot P_n^-(2\phi^s, \varepsilon)$.

For this, let an arbitrarily small $\varepsilon' > 0$ and find $n \ge 1$ and a collection $\Gamma \subset \mathcal{C}_n$ such that $P^-(2\phi^s, \varepsilon) \le \varepsilon' + \frac{\log(\sum_{C\in\Gamma} m_s(M(C,\varepsilon))}{n}$, where $C = (y, y_{-1}, ..., y_{-n}) \in \Gamma$. Since Λ is not a local repellor, there are no local stable manifolds of f contained in Λ, hence there exists a positive integer $N = N(\varepsilon)$ such that for any $z \in \Lambda$, we can cover the set $\Lambda \cap W_\varepsilon^s(z)$ with sets of the form $M(C', \varepsilon), C' \in \Gamma_z \subset \mathcal{C}_N$ such that

$$\sum_{C'\in\Gamma_z} m_s(M(C', \varepsilon) \cap W_\varepsilon^s(z)) \le \upsilon \cdot m_s(W_\varepsilon^s(z)),$$

for some $\upsilon \in (0,1)$. The collection Γ_z depends on z, but N is independent of z. Let now $\Gamma \subset \mathcal{C}_n(\Lambda)$ found above, which ε-covers Λ. For each prehistory $C = (y, y_{-1}, ..., y_{-n}) \in \Gamma$ we can cover the set $\Lambda \cap W_\varepsilon^s(y_{-n})$ with sets of the form $M(C', \varepsilon)$, where $C' \in \Gamma(C) \subset \mathcal{C}_N$, for N found above; this cover $\Gamma(C)$ is in fact the family $\Gamma_{y_{-n}}$, and hence satisfies:

$$\sum_{C'\in\Gamma(C)} m_s(M(C', \varepsilon) \cap W_\varepsilon^s(y_{-n})) \le \upsilon \cdot m_s(W_\varepsilon^s(y_{-n})) \tag{26}$$

Consider now a positive integer n and a prehistory $C \in \mathcal{C}_n(\Lambda), C = (y, y_{-1}, ..., y_{-n})$ like above; assume also that f_*^{-n} is the local inverse iterate of f, which takes y into y_{-n}; then $f_*^{-n}(M(C,\varepsilon) \cap W_\varepsilon^s(y)) \subset W_\varepsilon^s(y_{-n})$. We observe that the points in $M(C,\varepsilon) \cap W_\varepsilon^s(y)$ are taken by f_*^{-n} into $W_\varepsilon^s(y_{-n})$, while the points outside $W_\varepsilon^s(y)$ will be taken into points which are $(\lambda')^n$-close to $W_\varepsilon^s(y_{-n})$, for some $\lambda' \in (0,1)$ (λ' does not depend on n, y, C). Recall also that we cover each set $W_\varepsilon^s(y_{-n}) \cap \Lambda$ for $C = (y, y_{-1}, ..., y_{-n}) \in \Gamma$, with sets of the form $M(C', \varepsilon), C' \in \Gamma(C)$, where $\Gamma(C) \subset \mathcal{C}_N$. Thus from the above discussion, it follows that if n is large enough in comparison to N, i.e if $n > n(N)$, then $\bigcup_{C'\in\Gamma(C)} M(C', \varepsilon)$ is an open neighbourhood of $W_\varepsilon^s(y_{-n}) \cap \Lambda$, thus it contains the local inverse iterate $f_*^{-n}(M(C,\varepsilon))$. So we obtain a cover of Λ with sets of type $M(CC', 2\varepsilon), C \in \Gamma, C' \in \Gamma(C)$, where $\Gamma \subset \mathcal{C}_n(\Lambda)$, $\Gamma(C) \subset \mathcal{C}_N(\Lambda)$, and $n > n(N)$; CC' represents the prehistory obtained by concatenation of C and then C' ([61] for more details on the concatenation procedure). The new collection obtained from these concatenations CC' is called Γ', and $\Gamma' \in \mathcal{C}_{n+N}(\Lambda)$. Then after multiplying by $|Df_s(y_{-n})|^n$ in both sides of (26), we obtain from the fact that f is conformal on stable manifolds that:

$$\sum_{C'\in\Gamma(C)} m_s(M(CC', \varepsilon)) \le \upsilon \cdot m_s(M(C, \varepsilon))$$

So there exist integers $N \ge 1$ and $n(N) \ge 1$ such that for all $n > n(N)$ we have:

$$P_{n+N}^-(2\phi^s, \varepsilon) \le \upsilon \cdot P_n^-(2\phi^s, \varepsilon)$$

But then $P^-_{n+kN} \le \upsilon^k \cdot P^-_n(2\phi^s, \varepsilon), k \ge 1$, therefore $\log P^-_{n+kN}(2\phi^s, \varepsilon) \le k \log \upsilon + \log P^-_n(2\phi^s, \varepsilon)$, hence $P^-(2\phi^s, \varepsilon) \le \frac{\log \upsilon}{N} < 0$,. The last inequality follows since $\upsilon \in (0, 1)$. Therefore we obtained $t^s(\varepsilon) < 2, \varepsilon > 0$ small.

□

Thus the stable dimension is strictly less than 2, unless whole stable manifolds are contained inside Λ. The advantage of the condition of not being a local repellor is that it is stable under perturbations.

Theorem 4.2.4 ([45]). *Assume that f is a smooth endomorphism which is hyperbolic on a basic set Λ such that Λ is not a local repellor. Then for any smooth perturbation g close (in $\mathcal{C}^2$ topology) to f, it follows that the corresponding basic set Λ_g is not a local repellor for g either.*

This Theorem allows then to obtain large classes of maps and sets which are not local repellor, by perturbing some known examples, like basic sets for product maps, or skew products, etc.

By using Theorem 4.2.3, we showed in [45] that, also the *stable upper box dimension* (see [19], [35], [73], etc for definitions of upper box dimension) is strictly less than 2, in the case when Λ is not a local repellor.

Theorem 4.2.5 ([45]). *Assume we are in the same settings as in Theorem 4.2.3, that the dimension of stable tangent spaces over a basic set Λ is 2, and that Λ is not a local repellor. Then there exists small $\varepsilon > 0$ such that for any point $x \in \Lambda$ we have $\overline{dim}_B(W^s_r(x) \cap \Lambda) \le t^s(\varepsilon) < 2$.*

4.3 Relations between stable dimension and the preimage counting function

As we saw above, in the case of non-invertible dynamics, there is in general no formula for the stable dimension, instead it depends on the number of preimages that points in the basic set Λ have in Λ (some of the preimages may be outside Λ). We studied this problem in [57] where we gave a lower bound for the stable dimension in terms of a continuous function ω that bounds above the upper semi-continuous preimage counting function $d(\cdot)$. This result has interesting consequences, namely if the stable dimension takes it minimal possible value at some poingt, then the function is constant-to-1 on Λ and the stable dimension is constant over Λ.

Definition 4.3.1. *Let a smooth endomorphism f, and a basic set Λ for f, which contains no critical points of f. Then the preimage counting function over Λ is defined by:*

$$d(x) := Card\{f^{-1}(x) \cap \Lambda\}, \; x \in \Lambda$$

Definition 4.3.2. *Let a smooth endomorphism f defined on a Riemannian manifold M, and Λ a basic set for f. Then we say that f is c-hyperbolic on Λ if f is hyperbolic on Λ, conformal on local stable manifolds of points in Λ and the critical set $\mathcal{C}_f$ does not intersect Λ.*

Theorem 4.3.1 ([57]). *Assume f is a smooth endomorphism which is c-hyperbolic on a basic set Λ, and that there exists a continuous function ω on Λ such that for any point $z \in \Lambda$, we have $d(z) \le \omega(z)$. Then $\delta^s(x) \ge t^s_\omega$ for any $x \in \Lambda$, where t^s_ω is the unique zero of the pressure function $t \to P(t\Phi^s - \log \omega)$.*

Proof. Let us fix a point $x \in \Lambda$ and denote by $W := W^s_r(x) \cap \Lambda$. Let also $\varepsilon > 0$. We assume first that the function ω is locally constant (but not constant) over Λ, and then treat the general case at the end. From the transitivity property of f on Λ, it follows that there exists $m = m(\varepsilon)$ such that any local unstable manifold of type $W^u_\varepsilon(\hat{y})$ intersects the set $f^{-m}(W) \cap \Lambda$, for all $\hat{y} \in \hat{\Lambda}$. As f is locally bi-Lipschitz near Λ (f being c-hyperbolic), we obtain that $HD(W) = HD(f^{-m}W \cap \Lambda)$. Take an arbitrary number $t > \delta^s(x)$; then there exists a covering $\{U_i\}_{i \in I}$ of $f^{-m}W \cap \Lambda$ so that

$$\sum_{i \in I} (diamU_i)^t < \frac{1}{2} \tag{27}$$

Now consider $i \in I$ and suppose that $diamU_i > 0$. We can assume in fact that U_i is contained in a local stable manifold. Let us introduce a type of tubular unstable set used in [61] for the inverse pressure: for a finite prehistory $C = (x, x_{-1}, \dots, x_{-n})$ of x in Λ, define

$$\Lambda(C, \varepsilon) := \{y \in U, \text{there exists a prehistory of } y, (y, y_{-1}, \dots, y_{-n}), \text{s.t } d(y_{-j}, x_{-j}) < \varepsilon, j = 0, \dots, n\}$$

By *stable diameter* of $\Lambda(C, \varepsilon)$ we will understand the diameter of the intersection $\Lambda(C, \varepsilon) \cap W^s_r(x)$. For a point $y \in U_i$, consider a prehistory C of y in Λ of length n such that if $C = (y, \dots, y_{-n})$, then n is the largest integer such that $\varepsilon |Df^n_s(y_{-n})| > diamU_i$. We will call such a prehistory C a *maximal prehistory* relative to U_i and its length will be denoted also by $n(C)$. Obviously we cannot have just any length for such a maximal prehistory, so let us denote by $n_{i1}, \dots, n_{iq_i}$ all the different lengths of U_i-maximal prehistories. From construction it is clear that $U_i \subset \Lambda(C, \varepsilon)$ for C as above.

Now let us denote the set of U_i-maximal prehistories by $\mathcal{C}_i$ and let us assume that $\mathcal{F}_i$ is a minimal set of points of type $y_{-n(C)}$ for $C \in \mathcal{C}_i$, such that for any $C \in \mathcal{C}_i$, there exists $z \in \mathcal{F}_i$ with $y_{-n(C)} \in B_{n(C)}(z, \varepsilon)$ (where in general $B_m(z, \varepsilon)$ denotes the Bowen ball, i.e the set of points whose orbits are within ε distance of the orbit of z up to order m).

Denote the corresponding set of prehistories from $\mathcal{C}_i$ ending with the points of $\mathcal{F}_i$, by $\mathcal{C}^*_i$. Hence $\mathcal{C}^*_i \subset \mathcal{C}_i, i \in I$. If $z \in \mathcal{F}_i$, we will denote also by $n(z)$ the length of the corresponding prehistory $C \in \mathcal{C}^*_i$ having z as final preimage. Without loss of generality we may assume that the preimage counting function itself is locally constant, this giving in fact the worst case scenario. Since $d(\cdot)$ takes only finitely many values on Λ, we denote them by $d_1, \dots, d_p$.

Denote by $V_j := \{z \in \Lambda, d(z) = d_j\}, j = 1, \dots, p$; thus these sets are closed and mutually disjointed and assume that $d(V_j, V_k) > \varepsilon_0 > 0, j \neq k$, for some positive constant ε_0. Also, since the critical set of f does not intersect Λ, different f-preimages of any arbitrary point $x \in \Lambda$ are at a positive distance apart; without loss of generality, we can assume that this distance is also larger than ε_0.

Let us take now a point $\xi \in V_1$, hence ξ has d_1 f-preimages denoted by $\xi_1, \ldots, \xi_{d_1}$. These are simple preimages due to the fact that $\mathcal{C}_f \cap \Lambda = \emptyset$. Assume that there exists a sequence of points y from Λ which converges towards ξ, and let $y_1, \ldots, y_{d_1}$ be the d_1 preimages of y. Assume also that $d(\{y_1, \ldots, y_{d_1}\}, \{\xi_1, \ldots, \xi_{d_1}\}) > \alpha > 0$, for all points y in this sequence. Then the points $y_1, \ldots, y_{d_1}$ accumulate (eventually for a subsequence) to some points $y_1^*, \ldots, y_{d_1}^*$ which are preimages of ξ. But due to the condition on the distances between the sets of preimages, it follows that there exists at least a point y_j^* which is not in the set $\{\xi_1, \ldots, \xi_{d_1}\}$. This implies then that ξ has more than d_1 preimages in Λ, hence contradiction.

So each point $\xi \in \Lambda$ has a neighbourhood $V(\xi)$ such that any point $y \in V(\xi)$ has d_1 preimages in Λ close to the preimages $\xi_1, \ldots, \xi_{d_1}$ of ξ. Now, if for any $\eta > 0, \eta << \varepsilon_0$ there exists a point $y(\eta) \in \Lambda$ such that there exists a point $z(\eta) \in B(y(\eta), \eta)$ with the preimages of $z(\eta)$ in Λ far from the preimages of $y(\eta)$ in Λ, then we can take a subsequence of $y(\eta)$ converging towards a point $w \in \Lambda$ which has the property that in any neighbourhood there are points $z(\eta)$ with preimages far from the preimages of w, hence a contradiction with the fact proved earlier. So there exists a positive ε_1 such that if $d(y, z) < \varepsilon_1$, then the preimages of y in Λ are close (i.e closer than $d(y, z) \cdot \sup\limits_{\Lambda} |Df_s|^{-1}$) to the preimages of z in Λ. In this we used implicitly the fact that the preimages of any point from Λ have multiplicity 1, since $\mathcal{C}_f \cap \Lambda = \emptyset$.

In particular, for $C \in \mathcal{C}_i, C = (y, \ldots, y_{-n(C)})$, and $z \in B_{n(C)}(y_{-n(C)}, \varepsilon)$ we have that $f^k(z)$ has the same number of f-preimages in Λ as $f^k(y_{-n(C)})$ and moreover, these preimages are close to the f-preimages of $f^k(y_{-n(C)})$, for $k = 0, \ldots, n(C)$ (namely $\varepsilon \sup\limits_{\Lambda} |Df_s|^{-1}$-close).

Consider now the set of points of the form $y_{-n(C)}$ for some $C \in \mathcal{C}_i$ a U_i-maximal prehistory; from the definition we know that $\mathcal{F}_i$ is minimal and for any $C \in \mathcal{C}_i$ there is a prehistory $C^* = (f^{n(C)}z, \ldots, z) \in \mathcal{C}_i^*$ such that $n(C) = n(C^*)$ and $y_{-n(C)} \in B_{n(C)}(z, \varepsilon)$.

The prehistories in $\mathcal{C}_i^*$ may have different lengths. But if for example $z \in \mathcal{F}_i$ and $f(z) \in V_j$ then there exists $d_j - 1$ other points in $f^{-1}(f(z)) \cap \Lambda$ and these points will generate other prehistories from $\mathcal{C}_i^*$. Due to the above considerations we can assume without loss of generality that the set $\mathcal{F}_i$ is given by prehistories of a single point $y \in U_i$. Also we may assume that these points $y \in U_i$ do not belong to other sets $U_j, j \neq i$.

Let us arrange now the lengths of prehistories from $\mathcal{C}_i^*$ as $n_{i,q_i} > n_{i,q_i-1} > \ldots > n_{i,1}$.

Then denote by $\mathcal{F}_{i,n_{i,q_i}}$ the set of points $z \in \mathcal{F}_i$ which correspond to prehistories in $\mathcal{C}_i^*$ of length n_{i,q_i}. Denote also the cardinality of $\mathcal{F}_{i,n_{i,q_i}}$ by $N_{i,n_{i,q_i}}$.

Then let us take the set $\mathcal{F}_{i,n_{i,q_i}-1}$ as the union of $f(\mathcal{F}_{i,n_{i,q_i}})$ and the set of points $z \in \mathcal{F}_i$ which correspond to prehistories of length $n_{i,q_i} - 1$. The cardinality of $\mathcal{F}_{i,n_{i,q_i}-1}$ is denoted by $N_{i,n_{i,q_i}-1}$. We do this until reaching $N_{i,0}$ which is equal to 1, since these are considered as prehistories of a single point y from U_i. Define now:

$$N_{i,n_{i,q_i}}(j_1, \ldots, j_{n_{i,q_i}}) := Card\{z \in \mathcal{F}_{i,n_{i,q_i}}, f(z) \in V_{j_1}, \ldots, f^{n_{i,q_i}}(z) \in V_{j_{n_{i,q_i}}}\}$$

and similarly $N_{i,n_{i,q_i}-1}(j_1, \ldots, j_{n_{i,q_i}-1}) := Card\{\zeta \in \mathcal{F}_{i,n_{i,q_i}-1}, f(\zeta) \in V_{j_1}, \ldots, f^{n_{i,q_i}-1}(z) \in V_{j_{n_{i,q_i}-1}}\}$, etc. Then from the above construction we have that

$$\frac{N_{i,n_{i,q_i}}(1,j_2,\ldots,j_{n_{i,q_i}})}{d_1}+\ldots+\frac{N_{i,n_{i,q_i}}(p,j_2,\ldots,j_{n_{i,q_i}})}{d_p}\le N_{i,n_{i,q_i}-1}(j_2,\ldots,j_{n_{i,q_i}}) \tag{28}$$

Next we obtain

$$\frac{N_{i,n_{i,q_i}-1}(1,j_3,\ldots,j_{n_{i,q_i}})}{d_1}+\ldots+\frac{N_{i,n_{i,q_i}-1}(p,j_3,\ldots,j_{n_{i,q_i}})}{d_p}\le N_{i,n_{i,q_i}-2}(j_3,\ldots,j_{n_{i,q_i}}), \tag{29}$$

and we can combine this inequality with (5.4.1). By induction we obtain then that for all $i \in I$,

$$\Sigma_i := \sum_{z\in\mathcal{F}_i}\frac{1}{d_1^{m_1(z)}\cdot\ldots\cdot d_p^{m_p(z)}}\le 1, \tag{30}$$

where for each $z \in \mathcal{F}_i$, $m_1(z)$ represents the number of times that the orbit $z, f(z), \ldots, f^{n(z)}z$ hits $V_1, \ldots$, and $m_p(z) :=$ number of times that the above orbit hits V_p. We assumed that the points y chosen inside U_i do not belong to other $U_j, j \neq i$, and that the points of $\mathcal{F}_i$ are preimages (of different orders) of $y \in U_i$.

Let us assume also that N is the largest integer $n_{i,j}, 1 \le j \le q_i, i \in I$; since I is finite, it follows that $N < \infty$. We know from construction of $\mathcal{F}_i$ that any preimage of type $y_{-n(C)}$ for C a maximal prehistory associated to U_i belongs to a Bowen ball of type $B_{n(C)}(z,\varepsilon)$, for some $z \in \mathcal{F}_i$. Any local unstable manifold of size ε is contained in the union $\bigcup_{C\in\mathcal{C}_i^*}\Lambda(C,\varepsilon)$, and we want to extend these prehistories as to obtain in the end a common (or close) length for all of them. More precisely we will extend these prehistories until we reach a length between n and $n+N$, for a large integer n.

The idea is the following: let $z \in \mathcal{F}_i$ corresponding to a prehistory $C \in \mathcal{C}_i^*$ of length $n(C)$; then z itself is covered by $\bigcup_{j\in I}\bigcup_{C\in\mathcal{C}_j^*}\Lambda(C,\varepsilon)$, hence there exists $j \in I$ and a prehistory $D \in \mathcal{C}_j^*$ such that $z \in \Lambda(D,\varepsilon)$. We will concatenate now the prehistories C and D and will obtain $\Lambda(CD,\varepsilon) := \{y, \exists(y,\ldots,y_{-n(C)})\text{prehistory of } y\ \varepsilon-\text{shadowing } C, \text{and } y_{-n(C)} \in \Lambda(D,\varepsilon)\}$; so we follow the prehistories of preimages until we reach a length between n and $n+N$ for some large n.

To this end, consider the set $\mathcal{S}_n$ of all the multiples $(s, j_1, \ldots, j_s, p_1, \ldots, p_s)$ such that $s \in \mathbb{N}^*, j_1, \ldots, j_s \in I, 1 \le p_k \le q_{j_k}, k = 1, \ldots, s$ and $n \le n_{j_1,p_1} + \ldots + n_{j_s,p_s} < n + N$.

For such an element of $\mathcal{S}_n$, we start with a prehistory $C_1 = (\zeta, \ldots, \zeta_{-n_{j_1,p_1}}), \zeta \in U_{j_1}$, then we assume $\zeta_{-n_{j_1,p_1}} \in \Lambda(C_2,\varepsilon)$ with C_2 a prehistory of length n_{j_2,p_2} of a point in U_{j_2}, etc. This procedure will give in the end a final preimage $\zeta_{-n_{j_1,p_1}-\ldots-n_{j_s,p_s}} \in \Lambda$ and we denote by F_n the set of all such final points obtained by the above procedure.

Since for any $\mathcal{F}_i, i \in I$ we covered all the possible preimages $y_{-n(C)}$ corresponding to maximal U_i-prehistories C in Λ from $\mathcal{C}_i$, it follows that F_n is (n,ε)-spanning for Λ.

For $1 \le k \le q_i$, denote by $\tilde{N}_{ik}(m_1, \ldots, m_p)$ the number of elements ξ of $\mathcal{F}_i$ such that $n(\xi) = n_{i,k}$ and so that in the $n_{i,k}$-forward orbit of ξ there are exactly m_1 iterates belonging to $V_1, \ldots, m_p$ iterates belonging to V_p. By taking the product of inequalities from (30) for $j_1, \ldots, j_s$ we obtain

$$\sum_{1\le p_1\le q_{j_1},1\le p_s\le q_{j_s}} \sum_{m_1+\ldots+m_p=n_{j_1,p_1}} \frac{\tilde{N}_{j_1p_1}(m_1,\ldots,m_p)}{d_1^{m_1}\ldots d_p^{m_p}}\cdot\ldots\cdot \sum_{l_1+\ldots+l_p=n_{j_s,p_s}} \frac{\tilde{N}_{j_sp_s}(l_1,\ldots,l_p)}{d_1^{l_1}\ldots d_p^{l_p}} \le 1 \tag{31}$$

So, if $P_n(t\Phi^s - \log d(\cdot)) := \inf\{\sum_{z\in F} \exp(S_n(t\Phi^s - \log d(\cdot))(z), F(n,\varepsilon)-\text{spanning for } \Lambda\}$ and since F_n is (n,ε)-spanning, we obtain:

$$\begin{aligned}
&P_n(t\Phi^s - \log d(\cdot)) \le \sum_{z\in F_n} \exp(S_n(t\Phi^s - \log d(\cdot))(z) \le \\
&\le \sum_{(s,j_1,\ldots,j_s,p_1,\ldots,p_s)\in\mathcal{S}_n} \sum_{m_1+\ldots+m_p=n_{j_1,p_1}} \frac{\tilde{N}_{j_1p_1}(m_1,\ldots,m_p)}{d_1^{m_1}\ldots d_p^{m_p}}\cdots \sum_{l_1+\ldots+l_p=n_{j_s,p_s}} \frac{\tilde{N}_{j_sp_s}(l_s,\ldots,l_p)}{d_1^{l_1}\ldots d_p^{l_p}}\cdot \\
&\cdot (diamU_{j_1})^t\ldots(diamU_{j_s})^t \le \sum_{s,j_1,\ldots,j_s} (diamU_{j_1})^t\ldots(diamU_{j_s})^t,
\end{aligned} \tag{32}$$

after using (31). Therefore, by using (4.5.1)

$$P_n(t\Phi^s - \log d(\cdot)) \le \sum_s \sum_{j_1,\ldots,j_s} (diamU_{j_1})^t\ldots(diamU_{j_s})^t = \sum_s (\sum_{j\in I} (diamU_j)^t)^s \le \sum_s (\frac{1}{2})^s < 2 \tag{33}$$

But $P(t\Phi^s - \log d(\cdot)) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n}\log P_n(t\Phi^s - \log d(\cdot))$. This implies that $t \ge t_{d(\cdot)}$, where $t_{d(\cdot)}$ is the unique zero of the pressure function $t \to P(t\Phi^s - \log d(\cdot))$. If we have that the preimage counting function $d(\cdot)$ is only smaller or equal than $\omega(\cdot)$ at any point $x\in\Lambda$, it follows that $t \ge t_\omega$ in the same way. But t was taken arbitrarily larger than $HD(W_r^s(x)\cap\Lambda)$, so we obtain

$$HD(W_r^s(x)\cap\Lambda) \ge t_\omega, \forall x\in\Lambda$$

We now want to extend the proof above to the case when ω is an arbitrary continuous function on Λ. We consider as before the set of U_i-maximal prehistories $\mathcal{C}_i$ and an associated minimal set $\mathcal{F}_i$ of final preimages given by these prehistories.

Using the fact that the preimage counting function $d(\cdot)$ is upper semicontinuous on Λ we find again that for each point $z\in\Lambda$ there exists a neighbourhood of z such that each point y in this neighbourhood has at most $d(z)$ preimages and they are close to some of the preimages of z (however the point y may have strictly less than $d(z)$ preimages in Λ).

Again we will have that $N_{i0} = 1$ since in the minimal set $\mathcal{F}_i$ we can take only preimages of a point $y\in U_i$ where $\omega(\cdot)$ is largest on U_i. If not, then we can complete the prehistories of y with prehistories of other points but the total number will be the same as if we were considering prehistories of a single point from U_i. From the continuity of ω on Λ there exists a positive function $\rho(\varepsilon)$ defined for small $\varepsilon > 0$, with the following property:

$$\text{if } y,z\in\Lambda, \text{ and } d(y,z)<\varepsilon, \text{ then } |\omega(y)-\omega(z)| \le \rho(\varepsilon) \tag{34}$$

Since ω is continuous it follows that $\rho(\varepsilon) \to 0$ when $\varepsilon \to 0$, and we can assume that ρ has been taken such that it is an increasing function.

Now we notice that, if $y \in B_n(z, \varepsilon)$, then for any $0 \le j \le n$, $d(f^j y) \le \omega(f^j z) + \rho(\varepsilon)$, since by assumption $d(f^j y) \le \omega(f^j y)$. Thus the number of preimages of $f^j y$, $d(f^j y)$ may differ from $d(f^j z)$ by at most 1, but still $d(f^j y)$ is less or equal than $\omega(f^j z) + \rho(\varepsilon)$, where $\rho(\varepsilon) \underset{\varepsilon \to 0}{\to} 0$.

We take as before the set F_n of final preimages of type $y_{-n_{j_1,p_1} - \ldots n_{j_s,p_s}}$, over all sequences $(s, j_1, \ldots, j_s, p_1, \ldots, p_s)$ such that $j_1, \ldots, j_s \in I$ and $1 \le p_1 \le q_{j_1}, 1 \le p_s \le q_{j_s}$ with $n \le n_{j_1,p_1} + \ldots + n_{j_s,p_s} < n + N$. This set of sequences is denoted again by $\mathcal{S}_n$ as before.

Now as we mentioned, the preimage counting function is smaller or equal than ω and ω varies with at most $\rho(\varepsilon)$ on a ball of radius ε, thus we can apply this at every iterate (up to order n) for points in a Bowen ball $B_n(z, \varepsilon)$. We have then the analogues of inequalities (30) and (31), namely:

$$\Sigma_i := \sum_{z \in \mathcal{F}_i} \frac{1}{(\omega(fz) + \rho(\varepsilon)) \ldots (\omega(f^{n(C)} z) + \rho(\varepsilon))} \le 1, \tag{35}$$

where we assumed that $C = (f^{n(C)}(z), \ldots, z)$ is the prehistory from $\mathcal{C}_i^*$ whose final preimage is z, for $z \in \mathcal{F}_i$. We will denote the length $n(C)$ associated to the above C, by $n(z)$. Since ω is continuous on Λ, it takes finitely many positive integer values, denoted by $d_1, \ldots, d_p$ arranged as $d_1 < \ldots < d_p$. And similarly, by taking the product of the inequalities (35) for $j_1, \ldots, j_s$ we shall obtain:

$$\begin{aligned} &\sum_{1 \le p_1 \le q_{j_1}, 1 \le p_s \le q_{j_s}} \sum_{z \in \mathcal{F}_{j_1}, n(z) = n_{j_1,p_1}} \frac{1}{(\omega(fz) + \rho(\varepsilon)) \ldots (\omega(f^{n(z)} z) + \rho(\varepsilon)} \cdot \ldots \\ &\cdot \sum_{z \in \mathcal{F}_{j_s}, n(z) = n_{j_s,p_s}} \frac{1}{(\omega(fz) + \rho(\varepsilon)) \ldots (\omega(f^{n(z)} z) + \rho(\varepsilon))} \le 1 \end{aligned} \tag{36}$$

Then since by construction the set F_n is (n, ε)-spanning for Λ with respect to f (since we cover all final preimages with $\mathcal{F}_i$), we can finish the proof by using (36) in the same way as before. So we obtain that $t \ge t(\varepsilon)$ for $\varepsilon > 0$ small, with $t(\varepsilon)$ being the unique zero of the pressure function $t \to P_\varepsilon(t\Phi^s - \log(\omega + \rho(\varepsilon)))$, where in general $P_\varepsilon(g) := \limsup_n \frac{1}{n} \log \inf\{\sum_{z \in F} \exp(S_n(g)(z)), F(n, \varepsilon)-$ spanning in$\Lambda\}$ for g continuous on Λ.

Let us take now some T arbitrarily larger than t and $\eta > 0$ small; then $T > t \ge t(\eta)$. But if $0 < \varepsilon < \eta$, we get that $\rho(\varepsilon) \le \rho(\eta)$, so $t\Phi^s - \log(\omega + \rho(\varepsilon)) \ge T\Phi^s - \log(\omega + \rho(\eta))$. Now since $t \ge t(\varepsilon)$ for all ε small, it follows that $0 \ge P_\varepsilon(t\Phi^s - \log(\omega + \rho(\varepsilon))) \ge P_\varepsilon(T\Phi^s - \log(\omega + \rho(\eta)))$ for all $\varepsilon > 0$ small enough. But recalling the definition of the topological pressure $P(g) = \lim_{\varepsilon \to 0} P_\varepsilon(g)$, for all g continuous, we obtain that

$$P(T\Phi^s - \log(\omega + \rho(\eta))) \le 0 \tag{37}$$

Now let t_ω be the unique zero of the pressure function $t \to P(t\Phi^s - \log \omega)$. From the continuity of the pressure with respect to the potential, it follows that t_ω is the limit of the zeros of the pressure functions $t \to P(t\Phi^s - log(\omega + \rho(\eta)))$ when η converges to 0. Hence from (37), $T \ge t_\omega$.

Therefore since T was chosen arbitrarily larger than t which in turn was chosen arbitrarily larger than $HD(W_r^s(x) \cap \Lambda)$, we obtain the conclusion, $HD(W_r^s(x) \cap \Lambda) \geq t_\omega$.

□

Then in [52] Mihailescu and Stratmann found an upper estimate for the stable dimension on a folded hyperbolic fractal, using lower continuous bound for the preimage counting function; the proof in this case is very different from the lower estimate above. We denoted here the preimage counting function by $\Delta(x) = Card(f^{-1}(x) \cap \Lambda),\ x \in \Lambda$.

Theorem 4.3.2 ([52]). *Consider a $\mathcal{C}^2$-endomorphism f on the Riemannian manifold M, so that f is c-hyperbolic on a basic set Λ of f, and there exists a continuous function $\omega : \Lambda \to \mathbb{R}$ with $\Delta(x) \geq \omega(x)$, for all $x \in \Lambda$. It then follows that*

$$\delta^s(x) \leq t_\omega,$$

where t_ω refers to the unique zero of the pressure function $t \mapsto P(t\Phi^s - \log \omega)$ associated to the potential function $t\Phi^s - \log \omega$.

Proof. We will first consider the case when ω is locally constant and has only two different positive integer values on Λ, namely d_1 on the set V_1 and d_2 on the set V_2. We then have that $V_1 \cup V_2 = \Lambda$ and that V_1 and V_2 are two disjoint compact subsets of Λ. Thus there exists some small $\varepsilon_0 > 0$ such that the distance $d(V_1, V_2)$ between V_1 and V_2 is greater than ε_0. For $x \in \Lambda$ and $n \in \mathbb{N}$, let $B_n(x, \varepsilon) := \{y \in \Lambda : d(f^i(y), f^i(x)) < \varepsilon, 0 \leq i \leq n-1\}$ the n-Bowen ball centred at x, of radius $\varepsilon > 0$. For $0 < \varepsilon < \varepsilon_0$ we have that if $y \in B_n(x, \varepsilon)$ then $f^i(y)$ and $f^i(x)$ both belong to either V_1 or V_2, for each $0 \leq i \leq n-1$. Recall that $\Phi^s(x) := \log |Df_s(x)|, x \in \Lambda$. Now let $t > t_\omega$ fixed. By definition of t_ω, there exists $\beta > 0$ such that $P(t\Phi^s - \log \omega) < -\beta$. So if $\varepsilon > 0$ is sufficiently small, there exists a constant $C > 0$ such that for each $n \in \mathbb{N}$ large enough, there exists a minimal (n, ε)-spanning set E_n for Λ such that

$$\sum_{z \in E_n} (\text{diam } U_n(z))^t \cdot \frac{1}{\Delta(f(z) \cdot \ldots \cdot \Delta(f^n(z))} < C\, e^{-\beta n} < 1, \tag{38}$$

where we have set $U_n(z) := f^n(B_n(z, \varepsilon)) \cap W_r^s(x) \cap \Lambda$. Here we used that the set $U_n(z)$ is the intersection of an unstable tubular neighbourhood with the fixed stable manifold $W_r^s(x)$. Also, that $|Df_s^n(z)|$ is uniformly comparable to diam $U_n(z)$, which follows from the fact that f is conformal on local stable manifolds.

Let us denote $W := W_r^s(x) \cap \Lambda$. We wish to show that $\dim_H(W) \leq t$, for each $t > t_\omega$. The main idea is to extract succesively suitable covers of W out of the large set of covers which are given by taking n-preimages, such that at each step some different sum will be minimised. We say that a point y is a *k-preimage of* x if $f^k(y) = x$. Each such n-preimage will be included in a Bowen ball of type $B_n(z, \varepsilon)$, for some $z \in E_n$. This procedure is delicate, since at each step the number of preimages of points belonging to Λ varies. The idea is to consider the k iterates of n-preimages, then to subdivide Λ into various different parts and finally, to find suitable covers of these parts which minimise certain sums at the k-th level.

First, since Λ is covered by the set of Bowen balls $\{B_n(z,\varepsilon) : z \in E_n\}$, it follows that $\{U_n(z) : z \in E_n\}$ covers W. However, this cover is far too rich and we will have to extract a suitable subcover. Indeed, by using a well known theorem by Besicovitch (see for e.g. [35]), there exists a subcover $\{5U_n(z) : z \in \mathcal{G}(0)\}$ of W such that $\{U_n(z) : z \in \mathcal{G}(0)\}$ consists of pairwise disjoint sets. (Note that, since f is conformal on local stable manifolds, we can assume that the sets $U_n(z)$ are in fact balls, and we denote the radii of these by $r(n,z)$; also, we write $5U_n(z)$ to denote the ball of radius $5r(n,z)$ centred at the centre of $U_n(z)$). The next step is to "inflate" this cover, that is, to enlarge it to a "richer" cover of W. For this, we consider an $(n-1)$-preimage of w in Λ which we denote by $w(n-1)$, for each point $w \in W$. Let us assume that $w(n-1) \in V_1$ and hence, that $w(n-1)$ has at least d_1 1-preimages in Λ. Now, since E_n is (n,ε)-spanning, for each point $\xi \in \Lambda$, there exists at least one point $y \in E_n$ such that $\xi \in B_n(y,\varepsilon)$. However, we cannot have two 1-preimages of some $w(n-1)$ belonging to different Bowen balls $B_n(y,\varepsilon)$ and $B_n(y',\varepsilon)$ such that y and y' are both in $\mathcal{G}(0)$. This is an immediate consequence of the fact that $\{U_n(z) : z \in \mathcal{G}(0)\}$ consists of pairwise disjoint sets.

Therefore, by way of successive eliminations, we can find d_1 pairwise disjoint families, denoted by $\mathcal{F}(1,d_1;1),\ldots,\mathcal{F}(1,d_1;d_1)$, such that $\{5U_n(z) : z \in \mathcal{F}(1,d_2;i)\}$ is a cover of the set $\{w \in W : w(n-1) \in V_1\}$, for each $1 \le i \le d_1$. Obviously, for $w(n-1) \in V_2$ we can proceed in a similar way, which then gives rise to d_2 mutually disjoint families $\mathcal{F}(1,d_2;1),\ldots,\mathcal{F}(1,d_2;d_2)$ for which we have that $\{5U_n(z) : z \in \mathcal{F}(1,d_2;j)\}$ is a cover of $\{w \in W : w(n-1) \in V_2\}$, for each $1 \le j \le d_2$. Note that, since $d(V_1,V_2) > 0$, we have that $\mathcal{F}(1,d_1;i) \cap \mathcal{F}(1,d_2;j) = \emptyset$, for all i and j, and that by construction we have that the so obtained disjoint families are all contained in E_n. Next, define $\mathcal{F}(1) := \bigcup_{i=1}^{2} \bigcup_{1\le j\le d_i} \mathcal{F}(1,d_i,j)$, and let $\mathcal{G}(1,d_k)$ be given, for $k \in \{1,2\}$, by

$$\sum_{z\in\mathcal{G}(1,d_k)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^2(z))\ldots\Delta(f^n(z))} = \min\left\{ \sum_{z\in\mathcal{F}(1,d_k;i)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^2(z))\ldots\Delta(f^n(z))} : i \in \{1,...,d_k\} \right\}.$$

For $\mathcal{G}(1) := \mathcal{G}(1,d_1) \cup \mathcal{G}(1,d_2)$, we then obtain, by adding the sums over $\mathcal{G}(1,d_1)$ and $\mathcal{G}(1,d_2)$,

$$\sum_{z\in\mathcal{G}(1)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^2(z))\ldots\Delta(f^n(z))} \le \sum_{z\in\mathcal{F}(1)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z))\ldots\Delta(f^n(z))}. \tag{39}$$

Note that here we have used the trivial fact that for each $x \in \Lambda$ we have that $\sum_{y\in\Lambda, f(y)=x} 1/\Delta(x) = 1$. Also, note that the sum over the family $\mathcal{G}(1)$ on the left hand side of the inequality in (39) is smaller than the sum over the larger family $\mathcal{F}(1)$ on the right hand side. However, and this is the crucial point, the summands on the right hand side have one more factor in their denominator than the summands on the left hand side.

Now, we shall enlarge the family $\mathcal{F}(1)$ as follows. Recall that for each $w \in W$ we have fixed an $(n-1)$-preimage $w(n-1) \in \Lambda$. We now define $w(n-2) := f(w(n-1))$ and consider not only $w(n-1)$ but also the other 1-preimages of $w(n-2)$ in Λ. Subsequently, we will then take the 1-preimages of these 1-preimages of $w(n-2)$ and obtain new covers of W. Indeed similarly as before, if $w(n-2) \in V_1$ then we can construct, by succesive eliminations, pairwise disjoint families $\mathcal{F}(2, d_1; 1), ..., \mathcal{F}(2, d_1; d_1)$ by selecting the 1-preimages of the i-th preimage of $w(n-2)$, for each $1 \leq i \leq d_1$. In fact one of these families is $\mathcal{F}(1)$. As in the first step, the sets $\{5U_n(z) : z \in \mathcal{F}(2, d_1; i)\}$ cover $\{w \in W : w(n-2) \in V_1\}$, for each i. Let us remark that the procedure of successive elimination works, since if we take for instance the family $\mathcal{F}(2, d_1; 1)$, then for an arbitrary $w \in W$ we cannot have two 1-preimages y and y' of $w(n-2)$ and 1-preimages ξ of y and ξ' of y' such that ξ and ξ' are both contained in either $B_n(z, \varepsilon)$ or $B_n(z', \varepsilon)$, for some $z, z' \in \mathcal{F}(2, d_1; 1)$. Indeed, since $f^2(B_n(z, \varepsilon)) \cap f^2(B_n(z', \varepsilon)) \neq \emptyset$, in this situation it would follow that $U_n(z) \cap U_n(z') \neq \emptyset$ and hence we would have a contradiction. This implies that there exist d_1 disjoint families $\mathcal{F}(2, d_1; i)$ corresponding to the d_1 1-preimages of $w(n-2) \in V_1$.

Continuing the above procedure, assume we have constructed a family $\mathcal{F}(k) \subset E_n$ and a subfamily $\mathcal{G}(k)$, so that the sets $(U_n(z))_{z \in \mathcal{G}(k)}$ 5-cover W and

$$\sum_{z \in \mathcal{G}(k)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^{k+1}(z)) \dots \Delta(f^n(z))} \leq \sum_{z \in \mathcal{F}(k)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z)) \dots \Delta(f^n(z))}.$$

For each $w \in W$, we then take the k-th iterate of $w(n-1)$ and denote it by $w(n-k-1)$; this is an $(n-k-1)$-preimage of w in Λ. Now, if $w(n-k-1) \in V_1$ then it has d_1 1-preimages in Λ and to each of these we can apply the same procedure from step k. In this way we obtain by succesive eliminations d_1 mutually disjoint families $\mathcal{F}(k+1, d_1; i), 1 \leq i \leq d_1$ and inside each of these a subfamily $\mathcal{G}(k+1, d_1; i)$ such that

$$\sum_{z \in \mathcal{G}(k+1, d_1; i)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^{k+1}(z)) \dots \Delta(f^n(z))} \leq \sum_{z \in \mathcal{F}(k+1, d_1; i)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z)) \dots \Delta(f^n(z))}.$$

The succesive elimination procedure works since we cannot have two differerent 1-preimages y and y' of $w(n-k-1)$ having $(n-k)$-preimages $\xi \in \Lambda$ and $\xi' \in \Lambda$ respectively, such that $\xi \in B_n(z, \varepsilon), \xi' \in B_n(z', \varepsilon)$, for some $z, z' \in \mathcal{F}(k+1, d_1; i)$. Indeed, it would then follow that the family $\{U_n(z) : z \in \mathcal{F}(k+1, d_1; i)\}$ does not consist of pairwise disjoint sets, which clearly is a contradiction. Moreover, since $V_1 \cap V_2 = \emptyset$, we must have $\mathcal{F}(k+1, d_1; i) \cap \mathcal{F}(k+1, d_2; j) = \emptyset$. Hence, there is no repetition of elements, when we consider the union $\mathcal{F}(k+1) := \bigcup_{1 \leq j \leq d_1} \mathcal{F}(k+1, d_1; j) \cup \bigcup_{1 \leq j \leq d_2} \mathcal{F}(k+1, d_2; j)$.

Now among the collections $\mathcal{G}(k+1, d_1; i)$, for $1 \leq i \leq d_1$, let us consider the one which gives rise to the least sum $\sum_{z \in \mathcal{G}(k+1, d_1; i)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^{k+2}(z)) \dots \Delta(f^n(z))}$. Denote this minimizing collection by $\mathcal{G}(k+1, d_1)$. Similarly, we obtain the collection $\mathcal{G}(k+1, d_2)$. We now have that

$$\begin{aligned}
\sum_{z \in \mathcal{G}(k+1, d_1)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^{k+2}(z)) \dots \Delta(f^n(z))} &\leq \sum_{z \in \bigcup_{1 \leq i \leq d_1} \mathcal{G}(k+1, d_1; i)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^{k+1}(z)) \dots \Delta(f^n(z))} \\
&\leq \sum_{z \in \bigcup_{1 \leq i \leq d_1} \mathcal{F}(k+1, d_1; i)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z)) \dots \Delta(f^n(z))}.
\end{aligned} \tag{40}$$

Of course, we can proceed similarly for $\mathcal{G}(k+1, d_2)$. With $\mathcal{G}(k+1) := \mathcal{G}(k+1, d_1) \cup \mathcal{G}(k+1, d_2)$, it follows from above that

$$\sum_{z \in \mathcal{G}(k+1)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f^{k+2}(z)) \ldots \Delta(f^n(z))} \leq \sum_{z \in \bigcup_{1 \leq i \leq d_1} \mathcal{F}(k+1)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z)) \ldots \Delta(f^n(z))}.$$

We obtain thus, by finite induction, a union $\mathcal{F}(n)$ of families in E_n, as well as one particular family $\mathcal{G}(n)$ such that $\{5U_n(z) : z \in \mathcal{G}(n)\}$ covers the set W and has the property that

$$\sum_{z \in \mathcal{G}(n)} (\operatorname{diam} U_n(z))^t \leq \sum_{z \in \mathcal{F}(n)} \frac{(\operatorname{diam} U_n(z))^t}{\Delta(f(z)) \ldots \Delta(f^n(z))}.$$

By combining this with (38) at the start of the proof, this shows that $\sum_{z \in \mathcal{G}(n)} (\operatorname{diam} U_n(z))^t < 1$. Since $\{5U_n(z) : z \in \mathcal{G}(n)\}$ is a covering of the set $W = W_r^s(x) \cap \Lambda$, we can now conclude that

$$\delta^s(x) \leq t < t_\omega.$$

When ω is a general continuous function on Λ with $\omega(x) \leq \Delta(x)$, for all $x \in \Lambda$, we proceed as follows: notice first that, by the continuity of ω, we have that there exists an increasing, positive function ρ on $(0, \infty)$ such that $\rho(\varepsilon)$ decreases to zero for ε converging to zero with positive values, and such that if $d(y, z) \leq \varepsilon$, then $|\omega(y) - \omega(z)| \leq \rho(\varepsilon)$. Since if $y \in B_n(z, \varepsilon)$ then $f^i(y) \in B(f^i z, \varepsilon)$, the latter implies that if $y \in B_n(z, \varepsilon)$ then $|\omega(f^i(y)) - \omega(f^i(z))| \leq \rho(\varepsilon)$. Hence, since $\Delta(x) \geq \omega(x)$ for all $x \in \Lambda$, it follows that for each $0 \leq i \leq n-1$ we have

$$\Delta(f^i(y)) \geq \omega(f^i(y)) \geq \omega(f^i(z)) - \rho(\varepsilon).$$

Now in order to proceed, let us define the ε-pressure function P_ε, for some arbitrary potential function ψ, by

$$P_\varepsilon(\psi) := \liminf_{n \to \infty} \frac{1}{n} \log \inf \left\{ \sum_{x \in E} \exp \left(\sum_{k=0}^{n-1} \psi(f^k(x)) \right) : E \text{ is a } (n, \varepsilon)\text{-spanning set for } \Lambda \right\}$$

and let t_ε denote the unique zero of $P_\varepsilon(t\Phi^s - \log(\omega - \rho(\varepsilon)))$. Then let $t > t_\varepsilon$ be fixed and note that the above proof goes through in the same way if in the sums appearing there, we replace the function Δ by the function $\omega - \rho(\varepsilon)$. This follows since for all $0 \leq i \leq n-1$ we have that $\Delta(f^i y) \geq \omega(f^i(y)) \geq \omega(f^i(z)) - \rho(\varepsilon)$, for each $y \in B_n(z, \varepsilon)$ and for some arbitrary fixed element z contained in some minimal (n, ε)-spanning set E_n for Λ. In this way, the above inductive procedure produces a family $\mathcal{F}(n) \subset E_n$ and also a family $\mathcal{G}(n)$, such that $\{5U_n(z) : z \in \mathcal{G}(n)\}$ covers the set W and so that

$$\sum_{z \in \mathcal{G}(n)} (\operatorname{diam} U_n(z))^t \leq \sum_{z \in \mathcal{F}(n)} \frac{(\operatorname{diam} U_n(z))^t}{(\omega(f(z)) - \rho(\varepsilon)) \ldots (\omega(f^n(z)) - \rho(\varepsilon))} < 1.$$

Now, for $\eta > 0$ sufficiently small and $0 < \varepsilon < \eta$, let $\tau_{\varepsilon,\eta}$ refer to the unique zero of the pressure function $P_\varepsilon(t\Phi^s - \log(\omega - \rho(\eta)))$ and let τ_η denote the unique zero of the pressure function $P(t\Phi^s - \log(\omega - \rho(\eta)))$. Since $\lim\limits_{\varepsilon\to 0} P_\varepsilon(\psi) = P(\psi)$ for each continuous function ψ, it follows that $\lim_{\varepsilon\to 0}\tau_{\varepsilon,\eta} = \tau_\eta$. On the other hand, for every $0 < \varepsilon < \eta$ we have that $\rho(\varepsilon) < \rho(\eta)$ and therefore, $t\Phi^s - \log(\omega - \rho(\varepsilon)) \leq t\Phi^s - \log(\omega - \rho(\eta))$. Thus $\tau_\varepsilon \leq \tau_{\varepsilon,\eta}$. Now, consider some arbitrary fixed $t > \tau_\eta$. For $\varepsilon > 0$ sufficiently small, we have $t > \tau_{\varepsilon,\eta} \geq \tau_\varepsilon$. Hence, from above we have that for every t in this range and for n large enough, there exists a cover $\{5U_n(z) : z \in \mathcal{G}(n)\}$ of W such that $\sum\limits_{z\in\mathcal{G}(n)} (\operatorname{diam} U_n(z))^t < 1$. This shows that $t \geq \dim_H(W)$ and therefore, since $t > \tau_\eta$ was chosen to be arbitrary, it follows that $\tau_\eta \geq \dim_H(W)$. Now, from the continuity of the pressure function, we obtain $\lim_{\eta\to 0}\tau_\eta = t_\omega$, which gives $\dim_H(W) \leq t_\omega$. $\square$

As before, note that for an endomorphism f in higher dimension, a hyperbolic basic set is not necessarily totally invariant.

In [52], we considered the situation in which δ^s attains a maximal value and show that in this case, δ^s must be constant on Λ and that the preimage counting function $\Delta(\cdot)$ is equal to its least value d on an open dense subset.

Proposition 4.3.1 ([52])**.** *Assume we are in the same setting as in Theorem 4.3.2, that the minimal value of Δ on Λ is equal to d and that there exists a point $x \in \Lambda$ at which δ^s is equal to the unique zero t_d of the pressure function $t \mapsto P(t\Phi^s - \log d)$. Then Δ is equal to d on an open dense set of Λ, and $\delta^s(y)$ is equal to t_d, for all $y \in \Lambda$.*

This proposition can be applied in particular in the case in which d is equal to 1, and where there is no overlap. In this case the stable dimension is equal to the similarity dimension, and the proposition then guarantees that there exists an open dense set of points in Λ at which f has precisely one preimage in Λ. Thus, in this case the map behaves almost like a homeomorphism when restricted to Λ. This situation is relatively parallel to a result of Schief [86], although the setting and proofs are completely different.

Corollary 4.3.1 ([52])**.** *Let $f : M \to M$ be a $\mathcal{C}^2$-endomorphism which is c-hyperbolic on a basic set Λ of f and for which there exists a point $x \in \Lambda$, such that $\delta^s(x)$ is equal to the unique zero t_1 of the pressure function $t \mapsto P(t\Phi^s)$. Then there exists an open dense set of points in Λ at which f has precisely one preimage in Λ. Moreover, we have that $\delta^s(y) = t_1$, for all $y \in \Lambda$.*

Also, in [52] we applied Corollary 4.3.1 to a class of translations of horseshoes with overlaps, previously studied by Simon and Solomyak in [89].

Moreover in [52], we considered the *stable upper box dimension* $\beta^s(x)$ which is the upper box-counting dimension $\overline{\dim}_B(W^s_r(x) \cap \Lambda)$ of the intersection $W^s_r(x) \cap \Lambda$, for each $x \in \Lambda$. We showed that this function is constant throughout Λ and that in the situation in which Δ is bounded from below, similarly as in Theorem 4.3.2, one derives an upper bound for its value.

Proposition 4.3.2 ([52]). *Consider $f : M \to M$ a $\mathcal{C}^2$-endomorphism which is c-hyperbolic on a basic set Λ of f. Then the following statements are true:*

(a) If there exists a continuous function $\omega : \Lambda \to \mathbb{R}$ such that $\Delta(x) \geq \omega(x)$, for all $x \in \Lambda$, then we have, with t_ω given as in Theorem 4.3.2,

$$\beta^s(y) \leq t_\omega, \text{ for all } y \in \Lambda.$$

(b) The function β^s is constant on Λ.

In particular the above results can be applied for hyperbolic basic sets of saddle type for holomorphic maps $f : \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$.

In [20] (see also [92] and [70]) Falconer studied self-affine fractals with overlaps obtained from finitely many linear contractions $T_i(x) = \lambda_i x, i = 1, ..., \ell$ in $\mathbb{R}$ satisfying $0 < |\lambda_i| < 1$ and $\sum_{1 \leq i \leq \ell} |\lambda_i| < 1$. He showed that the Hausdorff dimension of the invariant set of the family of translated contractions $\{T_i + a_i, : 1 \leq i \leq \ell\}$ is equal to s, for Lebesgue almost all $(a_1, \ldots, a_\ell) \in \mathbb{R} \times ... \times \mathbb{R}$, where s represents the *similarity dimension* defined as the solution of $\sum_{1 \leq i \leq \ell} |\lambda_i|^s = 1$. However, the result fails if the condition $\sum_{1 \leq i \leq \ell} |\lambda_i| < 1$ is not satisfied, as observed by Edgar, who based his argument on a result by Przytycki and Urbański. Indeed, if $T_1 = T_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & \lambda \end{pmatrix}$ and if $|\lambda| > \frac{1}{2}$, then for Lebesgue almost every $a = (a_1, a_2) \in \mathbb{R}^2$ the attractor $\Lambda(a)$ of the system $\{T_1 + a_1, T_2 + a_2\}$ stays to be the same; and moreover if $1/\lambda$ is a Pisot number (that is, an algebraic integer such that the absolute value of all its algebraic conjugates is less than 1), then $\dim_H(\Lambda(a)) < 2 - (\log(1/\lambda))/\log 2$ (see e.g. [92]). This shows that fractals originating from overlapping constructions can have Hausdorff dimension *strictly less* than their similarity dimension.

In [86], Schief investigated self-similar fractal sets K and showed that, if for the similarity dimension σ of K one has that the σ-dimensional Hausdorff measure $\mathcal{H}^\sigma(K)$ is positive, then K satisfies the strong open set condition (so the system is similar to a homeomorphism on K). However the setting and the ideas of our proofs differ significantly from the approach in [86]. In addition, the assumptions in Proposition 4.3.1 are much weaker than the ones in [86]: in order to obtain the "almost injectivity" on Λ, we only require that the stable dimension $\delta^s(x)$ is equal to the zero t_1 of the pressure function $t \to P(t\Phi^s)$, for some $x \in \Lambda$; we do not require that $\mathcal{H}^{t_1}(W^s_r(x) \cap \Lambda) > 0$. In our case t_1 is the analogue of the similarity dimension in the stable direction, in the sense that it represents the dimension which one would obtain if the system would be invertible. Thus, if there exists a point $x \in \Lambda$, with $\mathcal{H}^{t_1}(W^s_r(x) \cap \Lambda) > 0$, then the stable dimension is everywhere equal to t_1 and there exists an open dense set of points in Λ with precisely one preimage remaining in Λ.

Recall also that in [57], Mihailescu and Urbański studied c-hyperbolic maps on Λ for which Δ is bounded from *above* by a continuous map η on Λ. The proof for the upper estimate above, is very different from the proof for lower estimates from [57]. Nevertheless, we can combine these two estimates to obtain that, if the preimage counting function Δ is locally constant on Λ, then the stable dimension is equal to t_Δ on the fractal Λ.

4.4 Transversality conditions for families of skew products with overlaps in fibers.

A particular class of endomorphisms with interesting behaviour is given by skew products with overlaps in fibers, which are hyperbolic on their respective limit sets.

A technique, introduced in a different setting by Peres, Solomyak, Simon, etc. (see [70], [89], [92], [93], and references therein) is that of using *transversality conditions* for families of parametrized dynamical systems. If the family satisfies certain transversality conditions with respect to its parameters, then it is possible to derive information about dimension, for Lebesgue-almost all parameters.

In [59], Mihailescu and Urbanski studied a family of parametrized skew products which are hyperbolic on their respective fractal limit sets, and proved a Bowen type formula for the stable dimension for Lebesgue-almost all parameters. Moreover we gave in that paper several concrete examples, some having iterated function systems in their base, others coming from higher dimensional complex dynamics.

In general, a continuous self-map $f : X \to X$ of a compact metric space (X, ρ) is called *open distance expanding*, if f is open, Lipschitz continuous, and there are constants $\eta > 0$, $\gamma > 1$ and an integer $k \geq 1$, such that $\rho(f^k(x), f^k(z)) \geq \gamma\rho(x, z)$ whenever $\rho(x, z) \leq \eta$. One can see that changing the metric ρ in a bi-Lipschitz manner, we may assume without loss of generality that $k = 1$.

Let now U be a bounded open subset of a Euclidean space $\mathbb{R}^p$, with $p \geq 1$. A map $g : U \to \mathbb{R}^p$ is called an *expanding repeller* if and only if the following conditions are satisfied:

i) $g : U \to \mathbb{R}^p$ is a $C^{1+\gamma}$ endomorphism.

ii) $X = \cap_{n=0}^{\infty} g^{-n}(U)$ is a compact g-invariant ($g(X) = X$) subset of U. The map $g : X \to X$ is transitive.

iii) The map $g : X \to X$ is infinitesimally expanding, i.e. there exists $k \geq 1$ such that for all $x \in X$ and for all $v \in \mathbb{R}^p$, we have $||D_x g^k(v)|| \geq 2||v||$.

Clearly, $g : X \to X$ is an open distance (with respect to the Euclidean metric) expanding map.

Let us now take $f : X \to X$ an open distance expanding map and suppose it is topologically transitive. Let V be a bounded convex open subset of $\mathbb{R}^q$, $q \geq 1$.

Definition 4.4.1 ([59]). *Suppose that for all $x \in X$ there exists a $C^{1+\gamma}$ conformal endomorphism $\phi_x : V \to V$ conformally extendable to a neighborhood of $\overline{V}$ with the following properties.*

(a) $\kappa := \sup\{|(\phi_x)'(y)| : (x, y) \in X \times \overline{V}\} < 1$.

(b) $\underline{\kappa}[[a := \inf\{|(\phi_x)'(y)| : (x, y) \in X \times \overline{V}\} > 0$.

If the conditions (a) and (b) are satisfied, then the map $F : U \times V \to \mathbb{R}^p \times V$, defined by

$$F(x, y) = (f(x), \phi_x(y))$$

is called a ***hyperbolic fiberwise conformal skew-product*** *if it is Lipschitz continuous (with respect to the sum metric on* $X \times \mathbb{R}^q$*) and the map* $(x, y) \mapsto (f(x), \phi'_x(y))$ *is also Lipschitz continuous; denote the common Lipschitz constant by* L_F*.*

Let us consider the fractal limit set

$$\Lambda = \cup_{x \in X} \cap_{n=0}^{\infty} \cup_{z \in f^{-n}(x)} \phi_z^n(\overline{V}),$$

where $\phi_z^n = \phi_{F^{n-1}(z)} \circ \phi_{f^{n-1}(z)} \circ \ldots \circ \phi_z : \overline{V} \to \overline{V}$ and $F^n(x, y) = (f^n(x), \phi_x^n(y))$; Λ is called the basic set of the endomorphism F. Obviously $F(\Lambda) \subset \Lambda$ and $F(Y_x) \subset Y_{f(x)}$, where $Y_x = \cap_{n=0}^{\infty} \cup_{z \in f^{-n}(x)} \phi_z^n(\overline{V})$. Let $\hat{f} : \hat{X} \to \hat{X}$ be the natural extension (inverse limit) of the endomorphism $f : X \to X$. For every $n \geq 0$ let $p_n : \hat{X} \to X$ be the projection onto nth coordinate of $\hat{X}$. Consider

$$\hat{\Lambda} = \cup_{x \in X} p_0^{-1}(x) \times Y_x$$

and define the map $\hat{F} : \hat{\Lambda} \to \hat{\Lambda}$ by the formula

$$\hat{F}(\hat{x}, y) = (\hat{f}(\hat{x}), \phi_{x_1}(y)).$$

Notice the map $\hat{F} : \hat{\Lambda} \to \hat{\Lambda}$ is a homeomorphism and the mapping $((x_n, y_n)_0^{\infty}) \mapsto ((x_n, y_0)_0^{\infty})$ is a homeomorphism from $\hat{\Lambda}$, the Rokhlin's natural extension of f_Λ, to $\hat{\Lambda}$ which establishes a canonical topological conjugacy between the map $\hat{F} : \hat{\Lambda} \to \hat{\Lambda}$ and the map $\hat{F} : \hat{\Lambda} \to \hat{\Lambda}$. Note that for every $\hat{x} \in \hat{X}$, $\{\phi_{x_n}^n(\overline{V})\}_{n=0}^{\infty}$ is descending (as $\phi_{x_{n+1}}^{n+1} = \phi_{x_n}^n \circ \phi_{x_{n+1}}$) sequence of compact sets whose diameters, by condition (e) converge to 0. Hence, the intersection

$$\cap_{n=0}^{\infty} \phi_{x_n}^n(\overline{V})$$

is a singleton, and denote its only element by $\pi(\hat{x})$. So, we have a map $\pi : \hat{X} \to \overline{V}$. For every $x \in X$, we have: $\pi(p_0^{-1}(x)) = Y_x$.. Endow $\hat{X}$ with a metric $\hat{\rho}$ defined as follows. $\hat{\rho}(\hat{x}, \hat{z}) = \sum_{n=0}^{\infty} \kappa^n \rho(x_n, z_n)$.. The map $\pi : \hat{X} \to \overline{V}$ is Lipschitz continuous.

For every continuous potential $g : \hat{X} \to \mathbb{R}$ let $P(g) = P(\hat{f}, g)$ be the topological pressure of g with respect to the dynamical system $\hat{f} : \hat{X} \to \hat{X}$. For the topological pressure and its basic properties see for ex. [5]. Now consider the potential $\zeta = \zeta_F : \hat{X} \to \mathbb{R}$ given by the formula

$$\zeta(\hat{x}) = \log \phi'_{x_0}(\pi(\hat{x}))|$$

This potential is Hölder continuous. Notice that the function $t \mapsto P(\hat{f}, t\zeta)$ is convex, Lipschitz continuous, strictly decreasing, and $\lim_{t \to -\infty} P(\hat{f}, t\zeta) = +\infty$ and $\lim_{t \to +\infty} P(\hat{f}, t\zeta) = -\infty$. So there exists exactly one $t \in \mathbb{R}$, denoted by h, such that $P(\hat{f}, h\zeta) = 0$. Since $P(\hat{f}, 0\zeta) = h_{top}(\hat{f}) > 0$, we see that $h > 0$. The number h is called Bowen's stable zero of the basic set Λ.

Now, endow the space $C^{1+\gamma}(\overline{V})$ of all $C^{1+\gamma}$ differentiable endomorphisms from $\overline{V}$ into $\overline{V}$ with the norm $|| \cdot ||_\gamma$ given by the formula

$$||\phi||_\gamma = ||\phi||_\infty + ||\phi'||_\infty + v_\gamma(\phi'),$$

where

$$v_\gamma(\phi') = \inf\{L > 0 : |\phi'(y) - \phi'(x)| \le L|y - x|^\gamma \text{for all } x, y \in \overline{V}\}.$$

Obviously $C^{1+\gamma}(\overline{V})$ endowed with this norm becomes a Banach space; and denote the metric induced by the norm $||\cdot||_\gamma$ by ρ_γ.

Definition 4.4.2. *In the above setting, fix $d \ge 1$ and an open set $W \subset \mathbb{R}^d$ and consider a family $\Phi = \{\phi_x^\lambda : \overline{V} \to V\}_{(\lambda,x)\in W\times X}$ of maps from $C^{1+\gamma}(\overline{V})$, satisfying the following conditions.*

(af) Conditions (a) and (b) with the same constants $\kappa, \underline{\kappa} \in (0,1)$.

(bf) The map $(\lambda, x) \mapsto \phi_x^\lambda \in C^{1+\gamma}(\overline{V})$ defined on $W \times X$ is continuous.

(cf) (Transversality Condition)

$$\forall (x \in X), \forall(\lambda_0 \in W)\, \exists(\delta(x,\lambda_0) > 0)\, \exists(C_1 > 0)\, \forall(\hat{x}, \hat{y} \in p_0^{-1}(x))\, \forall(r > 0)$$
$$x_1 \ne y_1 \Rightarrow l_d(\{\lambda \in B(\lambda_0, \delta(x,\lambda_0)) : ||\pi_\lambda(\hat{x}) - \pi_\lambda(\hat{y})|| \le r\}) \le C_1 r^q,$$

where l_d denotes the d-dimensional Lebesgue measure on $\mathbb{R}^d$ and $\pi_\lambda : \hat{X} \to \overline{V}$ is the canonical projection induced by the skew-product $F_\lambda : U \times \overline{V} : \mathbb{R}^p \times \overline{V}$, given by the formula

$$F_\lambda(x,y) = (f(x), \phi_x^\lambda(y)).$$

*Any such family Φ is said to be **transversal** and the canonically induced family $\overline{\Phi} = \{F_\lambda\}_{\lambda\in W}$ is also called **transversal**.*

For all $\lambda, \lambda' \in W$ define

$$||F_\lambda||_\gamma = \sup\{||\phi_x^\lambda||_\gamma : x \in X\} \text{ and } \overline{\rho}_\gamma(F_\lambda, F_{\lambda'}) = \sup\{\rho_\gamma(\phi_x^\lambda, \phi_x^{\lambda'}) : x \in X\}.$$

Condition (bf) can be now rephrased as:

(b'f) The function $\lambda \mapsto F_\lambda$, $\lambda \in W$, is continuous.

Lemma 4.4.1 ([59]). *Suppose that $\Phi = \{F_\lambda\}_{\lambda\in W}$ is a transversal family of hyperbolic fiberwise conformal skew-products. Then for all $x \in X$ we have*

(a)

$$\forall(\lambda_0 \in W)\, \forall(\varepsilon > 0)\, \exists(\delta > 0)$$
$$HD(Y_{\lambda,x}) \ge \min\{h_{\lambda_0}, q\} - \varepsilon$$

for l_d-a.e. $\lambda \in B(\lambda_0, \delta)$ and

(b) If $h_{\lambda_0} > q$, then there exists $\delta > 0$ such that

$$l_q(Y_{\lambda,x}) > 0$$

for l_d-a.e. $\lambda \in B(\lambda_0, \delta)$.

Theorem 4.4.1 ([59]). *Consider $\Phi = \{F_\lambda\}_{\lambda \in W}$ to be a transversal family of hyperbolic fiberwise conformal skew-product endomorphisms. Then the function $\lambda \mapsto h_\lambda$ is continuous on W, and for all $x \in X$ there exists a Borel set $W_x \subset W$ such that $l_d(W \setminus W_x) = 0$ and*

(a)

$$HD(Y_{\lambda,x}) = \min\{h_\lambda, q\} \text{ for all } \lambda \in W_x.$$

(b)

$$l_d(\{\lambda \in W : h_\lambda > q \text{ and } l_d(Y_{\lambda,x}) > 0\}) = l_d(\{\lambda \in W : h_\lambda > q\}).$$

Another way to guarantee the existence of a universal set W' as in the corollary above, is to strenghten the transversality condition (cf) as follows:

(c'f) (Uniform Transversality Condition) There exists $C_2 > 0$ such that for all $x \in X$, $\forall \hat{x}, \hat{y} \in p_0^{-1}(x), x_1 \neq y_1$, and $\forall r > 0$, we have

$$l_d(\lambda \in W : ||\pi_\lambda(\hat{x}) - \pi_\lambda(\hat{y})|| \leq r) \leq C_2 r^q.$$

All that has to be done then, is to replace $R_x(\lambda)$ in formula (**??**) by $\sup_{x \in X} R_x(\lambda)$. We thus get the following. For a uniformly transversal family we have the following:

Theorem 4.4.2 ([59]). *Consider $\Phi = \{F_\lambda\}_{\lambda \in W}$ to be a uniformly transversal family of hyperbolic fiberwise conformal skew-products. Then the function $\lambda \mapsto h_\lambda$ is continuous on W and there exists a measurable set $W' \subset W$ such that $l_d(W \setminus W') = 0$ and*

$$HD(Y_{\lambda,x}) = \min\{h_\lambda, q\}$$

for all $\lambda \in W'$ and all $x \in X$.

In [59] we gave also several classes of parametrized examples, satisfying transversality or uniform transversality conditions; so for them we can apply the above theorems to obntain the dimension of stable fibers a.e.

In order to do this, consider $f : X \to X$ a topologically exact open distance expanding map, for which there exist closed mutually disjoint sets $X_1, X_2, \ldots, X_d$ such that $X = \cup_{i=1}^d X_i$, $f(X_i) = X$ for all $i = 1, 2, \ldots, d$ and $f|_{X_i}$ is injective for all $i = 1, 2, \ldots, d$. The model that we have in mind is that of an expanding map $f : I_1 \cup \ldots \cup I_d \to [0, 1]$ where $I_1, \ldots, I_d$ are closed mutually disjoint subintervals of $[0, 1]$, $f(I_j) = [0, 1], \forall j$, and $f|_{I_j}$ is injective. Then we will take as the compact space X, the set $I_* = \{x \in I_1 \cup \ldots \cup I_d, f^m(x) \in I_1 \cup \ldots \cup I_d, \forall m \geq 0\}$. So, in this case, $X_i = I_* \cap I_i, i = 1, \ldots, d$.

Returning to the general case of the dynamical system $f : X \to X$ as above, consider $\lambda = (\lambda_1, \ldots, \lambda_d) \in B_d(0, \eta) \subset \mathbb{R}^d$, for some small enough $\eta > 0$, and fix Lipschitz continuous functions $\phi_1, \ldots, \phi_d : X \times [0, 1] \times B_d(0, \eta) \to (0, 1)$. So $\phi_1, \ldots, \phi_d$ are functions of $(x, y, \lambda) \in X^* := X \times [0, 1] \times B_d(0, \eta)$. Let us assume also that $\phi_1(x, \cdot, \cdot), \ldots, \phi_d(x, \cdot, \cdot)$ are C^2 differentiable functions of (y, λ), with derivatives in (y, λ) depending Lipschitz continuously on (x, y, λ), and that there exist constants $\alpha, \alpha' > 0$ with $0 < \alpha' < |\frac{\partial}{\partial y}\phi_i| < \frac{1}{4}$ on X^*, for all $i = 1, \dots, d$ and $|\frac{\partial}{\partial \lambda_j}\phi_i| < \alpha$ on X^*, for all $i, j = 1, \dots, d$. If $\phi_i \leq \beta$ on X^*, for $i = 1, \dots, d$, then we assume also that $\eta + \beta < 1$. Define now the parametrized maps $F_\lambda : X \times [0,1] \to X \times (0,1)$ by the formula

$$F_\lambda(x, y) = (f(x), \lambda_i + \phi_i(x, y, \lambda)),$$

if $x \in X_i, i = 1, \dots, d$. From the way we defined the functions $\phi_1, \dots, \phi_d$, one can see that F_λ is well defined and it is a hyperbolic fiberwise conformal skew product endomorphism. In this case, $\phi_x^\lambda(y) = \lambda_i + \phi_i(x, y, \lambda)$, for $x \in X_i, i = 1, \dots, d$. We see that $0 < \alpha' < |(\phi_x^\lambda)'| < \frac{1}{4}, x \in X, \lambda \in B_d(0, \eta)$, so condition (af) from the definition of a transversal family is satisfied automatically. For this family, the set of parameters is $W = B_d(0, \eta) \subset \mathbb{R}^d$.

Theorem 4.4.3 ([59]). *The family $\{F_\lambda\}_{\lambda \in B_d(0,\eta)}$ is uniformly transversal, and therefore, the assertions of Theorem 4.4.2 hold.*

Corollary 4.4.1 ([59]). *If $f : I_1 \cup \ldots \cup I_d \to [0,1]$ and $X = I_*$ satisfy the assumptions of Theorem 4.4.3, and if there exist constants a, b with $0 < a < b < \frac{1}{4}$ such that $a \leq |\frac{\partial}{\partial y}\phi_i(x, y, \lambda)| \leq b$ for all $(x, y, \lambda) \in X \times [0,1] \times B_d(0, \eta)$ and $i = 1, \dots, d$, then there exists a measurable set $W' \subset W$, with $l_d(W \setminus W') = 0$, such that for all $x \in X, \lambda \in W'$ we have:*

$$\min\left\{1, \frac{\log d}{|\log a|}\right\} \leq HD(Y_{\lambda,x}) \leq \min\left\{1, \frac{\log d}{|\log b|}\right\}$$

In particular, one obtains:

(a) $HD(Y_{\lambda,x}) > 0, x \in X, \lambda \in W'$.

(b) if $|a| \geq \frac{1}{d}$, then $HD(Y_{\lambda,x}) = 1$, for all $x \in X, \lambda \in W'$.

In [59] we gave also two other types of examples from higher dimensional complex dynamics, which satisfy the uniform transversality condition, and hence Theorem 4.4.2 can be applied to them. The first such example is the family

$$F_\lambda(z, w) = (f(z), h(z) + \frac{1}{2}w + \lambda z)$$

Here we assume that $(z, w) \in U \times V \subset \mathbb{C} \times \mathbb{C}$, the set $U = \Delta(0, 2)$ is the disk of center 0 and radius 2 in $\mathbb{C}$, the set $V \subset \mathbb{C}$ is open, bounded and convex; assume also that the function $f(z)$ is close enough to a map of the form $z \to z^2 + c$, with $|c|$ small, and that $X = J(f)$, is the Julia set of f (hence f can be considered expanding on X). Consider also h to be a complex valued Lipschitz continuous map defined in a neighbourhood of X; then since $|h|$ is bounded on X, we can take the bounded sets V and $W \subset \mathbb{C}$ in such a way that the map $F_\lambda : U \times \overline{V} \to \mathbb{C} \times V$ is well defined for all $\lambda \in W$; for example one can take $W = \Delta(0, 1), V = \Delta(0, M)$, where $M > 2(\sup_X |h| + 2)$.

Theorem 4.4.4 ([59]). *The parametrized family $\{F_\lambda\}_{\lambda \in W}$ defined above, satisfies the Uniform Transversality condition.*

Proof. We have $\pi_\lambda(\hat{z}) = \lim_{n\to\infty} \phi^\lambda_{z_1} \circ \phi^\lambda_{z_2} \circ \ldots \circ \phi^\lambda_{z_n}(\zeta)$, where $\phi^\lambda_z(w) := h(z) + \frac{1}{2}w + \lambda z$. Therefore,

$$\phi^\lambda_{z_1} \circ \phi^\lambda_{z_2}(\zeta) = h(z_1) + \frac{1}{2}(h(z_2) + \frac{1}{2}\zeta + \lambda z_2) + \lambda z_1 = h(z_1) + \frac{1}{2}h(z_2) + \lambda z_1 + \frac{1}{2}\lambda z_2 + \frac{1}{4}\zeta.$$

Then, by induction we obtain that

$$\pi_\lambda(\hat{z}) = [h(z_1) + \frac{1}{2}h(z_2) + \frac{1}{4}h(z_3) + \ldots] + \lambda(z_1 + \frac{1}{2}z_2 + \frac{1}{4}z_3 + \ldots).$$

Define also

$$A(\hat{z}) := h(z_1) + \frac{1}{2}h(z_2) + \frac{1}{4}h(z_3) + \ldots, \quad \text{and} \quad B(\hat{z}) = z_1 + \frac{1}{2}z_2 + \frac{1}{4}z_3 + \ldots.$$

Consider now two prehistories $\hat{z}, \hat{z}' \in p_0^{-1}(z)$, with $z_1 \neq z_1'$. Let $g(\lambda) := \pi_\lambda(\hat{z}) - \pi_\lambda(\hat{z}') = A(\hat{z}) + \lambda B(\hat{z}) - A(\hat{z}') - \lambda B(\hat{z}')$. Since f is close to the map $z \to z^2 + c$, we have $J(f)$ close to the circle S^1, if c is small enough, and also it follows that z_1' is close to $-z_1$; thus $z_2' \approx iz_2$ or $z_2' \approx -iz_2$. Hence $|z_2' - z_2| \approx \sqrt{2}$. Hence $|z_2' - z_2 + \frac{1}{2}(z_3' - z_3) + \ldots| \leq \sqrt{2.2} + \frac{1}{2}(2.1 + \frac{1}{2}2.2 + \ldots) \leq \sqrt{2.2} + 2.2$, where we assumed f to be so close to $z^2 + c$, and $|c|$ to be so small that $|z_2' - z_2| < \sqrt{2.2}$ and $X \subset \Delta(0, 1.1)$. Thus

$$|B(\hat{z}) - B(\hat{z}')| \geq 1.9 - \frac{1}{2}(\sqrt{2.2} + 2.2) > 0.2,$$

if $\hat{z}, \hat{z}' \in \hat{X}, z = z', z_1 \neq z_1'$. Therefore if $|g(\lambda)| = |A(\hat{z}) - A(\hat{z}') + \lambda(B(\hat{z}) - B(\hat{z}'))| < r$, then

$$\left|\lambda + \frac{A(\hat{z}) - A(\hat{z}')}{B(\hat{z}) - B(\hat{z}')}\right| < \frac{r}{|B(\hat{z}) - B(\hat{z}')|} < \frac{r}{0.2}$$

whenever $z = z'$ and $z_1 \neq z_1'$. This implies that $\lambda \in B(\frac{A(\hat{z}) - A(\hat{z}')}{B(\hat{z}) - B(\hat{z}')}, \frac{r}{0.2})$. Hence

$$l_2(\{\lambda : |g(\lambda)| < r\}) \leq 25\pi r^2$$

for all $r > 0$. Hence the Uniform Transversality Condition is satisfied for this family.

□

Another example from complex dynamics is presented below. Consider $f(z) = z^2 + c$, for $|c|$ small enough; f has a Julia set denoted by X, which is close to the unit circle; then f is expanding on X. Assume also that h is a complex valued Lipschitz continuous function defined on a neighbourhood of X, that $0.4 < |h(z)| < 0.6$, for $z \in X$, and that $|h(z) + h(z')| > \frac{3}{2}$ for $z^2 = -z'^2 - 2c, z \in X$, and $|c|$ small. Let then λ to be a complex parameter with $|\lambda| < \frac{1}{6}$, and consider the parametrized family

$$F_\lambda(z, w) = (f(z), h(z) + \frac{1}{5}w^2 + \lambda z^2)$$

Theorem 4.4.5 ([59])**.** *In the above setting, for any λ from $W := \{\lambda \in \mathbb{C}, |\lambda| < \frac{1}{6}\}$ and $z \in X$, the map $F_\lambda(z, \cdot)$ defined above, invariates the domain $V := \{w \in \mathbb{C}, \frac{1}{30} < |w| < 1\}$, and $\{F_\lambda\}_{\lambda \in W}$ satisfies the Uniform Transversality condition.*

Also, another example of a complex parametrized family with Uniform Transversality) given in [59], is the following

$$F_\lambda(z, w) = (z^2, z^2 + \lambda_1 z + \lambda_2 z w^2),$$

with $W = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2, |\lambda_1| < \frac{1}{50}, \frac{1}{10} < |\lambda_2| < \frac{1}{8}\}, V := \{w \in \mathbb{C}, \frac{1}{2} < |w| < 1.5\}$. We proved that $F_\lambda(z, \cdot) : \overline{V} \to V$ is well defined for $z \in S^1, \lambda \in W$, and $\exists\ \kappa, \underline{\kappa} \in (0, 1)$ such that $\underline{\kappa} \le |(\phi_z^\lambda)'| \le \kappa$ on V. For this example it can be proved similarly that $\{F_\lambda\}_{\lambda \in W}$ is a parametrized family with Uniform Transversality.

Thus the conclusions of Theorem 4.4.2 apply to the above examples and for Lebesgue almost all parameters λ, the Hausdorff dimension of **all** fibers (which represents the stable dimension in our case), can be given as solution of Bowen type equations on $\hat{X}$.

4.5 Global unstable sets in the non-invertible conformal saddle case.

Global unstable sets for hyperbolic basic sets of endomorphisms present a very complicated structure, due to the fact that there may exist many (even uncountably many) local unstable manifolds through most points of the basic set, and that these unstable manifolds may intersect each other both inside and outside the fractal set. By contrast to diffeomorphisms, for non-invertible maps the unstable manifolds *do not* form a foliation. Thus we cannot simply employ a procedure of writing the fractal as a product locally.

The global unstable sets are important in understanding the long time behaviour of a dynamical system. For instance for s-hyperbolic holomorphic endomorphisms on the complex projective space $\mathbb{P}^2$, the set K^- (the analogue of the set of points with bounded backward iterates from the Hénon automorphisms case) is equal to $W^u(\hat{S}_1) \cup S_0$, where S_i is the set of points with unstable index i of the non-wandering set (see Fornaess-Sibony, [22]). We answered a question of Fornaess and Sibony in [48] showing that the interior of K^- is empty, and in [45] we proved that in certain cases, the Hausdorff dimension of K^- is even strictly less than 4. Thus we proved that there exists a clear *dichotomy* between the dynamical behaviour of perturbations of Hénon maps (which may have basins of repelling periodic points in their respective sets K^-), and that of s-hyperbolic holomorphic endomorphisms.

We recall the definition of c-hyperbolic map given in 4.3.2. A particular case of a c-hyperbolic map on a basic set Λ is a holomorphic endomorphism which is hyperbolic on some basic set. In order to prove some of the results in [45] we employed the inverse pressure, a Laminated Distortion Theorem and thermodynamic formalism on the natural extension. In the next Theorem, an important role was played by Theorem 4.2.3. We showed in [45] also that an s-hyperbolic holomorphic map on $\mathbb{P}^2$, cannot have saddle sets which are local repellors (due basically to the Kontinuitätsatz).

Theorem 4.5.1 ([45]). *Let M be a compact Riemannian manifold of real dimension 4, and $f : M \to M$ be a smooth c-hyperbolic map on a basic set of saddle type Λ, which is not a local repellor.*

Assume also that the following condition on derivatives is satisfied:

$$\sup_{\hat\xi \in \hat\Lambda} |Df_u(\hat\xi)| \cdot |Df_s(\xi)| < 1 \tag{41}$$

Then $HD(W^u(\hat\Lambda)) < 4$. The same conclusion holds if $f : \mathbb{P}^2 \to \mathbb{P}^2$ is a holomorphic map which is s-hyperbolic on a basic set of saddle type Λ and satisfies (41).

Proof. Assume that $HD(W^u(\hat\Lambda)) = 4$ and we want to obtain a contradiction. If $HD(W^u(\hat\Lambda)) = 4$, then $\mathcal{H}^\sigma(W^u(\hat\Lambda)) = \infty, \forall \sigma < 4$. We can find then a subset of $W^u(\hat\Lambda)$ with Hausdorff dimension 4, and if it is not close enough to Λ, then we can take backward iterates until we get a set $\tilde\Delta_0$ close to Λ (for example so close that f can be approximated well with Df, and moreover $|Df_s| > 0$); the condition $HD(\tilde\Delta_0) = 4$ is preserved by taking backward iterates.

Then we construct inductively a sequence of Borel sets $\tilde\Delta_n$ such that $d(\tilde\Delta_n, \Lambda) \to 0$ when $n \to \infty$, and $f(\tilde\Delta_{n+1}) = \tilde\Delta_n, n \geq 1$. Let also $\delta_0 > 0$ be a small number so that we can apply the Mean Value Inequality for f on balls of diameter δ_0.

We shall estimate $\mathcal{H}^\sigma_\infty(\tilde\Delta_{n+1})$. Without loss of generality we can assume that $\tilde\Delta_{n+1}$ is covered with sets $E_i, i \in I$, which are cubes with side equal to $r_i, i \in I$. Then $\mathcal{H}^\sigma_\infty(\tilde\Delta_{n+1}) = \inf\{\sum\limits_{i\in I} r_i^\sigma, \tilde\Delta_{n+1} \subset \bigcup\limits_i E_i\}$. If there exists some i with $r_i > \delta_0$, then $\mathcal{H}^\sigma_\infty(\tilde\Delta_{n+1}) \geq \delta_0^\sigma$.

We notice also that, if $(E_i)_{i\in I}$ cover $\tilde\Delta_{n+1}$, then $(fE_i)_{i\in I}$ will cover $\tilde\Delta_n$. Now, fE_i will have its side in the stable direction of length $(|Df_s(\xi_i)| + \eta(n))r_i$, and the "unstable side" of length $(|Df_u(\hat\xi_i')| + \eta(n))r_i$, where $\eta(n) > 0$ is a small positive number which converges towards 0 when $n \to \infty$, and where $\xi_i, \xi_i' \in E_i$ and $\hat\xi_i'$ is an arbitrary prehistory of ξ_i'. So, $f(E_i)$ is approximately a box with a smaller side $(|Df_s(\xi_i)| + \eta(n))r_i$, and a larger side $(|Df_u(\hat\xi_i')| + \eta(n))r_i$. Assume also that n is large enough such that $|Df_s(\xi_i)| + \eta(n) < |Df_u(\hat\xi_i')| + \eta(n), i \in I$. Then the set $f(E_i)$ can be covered with m_i^2 cubes with side $(|Df_s(\xi_i)| + \eta(n)) \cdot r_i$, where m_i is a positive integer satisfying $m_i(|Df_s(\xi_i)| + \eta(n)) \cdot r_i \geq (|Df_u(\hat\xi_i')| + \eta(n)) \cdot r_i \geq (m_i - 1)(|Df_s(\xi_i)| + \eta(n)) \cdot r_i, i \in I$. Thus,

$$\mathcal{H}^\sigma_\infty(\tilde\Delta_n) \leq \sum_{i\in I} m_i^2 \cdot (|Df_s(\xi_i)| + \eta(n))^\sigma \cdot r_i^\sigma \leq \sum_{i \in I} r_i^\sigma (1 + \frac{|Df_u(\hat\xi_i')| + \eta(n)}{|Df_s(\xi_i)| + \eta(n)})^2 \cdot (|Df_s(\xi_i)| + \eta(n))^\sigma \tag{42}$$

But we can consider a finite iterate of f instead of f; assume this iterate is f^p for some p large enough. The basic set Λ remains the same, the stable/unstable local manifolds remain the same as before. But for p large, we will have $1 + \frac{|D(f^p)_u(x)|}{|D(f^p)_s(x)|} < 2\frac{|D(f^p)_u(x)|}{|D(f^p)_s(x)|}, x \in \Lambda$. Now recall that $d(\xi_i, \xi_i') < 3r_i, i \in I$. Hence there is a small $\delta_1 \in (0, \delta_0)$ such that if $r_i < \delta_1, i \in I$, and n is sufficiently, then (41) implies:

$$(|Df_s(\xi_i)| + \eta(n))^\sigma (1 + \frac{|Df_u(\hat\xi_i')| + \eta(n)}{|Df_s(\xi_i)| + \eta(n)})^2 < 2^2,$$

for σ close to 4, i.e $\sigma \in (\sigma_0, 4)$; also σ_0 independent of n. Thus for σ close to 4, we will obtain

$$\mathcal{H}^\sigma_\infty(\tilde\Delta_n) \leq \sum_{i\in I} r_i^\sigma,$$

in case $r_i < \delta_1, i \in I$. So in this case (i.e if $r_i < \delta_1, i \in I$), we got $\mathcal{H}^\sigma_\infty(\tilde{\Delta}_n) \le \mathcal{H}^\sigma_\infty(\tilde{\Delta}_{n+1})$. Therefore in general $\mathcal{H}^\sigma_\infty(\tilde{\Delta}_n) \ge \min\{\delta_1^\sigma, \mathcal{H}^\sigma_\infty(\tilde{\Delta}_0)\}, n \ge 1, \sigma \in (\sigma_0, 4)$. This means that there exists some number $\beta_0 > 0$ such that $\mathcal{H}^\sigma_\infty(\tilde{\Delta}_n) > \beta_0 > 0$, for $n \ge 1$ and $\sigma \in (\sigma_0, 4)$.

Since $\mathcal{H}^\sigma(\tilde{\Delta}_n) = \infty, n \ge 1$, we can apply Frostman Lemma, to get that for each $n \ge 1$, there exists a Radon measure μ_n on $\tilde{\Delta}_n$ with $\mu_n(\tilde{\Delta}_n) \ge c \cdot \mathcal{H}^\sigma_\infty(\tilde{\Delta}_n) > c \cdot \beta_0 > \beta_0' > 0$, (where c, β_0, β_0' are constants which do not depend on n). We also have that $\mu_n(B(y,r)) \le r^\sigma, y \in M, r > 0, n \ge 1$. The measure μ_n is compactly supported inside the Borel set $\tilde{\Delta}_n$.

But, since $d(\tilde{\Delta}_n, \Lambda) \to 0$, as $n \to \infty$, we see that there exists $R > 1$ large enough such that for each n, $\tilde{\Delta}_n \subset B(y_0, R)$, for some $y_0 \in \Lambda$. Hence $\mu_n(\tilde{\Delta}_n) \le R^4, n \ge 1$, so by a classical theorem in functional analysis, there exists a convergent subsequence of $(\mu_n)_n$. For brevity, we will denote this convergent subsequence also by $(\mu_n)_n$, and denote its limit by μ. We see also that, due to the fact that $d(\text{supp } \mu_n, \Lambda) \to 0$ when $n \to \infty$, it follows that supp $\mu \subset \Lambda$. But, since $\Lambda \subset B(y_0, R)$, it follows that $\mu(\Lambda) \le R^4 < \infty$; on the other hand, $\mu(\overline{B(y_0, R)}) \ge \overline{\lim_n} \mu_n(\overline{B(y_0, R)}) > \beta_0' > 0$, so $0 < \mu < \infty$. Notice also that for all $y \in M$ and all $r > 0$, the properties of the limit μ (Theorem 1.24 of [35]), imply that $\mu(B(y,r)) \le \underline{\lim_n} \mu_n(B(y,r)) \le r^\sigma$. In conclusion, μ is a Radon measure supported inside Λ, with $0 < \mu < \infty$ and such that $\mu(B(y,r)) \le r^\sigma, y \in M, r > 0$. Frostman's Lemma implies then that $\mathcal{H}^\sigma(\Lambda) > 0$, for $\sigma \in (\sigma_0, 4)$.

But recall that we showed in Section 2 that $\delta^s(x) = HD(W^s_r(x) \cap \Lambda) \le t^s(\varepsilon) < 2$, for all $x \in \Lambda$. Since Λ can be laminated locally with intersections of type $W^s_r(x) \cap \Lambda$, we conclude that there exists $\sigma_1 \le 2 + t^s(\varepsilon) < 4$ with $\mathcal{H}^\sigma(\Lambda) = 0, \forall \sigma \in (\sigma_1, 4)$. This leads then to a contradiction with the previous conclusion, and hence $HD(W^u(\hat{\Lambda})) < 4$.

□

Then we can use the Hölder estimates from [49] in order to prove a Theorem about the Hausdorff dimension of $W^u(\hat{\Lambda})$ by employing also the number of preimages of points in Λ. This condition can be verified on a number of examples.

Theorem 4.5.2 ([45])**.** *Let M be a compact Riemannian manifold of real dimension 4, and $f: M \to M$ be a smooth c-hyperbolic map on a basic set of saddle type Λ, which is not a local repellor. Let us denote by $\chi_s := \inf_\Lambda |Df_s|, \lambda_s := \sup_\Lambda |Df_s|$ and $\sup\limits_{\hat{x} \in \hat{\Lambda}} |Df_s(x)| \cdot |(Df|_{E^u(\hat{x})})^{-1}| =: \tau$. Suppose that every point from Λ has at most d f-preimages and at least d' f-preimages in Λ. If the condition:*

$$2 \inf\{1, \frac{-\log \tau}{|\log \chi_s|}\} - \frac{\log d}{|\log \chi_s|} \ge \frac{h_{top}(f|_\Lambda) - \log d'}{|\log \lambda_s|}$$

is satisfied, then $HD(W^u(\hat{\Lambda}) \cap \Delta) < 2$ for any disk Δ transversal to the unstable directions. Moreover we obtain $HD(W^u(\hat{\Lambda})) < 4$.

Let us discuss some aspects from higher dimensional complex dynamics related to the dimension of global unstable sets. For *Hénon automorphisms* on $\mathbb{C}^2$,

$$g(z, w) = (w, p(w) - az),$$

with p a monic polynomial of degree $d \geq 2$ and $a \neq 0$, Bedford and Smillie ([3]) proved that $K^-(g) = W^u(K(g))$, where $K^-(g) = \{x \in \mathbb{C}^2, (g^{-n}(x))_n$ is bounded in $\mathbb{C}^2\}$ and $K(g) := \{x \in \mathbb{C}^2, (g^{\pm n}(x))_n$ is bounded in $\mathbb{C}^2\}$. They proved that, if g is hyperbolic on its Julia set, it follows that for $|a| \leq 1$ the interior of $W^u(K(g))$ is empty, and if $|a| > 1$, then $Int(W^u(K(g))) = \bigcup_{i=1}^{m} B(p_i)$, where $B(p_i)$ are repelling basins for some repelling periodic points $p_1, \ldots, p_m$. This shows that the set K^- has non-empty ingterior, in fact contains repelling basins of periodic repelling points.

Let us look now at the case of *holomorphic endomorphisms* $f : \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$. Such maps are of the form $[P_0 : P_1 : P_2]$, where P_0, P_1, P_2 are homogeneous polynomials in coordinates z_0, z_1, z_2, all three having the same degree $d \geq 2$. For holomorphic endomorphisms $f : \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$, Fornaess and Sibony defined a measure μ_f obtained as the wedge product of the Green current T with itself, $\mu_f = T \wedge T$. This measure was proved by Briend and Duval to be the unique measure of maximal entropy in [8], namely its entropy is equal to $\log d^2$.

Also Fornaess and Sibony defined the set $U_f^- := \{z \in \mathbb{P}^2, \exists U_0$ neighbourhood of z, $f^{-n}(y) \to \text{supp}\mu_f, \forall y \in U_0\}$ and its complementary $K_f^- = \mathbb{P}^2 \setminus U_f^-$; the sets U_f^-, K_f^- are also denoted by U^-, K^- when no confusion can arise. For s-hyperbolic maps Fornaess and Sibony showed in [22] that $K^- = W^u(\hat{S}_1) \cup S_0$, where S_1 is the saddle part of the non-wandering set of f, and S_0 is the finite set of periodic attracting points. Thus the dimension of K^- is the same as the dimension of the global unstable set of S_1. The question asked by Fornaess and Sibony was to obtain more properties of the set K^-, and whether the properties of Hénon diffeomorphisms hold here too.

However for s-hyperbolic holomorphic endomorphisms of $\mathbb{P}^2\mathbb{C}$, we showed that this cannot be the case. In fact in [48] we showed that the interior of K^- is empty, by using methods from several complex variables. Then in [45], [49], we estimated the Hausdorff dimension of K^- (Theorems 4.5.1 and 4.5.2). We showed that in many instances the dimension of K^- is strictly less than 4, this depending on whether the map's contractions on stable manifolds are stronger than the dilations on unstable manifolds.

5 Geometric dynamics and relations to ergodic theory in chaotic systems.

5.1 Mixing, coding and asymptotic distribution of preimages for measure-preserving endomorphisms.

Measure-preserving endomorphisms have a very different behaviour than automorphic systems. This has been observed for instance by Parry and Walters, who showed in [68] that measurable endomorphisms of Lebesgue spaces behave **very differently** than automorphisms.

Theorem 5.1.1 ([68])**.** *There exist non-isomorphic exact endomorphisms S, T of a Lebesgue space $(X, \mathcal{B}, \mu)$ so that $S^2 = T^2$ (hence S, T have the same entropy w.r.t μ), $S^{-n}\mathcal{B} = T^{-n}\mathcal{B}, n \geq 0$ and s.t the Jacobians of S and T w.r.t μ are equal.*

Let us recall that for automorphisms, Ornstein proved in a famous result that two invertible Bernoulli shifts on Lebesgue spaces are isomorphic if and only if they have the same measure theoretic entropy (see eg. [34]). However as Parry and Walters showed in [68], for measure-preserving *endomorphisms* of Lebesgue spaces, $f : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$, the entropy alone $h_\mu(f)$ *does not* determine the conjugacy class.

The most chaotic (in a sense) endomorphism, is a 1-sided Bernoulli shift, whose definition we will give next.

Denote by $\Sigma_d^+ := \{1, \ldots, d\}^{\mathbb{Z}^+}$ the space of sequences ω of $1, \ldots, d$, indexed by the nonnegative integers. On Σ_d^+ we consider the shift map $\sigma_d : \Sigma_d^+ \to \Sigma_d^+$; also for a probability vector $p = (p_1, \ldots, p_d)$ we define the σ_d-invariant product measure ν_p, with the initial probabilities $\nu_p(\{\omega, \omega_0 = i\}) = p_i, i = 1, \ldots, d$. The triple $(\Sigma_d^+, \sigma_d, \nu_p)$ is called a (model) *1-sided Bernoulli shift.* By extension we call *1-sided Bernoulli shift* any triple (X, f, μ), with μ f-invariant, which is measure-theoretically isomorphic to $(\Sigma_d^+, \sigma_d, \nu_p)$, for some $d \geq 1$ and $p = (p_1, \ldots, p_d)$ a probabilistic vector.

There are quite a lot of papers dedicated to establishing whether certain measure-preserving endomorphisms on Lebesgue spaces, are 1-sided Bernoulli (see for instance [9] and references therein). The problem of coding for endomorphisms of Lebesgue spaces (in particular for 1-sided Bernoulli shifts) is subtle and there are no exhaustive classifications.

In [9], Bruin and Hawkins gave several criteria for maps to be 1-sided Bernoulli; in fact for interval/circle maps, there exist rigidity type results for 1-sided Bernoulli maps:

Theorem 5.1.2 ([9])**.** *Let $T : I \to I$ be a piecewise C^2 n-to-1 interval map preserving a probability measure μ equivalent to Lebesgue measure m s.t the Radon-Nikodym derivative $g = \frac{d\mu}{dm}$ is continuous and bounded away from 0. Then T is 1-sided Bernoulli on $(I, \mathcal{B}, \mu)$ if and only if T is C^1-conjugate to a map $S : I \to I$ whose graph consists of n linear pieces with slopes $\pm\frac{1}{p_i}$ s.t $h_\mu(T) = -\sum\limits_{i=1}^{n} p_i \log p_i$.*

Corollary 5.1.1 ([9])**.** *Let $T : S^1 \to S^1$ be an expanding C^2 degree $n \geq 2$ circle map with $T(0) = 0$; then T is 1-sided Bernoulli if and only if it is C^1-conjugate to $z \to z^n$.*

Also in [9] there were obtained several examples of non-Bernoulli n-to-1 maps.

In [43] we proved the following 1-sided Bernoullicity result, this time for expanding maps on general basic sets:

Theorem 5.1.3 ([43])**.** *Consider Λ to be a hyperbolic basic set for a smooth endomorphism f, such that $f|_\Lambda$ is d-to-1, $t_d = 0$ and $f|_\Lambda$ is expanding. Then (Λ, f, μ_0) is 1-sided Bernoulli, where μ_0 is the unique measure of maximal entropy.*

In fact for a system endowed with an equilibrium measure of a Hölder continuous potential, we can say more:

Theorem 5.1.4 ([37])**.** *a) Let f be a smooth endomorphism on a Riemannian manifold M such that f is hyperbolic on the basic set Λ and the critical set $\mathcal{C}_f$ does not intersect Λ. Then if the system (Λ, f, μ_0) given by the measure of maximal entropy μ_0 is 1-sided Bernoulli, it follows that f is expanding on Λ.*

b) Assume f is an expanding endomorphism on Λ. If μ_ϕ is the equilibrium measure of the Hölder potential ϕ and if (Λ, f, μ_ϕ) is 1-sided Bernoulli, then $\mu_\phi = \mu_0$, where μ_0 is the unique measure of maximal entropy for f on Λ.

Proof. a) We consider in this proof the restriction of f to Λ, $f|_\Lambda : \Lambda \to \Lambda$. For $\varepsilon > 0$ small enough,

$$\mu_0(B_n(x, \varepsilon)) \approx \frac{1}{e^{nh_{top}(f)}}, n > 0, x \in \Lambda,$$

and the comparability constants do not depend on n, x.

Assume that (Λ, f, μ_0) is isomorphic to $(\Sigma_d^+, \sigma_d, \nu_p)$ for a certain probability vector $p = (p_1, \ldots, p_d)$. Hence since the measure-theoretic entropy is preserved by isomorphisms (see [68]), it follows that

$$h_{\mu_0}(f) = h_{top}(f) = h_{\nu_p}(\sigma_d) \leq \log d \tag{43}$$

Also we know that the index with respect to an invariant measure, is preserved by isomorphisms (see [68], [96]), thus f is at least d-to-1 on Λ μ_0-a.e. Consider now a Rokhlin partition of (Λ, f, μ_0) with the sets $A_1, \ldots, A_d$ (see for example [67]); we have that $f|_{A_i} : A_i \to \Lambda$ is bijective (modulo μ_0) for any $i = 1, \ldots, d$. Denote $G := \{x \in \Lambda, |f^{-1}(x) \cap \Lambda| \geq d\}$. From above, we know that $\mu_0(G) = 1$. Let now $G_1 := f(G \cap A_1) \cap \ldots \cap f(G \cap A_d)$; this can be viewed also as the set of points x having at least d preimages in Λ, and such that each of its preimages has at least d preimages in turn.

Notice also that, since $\mu_0 \circ f$ is absolutely continuous with respect to μ_0 (see [67]), we obtain $\mu_0(f(G \cap A_i)) = \mu_0(f(A_i)) = 1, i = 1, \ldots, d$. Therefore $\mu_0(G_1) = 1$. In general define inductively

$$G_j := f(G_{j-1} \cap A_1) \cap \ldots \cap f(G_{j-1} \cap A_d), j \geq 2$$

Thus all points in G_j have at least d^{j+1} f^{j+1}-preimages in Λ, and by induction and a similar argument as above, we have $\mu_0(G_j) = 1, j \geq 1$. Also it is clear that $G_j \subset G_{j-1}, j \geq 1 \pmod{\mu_0}$, where $G_0 := G$.

But for any given $x \in \Lambda$, the set $f^{-n}(x) \cap \Lambda$ is an (n, ε)-separated set for $\varepsilon > 0$ small enough, since $\mathcal{C}_f \cap \Lambda = \emptyset$; so if $x \in G_n$, then there exist at least d^n f^n-preimages of x in Λ for $n > 2$. This implies that

$$h_{top}(f|_\Lambda) \geq \log d$$

This implies that $h_{\nu_p}(\sigma_d) = h_{\mu_0}(f) = \log d$, hence ν_p is the measure of maximal entropy on Σ_d^+. Therefore the probability vector p is equal to $(\frac{1}{d}, \ldots, \frac{1}{d})$. Hence $J_{\nu_p}(\sigma_d) = d, \nu_p - a.e$ But the Jacobians are preserved by measure-theoretic isomorphisms, hence

$$J_{\mu_0}(f) = d, \mu_0 - \text{a.e, and } J_{\mu_0}(f^n) = d^n, n > 0, \mu_0 - \text{a.e}$$

So from the properties of Jacobians from [67], we obtain that

$$\mu_0(f^n(B_n(x, \varepsilon))) = \int_{B_n(x,\varepsilon)} J_{\mu_0}(f^n) d\mu_0 = d^n \cdot \mu_0(B_n(x, \varepsilon)) \approx \frac{d^n}{e^{nh_{top}(f)}} = 1,$$

where the comparability constants do not depend on n, x.

Thus for $r > 0$ sufficiently small, the intersection $W_r^s(x) \cap \Lambda$ is equal to $\{x\}$, for $x \in \Lambda$. Hence f can be considered to be expanding on Λ since on Λ there are no points y close to x and forward-asymptotic to x, for any $x \in \Lambda$.

b) Since f is assumed expanding on Λ now, we have from [81] or [34] that the equilibrium measure μ_ϕ is the weak limit of the sequence of measures

$$\mu_n^x := \sum_{y \in f^{-n}(x) \cap \Lambda} \frac{\delta_y \cdot e^{S_n \phi(y)}}{e^{nP(\phi)}}, n > 1,$$

i.e $\mu_n^x \underset{n \to \infty}{\to} \mu_\phi$ for any $x \in \Lambda$. This implies then, that the Jacobian of μ_ϕ in the expanding case is

$$J_{\mu_\phi}(f)(x) = e^{-\phi(x) + P(\phi)}, \tag{44}$$

for μ_ϕ-almost all $x \in \Lambda$. However, the probability vector $p = (p_1, \ldots, p_d)$ gives the 1-sided Bernoulli measure ν_p on Σ_d^+, and we have the invariance of the Jacobians by the measure theoretic isomorphism. So $J_{\mu_\phi}(f) = J_{\nu_p}(\sigma_d)$ and $J_{\mu_\phi}(f)$ must take the values $\frac{1}{p_1}, \ldots \frac{1}{p_d}$ respectively, on the sets of a measurable partition of Λ. But we showed in (44) that $J_{\mu_\phi}(f)$ is in fact equal μ_ϕ-a. e with the continuous function $e^{-\phi + P(\phi)}$. Since μ_ϕ gives positive measure to open sets we obtain then that all the values $p_1, \ldots, p_d$ must be equal, i.e $p_1 = \ldots = p_d = \frac{1}{d}$. Also it follows that the continuous function ϕ must be constant a.e. Hence $\mu_\phi = \mu_0$, where μ_0 is the measure of maximal entropy.

□

From the above Theorem we can obtain immediately the following:

Corollary 5.1.2 ([37]). *Let f_A be a hyperbolic endomorphism of the torus $\mathbb{T}^m$ ($m \geq 2$), given by the integer valued matrix A. Assume that A has both eigenvalues of absolute value larger than 1 and eigenvalues of absolute value strictly less than 1. Then the measure-preserving system $(\mathbb{T}^m, f_A, m)$ is not 1-sided Bernoulli, where m is the Lebesgue (Haar) measure.*

In [37] we studied also other equilibrium measures μ_ϕ, in the case of hyperbolic toral endomorphisms (in which case the number of preimages is constant throughout the torus).

Theorem 5.1.5 ([37]). *Consider a hyperbolic non-expanding toral endomorphism $f_A : \mathbb{T}^m \to \mathbb{T}^m$ associated to the integer valued matrix A. Assume $|det(A)| = 2$, let $\alpha \neq (0, \ldots, 0)$ be a fixed point of f_A, and let ϕ be a periodic Holder continuous function of period α on $\mathbb{T}^m$. Then $(\mathbb{T}^m, f_A, \mu_\phi)$ is not isomorphic to $(\Sigma_2^+, \sigma_2, \nu_p)$, for $p = (p_1, p_2), p_1 \neq \frac{1}{2}$.*

An important notion related to the coding problem for endomorphisms on Lebesgue spaces is also that of *Rokhlin partition*. Let ϵ be the point partition on the Lebesgue space (X, f, μ), where μ is an f-invariant probability measure defined on the σ-algebra $\mathcal{B}$ on X. We denote by $\mathcal{P}_1 = \{E_1, \ldots, E_{m-1}\}$ a partition of X into measurable subsets so that $f|_{E_i}$ is a bijection a.e between E_i and X, $i = 0, \ldots, m-1$. Such a partition exists and it is called a **Rokhlin partition** (see [78], [67], [14]). Clearly such a partition is not uniquely defined.

Now, given a Rokhlin partition $\mathcal{P}_1$, we will define the measurable partition

$$\mathcal{P} := \bigvee_{i \geq 1} T^{-i} \mathcal{P}_1$$

The measurable partition $\mathcal{P}_1$ is called **a 1-sided generator** for (X, f, μ) if the smallest sub-σ-algebra of $\mathcal{B}(\Lambda)$ containing $\mathcal{P}$ and complete with respect to μ, is equal modulo μ to the borelian σ-algebra $\mathcal{B}(\Lambda)$. In this case we will say also that $\mathcal{P}_1$ is a *generating partition*. One of the questions related to coding, is the existence of generating Rokhlin partitions.

Corollary 5.1.3 ([37]). *a) Let an endomorphism f hyperbolic and non-expanding on a basic set Λ. Then there exists no generating Rokhlin partition $\mathcal{P}_1$ of (Λ, f, μ_0) s.t J_{μ_0} is piecewise constant a.e on the sets of $\mathcal{P}_1$ (where μ_0 is the measure of maximal entropy).*

Also if f is expanding on Λ but $\mu_\phi \neq \mu_0$, then there is no generating Rokhlin partition $\mathcal{P}_1$ of (Λ, f, μ_ϕ) s.t $J_{\mu_\phi}(f)$ piecewise constant a.e on the sets of $\mathcal{P}_1$.

b) A hyperbolic non-expanding toral endomorphism $f_A : \mathbb{T}^m \to \mathbb{T}^m, m \geq 2$, does not have generating Rokhlin partitions with respect to the Lebesgue measure.

Let us now turn our attention towards *mixing properties* of equilibrium measures on folded fractals. We prove mixing of any order (see [78] for definition) and Exponential Decay of Correlations (see [5], [10] for definitions) in general, for the triple (Λ, f, μ_ϕ).

Theorem 5.1.6 ([37]). *Let f be a smooth endomorphism on M, hyperbolic on a basic set Λ and let ϕ be a Holder continuous potential defined on Λ; let μ_ϕ be the unique equilibrium measure of ϕ. Then:*

a) the measure-preserving system (Λ, f, μ_ϕ) is mixing of any order.

b) the measure μ_ϕ has Exponential Decay of Correlations on Hölder observables.

Another problem is that of the *asymptotic distribution of preimages* on saddle basic sets, with respect to equilibrium measures. In the expanding case this asymptotic distribution of preimages follows from [81], by employing the Ruelle-Perron-Frobenius operator. However, if the basic set Λ is of saddle type, the problem is very different and needs new methods of proof. In this saddle case, the local inverse iterates act as dilations in the stable directions in backward time, which is changing completely the situation. In [38] we solved the above problem of the weighted preimage distribution with a Holder continuous weight ϕ, along a general hyperbolic saddle basic set (i. e *not necessarily a repellor*, and not necessarily for an expanding map):

Theorem 5.1.7 ([38]). *Let $f : M \to M$ be a smooth (say $\mathcal{C}^2$) map on a Riemannian manifold M, which is hyperbolic and finite-to-one on a basic set Λ so that $\mathcal{C}_f \cap \Lambda = \emptyset$. Consider also ϕ a Hölder continuous potential on Λ and μ_ϕ to be the equilibrium measure of ϕ on Λ. Then*

$$\int_\Lambda | < \frac{1}{n} \sum_{y \in f^{-n}x \cap \Lambda} \frac{e^{S_n\phi(y)}}{\sum\limits_{z \in f^{-n}x \cap \Lambda} e^{S_n\phi(z)}} \cdot \sum_{i=0}^{n-1} \delta_{f^i y} - \mu_\phi, g > | d\mu_\phi(x) \underset{n\to\infty}{\to} 0, \forall g \in \mathcal{C}(\Lambda, \mathbb{R})$$

The proof of this Theorem was based on a careful study of the measure μ_ϕ of various pieces of Bowen balls, and of iterates of Bowen balls. As a Corollary, we obtained in [38] the following result giving the weak convergence of the above discrete measures along the same subsequence, for all points in a set of full μ_ϕ-measure, in the case of a basic set of saddle type Λ and of a smooth non-invertible map f :

Corollary 5.1.4 ([38]). *In the same setting as in Theorem 5.1.7, for any Hölder potential ϕ, it follows that there exists a subset $E \subset \Lambda$, with $\mu_\phi(E) = 1$ and an infinite subsequence $(n_k)_k$ such that for any $z \in E$ we have the weak convergence of measures*

$$\mu^z_{n_k} \underset{k\to\infty}{\to} \mu_\phi$$

In particular, for Anosov endomorphisms we obtain the following result about the asymptotic distribution of preimages with respect to equilibrium measures of Hölder potentials:

Corollary 5.1.5 ([38]). *Let $f : M \to M$ be an Anosov endomorphism without critical points, and ϕ be a Hölder continuous potential on M, with μ_ϕ being its equilibrium measure of ϕ. Then*

$$\int_M | < \frac{1}{n} \sum_{y \in f^{-n}x} \frac{e^{S_n\phi(y)}}{\sum\limits_{z \in f^{-n}x} e^{S_n\phi(z)}} \cdot \sum_{i=0}^{n-1} \delta_{f^i y} - \mu_\phi, g > | d\mu_\phi(x) \underset{n\to\infty}{\to} 0, \forall g \in \mathcal{C}(\Lambda, \mathbb{R})$$

In particular, if μ_0 is the measure of maximal entropy, it follows that for μ_0-almost all points $x \in \Lambda$, $\frac{1}{n} \sum\limits_{y \in f^{-n}x} \frac{\sum_{i=0}^{n-1} \delta_{f^i y}}{Card(f^{-n}x)} \underset{n\to\infty}{\to} \mu_0$

5.2 Relations between 1-sided Bernoullicity and stable dimension.

In [43] and [37] we found relations between two apparently distant notions, namely the 1-sided Bernoullicity of certain equilibrium measures on folded fractals, and the stable dimension function. More precisely, we showed that 1-sided Bernoullicity represents a strong restriction on the ergodic theory of a non-invertible system, and also that the stable dimension (which is defined locally) is related to the global geometry of the invariant set.

In [43] we proved a *geometric flattening phenomenon* related to the stable dimension, namely if the stable dimension is zero at some point of a saddle basic fractal Λ, then Λ must be contained in a union of manifolds; hence the stable dimension influences strongly the geometry of the fractal.

Theorem 5.2.1 ([43])**.** *Let $f: M \to M$ be a smooth endomorphism which is hyperbolic on a basic set Λ with $C_f \cap \Lambda = \emptyset$, and such that f is conformal on local stable manifolds. Assume also that d is the maximum possible value of $d(\cdot)$ on Λ, and that there exists a point $x \in \Lambda$ where $\delta^s(x) = t_d = 0$. Then it follows that $d(\cdot) \equiv d$ on Λ and there exist a finite number of unstable manifolds whose union contains Λ. In particular if Λ is connected, then there exists an invariant unstable manifold containing Λ, and $f|_\Lambda$ is expanding.*

Proof. If d is the maximum value taken by the preimage counting function $d(\cdot)$ on Λ, and if $\delta^s(x) = t_d$, then we showed in [57] that $d(y) = d, \forall y \in \Lambda$. Thus $\delta^s(y) = t_d, y \in \Lambda$, from [58]. By definition of t_d, we have

$$P(t_d\Phi^s - \log d) = 0 \tag{45}$$

In the endomorphism case we obtain similarly as in the diffeomorphism case (see [5], [27]), estimates for the equilibrium measures on Bowen balls. If ϕ is a Hölder continuous potential on Λ, we denote the unique equilibrium measure for ϕ, by μ_ϕ. This follows from the bijection between f-invariant probabilities on Λ, and $\hat f$-invariant probabilities on $\hat\Lambda$ ([84]); μ is an equilibrium measure for ϕ on Λ if and only if its unique $\hat f$-invariant lifting to $\hat\Lambda$ is an equilibrium measure for $\phi \circ \pi$. We denote by $B_n(y, \varepsilon) := \{z \in \Lambda, d(f^i z, f^i y) < \varepsilon, i = 0, \ldots, n-1\}$ the (n, ε)-Bowen ball centered at y. Then given a Holder potential ϕ on Λ, one can show, similarly as for diffeomorphisms ([5], [90]) and by working in $\hat\Lambda$, that there exist constants $A_\varepsilon, B_\varepsilon > 0$ so that, for any $y \in \Lambda, n > 0$, we have:

$$A_\varepsilon e^{S_n(\phi - nP(\phi)} \le \mu_\phi(B_n(y, \varepsilon)) \le B_\varepsilon e^{S_n(\phi) - nP(\phi))}$$

Thus from (45) and since $t_d = 0$, we obtain :

$$\frac{A_\varepsilon}{d^n} \le \mu_0(B_n(y, \varepsilon)) \le \frac{B_\varepsilon}{d^n}, \tag{46}$$

where μ_0 is the measure of maximal entropy for $f|_\Lambda$.

For two quantities which depend on $y \in \Lambda, n > 0$ we will say that they are **comparable** if their quotient is bounded above and below by two positive numbers, independently on y, n; this is for example the case in (46 for the quantities $\mu_0(B_n(x, \varepsilon))$ and $\frac{1}{d^n}$).

Now we prove that the cardinality of $W^s_r(x) \cap \Lambda$ is finite. Indeed, let us assume that there are at least N different points inside $W^s_r(x) \cap \Lambda$ and denote their set by $F := \{y^1, \dots, y^N\}$. Let us take also a fixed, small $\varepsilon > 0$. There exists $n = n(N)$ sufficiently large so that any set of type $f^n(B_n(z,\varepsilon)) \cap W^s_r(x)$ is disjoint from any set of type $f^n(B_n(w,\varepsilon)) \cap W^s_r(x)$ if z, w are n-preimages in Λ of y^i, y^j respectively, and $i \neq j, 1 \leq i, j, \leq N$. But now the inclination of local unstable manifolds with respect to $W^s_r(x)$ is bounded below by some positive constant, since they are transversal to $W^s_r(x)$ and depend uniformly on prehistories from the compact space $\hat{\Lambda}$. This implies that given a point $y \in F$ and an n-preimage $z \in \Lambda$ of y, we have that the union $\underset{\xi \in F, \xi \neq y}{\cup} \underset{w \in f^{-n}\xi \cap \Lambda}{\cup} f^n(B_n(w,\varepsilon))$ does not contain the entire set $f^n(B_n(z,\varepsilon))$. This implies that, in the difference set

$$B_n(z,\varepsilon) \setminus \left[\underset{\xi \in F, \xi \neq y}{\cup} \underset{w \in f^{-n}\xi \cap \Lambda}{\cup} B_n(w,\varepsilon) \right],$$

there must exist at least $M(N,\varepsilon)$ mutually disjoint Bowen balls of type $B_{n+k(N,\varepsilon)}(\zeta, \varepsilon/l(N,\varepsilon))$, where $k(N,\varepsilon), l(N,\varepsilon)$ are positive integers. We also recall that, since $C_f \cap \Lambda = \emptyset$, there exists a positive constant ε_0 such that $d(z,w) > \varepsilon_0$ if $f(z) = f(w)$ and $z \neq w, z, w \in \Lambda$. Thus if w, z are different n-preimages of the same point from Λ, then $B_n(z, 4\varepsilon) \cap B_n(w, 4\varepsilon) = \emptyset$ if ε is small enough. By applying the estimates from (46) we obtain that there exists a positive constant D_ε, such that

$$1 \geq \mu_s(\underset{y \in F}{\cup} \underset{y_{-n} \in f^{-n}y \cap \Lambda}{\cup} B_n(y_{-n}, \varepsilon)) \geq D_\varepsilon \cdot \sum_{y \in F, z \in f^{-n}y \cap \Lambda} \mu_s(B_n(z,\varepsilon)) \geq D_\varepsilon A_\varepsilon N d^n \cdot \frac{1}{d^n}$$

So the number of points in $W^s_r(x) \cap \Lambda$ must be finite and bounded above by $N(\varepsilon)$, for any $x \in \Lambda$. We recall however that any hyperbolic basic set has local product structure, thus the intersection between any local stable manifold $W^s_r(x)$ and any local unstable manifold $W^u_r(\hat{y}), x \in \Lambda, \hat{y} \in \hat{\Lambda}$ must belong to Λ, for any $r > 0$ small. Hence Λ must be contained in the union of at most finitely many unstable manifolds, each of type $W^u(\hat{x}, T) := \overset{T}{\underset{i=0}{\cup}} f^i(W^u_r(\hat{x}))$ for some $T > 0$.

From the finiteness of $W^s_r(x) \cap \Lambda$, it follows that there exists a small $\rho > 0$ such that $W^s_\rho(x) \cap \Lambda = \{x\}, x \in \Lambda$. Also notice that Λ does not have isolated points, since it was assumed to be uncountable and topologically transitive.

Let us assume now that Λ is connected and contained in the union of the local unstable manifolds $W^u_1 \cup \dots W^u_N$. Let us consider for example the intersection between W^u_1 and W^u_2. If there would exist a point $y \in \Lambda \cap W^u_1 \setminus W^u_2$, close to $W^u_1 \cap W^u_2$, then $W^s_\rho(y) \cap W^u_2$ would be in Λ from the local product structure (since we work only near points from Λ, the above intersection is actually an intersection between $W^s_\rho(y)$ and $W^u_r(\hat{\zeta})$ for some $\hat{\zeta} \in \hat{\Lambda}$). But this is a contradiction since we saw that $W^s_\rho(y) \cap \Lambda = \{y\}$. So Λ must be contained in intersections of two or more manifolds $W^u_i, i = 1, \dots, N$. Now if there would exist only two such manifolds W^u_1 and W^u_2, and if $\Lambda \subset W^u_1 \cap W^u_2$, we are done, since it follows that $\Lambda \subset W^u_1$. If not, since we assumed that Λ is connected, there would exist at least another W^u_3 and $\Lambda_{12} := W^u_1 \cap W^u_2$ would intersect $\Lambda_{23} := W^u_2 \cap W^u_3$ in a point y. But then, again from the non-existence of isolated points in Λ, there must exist some point $z \in \Lambda_{23}$ as close as we want to y. Since $z \in \Lambda$ and f is hyperbolic on Λ, we can construct the local stable manifold

$W^s_\rho(z)$; as z is very close to $y \in W^u_1$, we will have that $W^s_\rho(z)$ intersects W^u_1 in a point $\xi \in \Lambda$. But then we obtain a contradiction since $W^s_\rho(z) \cap \Lambda$ would contain more than one point.

The other cases of intersections between the unstable manifolds W^u_i are treated similarly. Thus if Λ is connected, it must be contained in only one unstable manifold W^u, more precisely in the union of finitely many iterations of one local unstable manifold. In particular it follows that $f|_\Lambda$ is an expanding map in this case.

□

In the sequel we will use the important notions of *Jacobian of an invariant measure* introduced by Parry in [67], and that of *index* of a countable-to-one endomorphism of Lebesgue spaces (see [68]). In short, the **Jacobian** of the f-invariant probability measure μ on the Lebesgue space (X, f, μ) is the Radon-Nikodym derivative of $\mu \circ f$ with respect to μ. If (X, f, μ) is a measure-preserving system (with some σ-algebra $\mathcal{B}$), and if ϵ is the point partition, one can form the fiber partition $\xi = f^{-1}\epsilon$ which is a measurable partition if f is countable-to-1 on (X, μ); let also $\pi : X \to X/\xi$ be the canonical projection. This partition induces a *factor space* $(X/\xi, g, \nu)$, where an arbitrary point z of X/ξ is a fiber $f^{-1}(x), x \in X$, $g(z) := \pi(x), z \in X/\xi$ and $\nu(E) := \mu(\pi^{-1}(E)), E$ measurable in X/ξ. Now from the Rokhlin theory of measurable partitions (see [77], [67], etc.), ξ induces a family of *conditional measures* on the fibers of f, $\{\mu_z\}_{z \in X/\xi}$ such that $\mu(A) = \int_{X/\xi} \mu_z(A \cap z) d\nu(z)$, for A measurable in X. This family of conditional measures is unique modulo ν. Notice that μ_z is a probability measure on the (at most countable) fiber $z = f^{-1}x$; its support supp μ_z is a subset of $f^{-1}x$. Then the **index** of (X, f, μ) is the measurable function

$$ind_\mu(f)(x) := \text{card}(\text{supp } \mu_z), z = f^{-1}x, \text{ for } \mu - \text{a.e } x \in X$$

We look now at the opposite case from the one at the beginning of this section, namely when the *stable dimension is positive* at some point. We prove that an endomorphism with positive stable dimension at a point, cannot be 1-sided Bernoulli, if endowed with a certain equilibrium measure.

Theorem 5.2.2. *Let f be a smooth endomorphism, which is hyperbolic on a basic set Λ, such that $\Lambda \cap C_f = \emptyset$ and f is conformal on stable manifolds. Assume that there exists a point $x \in \Lambda$ with $\delta^s(x) > 0$, and denote by μ_s the equilibrium measure of the potential $\delta^s(x) \cdot \Phi^s(\cdot)$. Then the measure preserving system (Λ, f, μ_s) cannot be 1-sided Bernoulli.*

Proof. We assumed that $\delta^s(x) > 0$. As $\delta^s(x) \cdot \Phi^s$ is a Holder continuous potential on Λ, it follows that it has a unique equilibrium measure μ_s. From the definition of equilibrium measures:

$$P(\delta^s(x)\Phi^s) = h_{\mu_s} + \delta^s(x) \cdot \int_\Lambda \Phi^s(y) d\mu_s(y) \tag{47}$$

Assume now that (Λ, f, μ_s) is isomorphic to the 1-sided Bernoulli shift $(\Sigma^+_d, \sigma, \rho_{\mathbf{p}})$, where $\rho_{\mathbf{p}}$ is the probability measure induced on Σ^+_d by the probability vector $\mathbf{p}$. In our case, if (Λ, f, μ_s) is isomorphic to $(\Sigma_d^+, \sigma, \rho_{\mathbf{p}})$, then the index ind_{f,μ_s} is equal to d μ_s-almost everywhere. From definition we know that $d = ind_{f,\mu_s}(y) \leq \text{Card}(f^{-1}(y) \cap \Lambda)$, for μ_s-almost all $y \in \Lambda$. Now any non-empty open set must contain Bowen balls; so by using the estimates for the μ_s-measure of Bowen balls, we obtain that the μ_s-measure of any open non-empty set is strictly positive. Thus recalling the notion of preimage counting function from before, we obtain that:

$$d(y) \geq d, \text{for } y \text{ in a dense set in } \Lambda \tag{48}$$

This implies that

$$\delta^s(y) \leq t_d, y \in \Lambda$$

by a result from [58]; here again t_d is the unique zero of the pressure function $t \to P(t\Phi^s - \log d)$. Indeed in [58], when covering Λ with Bowen balls, one can take the centers of all these balls to be in the respective dense subset; this implies the last displayed inequality for any point $y \in \Lambda$. Therefore we have that $P(\delta^s(x)\Phi^s - \log d) \geq 0$; but since μ_s is the equilibrium measure of the potential $\delta^s(x)\Phi^s$ we obtain: $h_{\mu_s} + \delta^s(x) \cdot \int_\Lambda \Phi^s(y) d\mu_s(y) \geq \log d$. Recalling however that $\delta^s(x) > 0$ and that $\Phi^s < 0$ on the compact set Λ, we have:

$$h_{\mu_s} \geq \log d - \delta^s(x) \cdot \int_\Lambda \Phi^s(y) d\mu_s(y) > \log d \tag{49}$$

But since we assumed the existence of an isomorphism between (Λ, f, μ_s) and $(\Sigma_d^+, \sigma, \rho_{\mathbf{p}})$, we should have $h_{\mu_s} = h_{\rho_{\mathbf{p}}}$; hence from (49), it follows that $h_{\rho_{\mathbf{p}}} > \log d$. However by using the Variational Principle for entropy we obtain $h_{\rho_{\mathbf{p}}} \leq h_{top}(\sigma) = \log d$. This gives a contradiction. Therefore the measure preserving system (Λ, f, μ_s) cannot be 1-sided Bernoulli.

□

Moreover, in [43] we proved a *Classification Theorem* for the dynamical/ergodic behaviour of a class of perturbation maps, on their respective basic sets.

Theorem 5.2.3 ([43]). *For some small $|c|, c \in \mathbb{C} \setminus \{0\}$, let us consider the polynomial map $f(z, w) = (z^2 + c, w^2), (z, w) \in \mathbb{C}^2$. Let also a polynomial f_ε which is a smooth perturbation of f and let Λ_ε be the corresponding basic set of f_ε close to the set $\Lambda := \{p_c\} \times S^1$ (where p_c is the fixed attracting point of $z \to z^2 + c$). Then we may have exactly one of the following possibilities:*

a) There exists a point $x \in \Lambda_\varepsilon$ where $\delta^s(x) = 0$. Then there exists a manifold W such that $\Lambda_\varepsilon \subset W$, $f_\varepsilon|_{\Lambda_\varepsilon}$ is expanding and $f_\varepsilon|_{\Lambda_\varepsilon}$ is 2-to-1. In this case the stable dimension is 0 at any point from Λ_ε, and the measure preserving system $(\Lambda_\varepsilon, f_\varepsilon, \mu_{0,\varepsilon})$ is 1-sided Bernoulli (where $\mu_{0,\varepsilon}$ is the unique measure of maximal entropy for $f_\varepsilon|_{\Lambda_\varepsilon}$).

b) There exists a point $x \in \Lambda_\varepsilon$ with $0 < \delta^s(x) < 2$. Then the stable dimension is positive at any point of Λ_ε, and the measure preserving system $(\Lambda_\varepsilon, f_\varepsilon, \mu_{s,\varepsilon})$ cannot be 1-sided Bernoulli, where $\mu_{s,\varepsilon}$ is the equilibrium measure of the potential $\delta^s(x)\Phi^s_\varepsilon$. We have two subcases:

b1) $f_\varepsilon|_{\Lambda_\varepsilon}$ is a homeomorphism, and in this case the measure preserving system $(\Lambda_\varepsilon, f_\varepsilon, \mu_\phi)$ is 2-sided Bernoulli for any Holder continuous potential ϕ, where μ_ϕ is the equilibrium measure of ϕ.

b2) there exist both points with only one f_ε-preimage in Λ_ε, as well as points with two f_ε-preimages in Λ_ε; the set of points with one f_ε-preimage in Λ_ε has non-empty interior.

Now, we want to see what are the consequences of an arbitrary equilibrium measure on a folded fractal, being 1-sided Bernoulli.

For an f-invariant probability measure μ on Λ, let $\lambda_1(x) < \ldots < \lambda_{S(x)}(x) < 0$ be the negative Lyapunov exponents of μ with respect to f, which are defined for μ-a.e $x \in \Lambda$; let also the i-th partial stable manifold $W^s_i(x) := \{y \in M, \limsup_{n\to\infty} \frac{1}{n} \log d(f^n x, f^n y) \le \lambda_i(x)\}, 1 \le i \le S(x)$. It is clear that the (usual) stable manifold of x, namely $W^s(x)$ is actually $W^s_{S(x)}(x)$. We also denote for $r > 0$ small, by $W^s_{i,r}$ the i-th partial stable manifold of radius r. In our case since we work with uniformly hyperbolic maps, r can be chosen independent of x.

One can find a measurable partition ξ of Λ, subordinate to the partial stable manifolds W^s_i (see for instance [1], [31], [32]) and can define the i-th **pointwise stable dimension** of μ, or the dimension of μ on W^s_i-manifolds, as

$$\delta^s_i(\mu, x, \xi) := \liminf_{r\to 0} \frac{\log \mu^\xi_x(B^i(x,r))}{\log r},$$

where $\{\mu^\xi_x\}_x$ is the system of conditional measures of μ associated to the partition ξ and $B^i(x,r)$ is the ball of radius r centered at x inside W^s_i. It can be shown that $\delta^s_i(\mu, x, \xi)$ does not depend on ξ and it is constant along orbits. Moreover we have $\delta^s_i(\mu, x, \xi) = \limsup_{r\to 0} \frac{\log \mu^\xi_x(B^i(x,r))}{\log r}$. So if μ is ergodic, then the **pointwise i-th stable dimension of** μ, denoted by $\delta^s_i(\mu)$, is defined by $\delta^s_i(\mu) = \delta^s_i(\mu, x, \xi)$, μ-a.a $x \in \Lambda$, and $1 \le i \le S(x) = S$.

We show now that if the triple (Λ, f, μ_ϕ) is coded by a 1-sided Bernoulli shift, then f must be expanding on Λ from a certain measure-theoretical point of view. This is in contrast with the hyperbolic diffeomorphism case, where **all** equilibrium measures of Holder potentials can be coded with 2-sided Bernoulli shifts.

In general for a measurable partition ξ of Λ denote by $\xi(x)$ the unique (modulo μ) set of ξ which contains x. For a measurable partition ξ subordinated to the stable manifolds W^s_S, we can define the **stable dimension of** μ on $\xi(x)$ as:

$$HD^s(\mu, x) := HD(\mu^\xi_x) = \inf\{HD(Z), Z \subset \xi(x), \mu^\xi_x(Z) = 1\}, \mu - \text{a.e } x \in \Lambda$$

Theorem 5.2.4 ([43]). *Let f be a smooth hyperbolic endomorphism on a connected basic set Λ; let also ϕ be a Holder continuous potential on Λ and μ_ϕ the unique equilibrium measure of ϕ. Then, if the measure-preserving system (Λ, f, μ_ϕ) is 1-sided Bernoulli, it follows that either f is distance-expanding on Λ, or the stable dimension of μ_ϕ is zero, i.e $HD^s(\mu_\phi, x) = 0$ for μ_ϕ-a.e $x \in \Lambda$.*

Proof. Let us assume that (Λ, f, μ_ϕ) is 1-sided Bernoulli, i.e isomorphic to $(\Sigma^+_d, \sigma_d, \nu_p)$ for some $d > 1$ and probability vector p. Now the equilibrium measure of a Hölder potential μ_ϕ is supported everywhere, since the μ_ϕ-measure of any ball is positive, from its estimates on Bowen balls. Thus, as the index function is preserved by isomorphisms (see [68]) and since any point from Λ has finitely many preimages, it follows that the fiber $f^{-1}(x)$ must contain d points for μ_ϕ-almost all $x \in \Lambda$. Also since we have an isomorphism with a 1-sided Bernoulli shift, it follows from [67] that the Jacobian $J_{\mu_\phi}(f)$ of μ_ϕ, must be equal a.e with the Jacobian of the product measure ν_p.

Consider now a measurable partition ξ of Λ subordinated to the local stable manifolds W^s; by $\xi(x)$ we shall denote the set of ξ that contains x. We recall that $W^s_{S,r} = W^s_r$ notationally. Since f is uniformly hyperbolic on Λ and thus the local stable/unstable manifolds have a fixed positive radius, it follows that we may take the partition ξ to be with borelian subsets of the stable manifolds which contain a smaller stable set of fixed radius, i. e there exist $r_0, r_1 > 0$ s.t $W^s_{r_1}(x) \subset \xi(x) \subset W^s_{r_0}(x)$, μ_ϕ-a.a $x \in \Lambda$. To this measurable partition ξ, we can associate (uniquely) a family of conditional measures of μ_ϕ; a generic element of this family is denoted by $\mu^\xi_{\phi,x}$ and it is a probability measure on the subset $\xi(x)$ of ξ (containing the point x).

We want to show now that for μ_ϕ-almost all points $x \in \Lambda$ we have that the conditional measure $\mu^\xi_{\phi,x}$ gives positive measure to any non-empty open subset in the local stable manifold $\xi(x)$. First we notice that if A is the intersection of a Bowen ball $B_m(y,\varepsilon)$ with a neighbourhood of the local unstable manifold $W^u_\varepsilon(\hat\zeta)$, $\hat\zeta \in \hat\Lambda$, then the measure μ^ξ_ϕ induced on the factor space Λ/ξ has the property that: $\mu^\xi_\phi(A/\xi) = \mu_\phi(B_m(y,\varepsilon))$. But from the definition of conditional measures,

$$\mu_\phi(A) = \int_{A/\xi} \mu^\xi_{\phi,x}(A \cap \xi(x)) d\mu^\xi_\phi(\xi(x)),$$

where $\xi(x)$ are the leaves of the measurable partition ξ which intersect A (in the factor space Λ/ξ these leaves are identified with points). But $\mu_\phi(A) > 0$, since A is an open set in Λ (thus contains some Bowen ball); also $\mu^\xi_\phi(A/\xi) = \mu_\phi(B_m(y,\varepsilon) > 0$. Thus from the essential uniqueness of the conditional measures, and since the sets of type A as above form a basis for open sets, we obtain that for μ^ξ_ϕ-almost all partition leaves $\xi(x) \in \Lambda$, $\mu^\xi_{\phi,x}(V) > 0$, for V a neighbourhood of z and $z \in \xi(x)$. This implies that $\mathrm{supp}\mu^\xi_{\phi,x} = \xi(x) \cap \Lambda, \mu_\phi - a.e.$ In this case the Lyapunov exponents are all constant a.e and will be denoted simply by λ_i. Denote also by

$$\gamma_1 := \delta^s_1(\mu_\phi), \gamma_2 := \delta^s_2(\mu_\phi) - \delta^s_1(\mu_\phi), \ldots, \gamma_S := \delta^s_S(\mu_\phi) - \gamma_{S-1}$$

Recall now the notion of *folding entropy* $F_\mu(f)$ of an arbitrary f-invariant probability measure μ (see [80]), which is defined as the conditional entropy $F_\mu(f) := H_\mu(\epsilon|f^{-1}\epsilon)$, where ϵ is the partition of M into single points. We can consider thus the folding entropy $F_{\mu_\phi}(f)$ of an equilibrium measure μ_ϕ. From [80], [67] it follows that the folding entropy $F_{\mu_\phi}(f)$ is equal to the integral of the logarithm of the Jacobian of μ_ϕ, i. e $F_{\mu_\phi}(f) = \int_\Lambda \log J_{\mu_\phi}(f) d\mu_\phi$. We have also that:

$$h_{\mu_\phi}(f) = F_{\mu_\phi}(f) - \sum_{1 \le i \le S} \lambda_i \gamma_i(\mu_\phi), \tag{50}$$

Since (Λ, f, μ_ϕ) is isomorphic to $(\Sigma^+_m, \sigma_m, \nu_p)$ and since the Jacobian is preserved by isomorphisms of Lebesgue spaces (see [67]), it follows that

$$F_{\mu_\phi}(f) = \int_\Lambda \log J_{\mu_\phi}(f) d\mu_\phi = \int_{\Sigma^+_m} \log J_{\nu_p}(\sigma_m) d\nu_p = h_{\nu_p}(\sigma_m) = h_{\mu_\phi}(f)$$

Thus from (50) we obtain $\sum\limits_{1\le i\le S} \lambda_i\gamma_i(\mu_\phi) = 0$. But since we have a uniformly hyperbolic system, either f is distance-expanding on Λ (i. e it does not have stable directions), or $\lambda_i < 0, 1 \le i \le S$ and $\gamma_i(\mu_\phi) = \delta_i^s(\mu_\phi) = 0, 1 \le i \le S$. Thus for a measurable partition ξ subordinated to the stable manifolds $W^s = W^s_S$,

$$\delta_S^s = \limsup_{r\to 0} \frac{\log \mu^\xi_{\phi,x}(B(y,r))}{\log r} = 0, \text{for } \mu_\phi - \text{a.e } x, \text{and } \mu^\xi_{\phi,x} - \text{a.e } y \in \xi(x)$$

So there exists a set $E \subset \Lambda$ with $\mu_\phi(E) = 1$ so that for any small $\beta > 0$, there exists $r(y,\beta) > 0, y \in E$ such that

$$\mu^\xi_{\phi,x}(B(y,r)) > r^\beta, 0 < r < r(y,\beta), y \in E \cap \xi(x), \tag{51}$$

for μ_ϕ-a.e $x \in \Lambda$. From the definition of conditional measures (see [77], [67]), we deduce that if $\mu_\phi(E) = 1$ then for almost all x, $\mu^\xi_{\phi,x}(E \cap \xi(x)) = 1$. So for almost all leaves $\xi(x)$ of ξ, $\mu^\xi_{\phi,x}$-almost all points $y \in \xi(x)$ satisfy (51). Now, using the Vitali Covering Theorem, we can cover a set $E' \subset E \cap \xi(x)$ having $\mu^\xi_{\phi,x}(E') = 1$, with mutually disjoint balls $B(y,\rho(y))$ where $\rho(y) < r(y,\beta)$. Thus we obtain a cover with a family of mutually disjoint balls $B(y,\rho(y)), y \in F \subset E \cap \xi(x)$ and

$$1 \ge \sum_{y\in F} \mu^\xi_{\phi,x}(B(y,\rho(y))) \ge \sum_{y\in F} \rho(y)^\beta$$

Hence $HD(E') \le \beta$ for μ_ϕ-almost all $x \in \Lambda$. But $\beta > 0$ is arbitrarily small; hence recalling also that $\mu^\xi_{\phi,x}(E \cap \xi(x)) = \mu^\xi_{\phi,x}(E') = 1$ we obtain $HD^s(\mu_\phi, x) = 0$, $\mu_\phi - \text{a.e } x \in \Lambda$.

$\square$

5.3 Conditional measures of Gibbs states on local stable manifolds.

Given a hyperbolic structure for an endomorphism over a basic set Λ, it is natural to form a measurable partition subordinated to the foliation by local stable manifolds. Then, given an equilibrium measure for a Hölder potential over Λ, one can associate a family of conditional measures for this measurable partition by the method of Rokhlin ([77], [67], etc.)

In [42] we solved completely the problem of conditional measures for equilibrium measures in the case of c-hyperbolic maps (for instance holomorphic endomorphisms on $\mathbb{P}^2$, and proved that the equilibrium measure of a certain stable potential maximizes in the Variational Principle for the stable dimension, thus we can obtain the stable dimension as the pointwise stable dimension of this measure.Thus we obtain a relation between the purely metric stable dimension, and notion of thermodynamic formalism, like equilibrium measures.

In order to do this, we studied in [42] the measure of iterates of Bowen balls, which are actually unstable tubular neighbourhoods. This requires to compare the measures of various pieces coming from different local inverse iterates.

Proposition 5.3.1 ([42]). *Let f be a smooth endomorphism, which is hyperbolic on a basic set Λ. Consider also a Holder continuous potential ϕ on Λ and μ_ϕ be the unique equilibrium measure of ϕ. Let a small $\varepsilon > 0$, two disjoint Bowen balls $B_k(y_1, \varepsilon), B_m(y_2, \varepsilon)$ and a borelian set $A \subset f^k(B_k(y_1, \varepsilon)) \cap f^m(B_m(y_2, \varepsilon))$, s.t $\mu_\phi(A) > 0$; denote by $A_1 := f^{-k}A \cap B_k(y_1, \varepsilon)$, $A_2 := f^{-m}A \cap B_m(y_2, \varepsilon)$ and assume that $\mu_\phi(\partial A_1) = \mu_\phi(\partial A_2) = 0$. Then there exists a positive constant C_ε independent of k, m, y_1, y_2 such that*

$$\frac{1}{C_\varepsilon}\mu_\phi(A_2) \cdot \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} \cdot P(\phi)^{m-k} \leq \mu_\phi(A_1) \leq C_\varepsilon \mu_\phi(A_2) \cdot \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} \cdot P(\phi)^{m-k}$$

Proof. First we will fix a Hölder potential ϕ, and denote the uniquely determined equilibrium measure μ_ϕ of ϕ, by μ. We will consider the restriction $f|_\Lambda$. From construction we have $f^k(A_1) = f^m(A_2)$. Assume for example that $m \geq k$. Now if $P(f, \phi, n) := \sum_{x \in \mathrm{Fix}(f^n)} e^{S_n\phi(x)}, n \geq 1$, then the equilibrium measure μ can be considered as the limit of the sequence of measures (see [27]),

$$\tilde{\mu}_n := \frac{1}{P(f, \phi, n)} \cdot \sum_{x \in \mathrm{Fix}(f^n)} e^{S_n\phi(x)} \delta_x,$$

Hence

$$\tilde{\mu}_n(A_1) = \frac{1}{P(f, \phi, n)} \cdot \sum_{x \in \mathrm{Fix}(f^n) \cap A_1} e^{S_n\phi(x)}, n \geq 1 \tag{52}$$

Let us consider now a periodic point $x \in \mathrm{Fix}(f^n) \cap A_1$; by definition of A_1, it follows that $f^k(x) \in A$, so there exists a point $y \in A_2$ such that $f^m(y) = f^k(x)$. However the point y does not have to be periodic.

Now we will use the Specification Property ([5], [27]) on the hyperbolic compact locally maximal set Λ: if $\varepsilon > 0$ is fixed, then there exists a constant $M_\varepsilon > 0$ such that for all $n >> M_\varepsilon$, there exists a $z \in \mathrm{Fix}(f^{n+m-k})$ s.t z ε-shadows the $(n + m - k - M_\varepsilon)$-orbit of y.

Let now V be an arbitrary neighbourhood of the set A_2 s.t $V \subset B_m(y_2, \varepsilon)$. Consider two points $x, \tilde{x} \in \mathrm{Fix}(f^n) \cap A_1$ and assume the same periodic point $z \in V \cap \mathrm{Fix}(f^{n+m-k})$ corresponds to both x and $\tilde{x}$ by the above procedure. This means that the $(n - k - M_\varepsilon)$-orbit of $f^m z$, ε-shadows the $(n - k - M_\varepsilon)$-orbit of $f^k x$ and also the $(n - k - M_\varepsilon)$-orbit of $f^k \tilde{x}$. Hence the $(n - M_\varepsilon - k)$-orbit of $f^k x$, 2ε-shadows the $(n - M_\varepsilon - k)$-orbit of $f^k \tilde{x}$. But recall that we chose $x, \tilde{x} \in A_1 \subset B_k(y_1, \varepsilon)$, hence $\tilde{x} \in B_{n-M_\varepsilon}(x, 2\varepsilon)$.

But we can split the set $B_{n-M_\varepsilon}(x, 2\varepsilon)$ in at most N_ε smaller Bowen ball of type $B_n(\zeta, 2\varepsilon)$. In each of these $(n, 2\varepsilon)$-Bowen balls $B_n(\zeta, 2\varepsilon)$ we may have at most one fixed point for f^n. This holds since fixed points for f^n are solutions to the equation $f^n\xi = \xi$ and, on tangent spaces we have that $Df^n - Id$ is a linear map without eigenvalues of absolute value 1. Thus if $d(f^i\xi, f^i\zeta) < 2\varepsilon, i = 0, \ldots, n$ and if ε is small enough, we can apply the Inverse Function Theorem at each step. Therefore there exists only one fixed point for f^n in each Bowen ball $B_n(\zeta, 2\varepsilon)$. Hence there exist at most N_ε periodic points from $\mathrm{Fix}(f^n) \cap \Lambda$ having the same periodic point $z \in V$ attached to them by the above procedure.

Notice also that, if $x, \tilde{x}$ have the same point $z \in V \cap \text{Fix}(f^{n+m-k})$ attached to them, then as before, $\tilde{x} \in B_{n-M_\varepsilon}(x, 2\varepsilon)$. So the distances between iterates are growing exponentially in the unstable direction, and decrease exponentially in the stable direction. Thus we can use the Holder continuity of ϕ and a Bounded Distortion Lemma to prove that: $|S_n\phi(x) - S_n\phi(\tilde{x})| \le \tilde{C}_\varepsilon$, for some positive constant $\tilde{C}_\varepsilon$ depending on ϕ (but independent of n, x). This can be used then in the estimate for $\tilde{\mu}_n(A_1)$, according to (52). We use the fact that if $z \in B_{n+m-k-M_\varepsilon}(y, \varepsilon)$, then $f^m(z) \in B_{n-M_\varepsilon-k}(f^m y, \varepsilon)$; also recall that $f^k x = f^m y$, so $f^m z \in B_{n-M_\varepsilon-k}(f^k x, \varepsilon)$. Then from the Holder continuity of ϕ and the fact that $x \in A_1 \subset B_m(y_1, \varepsilon)$, it follows again by a Bounded Distortion Lemma that there exists a constant $\tilde{C}_\varepsilon$ (denoted as before without loss of generality) satisfying:

$$|S_{n+m-k}\phi(z) - S_n\phi(x)| \le |S_k\phi(y_1) - S_m\phi(y_2)| + \tilde{C}_\varepsilon, \tag{53}$$

for $n > n(\varepsilon, m)$. But from Proposition 20.3.3 of [27] (which extends immediately to endomorphisms), we have that there exists a positive constant c_ε such that for sufficiently large n:

$$\frac{1}{c_\varepsilon} e^{nP(\phi)} \le P(f, \phi, n) \le c_\varepsilon e^{nP(\phi)},$$

where the expression $P(f, \phi, n)$ was defined immediately before (52). Hence in our case, if $n > n(\varepsilon, m)$ we obtain:

$$\frac{1}{c_\varepsilon} e^{(n+m-k)P(\phi)} \le P(f, \phi, n+m-k) \le c_\varepsilon e^{(n+m-k)P(\phi)}, \text{and } \frac{1}{c_\varepsilon} e^{nP(\phi)} \le P(f, \phi, n) \le c_\varepsilon e^{nP(\phi)} \tag{54}$$

Recall also that there are at most N_ε points $x \in \text{Fix}(f^n)$ which have the same attached $z \in V \cap \text{Fix}(f^n)$. Therefore, by using (52), (53) and (54) we can infer that there exists a constant $C_\varepsilon > 0$ such that for n large enough ($n > n(\varepsilon, m)$),

$$\tilde{\mu}_n(A_1) \le C_\varepsilon \tilde{\mu}_{n+m-k}(V) \cdot \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} \cdot P(\phi)^{m-k}, \tag{55}$$

where recall that $A_1 \subset B_m(y_1, \varepsilon), A_2 \subset B_m(y_2, \varepsilon)$. But since $\partial A_1, \partial A_2$ have μ-measure zero, we have $\mu(A_1) \le C_\varepsilon \mu(V) \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} \cdot P(\phi)^{m-k}$. But V is an arbitrary neighbourhood of A_2, hence

$$\mu(A_1) \le C_\varepsilon \mu(A_2) \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} P(\phi)^{m-k}$$

Similarly we prove also the other inequality, hence we are done.

□

Next, we want to see what are the "induced" measures of equilibrium measures on local stable manifolds. In the case of the Lebesgue measure the induicved measures on stable manifolds are clear, but this is not the case for a general measure. Let us then recall a few notions about *measurable partitions* (see [77]). Let ζ be a partition of a Lebesgue space $(X, \mathcal{B}, \mu)$ with $\mathcal{B}$-measurable sets. Subsets of X that are unions of elements of ζ are called ζ-sets. For an arbitrary point $x \in X$

(modulo μ), we denote the unique set which contains x, by $\zeta(x)$. By *basis* for ζ we understand a countable collection $\{B_\alpha, \alpha \in A\}$ of measurable ζ-sets so that for any two elements $C, C' \in \zeta$, there exists some $\alpha \in A$ with $C \subset B_\alpha, C' \cap B_\alpha = \emptyset$ or viceversa, i.e $C \cap B_\alpha = \emptyset, C' \subset B_\alpha$. A partition ζ is called *measurable* if it has a basis as above.

Now recall the notion of *family of conditional measures*, associated to a measurable partition ζ. Assume we have an endomorphism f on a compact set Λ, and let a probability borelian measure μ on Λ which is f-invariant. If ζ is a measurable partition of $(\Lambda, \mathcal{B}, \mu)$ denote by $(\Lambda/\zeta, \mu_\zeta)$ the *factor space* of Λ relative to ζ. Then we can attach an *essentially unique* collection of *conditional measures* $\{\mu_C\}_{C\in\zeta}$ satisfying two conditions (see [77]):

i) (C, μ_C) is a Lebesgue space

ii) for any measurable set $B \subset \Lambda$, the set $B \cap C$ is measurable in C for μ_ζ-almost all points $C \in \Lambda/\zeta$, the function $C \to \mu_C(B \cap C)$ is measurable on Λ/ζ and $\mu(B) = \int_{\Lambda/\zeta} \mu_C(B \cap C) d\mu_\zeta(C)$.

Definition 5.3.1. *If f is a hyperbolic map on a basic set Λ and if μ is an f-invariant borelian measure on Λ, then a measurable partition ζ of $(\Lambda, \mathcal{B}(\Lambda), \mu)$ is said to be **subordinated to the local stable manifolds** if for μ-a. e $x \in \Lambda$, we have $\zeta(x) \subset W^s_{loc}(x)$, and $\zeta(x)$ contains an open neighbourhood of x in $W^s_{loc}(x)$ (with respect to the topology induced on the local stable manifold).*

Let us fix an f-invariant borelian measure μ on Λ. Since we work with a uniformly hyperbolic endomorphism, we can construct a measurable partition ξ (w. r. t μ) *subordinated to the local stable manifolds foliation*, in the following way: first, we know there is a small $r_0 > 0$ s. t for each $x \in \Lambda$ there exists a local stable manifold $W^s_{r_0}(x)$. Then it is possible to take a countable partition $\mathcal{P}$ of Λ (modulo μ) with open sets, each having diameter less than r_0 and such that the boundary of each set from $\mathcal{P}$ has μ-measure zero (see for example [27]). Now for every open set $U \in \mathcal{P}$, and $x \in U \subset \Lambda$, we consider the intersection between U and the unique local stable manifold going through x; denote this intersection by $\xi(x)$. It is clear that $\xi(x) = \xi(y)$ if and only if both x, y are in the same set $U \in \mathcal{P}$ and they are on the same local stable manifold $W^s_{r_0}(z)$ for some $z \in \Lambda$. Now take the collection ξ of all the borelian sets $\xi(x), x \in U, U \in \mathcal{P}$. We see easily that ξ is a partition of Λ (modulo sets of μ-measure zero) and that ξ is measurable, since $\mathcal{P}$ was assumed countable and, inside each member $U \in \mathcal{P}$, we can separate any two local stable manifolds with the help of a countable collection of ξ-sets (which are neighbourhoods of local stable manifolds).

Therefore we have concluded the construction of the measurable partition ξ which is subordinated to the local stable manifolds. Modulo a set of μ-measure zero we have thus a partition with pieces of local stable manifolds, $\xi(x) \subset W^s_{r(y(x))}(y(x)), x \in \Lambda$. In fact without loss of generality, we may assume that for each member $A \in \xi$, there exists some $x(A) \in \Lambda$ and $r(A) \in (0, r_0)$ so that $W^s_{r(A)/2}(x(A)) \cap \Lambda \subset A \subset W^s_{r(A)}(x(A)) \cap \Lambda$.

From the construction above it follows that, outside a set of μ-measure zero, the radius $r(A)$ can be taken to vary continuously, i.e there exists a constant $\chi > 0$ s. t for each x in a set of full μ-measure in Λ, there exists a neighbourhood $U(x)$ of x with $\frac{r(\xi(z))}{r(\xi(z'))} \le \chi, z, z' \in U(x)$.

Notation: Consider the measurable partition ξ constructed above, and denote the conditional measure μ_A by μ_A^s, for $W^s_{r(A)/2}(x(A)) \cap \Lambda \subset A \subset W^s_{r(A)}(x(A)) \cap \Lambda, A \in \xi$. We will also denote the set of centers $\{x(A), A \in \xi\}$ by S. In particular, if $\mu = \mu_s$, we denote the conditional measures by $\mu^s_{s,A}$ for $A \in \xi$, or by $\mu^s_{s,x}$ when $\xi(x) = A$ for μ_s-a.e $x \in \Lambda$. Also we shall denote the probability measure induced by μ_s on the factor space Λ/ξ by $(\mu_s)_\xi$. □

If f is a d-to-1 c-hyperbolic endomorphism on the basic set Λ, we showed in [58] that the stable dimension $\delta^s(x)$ at *any point* $x \in \Lambda$ is independent of x, and is equal to the unique zero of the pressure function $t \to P(t\Phi^s - \log d)$. Thus we can talk in this case about the *stable dimension of* Λ and will denote it by δ^s. The following Theorem from [42] shows that, the conditional measures of the equilibrium measure of a stable potential, and associated to the stable foliation, are in fact geometric measures if f is d-to-1 over Λ.

Theorem 5.3.1 ([42])**.** *Let f be a smooth endomorphism on a Riemannian manifold M, and assume that f is c-hyperbolic on a basic set of saddle type Λ. Assume moreover that f is d-to-1 on Λ, and denote $\Phi^s(y) := \log|Df_s(y)|, y \in \Lambda$. Denote also by δ^s the stable dimension of Λ, and by μ_s the equilibrium measure of the potential $\delta^s\Phi^s$ on Λ. Then the conditional measures of μ_s associated to the partition ξ, namely $\mu^s_{s,A}$, are geometric probabilities, i.e for $(\mu_s)_\xi$-almost all points $\pi_\xi(A)$ of Λ/ξ (corresponding to sets $A \in \xi$), there exists a positive constant C_A such that*

$$C_A^{-1}\rho^{\delta^s} \le \mu^s_{s,A}(B(y,\rho)) \le C_A\rho^{\delta^s}, y \in A \cap \Lambda, 0 < \rho < \frac{r(A)}{2}$$

Proof. By using the partition ξ subordinated to local stable manifolds from above, we can associate conditional measures of μ_s, denoted by $\mu^s_{s,A}$, $A \in \xi$. We want to estimate the measure $\mu^s_{s,A}$ of a small arbitrary ball $B(y,\rho)$ centered at some $y \in A$, where $W^s_{r(A)/2}(x) \cap \Lambda \subset A \subset W^s_{r(A)}(x) \cap \Lambda, x = x(A)$.

Let us first consider an arbitrary set $f^n(B_n(z,\varepsilon))$, where we remind that $B_n(z,\varepsilon)$ denotes a Bowen ball, and where $\varepsilon > 0$ is arbitrary but small. This set (i.e $f^n(B_n(z,\varepsilon))$) is actually a neighbourhood of the local unstable manifold $W^u_\varepsilon(\hat f^n z)$ corresponding to some prehistory $(f^n z, f^{n-1}z, \ldots, z, \ldots)$. We will estimate next the μ_s-measure of a cross-section of a set $f^n(B_n(z,\varepsilon))$, i.e an intersection of type

$$B(n,z;k,x;\varepsilon) := f^n(B_n(z,\varepsilon)) \cap B_k(x,\varepsilon),$$

for arbitrary $z, x \in \Lambda$ and positive integers n, k .

We shall now estimate the μ_s-measure of $B(n,z;k,x,\varepsilon)$. Notice that $B(n,z;k,x;\varepsilon)$ is contained in $f^n(B_{n+k}(z,\varepsilon))$. Without loss of generality we can assume that $z = x_{-n}$, i.e that z itself is the unique n-preimage of x inside $B_n(z,\varepsilon)$; if not, then we can replace z by a point x_{-n} which is ε-shadowed by z up to order $n+k$, and thus the dynamical behaviour of z up to order $n+k$ will be the same as that of x_{-n}.

Denote the positive quantity $|Df^n_s(z)| \cdot \varepsilon$ by ρ. Now, as f is conformal on local stable manifolds, the diameter of the intersection $f^n(B_n(z,\varepsilon)) \cap W^s_r(f^n z)$ is equal to 2ρ.

Recall also that we assumed $f^n z = x$, and consider all the finite prehistories of the point x, in Λ. We will call then **ρ-maximal prehistory** of x any finite prehistory $(x, x_{-1}, \ldots, x_{-p})$ so that: $|Df^{p-1}_s(x_{-p+1})| \cdot \varepsilon \ge \rho$ but $|Df^p_s(x_{-p})| \cdot \varepsilon < \rho$. Clearly, given any prehistory $\hat x = (x, x_{-1}, \ldots)$ of x, there exists some positive integer $n(\hat{x}, \rho)$ such that $(x, x_{-1}, \ldots, x_{-n(\hat{x},\rho)})$ is a ρ-maximal prehistory. Let us denote by

$$\mathcal{N}(x, \rho) := \{n(\hat{x}, \rho), \hat{x} \text{ prehistory of } x \text{ from } \Lambda\}$$

We consider now the various components of the p-preimages of $B(n, z; k, x; \varepsilon)$, when p ranges in $\mathcal{N}(x, \rho)$. We extended the stable diameter of $B(n, z; k, x; \varepsilon)$ in backward time until we reach a diameter of at most ε. As the maximum expansion in backward time is realized on the stable manifolds (local inverse iterates contract all the unstable directions), it follows that for any prehistory $\hat{x}$ of x, there exists a component of $f^{-n(\hat{x},\rho)}(B(n, z; k, x; \varepsilon))$ inside the Bowen ball $B_{n(\hat{x},\rho)}(x_{-n(\hat{x},\rho)}, \varepsilon)$; denote this component by $A(\hat{x}, \rho)$. We see that all these components $A(\hat{x}, \rho)$ are mutually disjoint if $\varepsilon << \varepsilon_0$, where ε_0 is the local injectivity constant of f on Λ (recall that there are no critical points in Λ). Indeed if the sets $A(\hat{x}, \rho)$ and $A(\hat{x}', \rho)$ would intersect for some prehistories $\hat{x} = (x, x_{-1}, \ldots), \hat{x}' = (x, x'_{-1}, \ldots)$ of x then, since they are contained in Bowen balls, their forward iterates would be 2ε-close. But then we get a contradiction since the prehistories $\hat{x}, \hat{x}'$ must contain different preimages $x_p, x - p'$ at some level p, and these different preimages must be at a distance of at least ε_0 from each other. Hence either $A(\hat{x}, \rho) = A(\hat{x}', \rho)$, or $A(\hat{x}, \rho) \cap A(\hat{x}', \rho) = \emptyset$.

Now we will use the f-invariance of the equilibrium measure μ_s in order to estimate the μ_s-measure of the set $B(n, z; k, x; \varepsilon)$. Recall that $f^n z = x$, and $\varepsilon|Df_s^n(z)| =: \rho$. Then we have

$$\mu_s(B(n, z; k, x; \varepsilon) = \sum_{\hat{x} \text{ prehistory of } x} \mu_s(A(\hat{x}, \rho)),$$

since we showed above that the sets $A(\hat{x}, \rho)$ either coincide or are disjoint.

Consider two sets $A(\hat{x}, \rho), A(\hat{x}', \rho)$, one of them with $n(\hat{x}, \rho) = p$ and the other with $n(\hat{x}', \rho) = p'$. We proved in [58] that for a d-to-1 c-hyperbolic endomorphism f on the basic set Λ, we have $\delta^s = t_d^s$, where t_d^s is the unique zero of the pressure function $t \to P(t\Phi^s - \log d)$. Therefore we can use that

$$P(\delta^s \Phi^s) = \log d \tag{56}$$

Then from the definition of $A(\hat{x}, \varepsilon)$ and by using Proposition 5.3.1 (since by taking n, z, k, x, ε appropriately, we can assume that the measure μ_s on the boundaries of $A(\hat{x}, \rho), A(\hat{x}', \rho)$ is zero), we can compare the measure μ_s on two sets $A(\hat{x}, \rho), A(\hat{x}', \rho)$ as follows:

$$\frac{1}{C_\varepsilon}\mu_s(A(\hat{x}', \rho))\frac{|Df_s^p(x_{-p})|^{\delta^s}}{|Df_s^{p'}(x'_{-p'})|^{\delta^s}} \cdot d^{p'-p} \leq \mu_s(A(\hat{x}, \rho)) \leq C_\varepsilon \mu_s(A(\hat{x}', \rho))\frac{|Df_s^p(x_{-p})|^{\delta^s}}{|Df_s^{p'}(x'_{-p'})|^{\delta^s}} \cdot d^{p'-p} \tag{57}$$

In general, if for two variable quantities Q_1, Q_2, there exists a positive universal constant c such that $\frac{1}{c}Q_2 \leq Q_1 \leq cQ_2$, we say that Q_1, Q_2 are **comparable**, and will denote this by $Q_1 \approx Q_2$; the constant c is called the *comparability constant.*

But from the definition of $n(\hat{x}, \rho), n(\hat{x}', \rho)$ above (as being the length of the ρ-maximal prehistory along $\hat{x}$, respectively $\hat{x}'$), and since $n(\hat{x}, \rho) = p, n(\hat{x}', \rho) = p'$, we obtain that $(x, x_{-1}, \ldots, x_{-p})$ and $(x', x'_{-1}, \ldots, x'_{-p'})$ are two ρ-maximal prehistories. So there exists a constant $C > 0$ *independent* of

$\hat{x}, \hat{x}'$, (for instance take $C = \sup\limits_{y\in\Lambda} \frac{1}{|Df_s(y)|}$, as we assumed that f has no critical points in Λ), such that:

$$\frac{1}{C}|Df_s^{p'}(x'_{-p'})| \le |Df_s^p(x_{-p})| \le C|Df_s^{p'}(x'_{-p'})|$$

Therefore, from relation (57) we obtain

$$\frac{1}{C_\varepsilon}\mu_s(A(\hat{x}', \rho))d^{p'-p} \le \mu_s(A(\hat{x}, \rho)) \le C_\varepsilon \mu_s(A(\hat{x}', \rho))d^{p'-p}, \tag{58}$$

where we used the same constant C_ε as in (57), without loss of generality. Hence the proof will now be reduced to a combinatorial argument about the different pieces/components, of the preimages of various orders of $B(n, z; k, x; \varepsilon)$.

However we assumed that every point from Λ has exactly d f-preimages inside Λ. We use (58) in order to compare the μ_s-measures of the different pieces $A(\hat{x}, \rho)$, which will then be added successively. Recall that one of these components $A(\hat{x}, \rho)$ is precisely $B_{n+k}(z, \varepsilon)$. The comparisons will always be made with respect to this component $B_{n+k}(z, \varepsilon)$. Let us order the integers from $\mathcal{N}(x, \rho)$ as: $n_1 > n_2 > \ldots > n_T$. We shall add first the measures $\mu_s(A(\hat{x}, \rho))$ over all the sets corresponding to $\hat{x}$ with $n(\hat{x}, \rho) = n_1$, then over those prehistories with $n(\hat{x}, \rho) = n_2$, etc. And will use that any point from Λ has exactly d^m m-preimages belonging to Λ for any $m \ge 1$. Therefore by such successive additions and by using (58) we obtain:

$$\mu_s(B_{n+k}(z, \varepsilon)) \cdot d^n \le \mu_s(B(n, z; k, x; \varepsilon)) = \sum_{\hat{x} \text{ prehistory of } x} \mu_s(A(\hat{x}, \rho)) \le \mu_s(B_{n+k}(z, \varepsilon)) \cdot d^n,$$

with the positive constant C_ε independent of n, k, z, x.

We use now Theorem 1 of [47] which gave estimates for equilibrium measures on Bowen balls, similar to those from the case of diffeomorphisms (see [27] for example); this was done by lifting to an equilibrium measure on $\hat{\Lambda}$. Hence from the last displayed formula and (56), we obtain:

$$\frac{1}{C_\varepsilon}\frac{|Df_s^{n+k}(z)|^{\delta^s}}{d^k} \le \mu_s(B(n, z; k, x; \varepsilon)) \le C_\varepsilon \frac{|Df_s^{n+k}(z)|^{\delta^s}}{d^k} \tag{59}$$

Let us prove now that, if we vary z, x, k, n, then we can write any open (borelian) set in Λ as a union of mutually disjoint sets (modulo μ_s), of type $B(n, z; k, x; \varepsilon)$. Consider sets of type $B(n, z; k, x; \varepsilon) = f^n(B_n(z, \varepsilon)) \cap B_k(x, \varepsilon)$, with $f^n(z) = x$ and such that the stable side $\varepsilon|Df_s^n(z)|$ is comparable to the unstable side $\varepsilon|Df_u^k(x)|^{-1}$, i.e more precisely such that:

$$\frac{1}{\lambda_u}|Df_u^k(x)|^{-1} \le |Df_s^n(z)| \le \lambda_u|Df_u^k(x)|^{-1}, \tag{60}$$

where $\lambda_u := \sup\limits_{y\in\Lambda}|Df_u(y)|$; such sets will be called **round**. Notice also that there exists a sufficiently large constant $M > 1$, independent of n, z, k, x such that, if $r_n(z) := \frac{|Df_s^n(z)|}{M}$, then we have $B(x, r_n(z)) \subset B(n, z; k, x; \varepsilon) \subset B(x, M \cdot r_n(z))$. We see now from (60) and since $\mathcal{C}_f \cap \Lambda = \emptyset$ that: if $B(n, z; k, x; \varepsilon)$ is round, then there exists a constant $C_1 > 0$ independent of n, z, x, k such that

$$C_1^{-1}k \le n \le C_1 k \tag{61}$$

Now let some $\ell \in \mathbb{Z}$, for which there exists another round set $B(n+\ell, z'; k', x'; \varepsilon)$ with $f^{n+\ell}(z') = x'$ and stable side $\varepsilon |Df_s^{n+\ell}(z')|$ comparable with $\varepsilon |Df_s^n(z)|$, with a fixed comparability constant, namely:

$$\inf_{y \in \Lambda}(|Df_s(y)|^2) \cdot |Df_s^n(z)| \le |Df_s^{n+\ell}(z')| \le \sup_{y\in\Lambda}(|Df_s(y)|^{-2}) \cdot |Df_s^n(z)| \tag{62}$$

In fact one sees from the uniform hiperbolicity of f, relation (62) and from $\mathcal{C}_f \cap \Lambda = \emptyset$, that: ℓ depends only on Df on Λ and that $|\ell|$ is smaller than some universal constant ℓ_0. Thus by applying (59), (61) and (62), we obtain that there exists a constant $C_2 > 1$, independent of n, z, z', x, x', k, ℓ, so that

$$C_2^{-1} \cdot \mu_s(B(n+\ell, z'; k', x'; \varepsilon)) \le \mu_s(B(n, z; k, x; \varepsilon)) \le C_2 \cdot \mu_s(B(n+\ell, z'; k', x'; \varepsilon)) \tag{63}$$

In other words, μ_s is a doubling measure on Λ. Now by varying n, we see that each point x from Λ is the center of round sets $B(n, z; k, x; \varepsilon)$ having arbitrarily small diameters. Therefore from (63), we can apply variants of the Vitali Covering Theorem (see Theorems 2.8.7 or 2.8.17 of [?]), for the family of round sets $B(n, z; k, x; \varepsilon)$ which cover Λ finely with respect to μ_s; in these variants of the Vitali Theorem, the covering sets are not necessarily balls. Therefore we conclude that we can cover Λ, modulo μ_s, with a union of mutually disjoint sets $B(n, z; k, x; \varepsilon)$.

Now, let us study in more detail the conditions from the definition of conditional measures.

From the construction of the measurable partition ξ we have that $W^s_{r(A)/2}(x) \cap \Lambda \subset A \subset W^s_{r(A)}(x) \cap \Lambda, x = x(A) \in S$ and the radii $r(A)$ vary continuously with A. So from Remark 1 we can split an arbitrary set $U \in \mathcal{P}$, modulo μ_s, into a disjoint union of open sets V, each being a ξ-set, so there exists $r = r(V) > 0$ s.t for all $A \in \xi$ intersecting V, we have $W^s_{r/2}(x(A)) \cap \Lambda \subset A \subset W^s_r(x(A)) \cap \Lambda$. Hence locally, on a subset $V \subset U \in \mathcal{P}$, we can consider that ξ is, modulo a set of μ_s-measure zero, a foliation with local stable manifolds $W^s_r(x)$ of the same size $r = r(V)$. The intersections of these local stable manifolds with Λ are then identified with points in the factor space Λ/ξ.

We will work for the rest of the proof on an open set V as above, i.e where the sets $A \in \xi$ can be assumed to be of type $W^s_r(x)$, of the same size $r = r(V)$. Take also $\varepsilon = r$.

Now, the $(\mu_s)_\xi$-measure induced on the quotient space Λ/ξ is given by $(\mu_s)_\xi(E) = \mu_s(\pi_\xi^{-1}(E))$, where $\pi_\xi : \Lambda \to \Lambda/\xi$ is the canonical projection which collapses a set from ξ to a point. We notice that the projection $\pi_\xi(B(n, z; k, x; r))$ in Λ/ξ has $(\mu_s)_\xi$-measure equal to $\mu_s(B_k(x, r))$, since $\pi_\xi^{-1}(\pi_\xi(B(n, z; k, x; r))$ is $B_k(x, r)$. Now since $P(\delta^s\Phi^s) = \log d$ (from relation (56)) and by using again the estimates of equilibrium states on Bowen balls, we obtain as in (59) that $\mu_s(B_k(x, r))$ is comparable to $\frac{|Df_s^k(x)|^{\delta^s}}{d^k}$ (with a comparability constant $c = c(V)$). Hence from this argument we obtain that

$$\begin{aligned}(\mu_s)_\xi(B_k(x, r)/\xi) =& (\mu_s)_\xi(\pi_\xi(B(n, z; k, x; r)) = \mu_s(\pi_\xi^{-1}(\pi_\xi(B(n, z; k, x; r))) = \\ &= \mu_s(B_k(x, r)) \approx \frac{|Df_s^k(x)|^{\delta^s}}{d^k},\end{aligned} \tag{64}$$

with the comparability constant C_V. Now by (59) and recalling that $f^n z = x$, and by taking $\rho := |Df_s^n(z)|r$ we obtain:

$$\mu_s(B(n, z; k, x; r)) \approx \frac{|Df_s^k(x)|^{\delta^s}}{d^k} \cdot \rho^{\delta^s}, \tag{65}$$

where the comparability constant can be taken again C_V (the size $r > 0$ is fixed for a set V fixed). So from (64) and (65) we see immediately that

$$\frac{\mu_s(B(n, z; k, x; r))}{(\mu_s)_\xi(B_k(x, r)/\xi)} \approx \rho^{\delta^s}, \tag{66}$$

where $\rho = |Df_s^n(z)|r$. But, from the definition of conditional measures we know that

$$\mu_s(B(n, z; k, x; r)) = \int_{B_k(x,r)/\xi} \mu^s_{s,A}(A \cap B(n, z; k, x; r)) d(\mu_s)_\xi(\pi_\xi(A)) \tag{67}$$

Recall now that we showed above that any borelian set in Λ can be written, modulo μ_s, as a countable union of disjoint sets of type $B(n, z; k, x; r)$; and these sets form a basis for the open sets in V. Also if we vary n, the radius $\rho = |Df_s^n(z)| \cdot r$ can be made arbitrarily small. Now we have the essential uniqueness of the system of conditional measures associated to (μ_s, ξ) given in [77]. Consider some fixed arbitrary local unstable manifold $W^u_r(\hat\zeta)$ which intersects any local stable manifold $A \subset V$ in some unique point $y = y_A$, from the local product structure of the basic hyperbolic set Λ; for instance $\hat\zeta$ can be taken as a continuation of the finite prehistory $(f^n z, \dots, z)$, for the point z appearing in (67). Now from (66), together with a Lebesgue type derivation theorem (see [35]) applied in formula (67) to the function $y_A \to \mu^s_{s,A}(B(y_A, \rho))$, $y_A \in \Lambda \cap W^u_r(\hat\zeta), y_A := \Lambda \cap W^u_r(\hat\zeta) \cap A$, we conclude that

$$\mu^s_{s,A}(B(y_A, \rho)) \approx \rho^{\delta_s},$$

for $(\mu_s)_\xi$-almost all points A in Λ/ξ. But our $\rho := |Df_s^n(z)|r$ becomes arbitrarily small when $n \to \infty$; and without loss of generality, by varying the unstable manifold $W^u_r(\hat\zeta)$ (i.e by varying z, n), we can take the point y arbitrarily inside A, since A is supposed to be the intersection of Λ with a local stable manifold. Thus we obtain that $\mu^s_{s,A}$ satisfies a geometric probability condition with a constant C_V, i.e:

$$\frac{1}{C_V}\rho^{\delta^s} \le \mu^s_{s,A}(B(y, \rho)) \le C_V \rho^{\delta^s}, y \in A, 0 < \rho < r/2,$$

for $(\mu_s)_\xi$-almost all $A \subset V, A \in \xi$. The comparability factor C_V is constant on V; in general it can be taken locally constant on the complement in Λ of a set of μ_s-measure zero. The proof of the Theorem is thus finished.

□

Definition 5.3.2 ([42]). *Let f be a hyperbolic endomorphism on the folded basic set Λ, μ a borelian probability measure on Λ and ξ a measurable partition subordinated to local stable manifolds. Then the conditional measure μ^s_A corresponding to $A \in \xi$ will be called the stable conditional measure of μ on A. When $\mu = \mu_s$ we denote this stable conditional measure by $\mu^s_{s,A}$.*

From the proof of Theorem 5.3.1 it follows that, the stable conditional measures of μ_s do not actually depend on the measurable partition ξ constructed above, subordinated to local stable manifolds. So there exists a set $\Lambda(\mu_s)$ of full μ_s-measure inside Λ, such that for every $x \in \Lambda(\mu_s)$ there exists some small $r(x) > 0$ so that $W^s_{r(x)}(x)$ is contained in a set A from a measurable partition of type ξ (subordinated to local stable manifolds); then one can construct the stable conditional measure $\mu^s_{s,A}$. We denote this conditional measure also by $\mu^s_{s,x}$, $x \in \Lambda(\mu_s)$.

Recall now the notions of *lower*, respectively *upper pointwise dimension* of a finite borelian measure μ on a compact space Λ (see for example [1], [71]). For $x \in \Lambda$, they are defined by

$$\underline{d}_\mu(x) := \liminf_{\rho \to 0} \frac{\log \mu(B(x,\rho))}{\log \rho}, \text{ and } \bar{d}_\mu(x) := \limsup_{\rho \to 0} \frac{\log \mu(B(x,\rho))}{\log \rho}$$

If the lower pointwise dimension at x coincides with the upper pointwise dimension at x, we denote the common value by $d_\mu(x)$ and call it simply the *pointwise dimension* at x.

The *Hausdorff dimension, lower box dimension and upper box dimension* of μ are defined respectively by:

$$HD(\mu) := \inf\{HD(Z), \mu(\Lambda \setminus Z) = 0\}$$

$$\underline{\dim}_B(\mu) := \lim_{\delta \to 0} \inf\{\underline{\dim}_B(Z), \mu(\Lambda \setminus Z) \le \delta\}$$

$$\overline{\dim}_B(\mu) := \lim_{\delta \to 0} \inf\{\overline{\dim}_B(Z), \mu(\Lambda \setminus Z) \le \delta\}$$

Assume now that f is a hyperbolic endomorphism on Λ and μ a probability measure on Λ, and let ξ be a measurable partition subordinated to local stable manifolds of f on Λ. We define then the *lower/upper stable pointwise dimension* of μ at y, for μ-a.e $y \in \Lambda$, as the lower/upper pointwise dimension of the stable conditional measure μ^s_A at y, for $y \in A$, namely:

$$\underline{d}^s_\mu(y) := \liminf_{\rho \to 0} \frac{\log \mu^s_A(B(y,\rho))}{\log \rho} \text{ and } \bar{d}^s_\mu(y) := \limsup_{\rho \to 0} \frac{\log \mu^s_A(B(y,\rho))}{\log \rho}$$

Similarly we defined in [42] the *stable Hausdorff dimension* of μ on $A \in \xi$, and the *stable lower/upper box dimension* of μ on A, respectively, as the quantities:

$$HD^s(\mu, A) := HD(\mu^s_A), \ \ \underline{\dim}^s_B(\mu, A) := \underline{\dim}_B(\mu^s_A), \ \ \overline{\dim}^s_B(\mu, A) := \overline{\dim}_B(\mu^s_A), A \in \xi$$

When $\mu = \mu_s$ we denote $HD^s(\mu_s, x) := HD(\mu^s_{s,x})$, $\underline{\dim}^s_B(\mu_s, x) := \underline{\dim}_B(\mu^s_{s,x})$, and $\overline{\dim}^s_B(\mu_s, x) := \overline{\dim}_B(\mu^s_{s,x})$, for $x \in \Lambda(\mu_s)$.

Corollary 5.3.1 ([42]). *Let f be a c-hyperbolic, d-to-1 endomorphism on a basic set Λ, and μ_s be the equilibrium measure of the potential $\delta^s \Phi^s$. Then the stable pointwise dimension of μ_s exists μ_s-almost everywhere on Λ and is equal to the stable dimension δ^s.*

Also the stable Hausdorff dimension of μ_s, stable lower box dimension of μ_s and stable upper box dimension of μ_s are all equal to δ^s.

Definition 5.3.3 ([42]). *We will say that a measure μ on Λ has maximal stable dimension on $A \in \xi, A \subset W^s_{r(x)}(x)$ if:*

$$HD^s(\mu, A) = \sup\{HD^s(\nu, A), \nu \text{ is an } f|_\Lambda - \text{invariant probability measure on } \Lambda\}$$

This definition is similar to that of measure of maximal dimension, see [1], where measures of maximal dimension on hyperbolic sets of surface diffeomorphisms were studied. Our setting/methods for the maximal *stable* dimension in the non-invertible case, are however different.

Now, since the stable Hausdorff dimension of any f-invariant probability measure ν on Λ is bounded above by $\delta^s := HD(W^s_r(x) \cap \Lambda)$, we see from Corollary 5.3.1 that the conditional measures studied above, satisfy the maximum in a *Variational Principle for the stable dimension*:

Corollary 5.3.2 ([42]). *In the setting of Theorem 5.3.1 it follows that the stable equilibrium measure μ_s of f, is of* ***maximal stable dimension*** *on $W^s_{r(x)}(x) \cap \Lambda$ among all f-invariant probability measures on Λ, for μ_s-a.e $x \in \Lambda$. And μ_s maximizes in a Variational Principle for stable dimension on Λ, i.e:*

$$\delta^s = HD^s(\mu_s, x) = \sup\{HD^s(\nu, x),\ \nu \text{ is an } f|_\Lambda - \text{invariant probability measure on } \Lambda\}, \mu_s - a.e\ x$$

As said before, a basic set Λ is called a repellor (or folded repellor), if there exists a neighbourhood U of Λ such that $\bar{U} \subset f(U)$. And that Λ is a local repellor if there are local stable manifolds of f contained inside Λ (see [45] for more on these notions in the case of endomorphisms). We proved in [42] that a basic fractal Λ is a folded repellor if and only if its stable conditional measures $\mu^s_{s,x}$ are absolutely continuous for almost all x.

Corollary 5.3.3 ([42]). *Let an open c-hyperbolic endomorphism f on a connected basic set Λ. Then we have that the stable conditional measures $\mu^s_{s,x}$ of μ_s, are absolutely continuous with respect to the induced Lebesgue measures on $W^s_{r(x)}(x), x \in \Lambda(\mu_s)$, if and only if Λ is a non-invertible repellor.*

In [42] we gave also several classes of examples of folded non-Anosov repellors, and of hyperbolic basic sets for which the above results apply.

5.4 Pointwise dimensions for equilibrium measures on saddle basic sets of holomorphic endomorphisms on $\mathbb{P}^2\mathbb{C}$.

The dynamics of holomorphic endomorphisms on complex projective space $\mathbb{P}^2$ can gather both methods from smooth ergodic theory and thermodynamic formalism, and from higher dimensional complex dynamics. It has also specific techniques and results, as well as examples. This has been observed first in the papers [49], [45], [21], etc.

In [21], J. E Fornaess and E. Mihailescu studied the problem of *pointwise dimension* for an important class of invariant measures, which are supported on basic sets of saddle type for holomorphic maps $f : \mathbb{P}^2 \to \mathbb{P}^2$. In that paper, we also gave a formula for the equilibrium measure μ_ϕ

of an arbitrary ball, for a Hölder continuosu potential ϕ. For terminal/minimal saddle sets we also proved that an invariant measure ν, which was obtained as a wedge product of two positive closed currents, is in fact the measure of maximal entropy of the *restriction* $f|_\Lambda$.

Now recall some definitions of pointwise dimensions for probabilities (see for eg. [1], [71], etc.)

Definition 5.4.1. *Given an arbitrary probability measure μ on a compact metric space X, one can define the **lower pointwise dimension** and the **upper pointwise dimension** at $x \in X$ respectively by:*

$$\underline{\delta}_\mu(x) := \liminf_{\rho \to 0} \frac{\log \mu(B(x,\rho))}{\log \rho}, \ \ \textit{and} \ \bar{\delta}_\mu(x) := \limsup_{\rho \to 0} \frac{\log \mu(B(x,\rho))}{\log \rho}$$

*In case they coincide, we call the common value $\delta_\mu(x)$ the **pointwise dimension of** μ at $x \in X$. Also one can define the Hausdorff dimension of μ by:*

$$HD(\mu) := \inf\{HD(Z), \ Z \ \textit{borelian set with} \ \mu(X \setminus Z) = 0\}$$

In [98] Young proved that for a hyperbolic measure μ (i.e without zero Lyapunov exponents) which is invariated by a smooth diffeomorphism f of a surface, we have μ-a.e the formula

$$\delta_\mu = h_\mu(\frac{1}{\chi_u(\mu)} - \frac{1}{\chi_s(\mu)}),$$

where $\chi_s(\mu), \chi_u(\mu)$ are the negative, respectively positive Lyapunov exponents of μ (see for eg. [34], [27], etc. for the definition of Lyapunov exponents).

In the case of analytic endomorphisms f on the Riemann sphere $\mathbb{P}^1\mathbb{C}$, Manning proved that, if f is hyperbolic on its Julia set $J(f)$ and has no critical points in $J(f)$, then for any ergodic f-invariant probability measure μ on $J(f)$ the Hausdorff dimension of μ is given by: $HD(\mu) = \frac{h_\mu}{\chi(\mu)}$, where $\chi(\mu)$ is the only Lyapunov exponent of μ. This formula was then extended by Mañe [34] to the case of all rational maps (i.e not only hyperbolic) and of ergodic probabilities with positive Lyapunov exponent (thus which are non-uniformly expanding).

Nevertheless, the situation for higher dimensional non-expanding case is very different, and requires other methods than in the above mentioned cases. Notice also that the behaviour to perturbation, of the preimage counting function in the higher dimensional case, is different.

The first result we will recall here, is about the formula for the equilibrium measure of an arbitrary iterate of a Bowen ball; this will be applied later to formulas for the equilibrium measure μ_ϕ of any arbitrary ball.

For arbitrary $z \in \Lambda, \ n > 0, k > 0$, define the iterate of a Bowen ball,

$$B(n,k,z,\varepsilon) := f^n(B_{n+k}(z,\varepsilon)), \tag{68}$$

When n and k vary, we can adjust the sides of $B(n,k,z,\varepsilon)$ arbitrarily, and can make this iterate have (almost) equal sides in the stable and unstable directions.

Theorem 5.4.1 ([21]). *Consider $f : \mathbb{P}^2 \to \mathbb{P}^2$ a holomorphic map of degree d, and Λ a basic set such that f is c-hyperbolic on Λ and such that the preimage counting function is constant and equal to d' on Λ. Let also a Hölder continuous potential ϕ on Λ, which satisfies $\phi(x) + \log d' < P(\phi), \forall x \in \Lambda$. Then for any integers n, k, we have the formula:*

$$\mu_\phi(B(n,k,z,\varepsilon)) \approx \frac{e^{S_{n+k}\phi(z)}}{(d')^k}$$

Moreover the pointwise dimension of μ_ϕ exists μ_ϕ-a.e, is denoted by δ_{μ_ϕ}, and we have μ_ϕ-a.e:

$$\delta_{\mu_\phi} = HD(\mu_\phi) = h_{\mu_\phi}(\frac{1}{\chi_u(\mu_\phi)} - \frac{1}{\chi_s(\mu_\phi)}) + \log d' \cdot \frac{1}{\chi_s(\mu_\phi)}$$

Moreover, even when the preimage counting function $d(\cdot)$ is *not constant* on Λ, we still can obtain bounds for the measure of iterates of Bowen balls, and hence estimates for the lower pointwise dimension:

Corollary 5.4.1 ([21]). *In the setting of Theorem 5.4.1 assume the preimage counting function satisfies $d(x) \le d'$ for μ_ϕ-a.e $x \in \Lambda$ and that $\phi(x) + \log d' < P(\phi)$, $\forall x \in \Lambda$. Then for μ_ϕ-a.e $x \in \Lambda$,*

$$\underline{\delta}_{\mu_\phi}(x) \ge h_{\mu_\phi}(\frac{1}{\chi_u(\mu_\phi)} - \frac{1}{\chi_s(\mu_\phi)}) + \log d' \cdot \frac{1}{\chi_s(\mu_\phi)}$$

Theorem 5.4.1 was afterwards applied to equilibrium states of stable potentials of type $t\Phi^s$ on folded fractals Λ, where $\Phi^s(x) := \log |Df_s(x)|, x \in \Lambda$; we obtained a result parallel to [2], but in the case of non-invertible maps.

Corollary 5.4.2 ([21]). *In the same setting as in Theorem 5.4.1, denote by μ_s the equilibrium measure of $\delta^s\Phi^s$, where $\delta^s(x) := HD(W^s_r(x) \cap \Lambda)$ is assumed to be positive for some $x \in \Lambda$. Then $\delta^s = \delta^s(x)$ does not depend on x and for μ_s-a.e $x \in \Lambda$,*

$$\delta_{\mu_s}(x) = \frac{h_{\mu_s}}{\chi_u(\mu_s)} + \delta^s$$

The above results can be applied to many examples of holomorphic maps on $\mathbb{P}^2$, which are hyperbolic on saddle basic sets, for instance to perturbations of product maps or to perturbations of maps obtained from Ueda's method.

5.5 Relations between geometric positive currents and invariant measures on minimal sets.

In [21] we studied also the measure of maximal entropy for the restriction $f|_\Lambda$ of a holomorphic endomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ on a saddle basic set Λ. We considered the special case when Λ is a minimal, or more general, a terminal basic set. Such minmal sets were considered also in [22] where certain positive closed currents were constructed on them. We obtained in [21] a geometric description of the measure of maximal entropy of $f|_\Lambda$ in terms of positive closed currents.

First we recall some properties of the associated positive closed **Green current** T (see [23] for more details). For a given holomorphic endomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$, there exists a continuous plurisubharmonic function G on $\mathbb{C}^3 \setminus \{0\}$ called the *Green function* of f, satisfying $G(F(z)) = d \cdot G(z)$ where $F : \mathbb{C}^3 \setminus \{0\} \to \mathbb{C}^3 \setminus \{0\}$ is the lift of f relative to the canonical projection $\pi_2 : \mathbb{C}^3 \to \mathbb{P}^2$. Thus $G \in \mathcal{P}_1$, where $\mathcal{P}_1$ is the cone of plurisubharmonic functions u on $\mathbb{C}^3 \setminus \{0\}$ satisfying the homogeneity condition $u(\lambda z) = \log |\lambda| + u(z)$, $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^3 \setminus \{0\}$. Recall that

$$\pi_2^* T = dd^c G,$$

and that the Green measure $\mu = T \wedge T$ is mixing.

In [22] there were studied *s-hyperbolic* maps on $\mathbb{P}^2$ and *minimal* saddle basic sets, for the ordering $\Lambda_i \succ \Lambda_j$ if $W^u(\hat{\Lambda}_i) \cap W^s(\Lambda_j) \neq \emptyset$. A related notion introduced in [15] is that of a *terminal* set in the case of a holomorphic map f on $\mathbb{P}^2$. Here f is not assumed to have Axiom A and the condition refers only to Λ itself. A saddle set Λ is called *terminal* if for any $\hat{x} \in \hat{\Lambda}$, the iterates of f restricted to $W^u_{loc}(\hat{x}) \setminus \Lambda$ form a normal family. Clearly, if f f is Axiom A and if Λ is minimal, then for any $\hat{x} \in \hat{\Lambda}$ the global unstable set $W^u(\hat{x})$ does not intersect any global stable set of any other basic set, thus $W^u(\hat{\Lambda}) \setminus \Lambda$ is contained in the union of basins of attraction of attracting cycles; so in this case minimal sets are also terminal.

Examples of minimal sets for holomorphic maps on $\mathbb{P}^2$ were given in [22], and examples of terminal sets in [15].

In [22], Fornaess and Sibony constructed positive closed currents σ on minimal sets for s-hyperbolic maps, by using iterated images of unstable disks (or equivalently of disks which are transverse to local stable directions). If D is an unstable disk then

$$\frac{f^n_\star([D])}{d^n} \to \sigma \cdot \int D \wedge T$$

Using the positive closed $(1,1)$ current σ, they constructed thus an invariant measure ν on Λ as

$$\nu = \sigma \wedge T$$

We remind now some properties of the **transversal measures** $\hat{\mu}^s_x$ associated to a hyperbolic structure on Λ; they are built as in [85] (see also [90]), but on the natural extension $\hat{\Lambda}$. In our endomorphism case, we employ a Markov partition on the inverse limit $\hat{\Lambda}$ (see [84]). Moreover the inverse limit $\hat{\Lambda}$ has local product structure, as it is a Smale space (see [84]).

One obtains then a system of *transversal measures* $\hat{\mu}^s_x$ on $\hat{W}^s_{loc}(x)$, where $\hat{W}^s_{loc}(x)$ and $\hat{W}^u_{loc}(\hat{x})$ are the lifts to $\hat{\Lambda}$ of the local stable intersection $W^s_{loc}(x) \cap \Lambda$, respectively of the local unstable intersection $W^u_{loc}(\hat{x}) \cap \Lambda$. So $\hat{W}^s_{loc}(x) := \pi^{-1}(W^s_{loc}(x) \cap \Lambda)$ and $\hat{W}^u_{loc}(\hat{x}) := \pi^{-1}(W^u_{loc}(\hat{x}) \cap \Lambda), \hat{x} \in \hat{\Lambda}$. Then the family of measures $\hat{\mu}^s_x$, satisfies the following properties:

i) if $\chi^s_{x,y} : \hat{W}^s_r(x) \to \hat{W}^s_r(y)$ is the holonomy map given by $\chi^s_{x,y}(\hat{\xi}) = \hat{W}^u_r(\hat{\xi}) \cap \hat{W}^s_r(y)$, then $\hat{\mu}^s_x(A) = \hat{\mu}^s_y(\chi^s_{x,y}(A))$ for any borelian set A.

ii) $\hat{f}_\star \hat{\mu}^s_x = e^{h_{top}(f|_\Lambda)} \hat{\mu}^s_{f(x)}|_{\hat{f}(\hat{W}^s_r(x))}$

iii) supp $\hat{\mu}^s_x = \hat{W}^s(x)$.

In fact from [85] and [90] applied to our case on $\hat{\Lambda}$, it follows that there exist also unstable transversal measures, denoted by $\hat{\mu}^u_{\hat{x}}$ on $\hat{W}^u_r(\hat{x}), \hat{x} \in \hat{\Lambda}$ with similar properties. Also, the measure of maximal entropy on $\hat{\Lambda}$ denoted by $\hat{\mu}_0$, can be written as the product of transversal stable measures $\hat{\mu}^s_y$ with transversal unstable measures $\hat{\mu}_{\hat{x}}$ i.e

$$\hat{\mu}_0(\phi) = \int_{\hat{W}^s_r(x)} (\int_{\hat{W}^u_r(\hat{y})} \phi \, d\hat{\mu}^u_{\hat{y}}) \, d\hat{\mu}^s_x(\hat{y}), \tag{69}$$

for any function ϕ defined on a neighbourhood of $\hat{x} \in \hat{\Lambda}$.

Transversal measures associated to stable/unstable foliations are subject to a result by Bowen and Marcus ([7]), which can be applied on the natural extension $\hat{\Lambda}$.

Also, in [15] Diller and Jonsson introduced a positive current σ^u by using transversal measures (see also the diffeomorphism case in [85], [90]); in a neighbourhood of $x \in \Lambda$,

$$< \sigma^u, \chi >= \int_{\hat{W}^s_{loc}} (\int_{W^u_{loc}(\hat{y})} \chi) d\hat{\mu}^s_x(\hat{y}),$$

where $\hat{\mu}^s_x$ are transversal measures on $\hat{W}^s_{loc}(x) := \pi^{-1}(W^s_{loc}(x)$.

If Λ is terminal, then they defined an invariant probability measure on Λ,

$$\nu_i = \sigma^u \wedge T$$

We used in [21] the notation ν_i in order to emphasize the way the current σ^u was constructed with the help of the inverse limit.

In Theorem 5.5.1 we will prove that the measures ν, ν_i defined above are both equal to the measure of maximal entropy of $f|_\Lambda$ if Λ is (topologically) mixing (which can be arranged by taking some iterate).

We proved in [21] that the measure of maximal entropy of the restriction to Λ can be written as a wedge product of two positive closed currents, which can be described geometrically. Our result connects thus, the geometric properties of the fractal set Λ to the ergodic ones.

Theorem 5.5.1 ([21]). *a) Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic map of degree d and Λ be a terminal mixing saddle set. Then ν_i is equal to the measure of maximal entropy μ_0 on Λ.*

b) Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be an Axiom A holomorphic map of degree d, which is c-hyperbolic on the mixing minimal saddle set Λ. Then $\nu_i = \nu = \mu_0$, where μ_0 is the maximal entropy measure of $f|_\Lambda$.

Proof. a) In [15], the measure ν_i was defined as the wedge product $\sigma^u \wedge T$, where the positive closed current σ^u is constructed with the help of the stable transversal measures $\hat{\mu}^s_x, x \in \Lambda$. Recall also that $\pi|_{\hat{W}^u_r(\hat{x})} : \hat{W}^u_r(\hat{x}) \to W^u_r(\hat{x})$ is a bijection (see [47]), so any function ϕ on $\hat{W}^u_r(\hat{x})$ determines uniquely a function denoted again with ϕ on $W^u_r(\hat{x})$. Then from ν_i, we can form a system of measures on the lifts of local unstable manifolds $\hat{W}^u_r(\hat{x}), \hat{x} \in \hat{\Lambda}$ in the following way:

$$\hat{\nu}^u_{\hat{x}}(\phi) = \int_{W^u_r(\hat{x})} \phi T|_{W^u_r(\hat{x})}$$

We assumed that f is mixing on Λ; in fact (topological) mixing of f on Λ is equivalent to mixing of $\hat{f}$ on $\hat{\Lambda}$. Define stable holonomy maps between lifts to $\hat{\Lambda}$ of local unstable manifolds, namely

$$\chi^u_{\hat{x},\hat{y}} : \hat{W}^u_r(\hat{x}) \to \hat{W}^u_r(\hat{y}),\ \chi^u_{\hat{x},\hat{y}}(\hat{\xi}) := \hat{W}^u_r(\hat{y}) \cap \hat{W}^s_r(\hat{\xi}),\ \hat{\xi} \in \hat{W}^u_r(\hat{x})$$

We want to prove that the measures $\hat{\nu}^u_{\hat{x}}$ are transversal and invariant with respect to stable holonomy maps in the Smale space structure of $\hat{\Lambda}$, in the sense of Bowen and Marcus ([7]). From the way the local unstable manifolds were constructed as determined by prehistories, it follows that there is a bijection between $W^u_r(\hat{x}) \cap \Lambda$ and its lift $\hat{W}^u_r(\hat{x})$ (see also [47]). Given a borelian set $\hat{A} \subset \hat{W}^u_r(\hat{x})$, there exists a unique borelian set $A \subset W^u_r(\hat{x}) \cap \Lambda$ such that π is a bijection between $\hat{A}$ and A. From the definition of $\hat{\nu}^u_{\hat{x}}$, we know that

$$\hat{\nu}^u_{\hat{x}}(\hat{A}) = \int_{A \cap W^u_r(\hat{x}) \cap \Lambda} dd^c G|_{W^u_r(\hat{x})}$$

Denote now the unstable intersection $W^u_r(\hat{x}) \cap \Lambda$ by $Z(\hat{x})$ for $\hat{x} \in \hat{\Lambda}$. Consider points x, y in a subset of Λ belonging to an open set $V \in \mathbb{P}^2$ so there exists a holomorphic inverse $s : V \to \mathbb{C}^3 \setminus \{0\}$ of π_2. Then for r small we can identify $Z(\hat{x}), Z(\hat{y})$ with their respective lifts to $\mathbb{C}^3 \setminus \{0\}$ for any prehistories $\hat{x}, \hat{y} \in \hat{\Lambda}$. Since there are no critical points of f in Λ and since we work on Λ, it follows that $Z(\hat{x})$ can be split into mutually disjoint subsets on which f^n is injective, i.e $Z(\hat{x}) = \bigcup_i Z_{i,n}(\hat{x})$, $f^n|_{Z_{i,n}(\hat{x})} : Z_{i,n}(\hat{x}) \to Z^n_i(\hat{x})$ is bijective, and moreover $Z^n_i(\hat{x}), i$ are mutually disjoint. It follows that $f^n(Z(\hat{x})) = \cup_i Z^n_i(\hat{x})$. Now if $Z(\hat{x})$ is contained in V, then $f^n(Z(\hat{x}))$ may not be contained in V; but, if $f^n(Z(\hat{x}))$ is contained say in $V_1 \cup V_2$ where V_1, V_2 are open sets in $\mathbb{P}^2$ as above, with respective local inverses s_1, s_2 of π_2, and if $V_1 \cap V_2 \neq \emptyset$, then there exists a holomorphic function ρ on $V_1 \cap V_2$ so that $s_1 = \rho s_2$ on $V_1 \cap V_2$. So

$$dd^c(G \circ s_1) = dd^c(G(\rho s_2)) = dd^c \log|\rho| + dd^c(G \circ s_2) = dd^c(G \circ s_2)$$

This implies that working with $dd^c G$ on $\mathbb{C}^3 \setminus \{0\}$ is the same as working on $\mathbb{P}^2$.

Now $G \circ F = d \cdot G$ and $f^n : Z_{i,n}(\hat{x}) \to Z^n_i(\hat{x})$ is bijective hence $\int_{Z^n_i(\hat{x})} dd^c G = d^n \int_{Z_{i,n}(\hat{x})} dd^c G$. Thus by adding over all the indices i we obtain:

$$\int_{f^n(Z(\hat{x}))} dd^c G = d^n \int_{Z(\hat{x})} dd^c G \tag{70}$$

Now let $x, y \in \Lambda$ closer than $r/2$ and iterate $Z(\hat{x})$ and $Z(\hat{y})$ for some prehistories $\hat{x}, \hat{y} \in \hat{\Lambda}$. Consider the subsets $Z_{i,n}(\hat{y})$ such that $f^n : Z_{i,n}(\hat{y}) \to Z^n_i(\hat{y})$ is a bijection, $Z^n_i(\hat{y}), i$ are mutually disjoint and $f^n(Z(\hat{y}) = \cup Z^n_i(\hat{y})$. If $Z_{i,n}(\hat{x}), Z_{i,n}(\hat{y})$ has diameter small enough, then it follows that $Z^n_i(\hat{x}), Z^n_i(\hat{y})$ both have diameter bounded above by r and they are very close to each other, in fact $d(Z^n_i(\hat{x}), Z^n_i(\hat{y})) \to 0$ for each i, when $n \to \infty$. This follows as in the Laminated Distortion Lemma (see [45]) since the distances between iterates of points on stable manifolds decrease exponentially, and $|Df_u|$ is Hölder continuous.

Now if ψ is a smooth test function equal to 1 on a fixed neighbourhood of Z_i^n we have $\int_{Z_i^n(\hat{x})} dd^c G = \int_{Z_i^n(\hat{x})} \psi dd^c G = \int_{Z_i^n(\hat{x})} G dd^c \psi$ hence since $dd^c\psi$ is continuous and $Z_i^n(\hat{x})$ and $Z_i^n(\hat{y})$ are close, we obtain similar to [22] that for n large enough

$$|\int_{f^n(A)\cap Z_i^n(\hat{x})} dd^c G - \int_{f^n(\chi^u_{\hat{x},\hat{y}}(A))\cap Z_i^n(\hat{y})} dd^c G| \le \varepsilon m_2(Z_i^n(\hat{x})),$$

where m_2 is the Lebesgue measure on $\mathbb{P}^2$. Now we add these inequalities over i and use the fact proved in Proposition 5.3 of [22] that $m_2(f^n(Z(\hat{x}))) \le Cd^n, n > 0$. Hence by dividing with d^n, using (70), and letting $n \to \infty$ we obtain $\int_{A\cap Z(\hat{x})} dd^c G = \int_{\chi^u_{\hat{x},\hat{y}}(A)\cap Z(\hat{y})} dd^c G$. We lift then to the natural extension, keeping in mind that there exists a homeomorphism between $Z(\hat{x})$ and $\hat{W}^u_r(\hat{x})$. Hence on $\hat{\Lambda}$ we have:

$$\hat{\nu}^u_{\hat{x}}(\hat{A}) = \hat{\nu}_{\hat{y}}(\chi^u_{\hat{x},\hat{y}}(\hat{A})),\ \hat{A} \text{ borelian set in } \hat{W}^u_r(\hat{x})$$

This can be extended also to general borelian sets contained in global unstable sets $\hat{W}^u(\hat{x}) = \bigcup_{n\ge 0} \hat{f}^n(\hat{W}^u_r(\hat{x})), \hat{x} \in \hat{\Lambda}$. Thus by a theorem of Bowen and Marcus (see the main result of [7]), extended to the mixing homeomorphism $\hat{f}$ on $\hat{\Lambda}$, it follows that there exists a positive constant γ such that $\hat{\nu}^u_{\hat{x}} = \gamma \cdot \hat{\mu}^u_{0,\hat{x}}$, for any $\hat{x} \in \hat{\Lambda}$, where $\hat{\mu}^u_{0,\hat{x}}$ are the transversal measures given by the measure of maximal entropy $\hat{\mu}_0$ on $\hat{\Lambda}$ (as in [85], [90]); see also (69). In fact if μ_0 is the unique measure of maximal entropy on Λ and if $\hat{\mu}_0$ is the unique measure of maximal entropy on $\hat{\Lambda}$, then

$$\mu_0 = \pi_*\hat{\mu}_0 \text{ and } h_{\mu_0} = h_{top}(f|_\Lambda) = h_{top}(\hat{f}|_{\hat{\Lambda}}) = h_{\hat{\mu}_0}$$

The measure ν_i is constructed with the transversal stable measures $\hat{\mu}^s_x$ (which we denote also by $\hat{\mu}^s_{0,x}$). Now from [84] we know that any f-invariant measure μ on Λ can be lifted uniquely to an $\hat{f}$-invariant measure $\hat{\mu}$ on $\hat{\Lambda}$ such that $\pi_*\hat{\mu} = \mu$. In our case we denote by $\hat{\nu}_i$ this unique lift of ν_i to $\hat{\Lambda}$. Since both $\hat{\nu}_i$ and $\hat{\mu}_0$ are ergodic probabilities on $\hat{\Lambda}$, it follows that $\gamma = 1$ and that $\hat{\mu}_0 = \hat{\nu}_i$, hence $\mu_0 = \nu_i$.

b) For this item, assume that f has Axiom A, that Λ is a minimal basic set, and that f is c-hyperbolic on Λ. Then from Section 2 there exists a positive closed $(1,1)$ current σ supported on the global unstable set $W^u(\hat{\Lambda})$ such that if D is a local disk transverse to the stable direction, then $\frac{f^n_*([D])}{d^n} \to (\int [D] \wedge T)\sigma$. Without loss of generality assume that the disk D is chosen such that $\int [D] \wedge T = 1$; and also that T has no mass on the boundary ∂D of D.

We have from [22] that on a neighbourhood Δ of a point $x \in \Lambda$ there exists a measure λ on the space of holomorphic maps from a local unstable disk Δ_1 to a local stable disk Δ_2 such that

$$\sigma = \int [W^u_r(\hat{y})]\ d\lambda(g_{\hat{y}}),$$

where $W^u_r(\hat{y})$ are local unstable manifolds intersecting Δ, $[W^u_r(\hat{y})]$ are the respective currents of integration and $g_{\hat{y}} : \Delta_1 \to \Delta_2$ is an arbitrary holomorphic map whose graph is $W^u_r(\hat{y})$. Then $\nu = \sigma \wedge T$ is supported only on Λ; hence we can define measures $\hat{\nu}^s_x$ on $\hat{W}^s_r(x)$ by

$$\hat{\nu}^s_x(\hat{A}) = \lambda(\{g_{\hat{y}},\ \hat{y} \in \hat{A}\}$$

Hence from the definition of $g_{\hat{y}}$ as a function whose graph is $W^u_r(\hat{y})$, it follows that these measures are invariant to the local holonomy map between $\hat{W}^s_r(x)$ and $\hat{W}^s_r(y)$ for x, y close. Also by covering with small flow boxes it follows we can extend this property globally. Therefore from [7] we obtain that $\hat{\nu}^s_x = \gamma \cdot \hat{\mu}^s_{0,x}$, where the constant $\gamma > 0$ does not depend on $x \in \Lambda$. Now ν was defined as integration of T on local unstable manifolds followed by integration with respect to transversal measures; by using a) we obtain that $\hat{\nu} = \hat{\mu}_0$, and thus $\nu = \mu_0$. Hence on minimal saddle basic sets the measure ν is equal to ν_i, and both are equal to the measure of maximal entropy μ_0 on Λ.

$\square$

Now we can combine Theorem 5.4.1 on pointwise dimension from last subsection, with the above result about the measure ν_i, to obtain the pointwise dimension of ν_i.

Corollary 5.5.1 ([21]). *a) Let Λ be a mixing terminal saddle set for a holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$ of degree d, s.t Λ does not intersect the critical set C_f of f. If each point in Λ has at most d' f-preimages in Λ and if $d' < d$, then for μ_ϕ-a.e z,*

$$\underline{\delta}_{\nu_i}(z) \geq \log d \cdot (\frac{1}{\chi_u(\nu_i)} - \frac{1}{\chi_s(\nu_i)}) + \log d' \cdot \frac{1}{\chi_s(\nu_i)}$$

b) If Λ is a mixing terminal saddle set for a holomorphic map f on $\mathbb{P}^2$ of degree d, if $C_f \cap \Lambda = \emptyset$ and if the preimage counting function is constant equal to d' on Λ for $d' \leq d$, then we have:

$$\delta_{\nu_i} = HD(\nu_i) = \log d \cdot (\frac{1}{\int \log |Df_u| d\nu_i} - \frac{1}{\int \log |Df_s| d\nu_i}) + \log d' \cdot \frac{1}{\int \log |Df_s| d\nu_i},$$

c) Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic Axiom A map of degree d, which is c-hyperbolic on a connected minimal saddle set Λ. Then the preimage counting function is constant on Λ, with value denoted d', and

$$\delta_\nu = HD(\nu) = \log d \cdot (\frac{1}{\chi_u(\nu)} - \frac{1}{\chi_s(\nu)}) + \log d' \cdot \frac{1}{\chi_s(\nu)}$$

In the case of minimal c-hyperbolic sets of maps of degree 2, we can then determine all the possible values of the pointwise dimension of ν. The preimage counting function is constant if Λ is connected.

Corollary 5.5.2 ([21]). *Let f be an Axiom A holomorphic map on $\mathbb{P}^2$ of degree 2, which is c-hyperbolic on a connected minimal saddle set Λ and let ν be the measure of maximal entropy of $f|_\Lambda$. Then we have exactly one of the following two possibilities:*

1) the preimage counting function of f is equal to 1 on Λ; then $f|_\Lambda$ is a homeomorphism and

$$\delta_\nu = \log 2 \cdot (\frac{1}{\int \log |Df_u| d\nu} - \frac{1}{\int \log |Df_s| d\nu})$$

2) or, the preimage counting function of f is equal to 2 on Λ; then $f|_\Lambda$ is expanding and

$$\delta_\nu = \log 2 \cdot \frac{1}{\int \log |Df_u| d\nu}$$

6 Conformal iterated function systems, and applications.

6.1 Dimension estimates for attractors of finite iterated function systems with overlaps.

Iterated function systems play a central role in fractal geometry (see for eg. [19], [26], etc.) and many Cantor sets can be obtained as limit sets of iterated function systems (or IFS for short).

In general, a *conformal iterated function system* is given by an arbitrary finite set E called in the sequel an alphabet, a bounded connected open set $V \subset \mathbb{R}^q$, a compact set $X \subset V$ and a collection $\mathcal{S} = \{\phi_\rceil : \mathcal{V} \to \mathcal{V}\}_{\rceil \in \mathcal{E}}$ of $C^{1+\varepsilon}$ conformal injective maps from V to V such that $||\phi'_e|| = \sup\{|\phi'_e(x)| : x \in V\} \leq s < 1$, and $\phi_e(X) \subset X$, for all $e \in E$. Here, $\phi'_e(x) : \mathbb{R}^q \to \mathbb{R}^q$ is the derivative of the map $\phi_e : V \to V$ evaluated at the point x, it is a similarity map, and $|\phi'_e(x)|$ is its operator norm, or equivalently, its scaling factor.

The system $\mathcal{S}$ is said to satisfy the *Open Set Condition* if there exists an open non-empty set U such that $\phi_i(U) \subset U$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$, $\forall i, j \in E$.

Let us denote now the space of sequences with values in E by E^∞, and for a sequence $\omega = (\omega_1, \omega_2, \ldots) \in E^\infty$ and an integer $n \geq 1$, denote by $\omega|_n = (\omega_1, \ldots, \omega_n)$ the n-th truncation of ω. Also for a finite sequence $\eta = (\eta_1, \ldots, \eta_n)$, denote by $\phi_\eta = \phi_{\eta_1} \circ \ldots \circ \phi_{\eta_n}$ the succesive composition of the above contractions. And let E^* the space of all finite sequences with elements in E. We will define now the *limit set* (or sometimes called *attractor*) of the iterated function system $\mathcal{S}$:

Definition 6.1.1. *For the iterated function system $\{\phi_i\}_{i \in E}$ as above, define the limit set*

$$J_{\mathcal{S}} := \bigcup_{\omega \in E^\infty} \bigcap_{n \geq 1} \phi_{\omega|_n}(X)$$

It can be shown that J_S is the unique compact set contained in X such that $J_S = \bigcup_{e \in E} \phi_e(J_S)$.

The theory of conformal iterated function systems satisfying the Open Set Condition is well studied and understood (see for eg. [19], [26], [36], [86], etc.) In this case the Hausdorff dimension of the limit set J_S is given as the unique zero of the pressure function of the potential obtained from the derivatives of $\phi_e, e \in E$.

However, in the case of iterated function systems without Open Set Condition, little is known about the dimension of the limit sets, and the methods and results are *very different.* This was the object of our study in [56], namely IFS with arbitrary overlaps for which we do not assume any kind of separation condition (in particular no Open Set Condition).

If the system $\mathcal{S}$ of contractions is fixed, we will denote the set J_S also simply by J. In order to present our results, let us define a function $d : J \to \mathbb{N}$ by the following formula,

$$d(x) = \#\{e \in E : x \in \phi_e(J)\}.$$

From this definition we get the following trivial, but useful formula, which is true for all $x \in J$,

$$\sum_{e\in E: x\in\phi_e(J)} d^{-1}(x) = 1 \tag{71}$$

Let now $\kappa : E^\infty \to [1, +\infty)$ to be a Hölder continuous function and, for an arbitrary parameter $t \in \mathbb{R}$, consider the potentials $\psi_{\kappa,t} : E^\infty \to \mathbb{R}$ defined as follows:

$$\psi_{\kappa,t}(\omega) = t\psi(\omega) - \log\kappa(\omega) = t\log\phi'_{\omega_1}(\pi(\sigma(\omega)))| - \log\kappa(\omega),\ \omega \in E^\infty$$

One can check easily that $\psi_{\kappa,t}$ is Hölder continuous, by using the Hölder continuity of κ.

Let $P(t) := P(\psi_{\kappa,t})$ be the topological pressure of the potential $\psi_{\kappa,t}$ with respect to the dynamical system on the space of sequences, $\sigma : E^\infty \to E^\infty$. Since $\log|\phi'_{\omega_1}(\pi(\sigma(\omega)))| \le \log s < 0$, there exists a unique $h_\kappa \in \mathbb{R}$ such that $P(\psi_{\kappa,h_\kappa}) = 0$. Let $\hat{\mu}_t$ be the unique shift-invariant Gibbs (equilibrium) state of the Hölder continuous potential $\psi_{\kappa,t} : E^\infty \to \mathbb{R}$, and let

$$\mu_t = \hat{\mu}_t \circ \pi^{-1}$$

It follows that $\mu_t(J) = 1$. For every $\omega \in E^*$, say $\omega \in E^n$, define

$$[\omega] = \{\tau \in E^\infty : \tau|_n = \omega\},$$

and call this set, the (initial) cylinder generated by ω. From the fact that μ_t is a Gibbs measure, we obtain that

$$\hat{\mu}_t([\omega|_n]) \approx e^{-P(t)n}||\phi'_{\omega|_n}||^t \cdot \Pi_{j=0}^{n-1}\kappa^{-1}(\pi(\sigma^j(\omega))). \tag{72}$$

If A is an arbitrary Borel subset of J and $F \subset E^*$ is a family of mutually incomparable words such that $\pi^{-1}(A) \subset \bigcup_{\omega\in F}[\omega]$, then

$$\mu_t(A) \le \sum_{\omega\in F}\hat{\mu}_t([\omega]). \tag{73}$$

Then, Mihailescu and Urbański showed in [56] that the dimension of the limit set can be estimated from below by the zero of a certain pressure function.

Theorem 6.1.1 ([56])**.** *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a conformal iterated function system, and $\hat{\kappa} : J \to [1, +\infty)$ be a continuous function such that $d(x) \le \hat{\kappa}(x)$ for all $x \in J$. Then $HD(J) \ge h_\kappa$, where $\kappa = \hat{\kappa} \circ \pi : E^\infty \to \mathbb{R}$.*

The above result permits good *lower estimates* for the dimension, by using *continuous functions $\hat{\kappa}$ as upper bounds* for $d(\cdot)$. In general, we cannot write an exact formula, as the dimension is strongly influenced by the oscillation in the number of "preimages" that a point may have in the limit set, i.e by the values of $d(\cdot)$ over J.

Moreover, in the same paper we studied an *upper bound* for the dimension of the limit set, which is however not as flexible, given that we bound $d(\cdot)$ below only by a constant number.

Theorem 6.1.2 ([56]). *If $\mathcal{S} = \{\phi_e\}_{e\in E}$ is a conformal iterated function system and $\kappa \geq 1$ is an integer satisfying $d(x) \geq \kappa$ for all $x \in J$, then $HD(J) \leq h_\kappa$.*

As a consequence of Theorem 6.1.1 and Theorem 6.1.2, we obtained in [56] the following Corollary, which says that if the dimension is equal to the *minimal value* that it can take as the zero of the pressure, i.e corresponding to the maximal value D of $d(\cdot)$, then $d(\cdot)$ must be constant on J and equal to D. This shows that the relation between the dimension of the limit set, and the function $d(\cdot)$, is *reciprocal*.

Corollary 6.1.1 ([56]). *Assume that $\mathcal{S} = \{\phi_e\}_{e\in E}$ is a conformal iterated function system, and let $D := \max\{d(x) : x \in J_S\}$. Then $HD(J_S) = h_D$ if and only if $d(x) = D$ for all $x \in J_S$.*

Proof. If $d(x) = D$ for all $x \in J_S$, then the equality $HD(J_S) = h_D$ is a direct consequence of Theorem 6.1.1 and Theorem 6.1.2.

We want now to prove the other way around; suppose then that $h := HD(J_S) = h_D$. By contradiction, assume there exists $z \in J$ such that $d(z) \leq D - 1$. Since the alphabet E is finite, there must exist an open neighborhood V of z such that $d(x) \leq D - 1$ for all $x \in V$. Fix a non-empty open set $U \subset J$ such that $\overline{U} \subset V$. There then exists a Lipschitz function $\hat{\kappa} : J \to [1, +\infty)$ such that $\hat{\kappa}(x) = D - 1$ for all $x \in \overline{U}$ and $\hat{\kappa}(x) = D$ for all $x \in J \setminus V$. In particular, $d(y) \leq \hat{\kappa}$ for all $y \in J$, and it therefore follows from Theorem 6.1.1 that $h_D = h \geq h_\kappa$; recall that $\kappa = \hat{\kappa} \circ \pi$. But we also have

$$\kappa \leq D \quad \text{on} \quad E^\infty, \tag{74}$$

and thus $h_D \leq h_\kappa$. Hence,

$$h_\kappa = h_D \tag{75}$$

Let now $\hat{\mu}_D$ be the unique equilibrium state on the metric space E^∞, of the potential $h_d\psi - \log D$. Since $P(h_D\psi - \log D) = 0$, we have

$$\int_{E^\infty} (h_D\psi - \log D) d\hat{\mu}_D + h_{\hat{\mu}_D}(\sigma) = 0, \tag{76}$$

where $h_{\hat{\mu}_D}(\sigma)$ is the Kolmogorov-Sinai metric entropy of the dynamical system $\sigma : E^\infty \to E^\infty$ with respect to the σ-invariant measure $\hat{\mu}_D$. Due to the Variational Principle, we also have,

$$\int_{E^\infty} (h_D\psi - \log \kappa) d\hat{\mu}_D + h_{\hat{\mu}_D}(\sigma) = \int_{E^\infty} (h_\kappa\psi - \log D) d\hat{\mu}_D + h_{\hat{\mu}_D}(\sigma) \leq P(h_\kappa\psi - \log \kappa) = 0$$

But this combined with (76), imply that

$$\int_{E^\infty} (\log D - \log \kappa) d\hat{\mu}_D \leq 0. \tag{77}$$

As the function $\log D - \log \kappa$ is continuous and as the equilibrium state $\hat{\mu}_D$ is positive on non-empty open subsets of E^∞ (they must contain Bowen balls), it follows from (77) and (74) that $\log \kappa = \log D$ on E^∞. So, $\hat{\kappa} = D$ on J and this contradiction finishes the proof.

□

The above methods present the advantage that are flexible for the large spectrum of overlaps that the IFS may have; they also show that isolated points, where the function $d(\cdot)$ has different values, do not really matter in the dimension of the limit set. We also have the liberty in chosing better and better continuous upper bounds for $d(\cdot)$ in Theorem 6.1.1, depending on the set of points with a certain number of "preimages" in J.

6.2 Limit sets for infinite conformal iterated function systems with arbitrary overlaps.

Iterated function systems with countably many generators form a rich and deep chapter in the geometric theory of fractals. Such infinite systems present many new features when compared to finite systems, for example the limit set is not compact anymore, the dimension formula is different in the Open Set Condition case, the boundary at infinity plays an important role, etc. (see for instance [36], [62]).

In [55] we studied infinite IFS with arbitrary overlaps, and obtained estimates for the dimension of the limit set. Some of the notations and techniques are similar to those in [57], but the differences in results and phenomena are significant, due to the fact that the system is now countable.

Let us fix thus an integer $q \geq 1$ and a real number $s \in (0,1)$, and let X a compact subset of $\mathbb{R}^q$ such that $X = \overline{IntX}$. As before we take V a bounded connected open subset of $\mathbb{R}^q$ such that $X \subset V$. Also we fix an arbitrary coutable, either finite or infinte, set E called *an alphabet*. A system $\mathcal{S} = \{\phi_\rceil : \mathcal{V} \to \mathcal{V}\}_{\rceil \in \mathcal{E}}$, of $C^{1+\varepsilon}$ conformal injective maps is called an *countable conformal iterated function system* if the following conditions are satisfied.

(a)
$$\phi_e(X) \subset X$$
for all $e \in E$.

(b) There exists $s \in (0,1)$ such that
$$||\phi_e'|| = \sup\{|\phi_e'(x)| : x \in X\} \leq s < 1$$
for all $e \in E$. Here, $\phi_e'(x) : \mathbb{R}^q \to \mathbb{R}^q$ is the derivative of the map $\phi_e : V \to V$ evaluated at the point x, it is a similarity map, and $|\phi_e'(x)|$ is its operator norm, or equivalently, its scaling factor.

(c) (Refined Distortion Property) There are two constants $L \geq 1$ and $\alpha > 0$ such that
$$\big||\phi_e'(y)| - |\phi_e'(x)|\big| \leq L\|(\phi_e')^{-1}\|^{-1}\|y - x\|^\alpha$$
for all $x, y \in V$ and all $e \in E$.

(d) If the alphabet E is infinite, then
$$\lim_{e \to \infty} diam(\phi_e(X)) = 0.$$

Moreover we assumed in [55], that the system $\mathcal{S}$ is *irreducible*, i.e

$$J_S \not\subset \partial X \quad \text{or equivalently} \quad J_S \cap Int(X) \neq \emptyset.$$

If, in addition, the system $\mathcal{S}$ satisfies the Open Set Condition (OSC), meaning that the interiors of the sets $\phi_e(X)$, $e \in E$, are mutually disjoint, then there is a fairly complete account of the fractal properties of its limit set ([36], [62], etc.)

However we did not assume any kind of such conditions, so we allow any overlaps of the sets $\phi_a(X)$ and $\phi_b(X)$, where $a, b \in E$ with $a \neq b$. Thus this theory of IFS with arbitrary overlaps is profoundly different from the one of IFS with Open Set Condition. As before, let

$$E^* = \bigcup_{n=0}^{\infty} E^n \text{ and } E^\infty = \{(\omega_n)_{n=1}^\infty : \forall (n \geq 1)\, \omega_n \in E\}$$

We fix $\omega \in E^\infty$ and notice that $\{\phi_{\omega|_n}(X)\}_{n=1}^\infty$ is a descending sequence of compact sets such that

$$diam(\phi_{\omega|_n}(X)) \leq \tilde{D} s^n diam(X),$$

where the number $\tilde{D} \geq 1$ is due to the non-convexity of X. Thus, $\bigcap_{n=1}^\infty \phi_{\omega|_n}(X)$ is a singleton, and we denote it by $\pi(\omega)$. So we have defined a map $\pi : E^\infty \to X$, which is Lipschitz continuous.

The *limit set* (or the *attractor*) $J = J_S$ of the system $\mathcal{S}$ is defined to be the projection $\pi(E^\infty)$. It can be seen easily that $J_{\mathcal{S}}$ satisfies the following self-conformality condition:

$$J_S = \bigcup_{e \in E} \phi_e(J_S),$$

and, by induction, $J_S = \bigcup_{|\omega|=n} \phi_\omega(J_S)$, for all $n \geq 1$.

Let $\sigma : E^\infty \to E^\infty$ be the (one sided) shift map on E^∞. By the definition of J_S we have that

$$J_S = \bigcup_{\omega \in E^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|_n}(X).$$

However the order of the union and the intersection **cannot** be exchanged always, i.e. in general it is not true that $J_S = \bigcap_{n=1}^\infty \bigcup_{\omega \in E^n} \phi_\omega(X)$. The former is contained in the latter, and equality holds if, for example the families $\{\phi_\omega(X) : \omega \in E^n\}$ are pointwise bounded for all $n \geq 1$. This is in particular the case if the system $\mathcal{S}$ satisfies the Open Set Condition. However in our case this condition is not satisfied.

Let now $\psi : E^\infty \to \mathbb{R}$ be the function defined by the following formula,

$$\psi(\omega) = \log |\phi'_{\omega_1}(\pi(\sigma(\omega)))|, \ \omega \in E^\infty.$$

We proved in [55] that the function $\psi : E^\infty \to \mathbb{R}$ is Hölder continuous. Now, for an arbitrary $\omega \in E^*$, say $\omega \in E^n$, one can define the *cylinder* initiated by ω by:

$$[\omega] = \{\tau \in E^\infty : \tau|_n = \omega\}.$$

Let also $Fin(\mathcal{S})$ denote the set of all $t \in \mathbb{R}$ such that

$$\sum_{e \in E} \|\phi'_\omega\|_\infty^t < +\infty.$$

We say then that the potential $t\psi$ is *summable*. And will denote $\theta_S := \inf(Fin(\mathcal{S}))$..

Lemma 6.2.1 ([55]). *If $g : E^\infty \to \mathbb{R}$ is Hölder continuous, then there exists a constant $C_g > 0$ such that*

$$\left| \sum_{j=0}^{n-1} g(\sigma^j(\omega)) - \sum_{j=0}^{n-1} g(\sigma^j(\tau)) \right| \le C_g$$

for all $n \ge 1$ and all $\omega, \tau \in E^\infty$ such that $\omega|_n = \tau|_n$.

As in the finite case, we defined in [55] a function $d : J \to \mathbb{N}$ by the following formula,

$$d(x) = \#\{e \in E : x \in \phi_e(J)\},$$

which entails the formula

$$\sum_{e \in E:\, x \in \phi_e(J)} \frac{1}{d(x)} = 1$$

for all $x \in J$. Now let $\kappa : E^\infty \to [1, +\infty)$ be a (not necessarily bounded) Hölder continuous function and, for an arbitrary parameter $t \in \mathbb{R}$, consider the potentials $\psi_{\kappa,t} : E^\infty \to \mathbb{R}$ defined as follows:

$$\psi_{\kappa,t}(\omega) = t\psi(\omega) - \log \kappa(\omega) = t \log |\phi'_{\omega_1}(\pi(\sigma(\omega)))| - \log \kappa(\omega),$$

for all $\omega \in E^\infty$. One can check easily that $\psi_{\kappa,t}$ is Hölder continuous. Since the function $\log \kappa$ is non-negative, the set

$$Fin^\kappa(S) = \{t \in \mathbb{R} : \sum_{i \in E} \exp(\sup(\psi_{\kappa,t}|_{[i]})) < \infty\},$$

that is the set of those parameters $t \in \mathbb{R}$ for which the potential $\psi_{\kappa,t}$ is summable, contains $Fin(\mathcal{S})$.

For any $t \ge 0$, let $P(\psi_{\kappa,t})$ be the *topological pressure*, of the potential $\psi_{\kappa,t}$ with respect to the dynamical system $\sigma : E^\infty \to E^\infty$; namely:

$$P(\psi_{\kappa,t}) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in E^n} \exp\left(\sup \left(\sum_{j=0}^{n-1} \psi_{\kappa,t}|_{\sigma^j[\omega]} \right) \right).$$

Since $\log |\phi'_{\omega_1}(\pi(\sigma(\omega)))| \le \log s < 0$, it is straightforward to check that the function $Fin^\kappa(\mathcal{S}) \ni \sqcup \to \mathcal{P}(\psi_{\kappa,\sqcup}) \in \mathbb{R}$ is convex, continuous, strictly decreasing, and $\lim_{t \to +\infty} P(\psi_{\kappa,t}) = -\infty$. We denoted $P(\psi_{\kappa,t})$ simply by $P(t)$. If it will be needed to be more specific, we will write also $P_{\mathcal{S}}(t)$ or $P_E(t)$ for $P(t)$. Define now

$$h_\kappa := \inf\{t \ge 0 : P(\psi_{\kappa,t}) \le 0\}.$$

Like with the pressure, we write $h_\kappa(\mathcal{S})$ or $h_\kappa(E)$ if we want to be more specific. If there exists $t \ge 0$ such that $P(\psi_{\kappa,t}) = 0$, then such a t is unique and is equal to h_κ.

If $t \in Fin^{\kappa}(\mathcal{S})$, then there exists a unique shift-invariant Gibbs (equilibrium) state $\hat{\mu}_t$ of the Hölder continuous potential $\psi_{\kappa,t} : E^\infty \to \mathbb{R}$, which is uniquely characterized by the (Gibbs) property that

$$\hat{\mu}_t([\omega|_n]) \approx e^{-P(t)n} ||\phi'_{\omega|_n}||^t \Pi_{j=0}^{n-1} \frac{1}{\kappa(\pi(\sigma^j(\omega)))} \tag{78}$$

for every $\omega \in E^\infty$ and every $n \geq 1$. Let

$$\mu_t = \hat{\mu}_t \circ \pi^{-1}. \tag{79}$$

Clearly, $\mu_t(J) = 1$. If A is an arbitrary Borel subset of J and $F \subset E^*$ is a family of mutually incomparable words, meaning that none is extension of another, such that $\pi^{-1}(A) \subset \bigcup_{\omega \in F} [\omega]$, then $\mu_t(A) \leq \sum_{\omega \in F} \hat{\mu}_t([\omega])$. We say that a set $F \subset X$ is $\mathcal{S}$-invariant if

$$\bigcup_{e \in E} \phi_e(F) \subset F.$$

We say that a Borel probability measure μ on X is $\mathcal{S}$-invariant if there exists a Borel probability shift-invariant measure $\tilde{\mu}$ on E^∞ such that

$$\mu = \tilde{\mu} \circ \pi^{-1}.$$

Then obviously $\mu(J_S) = 1$. Also such a measure μ is called *ergodic* iff the measure $\tilde{\mu}$ is; that is μ is ergodic if and only if for an $\mathcal{S}$-invariant Borel subset F of X either $\mu(F) = 0$ or $\mu(F) = 1$. Define also the *boundary at infinity*

$$\partial_\infty X := \bigcup_{\omega \in E^*} \phi_\omega(\partial X).$$

Of course $\partial_\infty X$ is an $\mathcal{S}$–invariant subset of X.

Recall that $\mathcal{S}(\infty)$, the *boundary at infinity of the system*, is defined to consist of all accumulation points of all sequences $(x_n)_{n=1}^\infty$, where $x_n \in \phi_{e_n}(X)$ with some $e_n \in E$, and all elements e_n, $n \geq 1$, are mutually distinct. Obviously $\mathcal{S}(\infty)$ is a closed subset of X. We put

$$\mathcal{S}^+(\infty) := \bigcup_{\omega \in \mathcal{E}^*} \phi_\omega(\mathcal{S}(\infty)).$$

So, $\mathcal{S}^+(\infty)$ is a Borel $\mathcal{S}$-invariant subset of X. We say that the system $\mathcal{S}$ is *small at infinity* if

$$\mu(S(\infty)) = 0$$

for every Borel $\mathcal{S}$-invariant probability measure μ on J_S such that $\mu(\partial X) = 0$.

Conditions in which the system $\mathcal{S}$ is small at infinity are not difficult to find. Assume that $\mathcal{S}$ is a conformal irreducible IFS. If any of the following conditions holds, then it can be shown easily (see [55]) that $\mathcal{S}$ is small at infinity.

(a) $\mathcal{S}(\infty) \subset \partial_\infty \mathcal{X}$.

(b) $\mathcal{S}(\infty)$ is countable.

(c) $\mathcal{S}(\infty) \cap \mathcal{J}_{\mathcal{S}} = \emptyset$

(d) $\mathcal{S}(\infty) = \emptyset$ meaning that the alphabet E is finite.

We proved in [55], an upper bound for the Hausdorff dimension of the limit set $J_{\mathcal{S}}$ from which an invariant subset is taken away.

Theorem 6.2.1 ([55])**.** *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a conformal iterated function system. Let H be an $\mathcal{S}$-invariant subset of $J_{\mathcal{S}}$. If $k \geq 1$ is an integer satisfying $d(x) \geq k$ for all $x \in J_{\mathcal{S}} \setminus H$, then $HD(J_{\mathcal{S}} \setminus H) \leq h_k$.*

The main result of [55] was the proof of the **lower bound for the Hausdorff dimension** of the limit set of an infinite IFS with overlaps. The difference from the finite case, in the formulation and in the proof, is now larger than in the case of the upper bound that we gave above. In this countable case we have to assume that the system is small at infinity; also the boundary points $\partial_\infty X$ and $\mathcal{S}^+(\infty)$ play an important role.

Theorem 6.2.2 ([55])**.** *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be an irreducible conformal iterated function system which is small at infinity. If $\hat{\kappa} : J_{\mathcal{S}} \to [1, +\infty)$ is a Hölder continuous function such that $d(x) \leq \hat{\kappa}(x)$ for all $x \in J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))$, then we obtain the lower estimate for the dimension,*

$$HD(J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) \geq h_\kappa,$$

where $\kappa = \hat{\kappa} \circ \pi : E^\infty \to \mathbb{R}$.

We also obtained results in the case when the dimension of the limit set attains its minimal possible value as the zero of a pressure function that involves the number of overlaps $d(\cdot)$.

Theorem 6.2.3 ([55])**.** *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be an irreducible conformal iterated function system which is small at infinity. Assume that*

$$D := \sup\{d(x) : x \in J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))\}$$

is finite; in particular the supremum becomes a maximum. Then we obtain:

$$HD(J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) = h_D \quad \Leftrightarrow \quad d(x) = D, \ \forall \ x \in J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty)).$$

We also gave in [55] many examples of countable IFS with overlaps, which are small at infinity and for which we can apply the Theorems above. Here are some of these examples.

Example 1. Let $X = [-1, 1]$, and for every $n \in \mathbb{Z} \setminus \{0\}$, define $\phi_n : [-1, 1] \to [-1, 1]$ by:

$$\phi_n(x) = \frac{n}{|n|} \left(\frac{x}{4n^2} + 1 - \frac{1}{|n|} \right).$$

Then $\phi_n([-1,1]) \subset [-1,1]$, the system $\mathcal{S} := \{\phi_\backslash\}_{\backslash\in\mathbb{Z}\backslash\{\prime\}}$ is irreducible, $\mathcal{S}(\infty) = \{-\infty, \infty\}$, so $\mathcal{S}$ is small at infinity as it satisfies condition (c) of the above conditions for small at infinity, and $d(x) = 1$ for all $x \in J_\mathcal{S} \setminus \{0\}$ while $d(0) = 2$. Therefore, from the above results we obtain that $HD(J_\mathcal{S}) = h_1$.

Example 2. Let again $X = I = [0,1]$, and $\phi_1 : X \to X$ be a strictly increasing differentiable (ex. linear) contraction such that $\phi_1(0) = 0 \quad \text{and} \quad \phi_1(1) < 1/2$. Then define inductively $\phi_{2(n+1)} : X \to X$ to be a strictly increasing differentiable (ex. linear) contraction such that

$$\phi_{2(n+1)}(0) = \phi_{2n+1}(1) \quad \text{and} \quad \phi_{2(n+1)}(1) < 1/2.$$

and

$$\phi_{2n+1}(0) > \phi_{2n}(1) \quad \text{and} \quad \phi_{2n+1}(1) < 1/2.$$

Similarly, let $\phi_{-1} : X \to X$ be a strictly increasing differentiable (ex. linear) contraction such that $\phi_0(1) = 1 \quad \text{and} \quad \phi_0(0) > 1/2$. Then define recursively $\phi_{2(n+1)} : X \to X$ to be a strictly increasing differentiable (ex. linear) contraction such that

$$1/2 < \phi_{-(n+1)}(1) < \phi_{-n}(0).$$

Consider the system

$$\mathcal{S} = \{\phi_n : n \in \mathbb{Z}\}.$$

We now notice that both points $0 = \pi_\mathcal{S}(1^\infty)$ and $1 = \pi_\mathcal{S}(0^\infty)$ belong to $J_\mathcal{S}$. Thus all the points of type $\phi_j(0), \phi_j(1),\ j \in \mathbb{Z}$, belong to $J_\mathcal{S}$. But for the contact points of type $\phi_{2j+1}(1),\ j > 0$ we see that the function d is equal to 2, whereas for all other points in J_S it is equal to 1. Also notice that $S(\infty)$ is countable, thus the system is small at infinity. Moreover, S is an irreducible system, since $J_S \not\subset \partial X$. Thus since these contact points are in $\partial_\infty(X)$, we can apply the above Corollary and obtain that $HD(J_S) = h_1$.

Example 3. Let $X = \overline{B}(0,1)$ be the closed unit disk in the plane. For every $n \in \mathbb{Z}$ let $\phi_n : \overline{B}(0,1) \to B(0,1)$ be a contracting similarity $z \mapsto a_n z + b_n$, where both a_n and b_n are real and $0 < a_n < 1$. Then $\phi_n([-1,1]) \subset (-1,1)$ and hence we obtain $J_\mathcal{S} \subset [-1,1]$, where $\mathcal{S} = \{\phi_n : n \in \mathbb{Z}\}$. We may select the numbers a_n and $b_n, n \in \mathbb{Z}$, so that $\phi_n([-1,1]) \cap \phi_k([-1,1]) \neq \emptyset$ if and only if $|n-k| = 1$, and when this does hold then in addition $\phi_n((-1,1)) \cap \phi_k((-1,1)) \neq \emptyset$. We further require that $|a_n| \underset{n\to\infty}{\to} 0$, and assume also that the sequence $(\phi_n(0))_{n\in\mathbb{Z}}$ is increasing and

$$\lim_{n\to+\infty} \phi_n(0) = 1, \text{ while } \lim_{n\to-\infty} \phi_n(0) = -1$$

In this example $S(\infty)$ is equal to the finite set $\{-1,1\}$, hence the system is small at infinity. Thus we obtain that $(-1,1) \subset J_S$. However the function $d(\cdot)$ is oscillating in $(-1,1)$, from the value 1 to the value 2, and thus is not continuous on J_S.

By choosing now a Hölder continuous function $\tilde{\kappa}$ such that $d(x) \leq \tilde{\kappa}(x)$, we obtain from Theorem 4.1 that $HD(J_\mathcal{S}) = 1 \leq h_\kappa$.

6.3 Families of non-stationary Moran fractals determined by asymptotic frequencies, and their pressure functions.

A class of special iterated function systems is given by non-stationary Moran fractals. We considered in [51], Moran fractals with countably many generators and, this time, satisfying the Open Set Condition. Our fractals are determined by certain asymptotic frequencies of letters, and thus, as shall be seen in the sequel, prove to be useful for number theoretic applications.

First, let us take $\{\Phi_k\}_{k\geq 1}$ a sequence of positive vectors in the infinite dimensional space $\mathbb{R}^{\mathbb{N}^*}$, such that for all $k \geq 1$, $\Phi_k = (c_{k1}, c_{k2}, \cdots)$, $c_{kj} \geq 0, k, j \geq 1$ satisfying

$$\sum_{j=1}^{\infty} c_{kj} \leq 1, \text{and} \sup\{c_{kj} : j \in DN\} < 1$$

Let D_0 be the empty set. For $k \geq 1$ write

$$D_{m,k} = \{(i_m, i_{m+1}, \cdots, i_k) : i_j \in \mathbb{N},\, m \leq j \leq k\},$$

and $D_k = D_{1,k}$, $D_\infty = \lim_{k\to\infty} D_k$. Define $D := \bigcup_{k\geq 0} D_k$. Elements of D are called words. For any $\sigma \in D$ if $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in D_n$, we write $\sigma^- = (\sigma_1, \sigma_2, \cdots, \sigma_{n-1})$ to denote the word obtained by deleting the last letter of σ, $|\sigma| = n$ to denote the length of σ, and $\sigma|_k := (\sigma_1, \sigma_2, \cdots, \sigma_k)$, $k \leq n$, to denote the truncation of σ to the length k. If $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_k) \in D_k$ and $\tau = (\tau_1, \tau_2, \cdots, \tau_m) \in D_{k+1,m}$, then we write $\sigma\tau = \sigma * \tau = (\sigma_1, \cdots, \sigma_k, \tau_1, \cdots, \tau_m)$ to denote the juxtaposition of $\sigma, \tau \in D$. For $\sigma \in D$ and $\tau \in D \bigcup D_\infty$ we say τ is an extension of σ, written as $\sigma \prec \tau$, if $\tau|_{|\sigma|} = \sigma$.

Let J a nonempty compact subset of $\mathbb{R}^d$ such that $J = \mathrm{cl}(\mathrm{int} J)$, where $\mathrm{int}(A)$ denotes the interior of a set A, and let $\mathcal{F} = \{J_\sigma : \sigma \in D\}$ be an infinite collection of nonempty subsets of $\mathbb{R}^d$. We say that $\mathcal{F}$ fulfills the conditions for the **infinite Moran structure with non-stationary contraction rates** (**IMSNC**), if it satisfies the following:

1) $J_\emptyset = J$.

2) For any $\sigma \in D$, J_σ is similar to J, i.e there exists a similarity $S_\sigma : \mathbb{R}^d \to \mathbb{R}^d$ so that $J_\sigma = S_\sigma(J)$.

3) For any $k \geq 0$ and $\sigma \in D_k$, $J_{\sigma*1}, J_{\sigma*2}, \cdots$ are subsets of J_σ, and $\mathrm{int}(J_{\sigma*i}) \cap \mathrm{int}(J_{\sigma*j}) = \emptyset$ for $i, j \geq 1$ and $i \neq j$.

4) For any $k \geq 1$ and $\sigma \in D_{k-1}$, $j \geq 1$, $\frac{|J_{\sigma*j}|}{|J_\sigma|} = c_{kj}$.

Since the cardinality of D_k is infinity for any $k \geq 1$, the set $\bigcup_{\sigma\in D_k} J_\sigma$ may not be closed. For example: if $k = 1$, then $D_k = D_1 = \mathbb{N}$. Take $J = [0,1]$ and $J_j = [1 - \frac{1}{2^{j-1}}, 1 - \frac{1}{2^j}]$, and then $\bigcup_{j=1}^{\infty} J_j = [0, 1)$ which is not closed.

The nonempty compact set $E := E(\mathcal{F})$ given by

$$E = \overline{\bigcup_{\sigma\in D_\infty} \cap_{1\leq i<\infty} J_{\sigma|_i}}$$

is called the Moran set (or Moran fractal) associated with the collection $\mathcal{F}$. As $\mathcal{F}$ satisfies the infinite Moran structure conditions, we call E an *infinitely generated Moran set with non-stationary contraction coefficients*, or simply *an infinitely generated non-stationary Moran fractal.*

Let now $\mathcal{F}_k = \{J_\sigma : \sigma \in D_k\}$, and $\mathcal{F} = \bigcup_{k\geq 0} \mathcal{F}_k$. The elements of $\mathcal{F}_k$ are called the basic sets of order k, and the elements of $\mathcal{F}$ are called the basic elements of the Moran set E.

If $\lim_{k\to\infty} \sup_{\sigma\in D_k} |J_\sigma| > 0$, then E contains interior points. Then the measure and dimension properties will be trivial. We assume therefore $\lim_{k\to\infty} \sup_{\sigma\in D_k} |J_\sigma| = 0$.

By the definition of the Moran set E associated with $\mathcal{F}$, we see that $\mathcal{F}$ is a net of E, i.e., for any $x \in E$ and any $\varepsilon > 0$, there is $G \in \mathcal{F}$ such that $x \in G$ and $|G| < \varepsilon$. Two basic elements are said to be disjoint if their interiors are disjoint.

Suppose that the set J and $\{\Phi_k\}_{k\geq 1}$ are given; we denote by $\mathcal{M}(J, \mathbb{N}, \{\Phi_k\})$ the class of Moran sets satisfying the IMSNC. We call $\mathcal{M}(J, \mathbb{N}, \{\Phi_k\})$ the Moran class associated with the triplet $(J, \mathbb{N}, \{\Phi_k\})$; this class is obtained by considering all the possible similitudes $S_\sigma, \sigma \in D$, which satisfy the conditions IMSNC above. Without loss of generality we can assume that the diameter of the set J is one.

Consider $A := \{a_1, a_2, \cdots, a_m\}$ to be a finite set of distinct positive integers, and define

$$A^{\mathbb{N}} := \{(t_j)_{j=1}^\infty \in D_\infty : t_j \in A\}$$

Consider an element $\omega = (s_1, s_2, \cdots) \in A^{\mathbb{N}}$, and let $\omega|_k = (s_1, s_2, \cdots, s_k)$ and $\|\omega|_k\|_{a_i} := \#\{s_j = a_i : s_j \text{ appears in } \omega|_k\}$ denote the number of times that a_i appears in the truncated word $\omega|_k$. In addition assume that the infinite sequence $\omega \in A^{\mathbb{N}}$ satisfies:

$$\lim_{k\to\infty} \frac{\|\omega|_k\|_{a_i}}{k} = \eta_i > 0,$$

for every $a_i \in A, 1 \leq i \leq m$; then we say that ω has the frequency vector $\eta = (\eta_1, \eta_2, \cdots, \eta_m)$. Let T be the left shift on $A^{\mathbb{N}}$, i.e., $T(\omega) = (s_2, s_3, \cdots)$ where $\omega = (s_1, s_2, \cdots) \in A^{\mathbb{N}}$. Note that $\sum_{i=1}^m \|\omega|_k\|_{a_i} = k$, and $\sum_{i=1}^m \eta_i = 1$. For $\eta = (\eta_1, \eta_2, \cdots, \eta_m) \in [0,1]^m$ define

$$A_\eta^{\mathbb{N}} := \{\rho = (\rho_j)_{j\geq 1} : \rho_j \in A,\ \lim_{k\to\infty} \frac{\|\rho|_k\|_{a_i}}{k} = \eta_i,\ 1 \leq i \leq m\}$$

We fix now $\Psi_i = (\gamma_{i1}, \gamma_{i2}, \cdots),\ 1 \leq i \leq m$ an infinite positive vector satisfying the conditions:

$$\sum_{j=1}^\infty \gamma_{ij} \leq 1, \text{and } \sup\{\gamma_{ij} : j \in \mathbb{N}\} < 1, \text{ for all } 1 \leq i \leq m \tag{80}$$

If $\omega = (s_1, s_2, \cdots) \in A_\eta^{\mathbb{N}}$, then for $k \geq 1$ arbitrary, if it happens that the element s_k of ω is equal to some $a_i \in A$, then we define the infinite positive vector

$$\Phi_k = (c_{k1}, c_{k2}, \ldots) := \Psi_i = (\gamma_{i1}, \gamma_{i2}, \cdots)$$

In this way by using our construction above, we obtain a class of Moran sets associated with $\omega \in A_\eta^{\mathbb{N}}$, by taking all possible similitudes that satisfy IMSNC. Denote such a generic Moran set by $E := E(\omega)$, and call it a **Moran set with infinitely many generators and non-stationary contraction rates** associated to $\Psi_i, 1 \leq i \leq m$, to η and to $\omega \in A_\eta^{\mathbb{N}}$. Later in the paper, we extend this construction also to the case of countably many infinite vectors Ψ_i.

Let $C = \sup\{\gamma_{ij} : j \in \mathbb{N},\, 1 \leq i \leq m\}$. Then by our assumption, $0 < C < 1$. Let us define

$$\theta_i = \inf\{t > 0 : \sum_{j=1}^{\infty} \gamma_{ij}^t < \infty\}, 1 \leq i \leq m, \text{ and } \theta = \max\{\theta_i : 1 \leq i \leq m\}$$

Note that $\theta_i \geq 0$ for $1 \leq i \leq m$, and so $\theta \geq 0$, and thus one of the following two mutually exclusive cases can occur:

$(C1)$ $\sum_{j=1}^{\infty} \gamma_{ij}^{\theta} = \infty$ for some i;
$(C2)$ $\sum_{j=1}^{\infty} \gamma_{ij}^{\theta} < \infty$ for all i.

We assume that $(C1)$ happens, and then $\sum_{j=1}^{\infty} \gamma_{ij}^t < \infty$ for all $t \in (\theta, +\infty)$ and for all $1 \leq i \leq m$. If $(C2)$ happens, then $\sum_{j=1}^{\infty} \gamma_{ij}^t < \infty$ for all $t \in [\theta, +\infty)$ and for all $1 \leq i \leq m$.

We will study now **pressure functions** for the fractals introduced before. Recall that we fixed an element $\omega \in A_\eta^{\mathbb{N}}$, and a sequence of infinite vectors $\{\Phi_k\}_{k \geq 1}$, with $\Phi_k = (c_{k1}, c_{k2}, \ldots), k \geq 1$ defined in terms of ω and of $\Psi_1, \ldots, \Psi_m$. For an arbitrary $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_k) \in D_k$, let us define

$$c_\sigma = \begin{cases} c_{1\sigma_1} c_{2\sigma_2} \cdots c_{k\sigma_k} & \text{if } k \geq 1 \\ 1 & \text{if } k = 0. \end{cases}$$

Let $t \in (\theta, +\infty)$ be given, and define

$$c_k(t) = \frac{1}{k} \log \sum_{\sigma \in D_k} c_\sigma^t = \frac{1}{k} \log \prod_{i=1}^{m} \left(\sum_{j=1}^{\infty} \gamma_{ij}^t \right)^{\|\omega|_k\|_{a_i}} = \sum_{i=1}^{m} \frac{\|\omega|_k\|_{a_i}}{k} \log \sum_{j=1}^{\infty} \gamma_{ij}^t,$$

for $k \in \mathbb{N}$. Let us denote by $L(t) = \sum_{i=1}^{m} \left| \log \sum_{j=1}^{\infty} \gamma_{ij}^t \right|$. Note that $0 < \sum_{j=1}^{\infty} \gamma_{ij}^t < \infty$ for all $t \in (\theta, +\infty)$ and for all $1 \leq i \leq m$. Thus $L(t)$ is a positive number for any $t \in (\theta, +\infty)$.

We claim that $\{c_k(t)\}_{k=1}^{\infty}$ is a Cauchy sequence. For all $1 \leq i \leq m$, $\left\{ \frac{\|\omega|_k\|_{a_i}}{k} \right\}_{k=1}^{\infty}$ is convergent, and so for any $\varepsilon > 0$ there exists a positive integer N such that for all $n, p \geq N$ and for all $1 \leq i \leq m$,

$$\left| \frac{\|\omega|_n\|_{a_i}}{n} - \frac{\|\omega|_p\|_{a_i}}{p} \right| < \frac{\varepsilon}{L(t)}$$

Using this fact, and noting that

$$c_n(t) - c_p(t) = \sum_{i=1}^{m} \left(\frac{\|\omega|_n\|_{a_i}}{n} - \frac{\|\omega|_p\|_{a_i}}{p} \right) \log \sum_{j=1}^{\infty} \gamma_{ij}^t,$$

for all $n, p \geq N$, we obtain:

$$|c_n(t) - c_p(t)| \leq \sum_{i=1}^{m} \left| \frac{\|\omega|_n\|_{a_i}}{n} - \frac{\|\omega|_p\|_{a_i}}{p} \right| \left| \log \sum_{j=1}^{\infty} \gamma_{ij}^t \right| < \frac{\varepsilon}{L(t)} \sum_{i=1}^{m} \left| \log \sum_{j=1}^{\infty} \gamma_{ij}^t \right|$$

which implies $|c_n(t) - c_p(t)| < \varepsilon$, and thus the claim is true. Notice also that from above, the sequence $\{c_k(t)\}_{k=1}^{\infty}$ is bounded. Hence $\{c_k(t)\}_{k=1}^{\infty}$ is a convergent sequence. Then, we can define a function $P(t)$:

$$P(t) = \lim_{k\to\infty} c_k(t) = \lim_{k\to\infty} \sum_{i=1}^{m} \frac{\|\omega|_k\|_{a_i}}{k} \log \sum_{j=1}^{\infty} \gamma_{ij}^t = \sum_{i=1}^{m} \eta_i \log \sum_{j=1}^{\infty} \gamma_{ij}^t, \tag{81}$$

where $t \in (\theta, +\infty)$. The function $P(t)$ will be called the *pressure function* corresponding to the Moran set $E := E(\omega)$, by analogy to the usual topological pressure for continuous functions [96], where the sets D_k play the role of Bowen sets of order k. The pressure function associated to $E(\omega)$ depends only on the contraction rates $c_{kj}, k, j \geq 1$, which in turn depend on ω and $\Psi_1, \ldots, \Psi_m$.

We will assume also that there exists some $u \in (\theta, +\infty)$ with $0 < P(u) < \infty$. We proved in [51] two lemmas that give some properties of the function $P(t)$, similar to the classical notion of pressure for a dynamical system ([5], [71], etc.)

Lemma 6.3.1 ([51])**.** *The function $P(t)$ is strictly decreasing, convex and continuous on $(\theta, +\infty)$.*

We also showed that there exists a unique zero for the pressure function in this case; this zero will prove useful later when dealing with the dimension of the limit set.

Lemma 6.3.2 ([51])**.** *There exists a unique $h \in \mathbb{R}$ such that $P(h) = 0$. In addition $h \in (\theta, +\infty)$.*

6.4 Representations of real numbers in m-ary expansions, beta-expansions, Lüroth expansions, continued fractions, and f-expansions. Besicovitch-Eggleston sets.

The ergodic theory of numbers and that of representations of real numbers in various expansions is a well established and rich field, at the crossroads of number theory, real analysis and ergodic theory (see for instance [13], [25], [76], etc.)

In this section we will recall some properties of expansions, starting with well-known expansions such as the m-ary and β-expansions, and then expansions with infinitely many digit values, such as the continued fractions. At the end of this section we will recall also some interesting ergodic properties of f-expansions. We will recall also the notion of (quasi)-normal numbers for certain expansions, and that of Besicovitch-Eggleston sets.

Let us take an integer number $m \geq 2$; any number $x \in [0,1) \setminus \mathbb{Q}$ has a unique **m-ary expansion**:

$$x = \sum_{k=1}^{\infty} \frac{d_k(x)}{m^k}, \quad d_k(x) \in \{0, 1, \ldots, m-1\} \tag{82}$$

The coefficients $d_k(x)$ may also be denoted by d_k when no confusion arises, and are called **the digits** of x in the m-ary expansion; they are obtained from iterations of the piecewise linear map $T : [0,1) \to [0,1),\ T(x) = mx \pmod 1,\ x \in [0,1)$. Hence,

$$x = \frac{d_1(x)}{m} + \frac{Tx}{m} = \frac{d_1(x)}{m} + \frac{d_2(x)}{m^2} + \ldots + \frac{d_k(x)}{m^k} + \frac{T^k x}{m^k}$$

It is well-known that the Lebesgue measure λ is T-invariant, and that the random variables X_k given by the digits $d_k(\cdot),\ k \geq 1$ are independent identically distributed with respect to Lebesgue measure on $[0, 1)$ (see [13], etc.) Then, for any $j \in \{0, \dots, m-1\}$ it follows from the Strong Law of Large Numbers that Lebesgue-a.e number $x \in [0, 1)$ is *normal*, i.e for Lebesgue-a.e $x \in [0, 1)$ and $0 \leq j \leq m-1$, the asymptotic frequency of the digit j is in this case $\frac{1}{m}$,

$$\lim_{n\to\infty} \frac{1}{n} Card\{1 \leq k \leq n, d_k(x) = j\} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i(x) = \lambda(y \in [0, 1), d_1(y) = j) = \frac{1}{m}$$

Denote now by $\mathcal{N}(m)$ the set of normal numbers in the m-ary expansion.

Nevertheless, there are also other real numbers $x \in [0, 1) \setminus \mathbb{Q}$ which are **not** normal, but for which the asymptotic frequency p_j of digit j exists, for every $0 \leq j \leq m-1$; such numbers are called $(p_0, \dots, p_{m-1})$-quasinormal, and let us denote their set by

$$F(p_0, \dots, p_{m-1}) := \left\{x = \sum_{k\geq 1} \frac{d_k(x)}{m^k},\ \frac{Card\{j,\ 1 \leq j \leq n,\ d_j(x) = i\}}{n} \underset{n\to\infty}{\longrightarrow} p_i,\ 0 \leq i \leq m-1\right\}$$

Definition 6.4.1. *A set $F(p_0, \dots, p_{m-1})$ as above, is called a Besicovitch-Eggleston set associated to the frequency probability vector $(p_0, \dots, p_{m-1})$ in the m-ary expansion.*

From the work of Besicovitch (for the dyadic case) and Eggleston (for the general case), we have the dimension formula (see also [19]):

$$\dim_{\mathrm{H}}(F(p_0, \dots, p_{m-1})) = -\frac{1}{\log m} \sum_{0\leq j\leq m-1} p_j \log p_j$$

Also the sets $F(p_1, \dots, p_m)$ are dense in the unit interval, as can be seen since the frequencies are not affected by the first n digits, for any fixed n.

Let us denote by $\mathcal{Q}_m$ the union of all sets $F(p_0, \dots, p_{m-1})$, over all probability vectors $(p_0, \dots, p_{m-1})$; a number from $\mathcal{Q}_m$ is called *quasinormal* in the m-ary expansion.

Another expansion where digits take finitely many values, is the **beta-expansion** for $\beta > 1$, $\beta \notin \mathbb{Z}$. It was introduced by Rényi ([76]) and uses iterations of the map

$$T_\beta : [0, 1) \to [0, 1),\ T_\beta(x) = \beta x \pmod 1$$

In this case the main generating equation is $\beta x = T_\beta(x) + [T_\beta(x)]$, where $[y]$ and $\{y\} := y - [y]$, denote respectively the *integer part*, and the *fractional part* of the real number y. Thus one obtains the β-expansion of x,

$$x = \frac{d_1(x)}{\beta} + \frac{d_2(x)}{\beta^2} + \dots,$$

where the digits $d_i(x), i \geq 1$ may be denoted also by $d_i, i \geq 1$ and where $d_i \in \{0, \dots, [\beta]\}, i \geq 1$.

However as it can be seen, the map T_β does not preserve Lebesgue measure on $[0, 1)$, unlike the map T_m for $m \in \mathbb{Z}$. Rényi proved in [76] that there exists however a unique probability measure ν_β

equivalent to the Lebesgue measure, and which is T_β-invariant. The probability ν_β is ergodic with respect to T_β (see [76], [13]). Moreover Parry ([67]) gave an **explicit form** for the density function h_β of ν_β, $h_\beta(x) := \frac{d\nu_\beta}{d\lambda}(x)$ for Lebesgue-a.e x,

$$h_\beta(x) = \frac{1}{\mathcal{I}(\beta)} \sum_{n\geq 0,\ x<T_\beta^n 1} \frac{1}{\beta^n}, \text{ and } \mathcal{I}(\beta) = \int_0^1 \Big(\sum_{n\geq 0,\ x<T_\beta^n 1} \frac{1}{\beta^n}\Big)\, d\lambda(x) \tag{83}$$

From the Ergodic Theorem ([96]), it follows that for ν_β-a.e $x \in [0,1)$ (hence for Lebesgue-a.e $x \in [0,1)$) and for $j \in \{0, 1, \ldots, [\beta]\}$,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} Card\{1 \leq i \leq n,\ d_i = j\} &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{[\frac{j}{\beta}, \frac{j+1}{\beta})}(T_\beta^i(x)) \\ &= \nu_\beta([\frac{j}{\beta}, \frac{j+1}{\beta})) = \eta_j\end{aligned} \tag{84}$$

As before, we will denote the set of numbers $x \in [0,1)$ for which the above holds, by $\mathcal{N}(\beta)$ and call it the set of *normal* numbers for the β-expansion; $\mathcal{N}(\beta)$ has full Lebesgue measure in $[0,1)$. Hence for $x \in \mathcal{N}(\beta)$, the sequence $\omega = (d_1(x), d_2(x), \ldots)$ given by the digits of x, belongs to $A_\eta^{\mathbb{N}}$, for the stochastic vector $\eta = (\eta_0, \ldots, \eta_{[\beta]})$ specified in (84).

In general, for a stochastic vector $(p_0, \ldots, p_{[\beta]})$ we define the set of $(p_0, \ldots, p_{[\beta]})$-*quasinormal numbers*:

$$F(p_0, \ldots, p_{[\beta]}) := \{x = \sum_{k\geq 1} \frac{d_k(x)}{\beta^k} \ \in [0,1),\ \frac{Card\{j,\ 1 \leq j \leq n,\ d_j(x) = i\}}{n} \underset{n\to\infty}{\to} p_i,\ 0 \leq i \leq [\beta]\}$$

We obtain now the dimension of a Besicovitch-Eggleston set for the β-expansion $F(p_0, \ldots, p_{[\beta]})$, similarly as in case of the m-ary expansion.

Lemma 6.4.1 ([51])**.** *Consider a number $\beta > 1, \beta \notin \mathbb{Z}$ and a stochastic vector $(p_0, \ldots, p_{[\beta]})$. Then the set of $(p_0, \ldots, p_{[\beta]})$-quasinormal numbers for the β-expansion, has Hausdorff dimension*

$$dim_H(F(p_0, \ldots, p_{[\beta]})) = -\frac{\sum_{i=0}^{[\beta]} p_i \log p_i}{\log \beta - p_{[\beta]} \log\{\beta\}}$$

Proof. We use the mass distribution principle (see for eg. [19]), in order to obtain a probability measure and then to compare it with Lebesgue measure on basic intervals. Let us denote by $I_0 = [0, \frac{1}{\beta}), \ldots, I_{[\beta]} = [\beta, 1)$ the basic intervals for the expansive map T_β, and let $I_{i_1\ldots i_k} := \{x \in I_{i_1}, T^j(x) \in I_{i_j}, 1 \leq j \leq k\},\ k \geq 1$.

For a given $j \in \{0, \ldots, [\beta]\}$, let us define the random variables $X_i, i \geq 1$ defined by

$$X_i(x) = \begin{cases} 1, \text{ if } d_i(x) = j, \\ 0, \text{ if } d_i(x) \neq j \end{cases}$$

We will take now a probability measure μ which makes the random variables $X_i, i \geq 1$ independent. This is defined such that $\nu(I_{i_1 \ldots i_k}) = p_{i_1} \ldots p_{i_k}$ for any $i_1, \ldots, i_k$ and $k \geq 1$. For this probability we thus have a sequence of independent identically distributed random variables $X_i, i \geq 1$. From the Strong Law of Large Numbers it follows as in [19] that, if we denote by $n_j(x,k)$ the number of occurences of the digit j in the first k digits of x, then

$$\lim_{k\to\infty} \frac{n_j(x,k)}{k} = p_j,\ j = 0, \ldots, [\beta]$$

Hence we see that μ gives measure 1 to the set $F(p_0, \ldots, p_{[\beta]})$. Now let us notice that the length of the interval $I_{i_1 \ldots i_k}$ is given by: $\lambda(I_{i_1 \ldots i_k}) = \frac{\{\beta\}^{n_{[\beta]}(x,k)}}{\beta^k}$. So by denoting the interval $I_{i_1 \ldots i_k}$ containing x by $I_k(x)$, we have for any $\delta > 0$,

$$\frac{1}{k} \log \frac{P(I_k(x))}{\lambda(I_k(x))^\delta} \underset{k\to\infty}{\to} \sum_{0 \leq i \leq [\beta]} p_i \log p_i + \delta(\log \beta - p_{[\beta]} \log\{\beta\})$$

From the mass distribution principle ([19], [34]) it follows that the Hausdorff dimension is:

$$\dim_{\mathrm{H}}(F(p_0, \ldots, p_{[\beta]})) = -\frac{\sum\limits_{0\leq i\leq[\beta]} p_i \log p_i}{\log \beta - p_{[\beta]} \log\{\beta\}}$$

$\square$

Another important representation for real numbers, which now has an infinite set of possible digit values, is that in terms of *continued fractions* (see for eg. [13], [25], etc.)

In this case, $T : [0,1) \to [0,1)$ is given by the formula $T(x) = \frac{1}{x} - \left[\frac{1}{x}\right],\ x \neq 0,$ and $T(0) = 0$. Any irrational number $x \in [0,1)$ can be represented uniquely in its continued fraction form as

$$x = \cfrac{1}{d_1(x) + \frac{1}{d_2(x) + \frac{1}{d_3(x) + \ldots}}}$$

The integers $d_k(x)$ are called *the digits* of x in its continued fraction expansion and are defined by: $d_n(x) := \left[\frac{1}{T^{n-1}(x)}\right], n \geq 1$. The map T does not preserve the Lebesgue measure λ, but there exists a T-invariant ergodic measure μ_G, i.e the *Gauss measure* which is absolutely continuous with respect to λ (see for eg. [13]). It is well-known that the Gauss measure satisfies

$$\mu_G(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx$$

It follows that we can apply the Ergodic Theorem for T and μ_G, and obtain the set of normal numbers for the continued fraction expansion $\mathcal{N}_G \subset [0,1)$ which has μ_G-measure 1 (and thus also Lebesgue measure 1). For any $x \in \mathcal{N}_G$ and $k \geq 1$ we have

$$\frac{1}{n} Card\{1 \leq i \leq n, d_i(x) = k\} \underset{n\to\infty}{\to} \mu_G(x \in [0,1), d_1(x) = k) := p_{G,k}$$

Let us consider now an infinite stochastic vector $\eta = (\eta_1, \eta_2, \ldots)$ and consider, as in [30] the probability ν_η which makes the digits $\{X_i\}_i$ in the continued fraction expansion to be independent random variables. Namely if the probability measure μ is defined by $\mu(d_n = j) = \eta_j,\ \forall n, j \geq 1$, then the probability measure

$$\nu_\eta(A) := \mu(X \in A), \tag{85}$$

gives the distribution of the random variable $X = \frac{1}{d_1 + \frac{1}{d_2 + \frac{1}{\ddots}}}$.

The measure ν_η is useful in our case since it gives the asymptotic frequencies of appearance of digits for quasinormal numbers. Clearly ν_η is singular with respect to the Lebesgue measure if $\eta \neq \mathbf{p}_G := (p_{G,1}, p_{G,2}, \ldots)$. For a stochastic vector $\eta = (\eta_1, \eta_2, \ldots)$ define the set of η-*quasinormal numbers* $F(\eta)$ to be

$$F(\eta) := \{y \in [0,1), \frac{Card\{1 \leq i \leq n,\ d_i(y) = k\}}{n} \underset{k \to \infty}{\to} \eta_k,\ k \geq 1\}$$

Now, from the Ergodic Theorem applied to the T-invariant ergodic probability ν_η (or from the Strong Law of Large Numbers), we know that $\nu_\eta(F(\eta)) = 1$.

We denote by $\mathcal{Q}_G$ the union of all sets $F(\eta)$ over stochastic vectors η with $\left|\sum_{i \geq 1} \eta_i \log \eta_i\right| < \infty$, and call it the set of *quasinormal numbers* for the continued fraction expansion. Then, Kinney and Pitcher proved that the dimension of the probability ν_η is given by: $\dim_{\mathrm{H}}(\nu_\eta) = \frac{-\sum_{i \geq 1} \eta_i \log \eta_i}{2 \int_0^1 |\log x| \ d\nu_\eta(x)} > 0$, hence from the fact that $F(\eta)$ has ν_η-measure equal to 1, it follows that

$$\dim_{\mathrm{H}}(F(\eta)) > 0 \tag{86}$$

In [30], Kifer, Peres and Weiss proved that there exists a constant $\varepsilon_0 > 0$ with $\dim_{\mathrm{H}}(\nu_\eta) \leq 1 - \varepsilon_0$.

In the end of this section we shall recall also the general notion of f-**expansions**, introduced by Rényi in [76]. General f-expansions have been studied by several authors (see also [30]), and are important in the ergodic theory of numbers.

Let first an integer $M \geq 2$ or $M = \infty$, and assume f is either a strictly decreasing continuous function on $[1, M+1]$ with $f(1) = 1, f(M+1) = 0$, or f is strictly increasing continuous on $[0, M]$ with $f(0) = 0, f(M) = 1$. In f-expansion, one represents real numbers by a repeated application of f and extraction of integer parts at each step; clearly one has the identities $x = f(f^{-1}(x))$ and $f^{-1}(x) = [f^{-1}(x)] + \{f^{-1}(x)\}$. Thus one can define inductively the **digits** $d_k(x)$ and the remainders $r_k(x)$ of x:

$$d_0(x) = 0,\ r_0(x) = x,\ d_{k+1}(x) = [f^{-1}(r_k(x))],\ r_{k+1}(x) = \{f^{-1}(r_k(x))\},\ k \geq 0$$

Then for any $x \in [0,1)$, the series

$$f(d_1(x) + f(d_2(x) + f(d_3(x) + \cdots) \cdots))),$$

converges to x, if certain technical conditions are satisfied (see [76]) namely:

(C) in case f is strictly decreasing, there exists $\alpha \in (0,1)$ so that $|f(t) - f(s)| \le \alpha|t-s|$ if $1 + f(2) < s < t$ and $|f(t) - f(s)| \le |t-s|$ if $0 \le s < t$; and, in case f is strictly increasing, we should have $|f(t) - f(s)| < |t-s|, \ 0 \le s < t$.

Now consider the associated transformation of $[0,1)$,

$$T(x) := f^{-1}(x) - [f^{-1}(x)], \ x \in [0,1),$$

which is injective on the intervals $I_k := f(k, k+1), 1 \le k \le M$ when f decreases, or $0 \le k \le M-1$ when f increases.

The map T is assumed to satisfy the following regularity and expansiveness conditions ([97]):

i) $T|_{I_k}$ is $\mathcal{C}^2$, for all k;

ii) there exists some $p \in \mathbb{N}$ such that $\inf\limits_{x \in I_k, k \ge 1} |(T^p)'(x)| > 1$;

iii) $\sup\limits_{x,y,z \in I_k, k \ge 1} |\frac{T''(x)}{T'(y)T'(z)}| < \infty$.

Then by using the Perron-Frobenius operator, Walters showed in [97] that T has a *unique absolutely continuous invariant probability* μ_T, which is in fact the equilibrium measure of the potential $-\log|T'|$. Moreover, the probability μ_T is exponentially mixing. In particular when $f(x) = \frac{1}{x}$, we obtain the continued fraction expansion and its Gauss measure μ_G.

For a general f-expansion, consider now a potential ϕ on $\bigcup\limits_k I_k$, satisfying the following growth conditions:

a1) $\sum\limits_{y \in T^{-1}(x)} e^{\phi(y)} \le K_{T,\phi} < \infty, \ x \in \bigcup\limits_K I_k$;

a2) there exists some $\gamma > 0$ and constants $C_k, k \ge 1$ such that $|\phi(x) - \phi(y)| \le C_k|x-y|^\gamma, \ x, y \in I_k$ and $\sup\limits_{x \in I_k, k \ge 1} C_k|T'(x)|^{-\gamma} < \infty$.

Under conditions i), ii), iii), a1), a2), (C), it follows from [97] that there exists a unique T-invariant equilibrium measure μ_ϕ corresponding to ϕ, which is positive on open nonempty sets and has no atoms. Consider now the infinite stochastic vector

$$\eta(\phi) = (\mu_\phi(I_1), \mu_\phi(I_2), \ldots),$$

then from Birkhoff Ergodic Theorem applied to the transformation T and to the ergodic measure μ_ϕ, there must exist a Borel set $F(\phi) \subset [0,1)$ with $\mu_\phi(F(\phi)) = 1$, such that every $x \in F(\phi)$ has asymptotic frequency vector $\eta(\phi)$ in its f-expansion.

6.5 Connections between families of infinitely generated Moran fractals, and the ergodic theory of f-expansions.

In [51], we studied also connections between the families of non-stationary fractals constructed above in 6.3, and the ergodic theory of f-expansions; namely the relations between dimension for the above infinitely generated Moran fractals, and the speed of convergence in the asymptotic frequencies of digits in f-expansions with countably many digit values.

We were able to relate the Hausdorff dimension of the non-stationary Moran fractal constructed above with the pressure function, in the following:

Theorem 6.5.1 ([51]). *Let $E := E(\omega) \in \mathcal{M}(J, \mathbb{N}, \{\Phi_k\})$ for $\omega \in A_\eta^{\mathbb{N}}$, and consider $h \in (\theta, +\infty)$ to be the unique number which satisfies the equation $P(h) = 0$. Then*

$$dim_H(E) = h, \ \text{and } \mathcal{H}^h(E) < \infty$$

Moreover if $h = d$, then it follows that $\mathcal{H}^d(E) > 0$.

In the proof of the Theorem above, we used the following result about a natural dynamically defined distribution on the fractal:

Proposition 6.5.1 ([51]). *Let $h \in (\theta, +\infty)$ be determined by $P(h) = 0$. Then there exists a Borel probability measure μ_h supported on $\bar{E}$ such that for any $k \geq 1$ and $\sigma_0 \in D_k$,*

$$\mu_h(J_{\sigma_0}) = \frac{c_{\sigma_0}^h}{\sum_{\sigma \in D_k} c_\sigma^h}$$

In particular if k is large enough, then $\mu_h(J_\sigma) = c_\sigma^h = |J_\sigma|^h$, for $\sigma \in D_k$.

From Theorem 6.5.1 above we can conclude that the dimension of $E(\omega)$ depends only on the frequency vector η and on the contraction rates given by the infinite vectors $\Psi_i, 1 \leq i \leq m$, for any sequence ω belonging to the space $A_\eta^{\mathbb{N}}$.

We then proved in [51] that the Hausdorff dimension of $E(\omega)$, $\omega \in A_\eta^{\mathbb{N}}$, depends in fact *real analytically on the frequency vector $\eta = (\eta_1, \ldots, \eta_m)$*, in case the contraction rates are fixed.

Theorem 6.5.2 ([51]). *Let us fix the infinite positive vectors $\Psi_i, 1 \leq i \leq m$ satisfying (80), and denote by $\mathcal{V}$ the interior of the set $\{(\eta_1, \ldots, \eta_m) \in [0,1]^m, \ \eta_1 + \ldots + \eta_m = 1\}$. Consider now $\eta = (\eta_1, \ldots, \eta_m) \in \mathcal{V} \subset \mathbb{R}^m$ and $\omega \in A_\eta^{\mathbb{N}}$, and let $E(\omega)$ be an associated Moran fractal with contraction rates given by $\{\Psi_i\}_{1 \leq i \leq m}$ as in Section 6.3. Then, function $\Delta : \mathcal{V} \to \mathbb{R}$,*

$$\Delta(\eta_1, \ldots, \eta_m) := dim_H(E(\omega)),$$

is real analytic on $\mathcal{V}$.

From the above Theorem 6.5.2 and the **Lojaciewicz Vanishing Theorem**, we obtained then the following estimation from below of the dimension oscillation, in terms of distances to certain real-analytic subvarieties in $\mathbb{R}^m$:

Corollary 6.5.1 ([51]). *In the setting of Theorem 6.5.2, it follows that for any compact set $K \subset \mathcal{V} \subset \mathbb{R}^m$ there exists a constant $C > 0$ and an integer q depending on K, such that for any $\eta = (\eta_1, \ldots, \eta_m) \in K$ and any $\omega \in A_\eta^{\mathbb{N}}$ we obtain:*

$$|dim_H(E(\omega)) - dim_H(E(\rho^0))| \geq C \cdot dist(\eta, Z_{\rho^0})^q, \ \text{where}, \tag{87}$$

$$Z_{\rho^0} := \{\omega' \in \mathcal{V},\ \Delta(\omega') = dim_H(E(\rho^0))\}$$

is a real-analytic subvariety in $\mathbb{R}^m$*, for any fixed stochastic vector* $\rho^0 \in \mathcal{V}$*.*

We then studied the dependence of dimension of these Moran fractals on quasi-normal numbers. Fix first a finite collection of infinite positive vectors $\Psi_i = (\gamma_{i1}, \gamma_{i2}, \ldots), 0 \le i \le m-1$, satisfying the conditions (80) in the construction from Section 2, i.e

$$\sum_{j=1}^{\infty} \gamma_{ij} < \infty, 1 \le i \le m \text{ and } \sup\{\gamma_{ij}, 1 \le i \le m, j \in \mathbb{N}\} < 1$$

Given the infinite vectors $\Psi_i,\ 0 \le i \le m-1$ as above, we obtain then a functional $E(\cdot; m)$ assigning to every $x \in \mathcal{Q}_m$, a Moran set $E(x; m) \subset \mathbb{R}^d$ with infinitely many generators and non-stationary contractions as in Section 6.3, by using the sequence of digits $\omega = (d_1(x), d_2(x), \ldots)$. This gives a function describing the Hausdorff dimensions of these Moran sets,

$$H_m : \mathcal{Q}_m \to [0, \infty),\ H_m(x) := \dim_{\mathrm{H}}(E(x; m)),\ x \in \mathcal{Q}_m$$

Similarly, for β-expansions, given a collection of infinite vectors $\Psi_i = (\gamma_{i1}, \gamma_{i2}, \ldots), 0 \le i \le [\beta]$ satisfying (80) we construct infinitely generated non-stationary Moran fractals $E(x; \beta) \subset \mathbb{R}^d$ for every $x \in F(p_0, \ldots, p_{[\beta]})$, and obtain a dimension function

$$H_\beta : \mathcal{Q}_\beta \to [0, \infty),\ H_\beta(x) := \dim_{\mathrm{H}}(E(x; \beta))$$

The dependence of the dimension of the non-stationary fractal on the quasi-normal numbers is given by:

Theorem 6.5.3 ([51])**.** *Consider integer* $m \ge 2$ *and a collection of infinite vectors* $\Psi_i = (\gamma_{i1}, \ldots), 0 \le i \le m-1$ *satisfying (80), and consider the dimension function* $H_m(\cdot)$ *defined above for the associated infinitely generated non-stationary Moran fractals* $E(\cdot; m) \subset \mathbb{R}^d$ *.*

Then, H_m *is measurable, everywhere discontinuous on* $\mathcal{Q}_m$*, and it is constant on every set of positive dimension* $F(p_0, \ldots, p_{m-1})$*. Moreover for any* $x \in \mathcal{Q}_m$ *and any neighbourhood* U *of* x*, it follows that* H_m *attains all its possible values inside* U*, i.e*

$$H_m(\mathcal{Q}_m) = H_m(U \cap \mathcal{Q}_m)$$

Same conclusions hold, for the function H_β *associated to* $\beta > 1, \beta \notin \mathbb{Z}$ *and to a fixed collection of infinite vectors* $\Psi_i, 0 \le i \le [\beta]$ *satisfying (80).*

Then, we studied the problem of dimension for non-stationary Moran fractals determined by expansions with infinitely many digit values (for example continued fractions).

Let us consider now an infinite stochastic vector $\eta = (\eta_1, \eta_2, \ldots)$ and consider, as in [30] the probability ν_η which makes the digits $\{X_i\}_i$ in the continued fraction expansion to be independent random variables. Hence the probability μ is defined by $\mu(d_n = j) = \eta_j, \ \forall n, j \geq 1$, then the probability measure

$$\nu_\eta(A) := \mu(X \in A), \tag{88}$$

gives the distribution of the random variable $X = \frac{1}{d_1 + \frac{1}{d_2 + \frac{1}{\dots}}}$. The measure ν_η is useful in our case since it gives the asymptotic frequencies of appearance of digits for quasinormal numbers. Also it can be seen that ν_η is singular with respect to the Lebesgue measure if $\eta \neq \mathbf{p}_G := (p_{G,1}, p_{G,2}, \dots)$.

For a stochastic vector $\eta = (\eta_1, \eta_2, \dots)$ define the set of *η-quasinormal numbers* $F(\eta)$ to be

$$F(\eta) := \{y \in [0,1), \frac{Card\{1 \leq i \leq n, \ d_i(y) = k\}}{n} \underset{k\to\infty}{\to} \eta_k, \ k \geq 1\}$$

From the Ergodic Theorem applied to the T-invariant ergodic probability ν_η, we know that $\nu_\eta(F(\eta)) = 1$.

As before we denote by $\mathcal{Q}_G$ the union of all sets $F(\eta)$ over stochastic vectors η with $\left|\sum_{i\geq 1} \eta_i \log \eta_i\right| < \infty$, and call it the set of *quasinormal numbers* for the continued fraction expansion. In this setting, Kinney and Pitcher proved that the dimension of the probability ν_η is given by: $\dim_{\mathrm{H}}(\nu_\eta) = \frac{-\sum_{i\geq 1} \eta_i \log \eta_i}{2\int_0^1 |\log x| \ d\nu_\eta(x)} > 0$, hence from the fact that $F(\eta)$ has ν_η-measure equal to 1, it follows that $\dim_{\mathrm{H}}(F(\eta)) > 0$. Also, in [30] it was showed that there exists a constant $\varepsilon_0 > 0$ with $\dim_{\mathrm{H}}(\nu_\eta) \leq 1 - \varepsilon_0$.

We considered then a **countable collection** of fixed infinite vectors $\Psi_i = (\gamma_{i1}, \gamma_{i2}, \dots), i \geq 1$, satisfying the condition

$$\sup\{\sum_{j\geq 1} \gamma_{ij}, i \geq 1\} < \infty, \text{ and } \sup\{\gamma_{ij}, i, j \geq 1\} < 1 \tag{89}$$

For any stochastic infinite vector $\eta = (\eta_1, \eta_2, \dots)$ satisfying $\left|\sum_{i\geq 1} \eta_i \log \eta_i\right| < \infty$ and any quasinormal number $x \in F(\eta)$, construct an associated infinitely generated fractal $E(\omega) \subset \mathbb{R}^d$ with non-stationary contraction rates given by $(\Psi_i)_{i\geq 1}$ and η, where $\omega := (d_1(x), d_2(x), \dots)$. The set $E(\omega)$ will be denoted also by $E(x; G)$. We defined as before,

$$H_G : \mathcal{Q}_G \to [0, \infty), \ H_G(x) := \dim_{\mathrm{H}}(E(x; G)),$$

and proved a Theorem similar to the one in the β-expansion case:

Theorem 6.5.4 ([51]). *Consider a sequence of infinite positive vectors $\Psi_i, i \geq 1$ satisfying condition (89), and the associated infinitely generated non-stationary Moran fractals $E(x; G) \subset \mathbb{R}^d, \ x \in \mathcal{Q}_G$. Then the function H_G is constant on dense subsets of positive Hausdorff dimension, and for any $x \in \mathcal{Q}_G$ and any interval $U \subset [0,1), \ x \in U$, it follows that*

$$H_G(\mathcal{Q}_G) = H_G(U \cap \mathcal{Q}_G)$$

We finish this section with some results about dimension of non-stationary Moran fractals associated to general f-expansions; in this case, unlike in the cases discussed above, we do not have a specific form of the unique absolutely continuous invariant measure. Generalities about ergodic properties of f-expansions were given in Section 6.4.

Assume then, that the contraction vectors $\Psi_i = (\gamma_{i1}, \gamma_{i2}, \ldots),\ i \geq 1$ are *fixed.* As in Section 6.3, we obtain for $x \in F(\phi)$ a *family of non-stationary Moran fractals* $E(x; f, \phi) \subset \mathbb{R}^d$, constructed by using the digits of x in its f-expansion. From Theorem 6.5.1, for $x \in F(\phi)$, the Hausdorff dimension of $E(x; f, \phi)$ is equal to the unique number $h(\phi)$ satisfying:

$$\sum_{i \geq 1} \mu_\phi(I_i) \cdot \log \sum_{j \geq 1} \gamma_{ij}^{h(\phi)} = 0 \tag{90}$$

In general, for *any* stochastic vector $\eta = (\eta_1, \eta_2, \ldots)$, there exists a probability measure ν_η as follows: first consider the measure μ_η for which $d_i(\cdot)$ are i.i.d. random variables and $\mu_\eta(d_n(\cdot) = k) = \eta_k,\ \forall n, k \geq 1$; then the measure ν_η is defined as the μ_η-distribution of the random variable $X,\ X(\cdot) := f(d_1(\cdot) + f(d_2(\cdot) + f(d_3(\cdot) + \ldots)))$. From the definition of ν_η, for any arbitrary Borel set $B \subset [0, 1)$, we have $\nu_\eta(B) = \mu_\eta(X \in B)$. The digits $d_i(\cdot)$ in the f-expansion, become independent random variables with respect to ν_η.

By fixing the contraction vectors, and by taking as stochastic asymptotic frequency vector $\eta = (\mu_\phi(I_1), \mu_\phi(I_2), \ldots)$, we obtained in [51] a dimension function $H_f(x) = \dim_{\mathrm{H}}(E(x; f))$ on the set of quasinormal numbers x in the f-expansion, that can be studied as before.

7 Plans for future research.

We will continue to investigate several directions in smooth ergodic theory and the thermodynamic formalism of systems with certain degrees of hyperbolicity, and mainly for systems with overlaps (self-intersections). This theory is very rich and has connections with several other fields in mathematics. Thermodynamic formalism for non-invertible systems is different both in methods and in results from the case of diffeomorphisms

Some of our concrete objectives in the future are: Entropy production for various invariant probability measures in the context of non-invertible hyperbolic dynamics. Pointwise dimensions for equilibrium measures on fractal saddle sets. Hausdorff dimension for non-conformal basic sets. Conformal iterated function systems with overlaps, and applications. Estimation of topological entropy for dynamical systems associated to approximation operators. Examples of endomorphisms far from diffeomorphisms, and invariant measures with non-zero Lyapunov exponents. Conformal measures on saddle folded fractals. Perron-Frobenius operators. Connections with ergodic number theory. Applications in economics, statistical physics, chaos theory, etc.

The elements of originality and innovation in our approach will be: Relating the entropy production of equilibrium measures of arbitrary Holder potentials for smooth non-invertible systems on a notion of asymptotic logarithmic degree which, instead of considering all n-preimages of a point in the basic set through the function, considers only those good n-preimages from the point of view of the measure. We plan to find also concrete examples of invariant measures for endomorphisms for which we can determine whether the entropy production is positive or not. This is a central problem in the theory of entropy production (see also Ruelle [79])). We plan to investigate the entropy production of inverse SRB measures (introduced for non-invertible repellers in [46]).

The study of pointwise dimension for equilibrium measures on folded fractals of saddle type. This is a new approach, until now there were studied measures invariated by diffeomorphisms on surfaces, probabilities with positive Lyapunov exponent preserved by rational maps or the case of the measure of maximal entropy $T \wedge T$ for polynomial endomorphisms. Our approach will however be completely different and will use some deep properties of equilibrium measures on folded basic sets of saddle type, in the sense of estimations of measure of various intersections of unstable tubular neighbourhoods.

We also plan to show that certain invariant measures obtained as wedge products of positive closed currents on minimal/terminal sets are in fact equilibrium (Gibbs) measures, and to estimate their pointwise and Hausdorff dimensions. We plan to extend also the results about conditional measures of certain equilibrium states.

The study of non-conformal fractal sets presents also originality and difficulties; so far the conformal case received much more attention. In the non-conformal case the Hausdorff dimension and the box dimension may differ, and new interesting phenomena appear. We plan to investigate these issues in the setting of non-conformal iterated function systems, and in the case of non-conformal repellers with overlaps.

Our research will investigate an area with great potential in smooth ergodic theory, namely that of the thermodynamics of systems where we do not have the usual stable and unstable foliations from the case of Anosov or Axiom A diffeomorphisms, and especially the thermodynamics of non-invertible systems with a certain degree of uniform or non-uniform hyperbolicity.

We will investigate new research directions, like the entropy production for non-reversible systems, conformal finite or infinite iterated function systems with overlaps, thermodynamic formalism for endomorphisms with various levels of hyperbolicity (or non-uniform hyperbolicity), Hausdoff dimension in the context of iterated function systems with overlaps, estimations of topological entropy for chaotic dynamical systems given by approximation operators (like those obtained from Newton's method), and applications of the above in chaos theory, economics, statistical physics. Thus there will be forged new connections between these fields and we shall investigate also several classes of examples of non-reversible systems and measures for them.

In the area of entropy production for non-invertible systems we will investigate new connections between entropy production, folding entropy and a notion of asymptotic logarithmic degree associated to an equilibrium measure for a Holder potential. Also in the study of pointwise dimension of equilibrium measures on folded fractals, we plan to open new lines of research for instance by investigating this notion for equilibrium measures supported on minimal basic sets for endomorphisms on Riemannian manifolds, and also to make connections with the theory of positive, closed currents.

In the field of iterated function systems most of the research so far was by using the Open Set Condition or its variants or to show that certain classes of systems fall close to this situation. In our research we want to explore iterated function systems where it is not possible to get rid of overlaps, instead they influence strongly the dynamics and the dimension of the limit set. These iterated function systems with overlaps present the potential for new phenomena and a rich theory, different than in the case of iterated function systems with Open Set Condition (or its variants).

Collaborations and advising.

In the future I plan to continue my ongoing collaborations with several well-known mathematicians, M. Urbanski (North Texas, USA), J. E Fornaess (now in Trondheim, Norway), B. Stratmann (Bremen, Germany), M. Roychowdhury (Texas, USA), etc.

In addition I plan to start new research projects with the above mathematicians and also with others. This will include collaborative work, mutual research visits and participation in important conferences in the field. I also plan to work in dynamical systems and thermodynamic formalism with professors at the Institut des Hautes Études Sciéntifiques (IHÉS)-Paris, in the Fall of 2013 and Summer of 2014.

We plan to publish research papers on subjects of high current interest, in which we will use sophisticated methods, not only from a very narrow subfield, but instead various methods and ideas from several fields. We plan to publish in high caliber journals such as Advances in Math., Ergodic Theory and Dynam. Syst., J. Stat. Physics, International Math Res Notices, Math. Annalen, Commun. Math. Physics, Inventiones Math., Discrete and Cont. Dynam. Syst., Math Proceed Cambridge Phil Soc., Commun Contemp Math, Bulletin and Journal of London Math Soc, Dynam Systems, etc.

From the point of view of advising and teaching advanced courses, I plan to continue to teach Masters courses in Dynamical Systems and Ergodic Theory, at the Normal Superior School of Bucharest (SNSB), and possibly also at Univ. of Bucharest.

I also plan to attract students who want to pursue a PH.D in Dynamical Systems and Ergodic Theory at IMAR. One possibility is to attract students from the Normal Superior School of Bucharest through courses and seminars. I also plan to have common research projects with postdocs and PH.D students at Univ. of Bremen, who may visit me at IMAR, and I could visit them in order to work together.

Jointly with B. Stratmann (Germany) and B. Schapira (France), I will also organize a conference on Hyperbolic Dynamics, Thermodynamic Formalism and Stable Foliations, at Mathematische ForschungsInstitut Oberwolfach, Germany, in 2014.

In addition, I plan to organize in the future a conference in Dynamical Systems and Smooth Ergodic Theory in Romania too, in case there will be enough funds available.

I am also currently preparing a book on Dimension Theory for Folded Dynamical Systems.

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