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Relations between stable dimension and the preimage counting function on basic sets with overlaps

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Abstract

In this paper we study non-invertible hyperbolic maps f and the relation between the stable dimension (that is, the Hausdorff dimension of the intersection between local stable manifolds of f and a given basic set Λ) and the preimage counting function of the map f restricted to the fractal set Λ. The case of diffeomorphisms on surfaces was considered in [A. Manning and H. McCluskey, 'Hausdorff dimension for horseshoes', *Ergodic Theory Dynam. Systems* 3 (1983) 251–260], where thermodynamic formalism was used to study the stable/unstable dimensions. In the case of endomorphisms, the non-invertibility generates new phenomena and new difficulties due to the overlappings coming from the different preimages of points, and also due to the variations of the number of preimages belonging to Λ (when compared with [E. Mihailescu and M. Urbanski, 'Estimates for the stable dimension for holomorphic maps', *Houston J. Math.* 31 (2005) 367–389]). We show that, if the number of preimages belonging to Λ of any point is less than or equal to a continuous function $\omega(\cdot)$ on Λ, then the stable dimension at every point is greater than or equal to the zero of the pressure function $t \to P(t\Phi^s - \log \omega(\cdot))$. As a consequence we obtain that, if d is the maximum value of the preimage counting function on Λ and if there exists $x \in \Lambda$ with the stable dimension at x equal to the zero t_d of the pressure function $t \to P(t\Phi^s - \log d)$, then the number of preimages in Λ of any point y is equal to d, and the stable dimension is t_d everywhere on Λ. This has further consequences to estimating the stable dimension for non-invertible skew products with overlaps in fibers.

1. *Introduction*

The relations between Hausdorff dimension and the zero of topological pressure have been first found in the case of rational maps of one variable, by Bowen [3] and Ruelle [11].

Theorem A (Ruelle). *Let f be a rational map that is hyperbolic on its Julia set $J(f)$. Then the Hausdorff dimension of $J(f)$ is equal to the zero of the pressure function $t \to P(t\Phi^u)$, where $\Phi^u(z) := -\log |Df(z)|$, with $z \in J(f)$. In particular the Hausdorff dimension of the Julia set depends real analytically on parameters when the parameters (that is, the map f) are perturbed holomorphically.*

Furthermore, for surface diffeomorphisms, Manning and McCluskey [5] proved the following theorem.

Theorem B (Manning–McCluskey). *Let Λ be a basic set for a $\mathcal{C}^1$ axiom A diffeomorphism $f : M^2 \to M^2$ with a $(1, 1)$ splitting $T_\Lambda = E^s \oplus E^u$. Then $\mathrm{HD}(W^s(x) \cap \Lambda) = t_s$ and $\mathrm{HD}(W^u(x) \cap \Lambda) = t_u$, where t_s and t_u are the unique zeros of the pressure functions*

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$t \to P(t\Phi^s)$ and $t \to P(t\Phi^u)$, respectively. Moreover, t_s depends continuously on f in the $\mathcal{C}^1$ topology on diffeomorphisms.

In [**8**], Mihailescu and Urbanski studied the Hausdorff dimension of the intersection between local stable manifolds and basic sets for *non-invertible* holomorphic maps of several variables. Here the multidimensional setting and the fact that the map is non-invertible generate new phenomena and obstacles. In [**13**], Simon studied a certain class of skew products exhibiting a type of transversality condition giving that the attractor Λ is the union of smooth curves that intersect each other in at most one point and that at this point the angle between their tangents is greater than a positive constant, if their first preimages are different.

Transversality-type conditions were studied also in [**14**]. In [**9**] we introduced a different form of transversality, for parametrized families of skew products in order to prove a Bowen-type formula for the stable dimension for almost all parameters. In that paper there are many examples that satisfy this transversality condition including some skew products with iterated function systems (IFS) in their base and examples from higher-dimensional complex dynamics. However, we do not know if transversality (in any form) is generic in some way. Also, in [**12**], Schmeling studied attractors for the Belykh family depending on three parameters; there exists an open subset of parameters for which the corresponding maps are not injective and we have a bifurcation picture of invertibility according to the parameters. There are also many other examples of hyperbolic non-invertible maps; for instance, holomorphic maps on $\mathbb{P}^2\mathbb{C}$ obtained from perturbations of hyperbolic product maps $(P(z), Q(w))$, skew products $(P(z), Q(z,w))$ (we shall talk about these in the end), solenoids with overlaps or the family of horseshoes with overlaps introduced by Bothe [**1**]. Bothe proved in fact that the set of such non-invertible horseshoes with overlaps has non-empty interior in some sense.

This paper answers to the case when the transversality condition is not present or, even if it is present, to the case of those parameters for which we do not necessarily have a Bowen-type equation for the stable dimension. In particular our work gives estimates for the stable dimension based on the number of preimages that points in the basic set Λ have in Λ. This allows us flexibility in choosing continuous functions that bound the number of preimages and thus, it allows using thermodynamical formalism of equilibrium states (see [**2**, **4**] for background) in Corollary 1 in order to prove a rigidity-type result about the stable dimension.

Moreover, in this paper we *do not* assume in general that Λ is an attractor (unlike in [**13**] or [**12**]); instead Λ is just a *basic set* (as defined below). First we recall some definitions.

Definition 1. (a) Let X be a compact metric space and let $f : X \to X$ be a continuous map. For a point x from X, we say that a point $y \in X$ is an *f-preimage* of x if $f(y) = x$; we call such a point y also a *1-preimage* of x. If $f^k(z) = x$ for some $z \in X$ and $k \geqslant 1$, then we say that z is a *k-preimage* of x.

(b) We say that a finite sequence $C = (x, x_{-1}, \ldots, x_{-n})$, with $n \geqslant 1$ is a *finite prehistory* of x if $f(x_{-n}) = x_{-n+1}, \ldots, f(x_{-1}) = x$; in this case we say that n is the length of C and x_{-n} will be called the *final preimage* associated to C.

(c) We say that an infinite sequence $C = (x, x_{-1}, \ldots)$ is a *full prehistory* (or simply a *prehistory*) of x if we have $f(x_{-i-1}) = x_{-i}$, with $i \geqslant 0$. We assume that notationally $x = x_0$.

(d) A full prehistory of x will also be denoted by $\hat{x} = (x, x_{-1}, x_{-2}, \ldots)$. The space of all prehistories from X is denoted by $\hat{X}$ and we have the shift map $\hat{f}(\hat{x}) = (f(x), x, x_{-1}, \ldots)$. The map $\hat{f}$ is a homeomorphism on $\hat{X}$. The pair $(\hat{X}, \hat{f})$ is called *the natural extension* (or *inverse limit*) of (X, f).

It can be remarked that $\hat{X}$ has a compact metric space structure (see [**6**] for more on these notions).

DEFINITION 2. (a) Let $f : U \to M$ be a smooth (say $\mathcal{C}^2$) map defined on an open set U in a smooth Riemannian manifold M. Consider also a *basic set* Λ for f, that is, a compact subset of U with the following properties:

(1) $f(\Lambda) = \Lambda$ and f is transitive on Λ;
(2) there exists an open neighborhood V of Λ such that $\Lambda = \cap_{n\in\mathbb{Z}} f^n(V)$.

(b) We say that f is *hyperbolic* on Λ if there exists a continuous splitting of the tangent bundle over $\hat{\Lambda}$, that is $T_{\hat{\Lambda}}(M)$, as $T_{\hat{x}}M = E^s_x \oplus E^u_{\hat{x}}$, where $T_{\hat{x}}M = \{(\hat{x}, v), v \in T_x M\}$; the subspaces E^s_x and $E^u_{\hat{x}}$ are invariant, that is, $Df_x(E^s_x) \subset E^s_{fx}$ and $Df_x(E^u_{\hat{x}}) \subset E^u_{\hat{f}\hat{x}}$, and the derivative of f contracts and expands uniformly on E^s_x and $E^u_{\hat{x}}$, respectively.

(c) If f is hyperbolic on Λ, then there exist *local stable and local unstable manifolds*, namely $W^s_r(x, f) := \{y \in U, d(f^i y, f^i x) \leqslant r, i \geqslant 0\}$ and $W^u_r(\hat{x}, f) = \{y \in U$, there exists a full prehistory of y in $\Lambda, \hat{y}$, s.t. $d(y_{-j}, x_{-j}) \leqslant r, j \geqslant 0\}$, respectively. They will also be denoted by $W^s_r(x)$ and $W^u_r(\hat{x})$, respectively, when no confusion arises on f.

(d) By *stable dimension* at $x \in \Lambda$ we mean the Hausdorff dimension $\mathrm{HD}(W^s_r(x) \cap \Lambda)$; it is denoted by $\delta^s(x)$.

The sets $W^s_r(x)$ and $W^u_r(\hat{x})$ have indeed the structure of manifolds of dimensions equal to the respective dimensions of E^s_x and $E^u_{\hat{x}}$; the local unstable manifolds depend in general on the whole prehistories, whereas the local stable manifolds depend only on their base point. In general for a non-invertible map we may have infinitely many local unstable manifolds passing through a point $x \in \Lambda$ (as was proved in [**10**]); this complicates the situation further.

Also, we note that we work with general basic sets as defined above, that is, intersections of $f^n(V)$ for all $n \in \mathbb{Z}$, and not just with attractors (which require only intersections of $f^n(V)$ for $n \geqslant 0$). Clearly, any attractor Λ, for which there exists a neighborhood V such that $f(V) \subset V$, is also a basic set.

Coming back to the hyperbolic non-invertible higher-dimensional case, in [**8**] we have shown the following theorem.

THEOREM C. *Assume that f is a holomorphic endomorphism on $\mathbb{P}^2\mathbb{C}$ and that f is hyperbolic on a basic set Λ of unstable index* 1; *suppose also that the critical set of f, denoted by $\mathcal{C}_f$, does not intersect Λ, and that each point x from Λ has at least d f-preimages in Λ. Then* $\mathrm{HD}(W^s_r(x) \cap \Lambda) \leqslant t^s_0$, *where t^s_0 is the unique zero of the function $t \to P(t \log |Df_s(y) - \log d)$. Therefore this estimate is independent of $x \in \Lambda$.*

In the conformal hyperbolic non-invertible case (for instance, for hyperbolic holomorphic maps on projective spaces), the situation is different from that in the diffeomorphism case; this is due to the non-existence of inverse iterates and also to the fact that when taking forward images of balls centered at different preimages of the same point, these images may overlap. Also since the number of preimages belonging to Λ of a point x can vary when x ranges in Λ, it follows that the multiplicities of covers from [**8**] are not constant and hence we cannot apply the successive elimination process from [**8**].

To illustrate some of the new phenomena/difficulties that appear in the non-invertible case, in [**8**] we have proved the following theorem.

THEOREM D (Behavior of endomorphisms at perturbation). *Given the map* $f_\varepsilon(z, w) = (z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2, w^2)$, *there exist small positive constants $c(a, b, d, e)$ and $\varepsilon(a, b, c, d, e)$ such that, for $b \neq 0, 0 \neq |c| < c(a, b, d, e)$ and $0 < \varepsilon < \varepsilon(a, b, c, d, e)$, we have that f_ε is injective on its basic set Λ_ε close to $\Lambda := \{p_0(c)\} \times S^1$ (where $p_0(c)$ is the attracting fixed point for $z \to z^2 + c$).*

In particular there exists a positive constant $\alpha(c)$ *such that* $\mathrm{HD}(W^s_r(y, f_\varepsilon) \cap \Lambda_\varepsilon) > \alpha(c)$ *for all* $\varepsilon > 0$ *small enough and all* $y \in \Lambda_\varepsilon$.

This result implies that the stable dimension for f_ε does not depend real analytically (not even continuously) on the parameters when we perturb the map $f(z, w) = (z^2 + c, w^2)$, since the stable dimension of f relative to Λ is equal to zero (as the intersection $W^s_r(x, f) \cap \Lambda$ consists of only one point). However, as Theorem D proves, $\mathrm{HD}(W^s_r(y, f_\varepsilon) \cap \Lambda_\varepsilon) > \alpha(c) > 0$, for all $\varepsilon > 0$ small.

Therefore, for non-invertible maps the situation is significantly different and the methods from the one variable case or from the diffeomorphism case do not apply in general.

One must be careful also about the different preimages belonging to Λ, whose number may vary. Locally near Λ a point x may have a constant number of f-preimages, but some of these preimages may not be in Λ. However, for the estimate of stable dimension we need only those preimages from Λ, since Λ is f-invariant.

In what follows we employ maps of the following type.

Definition 3. Let M be a smooth (say $\mathcal{C}^2$) Riemannian manifold and let $f : U \to M$ be a smooth finite-to-one map defined on an open set U of M. Assume that f is hyperbolic on the basic set $\Lambda \subset U$ and that f is conformal on stable manifolds. Also suppose that the critical set $\mathcal{C}_f$ of f does not intersect Λ. We then say that f is a *c-hyperbolic map* on Λ.

Definition 4. Let f be a c-hyperbolic function on a basic set Λ, and let an arbitrary point x from Λ. We denote by $d(x)$ the number of f-preimages of x belonging to Λ and the function $d(\cdot)$ will be called the *preimage counting function* on Λ.

We remark that the number of preimages $d(x)$ may vary when x ranges in Λ. This brings, as we mentioned, additional significant difficulties for the estimate of the stable dimension.

In Theorem 1 we will prove that if the function $d(\cdot)$ is less than or equal to a locally constant function $\omega(\cdot)$ on Λ, then the stable dimension $\delta^s(x)$ at any point $x \in \Lambda$ is greater than or equal to the unique zero t_ω of the pressure function $t \to P(t\Phi^s - \log \omega)$, where $\Phi^s(y) := \log |Df_s(y)|$, with $y \in \Lambda$.

We refine this result in Theorem 2 for the case when ω is any continuous function on Λ; this is a large extension of the class of maps for which we can estimate the stable dimension and it improves the estimate from [**8**].

Then, in Corollary 1, we will prove that if there exists at least a point $x \in \Lambda$ where the stable dimension $\delta^s(x)$ is equal to the unique zero t_d of the pressure function $t \to P(t\Phi^s - \log d)$, where d is the maximum value for $d(\cdot)$ on Λ, then $d(\cdot)$ is identically equal to d on Λ, and the stable dimension will be t_d everywhere on Λ. In Corollary 2 we obtain an estimate for the stable dimension of fractal sets in the fibers of some non-invertible hyperbolic skew products, having finite IFS in the base, and related to [**9**]. Furthermore, in Corollaries 3 and 4 we give cases when the stable dimension is non-zero.

2. *Main results for the non-invertible case*

For the rest of the paper, we work with a c-hyperbolic mapping f on a basic set Λ. We recall that by $d(x)$ we denoted the number of f-preimages of x belonging to the fixed basic set Λ and $d(\cdot)$ is called the *preimage counting function* associated to f and Λ. It is important to first know some simple topological properties of $d(\cdot)$.

Lemma 1. *Let f be a c-hyperbolic map on a basic set Λ. Then the preimage counting function $d(\cdot)$ is upper semicontinuous and bounded on Λ.*

Proof. Indeed we take a point $x \in \Lambda$ and a sequence x_n converging toward x in Λ. Then let an integer value d' be such that, for any n large enough, there are at least d' f-preimages of x_n denoted by $y(n,1), \dots, y(n,d')$ in Λ. By taking eventually a subsequence of $(x_n)_n$, it happens that the respective preimages will accumulate to certain points $y(1), \dots, y(d')$ in Λ. Also, since the critical set $\mathcal{C}_f$ does not intersect Λ, it follows that there exists a positive ε_0 such that the mutual distances between $y(1), \dots, y(d')$ are greater than ε_0. Now, since f is continuous on Λ, it follows that $y(1), \dots y(d')$ are different preimages of x, and hence $d(x) \geqslant d'$. Since this is true for any subsequence $(x_n)_n$ converging to x, it implies that $d(\cdot)$ is upper semicontinuous on Λ. Furthermore, since Λ is compact, this means that $d(\cdot)$ is bounded. □

We will now prove Theorem 1 of the paper about the case when the preimage counting function is bounded above by a locally constant function. After this we shall state and also prove the theorem in the case when the preimage counting function is bounded above by a continuous function; the idea of proof is essentially the same, but in the first case it is easier to see the method of proof.

Theorem 1. *Let a smooth function $f : U \to M$ be defined on an open set of a smooth Riemannian manifold M and assume that f is c-hyperbolic on a basic set $\Lambda \subset U$. Assume that there exists a locally constant function ω on Λ such that $d(x) \leqslant \omega(x)$, with $x \in \Lambda$. Then $\delta^s(x) \geqslant t_\omega$, where t_ω is the unique zero of the pressure function $t \to P(t\Phi^s - \log \omega)$.*

Proof. We fix a point $x \in \Lambda$ and define $W := W^s_r(x) \cap \Lambda$. Also let $\varepsilon > 0$ be small. Since Λ is compact, it follows that we can cover it with a finite number of balls $B(z_1, \varepsilon/2), \dots, B(z_k, \varepsilon/2)$. From the transitivity property of f on Λ, it follows that, for all $j \in \{1, \dots k\}$, there exists $m_j = m_j(\varepsilon)$ such that any local unstable manifold of type $W^u_\varepsilon(\hat{y})$ intersects the set $f^{-m_j}(W) \cap \Lambda$ for all $\hat{y} \in \hat{\Lambda}$ and $y \in B(z_j, \varepsilon/2)$.

Since f is locally bi-Lipschitz near Λ (f being smooth), we obtain that $\mathrm{HD}(W) = \mathrm{HD}(f^{-m_j} W \cap \Lambda)$. We take an arbitrary number $t > \delta^s(x)$; then there exists a covering $\{U_i\}_{i \in I_j}$ of $f^{-m_j} W \cap \Lambda$ such that

$$\sum_{i \in I_j} (\mathrm{diam}\ U_i)^t < \frac{1}{2k}. \tag{1}$$

Then we consider the union $I := \cup_{j=1}^k I_j$. Thus we obtain a collection of sets U_i, with $i \in I$ such that any local unstable manifold $W^u_\varepsilon(\hat{y})$ intersects at least one such U_i and from (1) we obtain

$$\sum_{i \in I} (\mathrm{diam}\ U_i)^t < \frac{1}{2}. \tag{2}$$

Now consider $i \in I$ and suppose that $\mathrm{diam}\ U_i > 0$. We can assume in fact that U_i is contained in a local stable manifold. We introduce a type of tubular unstable set used in [**7**] for the inverse pressure: for a finite prehistory $C = (x, x_{-1}, \dots, x_{-n})$ of x in Λ, we define

$$\Lambda(C, \varepsilon) := \{y \in U, \text{there exists a prehistory of } y, (y, y_{-1}, \dots, y_{-n}), \text{s.t. } d(y_{-j}, x_{-j}) < \varepsilon, j = 0, \dots, n\}.$$

By *stable diameter* of $\Lambda(C, \varepsilon)$ we understand the diameter of the intersection $\Lambda(C, \varepsilon) \cap W^s_r(x)$.

We now detail how to take some special prehistories C of points in U_i. For a point $y \in U_i$, consider a prehistory C of y in Λ of length n such that if $C = (y, \dots, y_{-n})$, then n is the largest integer such that $\varepsilon|Df_s^n(y_{-n})| > \text{diam } U_i$. We call such a prehistory C a *maximal prehistory* relative to U_i and its length will also be denoted by $n(C)$. Obviously we cannot have just any length for such a maximal prehistory, and hence we denote by $n_{i1}, \dots, n_{iq_i}$ all the different lengths of U_i-maximal prehistories. From construction it is clear that $U_i \subset \Lambda(C, \varepsilon)$ for C as above.

Now we denote the set of U_i-maximal prehistories by $\mathcal{C}_i$ and we assume that $\mathcal{F}_i$ is a minimal set of points of type $y_{-n(C)}$ for $C \in \mathcal{C}_i$ such that, for any $C \in \mathcal{C}_i$, there exists $z \in \mathcal{F}_i$ with $y_{-n(C)} \in B_{n(C)}(z, \varepsilon)$ (where in general $B_m(z, \varepsilon)$ denotes the Bowen ball, that is, the set of points whose orbits are within ε distance of the orbit of z up to order m).

Denote the corresponding set of prehistories from $\mathcal{C}_i$ ending with the points of $\mathcal{F}_i$, by $\mathcal{C}_i^*$. Hence $\mathcal{C}_i^* \subset \mathcal{C}_i$, with $i \in I$. If $z \in \mathcal{F}_i$, then we also denote by $n(z)$ the length of the corresponding prehistory $C \in \mathcal{C}_i^*$ having z as final preimage.

Without loss of generality we may assume that the preimage counting function is equal to ω and thus locally constant, this giving in fact the case when the stable dimension is minimal under the assumption $d(\cdot) \leqslant \omega(\cdot)$ on Λ . Since $d(\cdot)$ takes only finitely many values on Λ, we denote them by $d_1, \dots, d_p$. In this setting, denote by $V_j := \{z \in \Lambda, d(z) = d_j\}$, with $j = 1, \dots, p$; thus these sets are closed and mutually disjoint. In general, the sets V_j may be taken to be the level sets of the locally constant map ω. Assume that $d(V_j, V_k) > \varepsilon_0 > 0$, with $j \neq k$, for some positive constant ε_0. Since the critical set of f does not intersect Λ, it follows that different f-preimages of any arbitrary point $x \in \Lambda$ are at a positive distance apart; this distance may be assumed to be greater than ε_0 too.

We take now a point $\xi \in V_1$, and hence ξ has d_1 f-preimages denoted by $\xi_1, \dots, \xi_{d_1}$. These are simple preimages due to the fact that $\mathcal{C}_f \cap \Lambda = \emptyset$. Assume that there exists a sequence of points y from Λ that converges toward ξ, and let $y_1, \dots, y_{d_1}$ be the d_1 preimages of y. Also assume that $d(\{y_1, \dots, y_{d_1}\}, \{\xi_1, \dots, \xi_{d_1}\}) > \alpha > 0$, for all points y in this sequence. Then the points $y_1, \dots, y_{d_1}$ accumulate (eventually for a subsequence) to some points $y_1^*, \dots, y_{d_1}^*$ that are preimages of ξ. However, due to the condition on the distances between the sets of preimages, it follows that there exists at least a point y_j^* which is not in the set $\{\xi_1, \dots, \xi_{d_1}\}$. This then implies that ξ has more than d_1 preimages in Λ, thus giving a contradiction.

Thus each point $\xi \in \Lambda$ has a neighborhood $V(\xi)$ such that any point $y \in V(\xi)$ has d_1 preimages in Λ close to the preimages $\xi_1, \dots, \xi_{d_1}$ of ξ. Now, if, for any $\eta > 0, \eta \ll \varepsilon_0$, there exists a point $y(\eta) \in \Lambda$ such that there exists a point $z(\eta) \in B(y(\eta), \eta)$ with the preimages of $z(\eta)$ in Λ far from the preimages of $y(\eta)$ in Λ, then we can take a subsequence of $(y(\eta))_{\eta>0}$ converging toward a point $w \in \Lambda$ that has the property that in any neighborhood there are points $z(\eta)$ with preimages far from the preimages of w, which is a contradiction with the fact proved earlier. Thus there exists a positive ε_1 such that if $d(y, z) < \varepsilon_1$, then the preimages of y in Λ are close (that is, closer than $d(y, z) \cdot \sup_\Lambda |Df_s|^{-1}$) to the preimages of z in Λ. In this we have implicitly used the fact that the preimages of any point from Λ have multiplicity 1, since $\mathcal{C}_f \cap \Lambda = \emptyset$.

In particular, for $C \in \mathcal{C}_i, C = (y, \dots, y_{-n(C)})$, and $z \in B_{n(C)}(y_{-n(C)}, \varepsilon)$ we have that $f^k(z)$ has the same number of f-preimages in Λ as $f^k(y_{-n(C)})$ and, moreover, these preimages are close to the f-preimages of $f^k(y_{-n(C)})$, for $k = 0, \dots, n(C)$ (namely $\varepsilon \sup_\Lambda |Df_s|^{-1}$-close).

Now consider the set of points of the form $y_{-n(C)}$ for some $C \in \mathcal{C}_i$ with a U_i-maximal prehistory; from the definition we know that $\mathcal{F}_i$ is minimal and for any $C \in \mathcal{C}_i$ there is a prehistory $C^* = (f^{n(C)}z, \dots, z) \in \mathcal{C}_i^*$ such that $n(C) = n(C^*)$ and $y_{-n(C)} \in B_{n(C)}(z, \varepsilon)$.

The prehistories in $\mathcal{C}_i^*$ may have different lengths. However, if, for example, $z \in \mathcal{F}_i$ and $f(z) \in V_j$, then there exists $d_j - 1$ other points in $f^{-1}(f(z)) \cap \Lambda$ and these points will generate other prehistories from $\mathcal{C}_i^*$. Due to the above considerations we can assume without loss of generality that the set $\mathcal{F}_i$ is given by the prehistories of a single point $y \in U_i$. Also we may assume that these points $y \in U_i$ do not belong to other sets U_j, with $j \neq i$.

We now arrange the lengths of the prehistories from $\mathcal{C}_i^*$ as follows:

$$n_{i,q_i} > n_{i,q_i-1} > \ldots > n_{i,1}.$$

Then denote by $\mathcal{F}_{i,n_{i,q_i}}$ the set of points $z \in \mathcal{F}_i$ which correspond to the prehistories in $\mathcal{C}_i^*$ of length n_{i,q_i}. Denote also the cardinality of $\mathcal{F}_{i,n_{i,q_i}}$ by $N_{i,n_{i,q_i}}$.

Then we take the set $\mathcal{F}_{i,n_{i,q_i}-1}$ as the union of $f(\mathcal{F}_{i,n_{i,q_i}})$ and the set of points $z \in \mathcal{F}_i$ which correspond to the prehistories of length $n_{i,q_i} - 1$. The cardinality of $\mathcal{F}_{i,n_{i,q_i}-1}$ is denoted by $N_{i,n_{i,q_i}-1}$. We do this until reaching $N_{i,0}$ which is equal to 1, since these are considered as the prehistories of a single point y from U_i. We now define

$$N_{i,n_{i,q_i}}(j_1, \ldots, j_{n_{i,q_i}}) := \text{Card}\{z \in \mathcal{F}_{i,n_{i,q_i}}, f(z) \in V_{j_1}, \ldots, f^{n_{i,q_i}}(z) \in V_{j_{n_{i,q_i}}}\}$$

and similarly $N_{i,n_{i,q_i}-1}(j_1, \ldots, j_{n_{i,q_i}-1}) := \text{Card}\{\zeta \in \mathcal{F}_{i,n_{i,q_i}-1}, f(\zeta) \in V_{j_1}, \ldots, f^{n_{i,q_i}-1}(z) \in V_{j_{n_{i,q_i}-1}}\}$, etc.

Then, from the above construction, we have that

$$\frac{N_{i,n_{i,q_i}}(1, j_2, \ldots, j_{n_{i,q_i}})}{d_1} + \ldots + \frac{N_{i,n_{i,q_i}}(p, j_2, \ldots, j_{n_{i,q_i}})}{d_p} \leqslant N_{i,n_{i,q_i}-1}(j_2, \ldots, j_{n_{i,q_i}}). \quad (3)$$

Next we obtain

$$\frac{N_{i,n_{i,q_i}-1}(1, j_3, \ldots, j_{n_{i,q_i}})}{d_1} + \ldots + \frac{N_{i,n_{i,q_i}-1}(p, j_3, \ldots, j_{n_{i,q_i}})}{d_p} \leqslant N_{i,n_{i,q_i}-2}(j_3, \ldots, j_{n_{i,q_i}}), \quad (4)$$

and we can combine this inequality with (3). By induction, for all $i \in I$, we then obtain that

$$\Sigma_i := \sum_{z \in \mathcal{F}_i} \frac{1}{d_1^{m_1(z)} \cdot \ldots \cdot d_p^{m_p(z)}} \leqslant 1, \quad (5)$$

where, for each $z \in \mathcal{F}_i$, $m_1(z)$ represents the number of times that the orbit $z, f(z), \ldots, f^{n(z)}z$ hits $V_1, \ldots$, and $m_p(z)$ represents the number of times that the same orbit hits V_p. We assumed that the points y chosen inside U_i do not belong to other U_j, with $j \neq i$, and that the points of $\mathcal{F}_i$ are preimages (of different orders) of $y \in U_i$.

We also assume that N is the largest integer $n_{i,j}$, with $1 \leqslant j \leqslant q_i$, and $i \in I$; since I is finite, it follows that $N < \infty$.

We know from the construction of $\mathcal{F}_i$ that any preimage of type $y_{-n(C)}$ for C a maximal prehistory associated to U_i belongs to a Bowen ball of type $B_{n(C)}(z, \varepsilon)$ for some $z \in \mathcal{F}_i$.

Any local unstable manifold of size ε is contained in the union $\cup_{C \in \mathcal{C}_i^*} \Lambda(C, \varepsilon)$, and we want to extend these prehistories so as to obtain in the end a common (or close) length for all of them. More precisely we extend these prehistories until we reach a length between n and $n + N$, for a large integer n. The idea is the following. Let $z \in \mathcal{F}_i$ corresponding to a prehistory $C \in \mathcal{C}_i^*$ of length $n(C)$; then z itself is covered by $\cup_{j \in I} \cup_{C \in \mathcal{C}_j^*} \Lambda(C, \varepsilon)$, and hence there exists $j \in I$ and a prehistory $D \in \mathcal{C}_j^*$ such that $z \in \Lambda(D, \varepsilon)$. We now concatenate, as in [7], the prehistories C and D and will obtain $\Lambda(CD, \varepsilon) := \{y, \exists (y, \ldots, y_{-n(C)})$ a prehistory of y, ε-shadowing C, and $y_{-n(C)} \in \Lambda(D, \varepsilon)\}$; thus we follow the prehistories of preimages until we reach a length between n and $n + N$ for some large n.

To this end, consider the set $\mathcal{S}_n$ of all the multiples $(s, j_1, \ldots, j_s, p_1, \ldots, p_s)$ such that $s \in \mathbb{N}^*, j_1, \ldots, j_s \in I, 1 \leqslant p_k \leqslant q_{j_k}, k = 1, \ldots, s$ and $n \leqslant n_{j_1,p_1} + \ldots + n_{j_s,p_s} < n + N$.

For such an element of $\mathcal{S}_n$, we start with a prehistory $C_1 = (\zeta, \ldots, \zeta_{-n_{j_1,p_1}})$, with $\zeta \in U_{j_1}$, then we assume that $\zeta_{-n_{j_1,p_1}} \in \Lambda(C_2, \varepsilon)$ with C_2 a prehistory of length n_{j_2,p_2} of a point in U_{j_2}, etc. This procedure will give in the end a final preimage $\zeta_{-n_{j_1,p_1} - \ldots - n_{j_s,p_s}} \in \Lambda$ and we denote by F_n the set of all such final points obtained by the above procedure.

Since, for any $\mathcal{F}_i$, with $i \in I$, we covered all the possible preimages $y_{-n(C)}$ corresponding to the maximal U_i-prehistories C in Λ from $\mathcal{C}_i$, it follows that F_n is (n, ε)-spanning for Λ.

For $1 \leqslant k \leqslant q_i$, denote by $\tilde{N}_{ik}(m_1, \ldots, m_p)$ the number of elements ξ of $\mathcal{F}_i$ such that $n(\xi) = n_{i,k}$, and so that in the $n_{i,k}$-forward orbit of ξ there are exactly m_1 iterates belonging to $V_1, \ldots, m_p$ iterates belonging to V_p. By taking the product of the inequalities from (5) for $j_1, \ldots, j_s$, we obtain that

$$\sum_{1\leqslant p_1\leqslant q_{j_1},\ldots,1\leqslant p_s\leqslant q_{j_s}} \sum_{m_1+\ldots+m_p=n_{j_1,p_1}} \frac{\tilde{N}_{j_1p_1}(m_1,\ldots,m_p)}{d_1^{m_1}\cdot\ldots\cdot d_p^{m_p}} \cdot \ldots \cdot \sum_{l_1+\ldots+l_p=n_{j_s,p_s}} \frac{\tilde{N}_{j_sp_s}(l_1,\ldots,l_p)}{d_1^{l_1}\cdot\ldots\cdot d_p^{l_p}} \leqslant 1. \tag{6}$$

So, if $P_n(t\Phi^s - \log d(\cdot)) := \inf\{\sum_{z\in F}\exp(S_n(t\Phi^s - \log d(\cdot))(z), F(n,\varepsilon)\text{-spanning for } \Lambda\}$ and since F_n is (n,ε)-spanning, we obtain

$$\begin{aligned}
P_n(t\Phi^s - \log d(\cdot)) &\leqslant \sum_{z\in F_n}\exp(S_n(t\Phi^s - \log d(\cdot))(z) \\
&\leqslant \sum_{(s,j_1,\ldots,j_s,p_1,\ldots,p_s)\in\mathcal{S}_n} \sum_{m_1+\ldots+m_p=n_{j_1,p_1}} \frac{\tilde{N}_{j_1p_1}(m_1,\ldots,m_p)}{d_1^{m_1}\cdot\ldots\cdot d_p^{m_p}} \\
&\quad \cdot\ldots\cdot \sum_{l_1+\ldots+l_p=n_{j_s,p_s}} \frac{\tilde{N}_{j_sp_s}(l_s,\ldots,l_p)}{d_1^{l_1}\cdot\ldots\cdot d_p^{l_p}} \cdot (\text{diam } U_{j_1})^t\cdot\ldots\cdot(\text{diam } U_{j_s})^t \\
&\leqslant \sum_{s,j_1,\ldots,j_s} (\text{diam } U_{j_1})^t\cdot\ldots\cdot(\text{diam } U_{j_s})^t,
\end{aligned} \tag{7}$$

after using (6).

Therefore, by using (2) we obtain

$$\begin{aligned}
P_n(t\Phi^s - \log d(\cdot)) &\leqslant \sum_s \sum_{j_1,\ldots,j_s} (\text{diam } U_{j_1})^t\cdot\ldots\cdot(\text{diam } U_{j_s})^t \\
&= \sum_s \left(\sum_{j\in I}(\text{diam } U_j)^t\right)^s \\
&\leqslant \sum_s \left(\frac{1}{2}\right)^s < 2
\end{aligned} \tag{8}$$

Nevertheless, $P(t\Phi^s - \log d(\cdot)) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}(1/n)\log P_n(t\Phi^s - \log d(\cdot))$. This implies that $t \geqslant t_{d(\cdot)}$, where $t_{d(\cdot)}$ is the unique zero of the pressure function $t \to P(t\Phi^s - \log d(\cdot))$.

If we have that the preimage counting function $d(\cdot)$ is only less than or equal to $\omega(\cdot)$ at any point $x \in \Lambda$, it follows that $t \geqslant t_\omega$ in the same way.

However, t was taken arbitrarily greater than $\text{HD}(W_r^s(x)\cap\Lambda)$, and hence we obtain the announced inequality as follows:

$$\text{HD}(W_r^s(x)\cap\Lambda) \geqslant t_\omega \quad \forall x \in \Lambda. \qquad \square$$

THEOREM 2. *In the same setting as in Theorem 1, assume that there exists a continuous function ω on Λ such that, for any point $z \in \Lambda$, we have $d(z) \leqslant \omega(z)$. Then $\delta^s(x) \geqslant t_\omega$ for any $x \in \Lambda$, where t_ω is the unique zero of the pressure function $t \to P(t\Phi^s - \log\omega)$.*

Proof. The proof is similar to the one of the previous Theorem. As before we consider the set of U_i-maximal prehistories $\mathcal{C}_i$ and an associated minimal set $\mathcal{F}_i$ of final preimages given by these prehistories (see Definition 1 and previous proof).

Using the fact that the preimage counting function $d(\cdot)$ is upper semicontinuous on Λ we again find that, for each point $z \in \Lambda$, there exists a neighborhood of z such that each point y in this neighborhood has at most $d(z)$ preimages and they are close to some of the preimages of z (however, the point y may have strictly less than $d(z)$ preimages in Λ).

Again we have that $N_{i0} = 1$, since in the minimal set $\mathcal{F}_i$ we can take only the preimages of a point $y \in U_i$, where $\omega(\cdot)$ is largest on U_i. If not, then we can complete the prehistories of y with the prehistories of other points but the total number will be the same as if we were considering the prehistories of a single point from U_i.

From the continuity of ω on Λ there exists a positive function $\rho(\varepsilon)$ defined for small $\varepsilon > 0$, with the following property:

$$\text{if } y, z \in \Lambda, \text{ and } d(y, z) < \varepsilon, \text{ then } |\omega(y) - \omega(z)| \leqslant \rho(\varepsilon). \tag{9}$$

Since ω is continuous it follows that $\rho(\varepsilon) \to 0$ when $\varepsilon \to 0$, and we can assume that ρ has been taken such that it is an increasing function.

Now we notice that if $y \in B_n(z, \varepsilon)$, then, for any $0 \leqslant j \leqslant n$, we have $d(f^j y) \leqslant \omega(f^j z) + \rho(\varepsilon)$, since by assumption $d(f^j y) \leqslant \omega(f^j y)$. Thus the number of preimages of $f^j y$, $d(f^j y)$ may differ from $d(f^j z)$ by at most 1, but still $d(f^j y)$ is less than or equal to $\omega(f^j z) + \rho(\varepsilon)$, where $\rho(\varepsilon) \underset{\varepsilon \to 0}{\to} 0$.

As before, we take the set F_n of the final preimages of type $y_{-n_{j_1,p_1} - \ldots - n_{j_s,p_s}}$, over all sequences $(s, j_1, \ldots, j_s, p_1, \ldots, p_s)$ such that $j_1, \ldots, j_s \in I$, with $1 \leqslant p_1 \leqslant q_{j_1}$ and $1 \leqslant p_s \leqslant q_{j_s}$, with $n \leqslant n_{j_1,p_1} + \ldots + n_{j_s,p_s} < n + N$. This set of sequences is denoted again by $\mathcal{S}_n$ as in the proof of Theorem 1.

Now, as we mentioned, the preimage counting function is less than or equal to ω and ω varies with at most $\rho(\varepsilon)$ on a ball of radius ε; thus we can apply this at every iterate (up to order n) for points in a Bowen ball $B_n(z, \varepsilon)$. We then have the analogs of inequalities (5) and (6), namely

$$\Sigma_i := \sum_{z \in \mathcal{F}_i} \frac{1}{(\omega(fz) + \rho(\varepsilon)) \cdot \ldots \cdot (\omega(f^{n(C)} z) + \rho(\varepsilon))} \leqslant 1, \tag{10}$$

where we have assumed that $C = (f^{n(C)}(z), \ldots, z)$ is the prehistory from $\mathcal{C}_i^*$ whose final preimage is z for $z \in \mathcal{F}_i$. We denote the length $n(C)$ associated to the above C by $n(z)$.

Since ω is continuous on Λ, it takes only finitely many positive integer values, denoted again by $d_1, \ldots, d_p$ arranged as $d_1 < \ldots < d_p$. Furthermore, similarly, by taking the product of the inequalities (10) for $j_1, \ldots, j_s$ we obtain

$$\sum_{1 \leqslant p_1 \leqslant q_{j_1}, \ldots, 1 \leqslant p_s \leqslant q_{j_s}} \sum_{z \in \mathcal{F}_{j_1}, n(z) = n_{j_1,p_1}} \frac{1}{(\omega(fz) + \rho(\varepsilon)) \cdot \ldots \cdot (\omega(f^{n(z)} z) + \rho(\varepsilon))} \cdot \ldots \cdot \sum_{z \in \mathcal{F}_{j_s}, n(z) = n_{j_s,p_s}} \frac{1}{(\omega(fz) + \rho(\varepsilon)) \cdot \ldots \cdot (\omega(f^{n(z)} z) + \rho(\varepsilon))} \leqslant 1. \tag{11}$$

Then since, by construction, the set F_n is (n, ε)-spanning for Λ with respect to f (since we cover all final preimages with $\mathcal{F}_i$), we can finish the proof by using (11) in the same way as in the proof of Theorem 1.

Therefore we obtain that $t \geqslant t(\varepsilon)$ for $\varepsilon > 0$ small, with $t(\varepsilon)$ being the unique zero of the pressure function $t \to P_\varepsilon(t\Phi^s - \log(\omega + \rho(\varepsilon)))$, where, in general, $P_\varepsilon(g) := \limsup_n (1/n) \log \inf\{\sum_{z \in F} \exp(S_n(g)(z)), F\ (n, \varepsilon)\text{-spanning for } \Lambda\}$ for g continuous on Λ.

We now take some T arbitrarily greater than t and $\eta > 0$ small; then $T > t \geqslant t(\eta)$. However, if $0 < \varepsilon < \eta$, then we get that $\rho(\varepsilon) \leqslant \rho(\eta)$, so $t\Phi^s - \log(\omega + \rho(\varepsilon)) \geqslant T\Phi^s - \log(\omega + \rho(\eta))$. Now, since $t \geqslant t(\varepsilon)$ for all ε small, it follows that $0 \geqslant P_\varepsilon(t\Phi^s - \log(\omega + \rho(\varepsilon))) \geqslant P_\varepsilon(T\Phi^s - \log(\omega + \rho(\eta)))$ for all $\varepsilon > 0$ small enough. Nevertheless, recalling the definition of the topological pressure $P(g) = \lim_{\varepsilon \to 0} P_\varepsilon(g)$, for all g continuous, we obtain that

$$P(T\Phi^s - \log(\omega + \rho(\eta))) \leqslant 0. \tag{12}$$

Now let t_ω be the unique zero of the pressure function $t \to P(t\Phi^s - \log \omega)$. From the continuity of the pressure with respect to the potential, it follows that t_ω is the limit of the zeros of the pressure functions $t \to P(t\Phi^s - \log(\omega + \rho(\eta)))$ when η converges to 0. Hence from (12), we have

$$T \geqslant t_\omega.$$

Therefore since T was chosen arbitrarily greater than t, which in turn was chosen arbitrarily greater than $\mathrm{HD}(W^s_r(x) \cap \Lambda)$, we obtain the conclusion as follows:

$$\mathrm{HD}(W^s_r(x) \cap \Lambda) \geqslant t_\omega. \qquad \square$$

We are now ready to prove some consequences of these results.

More precisely, we first consider what happens if there exists a point $x \in \Lambda$ such that the stable dimension at x is the smallest one possible.

Corollary 1. *Assume that f is c-hyperbolic on a basic set Λ and that the preimage counting function $d(\cdot)$ reaches a maximum value of d on Λ. If there exists a point $x \in \Lambda$ such that $\delta^s(x) = t_d$, where t_d is the unique zero of the pressure function $t \to P(t\Phi^s - \log d)$, then $d(y) = d, \forall y \in \Lambda$. Hence the stable dimension at every point of Λ is equal to t_d.*

Proof. We know that there exists a point $x \in \Lambda$ with $\delta^s(x) = t_d$. Assume that there exists an open set $V \subset \Lambda$ such that $d(y) \leqslant d - 1$, with $y \in V$ and also let W open inside V such that $\bar{W} \subset V$.

Then we can take a Lipschitz function Ψ with $\Psi(z) = d - 1$ for $z \in \bar{W}$, $\Psi(z) = d$ for z outside V, and $d - 1 \leqslant \Psi \leqslant d$ on $V \setminus \bar{W}$. Thus we have $d(y) \leqslant \Psi(y)$, with $y \in \Lambda$.

But then, from Theorem 2, it follows that $\delta^s(x) \geqslant t_\Psi$, where t_Ψ denotes the unique zero of the function $t \to P(t\Phi^s - \log \Psi)$; hence, since $\delta^s(x) = t_d$, it means that $t_d \geqslant t_\Psi$. However, also $\Psi \leqslant d$ on Λ and hence $t_\Psi \geqslant t_d$; therefore $t_d = t_\Psi$.

We now consider an equilibrium measure μ_d for the Holder continuous potential $t_d\Phi^s - \log d$. Thus, from the definition of equilibrium measures (see [**4**]) and since $P(t_d\Phi^s - \log d) = 0$, we have

$$\int (t_d\Phi^s - \log d)\, d\mu_d + h_{\mu_d} = 0, \tag{13}$$

where h_μ denotes in general the metric entropy of the f-invariant probability measure μ on Λ.

But then, from the Variational Principle applied to the potential $t_\Psi\Phi^s - \log \Psi$ (see [**4**]), we obtain

$$\int (t_\Psi\Phi^s - \log \Psi)\, d\mu_d + h_{\mu_d} \leqslant P(t_\Psi\Phi^s - \log \Psi) = 0. \tag{14}$$

Also recall that we have proved above that $t_d = t_\Psi$, and hence consequently we have

$$\int (t_d\Phi^s - \log \Psi)\, d\mu_d + h_{\mu_d} \leqslant \int (t_d\Phi^s - \log d)\, d\mu_d + h_{\mu_d} = 0.$$

This implies that

$$\int \log \Psi\, d\mu_d \geqslant \int (\log d)\, d\mu_d. \tag{15}$$

On the other hand, μ_d is an equilibrium measure and hence it is positive on open sets, since any open set contains a Bowen ball and thus one can use the estimates for the equilibrium measures on Bowen balls similar to the ones for homeomorphisms from [4]. These estimates were proved in [4, Lemma 20.3.4] for homeomorphisms with specification; we know, however, that hyperbolicity implies specification [4]. For the case of non-invertible maps they follow by using the lift to the inverse limit $\hat{\Lambda}$. Indeed we denote by $B_n(z,\varepsilon,f) := \{w \in \Lambda, d(f^i z, f^i w) < \varepsilon,\ i = 0,\dots,n-1\}$ a Bowen ball relative to $f|_\Lambda$, by $B_n(\hat{z},\varepsilon,\hat{f}) := \{\hat{w} \in \hat{\Lambda}, d(\hat{f}^i \hat{z}, \hat{f}^i \hat{w}) < \varepsilon,\ i = 0,\dots,n-1\}$ a Bowen ball relative to $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ and by $\pi : \hat{\Lambda} \to \Lambda$ the canonical projection. Then we have that there exists a $k = k(\varepsilon) \geqslant 1$ such that $\hat{f}^k(\pi^{-1}B_n(y,\varepsilon,f)) \subset B_{n-k}(\hat{f}^k\hat{y}, 2\varepsilon, \hat{f}) \subset \hat{\Lambda}$. Then if ϕ is some Holder potential and μ_ϕ denotes the equilibrium state of ϕ with $\hat{\mu}_\phi$ its unique lifting to $\hat{\Lambda}$, it follows that $\mu_\phi(B_n(y,\varepsilon,f)) = \hat{\mu}_\phi(\hat{f}^k\pi^{-1}(B_n(y,\varepsilon,f))) \leqslant \hat{\mu}_\phi(B_{n-k}(\hat{f}^k\hat{y},2\varepsilon,\hat{f})) \leqslant A_{2\varepsilon} e^{S_{n-k}\phi(f^k y)-(n-k)P(\phi)} \leqslant \tilde{A}_\varepsilon e^{S_n\phi(y)-nP(\phi)}$, for any $y \in \Lambda$, with $n \geqslant 1$, for some positive constant $\tilde{A}_\varepsilon$ depending on ε, ϕ, f. Conversely we have $\pi(B_n(\hat{y},\varepsilon,\hat{f})) \subset B_n(y,\varepsilon,f)$, and thus we get the lower bound for $\mu_\phi(B_n(y,\varepsilon,f))$ from the lower bound for $\hat{\mu}_\phi(B_n(\hat{y},\varepsilon,\hat{f}))$, since $\hat{f}$ is an expansive homeomorphism with specification on $\hat{\Lambda}$. In conclusion the estimates given by

$$B_\varepsilon e^{S_n\phi(y)-nP(\phi)} \leqslant \mu_\phi(B_n(y,\varepsilon,f)) \leqslant A_\varepsilon e^{S_n\phi(y)-nP(\phi)}, \forall y \in \Lambda, n \geqslant 1,$$

hold also for hyperbolic non-invertible maps f on Λ. Thus we have shown indeed that the equilibrium measure μ_d (of the potential $t_d\Phi^s - \log d$) is positive on Bowen balls, and hence also on non-empty open sets.

Coming back to the proof of the Corollary, we chose Ψ with $0 \leqslant \Psi \leqslant d$ on Λ, and $\log\Psi \leqslant \log(d-1) < \log d$ on W. Hence if μ_d is positive on open sets, then we obtain a contradiction with (15).

Hence the preimage counting function $d(\cdot)$ must be equal to d on a dense set in Λ. However, recall that $d(\cdot)$ is upper semicontinuous (Lemma 1), and therefore $d(y) = d, \forall y \in \Lambda$. □

Finally we can apply the above theorems for the case of hyperbolic skew products with overlaps in the stable fibers and having a finite IFS in the base.

We consider a finite union of compact sets $X_1,\dots,X_m$ in an open set $S \subset \mathbb{R}^l$ and denote it by $X := X_1 \cup \ldots \cup X_m$. Also consider a continuous expanding topologically transitive function $f : X \to X$. Also assume that f is injective on each X_i and that $f(X_i) = X(i,1) \cup \ldots \cup X(i,m_i)$, with $i = 1,\dots,m$, where $X(i,j)$ are sets from the same collection $\{X_1,\dots,X_m\}$.

The source model for this is the case of an expanding map $f : I_1 \cup \ldots \cup I_m \to I_1 \cup \ldots \cup I_m$, with $I_1,\dots,I_m$ compact subintervals in $[0,1]$, such that $f(I_j)$ is a union of some of the same subintervals, that is, $f(I_j) = I(j,1) \cup \ldots \cup I(j,m_j)$, with $j = 1,\dots,m$.

We also take functions $g(x,y) : X \times \tilde{W} \to X \times \tilde{W}$, with $\tilde{W} \subset \mathbb{R}^k$ a neighborhood of the closure of an open set W, such that g is smooth (say $\mathcal{C}^2$) in (x,y), and such that for every $x \in X$, the function $g(x,\cdot) : W \to W$ is contracting uniformly in x, and it is injective and conformal. We shall denote the function $g(x,\cdot)$ also by g_x; due to the contraction, $g_x(\bar{W})$ is strictly contained in V.

Then we take the compact f-invariant set $X^* := \{y \in X, f^j y \in X, j \geqslant 0\}$ and for each $x \in X^*$, we consider the fiber $\Lambda_x := \cap_{n\geqslant 0} \cup_{z \in f|_{X^*}^{-n}(x)} g_{f^n z} \circ \ldots \circ g_z(\bar{W})$. Then define

$$\Lambda := \bigcup_{x \in X^*} \Lambda_x$$

(see, for example, [9] for a similar type of skew products).

The set Λ is an invariant set for the skew product $F(x,y) = (f(x), g(x,y))$ defined on $X^* \times W$, and because of the expansion on X^* and the contraction on vertical fibers, F is hyperbolic on Λ. Thus we see that F is c-hyperbolic on Λ. The local stable manifolds of F are contained in the vertical fibers $\{x\} \times W, x \in X^*$. Then we call (F,Λ) a *c-hyperbolic skew-product pair*.

The important thing to note here is that we allow the images $g_y(W)$, coming from *different preimages* y of a point $x \in X^*$, *to overlap*.

In this case we can apply the above Theorem 1. Indeed for each j, with $1 \leqslant j \leqslant m$, we know that a point $z \in X^* \cap X_j$ has at most q_j preimages in Λ, where q_j is the number of subsets X_i, with $1 \leqslant i \leqslant m$ such that $f(X_i) \supset X_j$. Then we have that the preimage counting function associated to F and Λ is less than or equal to a locally constant function ω given by $\omega(x, y) := q_j$ if $x \in X_j$, with $1 \leqslant j \leqslant m$. However, points in $\Lambda \cap (\{x\} \times W)$ may have strictly less than q_j F-preimages in Λ for $x \in X_j \cap X^*$. Thus we obtain the following corollary which gives a lower estimate for the stable dimension.

Corollary 2. *Let (F, Λ) be a c-hyperbolic skew-product pair as above. Then the stable dimension of Λ, that is, the Hausdorff dimension of the fibers Λ_x, with $x \in X^*$, is greater than or equal to the unique zero of the pressure function $t \to P(t\Phi^s - \log \omega)$, where $\omega|_{(X^* \cap X_j) \times W} = q_j$, with $1 \leqslant j \leqslant m$.*

Corollary 1 also has the following consequence.

Corollary 3. *In the setting of Corollary 2, if $f(I_j)$ contains all subintervals $I_1, \ldots, I_m$, with $1 \leqslant j \leqslant m$, it follows that F is m-to-1 on Λ if and only if $\exists z \in \Lambda$ with $\delta^s(z) = 0$. In this case we obtain $\delta^s(y) = 0, \forall y \in \Lambda$.*

Proof. Since $f(I_j) \supset I_1 \cup \ldots \cup I_m$ for all j, with $1 \leqslant j \leqslant m$, we can model the dynamics of f on X^* after the one-sided shift on m symbols Σ_m^+, whose topological entropy is equal to $\log m$. Also we note that the topological entropy of F on Λ is equal to the topological entropy of f on X^* since on vertical fibers we have contractions that do not add to the entropy. Thus $h_{\text{top}}(F|_\Lambda) = h_{\text{top}}(f|X^*) = \log m$.

From Corollary 1 and [8], it follows also that F is m-to-1 if and only if $\delta^s(z) = t_m$ for some point $z \in \Lambda$. Hence if F is m-to-1 on Λ, then we have $\delta^s(z) = t_m$; but $t_m = 0$ since $P(0 - \log m) = h_{\text{top}}(f|_{X^*}) - \log m = \log m - \log m = 0$.

Conversely, $P(0 \cdot \Phi^s - \log m) = h_{\text{top}}(f|_{X^*}) - \log m = 0$, and hence we have $t_m = 0$ as being the unique zero of the pressure. Thus if there exists a point $z \in \Lambda$ with $\delta^s(z) = 0$, then $\delta^s(z) = t_m$. Hence, from Corollary 1 we obtain that F is m-to-1 on Λ. □

Corollary 4. *In the setting of Corollary 2, assume that $f(I_j)$ contains all the subintervals $I_1, \ldots, I_m$ for $j = 1, \ldots, m$ and that there exists $x \in X^*$ such that $g_\xi(W) \cap g_\zeta(W) = \emptyset$ for some 1-preimages $\xi, \zeta \in X^*$ of x, with $\xi \neq \zeta$. Then it follows that $\delta^s(z) > 0, \forall z \in \Lambda$.*

Proof. Since $h_{\text{top}}(F|_\Lambda) = h_{\text{top}}(f|_{X^*}) = \log m$, we have that $P(0 \cdot \Phi^s - \log m) = h_{\text{top}}(f|_{X^*}) - \log m = 0$, and hence $t_m = 0$. Now if there would exist a point $z \in \Lambda$ with $\delta^s(z) = 0$, then $\delta^s(z) = t_m$ and from Corollary 1 we have that F is m-to-1 on Λ. Nevertheless, since g_y is injective for all $y \in X^*$ and since there exist 1-preimages ξ, ζ of x such that $g_\xi(W) \cap g_\zeta(W) = \emptyset$, we conclude that F cannot be m-to-1 on Λ. Therefore we have

$$\delta^s(z) > 0 \quad \forall z \in \Lambda.$$ □

Remark. The dynamics of f on X^* can be modeled in general after shifts of finite type. Indeed we define the matrix $A = (a_{ij})_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m}$, with $a_{ij} = 1$ if and only if $f(I_i) \supset I_j$. We can assume that A is irreducible such that σ_A becomes topologically transitive on Σ_A. The entropy of this dynamical system is equal to $\log |\lambda_A|$, where $|\lambda_A|$ is the spectral radius of A (equal to the maximum eigenvalue of A). A minimal (n, ε)-spanning set is obtained from cylinders of rank $n+p$, where p depends only on ε (see [4]). This spanning set can then be used to estimate the pressure of the function $t\Phi^s - \log \omega$, where $\omega(z) = q_j$ for $z \in \Lambda \cap ((X^* \cap X_j) \times W)$ and $q_j := \sum_{1 \leqslant i \leqslant m} a_{ij}$, with $1 \leqslant j \leqslant m$.

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