Geometric Structures on Manifolds

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Contents

Chapter 1

An overview

1.1 Abstract

This thesis describes my main academic achievements after I obtained my Ph.D Degree, from the University of Edinburgh, in 2001. It begins with a short description of my research profile, with information on: the areas in which I have worked, the (most important) journals of mathematics where I published my research results, my collaborators, the research grants which I won and the institutions where I held post-doctoral and visiting positions. Details on all of these may be found in my CV and list of publications.

It follows a description of my main research results in the four research fields in which I worked *after I obtained my Ph.D Degree*: Kähler and Sasaki Geometry; Quaternionic Geometry; Generalized Complex Geometry; Frobenius Manifolds. For each of these areas, we present an introduction, which contains details on the most important original results and the way they fit into the research field. Then we give a short account on various notions and well-known results from the field, which are relevant from the point of view of the original results. Afterwards, we describe these original results. To keep the text of reasonable length, only the ideas of the proofs are given, and, sometimes, the proofs are skipped completely. More details can be found in the papers published by the author of this thesis.

Below we briefly describe the relevant results of our research.

Kähler and Sasaki Geometry. We define and study a class of natural connections (the so called Tanaka connections) on a CR manifold, which may be viewed as the analogue of Weyl connections from conformal geometry. This is the starting point in our treatment of the relations between Kähler and Sasaki geometry, using the Webster's correspondence and a generalized cone construction. We give a new proof of the Guillemin's formula for the potential of the canonical Kähler metric of a compact toric symplectic manifold.

Quaternionic Geometry. We study in detail the quaternionic con-

nections using twistor theory and Penrose operators. We study, from the local and global point of view, the conformal-Killing operator acting on (compatible) 2-forms on a quaternionic-Kähler manifold. We determine the G-structures ($G := Sp(1)Sp(n), G_2$ and $Spin_7$) whose fundamental forms are conformal-Killing.

Generalized Complex Geometry. We develop a detailed study of a class (called regular) of invariant generalized complex structures on a real semisimple Lie group G. The problem reduces to the description of admissible pairs $(\mathfrak{k}, \omega)$, where $\mathfrak{k} \subset \mathfrak{g}^{\mathbb{C}}$ is an appropriate regular subalgebra of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ associated to G and ω is a closed 2-form on $\mathfrak{k}$, such that $\mathrm{Im}\left(\omega|_{\mathfrak{k}\cap\mathfrak{g}}\right)$ is non-degenerate. When G is a semisimple Lie group of inner type (in particular, when G is compact semisimple) a classification of regular generalized complex structures on G is given. When G is a semisimple Lie group of outer type, we describe the subalgebras $\mathfrak{k}$ in terms of appropriate root subsystems of a root system of $\mathfrak{g}^{\mathbb{C}}$ and we construct a large class of admissible pairs $(\mathfrak{k}, \omega)$ (hence, regular generalized complex structures of G).

Frobenius Manifolds. We develop a generalization of the construction of K. Saito of adding a variable to a Frobenius manifold. We prove a duality theorem for F-manifolds with eventual identities, a problem raised by Y. Manin. We relate this duality to several notions and constructions from the theory of Frobenius and F-manifolds: compatible connections, Riemannian F-manifolds (by introducing an invariant metric) and tt^*-geometry (by introducing a Hermitian metric).

The final part of the thesis contains an account on the future academic plans of the author. My main activity will continue to be the research at IMAR, supported also in the future, I hope, by the research grants I will win. The research directions I intend to follow and the new background I will acquire are described. In the same chapter it is described briefly the contents of a book I am writing with D. V. Alekseevsky and on which I intend to work also in the following years. There are described various topics I would like to teach, in order to attract young mathematicians to differential geometry and, eventually, to coordinate their doctoral studies.

1.2 Rezumat

Aceasta teza descrie realizarile mele stiintifice dupa ce am obtinut titlul de Doctor in Matematica, la Universitatea din Edinburgh, in 2001. Ea incepe cu o scurta prezentare a profilului cercetarii, incluzand informatii despre: domeniile in care am lucrat, revistele de matematica (semnificative) in care am publicat, despre colaboratori, granturi de cercetare obtinute prin concurs si institutiile unde am efectuat stagii de cercetare. Detalii despre toate acestea se gasesc in CV si in lista de publicatii.

Urmeaza o descriere a rezultatelor principale obtinute in cele patru directii de cercetare abordate *dupa doctorat*: Geometria Kähler si Sasaki; Geometria Cuaternionica; Geometria Complexa Generalizata; Varietati Frobenius. Pentru fiecare din aceste domenii se incepe cu o introducere, care contine detalii si precizari asupra rezultatelor originale mai importante si a felului in care se incadreaza ele in literatura de specialitate. Se mentioneaza apoi diverse notiuni si rezultate de baza din domeniu, relevante din punctul de vedere al rezultatelor originale obtinute de catre autoare. In continuare sunt descrise aceste rezultate originale. Pentru a nu depasi o anumita lungime a textului, se prezinta numai ideile demonstratiilor, iar uneori demonstratiile se omit in intregime. Detalii suplimentare pot fi gasite in lucrarile publicate de catre autoare.

Mai jos descriem pe scurt rezultatele relevante ale cercetarii:

Geometria Kähler si Sasaki. Se studiaza o clasa de conexiuni naturale pe o varietate CR (asa numitele conexiuni Tanaka), care sunt similare conexiunilor Weyl din geometria conforma. Acesta este punctul de plecare al studiului pe care il dezvoltam asupra legaturilor dintre varietatile Kähler si Sasaki, prin intermediul corespondentei lui Webster si constructiei conului generalizat. De asemenea, se da o noua demonstratie a formulei lui Guillemin pentru potentialul metricii canonice Kähler pe o varietate simplectica torica compacta.

Geometria Cuaternionica. Se trateaza detaliat conexiunile cuaternionice folosind teoria twistor si operatori Penrose. Se studiaza, din punct de vedere local si global, operatorul conform-Killing ce actioneaza pe spatiul 2-formelor (compatibile) pe o varietate cuaternionica-Kähler. Se determina G-structurile (unde $G = Sp(1)Sp(n)$, G_2 sau $Spin_7$) cu forma fundamentala conforma-Killing.

Geometria Complexa Generalizata. Se dezvolta un studiu detaliat al unei clase (numita regulata) de structuri complexe generalizate invariante pe un grup Lie real semisimplu G. Problema se reduce la descrierea perechilor admisibile $(\mathfrak{k}, \omega)$, unde $\mathfrak{k} \subset \mathfrak{g}^{\mathbb{C}}$ este o subalgebra regulata a algebrei complexe $\mathfrak{g}^{\mathbb{C}}$ asociata lui G si ω este o 2-forma pe $\mathfrak{k}$, astfel ca $\mathrm{Im}\left(\omega|_{\mathfrak{g}\cap\mathfrak{k}}\right)$ sa fie nedegenerata. Cand grupul Lie semisimplu G este de tip interior (in particular, cand G este compact semisimplu), se dezvolta o clasificare a structurilor complexe generalizate invariante regulate pe G. Cand G este un grup Lie semisimplu de tip exterior, se descrie subalgebra $\mathfrak{k}$ in termeni de sisteme de radacini ale algebrei Lie complexe semisimple $\mathfrak{g}^{\mathbb{C}}$ si se construieste o larga clasa de perechi admisibile $(\mathfrak{k}, \omega)$ (si, deci, de structuri complexe generalizate invariante regulate pe G).

Varietati Frobenius. Se dezvolta o generalizare a constructiei lui K. Saito de adaugare a unei variabile la o varietate Frobenius si o teorema de dualitate pentru F-varietati cu identitati eventuale, o problema sugerata de Y. Manin. De asemenea, se pune in evidenta legatura acestei dualitati cu alte notiuni si constructii din teoria varietatilor Frobenius si a F-varietatilor: conexiuni compatibile, F-varietati Riemanniene (introducand o metrica invarianta) si geometria tt^* (introducand o metrica Hermitiana).

In partea finala a tezei sunt prezentate planurile academice ale autoarei. Activitatea mea principala va continua sa fie cercetarea la IMAR, sprijinita, sper, si in viitor, de granturile pe care le voi obtine prin concurs. Se prezinta directiile de cercetare pe care intentionez sa le urmez si noi teme in care intentionez sa ma introduc. Este rezumat continutul unei carti pe care o scriu cu D. V. Alekseevsky si la care voi lucra si in anii urmatori. De asemenea, se descriu aici unele tematici pe care mi-ar placea sa le predau, pentru a atrage tineri matematicieni inspre geometrie diferentiala si, eventual, pentru a le coordona studiile doctorale.

1.3 Introduction

This thesis presents an overview of my scientific activity after I obtained my Ph.D from the University of Edinburgh, in 2001.

My research from this period can be divided into four main directions: Kähler and Sasaki Geometry; Quaternionic Geometry; Generalized Complex Geometry; Frobenius Manifolds.

I published research papers in international journals of mathematics of high influence score, between which I mention: Proc. London Math. Soc. (2012), Adv. Math (2011), Compositio Math. (2008), , J. London Math. Soc (2009), Internat. Math. Res. Notices (2004), Ann. Math. Pura Applic. (2011), J. Geom. Physics (2006, 2010, 2011), Int. J. Math. (2008), Ann. Global Anal. Geom. (2004), Diff. Geom. Aplic. (2008).

In 2008 I was awarded the "Gheorghe Titzeica Prize of the Romanian Academy", for research in Riemannian Geometry.

I presented my research results at various international workshops from Romania and abroad and I collaborated with internationally recognized mathematicians, like D. V. Alekseevsky, D. M. J. Calderbank, P. Gauduchon, M. Pontecorvo, I. A. B. Strachan.

I did research stages (as post-doctoral or visiting fellow) at some of the most well-known institutions from abroad, namely: Ecole Polytechnique (Palaiseau), Scuola Normale Superiore di Pisa, Institut des Hautes Etudes Scientifiques (Paris), International Center for Theoretical Physics (Trieste) University of Glasgow (UK).

I was the Director of a CNCS-UEFISCDI research project (project no. 1187/2008) during 2008-2011. I am currently the Director of a CNCS-UEFISCDI research project (project no. PNII-ID-PCE-2011-0362), which will last until December 2013. I was also a member of the research teams of two other (CNCSIS and CEX) research projects at IMAR. For more details on my scientific activity, see the CV and list of publications.

Chapter 2

Contributions to Bochner-flat Kähler Geometry and Sasaki Geometry

2.1 Introduction

In this chapter is concerned with the geometry of Bochner-flat Kähler manifolds and its interactions with CR and Sasaki geometry and summarizes the results published by the author of this thesis in **Compositio Mathematica** [36], **Annals of Global Analysis and Geometry** [40], **Journal of Symplectic Geometry** [25], **C.R.M Proceedings and Lecture Notes** [43]. It includes three main related topics: a detailed study of a class of natural connections on a CR manifold (the so called Tanaka connections) and two constructions which relate Kähler and Sasaki geometry, namely the Webster's correspondence and a generalized cone construction. At the end of the chapter we also derive a simple proof for the Guillemin formula on compact toric symplectic manifolds.

We now present with details the contents of each section separately, explaining also how the original results fit into the general research field.

In **Section 2.2** we recall basic definitions and well-known facts from Kähler and Sasaki geometry: the Bochner tensor, CR and Sasaki manifolds, the Webster's correspondence (which states that any Kähler manifold is locally isomorphic to the quotient of a Sasaki manifold by its Reeb vector field).

In **Section 2.3** we define and study a class of natural connections on a CR manifold. They will be refered as Tanaka connections, because they generalize, in a framework similar to conformal geometry, a class of connections already considered by Tanaka in [106]. On a CR manifold the Tanaka connections are parametrized by 1-forms, and for this reason they may be seen as the analogue of Weyl connections from conformal geometry. However, unlike Weyl connections from conformal geometry, they always have torsion. A detailed study of the curvature of Tanaka connections was developed in [40]. In this section we content ourselves to give a description of the Chern-Moser tensor of a CR manifold, in terms of the curvature of Tanaka connections (see Theorem 9). The advantage of this description is that it makes more transparent the well-known identification between the Chern-Moser tensor and the Bochner tensor in the Webster's correspondence, which is the starting point of our treatment from Section 2.4. Recall that the Chern-Moser tensor in CR geometry plays the same role as the Weyl tensor in conformal geometry, and measures the non-flatness of a CR manifold. More precisely, it is known that the Chern-Moser tensor of a CR manifold of dimension $2n+1 \geq 5$ is zero if and only if the CR manifold is locally isomorphic to the standard CR sphere $(S^{2n+1}, H_{\mathrm{can}}, I_{\mathrm{can}})$.

In **Section 2.4** we develop a local study of Bochner-flat Kähler manifolds, using the Webster's correspondence. The Bochner tensor of a Kähler manifold is the "biggest" irreducible component of the curvature tensor under the action of the unitary group. In complex dimension two, the Bochner tensor coincides with the anti-self-dual Weyl tensor. A Kähler manifold is Bochner-flat if its Bochner tensor vanishes. Bochner-flat Kähler manifolds represent an important class of Kähler manifolds and have been intensively studied by mathematicians like V. Apostolov, R. Bryant, D. Calderbank, P. Gauduchon, etc. Despite the formal similarity between the Bochner tensor and the Weyl tensor from conformal geometry (or the Chern-Moser tensor from CR geometry), the local geometry of Bochner-flat Kähler manifolds is very rich. In this section we show how the geometry of Bochner-flat Kähler manifolds can be read from the standard CR sphere by means of the Webster's correspondence. Because in the Webster's correspondence the Chern-Moser tensor of the Sasaki manifold is the pull-back of the Bochner tensor of the Kähler manifold and the Chern-Moser tensor is zero if and only if the CR manifold is locally isomorphic to the standard CR sphere (in dimension at least 5), any Bochner-flat Kähler manifold of dimension $2m \geq 4$ is locally isomorphic to the quotient of the standard CR sphere $(S^{2m+1}, H_{\mathrm{can}}, I_{\mathrm{can}})$ by the Reeb vector field of a Sasaki structure. Therefore the problem of classifying locally the Bochner-flat Kähler manifolds reduces to the problem of describing all compatible Sasaki structures on the standard CR sphere. It turns out that any compatible Sasaki structure on the standard CR sphere $(S^{2m+1}, H_{\mathrm{st}}, I_{\mathrm{st}})$ is determined by an element $iA \in \mathfrak{su}_{m+1,1}$ and there are four types of compatible Sasaki structures on $(S^{2m+1}, H_{\mathrm{st}}, I_{\mathrm{st}})$ determined by the four types of conjugacy classes in the Lie algebra $\mathfrak{su}_{m+1,1}$ (see Theorem 13). For any $iA \in \mathfrak{su}_{m+1,1}$, we consider the associated Bochner-flat Kähler manifold M_A and we compute its Ricci and scalar curvatures, see Section 2.4. As an example on how the geometry of M_A can be described in terms of A, we determine the operators A such that M_A is Kähler-Einstein. (This will also be a key fact in the generalized cone construction from the next section). Finally, we show how the type of a Bochner-flat Kähler manifold can be read from its Ricci tensor (with no reference to the standard CR sphere) by means of the so called Bryant minimal and characteristic polynomials, hence we relate our approach of Bochner-flat Kähler manifolds with Bryant's approach [19]. More details on the material presented in this section may be found in [43]. Besides complementing the paper of R. Bryant [19], the results from this section are also closely related to the more general theory of Kähler manifolds which admit a Hamiltonian 2-form, studied by V. Apostolov, D. Calderbank, P. Gauduchon in [13, 14, 59]. The relation is given by the remark that a Kähler manifold is Bochner-flat if and only if a certain modified Ricci tensor is Hamiltonian.

In **Section 2.5** we develop a new construction which relates CR and Bochner-flat Kähler manifolds, called a generalized cone construction. Recall that an important class of Kähler manifolds is represented by the Kähler cones over Sasaki manifolds. Unfortunately, except when the Sasaki manifold is an open subset of the standard CR sphere with its standard metric as the Sasaki metric, the Kähler cones are not Bochner-flat. With this motivation, we develop a natural generalization of the Kähler cone construction, which produces, from a CR manifold (N, H, I) endowed with a family of compatible Sasaki-Reeb vector fields, a Kähler structure on an open subset of the cone $N \times \mathbb{R}^{>0}$. We refer to such a Kähler structure as a generalized Kähler cone. It coincides with the usual Kähler cone of a Sasaki manifold when the family of Reeb vector fields is constant. Besides the usual Kähler cone, another strong motivation for this construction comes from the fact that the Bryant family of Bochner-flat Kähler structures (which has been discovered by R. Bryant in his classification theorem of complete Bochner-flat Kähler structures on simply connected manifolds [19] and has been further studied in [43]) are generalized Kähler cones. The main result from this section states that any Bochner-flat Kähler manifold of complex dimension bigger than two is locally isomorphic to a generalized Kähler cone (see Theorem 27).

Based on the straightforward behaviour of dual potentials under Kähler reduction, in **Section 2.6** we develop an alternative proof of the Guillemin's formula for the Kähler potential of the canonical Kähler metric of a compact toric symplectic manifold (see Theorem 28).

2.2 Preliminary material

2.2.1 The Bochner tensor of a Kähler manifold

In this section we recall the definition of the Bochner tensor of a Kähler manifold. We use the formalism developed in [13, 59].

Let (V, g, J) be a real vector space together with a complex structure J and a J-invariant positive definite metric g. We identify vectors and covectors of V using the metric g. Let $\omega := g(J\cdot, \cdot)$ be the Kähler form. Recall that the space $\mathcal{K}(V)$ of Kähler curvature tensors of (V, g, J), defined as those curvature tensors which annihilate all J-anti-invariant 2-forms on V, decomposes into a g-orthogonal sum

$$\mathcal{K}(V) := c^*_{\mathcal{K}}\left(\mathrm{Sym}^{1,1}(V)\right) \oplus \mathcal{W}^{\mathcal{K}}(V), \tag{2.1}$$

where $c^*_{\mathcal{K}} : \mathrm{Sym}^{1,1}(V) \to \mathcal{K}(V)$ is the adjoint of the Ricci contraction

$$c_{\mathcal{K}} : \mathcal{K}(V) \to \mathrm{Sym}^{1,1}(V), \quad c_{\mathcal{K}}(R)(v, w) := \mathrm{trace} R(v, \cdot, w, \cdot), \quad v, w \in V$$

and has the following expression [59]

$$c^*_{\mathcal{K}}(S) = \frac{1}{2}[\frac{S \wedge \mathrm{Id} + (J \circ S) \wedge J}{2} + \omega \otimes S + \beta \otimes J], \tag{2.2}$$

where $S \in \mathrm{Sym}^{1,1}(V)$ is a symmetric J-invariant endomorphism of V, "Id" is the identity endomorhism, $\beta \in \Lambda^{1,1}(V)$ is the J-invariant 2-form on V, related to S by $\beta(v, w) := g(SJv, w)$, and, for two endomorphisms S and T of V, $S \wedge T$ is the endomorphism of $\Lambda^2(V)$ defined by the formula

$$(S \wedge T)(v \wedge w) := S(v) \wedge T(w) - S(w) \wedge T(v), \quad v, w \in V.$$

According to the decomposition (2.1), a Kähler curvature tensor $R \in \mathcal{K}(V)$ decomposes into the sum

$$R = c^*_{\mathcal{K}}(S) + W^K,$$

where $W^K \in \mathcal{W}^{\mathcal{K}}(V)$ is the *principal part* (or the *Bochner part* of R) and $S \in \mathrm{Sym}^{1,1}(V)$ is a modified Ricci tensor.

The curvature R^g of a Kähler metric g on a complex manifold (M, J), is, at any point $p \in M$, a Kähler curvature tensor of the tangent space (T_pM, J_p, g_p).

Definition 1. *The principal part of the curvature R^g of a Kähler manifold (M, J, g) is called the Bochner tensor of (M, J, g). The Kähler manifold (M, J, g) is Bochner-flat if its Bochner tensor vanishes.*

2.2.2 CR and Sasaki manifolds

Recall that a (strongly pseudo-convex) CR manifold (N, H, I) (always assumed to be connected and oriented) has a codimension one oriented subbundle H of the tangent bundle TN, called the contact bundle, and a bundle homomorphism $I : H \to H$ with $I^2 = -\mathrm{Id}$, such that, for every smooth sections $X, Y \in \Gamma(H)$, $[IX, IY] - [X, Y]$ is also a section of H and the integrability condition

$$[IX, IY] - [X, Y] = I\left([IX, Y] + [X, IY]\right) \tag{2.3}$$

is satisfied. Since N and H are oriented, the co-contact line bundle $L := TN/H$ is also oriented, hence trivialisable. A positive section μ of L defines a contact form $\theta := \eta\mu^{-1}$ on M, where $\eta : TN \to L$ is the natural projection and $\mu^{-1} \in \Gamma(L^*)$ is the dual section of μ, i.e. the natural contraction between μ and μ^{-1} is the function on N identically equal to one. The bilinear form

$$g(X, Y) := \omega(X, IY) := \frac{1}{2} d\theta(X, IY), \quad X, Y \in TM \tag{2.4}$$

of the bundle H is independent, up to a positive multiplicative function, of the choice of the contact form and is positive definite – the strongly pseudo-convexity condition. The contact form θ determines a Reeb vector field T, uniquely defined by the conditions $\theta(T) = 1$ and $i_T d\theta = 0$. Note that the Reeb vector field preserves the bundle H, i.e. $[T, X] \in \Gamma(H)$ when $X \in \Gamma(H)$ and hence the Lie derivative $L_T(I)$ is a well-defined endomorphism of H. The contact form θ determines a Riemannian metric g on the manifold N, which on H is defined by (2.4) and such that T is of norm one and g-orthogonal to H.

Definition 2. *The metric g is called Sasaki if T is a Killing vector field on (M, g).*

It is easy to see that the Sasaki condition $L_T(g) = 0$ is equivalent to $L_T(I) = 0$.

2.2.3 The Webster's correspondence

Let (N, H, I) be a CR manifold, together with a fixed Reeb vector field T, which determines a Sasaki metric g on M. Since g is Sasaki, the complex structure I and the metric g of the bundle H descend on the quotient $M := N/T$ (assumed to be smooth), and determine an almost complex structure $\hat{I}$ and an $\hat{I}$-invariant Riemannian metric $\hat{g}$ on M. The Webster's correspondence is stated as follows:

Theorem 3. *The quotient $M := N/T$ of a Sasaki manifold (N, H, I, T, g) by its Reeb vector field is a Kähler manifold. Moreover, any Kähler manifold can be locally obtained in this way (via the choice of a primitive of the Kähler form).*

2.3 Tanaka connections on CR-manifolds

In this section we define and study the notions of Weyl connections and associated Tanaka connections on CR manifolds. All our CR manifolds are assumed to be oriented and strongly pseudo-convex. Our main results are Theorem 6 (which describes the Tanaka connections) and Theorem 9 (which describes the Chern-Moser tensor in terms of Tanaka connections). Theorem 10 is well-known and recalls the importance of the Chern-Moser tensor in CR geometry.

Definition 4. *A Weyl connection on a CR manifold (N, H, I) is a connection D on the line bundle $L = TN/H$. Its curvature F^D is called the Faraday curvature. The Weyl connection D is said to be closed if $F^D = 0$ and exact if there is a global non-vanishing D-parallel section of L.*

Let $\eta : TN \to L$ be the natural projection. If D^0 and $D := D^0 + \gamma$ are two Weyl connections, then $d^D\eta = d^{D^0}\eta + \gamma \wedge \eta$ and the restriction of the 2-form $\omega := \frac{1}{2}d^D\eta$ to $H \times H$ is independent of the choice of Weyl connection D. Remark that the bilinear form $g := \omega(\cdot, I\cdot)$ of H is a positive definite metric (with values in the oriented bundle L).

Lemma 5. *Let D be a Weyl connection on N. There is a unique $\psi^D : L \to TN$ (called the Reeb vector field associated to D) such that $\eta \circ \psi^D = \mathrm{Id}$ and $i_{\psi^D} d^D\eta = 0$. (Here and elsewhere 'Id' denotes the identity operator).*

As has already been done in the context of conformal geometry (see [23]) we define a Weyl-Lie derivative L_{ψ^D} associated to D in the following way: we first introduce a Weyl-Lie derivative L_X along any (genuine) vector field X, acting on tensor fields tensored with any section of L, or L^k (for any integer k) by saying that $L_X(\mu) = D_X(\mu)$ for any section μ of L and then combining with the usual Lie derivative. Then $L_X(\psi^D)$ is well-defined (since ψ^D is a section of $L^{-1}TN$) and we set $L_{\psi^D}(X) := -L_X(\psi^D)$.

Theorem 6. *Let (N, H, I) be a CR manifold with $\eta : TN \to L$ the natural contact form. Let D be a Weyl connection on N, $\psi^D : L \to TN$ the corresponding Reeb vector field and J the endomorphism of TN which extends I, being zero on the image of ψ^D. There is a unique connection ∇^D on $TN = H \oplus \mathrm{Im}\psi^D$, called the Tanaka connection associated to D, which has the following properties:*

i) The connection ∇^D preserves the bundle H and coincides with D on $\mathrm{Im}\psi^D \cong L$. (The second condition is equivalent to $(D \otimes \nabla^D)\psi^D = 0$, when ψ^D is considered as a section of $L^{-1}TM$.)

ii) The connection ∇^D preserves J. The covariant derivative of the coupled connection $D \otimes \nabla^D$, in the directions of H, preserves the canonical

L-valued metric

$$g(X,Y)=\frac{1}{2}d^D\eta(X,JY) \tag{2.5}$$

on H.

iii) The torsion T^{∇^D} of ∇^D satisfies

$$T^{\nabla^D}_X Y=(d^D\eta)(X,Y)\psi^D \tag{2.6}$$

for every $X,Y\in H$ and

$$T^{\nabla^D}_{\psi^D}JX+JT^{\nabla^D}_{\psi^D}X=0 \tag{2.7}$$

for every $X\in H$.

Proof. We only explain the uniqueness of ∇^D (for a complete proof, see [40]). From *ii)*, ∇^D is determined on H by a Koszul type formula

$$\begin{aligned}2g(\nabla^D_X Y,Z)&=D_X\left(g(Y,Z)\right)+D_Y\left(g(X,Z)\right)-D_Z\left(g(X,Y)\right)\\&\quad+g([X,Y]^H,Z)-g([X,Z]^H,Y)-g([Y,Z]^H,X)\end{aligned} \tag{2.8}$$

where $X,Y,Z\in\Gamma(H)$ and for a vector field W of N, $W^H:=W-\eta(W)\psi^D$ is its g-orthogonal projection on the bundle H. Relation (2.7) implies that

$$T^{\nabla^D}_{\psi^D}X=-\frac{1}{2}JL_{\psi^D}(J)(X),\quad X\in H$$

which, together with the $D\otimes\nabla^D$-flatness of ψ^D, implies that

$$\nabla^D_{\psi^D}X=L_{\psi^D}(X)-\frac{1}{2}JL_{\psi^D}(J)(X). \tag{2.9}$$

From (2.8) and (2.9), ∇^D is uniquely determined on the bundle H, hence on all of TN, from condition *i)*.

□

Remark 7. When D is an exact Weyl connection, it determines a genuine contact form $\theta=\mu^{-1}\eta$ (where $D(\mu)=0$, $\mu\in\Gamma(L)$), unique up to the multiplication by a constant factor. The connection ∇^D was already defined in [106] and is known in the literature as the Tanaka connection associated to θ.

Notations 8. In the next theorem we identify H with $H^*\otimes L$ using the canonical (i.e. independent of D) L-valued metric g, defined by (2.5). For a fixed Weyl connection D, we denote by $(R^{\nabla^D})^{\mathrm{skew}}$ the 2-form on H with values in $\mathrm{End}(H)$, whose value at a pair $(X,Y)\in H\times H$ is the skew-part with respect to g of the endomorphism $R^{\nabla^D}(X,Y)$ of H (note that $R^{\nabla^D}(X,Y)$ preserves H because so does ∇^D). Similarly, $F^{D,+}$ is the I-invariant part of the restriction of the Faraday form F^D to H. Since we identify H with $H^* \otimes L$, the expression $F^{D,+} \wedge \mathrm{Id} + F^{D,+} \circ J \wedge J$ is a genuine 2-form on H with values in $\mathrm{End}(H)$. In the next theorem the expression (2.10) is viewed as a vector valued 2-form on H and its I-invariant part is a vector valued $(1,1)$-form on H.

In [40] it was developed a detailed study of the curvature of Tanaka connections. Here we only state the result which allows to define the Chern-Moser tensor in terms of Tanaka connections, in a similar way as the Weyl tensor of a conformal manifold is defined in terms of Weyl connections. We use the conventions from the previous paragraph.

Theorem 9. *Let D be a Weyl connection on a CR-manifold (N, H, I) and ∇^D the associated Tanaka connection. The I-invariant part of*

$$(R^{\nabla^D})^{\mathrm{skew}} - \frac{1}{4}\left(F^{D,+} \wedge \mathrm{Id} + F^{D,+} \circ J \wedge J\right) \tag{2.10}$$

is a Kähler curvature tensor on (H, I, g). Its principal part is independent of the choice of Weyl connection and coincides with the Chern-Moser tensor of (N, H, I).

We end this section by recalling the well-known importance of the Chern-Moser tensor in CR-geometry [30, 16].

Theorem 10. *Let (N, H, I) be a CR manifold of dimension $2m + 1 \geq 5$. Then (N, H, I) is flat (i.e. its Chern-Moser tensor is zero) if and only if (M, H, I) is locally isomorphic to the standard CR sphere $(S^{2m+1}, H_{\mathrm{can}}, I_{\mathrm{can}})$, with CR structure $(H_{\mathrm{can}}, I_{\mathrm{can}})$ induced from the standard embedding $S^{2m+1} \subset \mathbb{C}^{m+1}$.*

2.4 Bochner-flat Kähler manifolds

The following local characterization of Bochner-flat Kähler manifolds holds.

Theorem 11. *Any Bochner-flat Kähler manifold of dimension $2m \geq 4$ is locally isomorphic to the quotient of the standard CR sphere $(S^{2m+1}, H_{\mathrm{can}}, I_{\mathrm{can}})$ by the Reeb vector field of a Sasaki structure, compatible with $(H_{\mathrm{can}}, I_{\mathrm{can}})$.*

Proof. It may be shown that in the Webster's correspondence (see Theorem 3), the Tanaka connection of the Sasaki manifold (N, H, I, T, g) descends to the Levi-Civita connection of the Kähler manifold $(M, \hat{I}, \hat{g})$. Similarly, the Bochner-tensor of $(M, \hat{I}, \hat{g})$ pulls back to the Chern-Moser tensor of (N, H, I). Hence, the Chern-Moser tensor is zero (and (N, H, I) is locally isomorphic to the standard CR sphere, see Theorem 10) if and only if $(M, \hat{I}, \hat{g})$ is Bochner-flat. This implies our claim. □

Therefore, the local classification of Bochner-flat Kähler manifolds of dimension $2m$ reduces to the description of Sasaki structures on S^{2m+1}, compatible with $(H_{\mathrm{can}}, I_{\mathrm{can}})$. In the next section we describe these Sasaki structures.

2.4.1 Sasaki structures on odd-dimensional sphere

In order to describe the Sasaki structures on $(S^{2m+1}, H_{\mathrm{can}}, I_{\mathrm{can}})$, we need an alternative description of the standard CR sphere, as follows. Let W be a complex vector space of dimension $m+2$ and h a Hermitian form on W, of signature $(m+1, 1)$.

Definition 12. *The space Σ^{2m+1} of all complex null lines of the Hermitian vector space (W, h) is called the Hermitian $2m+1$-dimensional sphere.*

Our first remark is that Σ^{2m+1} is diffeomorphic to S^{2m+1}. Indeed, chose a basis $\{e_0, \cdots, e_{m+1}\}$ of W which is orthonormal with respect to h, i.e. $h(e_i, e_j) = 0$ for any $i \neq j$, $h(e_0, e_0) = -1$ and $h(e_j, e_j) = +1$, for any $j = 1, \ldots, m+1$. A null line $l \in \Sigma^{2m+1}$ has a unique representative of the form $e_0 + u$, where $u \in \mathrm{Span}\{e_1, \cdots, e_{m+1}\}$ satisfies $h(u, u) = 1$, i.e. it belongs to the unit sphere S^{2m+1} of the positive definite Hermitian vector space $\mathrm{Span}\{e_1, \cdots, e_{m+1}\}$. Then the map

$$\Sigma^{2m+1} \ni l \to u \in S^{2m+1} \tag{2.11}$$

is a diffeomorphism, whose inverse maps the standard CR structure $(H_{\mathrm{can}}, I_{\mathrm{can}})$ of S^{2m+1} to the CR-structure (H, I) of Σ^{2m+1}, where $H \to \Sigma^{2m+1}$ is given by $H_l := \mathrm{Hom}_{\mathbb{C}}(l, l^{\perp}/l)$, for any $l \in \Sigma^{2m+1}$ (where $l^{\perp}$ denotes the Hermitian orthogonal of l) and I is induced by the natural complex structure of $l^{\perp}/l$.

The classification of Sasaki structures of (Σ^{2m+1}, H, I) is stated as follows:

Theorem 13. *i) Any element $iA \in \mathfrak{su}_{m+1,1}$, where A is a Hermitian trace-free endomorphism of W defines a Sasaki structure (g_A, T_A), compatible with the CR-structure (H, I), on the open subset*

$$\Sigma_A^{2m+1} := \{l \in \Sigma^{2m+1} : \quad h(Av, v) > 0, \quad \forall v \in l, \quad \forall v \neq 0\}$$

of Σ^{2m+1}, where

$$T_{A,l}(v) := iAv \,(\mathrm{mod} v)\,, \quad \forall l \in \Sigma_A^{2m+1}, \quad \forall v \in l$$

is the velocity vector field on Σ_A^{2m+1} generated by iA and

$$g_A = \theta_A^2 + \frac{1}{2} d\theta_A(\cdot, I\cdot).$$

Here

$$\theta_A(X) = \frac{\mathrm{Im} h(Xv, v)}{h(Av, v)}, \quad X \in T_l\Sigma^{2m+1} \subset \mathrm{Hom}(l, W/l), \quad 0 \neq v \in l, \quad l \in \Sigma_A^{2m+1}$$

is the contact form of (H, I) which corresponds to T_A.

ii) Any Sasaki structure compatible with the CR structure (H, I) is obtained as above.

iii) Two Sasaki structures (g_A, T_A) and $(g_{A'}, T_{A'})$ are isomorphic (in the obvious way) if and only if iA and iA' belong to the same adjoint orbit of $\mathfrak{su}_{m+1,1}$.

Using the above classification of Sasaki structures on the standard CR sphere, we obtain the following local classification of Bochner-flat Kähler manifolds.

Corollary 14. *Any Bochner-flat Kähler manifold of dimension $2m \geq 4$ is locally isomorphic, as a Kähler manifold, with the quotient of a Sasaki manifold (Σ_A^{2m+1}, g_A) by its Reeb vector field T_A, for a Hermitian endomorphism A of W. In particular, there are four types of classes of Bochner-flat Kähler manifolds, which correspond to the four types of conjugacy classes in the Lie algebra $\mathfrak{su}_{2m+1,1}$.*

Proof. The first statement follows from Theorems 11 and 13. For the second statement, recall that there are four types of conjugacy classes in the Lie algebra $\mathfrak{su}_{2m+1,1}$ (see [65]) - elliptic, hyperbolic, 1 and 2-step parabolic, which can be described as follows. If $iA \in \mathfrak{su}_{2m+1,1}$, then $A \in \mathrm{End}(W)$ is Hermitian with respect to h. If A has a time-like eigenvector then it is diagonalisable with respect to an orthonormal basis of W. In this case A is called of elliptic type. Suppose now that A does not have any time-like eigenvectors and let W_0 be the complement of the maximal subspace of W generated by the space-like eigenvectors of A. There are three possibilities. The first possibility happens when $A|_{W_0}$ has an eigenvector, which can't be timelike or spacelike, hence must be null. It turns out that W_0 is of dimension two, and is generated by two independent null eigenvectors of A, with eigenvalues δ and $\bar{\delta}$, for a non-real number δ. The remaining two possibilities happen when $A|_{W_0}$ doesn't have any eigenvectors. Then A must be of the form $\lambda I + N$, where $\lambda \in \mathbb{R}$ and N is nilpotent. It turns out that either $\dim(W_0) = 2$ (in which case A is 1-step parabolic) or $\dim(W_0) = 3$ (in which case A is 2-step parabolic). □

Example 15. i) The weighted projective space $\mathbb{P}^m_{\vec{a}}$ of weights $\vec{a} = (a_1, \cdots, a_{m+1})$ (where a_i are positive integers) is defined as the quotient of $\mathbb{C}^{m+1} \setminus \{0\}$ by the weighted diagonal action

$$\tau \cdot (z_1, \cdots, z_{m+1}) := (\tau^{a_1} z_1, \cdots, \tau^{a_{m+1}} z_{m+1}), \quad \tau \in \mathbb{C} \setminus \{0\}.$$

It may be shown that $\mathbb{P}^m_{\vec{a}}$ is isomorphic to the space of trajectories of the vector field T^A on Σ^{2m+1}, where

$$A = \operatorname{diag}\left(-\sum_{j=1}^{m+1}\lambda_j, \lambda_1, \cdots, \lambda_{m+1}\right)$$

and

$$\lambda_j := a_j - \frac{1}{m+1}\sum_{r=1}^{m+1} a_r, \quad 1 \leq j \leq m+1.$$

Hence $\mathbb{P}^m_{\vec{a}}$ inherits a Bochner-flat Kähler structure of elliptic type.

ii) The Bryant's family of Bochner-flat Kähler structures [19] is a family of complete Bochner-flat Kähler structures on the standard $\mathbb{C}^m$, parametrized by systems of m of positive numbers. Let $\vec{k} = (k_1, \cdots, k_m)$ be such a system. The associated Bochner-flat Kähler structure $\mathbb{M}^m_{\vec{k}}$ has a globally defined Kähler potential $s_{\vec{k}}$, defined implicitly by

$$s_{\vec{k}}(w) = \sum_{k=1}^{m}\frac{|w_i|^2}{e^{k_i s_{\vec{k}}(w)}}, \quad w = (w_1, \cdots, w_m).$$

It can be shown that $\mathbb{M}^m_{\vec{k}}$ is of 1-step parabolic type, determined by a Hermitian endomorphism of W which in an orthonormal basis is given by

$$A = \begin{pmatrix} \lambda-1 & 1 & 0 & \cdots & 0 \\ -1 & \lambda+1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_m \end{pmatrix}$$

and λ_i are related to k_i by

$$\lambda_i := k_i - \frac{\sum_{r=1}^m k_r}{m+2}, \quad 1 \leq i \leq m \quad , \lambda = -\frac{\sum_{r=1}^m k_r}{m+2}.$$

2.4.2 The local geometry of Bochner-flat Kähler manifolds

From the theory developed in the previous section, any Bochner-flat Kähler manifold of dimension $2m$ is locally isomorphic to a quotient $M_A = \Sigma^{2m+1}_A / T_A$ and the entire geometry of the Kähler manifold may be expressed only in terms of A. We shall denote by $\bar{g}_A$ and $\bar{\omega}_A$ the Kähler metric and Kähler form of M_A. In this section we state the expression of the curvature of $\bar{g}_A$ and its various contractions (Ricci and scalar curvatures). The proof is based on the remark (from the Webster's correspondence) that the curvature of $\bar{g}_A$

pulls back to the horizontal part of the curvature of the Tanaka connection ∇^A associated to T_A. Thus, one needs to find ∇^A and then compute its curvature. This was done in [43]. Here we only give the final formulae.

We identify vector fields on M_A with T^A-invariant sections of H. In particular, we view $\bar{g}_A$ and $\bar{\omega}_A$ as a (T_A-invariant) metric and 2-form on H. Define a complex linear operator $\Theta_A : H \to H$ by

$$\Theta_A(X)(w) := AX(w) - \frac{(AX(w), w)}{(Aw, w)} Aw, \quad \mathrm{mod}(l) \quad \forall X \in H_l = \mathrm{Hom}(l, l^{\perp}/l), \tag{2.12}$$

for any $w \in l \setminus \{0\}$. One may check that the complex trace of Θ_A is given by

$$s_A(x) = -\frac{(A^2 w, w)}{(Aw, w)}, \quad \forall x \in \Sigma_A^{2m+1}, \quad \forall w \in l \setminus \{0\}.$$

Proposition 16. *i) The curvature $R^{\bar{g}_A}$ has the following expression: for any $X, Y, Z \in \Gamma(H)$,*

$$\begin{aligned} R^{\bar{g}_A}(X,Y)(Z) = {} & 2\bar{\omega}_A\left((\Theta_A + \mathrm{tr}\Theta_A \mathrm{Id})X, Y\right) iZ + 2\bar{\omega}_A(X,Y) i\Theta_A(Z) \\ & - h_A\left((\Theta_A + s_A)Z, Y\right) X - h_A(Z,Y)\Theta_A(X) \\ & + h_A\left((\Theta_A + s_A)Z, X\right) Y + h_A(Z,X)\Theta_A(Y), \end{aligned}$$

where

$$h_A(X,Y) = \frac{(Xw, Yw)}{(Aw, w)}, \quad X, Y \in \Gamma(H).$$

ii) the Ricci curvature $\mathrm{Ric}^{\bar{g}_A}$ *has the following expression:*

$$\mathrm{Ric}^{\bar{g}_A} = 2(m+2)\left(\Theta_A + s_A \mathrm{Id}\right).$$

Proof. Straightforward curvature computations. See [43]. □

Definition 17. *The operator Θ_A is called the modified Ricci operator of $(M_A, \bar{g}_A, \bar{\omega}_A)$. Its complex trace s_A is called the modified scalar curvature of $(M_A, \bar{g}_A, \bar{\omega}_A)$.*

Bochner-flat Einstein-metrics

It is natural to ask how various curvature conditions on a Bochner-flat Kähler manifold $(M_A, \bar{g}_A, \bar{\omega}_A)$ determine the endomorphism A. In such questions, the next theorem turns out to be very useful. We first need to introduce some notations.

Remark 18. Preserving the setting from the previous section, let Q_A and q_A be the minimal and characteristic polynomials of A and $r_A := Q_A/q_A$ their quotient polynomial. Denote by $\tilde{A}$ and $\tilde{a}$ the operator polynomials defined by

$$(t\mathrm{Id} - A)\tilde{A}(t) = Q_A(t)\mathrm{Id}, \quad (t\mathrm{Id} - A)\tilde{a}(t) = q_A(t)\mathrm{Id}.$$

Hence,

$$\tilde{A}(t) = r_A(t)\tilde{a}(t).$$

The following theorem holds.

Theorem 19. *For any $x \in M_A$, the characteristic polynomial, $P_{A,x}$, of the modified Ricci operator Θ_A at x is given by*

$$P_{A,x}(t) = \frac{(\tilde{A}(t)w, w)}{(Aw, w)} = r_A(t)p_{A,x}(t)$$

where $p_{A,x}$, called the reduced characteristic polynomial at Θ_A at x, is the monic polynomial of degree l, defined by

$$p_{A,x}(t) := \frac{(\tilde{a}(t)w, w)}{(Aw, w)}.$$

In particular, $P_{A,x}$ has at least l constant roots, namely the roots of r_A. Moreover, on a dense open subset, $P_{A,x}$ has precisely l constant roots, and the other roots are smooth and functionally independent.

As an application of Theorem 19, we determine the operators A such that M_A with its canonical metric $\bar{g}_A$ is Einstein. For this, we remark that if $(M_A, \bar{g}_A)$ is Einstein, then the characteristic polynomial $P_{A,x}$ of Θ_A has only one eigenvalue, which is constant. It follows that $r_A(t) = (t-\lambda)^m$, where m is the complex dimension of M_A and $\lambda \in \mathbb{R}$, and we are left with three cases, described in the following proposition.

Proposition 20. *A Bochner-flat Kähler manifold $(M_A, \bar{g}_A)$ is Kähler-Einstein if and only if, with respect to an orthonormal basis of W, one of the following three situations hold:*

i) If $\lambda > 0$, $A = \lambda \mathrm{diag}(-(m+1), 1, \cdots, 1)$ is elliptic and $M_A = \mathbb{C}P^m$, with the Fubini Study metric of modified scalar curvature λ.

ii) If $\lambda < 0$, $A = \lambda \mathrm{diag}(1, -(m+1), 1, \cdots, 1)$ is again elliptic and $M_A = \mathbb{C}\mathbb{H}^m$, with the hyperbolic metric of modified scalar curvature λ.

iii) If $\lambda = 0$, A is 1-step parabolic and all its eigenvalues are equal to zero.

We end this section with a remark, very useful in explicit computations. It explains how to "read" the type of a Bochner-flat Kähler manifold directly from the geometry of the manifold, with no reference to the standard CR sphere. It also makes the link between our approach of Bochner-flat Kähler manifolds via the Webster's correspondence and the Bryant's approach [20].

Remark 21. Given a Bochner-flat Kähler manifold (M, g, J), it is difficult in general to write it down explicitly as a quotient of the standard CR sphere by the Reeb vector field of a compatible Sasaki structure. However, there is another way to determine the local type of (M, J, g), by means of the Bryant's minimal and characteristic polynomials, defined as follows [20]. Let S be a symmetric, J-invariant endomorphism of TM such that $c^*_{\mathcal{K}}(S) = R^g$ (where $c^*_{\mathcal{K}}$ is the adjoint of the Ricci contraction $c_{\mathcal{K}}$, see (2.2)) and $P(t)$ the characteristic polynomial of a new modified Ricci operator Θ, related to S by

$$\Theta := \frac{1}{4}\left(S - \frac{\mathrm{trace}_{\mathbb{R}}(S)}{2(m+2)}\mathrm{Id}\right), \tag{2.13}$$

where m is the complex dimension of M. The Ricci operator Θ will be considered as a complex linear operator on the complex vector bundle (TM, J). Its trace is called the modified scalar curvature of (M, g, J). Denote by $\xi_1, \cdots, \xi_l$ the non-constant roots of P and by P_n its non-constant part, defined by $P_n(t) := (t - \xi_1)\cdots(t - \xi_l)$. The number l is called the order of (M, g, J). On a dense open subset M^0 of M, the eigenvalues ξ_j (for any $j \in \{1, \cdots, l\}$) are simple, different from each other at any point and different, at any point, from any constant eigenvalue of Θ; the functions $\xi_1, \cdots, \xi_l$ are functionally independent on M^0 and

$$|\mathrm{grad}_g(\xi_j)|^2 = -4\frac{p_m(\xi_j)}{P_n'(\xi_j)}, \quad j \in \{1, \cdots, l\} \tag{2.14}$$

where p_m is a monic polynomial of degree $l + 2$, with constant coefficients, independent of j, called the *Bryant minimal polynomial* of (M, g, J). The *Bryant characteristic polynomial* p_c of (M, g, J) is by definition the product of p_m with the constant part P/P_n of P.

When the Kähler manifold is $(M_A, \bar{g}_A, \bar{\omega}_A)$, for $iA \in \mathfrak{su}_{m+1,1}$, the Ricci operator Θ defined by (2.13) coincides with the operator Θ_A from (2.12) and the Bryant characteristic and minimal polynomials of $(M_A, \bar{g}_A, \bar{\omega}_A)$ coincide with the characteristic and minimal polynomials of A (which determine A up to conjugacy). Therefore, the local type of a Kähler manifold (i.e. the conjugacy class of the associated operator A) may be read directly from its Ricci tensor.

2.5 The generalized cone construction

In Section 2.5.1 we define the generalized Kähler cones and we prove our main result (Theorem 27). Section 2.5.2 is concerned with examples: we describe the weighted projective spaces (with their canonical Bochner-flat Kähler structure) and the Bryant family of Bochner-flat Kähler metrics as generalized Kähler cones.

2.5.1 The main result

Let (N, H, I) be an (oriented strongly pseudo-convex) CR manifold and $\{T_r\}_{r\in\mathcal{J}}$, with $\mathcal{J} \subseteq \mathbb{R}^{>0}$ a connected open interval, a family of Reeb vector fields of Sasaki structures on (N, H, I). On the cone manifold $N \times \mathcal{J}$ we define the vector fields T and V, by

$$T_{(p,r)} := T_r(p), \quad V_{(p,r)} := r\frac{\partial}{\partial r}, \quad \forall (p,r) \in N \times \mathcal{J},$$

an almost complex structure J by

$$J|_H := I, \quad J(V) := T,$$

a 2-form ω and a bilinear form g by

$$\omega := \frac{1}{2}dd^J r^2, \quad g := \omega(\cdot, J\cdot).$$

Let $M \subset N \times \mathcal{J}$ be the open subset where $g(T,T)$ is positive. It turns out that g is positive definite on M.

Lemma 22. *The pair (ω, J) defines a Kähler structure on M.*

Proof. The integrability of J follows from the integrability of I and the Sasaki condition $L_{T_r}(I) = 0$, for any r. Since ω is closed and g is positive definite, (ω, J) is Kähler.

□

Definition 23. *The Kähler manifold (M, ω, J) is a generalized Kähler cone over the CR manifold (N, H, I). It is a restricted generalized Kähler cone if the function $g(T,T)$ on M is constant along the trajectories of the vector field T.*

Example 24. When the Sasaki-Reeb vector fields T_r are independent of r, the associated generalized Kähler cone is restricted and coincides with the usual Kähler cone over a Sasaki manifold.

Conventions 25. For simplicity, we consider only restricted generalized Kähler cones. When we refer to a generalized Kähler cone, we implicitly mean restricted generalized Kähler cone. The restricted condition simplifies considerably the curvature computations of the generalized Kähler cones.

A CR manifold (N, H, I) with a family of Sasaki-Reeb vector fields $\{T_r\}_r$ as above, with contact forms θ_r, comes equipped with a second family of contact forms, which plays an important role in finding the Bochner-flat generalized Kähler cones. They are defined as follows. Define a positive function $f : M \to \mathbb{R}^{>0}$ by $g(T,T) = r^2 f$. It is easy to see that

$$f = 1 + \frac{r\dot{\theta}_r(T_r)}{2}. \tag{2.15}$$

The restriction $f_r := f(\cdot, r)$ of f to $N_r := M \cap (N \times \{r\})$ is positive. We introduce a new family of contact forms

$$\tilde{\theta}_r = \frac{1}{f_r}\theta_r. \tag{2.16}$$

For any r, the contact form $\tilde{\theta}_r$ is defined on N_r (viewed as an open subset of N).

Main class of generalized Kähler cones. We shall be mainly concerned with generalized Kähler cones over (open subsets) of Hermitian CR spheres. Suppose that $N \subset \Sigma^{2m+1}$ is an open subset of the Hermitian CR sphere of complex null lines in a pseudo-Hermitian vector space $W = \mathbb{C}^{m+1,1}$ with pseudo-Hermitian metric h of signature $(m+1, 1)$, and let

$$\eta(X) = \mathrm{Im} h(Xw, w), \quad X \in T_x(N), \quad w \in x,$$

be the canonical (line bundle valued) contact form of Σ^{2m+1}, restricted to N (above X is considered as a tangent vector in x at Σ^{2m+1}, hence an element of $\mathrm{Hom}(x, W/x)$). Being the contact form of a Sasaki structure,

$$\theta_r(X) = \frac{\eta(X)}{(B_r w, w)}, \quad X \in T_x(N), \quad 0 \neq w \in x$$

for any r, where B_r is a Hermitian operator of (W, h) (see Theorem 13). From (2.15) and (2.16), it turns out that

$$\tilde{\theta}_r(X) = \frac{\eta(X)}{(A_r w, w)}$$

where

$$A_r := B_r - \frac{r}{2}\dot{B}_r. \tag{2.17}$$

The condition $T(f) = 0$ is equivalent to $[A_r, B_r] = 0$ for any r.

Our main result from this section is the following:

Theorem 26. *A generalized Kähler cone (M, g, J) defined by a family of Sasaki Reeb vector fields $\{T_r\}$ over a CR manifold (N, H, I) of dimension $2m+1 \geq 5$ is Bochner-flat if and only if (N, H, I) is locally isomorphic to the Hermitian CR sphere Σ^{2m+1} of complex null lines in a complex Hermitian vector space W of signature $(m+1, 1)$, and $\{T_r\}$ is defined by one of the following families of Hermitian operators B_r of W:*

i) $B_r = r^2(B - \mu(r^2)A)$. Here the real function μ satisfies $\mu' > 0$ and is a solution of the differential equation

$$\mu' = \frac{1}{2}\mu^2 + d, \tag{2.18}$$

where $d \in \mathbb{R}$ is an arbitrary real number. The operator A is Hermitian semisimple, with a positive definite eigenspace, of dimension $m+1$, which corresponds to the eigenvalue $\frac{1}{2(m+2)}$ and a 1-dimensional timelike eigenspace, which corresponds to the eigenvalue $-\frac{(m+1)}{2(m+2)}$.

ii) $B_r = r^2(B + \mu(r^2)A)$, where μ satisfies (2.18) and $\mu' < 0$. The operator A is Hermitian semisimple, with an eigenspace of signature $(m, 1)$, which corresponds to the eigenvalue $-\frac{1}{2(m+2)}$, and a 1-dimensional spacelike eigenspace, which corresponds to the eigenvalue $\frac{m+1}{2(m+2)}$.

iii) $B_r = r^2(B - r^2 A)$, where A is 1-step parabolic, with all eigenvalues equal to zero.

iv) $B_r = r^2\left(B - \frac{e^{\lambda r^2}}{\lambda}A\right)$, where $\lambda \in \mathbb{R} \setminus \{0\}$ and A is 1-step parabolic with all eigenvalues equal to zero.

In all these cases, B is any Hermitian, trace-free operator of W which commutes with A.

Proof. From straightforward computations one determines the Levi-Civita connection and the curvature of g. The Bochner-flatness condition on g applied to horizontal arguments (i.e. arguments which belong to H) readily implies that the Chern-Moser tensor of (N, H, I) is zero, i.e. (N, H, I) is locally isomorphic to the Hermitian CR sphere (Σ^{2m+1}, H, I) of complex null lines in a Hermitian vector space $W = \mathbb{C}^{m+1,1}$. Thus, we may assume that θ_r and $\tilde{\theta}_r$ (where $\tilde{\theta}_r$ is defined by (2.16)) are given by Hermitian operators B_r and A_r of W (see the main class of generalized Kähler cones described above), i.e.

$$\theta_r(X) := \frac{\eta(X)}{(B_r w, w)}, \quad \tilde{\theta}_r(X) := \frac{\eta(X)}{(A_r w, w)}, \quad X \in T_x(N), \quad 0 \neq w \in x,$$

where A_r and B_r are related by (2.17) and η is the canonical (line bundle valued) contact form of Σ^{2m+1}. The Bochner-flatness condition also implies that the operators A_r determine an Einstein Bochner-flat Kähler structure M_{A_r} with scalar curvature satisfying a certain differential equation. This allows to determine, using Proposition 20, the operators A_r. Once the A_r's are known, the operators B_r are then determined from (2.17), as in the statement of the theorem. For more details, see [36]. The condition $[A, B] = 0$ just means that (M, g, J) is a restricted generalized Kähler cone. □

Our main result from this section is the following.

Theorem 27. *Any Bochner-flat Kähler manifold of dimension $2m > 4$ is locally isomorphic to a Bochner-flat generalized Kähler cone.*

Proof. One shows (either by using the Webster's correspondence or by computing directly the Bryant minimal and characteristic polynomials) that the Bochner-flat generalized Kähler cones (M, J, g) which belong to case *i)* of Theorem 26 cover all hyperbolic (when $d > 0$), 1-step parabolic (when $d = 0$) and elliptic (when $d < 0$) types, while the Bochner-flat generalized Kähler cones which belong to the case *iii)* of Theorem 26 cover all 2-step parabolic types. Thus, all local types of Bochner-flat Kähler manifolds are covered by the generalized Kähler cones from Theorem 26.

□

2.5.2 Examples

We now consider some important classes of Bochner-flat Kähler manifolds and show how they can be realised locally as generalised Kähler cones.

i) *Weighted projective spaces as generalized Kähler cones.* Let $\mathbb{P}^{m+1}_{\vec{a}}$ be a weighted projective space, of weights $(a_1, \cdots, a_{m+2})$, where a_j are positive integers. As shown in [20, 43] (see also Example 15) $\mathbb{P}^{m+1}_{\vec{a}}$ has a canonical Bochner-flat Kähler structure, of elliptic type, isomorphic with M_C, where C is a Hermitian elliptic operator of $\mathbb{C}^{m+2,1}$, with eigenvalues $-\sum_{j=1}^{m+2} \lambda_j, \lambda_1, \cdots, \lambda_{m+2}$, where λ_j are related to the weights a_j by $\lambda_j = a_j - \frac{1}{m+2}\sum_{i=1}^{m+2} a_i$, for any $j \in \{1, \cdots, m+2\}$. As a Bochner-flat generalized Kähler cone, $\mathbb{P}^{m+1}_{a}$ belongs to the first case of Theorem 26; μ is any solution of equation (2.18), with $d = \frac{2a^2_{m+2}}{(m+3)^2}$, and the Hermitian operator B is elliptic, with eigenvalues $-\frac{\sum_{j=1}^{m+1} a_j}{m+2}, a_1 - \frac{\sum_{j=1}^{m+1} a_j}{m+2}, \cdots, a_{m+1} - \frac{\sum_{j=1}^{m+1} a_j}{m+2}$.

ii) *Bryant Bochner-flat Kähler structures.* They were defined in [19] and further studied in [43] (see also Example 15). Here we give an alternative description, as generalized Kähler cones. Let $N = S^{2m+1} \subset \mathbb{C}^{m+1}$ with its standard CR structure and $(k_1, \cdots, k_{m+1})$ a system of non-negative real numbers. Define, for every $r > 0$, the vector field

$$T_r(z) := \sum_{j=1}^{m+1} \left(1 + k_j r^2\right) \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}\right),$$

which is the Reeb vector field of a Sasaki structure on $S^{2m+1} \subset \mathbb{C}^{m+1}$. Here $z = (z_1, \cdots, z_{m+1})$ belongs to S^{2m+1}, $z_j = x_j + iy_j$ for any $j \in \{1, \cdots, m+1\}$ and $r^2 = |z_1|^2 + \cdots + |z_{m+1}|^2$. The family of Sasaki Reeb vector fields $\{T_r, r > 0\}$ defines a Bochner-flat generalized Kähler cone on $\mathbb{C}^{m+1} \setminus \{0\}$, which belongs to the first case of Theorem 26; the solution of equation (2.18) is $\mu(t) = -\frac{2}{t}$ and the Hermitian operator B is semisimple, with eigenvalues $k'_j = k_j - \frac{1}{m+2}\sum_{i=1}^{m+1} k_i$, for $j \in \{1, \cdots, m+1\}$, and $k_0 = -\frac{1}{m+2}\sum_{i=1}^{m+1}$. This Bochner-flat generalized Kähler cone extends to the entire $\mathbb{C}^{m+1}$ and as such it is isomorphic to Bryant Bochner-flat Kähler structure $\mathbb{M}_{\vec{k}}^{m+1}$, where $\vec{k} := (k_1, \cdots, k_{m+1})$ (see e.g. Example 15).

2.6 The Guillemin formula for toric symplectic manifolds

Dual potentials and Legendre transformations. In this section we often identify the standard vector space $\mathbb{R}^n$ with its dual, using the natural metric $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^n$. We denote by $\{e_i\}$ the standard basis of $\mathbb{R}^n$ and by $\{e_i^*\}$ the dual basis.

Let (M, ω) be a $2n$-dimensional symplectic manifold together with a free Hamiltonian action of a torus T^n. In a neighborhood of any point, one may find coordinates (x, t) such that $\omega = \sum_{r=1}^n dx^r \wedge dt^r$. The vector fields $\frac{\partial}{\partial t^i}$ are fundamental vector fields generated by the torus action and $(x^1, \cdots, x^n)$ is the moment map. In these coordinates, invariant compatible Kähler metrics are parametrized by functions G depending on (x^i) only, with positive definite Hessian (defining the induced metric on the image of the moment map). Such a function is called a dual potential. From the dual potential, one may obtain an invariant potential of the Kähler metric by means of a Legendre transformation

$$F + G = \sum_{k=1}^n x^k \frac{\partial G}{\partial x^k}. \tag{2.19}$$

The Guillemin formula. Let (M^{2n}, ω) be a compact toric symplectic manifold and $\nu : M \to (\mathbb{R}^n)^*$ the moment map, whose image is a Delzant polytope

$$\Delta = \{x \in (\mathbb{R}^n)^*, \quad l_j(x) := \langle x, u_j \rangle - \lambda_j \geq 0, \quad j = 1, \cdots, d\},$$

where $d > n$, $\lambda_j \in \mathbb{R}$, $u_j \in \mathbb{Z}^n$ and the u_j corresponding to the faces meeting any vertex of Δ is $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^n$. It is well-known that (M^{2n}, ω) can be recovered from Δ as a symplectic reduction of the standard symplectic vector space $\left(\mathbb{R}^{2d}, \omega_0 = \sum_{j=1}^d r_j dr_j \wedge d\theta_j\right)$ with the natural action of a $(d-n)$-subtorus N of T^d, where N is the kernel of the map $T^d \to T^n$, induced by the map $u : \mathbb{R}^d \to \mathbb{R}^n$, $e_j \to u_j$. More precisely, $M = \mu_N^{-1}(c)/N$, where $\mu_N = i^* \circ \mu$ is the moment map of the N-action on $\mathbb{R}^{2d}$, μ is the moment map of the natural action of T^d on $\mathbb{R}^{2d}$, i.e.

$$\mu(r_1, \theta_1, \cdots, r_d, \theta_d) = \frac{1}{2}(r_1^2, \cdots, r_d^2), \tag{2.20}$$

$i^* : (\mathbb{R}^d)^* \to \mathfrak{n}^*$ is the adjoint of the inclusion of $\mathfrak{n} = \mathrm{Lie}(N)$ in $\mathbb{R}^d = \mathrm{Lie}(T^d)$ and $c = i^*(-\lambda) \in \mathfrak{n}^*$, where $\lambda = (\lambda_1, \cdots, \lambda_d) \in (\mathbb{R}^d)^*$. Moreover, M comes with a Hamiltonian action of the quotient torus $T^n := T^d/N$, whose moment map ν fits into a commutative diagram

$$\begin{array}{ccc} \mu_N^{-1}(c) & \xrightarrow{\mu} & (\mathbb{R}^d)^* \\ q \downarrow & & \uparrow l \\ M & \xrightarrow{\nu} & \Delta \subset (\mathbb{R}^n)^* \end{array}$$

where $l : (\mathbb{R}^n)^* \to (\mathbb{R}^d)^*$ has components $(l_1, \cdots, l_d)$. The symplectic manifold (M, ω) has a canonical Kähler metric g_{can}, inherited from the standard Kähler metric of $\mathbb{R}^{2d} \cong \mathbb{C}^d$, which on $M_0 := \nu^{-1}(\Delta^0)$ (Δ^0 the interior of Δ) has a global invariant Kähler potential, computed by Guillemin [63]. We now give an alternative simple argument for the Guillemin formula.

Our argument is based on the natural behaviour of dual potentials under Kähler reduction [25], which in our case means that $G = l^*(G_0)$, where G_0 is the dual potential of the standard Kähler structure on $\mathbb{C}^{2d}$. Since $G_0 = \frac{1}{2}\sum_{j=1}^d \mu_j \mathrm{log} \mu_j$ (a simple computation) we obtain $G = \frac{1}{2}\sum_{j=1}^d l_j \mathrm{log} l_j$. The Kähler potential of g_{can} is now obtained from G, using a Legendre transformation (2.19). We obtain:

Theorem 28. *The canonical invariant Kähler metric g_{can} of the compact toric symplectic manifold (M, ω) admits an invariant Kähler potential which, read on Δ^0, is given by $F = \frac{1}{2}\sum_{j=1}^d (\lambda_j \mathrm{log} l_j + l_j)$.*

Chapter 3

Contributions to Quaternionic Geometry

3.1 Introduction

This chapter is concerned quaternionic connections and natural differential operators in quaternionic geometry and summarizes the results published by the author of this thesis in **Journal of the London Mathematical Society** [42], **International Journal of Mathematics** [37], **Annali di Matematica Pura ed Applicata** [31], **Journal of Geometry and Physics** [32, 33]. We are mainly interested in the geometry of two differential operators on quaternionic or quaternionic-Kähler manifolds, closely related to each other: the Penrose operator and the conformal-Killing operator.

We now present with details the contents of each section separately, explaining also how the original results fit into general research field.

In **Section 3.2** we recall the basic facts we need from quaternionic geometry: quaternionic connections, the $E-H$ formalism, twistor spaces, quaternionic-Kähler manifolds. At the end of this section we recall the definition of conformal-Killing forms in the general setting of Riemannian manifolds.

In **Section 3.3** we develop a detailed study of the geometry of quaternionic connections, their twistor theory and associated Penrose operators. The results from this section may be seen as an extension of the theory developed in [56], from the conformal setting to the quaternionic setting. We are concerned with the tangent vertical bundle Θ of the twistor fibration $\pi : Z \to M$ of a quaternionic manifold (M, Q). It is a complex vector bundle, with complex structure induced by the complex structure of the twistor lines (i.e. the fibers of π). Our main result is a classification of all holomorphic structures of Θ, in terms of self-dual quaternionic connections (i.e. quaternionic connections for which the skew part of the Ricci tensor is Q-Hermitian) and Maxwell fields (i.e. 1-forms with Q-Hermitian exterior derivative), see Theorem 30. In particular, any self-dual quaternionic connection D on (M, Q) defines an holomorphic structure on $\bar{\partial}^D$ on Θ and we develop a Penrose transformation from the kernel of the Penrose operator P^D defined by D to a class of (so called purely imaginary) holomorphic sections of $(\Theta, \bar{\partial}^D)$ (see Proposition 33). We find a criterion for the non-existence of global holomorphic sections of $(\Theta^s, \bar{\partial}^D)$, for any $s \in \mathbb{N} \setminus \{0\}$ (see Proposition 34).

In **Section 3.4** we develop a local treatment of a class (called compatible) of conformal-Killing 2-forms on quaternionic-Kähler manifolds. A 2-form on a quaternionic-Kähler manifold (M^{4n}, g) is called compatible if it is a section of the direct sum bundle $S^2H \oplus S^2E$, where E and H are the (locally defined) complex vector bundles associated to the standard representations of $Sp(n)$ and $Sp(1)$ on $\mathbb{C}^{2n}$ and $\mathbb{C}^2$, respectively. Compatible conformal-Killing 2-forms are closely related to Penrose operators - one may show that the S^2H-component ψ^{S^2H} of a compatible conformal-Killing 2-form ψ on (M, g) belongs to the kernel of the Penrose operator defined by the Levi-Civita connection of g. Our main result from this section is Theorem 35, which provides a prolongation $\mathcal{D}$ of the conformal-Killing operator acting on compatible 2-forms on (M, g). This means that $\mathcal{D}$ is a vector bundle connection whose space of parallel sections is isomorphic to the space $\mathcal{C}_2(M)$ of compatible conformal-Killing 2-forms on (M, g). The connection $\mathcal{D}$ acts on the bundle $S^2H \oplus S^2E \oplus TM$. We compute the curvature of $\mathcal{D}$ and we show that $\mathcal{D}$ is flat if and only if the quaternionic-Weyl tensor W^Q of (M, g) is zero (see Proposition 38). Several applications of this result are developed. Namely, we obtain an upper bound for the dimension of the space of compatible conformal-Killing 2-forms, which is reached on the standard quaternionic projective space (see Corollary 39). We also show that a quaternionic-Kähler manifold (M^{4n}, g), with non-zero scalar curvature, which admits a non-parallel compatible conformal-Killing 2-form ψ has holonomy group the entire $Sp(1)Sp(n)$ (see Proposition 40) and that ψ is uniquely determined by its S^2E-component (see Proposition 41). It is worth to mention that $\mathcal{D}$ may be seen as the quaternionic analogue of a prolongation of the conformal-Killing operator on a conformal manifold, constructed in [99], and whose curvature was identified with the Weyl tensor of the conformal manifold.

In **Section 3.5** we prove that any compact quaternionic-Kähler manifold (M, g) of dimension $4n \geq 8$ admitting a non-parallel conformal-Killing 2-form is isomorphic to the quaternionic projective space $\mathbb{H}P^n$, with its standard quaternionic-Kähler structure, and we find all conformal-Killing 2-forms on the standard $\mathbb{H}P^n$ - it turns out that all are compatible 2-forms. The idea of the proof is the following. First one reduces (easily) this statement to the case when the scalar curvature of the compact quaternionic-Kähler manifold (M, g) is positive. With this additional assumption, the proof of the statement consists in two steps. In a first stage, one shows that the codifferential X of a non-parallel conformal-Killing 2-form ψ on (M, g) is a (non-trivial) Killing vector field which belongs to the kernel of the quaternionic-Weyl tensor W^Q. In a second stage, one shows that a compact quaternionic-Kähler manifold with positive scalar curvature and which admits a Killing vector field X such that $W^Q(X, \cdot) = 0$ is isomorphic to the quaternionic projective space, with its standard quaternionic-Kähler metric. Then one determines the general form of conformal-Killing 2-forms on $\mathbb{H}P^n$ and this concludes the proof of the statement. The results of this section complement a theorem by A. Moroianu and U. Semmelman, which states that any Killing p-form ($p \geq 2$) on a compact quaternionic-Kähler manifold is parallel and fit into the systematic treatment of conformal-Killing or Killing forms on other compact Riemannian manifolds with special holonomy (e.g. U_n, G_2 and $Spin_7$), developed by the same authors in [81, 82, 100].

The initial motivation of **Section 3.6** is a result of U. Semmelmann [99], which states that an almost Hermitian manifold (M, J, g) whose Kähler form ω is conformal-Killing is necessarily nearly-Kähler (i.e. ω is a Killing form or, equivalently, $\nabla\omega = \frac{1}{2}d\omega$, where ∇ is the Levi-Civita connection). We prove the analogous statements for other G-structures, where $G := Sp(1)Sp(n)$, G_2 and $Spin_7$. More precisely, we prove that if the fundamental form of an almost quaternionic-Hermitian manifold (respectively, of a manifold with a G_2-structure or a manifold with a $Spin_7$-structure) is conformal-Killing, then the almost quaternionic-Hermitian manifold is quaternionic-Kähler (respectively, the G_2-structure is nearly parallel and the $Spin_7$-structure is parallel). The proofs of these statements are based on the Schur's lemma and representation theory of the groups in question. It is worth to mention that other similar results exist in the literature. Namely, A. Swann proved that an almost quaternionic-Hermitian manifold with closed fundamental form is necessarily quaternionic-Kähler [104]. In [105] he also considered almost quaternionic-Hermitian manifolds with co-closed fundamental form.

3.2 Preliminary material

3.2.1 Quaternionic connections

An almost quaternionic structure on a manifold M (always assumed of dimension $4n \geq 8$) is a rank three subbundle Q of $\mathrm{End}(TM)$ locally generated by three anti-commuting almost complex structures $\{J_1, J_2, J_3\}$ such that $J_3 = J_1 J_2$ (such a triple is called a local admissible basis of Q). The bundle Q has a natural Euclidian metric $\langle \cdot, \cdot \rangle$, with respect to which any local admissible basis is orthonormal. Q is called a quaternionic structure (and (M, Q) a quaternionic manifold) if it is preserved by a torsion-free linear connection on M, called a quaternionic connection. Alternatively, a quaternionic manifold may also be defined as a manifold with a $GL_n(\mathbb{H})Sp(1)$-structure.

On a quaternionic manifold (M, Q) the space of quaternionic connections is parametrized by 1-forms on M, i.e. any two quaternionic connections D and D' are related by $D' = D + S^{\alpha}$, where $\alpha \in \Omega^1(M)$ and S^{α} is an $\mathrm{End}(TM)$-valued 1-form on M whose expression may be found e.g. in [5]. We say that D and D' are equivalent if α is an exact 1-form. We say that D is a closed quaternionic connection (respectively, an exact quaternionic connection) if it induces a flat connection on $\Lambda^{4n}(T^*M)$ (respectively, if there is a volume form on M preserved by D). Since the curvature of the induced connection on $\Lambda^{4n}(T^*M)$ coincides (up to a multiplicative constant) with the skew part of the Ricci tensor [5], a quaternionic connection is closed if and only if its Ricci tensor is symmetric. It can be shown that any two exact quaternionic connections are equivalent. The family of exact quaternionic connections forms the canonical class of equivalent quaternionic connections of (M, Q). We shall meet a third class of connections, the so called self-dual quaternionic connections; a quaternionic connection is self-dual if the induced connection on the bundle $\Lambda^{4n}(T^*M)$ has Q-Hermitian curvature, i.e. its curvature is of type $(1, 1)$ with respect to any complex structure on M which is a section of the bundle Q.

A quaternionic curvature tensor of (M, Q) is a curvature tensor R of M (i.e. a section of $\Lambda^2(T^*M) \otimes \mathrm{End}(TM)$ in the kernel of the Bianchi map) which takes values in the normalizer of Q, i.e. for any $X, Y \in TM$, $[R(X, Y), Q] \subset Q$. The space $\mathcal{R}(N(Q))$ of quaternionic curvature tensors decomposes into the direct sum $\mathcal{W}^Q \oplus \mathcal{R}^{\mathrm{Bil}}$ where $\mathcal{W}^Q$, called the space of quaternionic Weyl curvatures, is the kernel of the Ricci contraction Ricci : $\mathcal{R}(N(Q)) \to \mathrm{Bil}(TM)$, defined by $\mathrm{Ricci}(R)(X, Y) := \mathrm{trace}\{Z \to R_{Z,X}Y\}$ and $\mathcal{R}^{\mathrm{Bil}}$ is its orthogonal complement, isomorphic to the space $\mathrm{Bil}(TM)$ of bilinear forms on TM (see [5]). The curvature R^D of a quaternionic connection D belongs to $\mathcal{R}(N(Q))$ and its projection to $\mathcal{W}^Q$ is independent of the choice of D and is called the quaternionic-Weyl tensor of (M, Q). The following result is well-known.

Theorem 29. *A quaternionic manifold (M, Q) of dimension $4n \geq 8$ is locally isomorphic to the quaternionic projective space $\mathbb{H}P^n$ with its standard quaternionic structure if and only if $W^Q = 0$, where W^Q is the quaternionic-Weyl tensor of (M, Q).*

3.2.2 Twistor theory for quaternionic manifolds

Recall that the quaternionic bundle Q of a quaternionic manifold (M, Q) has a natural Euclidian metric $\langle \cdot, \cdot \rangle$, for which any local admissible basis is orthonormal. The twistor space Z of (M, Q) is defined as the total space of the unit sphere bundle of Q and has a natural complex structure. In order to define it, we first consider a twistor line Z_p, i.e the fiber of the natural projection $\pi : Z \to M$ corresponding to a point $p \in M$. Then $T_J Z_p$

consists of all J-anti-linear endomorphisms of T_pM which belong Q_p, or to the orthogonal complement $J^\perp$ of J in Q_p, with respect to the metric $\langle\cdot,\cdot\rangle$. Note that Z_p is a Kähler manifold, with complex structure $\mathcal{J}$ defined by

$$\mathcal{J}(A) := J \circ A, \quad \forall A \in T_J Z_p, \quad \forall J \in Z_p, \quad \forall p \in M. \tag{3.1}$$

Note that $\mathcal{J}$ is well-defined, since

$$T_J Z_p = \{A \in Q_p : \quad A \circ J + J \circ A = 0\} = J^\perp \subset Q_p,$$

where $\perp$ denotes the orthogonal complement with respect to the metric $\langle\cdot,\cdot\rangle$. Now we are able to define the complex structure $\mathcal{J}$ of Z: chose a quaternionic connection D of (M, Q). Since D preserves Q and $\langle\cdot,\cdot\rangle$, it induces a connection on the twistor bundle $\pi : Z \to M$, i.e. a decomposition of every tangent space $T_J Z$ into the vertical tangent space $T_J Z_p$ and horizontal space Hor_J. On Hor_J, identified with T_pM by means of the differential π_*, $\mathcal{J}$ is equal to J. On $T_J Z_p$, $\mathcal{J}$ is defined as in (3.1). It can be shown that $\mathcal{J}$ so defined is independent of the choice of quaternionic connection and is integrable. The twistor space Z becomes a complex manifold of dimension $2n+1$ and the twistor lines are complex projective lines of Z with normal bundle $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$ (where $4n$ is the dimension of M).

The tangent vertical bundle Θ: The tangent vertical bundle Θ of the twistor projection $\pi : Z \to M$ is the bundle over Z whose fiber at a point $J \in Z$ is the tangent space $T_J Z_p$ at the twistor line Z_p where $p = \pi(J)$. It is a complex line bundle over the complex manifold Z, with complex structure of the fibers defined by the complex structure of the twistor lines. Moreover, it has a canonical Hermitian metric $h(X, Y) := \frac{1}{2}\left(\langle X, Y\rangle - i\langle \mathcal{J}X, Y\rangle\right)$, for any $X, Y \in \Theta_J = T_J Z_p \subset Q_p$.

Any section $A \in \Gamma(Q)$ defines a section $\tilde{A}$ of Θ, by the formula:

$$\tilde{A}(J) = \Pi_J(A) := \frac{1}{2}\left(A + J \circ A \circ J\right) = A - \langle A, J\rangle J, \quad J \in Z,$$

where the bundle homomorphism $\Pi : \pi^* Q \to \Theta$ is the orthogonal projection onto $\Theta \subset \pi^* Q$ with respect to the metric of $\pi^* Q$ induced by the natural Euclidian metric $\langle\cdot,\cdot\rangle$ of Q. Such sections of Θ will be called distinguished. The differential $\sigma_* : TZ \to TZ$ of the antipodal map $\sigma : Z \to Z$, $\sigma(J) = -J$ induces an involution on the space of smooth sections of Θ, which associates to a section s the section $\bar{s}$ defined as follows: for any $J \in Z$, $\bar{s}_J := \sigma_*\left(s_{\sigma(J)}\right)$. If $s := \tilde{A}$ is distinguished, then $\bar{s} = -s$. This is why the distinguished sections are also called purely imaginary. Moreover, $\mathcal{J}s$ is real, i.e. $\overline{\mathcal{J}s} = \mathcal{J}s$. The distinguished sections of Θ will play a fundamental role in our twistor theory for Penrose operators (see Section 3.3).

3.2.3 Quaternionic-Kähler manifolds

A quaternionic-Kähler manifold is a Riemannian manifold (M, g) of dimension $4n \geq 8$ with holonomy group included in $Sp(n)Sp(1)$. A quaternionic-Kähler manifold is quaternionic and the Levi-Civita connection is a quaternionic connection. The metric g of a quaternionic-Kähler manifold is Einstein and its curvature has the expression

$$R^g(X,Y) = -\frac{\nu}{4}\left(X \wedge Y + \sum_{i=1}^{3} J_iX \wedge J_iY + 2\sum_{i=1}^{3}\omega_i(X,Y)\omega_i\right) + W^Q(X,Y) \tag{3.2}$$

where $\nu := \frac{k}{4n(n+2)}$ is the reduced scalar curvature (k being the usual scalar curvature).

When the reduced scalar curvature ν of (M, g) is positive, the twistor space Z of the underlying quaternionic structure of (M, g) has an Einstein metric $\bar{g}$, which, together with the natural complex structure $\mathcal{J}$ of Z, makes Z a Kähler-Einstein manifold. In order to define $\bar{g}$, consider the horizontal bundle $H^\nabla \subset TZ$ associated to the Levi-Civita connection ∇ of g, acting on the twistor bundle $\pi : Z \to M$. On H^∇ $\bar{g}$ is the pull-back of g; the twistor lines are $\bar{g}$-orthogonal to H^∇; when $\nu = 1$ the restriction of $\bar{g}$ to the twistor lines is the standard metric on S^2 of curvature one; equivalently, the restriction of $\bar{g}$ to a twistor line Z_p is induced by the Euclidian metric $\langle\cdot,\cdot\rangle$ of the fiber Q_p of Q over p, by means of the inclusion $Z_p \subset Q_p$. The twistor projection $\pi : (Z, \bar{g}) \to (M, g)$ becomes a Riemannian submersion with totally geodesic fibers.

3.2.4 The $E - H$-formalism

Like for conformal 4-manifolds, on any quaternionic manifold (M, Q) there are two locally defined complex vector bundles H and E over M (sometimes called the spin bundles), of rank 2 and $2n$, associated to the standard representations of $Sp(1)$ and $GL(n, \mathbb{H})$ on $\mathbb{C}^2$ and $\mathbb{C}^{2n}$ respectively. The bundles E and H play the role of the spin bundles in conformal geometry, because

$$T_{\mathbb{C}}M = H \otimes E. \tag{3.3}$$

Since $Sp(1) \cong SU(2)$, H has a complex parallel symplectic form ω_H and a compatible quaternionic structure, i.e. a $\mathbb{C}$-anti-linear map $q_H : H \to H$, which satisfies $q_H^2 = -\mathrm{Id}_H$, $\omega_H(qv, qw) = \overline{\omega_H(v,w)}$ and $\omega_H(v, q_Hv) > 0$, for any $v, w \in H$. The 2-form ω_H together with q_H define an invariant Hermitian positive definite metric $\langle\cdot,\cdot\rangle_H := \omega_H(\cdot, q_H\cdot)$ on H. By means of the identification $H \ni h \to \omega(h, \cdot) \in H$ between H and its dual H^*, $S^2(H) \subset H \otimes H \cong H^* \otimes H \subset \mathrm{End}(H)$ acts on H and its real part (with respect to the real structure on $\mathrm{End}(H)$ induced by q_H) is isomorphic to the bundle Q. Similarly, S^2E is isomorphic to the complexification of the bundle of Q-Hermitian forms, i.e. 2-forms ψ which satisfy

$$\psi(AX, Y) = -\psi(X, AY), \quad \forall A \in Q, \quad \forall X, Y \in TM.$$

When Q is the quaternionic bundle of a quaternionic-Kähler manifold (M, g), the bundle E comes equipped with a parallel complex symplectic form ω_E and, by means of the decomposition (3.10),

$$g_{\mathbb{C}} = \omega_H \otimes \omega_E.$$

Moreover,

$$\Lambda^2(T^*_{\mathbb{C}}M) = S^2H \oplus S^2E \oplus (S^2H \otimes \Lambda^2_0 E), \tag{3.4}$$

where $\Lambda^2_0 E \subset \Lambda^2 E$ is the kernel of the natural contraction with the standard symplectic form on E. For a 2-form $\psi \in \Omega^2(M)$, we denote by ψ^{S^2H}, ψ^{S^2E} and $\psi^{S^2H\otimes\Lambda^2_0E}$ its projections on the three factors of the decomposition (3.4). By identifying vectors with covectors using g, for any decomposable 2-form $\psi = X \wedge Y$, the following identities hold:

$$(X \wedge Y)^{S^2H} = \frac{1}{2n}\sum_{i=1}^{3} \omega_i(X, Y)\omega_i \tag{3.5}$$

and

$$(X \wedge Y)^{S^2E} = \frac{1}{4}\left(X \wedge Y + \sum_{i=1}^{3} J_iX \wedge J_iY\right), \tag{3.6}$$

with respect to any local admissible basis $\{J_1, J_2, J_3\}$ of Q, with associated Kähler forms $\omega_i = g(J_i\cdot, \cdot)$.

3.2.5 Conformal-Killing forms on Riemannian manifolds

At the end of this preliminary section we briefly recall the definition of conformal-Killing forms on Riemannian manifolds. For a survey on this topic, see [99].

Let (M^m, g) be a Riemannian m-dimensional manifold. For any $1 \leq p \leq m$ consider the tensor product bundle $T^*M \otimes \Lambda^p(M)$ and its irreducible $O(m)$-decomposition:

$$T^*M \otimes \Lambda^p(M) = \Lambda^{p+1}(M) \oplus \Lambda^{p-1}(M) \oplus \mathcal{T}^{p,1}(M), \tag{3.7}$$

where the sub-bundle $\mathcal{T}^{p,1}(M)$ of $T^*M \otimes \Lambda^p(M)$ is the intersection of the kernels of the wedge product and inner contraction maps. The orthogonal projection $P_{p,m}$ onto the third component in the above decomposition is called the algebraic conformal-Killing operator and depends on the degree of the form and the dimension of the manifold. If $\psi \in \Omega^p(M)$ is a p-form, the covariant derivative $\nabla\psi$ with respect to the Levi-Civita connection ∇ of g is a section of $T^*M \otimes \Lambda^p(M)$. Its projection onto $\Lambda^{p+1}(M)$ and $\Lambda^{p-1}(M)$ is given, essentially, by the exterior derivative $d\psi$ and the codifferential $\delta\psi$, respectively. The p-form ψ is called conformal-Killing if $\nabla\psi$ is a section of $\Lambda^{p+1}(M) \oplus \Lambda^{p-1}(M)$, i.e. belongs to the kernel of $P_{p,m}$, i.e. the conformal-Killing equation

$$\nabla_Y\psi = \frac{1}{p+1} i_Y d\psi - \frac{1}{m-p+1} Y \wedge \delta\psi, \quad \forall Y \in TM \tag{3.8}$$

is satisfied (we identify, without mentioning explicitly, tangent vectors with 1-forms by means of the Riemannian duality). A co-closed conformal-Killing form is called Killing.

3.3 The geometry of quaternionic connections

3.3.1 Holomorphic structures on Θ

In this section we classify the holomorphic structures of the tangent vertical bundle Θ of the twistor fibration $\pi : Z \to M$ of a quaternionic manifold (M, Q). Recall that Θ is a complex Hermitian line bundle over the twistor space Z, which is a complex manifold. Due to this, there is an isomorphism between Chern connections of Θ (i.e. Hermitian connections with $\mathcal{J}$-invariant curvature) and holomorphic structures of Θ, i.e. operators

$$\bar\partial : \Gamma(\Theta) \to \Omega^{0,1}(Z, \Theta)$$

which satisfy the Liebniz rule

$$\bar\partial(fs) = f\bar\partial(s) + \bar\partial(f)s, \quad f \in C^\infty(Z, \mathbb{C}), \quad s \in \Gamma(\Theta)$$

and whose natural extension to the complex $\Omega^{0,*}(Z, \Theta)$ satisfy $\bar\partial^2 = 0$. The isomorphism associates to a Chern connection ∇ its $(0,1)$-part

$$\bar\partial_U s := \frac{1}{2}\left(\nabla_U s + \mathcal{J}\nabla_{\mathcal{J}U} s\right), \quad \forall U \in TZ, \quad \forall s \in \Gamma(\Theta).$$

Hence the study of holomorphic structures of Θ reduces to the study of Chern connections.

Remark now that if D is a quaternionic connection on (M, Q), then π^*D is a connection on the pull-back bundle π^*Q and $\nabla^D := \Pi \circ \pi^*D$ is a connection on the bundle Θ, where $\Pi : \pi^*Q \to \Theta$ is the orthogonal projection (with respect to the metric of π^*Q induced by the standard metric $\langle \cdot, \cdot \rangle$ of Q). One may show that ∇^D is a Chern connection (and defines an holomorphic structure $\bar\partial^D$) if and only if D is self-dual. More generally, one may prove:

Theorem 30. *i) Any holomorphic structure of* Θ *is equivalent (i.e. conjugated by an element of the gauge group* $C^\infty(Z, \mathbb{C}^*)$*) with an holomorphic structure* $\bar{\partial}^{D,\beta} := \bar{\partial}^D + \tilde{\beta}$*, where* $\bar{\partial}^D$ *is the* $(0,1)$*-part of the Chern connection* ∇^D *of* Θ *induced by a self-dual quaternionic connection* D *of* (M, Q)*,* $\beta \in \Omega^1(M)$ *has* Q*-Hermitian exterior differential and* $\tilde{\beta} \in \Omega^{0,1}(Z, \mathrm{End}_{\mathbb{C}}(\Theta))$ *is defined as follows: for any* $U \in T_J Z$ *with* $\pi_* U = X$ *and* $s \in \Gamma(\Theta)$*,*

$$\tilde{\beta}_U(s) := \frac{1}{2} \left(\beta(X) \mathcal{J} s - \beta(JX) s \right).$$

ii) Two holomorphic structures $\bar{\partial}^{D,\beta}$ *and* $\bar{\partial}^{D',\beta'}$ *are equivalent if and only if* D *and* D' *are equivalent as quaternionic connections of* (M, Q) *(see Section 3.3) and* $\beta - \beta'$ *is an exact* 1*-form.*

Recall now that any two exact quaternionic connections are equivalent. The following Corollary is a consequence of Theorem 84.

Corollary 31. *The tangent vertical bundle of the twistor fibration of a quaternionic manifold* (M, Q) *has a canonical class of equivalent holomorphic structures, determined by the exact quaternionic connections of* (M, Q)*.*

3.3.2 A Penrose transform

Let (M, Q) be a quaternionic manifold, with spin bundles E and H. The product $H \otimes S^2(H)$ decomposes into two irreducible subbundles: $S^3(H)$, which is the kernel of the map $F : H \otimes S^2(H) \to H$ defined by

$$F(h, h_1 h_2 + h_2 h_1) = \omega_H(h, h_1) h_2 + \omega_H(h, h_2) h_1, \quad h_1, h_2, h \in H, \tag{3.9}$$

and H, isomorphic to the Hermitian orthogonal of $S^3(H)$ in $H \otimes S^2(H)$ with respect to the Hermitian metric of $H \otimes S^2(H)$ induced by the Hermitian metric $\langle \cdot, \cdot \rangle_H$ of H (to simplify notations, we omit the tensor product signs, so that $h_1 h_2 + h_2 h_1$ denotes $h_1 \otimes h_2 + h_2 \otimes h_1$). Therefore,

$$T^* M \otimes Q = E^* \otimes S^3 H \oplus E^* \otimes H. \tag{3.10}$$

Definition 32. *The Penrose operator*

$$P^D : \Gamma(S^2 H) \to \Gamma(E^* \otimes S^3 H)$$

defined by a quaternionic connection D *on* (M, Q) *maps a section* σ *of* $S^2 H$ *to the projection of* $D\sigma \in \Gamma(T^* M \otimes S^2 H)$ *onto the first component of the irreducible decomposition (3.10).*

Our main result in this section is the following.

Proposition 33. *Let D be a self-dual quaternionic connection on (M,Q) and $A \in \Gamma(Q)$. Then the distinguished section $\tilde{A}$ of Θ, defined by*

$$\tilde{A}_J := A_p - \langle A_p, J\rangle J, \quad J \in Z, \quad p = \pi(J),$$

is $\bar{\partial}^D$-holomorphic if and only if $P^D(A) = 0$.

Proof. The proof is representation theoretic. The section $\tilde{A}$ is $\bar{\partial}^D$-holomorphic if and only if it satisfies

$$\nabla^D_{\mathcal{J}U}(\tilde{A}) = \mathcal{J}\nabla^D_U(\tilde{A}), \quad \forall U \in TZ, \tag{3.11}$$

where ∇^D is the Chern connection of Θ induced by D. It can be seen that (3.11) is equivalent with

$$D_{JX}A - \langle D_{JX}A, J\rangle J - JD_X A - \langle D_X A, J\rangle \mathrm{Id}_{TM} = 0, \quad \forall X \in TM, \quad \forall J \in Z. \tag{3.12}$$

Remark now that $\gamma := DA$ is a section of $E^* \otimes H^* \otimes S^2H$, which has two $\tilde{G}$-irreducible subbundles (where $\tilde{G} := GL_n(\mathbb{H}) \times Sp(1)$), namely $E^* \otimes H$ and $E^* \otimes S^3H$. In terms of γ, relation (3.12) becomes

$$\gamma_{JX} - \langle \gamma_{JX}, J\rangle J - J\gamma_X - \langle \gamma_X, J\rangle \mathrm{Id}_{TM} = 0, \quad \forall X \in TM, \quad \forall J \in Z. \tag{3.13}$$

The action of $\tilde{G}$ on $E^* \otimes H^* \otimes S^2H$ preserves the space of solutions of (3.13). Since obviously there are distinguished sections of Θ which are not $\bar{\partial}^D$-holomorphic, it follows that the space of solutions of (3.13) coincides either with $E^* \otimes H$ or with $E^* \otimes S^3H$. One then shows that any element of $E^* \otimes H$ satisfies (3.13). This means that $\tilde{A}$ is $\bar{\partial}^D$-holomorphic if and only if DA is a section of $E^* \otimes H$, i.e. $P^D(A) = 0$. □

In the compact case, when the quaternionic manifold has in addition a compatible quaternionic-Kähler metric, there is the following non-existence criterion on holomorphic sections of tensor powers of Θ. For details of the proof (which uses the so called conformal weight operator in the quaternionic setting, similar to the one from the conformal setting [56], and the Penrose transformation from the above proposition), see [37].

Proposition 34. *Let D be a closed quaternionic connection on a compact quaternionic-Kähler manifold (M,Q,g). If $\mathrm{Scal}^g < 0$ (respectively, $\mathrm{Scal}^g = 0$ and D is not exact) then Θ^s (for any $s \in \mathbb{N}\setminus\{0\}$) has no global non-trivial $\bar{\partial}^D$-holomorphic sections.*

3.4 A prolongation of the conformal-Killing operator on quaternionic-Kähler manifolds

In this section we find a prolongation of the conformal-Killing operator acting on the space of so called compatible 2-forms on a quaternionic-Kähler manifold and we deduce various consequences of this prolongation.

3.4.1 The prolongation $\mathcal{D}$

Given a linear differential operator D, it is sometimes useful to determine a vector bundle connection (called a prolongation of D) whose space of parallel sections is isomorphic with the kernel of D. In general, there are several connections with this property. However, if one prolongation is flat, then all are.

In this section we determine a prolongation $\mathcal{D}$ of the conformal-Killing operator acting on compatible 2-forms on a quaternionic-Kähler manifold (M, g) and we show that $\mathcal{D}$ is flat if and only if the quaternionic-Weyl tensor W^Q of (M, g) is zero. (Recall that a 2-form on a quaternionic-Kähler manifold is called compatible if it is a (real) section of $S^2H \oplus S^2E$.) The prolongation $\mathcal{D}$ acts on the direct sum bundle $S^2H \oplus S^2E \oplus TM$. More precisely, we state:

Theorem 35. *Let (M, g) be a quaternionic-Kähler manifold of dimension $4n \geq 8$, reduced scalar curvature ν and quaternionic-Weyl tensor W^Q. Define a connection $\mathcal{D}$ on $S^2H \oplus S^2E \oplus TM$, by*

$$\mathcal{D}_Z(\psi, X)^{S^2E\oplus S^2H} = \nabla_Z\psi - \frac{1}{4n-1}\left(X \wedge Z + \sum_{i=1}^{3} J_iX \wedge J_iZ - \sum_{i=1}^{3}\omega_i(X, Z)\omega_i\right)$$
$$\mathcal{D}_Z(\psi, X)^{TM} = \nabla_Z X - \frac{4n-1}{4} i_Z\left(\nu\psi^{S^2E} - 2\nu\psi^{S^2H} + \frac{1}{n+1}W^Q(\psi)\right),$$

where ∇ is the Levi-Civita connection, $\{J_1, J_2, J_3\}$ is a local admissible basis of the quaternionic bundle Q, with Kähler forms $\{\omega_1, \omega_2, \omega_3\}$, ψ is a section of $S^2H \oplus S^2E$ and X, Z are vector fields on M. Then $\mathcal{D}$ is a prolongation of the conformal-Killing operator acting on compatible 2-forms. Moreover, $\mathcal{D}$ is flat if and only if $W^Q = 0$.

We divide the proof of Theorem 35 into three propositions. In Proposition 36 we rewrite the conformal-Killing equation on compatible 2-forms in a way suitable for the prolongation procedure. In Proposition 37 we show that the connection $\mathcal{D}$ from Theorem 35 is a prolongation of the conformal-Killing operator acting on compatible 2-forms. Finally, in Proposition 38 we compute the curvature of $\mathcal{D}$ and we show that $\mathcal{D}$ is flat if and only if $W^Q = 0$. For detailed proofs, see [31].

Proposition 36. *With the notations from Theorem 35, a compatible 2-form ψ on (M, g) is conformal-Killing if and only if it satisfies*

$$\nabla_Y\psi = \frac{1}{4n-1}\left(X \wedge Y + \sum_{i=1}^{3} J_iX \wedge J_iY - \sum_{i=1}^{3}\omega_i(X, Y)\omega_i\right), \quad \forall Y \in TM, \tag{3.14}$$

where X is a vector field (necessarily equal to $\delta\psi$).

We remark that Proposition 36 implies that any compatible Killing 2-form on (M, g) is parallel, a result previously proved, in the compact case (and also for higher degree Killing forms), in [81]. It also implies that the S^2H-component of a compatible conformal-Killing 2-form is a solution of the Penrose operator defined by the Levi-Civita connection [31].

Proposition 37. *The map*

$$\psi \to (\psi, \delta\psi)$$

is an isomorphism from the vector space $\mathcal{C}_2(M)$ of compatible conformal-Killing 2-forms to the vector space of $\mathcal{D}$-parallel sections (where $\mathcal{D}$ is the connection from Theorem 35).

In order to conclude the proof of Theorem 35, we still need to compute the curvature of the connection $\mathcal{D}$. This is done in the following proposition.

Proposition 38. *The curvature $R^{\mathcal{D}}$ of the connection $\mathcal{D}$ defined in Theorem 35 has the following expression: for any section (ψ, X) of $S^2H \oplus S^2E \oplus TM$ and vector fields $Y, Z \in \mathcal{X}(M)$,*

$$\begin{aligned} R^{\mathcal{D}}_{Y,Z}(\psi, X)^{S^2H \oplus S^2E} &= [W^Q_{Y,Z}, \psi] - \frac{1}{n+1}\left(W^Q(\psi) \wedge \mathrm{Id}\right)^{S^2E}_{Y,Z} \\ R^{\mathcal{D}}_{Y,Z}(\psi, X)^{TM} &= \frac{n+2}{n+1} W^Q_{Y,Z} X + \frac{4n-1}{4(n+1)} C(\psi^{S^2E})_{Y,Z}, \end{aligned}$$

where

$$(W^Q(\psi) \wedge \mathrm{Id})^{S^2E}_{Y,Z} := \left(i_Y W^Q(\psi) \wedge Z - i_Z W^Q(\psi) \wedge Y\right)^{S^2E} \tag{3.15}$$

and

$$C(\psi^{S^2E})_{Y,Z} := i_Y \left(\nabla_Z W^Q\right)(\psi^{S^2E}) - i_Z \left(\nabla_Y W^Q\right)(\psi^{S^2E}). \tag{3.16}$$

In particular, $\mathcal{D}$ is flat if and only if $W^Q = 0$.

The proof of Theorem 35 is now completed.

The dimension of the space $\mathcal{C}_2(M)$

As a first application of Theorem 35 we determine a sharp estimate for the dimension of the vector space $\mathcal{C}_2(M)$ of compatible conformal-Killing 2-forms on a quaternionic-Kähler manifold (M, g). It is known that on an arbitrary Riemannian manifold (not necessarily compact) the space of conformal-Killing forms (of any degree) is finite-dimensional and an upper bound, which is realized on the standard sphere, was found in [99]. For the special class of compatible conformal-Killing 2-forms on quaternionic-Kähler manifolds, this upper bound can be improved, due to the following result:

Corollary 39. *Let (M, g) be a quaternionic-Kähler manifold of dimension $4n \geq 8$. Then*

$$\dim \mathcal{C}_2(M) \leq (n+1)(2n+3). \tag{3.17}$$

Equality holds on the standard $\mathbb{H}P^n$.

Proof. Notice that $(n+1)(2n+3)$ is the rank of the bundle $S^2H \oplus S^2E \oplus TM$ on which the connection $\mathcal{D}$ is defined. Therefore, inequality (3.17) follows from Theorem 35. It remains to show that equality holds on the quaternionic projective space $\mathbb{H}P^n$, with its standard quaternionic-Kähler structure. This follows from Theorem 42 (see the next section), where it is shown that $\mathcal{C}_2(\mathbb{H}P^n)$ is isomorphic to the space of Killing vector fields on $\mathbb{H}P^n$, which has dimension $(n+1)(2n+3)$. Our claim follows. $\square$

The case when $\nu \neq 0$

In this section we develop various properties (see Propositions 40 and 41) which hold under the additional assumption that the scalar curvature is non-zero. They are consequences of the prolongation $\mathcal{D}$ found in the previous section.

Proposition 40. *If a quaternionic-Kähler manifold (M, g) of dimension $4n \geq 8$ and non-zero scalar curvature has a non-parallel compatible conformal-Killing 2-form, then the holonomy group of (M, g) is $Sp(1)Sp(n)$.*

Proof. We will show that the holonomy algebra $\mathrm{hol}(M, g)$ of (M, g) coincides with $\mathrm{sp}(1) \oplus \mathrm{sp}(n)$. Let ψ be a non-parallel compatible conformal-Killing 2-form on (M, g) and $X := \delta\psi$, which is non-trivial (from Proposition 36). The key fact in the proof is the following relation

$$W^Q(V, X) = \frac{4n-1}{4(n+2)}(\nabla_V W^Q)(\psi), \quad \forall V, \tag{3.18}$$

which is a consequence of the curvature computation from Proposition 38. Using the Ambrose-Singer theorem (see Chapter 10 of [15]) and (3.18) we obtain that $W^Q(X, V) \in \mathrm{hol}(M, g)$. Since $R^g(Y, Z)$ belongs to the holonomy algebra as well, and $\mathrm{sp}(1) \subset \mathrm{hol}(M, g)$ (since the scalar curvature is non-zero, see Lemma 14.46 of [15]), we deduce that $R^g(Y, Z)^{S^2E}$ belongs to $\mathrm{hol}(M, g)$, for any $Y, Z \in TM$. It follows that

$$R^g(X, V)^{S^2E} - W^Q(X, V) = -\nu(X \wedge V)^{S^2E} \in \mathrm{hol}(M, g), \quad \forall V \tag{3.19}$$

and

$$R^g(J_iX, V)^{S^2E} - W^Q(J_iX, V) = -\nu(J_iX \wedge V)^{S^2E} \in \mathrm{hol}(M, g), \quad \forall V, \tag{3.20}$$

where $\{J_1, J_2, J_3\}$ is a local admissible basis of Q. We just proved that if Y or U belong to $\mathcal{V} := \mathrm{Span}\{X, J_1X, J_2X, J_3X\}$, then $(Y \wedge U)^{S^2E}$ belongs to $\mathrm{hol}(M,g)$. It remains to show that $(Y \wedge U)^{S^2E}$ belongs to the holonomy algebra when both Y and U are orthogonal to $\mathcal{V}$. Take such two tangent vectors Y and U. Since both $(X \wedge Y)^{S^2E}$ and $(X \wedge U)^{S^2E}$ belong to $\mathrm{hol}(M,g)$, also their Lie bracket does, and its S^2E projection as well. We obtain that $(Y \wedge U)^{S^2E} \in \mathrm{hol}(M,g)$. Our claim follows. □

Proposition 41. *On a quaternionic-Kähler manifold with non-zero scalar curvature, the map*

$$\mathcal{C}_2(M) \ni \psi \to u := \psi^{S^2E} \tag{3.21}$$

is an isomorphism from the vector space $\mathcal{C}_2(M)$ of compatible conformal-Killing 2-forms on (M,g) to the vector space of (real) sections of S^2E which satisfy

$$\nabla_Y u = \frac{1}{4n-1}\left(X \wedge Y + \sum_{i=1}^{3} J_i X \wedge J_i Y\right) \quad \forall Y \in TM, \tag{3.22}$$

where $X \in \mathcal{X}(M)$ is a vector field on M (necessarily equal to $\frac{4n-1}{4n+2}\delta u$) and $\{J_1, J_2, J_3\}$ is an admissible basis of Q. The inverse is the map

$$u \to \psi := u - \frac{1}{(2n+1)\nu}(\nabla \delta u)^{S^2H}. \tag{3.23}$$

Proof. For a proof, see [31]. □

3.5 Global classification of conformal-Killing 2-forms on quaternionic-Kähler manifolds

3.5.1 Statement of the main result

In this section we find a complete description of conformal-Killing 2-forms on compact quaternionic-Kähler manifolds. Our main result is the following [42].

Theorem 42. *i) A compact, connected, quaternionic-Kähler manifold (M,g) of dimension $4n \geq 8$ admits a non-parallel conformal Killing 2-form if and only if it is isomorphic to the quaternionic projective space $\mathbb{H}P^n$, with its standard quaternionic-Kähler structure.*

ii) Let $g_{\mathrm{can}}(\nu)$ be the standard metric of $\mathbb{H}P^n$, with reduced scalar curvature $\nu > 0$. The map which associates to a Killing vector field X on $(\mathbb{H}P^n, g_{\mathrm{can}}(\nu))$ the 2-form

$$\psi := -\frac{2}{\nu(4n-1)}(\nabla X)^{S^2H} + \frac{4}{\nu(4n-1)}(\nabla X)^{S^2E} \tag{3.24}$$

is an isomorphism from the space of Killing vector fields to the space of conformal-Killing 2*-forms on* $(\mathbb{H}P^n, g_{\mathrm{can}}(\nu))$*, with inverse the codifferential:* $\delta(\psi) = X$.

We divide the proof of Theorem 42 in two steps. In a first stage (Section 3.5.2) we discuss the structure of conformal-Killing 2-forms on compact quaternionic-Kähler manifolds (see Proposition 43). In a second stage (Section 3.5.3) we prove a property of Killing vector fields on quaternionic-Kähler manifolds, in relation with the quaternionic-Weyl tensor (see Proposition 46). Combining these two steps we conclude the proof of Theorem 42.

3.5.2 Structure of conformal-Killing 2-forms

Proposition 43. *Let* ψ *be a conformal-Killing* 2*-form on a compact quaternionic-Kähler manifold* (M, g) *of dimension* $4n \geq 8$ *and reduced scalar curvature* ν.

i) If $\nu > 0$ *then*

$$\psi = -\frac{2}{\nu(4n-1)}(\nabla X)^{S^2H} + \frac{4}{\nu(4n-1)}(\nabla X)^{S^2E} + \tilde{u}, \tag{3.25}$$

where $X := \delta(\psi)$ *is the codifferential of* ψ *and belongs to the kernel of the quaternionic-Weyl tensor,* $\tilde{u} \in \Omega^2(M)$ *is parallel and* ∇ *is the Levi-Civita connection.*

ii) If $\nu \leq 0$ *then* ψ *is parallel.*

Proof. On an Einstein manifold the (dual of the) codifferential of a conformal-Killing 2-form is a Killing vector field [99]. With this remark, the statement of Proposition 43 in the case when $\nu \leq 0$ is an easy consequence of the following facts: there are no (non-trivial) Killing vector fields on a compact quaternionic-Kähler manifold (M, g), with $\nu < 0$ (this is an application of the Weizenböck formula; see also [17], Theorem 1.84). Similarly, if $\nu = 0$ then any Killing vector field on (M, g) is harmonic, and, if coexact, it is identically zero (M being compact). Due to these facts, any conformal-Killing 2-form on a compact, quaternionic-Kähler manifold with non-positive scalar curvature is Killing, hence parallel [81]. This proves our claim when $\nu \leq 0$.

Now we assume that $\nu > 0$. Recall the decomposition (3.4) of the bundle of 2-forms into irreducible subbundles. A Weizenböck type argument together with the estimates on the eigenvalues of the Laplace operator on compact quaternionic-Kähler manifolds [101] allows to determine the S^2H and $S^2H \otimes \Lambda^2_0(E)$ components of ψ as follows:

$$\psi^{S^2H} = -\frac{2}{\nu(4n-1)}(\nabla X)^{S^2H}, \quad \psi^{S^2H\otimes\Lambda^2_0(E)} = 0$$

(to determine the S^2H-component one may also use Proposition 41). Therefore, we may write

$$\psi = -\frac{2}{\nu(4n-1)}(\nabla X)^{S^2H} + \frac{4}{\nu(4n-1)}(\nabla X)^{S^2E} + \tilde{u},$$

where $\tilde{u}$ is a section of S^2E. The conformal-Killing equation for ψ becomes

$$\nabla_Z \tilde{u} = -\frac{4}{\nu(4n-1)} W^Q(Z, X) \quad \forall Z \in TM. \tag{3.26}$$

Using (3.26), we get

$$\begin{aligned}(d\tilde{u})(Z_0, Z_1, Z_2) &= (\nabla_{Z_0}\tilde{u})(Z_1, Z_2) + (\nabla_{Z_2}\tilde{u})(Z_0, Z_1) + (\nabla_{Z_1}\tilde{u})(Z_2, Z_0)\\ &= \frac{4}{\nu(4n-1)}\big(-W^Q(Z_0, X, Z_1, Z_2) + W^Q(Z_1, X, Z_0, Z_2)\big)\\ &- \frac{4}{\nu(4n-1)} W^Q(Z_2, X, Z_0, Z_1).\end{aligned}$$

The symmetries of the curvature tensor W^Q imply that $d\tilde{u} = 0$. Similarly, we can write the codifferential of u in the form

$$\delta\tilde{u} = -\sum_{i=1}^{4n}(\nabla_{E_i} u)(E_i, \cdot) = \frac{4}{\nu(4n-1)}\sum_{i=1}^{4n} W^Q(E_i, X)(E_i) = 0,$$

because W^Q is in the kernel of the Ricci contraction. We proved that $\tilde{u}$ is harmonic. Recall now that the second Betti number $b_2(M)$ of (M, g) is zero, unless (M, g) is isomorphic to the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^{n+2})$ of complex 2-planes in $\mathbb{C}^{n+2}$, with its standard quaternionic-Kähler metric [75]; moreover, the space of harmonic 2-forms on $\mathrm{Gr}_2(\mathbb{C}^{n+2})$ is one dimensional, generated by the Kähler form, which is a parallel section of S^2E. This proves that $\tilde{u}$ is actually parallel. Our claim follows. □

3.5.3 Killing vector fields and the quaternionic-Weyl tensor

In this Section we conclude the proof of Theorem 84. We do this by proving Proposition 46 stated below, which in turn relies on the following Theorem of Obata (see [84], Theorem C).

Theorem 44. *Let (N^{2n}, J, g) be a complete, connected and simply connected Kähler manifold. Suppose there is a non-constant smooth function f on N which satisfies the Obata's equation*

$$\begin{aligned}4\nabla^2(df)(Y, U, V) &= -2df(Y)g(U, V) - df(U)g(Y, V) - df(V)g(Y, U)\\ &+ df(JU)\omega(Y, V) + df(JV)\omega(Y, U),\end{aligned}$$

for any vector fields $Y, U, V \in \mathcal{X}(N)$, where ∇ is the Levi-Civita connection and ω is the Kähler form. Then (N, J, g) is isometric to $(\mathbb{C}P^n, g_{\mathrm{FS}})$, where g_{FS} is the Fubini-Study metric of constant holomorphic sectional curvature equal to one.

Remark 45. In order to explain the geometric meaning of Obata's theorem, we recall some well-known facts about Hamiltonian functions and Hamiltonian Killing vector fields. A Killing vector field X on a connected Kähler manifold (N, J, g) is called Hamiltonian, if it is of the form $X = J\mathrm{grad}_g(f)$, for a smooth function $f \in C^\infty(N)$, uniquely determined up to addition by a constant and usually called the Hamiltonian function of X. There is a general result due to Matsushima (see [79] or [17], page 330) which states that any Killing vector field on a compact Kähler-Einstein manifold with positive scalar curvature is Hamiltonian; in fact, the Hamiltonian function can be chosen to be an eigenfunction of the Laplace operator Δ, with eigenvalue k/m (where k is the scalar curvature and m is the complex dimension) and the map $f \to X^f = J\mathrm{grad}_g(f)$ is an isomorphism between the eigenspace of Δ relative to k/m and the space of Killing vector fields.

Coming back to Obata's theorem, it is easy to verify that the Hamiltonian function of any Killing vector field on $(\mathbb{C}P^n, g_{\mathrm{FS}})$ satisfies the Obata's equation. Theorem 44 is a strong converse of this statement: the existence of a *single* smooth non-constant solution of Obata's equation on a complete, connected and simply connected Kähler manifold, insures that the Kähler manifold is isometric to the standard complex projective space, with Fubini-Study metric g_{FS}.

Proposition 46 below concerns compact quaternionic-Kähler manifolds with positive scalar curvature. Without loss of generality, we will normalize the quaternionic-Kähler metric to have reduced scalar curvature $\nu = 1$. We shall denote by $g_{\mathrm{can}} := g_{\mathrm{can}}(1)$ the standard quaternionic-Kähler metric of $\mathbb{H}P^n$, normalized in this way. The main result of this section, which will be used to conclude the proof of Theorem 84, is the following.

Proposition 46. *Let (M, g) be a compact, quaternionic-Kähler manifold, of dimension $4n \geq 8$ and reduced scalar curvature $\nu = 1$. Suppose there is a non-trivial Killing vector field X on M such that $W^Q(X, \cdot) = 0$, where W^Q is the quaternionic-Weyl tensor. Then $W^Q = 0$ and (M, g) is isometric to $(\mathbb{H}P^n, g_{\mathrm{can}})$.*

Proof. The idea of the proof is to show that the Hamiltonian function f^X of the natural lift X^Z of X to the Kähler-Einstein twistor space $(Z, \bar{g}, \mathcal{J})$ of (M, g) satisfies the Obata's equation stated above. Then Theorem 44 implies that $(Z, \bar{g}, \mathcal{J})$ is isomorphic to $(\mathbb{C}P^n, g_{FS})$ and hence (M, g) is isomorphic to $(\mathbb{H}P^n, g_{\mathrm{can}})$. Details can be found in [42]. Here we only explain how the natural lift X^Z is defined. The flow of X lifts in a natural way to a flow on all tensor bundles on M, in particular also on $\mathrm{End}(TM)$, and the associated vector field $X^{\mathrm{End}(TM)}$ is the natural lift of X to $\mathrm{End}(TM)$. Moreover, $X^{\mathrm{End}(TM)}$ is tangent to $Z \subset \mathrm{End}(TM)$ along Z, and defines a genuine vector field on Z, denoted by X^Z and called the natural lift of X to the twistor space Z. Note that X^Z is Killing, real holomorphic and Hamiltonian on the compact Kähler-Einstein manifold $(Z, \mathcal{J}, \bar{g})$ of positive scalar curvature. At a point $J \in Z_p$, it is given by [42]

$$X^Z_J = \bar{X}_J + [\nabla X, J], \tag{3.27}$$

where $\bar{X}_J$ is the horizontal lift of X to J (using the Levi-Civita connection ∇ of g) and $[\nabla X, J]$ is viewed as a tangent vertical vector at J (since ∇X is a section of $S^2H \oplus S^2E$ [73], $[\nabla X, J]$ anti-commutes with J, i.e. belongs to the tangent space of Z_p at J). The second covariant derivatives of X^Z on $(Z, \bar{g})$ and the proof that the Hamiltonian function of X^Z satisfies the Obata equation on $(Z, \bar{g}, \mathcal{J})$ can be found in [42]. $\square$

Theorem 84 can now be easily concluded as follows. From Proposition 43 and Proposition 46, a compact quaternionic-Kähler manifold (M, g) of real dimension $4n \geq 8$ and reduced scalar curvature ν, admits a non-parallel conformal-Killing 2-form if and only if (M, g) is isometric to $(\mathbb{H}P^n, g_{\mathrm{can}}(\nu))$. Recall now that there are no non-trivial parallel 2-forms on the standard quaternionic projective space. Using this fact, Proposition 43 implies that any conformal-Killing 2-form ψ on $(\mathbb{H}P^n, g_{\mathrm{can}}(\nu))$ must be of the form

$$\psi = -\frac{2}{\nu(4n-1)}(\nabla X)^{S^2H} + \frac{4}{\nu(4n-1)}(\nabla X)^{S^2E}, \tag{3.28}$$

where $X = \delta(\psi)$ is the codifferential of ψ and is Killing. Conversely, one may check that for any Killing vector field X, the 2-form (3.28) is conformal-Killing on $(\mathbb{H}P^n, g_{\mathrm{can}}(\nu))$ and the map $\psi \to X$ is an isomorphism from the space of conformal-Killing 2-forms to the space of Killing vector fields. This concludes the proof of Theorem 42.

3.6 The conformal-Killing equation and G-structures

3.6.1 Statement of the main result

Our main result from this section is the following [32, 33]:

Theorem 47. *i) Let (M^{4n}, Q, g) be a $4n$-dimensional manifold with an $Sp(1)Sp(n)$-structure (i.e. an almost quaternionic-Hermitian structure) and $n \geq 2$. Suppose that the fundamental 4-form*

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3,$$

where ω_i are the Kähler forms of an admissible basis $\{J_1, J_2, J_3\}$ of Q, is conformal-Killing. Then Ω is parallel (and (M, Q, g) is quaternionic-Kähler).

ii) Let M be a 7-manifold with a G_2-structure defined by $\phi \in \Omega^3_+(M)$. Then ϕ is conformal-Killing with respect to the associated metric g_ϕ if and only if the G_2-structure is nearly parallel (i.e. $\nabla\phi$ is a multiple of $*_\phi\phi$, where ∇ is the Levi-Civita connection of g_ϕ and $*_\phi$ is the Hodge star operator determined by g_ϕ and the orientation on M defined by ϕ).

iii) Let M be an 8-manifold with a $Spin_7$-structure defined by $\psi \in \Omega^4_+(M)$. Then ψ is conformal-Killing with respect to the associated metric g_ψ if and only if the $Spin_7$-structure is parallel (i.e. ψ is parallel with respect to the Levi-Civita connection of g_ψ).

Proof. The above statements are proved using, basically, the same conceptual argument for all three cases, $G = Sp(1)Sp(n)$, G_2 or $Spin_7$. For this reason we present the main ideas of the proofs in a unified way, for all statements at once. As usual, we identify vectors with covectors using the metric. The main idea of the proof is to use the well-known facts [20, 53, 104] that the covariant derivatives (with respect to the Levi-Civita connections of g, g_ϕ and g_ψ respectively) of the forms Ω, ϕ, ψ from the statements of the theorem, which are sections of $T^*M \otimes \Lambda^i(M)$ (where $i = 4$ in the first and third cases and $i = 3$ in the second case), actually belong to the tensor product $T^*M \otimes \mathrm{adj}(G)^\perp$, where $\mathrm{adj}(G) \subset \Lambda^2(M)$ is the adjoint bundle with typical fiber $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathrm{adj}(G)^\perp \subset \Lambda^2(M)$ is its orthogonal complement, considered as a subbundle of $\Lambda^i(M)$. (The embedding of $\mathrm{adj}(G)^\perp$ in $\Lambda^i(M)$ is described in [20, 104]). Then one shows, using Schur's lemma and standard arguments from representation theory, that the algebraic conformal Killing operator (see Section 3.2.5) is injective on $T^*M \otimes \mathrm{adj}(G)^\perp$ in the first and third case (i.e. when $G = Sp(1)Sp(n)$ and $Spin_7$). This implies that Ω and ψ are parallel, i.e. the first and third required statements. In the second case (when $G = G_2$) one shows that the kernel of the algebraic conformal-Killing operator restricted to $T^*M \otimes \mathrm{adj}(G)^\perp$ is the one dimensional subbundle L_ϕ generated by $*_\phi\phi$; the way L_ϕ is realized as a subbundle of $T^*M \otimes \mathrm{adj}(G)^\perp$ is described in [20]). This implies the second statement. For complete proofs, [32, 33]. $\square$

Chapter 4

Contributions to Generalized Complex Geometry

4.1 Introduction

Chapter 4 is concerned with the generalized complex geometry of Lie groups and describes the results published by the author of this thesis in **Proceedings of the London Mathematical Society** [3]. Generalized complex geometry was discovered by N. Hitchin as a unification of complex and symplectic geometries and has become one of the main streams of current research in mathematics. The idea is to replace the usual tangent bundle TM of a manifold M by the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$ and to consider both complex and symplectic structures as particular classes of a more general structure, the so called generalized complex structure. Many notions from complex and/or symplectic geometry were extended to generalized complex geometry. In this chapter we develop a systematic treatment of a class (called regular) of invariant generalized complex structures on semisimple Lie groups. The results of this chapter may be seen as complementary to some previous constructions of invariant generalized complex and Kähler structures on some classes of homogeneous manifolds (e.g. nilpotent or solvable Lie groups and their compact quotients [12, 28, 54], homogeneous manifolds G/K, where G is a compact Lie group and K a closed subgroup of maximal rank and adjoint orbits in semisimple Lie algebras [83]). They also complement a result of M. Gualtieri, namely that any compact semisimple Lie group of even dimension admits a H-twisted generalized Kähler structure (see Example 6.39 of [61]).

Section 4.2 is intended to fix the notations and conventions we shall use along the chapter. We recall basic facts we need about real and complex semisimple Lie algebras, invariant complex structures on homogeneous manifolds and generalized complex structures on manifolds. Our approach follows closely [61, 72, 109].

In **Section 4.3** we present an infinitesimal description of invariant generalized complex structures on Lie groups, in terms of the so called admissible pairs. The holomorphic bundle L of an invariant generalized complex structure $\mathcal{J}$ on a Lie group G, with Lie algebra $\mathfrak{g}$, can be defined in terms of a pair $(\mathfrak{k}, \omega)$ (called $\mathfrak{g}$-admissible), formed by a subalgebra $\mathfrak{k} \subset \mathfrak{g}^{\mathbb{C}}$ and a closed 2-form $\omega \in \Lambda^2(\mathfrak{k}^*)$, with the property that $\omega_{\mathfrak{l}} := \mathrm{Im}\left(\omega|_{\mathfrak{l}}\right)$ is non-degenerate, where $\mathfrak{l} = \mathfrak{k} \cap \mathfrak{g}$ is the real part of $\mathfrak{k} \cap \bar{\mathfrak{k}}$.

In **Section 4.4** we begin our treatment of invariant generalized complex structures on semisimple Lie groups. An invariant generalized complex structure $\mathcal{J}$ on a semisimple Lie group G and the associated $\mathfrak{g}$-admissible pair $(\mathfrak{k}, \omega)$ are called regular if $\mathfrak{k}$ is a regular subalgebra of $\mathfrak{g}^{\mathbb{C}}$, i.e. is normalized by a Cartan subalgebra $\mathfrak{h}_{\mathfrak{g}}$ of $\mathfrak{g} = \mathrm{Lie}(G)$. We describe all regular $\mathfrak{g}$-admissible pairs (hence also the associated generalized complex structures on G) where $\mathfrak{g}$ is a real form of inner type and $\mathfrak{h}_{\mathfrak{g}}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}$ (see Proposition 59 and Theorem 84). We study how these generalized complex structures can be reduced to the normal form by means of invariant B-field transformations (see Proposition 61). At the end of this section we show that any invariant generalized complex structure on a compact semisimple Lie group G, with Lie algebra $\mathfrak{g}$, is regular, provided that the Lie algebra $\mathfrak{k}$ of the associated $\mathfrak{g}$-admissible pair has the property that $\mathfrak{k} \cap \mathfrak{g}$ generates a closed subgroup of G (see Theorem 62).

Section 4.5 is concerned with regular generalized complex structures (or regular admissible pairs $(\mathfrak{k}, \omega)$) on simple Lie groups G of outer type, using the formalism of Vogan diagrams. The classification of the subalgebras $\mathfrak{k}$ reduces to the description of the so called σ-positive root systems (see Definition 57). We describe explicitly these systems and we obtain a large class of regular generalized complex structures on G.

4.2 Preliminary material

4.2.1 Invariant complex structures on Lie groups and homogeneous manifolds

Invariant complex structures on homogeneous manifolds

The Lie algebra of a Lie group will be identified as usual with the tangent space at the identity element or with the space of left-invariant vector fields.

Let G be a real Lie group, with Lie algebra $\mathfrak{g}$, and L a closed connected subgroup of G, with Lie algebra $\mathfrak{l}$. Suppose that the space $M = G/L$ of left cosets is reductive, i.e. $\mathfrak{g}$ has an Ad_L-invariant decomposition

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}. \tag{4.1}$$

We shall identify $\mathfrak{m}$ with the tangent space T_oM at the origin $o = eL \in G/L$. An invariant complex structure J on M is determined by its value J_o at o, which is an Ad_L-invariant complex structure on the vector space $\mathfrak{m} = T_oM$. Let $\mathfrak{m}^{1,0}$ and $\mathfrak{m}^{0,1} = \overline{\mathfrak{m}^{1,0}}$ be the holomorphic, respectively anti-holomorphic subspaces of J_o, so that

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{1,0} \oplus \mathfrak{m}^{0,1}. \tag{4.2}$$

The invariance and integrability of J mean that $\mathfrak{k} = \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{1,0}$ is a (complex) subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Conversely, any decomposition (4.2) of $\mathfrak{m}^{\mathbb{C}}$ into two vector spaces $\mathfrak{m}^{1,0}$ and $\mathfrak{m}^{0,1} = \overline{\mathfrak{m}^{1,0}}$ such that $\mathfrak{k} = \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{1,0}$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$, defines an invariant complex structure on M. We get the following well known algebraic description of invariant complex structures on reductive homogeneous manifolds.

Proposition 48. *Let $M = G/L$ be a reductive homogeneous manifold, with reductive decomposition (4.1). There is a natural one to one correspondence between:*

i) invariant complex structures on M;

ii) decompositions $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{1,0} \oplus \mathfrak{m}^{0,1}$, where $\mathfrak{m}^{0,1} = \overline{\mathfrak{m}^{1,0}}$ and $\mathfrak{k} = \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{1,0}$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$;

iii) subalgebras $\mathfrak{k} \subset \mathfrak{g}^{\mathbb{C}}$ such that

$$\mathfrak{k} + \bar{\mathfrak{k}} = \mathfrak{g}^{\mathbb{C}}, \quad \mathfrak{k} \cap \bar{\mathfrak{k}} = \mathfrak{l}^{\mathbb{C}}. \tag{4.3}$$

In particular, if $M = G$ is a Lie group, there is a one to one correspondence between invariant complex structures on G and decompositions

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1},$$

where $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ are subalgebras of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{g}^{0,1} = \overline{\mathfrak{g}^{1,0}}$.

Some basic facts on semisimple Lie algebras

We fix our notations from the theory of semisimple Lie algebras.

Complex semisimple Lie algebras. Recall that any complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ has a root space decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} + \mathfrak{g}(R) = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

with respect to a Cartan subalgebra $\mathfrak{h}$, where $R \subset \mathfrak{h}^*$ is the root system of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}$ and for any subset $P \subset R$ we denote by $\mathfrak{g}(P)$ the direct sum of root spaces

$$\mathfrak{g}(P) := \sum_{\alpha \in P} \mathfrak{g}_\alpha.$$

Along the paper we will denote by $E_\alpha \in \mathfrak{g}_\alpha$ the root vectors of a Weyl basis of $\mathfrak{g}(R)$, which have the following properties:

i)
$$\langle E_\alpha, E_{-\alpha} \rangle = 1, \quad \forall \alpha \in R,$$
where $B = \langle \cdot, \cdot \rangle$ is the Killing form of $\mathfrak{g}^{\mathbb{C}}$;

ii) the structure constants $N_{\alpha\beta}$ defined by
$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}, \quad \forall \alpha, \beta \in R$$
are real and satisfy
$$N_{-\alpha,-\beta} = -N_{\alpha\beta}, \quad \forall \alpha, \beta \in R.$$

We will identify the dual space $\mathfrak{h}^*$ with $\mathfrak{h}$ using the restriction $\langle \cdot, \cdot \rangle$ of the Killing form to $\mathfrak{h}$, which is non-degenerate on $\mathfrak{h}$ and positive definite on the real form $\mathfrak{h}(\mathbb{R})$ of $\mathfrak{h}$, spanned by the root system $R \subset \mathfrak{h}$. For a set of roots P, we will constantly use the notation $P^{\text{sym}} := P \cap (-P)$ for the symmetric part of P and $P^{\text{asym}} := P \setminus P^{\text{sym}}$ for the anti-symmetric part.

Real semisimple Lie algebras. Let $\mathfrak{g}$ be a real semisimple Lie algebra. Its complexification $\mathfrak{g}^{\mathbb{C}}$ is a complex semisimple Lie algebra and $\mathfrak{g} = (\mathfrak{g}^{\mathbb{C}})^\sigma$, that is, $\mathfrak{g}$ can be reconstructed from $\mathfrak{g}^{\mathbb{C}}$ as the fix point set of a complex conjugation or antiinvolution
$$\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}, \quad x \to \bar{x},$$
that is, σ is an involutive automorphism of $\mathfrak{g}^{\mathbb{C}}$, as a real Lie algebra, and is antilinear. One can always assume that the antiinvolution σ preserves a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$. Then $\mathfrak{h}_{\mathfrak{g}} = \mathfrak{h}^\sigma := \mathfrak{h} \cap \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$. The antiinvolution σ acts naturally on the set of roots R of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}$, by
$$\sigma(\alpha) := \overline{\alpha \circ \sigma}, \quad \forall \alpha \in R$$
and
$$\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma(\alpha)}, \quad \forall \alpha \in R.$$
The compact real form of a complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is unique (up to conjugation) and is defined by the antiinvolution τ (called compact) given by
$$\tau|_{\mathfrak{h}(\mathbb{R})} = -\mathrm{Id}, \ \tau(E_\alpha) = -E_{-\alpha}, \forall \alpha \in R.$$
Assume now that $\mathfrak{g}$ is a real form of $\mathfrak{g}^{\mathbb{C}}$, not necessarily compact. Consider a Cartan decomposition
$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

of $\mathfrak{g}$ and let θ be the associated Cartan involution, with $+1$ eigenspace $\mathfrak{k}$ and -1 eigenspace $\mathfrak{p}$. The real form

$$\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$$

is compact, the corresponding antiinvolution τ commutes with θ and $\mathfrak{g} = (\mathfrak{g}^{\mathbb{C}})^{\sigma}$, where $\sigma := \theta \circ \tau$. A θ-invariant Cartan subalgebra $\mathfrak{h}_{\mathfrak{g}}$ of $\mathfrak{g}$ decomposes as

$$\mathfrak{h}_{\mathfrak{g}} = \mathfrak{h}^{+} \oplus \mathfrak{h}^{-}, \quad \mathfrak{h}^{+} \subset \mathfrak{k}, \quad \mathfrak{h}^{-} \subset \mathfrak{p}$$

and any root of $\mathfrak{g}^{\mathbb{C}}$ relative to $(\mathfrak{h}_{\mathfrak{g}})^{\mathbb{C}}$ takes purely imaginary values on $\mathfrak{h}^{+} \oplus i\mathfrak{h}^{-}$. The Cartan subalgebra $\mathfrak{h}_{\mathfrak{g}}$ is called maximally compact (respectively, maximally non-compact) if the dimension of $\mathfrak{h}^{+}$ (respectively, $\mathfrak{h}^{-}$) is as large as possible. The real form $\mathfrak{g}$ is called of inner type if θ is an inner automorphism of $\mathfrak{g}$, that is, it belongs to $\mathrm{Int}(\mathfrak{g})$. The definition is independent of the choice of θ, since any two Cartan involutions of $\mathfrak{g}$ are conjugated via $\mathrm{Int}(\mathfrak{g})$, see e.g. [72, p. 301]. It can be shown that $\mathfrak{g}$ is of inner type if and only if there is a (maximally compact) Cartan subalgebra $\mathfrak{h}_{\mathfrak{g}}$ included in $\mathfrak{k}$, see e.g. [21, p. 24]. Then

$$\sigma(\alpha) = -\alpha$$

for any root α of $\mathfrak{g}^{\mathbb{C}}$ relative to $(\mathfrak{h}_{\mathfrak{g}})^{\mathbb{C}}$. Any compact real form is a real form of inner type.

A non-inner real form $\mathfrak{g}$ (and the corresponding antilinear involution) is called outer. For such a real form, the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$ admits a non-trivial automorphism of order two, defined by the Cartan involutions of $\mathfrak{g}$ (more precisely, any Cartan involution θ permutes a set of simple roots of $\mathfrak{g}^{\mathbb{C}}$ relative to $(\mathfrak{h}_{\mathfrak{g}})^{\mathbb{C}}$, where $\mathfrak{h}_{\mathfrak{g}} \subset \mathfrak{g}$ is a maximally compact θ-invariant Cartan subalgebra, giving rise to the resulting automorphism of the Dynkin diagram, see [72, p. 339]). In particular, if $\mathfrak{g}^{\mathbb{C}}$ is simple, then $\mathfrak{g}^{\mathbb{C}} = A_n$, $(n \geq 2)$, D_n $(n \geq 3)$ or E_6. The list of simple non-complex Lie algebras of outer type is short and it is given below (see [72], Appendix C):

$\mathfrak{sl}_n(\mathbb{R})\,(n > 2)$, $\mathfrak{sl}_n(\mathbb{H})\,(n \geq 2)$, $\{\mathfrak{so}_{2p+1,2q+1},\ 0 \leq p \leq q\} \backslash \{\mathfrak{so}_{1,1}, \mathfrak{so}_{1,3}\}$, $\mathfrak{e}_6(\mathfrak{f}_4)$, $\mathfrak{e}_6(\mathfrak{sp}_4)$.

For the real forms of the exceptional Lie algebra $\mathfrak{e}_6$ we indicate in the bracket the type of maximal compact subalgebras. A Lie group is called inner (respectively, outer) if its Lie algebra is inner (respectively, outer).

Invariant complex structures on homogeneous manifolds of compact semisimple Lie groups

Let $F = G/K = \mathrm{Ad}_G(x_0) \subset \mathfrak{g}$ be a flag manifold of a compact connected semisimple Lie group G, with Lie algebra $\mathfrak{g} = (\mathfrak{g}^{\mathbb{C}})^{\tau}$, and $K = T \cdot L$ a decomposition of the stabilizer K into a product of a torus T and a semisimple group L. Let T' be a subtorus of T, such that the quotient T/T' has even dimension. The total space of the fibration

$$\pi : M = G/(L \cdot T') \to F = G/K$$

admits an invariant complex structure, defined by an invariant complex structure on the fiber T/T' and an invariant complex structure on the base F, such that π is an holomorphic fibration. Moreover, any invariant complex structure on M is obtained in this way.

More precisely, let $\mathfrak{h}_{\mathfrak{g}}$ be a Cartan subalgebra of $\mathfrak{g}$ included in $\mathfrak{k} = \mathrm{Lie}(K)$. The complexification of the B-orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $F = G/K$ is given by

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h} + \mathfrak{g}([\Pi_0]), \quad \mathfrak{p}^{\mathbb{C}} = \mathfrak{g}(R'), \quad R' = R \setminus [\Pi_0]$$

where $\Pi_0 \subset \Pi$ is a subset of a system Π of simple roots of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h} = (\mathfrak{h}_{\mathfrak{g}})^{\mathbb{C}}$ and $[\Pi_0] := R \cap \mathrm{span}(\Pi_0)$ is the set of all roots from R spanned by Π_0. The Cartan subalgebra $\mathfrak{h}_{\mathfrak{g}}$ decomposes into a B-orthogonal direct sum

$$\mathfrak{h}_{\mathfrak{g}} = \mathfrak{t} \oplus \mathfrak{h}_0 = \mathfrak{t}' \oplus \mathfrak{a} \oplus \mathfrak{h}_0$$

where $\mathfrak{t} = \mathrm{Lie}(T)$ and $\mathfrak{t}' = \mathrm{Lie}(T')$. The complexification of the reductive decomposition of $M = G/(L \cdot T')$ is given by

$$\mathfrak{g}^{\mathbb{C}} = (\mathfrak{l} + \mathfrak{t}')^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$$

where $\mathfrak{l} = \mathrm{Lie}(L)$, $\mathfrak{l}^{\mathbb{C}} = \mathfrak{h}_0^{\mathbb{C}} + \mathfrak{g}([\Pi_0])$ and $\mathfrak{m}^{\mathbb{C}} = \mathfrak{a}^{\mathbb{C}} + \mathfrak{g}(R')$.

The following theorem was proved in [109].

Theorem 49. *i) The compact homogeneous manifold $M = G/(L \cdot T')$ described above admits an invariant complex structure defined by the decomposition*

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{1,0} \oplus \overline{\mathfrak{m}^{1,0}} = \mathfrak{m}^{1,0} \oplus \tau(\mathfrak{m}^{1,0}),$$

where

$$\mathfrak{m}^{1,0} = \mathfrak{a}^{1,0} + \mathfrak{g}(R'_+), \quad R'_+ := R^+ \cap (R \setminus [\Pi_0]),$$

R^+ is the positive root system defined by Π and $\mathfrak{a}^{1,0}$ is the holomorphic subspace of a complex structure $J^{\mathfrak{a}}$ on $\mathfrak{a}$.

ii) Conversely, any invariant complex structure on an homogeneous manifold of a compact, connected, semisimple Lie group G can be obtained by this construction.

4.2.2 Generalized complex structures on manifolds

Let M be a smooth manifold and g_{can} the canonical indefinite metric on the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$, given by

$$g_{\mathrm{can}}(X+\xi, Y+\eta) := \frac{1}{2}\left(\xi(Y)+\eta(X)\right), \quad \forall X+\xi, Y+\eta \in \mathbb{T}M. \tag{4.4}$$

A generalized almost complex structure [61, 71] on M is a g_{can}-skew-symmetric field of endomorphisms

$$\mathcal{J} : \mathbb{T}M \to \mathbb{T}M$$

with $\mathcal{J}^2 = -\mathrm{Id}$ (where "Id" denotes the identity endomorphism). A generalized almost complex structure $\mathcal{J}$ is said to be integrable (or is a generalized complex structure) if the i-eigenbundle $L = \mathbb{T}^{1,0}M \subset (\mathbb{T}M)^{\mathbb{C}}$ of $\mathcal{J}$ (called the holomorphic bundle of $\mathcal{J}$) is closed under the complex linear extension of the Courant bracket $[\cdot,\cdot]$, defined by

$$[X+\xi, Y+\eta] := [X,Y] + L_X\eta - L_Y\xi - \frac{1}{2}d\left(\eta(X)-\xi(Y)\right), \tag{4.5}$$

for any smooth sections $X+\xi$ and $Y+\eta$ of $\mathbb{T}M$. Complex and symplectic structures define, in a natural way, generalized complex structures.

In this chapter we consider only generalized complex structures of constant type [61]. According to M. Gualtieri [61], the holomorphic bundle $L = \mathbb{T}^{1,0}M \subset (\mathbb{T}M)^{\mathbb{C}}$ of a generalized complex structure (of constant type) $\mathcal{J}$ can be described in terms of a subbundle $E \subset (TM)^{\mathbb{C}}$ and a 2-form $\tilde{\omega} \in \Gamma(\Lambda^2(E^*))$ defined on E. We now recall this description.

Note that $L \subset (\mathbb{T}M)^{\mathbb{C}}$ is isotropic with respect to the complex linear extension of g_{can} and $L \oplus \bar{L} = (\mathbb{T}M)^{\mathbb{C}}$. Conversely, any isotropic subbundle $L \subset (\mathbb{T}M)^{\mathbb{C}}$ such that $L \oplus \bar{L} = (\mathbb{T}M)^{\mathbb{C}}$ defines a generalized almost complex structure $\mathcal{J}$, whose complex linear extension to $(\mathbb{T}M)^{\mathbb{C}}$ satisfies $\mathcal{J}|_L = i$ and $\mathcal{J}|_{\bar{L}} = -i$. These considerations play a key role in the proof of the following Proposition (see [61], page 49):

Proposition 50. *A complex rank n subbundle L of $(\mathbb{T}M)^{\mathbb{C}}$ is the holomorphic bundle of a generalized almost complex structure $\mathcal{J}$ if and only if it is of the form*

$$L = L(E,\tilde{\omega}) := \{X+\xi \in E \oplus (T^{\mathbb{C}}M)^*, \quad \xi|_E = \tilde{\omega}(X,\cdot)\}$$

where $E \subset (TM)^{\mathbb{C}}$ is a complex subbundle and $\tilde{\omega} \in \Gamma(\Lambda^2(E^*))$ is a complex 2-form on E such that the imaginary part $\mathrm{Im}\,(\tilde{\omega}|_\Delta)$ is non-degenerate. Here

$$\Delta = E \cap TM \subset TM$$

is the real part of $E \cap \bar{E}$, i.e.

$$\Delta^{\mathbb{C}} = E \cap \bar{E}.$$

Moreover, $\mathcal{J}$ is integrable if and only if E is involutive (i.e. its space of sections is closed under the Lie bracket) and $d_E\tilde{\omega} = 0$, where d_E is the exterior derivative along E.

The codimension of the subbundle $E \subset T^{\mathbb{C}}M$ is called the type of the generalized complex structure $\mathcal{J}$.

Any complex or symplectic structure defines a generalized complex structure (see e.g. [61]). Other examples of generalized complex structures can be obtained using B-field transformations, as follows. Any closed 2-form $B \in \Omega^2(M)$ (usually called a B-field) defines an automorphism of $\mathbb{T}M$, by

$$\exp(B)(X + \xi) = X + i_X B + \xi, \quad \forall X + \xi \in \mathbb{T}M$$

which preserves the Courant bracket (this follows from $dB = 0$). If $\mathcal{J}$ is a generalized complex structure on M, with holomorphic bundle $L(E, \tilde{\omega})$, then $L(E, \tilde{\omega} + i^*B)$, where $i^*B \in \Lambda^2(E^*)$ is the restriction of (the complexification of) B to E, is the holomorphic bundle of another generalized complex structure $\exp(B) \cdot \mathcal{J}$, called the B-field transformation of $\mathcal{J}$. Obviously, a B-field transformation preserves the type.

The last notion we need from generalized complex geometry is the normal form of generalized complex structures [86]. Recall first that an (almost) f-structure on a manifold M is an endomorphism F of TM satisfying $F^3 + F = 0$. Let T^0M, $T^{1,0}M$ and $T^{0,1}M$ be the eigenbundles of the complex linear extension of F, with eigenvalues 0, i and $-i$ respectively. A (real) 2-form $\eta \in \Omega^2(M)$ is called compatible with F if $\eta_{\mathbb{C}}|_{T^0M}$ is non-degenerate and $\mathrm{Ker}(\eta_{\mathbb{C}}) = T^{1,0}M \oplus T^{0,1}M$, where $\eta_{\mathbb{C}}$ is the complex linear extension of η. A generalized (almost) complex structure $\mathcal{J}$ on M, with holomorphic bundle L, is in **normal form** if $L = L(T^0M \oplus T^{1,0}M, i\eta_{\mathbb{C}})$ for some almost f-structure and compatible 2-form η. In the language of [61], this means that $\mathcal{J}_p$, at any $p \in M$, is the direct sum of a complex structure and a symplectic structure.

4.3 Infinitesimal description of invariant generalized complex structures on Lie groups

4.3.1 Admissible pairs: definition and general results

Generalized geometry of Lie groups and homogeneous spaces already appears in the literature (see e.g. [12, 28, 54, 83]). It is known that invariant generalized complex structures on a Lie group G are in bijective correspondence with invariant complex structures on the cotangent group T^*G, which are compatible with the standard neutral bi-invariant metric of T^*G, see [12]. The proof follows from the remark that the restriction of the Courant bracket to the space of invariant generalized vector fields (i.e. invariant sections of $\mathbb{T}G$) coincides with the Lie bracket in the Lie algebra $\mathfrak{t}^*(\mathfrak{g})$ of the cotangent group T^*G.

In this section we develop an infinitesimal description of invariant generalized complex structures on Lie groups in terms of the so called admissible pairs, which will be a main tool in this paper [3, 83].

Let G be a real Lie group. On the generalized tangent bundle $\mathbb{T}G$ we consider the natural action induced by left translations:

$$g \cdot (X + \xi) := (L_g)_* X + \xi \circ (L_g^{-1})_*, \quad \forall g \in G, \quad \forall X + \xi \in \mathbb{T}G.$$

Definition 51. *A generalized almost complex structure $\mathcal{J}$ on G is called invariant if*

$$\mathcal{J}(X + \xi) = g^{-1} \cdot \mathcal{J}\left(g \cdot (X + \xi)\right), \quad \forall g \in G, \quad \forall X + \xi \in \mathbb{T}G. \tag{4.6}$$

The holomorphic bundle $L = L(E, \tilde{\omega})$ of an invariant generalized almost complex structure $\mathcal{J}$ is determined by its fiber L_e at the identity element $e \in G$, i.e. by a subspace $\mathfrak{k} := E_e \subset \mathfrak{g}^{\mathbb{C}}$ and a complex 2-form $\omega := \tilde{\omega}_e \in \Lambda^2(\mathfrak{k}^*)$. It is easy to check that the bundle E is involutive if and only if $\mathfrak{k}$ is a complex subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Moreover, if E is involutive, then $d_E\tilde{\omega} = 0$ if and only if ω is closed as a 2-form on the Lie algebra $\mathfrak{k}$, i.e. for any $X, Y, Z \in \mathfrak{k}$,

$$(d_{\mathfrak{k}}\omega)(X, Y, Z) := \omega(X, [Y, Z]) + \omega(Z, [X, Y]) + \omega(Y, [Z, X]) = 0.$$

To simplify terminology we introduce the following definition.

Definition 52. *Let $\mathfrak{g}$ be a real Lie algebra, given by the fixed point set of an antiinvolution $\sigma(x) = \bar{x}$ of $\mathfrak{g}^{\mathbb{C}}$. A pair $(\mathfrak{k}, \omega)$ formed by a complex subalgebra $\mathfrak{k} \subset \mathfrak{g}^{\mathbb{C}}$ and a closed 2-form $\omega \in \Lambda^2(\mathfrak{k}^*)$ is called* **$\mathfrak{g}$-admissible** *if*

i) $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k} + \bar{\mathfrak{k}}$*;*

ii) $\omega_{\mathfrak{l}} := \operatorname{Im}\left(\omega|_{\mathfrak{l}}\right)$ *is a symplectic form on* $\mathfrak{l} = \mathfrak{k} \cap \mathfrak{g}$*, i.e. it is non-degenerate (and closed).*

The following result, which is a corollary of Proposition 50, reduces the description of invariant generalized complex structures on a Lie group G to the description of $\mathfrak{g}$-admissible pairs.

Theorem 53. *Let G be a Lie group, with Lie algebra $\mathfrak{g}$. There is a natural one to one correspondence between:*

i) invariant generalized complex structures on G;

ii) $\mathfrak{g}$-admissible pairs $(\mathfrak{k}, \omega)$.

More precisely, a $\mathfrak{g}$-admissible pair $(\mathfrak{k}, \omega)$ defines an invariant generalized complex structure $\mathcal{J}$, with holomorphic space at $e \in G$ given by

$$\mathbb{T}^{1,0}_e G = \mathfrak{w}^{1,0} := \{X + \xi \in \mathfrak{k} \oplus (\mathfrak{g}^{\mathbb{C}})^* : \quad \xi|_{\mathfrak{k}} = \omega(X, \cdot)\}.$$

Theorem 53 has the following important consequence.

Corollary 54. *Let $\mathcal{J}$ be an invariant generalized complex structure on a Lie group G, defined by a $\mathfrak{g}$-admissible pair $(\mathfrak{k}, \omega)$. Suppose that the real Lie algebra*

$$\mathfrak{l} = \mathfrak{g} \cap \mathfrak{k} \subset \mathfrak{g}$$

generates a closed, connected Lie subgroup L of G, such that the homogeneous space $M = G/L$ is reductive. Then $\mathcal{J}$ defines an invariant complex structure J on M.

Proof. Since $\mathfrak{k}$ belongs to an admissible pair, $\mathfrak{k} + \bar{\mathfrak{k}} = \mathfrak{g}^{\mathbb{C}}$. Moreover, $\mathfrak{k} \cap \bar{\mathfrak{k}} = \mathfrak{l}^{\mathbb{C}}$. From Proposition 48 *iii)*, $\mathfrak{k}$ defines an invariant complex structure J on M. $\square$

We end this section with a property of admissible pairs, which will be useful in our treatment of invariant generalized complex structures on semisimple Lie groups.

Proposition 55. *Let $\mathfrak{g}$ be a Lie algebra and $(\mathfrak{k}, \omega)$ a $\mathfrak{g}$-admissible pair. Suppose that $\mathfrak{l} = \mathfrak{k} \cap \mathfrak{g}$ is reductive. Then $\mathfrak{l}$ is abelian.*

Proof. The claim is a consequence of the general statement that any (real of complex) reductive Lie algebra which admits a symplectic form is abelian (see e.g. [3]). We apply this statement to the reductive Lie algebra $\mathfrak{l} = \mathfrak{k} \cap \mathfrak{g}$, which admits a symplectic form, namely $\omega_{\mathfrak{l}} := \mathrm{Im}\left(\omega|_{\mathfrak{l}}\right)$. $\square$

4.4 Invariant generalized complex structures of regular type on semisimple Lie groups

In the remaining part of the paper we define and study a class (called regular) of invariant generalized complex structures on semisimple Lie groups.

4.4.1 Regular $\mathfrak{g}$-admissible pairs: definition and general results

Let G be a real semisimple Lie group with Lie algebra $\mathfrak{g}$.

Definition 56. *A $\mathfrak{g}$-admissible pair $(\mathfrak{k}, \omega)$ and the associated invariant generalized complex structure $\mathcal{J}$ on G are called regular if the complex subalgebra $\mathfrak{k} \subset \mathfrak{g}^{\mathbb{C}}$ is regular, i.e. it is normalized by a Cartan subalgebra of $\mathfrak{g}$.*

We denote by

$$\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}, \quad x \to \bar{x}$$

the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$. Let $\mathfrak{k}$ be a regular subalgebra of $\mathfrak{g}^{\mathbb{C}}$. The complexification $\mathfrak{h}$ of the Cartan subalgebra $\mathfrak{h}_{\mathfrak{g}}$ of $\mathfrak{g}$ which normalizes $\mathfrak{k}$ is a σ-invariant Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Being regular, the subalgebra $\mathfrak{k}$ is of the form

$$\mathfrak{k} = \mathfrak{h}_{\mathfrak{k}} + \mathfrak{g}(R_0) \tag{4.7}$$

where $\mathfrak{h}_{\mathfrak{k}} := \mathfrak{k} \cap \mathfrak{h}$ and $R_0 \subset R$ is a closed subset of the root system R of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}$. The condition $\mathfrak{k} + \bar{\mathfrak{k}} = \mathfrak{g}^{\mathbb{C}}$ from the definition of admissible pairs is equivalent to

$$R_0 \cup \sigma(R_0) = R, \quad \mathfrak{h}_{\mathfrak{k}} + \bar{\mathfrak{h}}_{\mathfrak{k}} = \mathfrak{h}.$$

To simplify terminology we introduce the following definition:

Definition 57. *i) A subset $R_0 \subset R$ is called σ-parabolic if it is closed and*

$$R_0 \cup \sigma(R_0) = R.$$

ii) A σ-parabolic subset $R_0 \subset R$ is called a σ-positive system if it satisfies the additional condition

$$R_0 \cap \sigma(R_0) = \emptyset.$$

iii) Two σ-parabolic subsets R_0, R_0' are called equivalent if one of them can be obtained from the other by transformations $R \to -R$, $R \to \sigma(R)$ and a transformation from the Weyl group of R, which commutes with σ.

We remark that if $\mathfrak{g}$ is a real form of inner type of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{h}_{\mathfrak{g}}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}$, then

$$\sigma(\alpha) = -\alpha, \quad \forall \alpha \in R \tag{4.8}$$

and by a result of Bourbaki [18] (Chapter VI, Section 1.7), σ-parabolic subsets (respectively, σ-positive systems) of R are just parabolic subsets, that is, closed subsets which contain a positive root system (respectively, positive root systems).

Lemma 58. *Let $\mathfrak{k}$ be the regular subalgebra (4.7) of $\mathfrak{g}^{\mathbb{C}}$, such that*

$$(R_0 \cap \sigma(R_0))^{\mathrm{asym}} = \emptyset. \tag{4.9}$$

Suppose that $\mathfrak{k}$ can be included into a $\mathfrak{g}$-admissible pair $(\mathfrak{k}, \omega)$. Then

$$\mathfrak{k} = \mathfrak{l}^{\mathbb{C}} + \mathfrak{a}^{1,0} + \mathfrak{g}(R_0)$$

where R_0 is a σ-positive system of R, $\mathfrak{l} := \mathfrak{k} \cap \mathfrak{h}_{\mathfrak{g}}$ and $\mathfrak{a}^{1,0}$ is the holomorphic space of a complex structure $J^{\mathfrak{a}}$ on a complement $\mathfrak{a}$ of $\mathfrak{l}$ in $\mathfrak{h}_{\mathfrak{g}}$. In particular, the dimension of $\mathfrak{a}$ is even.

Proof. The complex conjugated subalgebra $\bar{\mathfrak{k}}$ has the form

$$\bar{\mathfrak{k}} = \bar{\mathfrak{h}}_{\mathfrak{k}} + \sigma(\mathfrak{g}(R_0)) = \bar{\mathfrak{h}}_{\mathfrak{k}} + \mathfrak{g}(\sigma(R_0)). \tag{4.10}$$

From (4.10),

$$\mathfrak{k} \cap \bar{\mathfrak{k}} = \mathfrak{h}_{\mathfrak{k}} \cap \bar{\mathfrak{h}}_{\mathfrak{k}} + \mathfrak{g}(R_0 \cap \sigma(R_0)).$$

Condition (4.9) means that $R_0 \cap \sigma(R_0)$ is symmetric. Thus the Lie algebra $\mathfrak{k} \cap \bar{\mathfrak{k}}$ is reductive, with semisimple part generated by $\mathfrak{g}(R_0 \cap \sigma(R_0))$ and the center which is the annihilator of $R_0 \cap \sigma(R_0)$ in $\mathfrak{h}_{\mathfrak{k}} \cap \bar{\mathfrak{h}}_{\mathfrak{k}}$. Since $\mathfrak{k} \cap \bar{\mathfrak{k}}$ is a reductive subalgebra with a symplectic form, from the proof of Lemma 55 it is commutative. It follows that $R_0 \cap \sigma(R_0) = \emptyset$. On the other hand, $\mathfrak{k} + \bar{\mathfrak{k}} = \mathfrak{g}^{\mathbb{C}}$ implies that $R_0 \cup \sigma(R_0) = R$. We proved that R_0 is a σ-positive system.

Let $\mathfrak{w}$ be a complement of $\mathfrak{h}_{\mathfrak{k}} \cap \bar{\mathfrak{h}}_{\mathfrak{k}}$ in $\mathfrak{h}_{\mathfrak{k}}$. Since $\mathfrak{h}_{\mathfrak{k}} + \bar{\mathfrak{h}}_{\mathfrak{k}} = \mathfrak{h}$, $\mathfrak{w} + \bar{\mathfrak{w}} = \mathfrak{a}^{\mathbb{C}}$ where $\mathfrak{a}$ is a complement of $\mathfrak{l}$ in $\mathfrak{h}_{\mathfrak{g}}$. Being transverse, $\mathfrak{w}$ and $\bar{\mathfrak{w}}$ are the holomorphic and anti-holomorphic spaces of a complex structure $J^{\mathfrak{a}}$ on $\mathfrak{a}$.

□

Note that if $\sigma(\alpha) = -\alpha$, for any $\alpha \in R$, the condition (4.9) is automatically satisfied. Our main result in this section is stated as follows.

Proposition 59. *Let $\mathfrak{g}$ be a real form of inner type of a complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Any regular subalgebra of $\mathfrak{g}^{\mathbb{C}}$ which is normalized by a maximally compact Cartan subalgebra $\mathfrak{h}_{\mathfrak{g}}$ of $\mathfrak{g}$ and can be included in a $\mathfrak{g}$-admissible pair is of the form*

$$\mathfrak{k} = \mathfrak{l}^{\mathbb{C}} + \mathfrak{a}^{1,0} + \mathfrak{g}(R^{+}) \tag{4.11}$$

where $\mathfrak{l} := \mathfrak{k} \cap \mathfrak{h}_{\mathfrak{g}}$, $\mathfrak{a}^{1,0}$ is the holomorphic space of a complex structure on a complement $\mathfrak{a}$ of $\mathfrak{l}$ in $\mathfrak{h}_{\mathfrak{g}}$ and R^{+} is a system of positive roots of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h} := (\mathfrak{h}_{\mathfrak{g}})^{\mathbb{C}}$.

Proof. Condition (4.9) is trivially satisfied and $R_0 = R^{+}$ is a positive system.

□

In the next section we find all 2-forms, which, together with the subalgebra (4.11), form $\mathfrak{g}$-admissible pairs.

4.4.2 Regular pairs on semisimple Lie groups of inner type

In this Section we assume that the real form $\mathfrak{g} = (\mathfrak{g}^{\mathbb{C}})^{\sigma}$ of a complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is of inner type. Then, preserving the notations of Proposition 59, a regular subalgebra $\mathfrak{k} \subset \mathfrak{g}^{\mathbb{C}}$ normalized by a maximally compact Cartan subalgebra $\mathfrak{h}_{\mathfrak{g}}$ of $\mathfrak{g}$ and which is part of a $\mathfrak{g}$-admissible pair $(\mathfrak{k}, \omega)$ has the form

$$\mathfrak{k} = \mathfrak{h}_0 + \mathfrak{g}(R^{+}) = \mathfrak{l}^{\mathbb{C}} + \mathfrak{a}^{1,0} + \mathfrak{g}(R^{+}), \tag{4.12}$$

where $R^+ \subset R$ is a positive root system of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h} = (\mathfrak{h}_{\mathfrak{g}})^{\mathbb{C}}$ and $\mathfrak{h}_0 \subset \mathfrak{h}$ satisfies $\mathfrak{h}_0 + \bar{\mathfrak{h}}_0 = \mathfrak{h}$. We now determine all 2-forms $\omega \in \Lambda^2(\mathfrak{k}^*)$ which, together with $\mathfrak{k}$, form a $\mathfrak{g}$-admissible pair. In the following theorem we fix root vectors $\{E_\alpha\}_{\alpha\in R}$ of a Weyl basis of $\mathfrak{g}(R)$ and we denote by $\omega_\alpha \in (\mathfrak{g}^{\mathbb{C}})^*$ the dual covectors defined by

$$\omega_\alpha|_{\mathfrak{h}} = 0, \quad \omega_\alpha(E_\beta) = \delta_{\alpha\beta}, \quad \forall \alpha, \beta \in R. \tag{4.13}$$

As usual $N_{\alpha\beta}$ will denote the structure constants, defined by $[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$.

Theorem 60. *Let $\mathfrak{k} \subset \mathfrak{g}^{\mathbb{C}}$ be the regular subalgebra defined by (4.12) and $\omega \in \Lambda^2(\mathfrak{k}^*)$. Then $(\mathfrak{k}, \omega)$ is a $\mathfrak{g}$-admissible pair if and only if*

$$\omega = \widehat{\omega}_0 + \sum_{\alpha\in R^+} \mu_\alpha \alpha \wedge \omega_\alpha + \frac{1}{2} \sum_{\alpha,\beta\in R^+} \mu_{\alpha+\beta} N_{\alpha\beta} \omega_\alpha \wedge \omega_\beta, \tag{4.14}$$

where $\mu_\alpha \in \mathbb{C}$, for any $\alpha \in R^+$, and $\widehat{\omega}_0 \in \Lambda^2(\mathfrak{h}_0^*)$ is any 2-form on $\mathfrak{h}_0$ (trivially extended to $\mathfrak{k}$), such that $\mathrm{Im}\left(\widehat{\omega}_0|_{\mathfrak{l}}\right)$ *is non-degenerate.*

Proof. The proof is based on the theory of (invariant) closed forms defined on Lie algebras, developed in [11]. More specifically, there is a general result which describes all closed 2-forms defined on a Lie algebra $\mathfrak{k}$ which admits a semidirect decomposition $\mathfrak{k} = \mathfrak{s} \oplus \mathfrak{p}$ into a subalgebra $\mathfrak{s}$ and ideal $\mathfrak{p}$. Namely, consider any 2-form $\rho \in \Lambda^2(\mathfrak{k}^*)$ and decompose it into three parts

$$\rho = \rho_0 + \rho_1 + \rho_2,$$

where $\rho_0 \in \Lambda^2(\mathfrak{s}^*)$ is the $\mathfrak{s}$-part, $\rho_1 \in \Lambda^2(\mathfrak{p}^*)$ is the $\mathfrak{p}$-part and $\rho_2 \in \mathfrak{s}^* \wedge \mathfrak{p}^* \subset \Lambda^2(\mathfrak{k}^*)$ is the mixed part of ρ (all trivially extended to $\mathfrak{k}$). It turns out that ρ is closed if and only if the forms ρ_0, ρ_1 are closed on $\mathfrak{s}$ and, respectively, on $\mathfrak{p}$, and the following two conditions are satisfied:

$$\rho_2(s, [p, p']) = \rho_1([s, p], p') + \rho_1(p, [s, p']) \tag{4.15}$$

and

$$\rho_2([s, s'], p) + \rho_2([s', p], s) + \rho_2([p, s], s') = 0, \tag{4.16}$$

for any $s, s' \in \mathfrak{s}$ and $p, p' \in \mathfrak{p}$. One applies this general result to the Lie algebra $\mathfrak{k}$ from the statement of the theorem, with commutative subalgebra $\mathfrak{s} := \mathfrak{h}_0 = \mathfrak{l}^{\mathbb{C}} + \mathfrak{a}^{1,0}$ and ideal $\mathfrak{p} := \mathfrak{g}(R^+)$. Any $\omega \in \Lambda^2(\mathfrak{k}^*)$ is given by

$$\omega = \widehat{\omega}_0 + \rho_1 + \rho_2$$

where $\widehat{\omega}_0 \in \Lambda^2(\mathfrak{h}_0^*)$, $\rho_1 \in \Lambda^2(\mathfrak{p}^*)$ and $\rho_2 \in \mathfrak{h}_0^* \wedge \mathfrak{p}^*$ are trivially extended to $\mathfrak{k}$. Since $\mathfrak{h}_0$ is abelian, $d_{\mathfrak{h}_0} \widehat{\omega}_0 = 0$ for any $\widehat{\omega}_0$. It may be shown that other conditions (4.15) and (4.16) together with the closedness of ρ_1 imply:

$$\rho_1 = \frac{1}{2} \sum_{\alpha,\beta\in R^+} N_{\alpha\beta} \mu_{\alpha+\beta} \omega_\alpha \wedge \omega_\beta \tag{4.17}$$

and

$$\rho_2 = \sum_{\alpha \in R^+} \mu_\alpha \alpha \wedge \omega_\alpha \tag{4.18}$$

for some constants μ_α. Combining (4.17) with (4.18) we obtain (4.14). If, moreover, Im $\left(\widehat{\omega}_0|_{\mathfrak{l}}\right)$ is non-degenerate, then $(\mathfrak{k}, \omega)$ is a $\mathfrak{g}$-admissible pair. □

For simplicity, the following proposition is stated for compact forms, but it holds also for any real form of inner type. We preserve the notations from Theorem 60.

Proposition 61. *Let $\mathcal{J}$ be a regular generalized complex structure on a compact semisimple Lie group G, with associated $\mathfrak{g}$-admissible pair $(\mathfrak{k}, \omega)$ defined by (4.12) and (4.14). Define a covector $\xi \in \mathfrak{g}^*$ by*

$$\xi := -\sum_{\alpha \in R^+} \mu_\alpha \omega_\alpha + \sum_{\alpha \in R^+} \bar{\mu}_\alpha \omega_{-\alpha}$$

and let $B := d\xi$. Then $\mathcal{J}$ is the B-field transformation of the regular generalized complex structure $\widehat{\mathcal{J}}$ whose associated $\mathfrak{g}$-admissible pair is $(\mathfrak{k}, \widehat{\omega}_0)$. Moreover, $\widehat{\mathcal{J}}$ is in normal form if and only if

$$i_H \widehat{\omega}_0 = 0, \quad \forall H \in \mathfrak{a}^{1,0}.$$

4.4.3 Invariant generalized complex structures on compact semisimple Lie groups

We now show that any invariant generalized complex structure $\mathcal{J}$ on a compact semisimple Lie group G is regular, provided that $\mathcal{J}$ satisfies an additional natural condition. More precisely, we prove:

Theorem 62. *Let G be a compact semisimple Lie group, with Lie algebra $\mathfrak{g}$, and $\mathcal{J}$ an invariant generalized complex structure on G defined by a $\mathfrak{g}$-admissible pair $(\mathfrak{k}, \omega)$. Suppose that $\mathfrak{l} := \mathfrak{k} \cap \mathfrak{g}$ generates a closed subgroup L of G. Then $\mathcal{J}$ is regular.*

Proof. Since G is semisimple and compact, $M = G/L$ is reductive and $\mathcal{J}$ induces an invariant complex structure J on M, defined by the subalgebra $\mathfrak{k}$ (see Corollary 54). By Theorem 49, $\mathfrak{k}$ is regular. □

4.5 Invariant generalized complex structures on semisimple Lie groups of outer type

In this section we construct a large class of regular generalized complex structures on semisimple Lie groups of outer type. The following Proposition simplifies considerably this task. It shows that we can chose (arbitrarily), for each complex Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_{2n}(\mathbb{C})$ $(n \geq 2)$, $\mathfrak{so}_{2n}(\mathbb{C})$ $(n \geq 3)$ and $\mathfrak{e}_6$, a real form of outer type and find regular pairs with respect to these real forms. They will provide invariant generalized complex structures on all real forms of outer type of $SL_{2n}(\mathbb{C})$ $(n \geq 2)$, $SO_{2n}(\mathbb{C})$ $(n \geq 3)$ and E_6.

Proposition 63. *Let $\mathfrak{g}^{\mathbb{C}}$ be a complex simple Lie algebra, $\mathfrak{g} = (\mathfrak{g}^{\mathbb{C}})^{\sigma}$ a real form of outer type of $\mathfrak{g}^{\mathbb{C}}$ and $(\mathfrak{k}, \omega)$ a regular $\mathfrak{g}$-admissible pair, with*

$$\mathfrak{k} = \mathfrak{h}_{\mathfrak{k}} + \mathfrak{g}(R_0) \tag{4.19}$$

normalized by a maximally compact Cartan subalgebra $\mathfrak{h}_{\mathfrak{g}}$ of $\mathfrak{g}$ and R_0 a σ-positive system. Then any other real form of outer type of $\mathfrak{g}^{\mathbb{C}}$ is conjugated to a real form $\mathfrak{g}'$ such that $(\mathfrak{k}, \omega)$ is $\mathfrak{g}'$-admissible.

Proof. To prove the claim, we recall the formalism of Vogan diagrams (see [72], page 344) which describes the real forms of a given complex simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$. An abstract Vogan diagram is an abstract Dynkin diagram, which represents a system of simple roots Π of $\mathfrak{g}^{\mathbb{C}}$ relative to a Cartan subalgebra $\mathfrak{h}$, together with two pieces of additional information: some arrows between vertices, which indicate the action of an involutive symmetry s of the Dynkin diagram (in the case of outer real forms) and a subset of painted nodes (in the fix point set of s), which indicates the non-compact simple roots. The symmetry s defines an involution θ of the Cartan subalgebra $\mathfrak{h}$, which can be canonically extended to an involutive automorphism θ of $\mathfrak{g}^{\mathbb{C}}$, by the conditions: for any $\alpha \in \Pi$,

$$\theta(E_\alpha) = \begin{cases} E_{\alpha'}, & \text{if } s(\alpha) = \alpha' \neq \alpha \\ E_\alpha, & \text{if } \alpha = s(\alpha) \text{ is not a painted root} \\ -E_\alpha, & \text{if } \alpha = s(\alpha) \text{ is a painted root,} \end{cases} \tag{4.20}$$

where $\{E_\alpha\}$ are root vectors of a Weyl basis.

The composition $\sigma := \theta \circ \tau$, where τ is the compact involution commuting with θ, defines the real form $\mathfrak{g} = (\mathfrak{g}^{\mathbb{C}})^{\sigma}$ associated to the Vogan diagram. Moreover, the real subalgebra

$$\mathfrak{h}_{\mathfrak{g}} = \mathfrak{h}^{+} \oplus \mathfrak{h}^{-} = \{x \in i\mathfrak{h}(\mathbb{R}), \quad \theta(x) = x\} \oplus \{x \in \mathfrak{h}(\mathbb{R}), \quad \theta(x) = -x\} \tag{4.21}$$

is a maximally compact (θ-invariant) Cartan subalgebra of $\mathfrak{g}$.

Consider now a regular $\mathfrak{g}$-admissible pair $(\mathfrak{k}, \omega)$, like in (4.19), where $\mathfrak{g}$ is a real form of outer type of a complex simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$. The defining conditions for $(\mathfrak{k}, \omega)$, namely

$$R_0 \cup \sigma(R_0) = R, \quad R_0 \cap \sigma(R_0) = \emptyset, \quad \mathfrak{h}_{\mathfrak{k}} + \bar{\mathfrak{h}}_{\mathfrak{k}} = \mathfrak{h}, \quad d_{\mathfrak{k}}\omega = 0$$

and $\mathrm{Im}(\omega|_{(\mathfrak{h}_{\mathfrak{k}} \cap \bar{\mathfrak{h}}_{\mathfrak{k}})^{\sigma}})$ non-degenerate, depend only on the symmetry s of the associated Vogan diagram. Remark now that the symmetry s is unique, for $\mathfrak{g}^{\mathbb{C}} \neq D_4$ and for D_4 there are three symmetries related by an outer automorphism of D_4. This proves our claim.

□

Remark 64. In the statement of Proposition 63 it is essential that R_0 is a σ-positive system. When R_0 is σ-parabolic but not necessarily σ-positive, $(\mathfrak{k} \cap \bar{\mathfrak{k}})^{\sigma}$ does not reduce in general to the Cartan part $(\mathfrak{h}_{\mathfrak{k}} \cap \bar{\mathfrak{h}}_{\mathfrak{k}})^{\sigma}$ and it may depend also on which nodes from the Vogan diagram are painted; therefore, the same is true for the condition $\mathrm{Im}(\omega|_{(\mathfrak{k}\cap\bar{\mathfrak{k}})^{\sigma}})$ from the definition of $\mathfrak{g}$-admissible pairs.

Motivated by Proposition 63, in the remaining part of this section we construct regular generalized complex structures (defined by associated admissible pairs) on $G = SL_n(\mathbb{H})$ $(n \geq 2)$, $SO_{2n-1,1}$ $(n \geq 3)$ and two real forms of outer type of E_6.

4.5.1 Generalized complex structures on $SL_n(\mathbb{H})$

a) Description of the antiinvolution σ which defines $\mathfrak{sl}_n(\mathbb{H})$

Let $V = \mathbb{C}^{2n}$ be a complex vector space of dimension $2n \geq 4$, with standard basis $\{e_1, \cdots, e_n, e_{1'}, \cdots, e_{n'}\}$ and $\mathfrak{sl}_{2n}(\mathbb{C})$ the Lie algebra of traceless endomorphisms of V. We denote by

$$E_{ij} = e_i \otimes e_j^*,\ E_{i'j'} = e_{i'} \otimes e_{j'}^*,\ E_{i'j} = e_{i'} \otimes e_j^*,\ E_{ij'} = e_i \otimes e_{j'}^*$$

the associated basis of $\mathfrak{gl}(V)$ and we choose a Cartan subalgebra

$$\mathfrak{h} = \{H = \sum_{i=1}^{n} x_i E_{ii} + \sum_{j'=1}^{n} x_{j'} E_{j'j'},\ \sum_{i=1}^{n} x_i + \sum_{j'=1}^{n} x_{j'} = 0\}$$

which consists of traceless diagonal matrices. Denote by $\epsilon_i, \epsilon_{j'}$ the linear forms on $\mathfrak{h}$ defined by

$$\epsilon_i(H) = x_i, \quad \epsilon_{j'}(H) = x_{j'}.$$

The roots of $\mathfrak{sl}(V)$ are

$$R := \{\epsilon_{ij} := \epsilon_i - \epsilon_j, \epsilon_{i'j'} := \epsilon_{i'} - \epsilon_{j'}, \epsilon_{i'j} := \epsilon_{i'} - \epsilon_j, \epsilon_{ij'} := \epsilon_i - \epsilon_{j'}\}.$$

The Lie algebra $\mathfrak{sl}_n(\mathbb{H})$ is a real form of outer type of $\mathfrak{sl}_{2n}(\mathbb{C})$, defined by the antilinear involution

$$\sigma(A) = -J\bar{A}J, \quad \forall A \in \mathfrak{sl}_{2n}(\mathbb{C}),$$

where J is the matrix

$$J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

The antilinear involution σ acts on roots, transforming unprime indices into prime indices and vice versa, i.e.

$$\sigma(\epsilon_{ij}) = \epsilon_{i'j'}, \quad \sigma(\epsilon_{i'j'}) = \epsilon_{ij}, \quad \sigma(\epsilon_{ij'}) = \epsilon_{i'j}, \quad \sigma(\epsilon_{i'j}) = \epsilon_{ij'}.$$

Alternatively, one may also define $\mathfrak{sl}_n(\mathbb{H})$ as the unique real form of outer type of $\mathfrak{sl}_{2n}(\mathbb{C})$, whose Vogan diagram has no painted node. As described above, $\mathfrak{sl}_n(\mathbb{H})$ can be obtained as in the proof of Theorem 6.88 of [72] (see also Proposition 63), by considering the Weyl basis (4.23) (see below) and assigning to the nodes of A_{2n-1} the simple roots $\epsilon_{12'}, \epsilon_{2'3}, \epsilon_{34'}, \cdots, \epsilon_{3'2}, \epsilon_{21'}$. The unique simple root fixed by the non-trivial automorphism of A_{2n-1} (given by the horizontal reversal) is $\epsilon_{n'n}$ (n-even) or $\epsilon_{nn'}$ (n-odd) and is a white root. The Cartan subalgebra

$$\mathfrak{h}^{\sigma} = \{H = \sum_{i=1}^{n} x_i E_{ii} + \sum_{j=1}^{n} \overline{x}_j E_{j'j'}, \quad \sum_{i=1}^{n} (x_i + \bar{x}_i) = 0\}$$

is a maximally compact Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{H})$. Any $\alpha \in R$ for which $\sigma(\alpha) = -\alpha$ is of the form $\epsilon_{ii'}$ or $\epsilon_{i'i}$ and is compact, since $\sigma(E_{\epsilon_{ii'}}) = -E_{\epsilon_{i'i}}$, where $E_{\epsilon_{ij}}$ are root vectors of the Weyl basis (4.23). It follows that $\mathfrak{h}^{\sigma}$ is also a maximally non-compact Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{H})$ (see [72, p. 335]).

b) σ-positive systems of the Lie algebra $\mathfrak{sl}_{2n}(\mathbb{C})$

Proposition 65. *Any σ-positive root system R_0 of the root system R is equivalent to one of the following systems:*
a) $\{\epsilon_i - \epsilon_j, \, \epsilon_i - \epsilon_{j'}, \; i, j = 1, 2, \cdots, n\}$;
b) $\{\epsilon_i - \epsilon_j, \, \epsilon_{i'} - \epsilon_j, \; i, j = 1, 2, \cdots, n\}$.

Proof. For a proof, see [3]. □

c) $\mathfrak{sl}_n(\mathbb{H})$-admissible pairs

Now we describe $\mathfrak{sl}_n(\mathbb{H})$-admissible pairs $(\mathfrak{k}, \omega)$, where the subalgebra $\mathfrak{k} \subset \mathfrak{sl}_{2n}(\mathbb{C})$ is regular with root system the σ-positive system R_0 of type *a)* from Proposition 65. The case *b)* is similar. The subalgebra $\mathfrak{k}$ can be written as

$$\mathfrak{k} = \mathfrak{h}_0 + \mathfrak{g}(R_0) = \mathfrak{h}_0 + \sum_{i,j} \mathfrak{g}_{\epsilon_{ij}} + \sum_{i,j'} \mathfrak{g}_{\epsilon_{ij'}} \subset \mathfrak{sl}_{2n}(\mathbb{C}), \tag{4.22}$$

where $\mathfrak{h}_0 \subset \mathfrak{h}$, with $\mathfrak{h}_0 + \bar{\mathfrak{h}}_0 = \mathfrak{h}$. The vectors

$$E_{\epsilon_{ij}} = \frac{1}{\sqrt{2n}} E_{ij}, \quad E_{\epsilon_{i'j'}} = \frac{1}{\sqrt{2n}} E_{i'j'}, \quad E_{\epsilon_{i'j}} = \frac{1}{\sqrt{2n}} E_{i'j}, \quad E_{\epsilon_{ij'}} = \frac{1}{\sqrt{2n}} E_{ij'} \tag{4.23}$$

are root vectors of a Weyl basis and the associated structure constants are given by

$$N_{\epsilon_{ij},\epsilon_{js}} = -N_{\epsilon_{ji},\epsilon_{sj}} = \frac{1}{\sqrt{2n}}, \quad \forall i \neq j \neq s$$

and their prime analogues (obtained by replacing any of the i, j, s by its prime analogue). Below the covectors $\omega_{\epsilon_{ij}}, \omega_{\epsilon_{i'j}}, \omega_{\epsilon_{ij'}}, \omega_{\epsilon_{i'j'}}$ are dual to the root vectors (4.23) and annihilate $\mathfrak{h}$. We assume that $n \geq 3$ (when $n = 2$ the theorem still holds, under the additional assumption that $\epsilon_i + \epsilon_r - \epsilon_{j'} - \epsilon_{s'}$ is not identically zero on $\mathfrak{h}_0$, for any i, r, j', s', with $(i, j') \neq (r, s')$).

Theorem 66. *Any closed 2-form ω on the Lie algebra $\mathfrak{k}$ defined in (4.22) is given by*

$$\omega = \widehat{\omega}_0 + \sum_{i\neq j} \lambda_{(ij)}\epsilon_{ij} \wedge \omega_{\epsilon_{ij}} + \frac{1}{\sqrt{2n}} \sum_{i\neq j\neq k} \lambda_{(ik)}\omega_{\epsilon_{ij}} \wedge \omega_{\epsilon_{jk}} + \sum_{i\neq j} \eta_{(ij)}\omega_{\epsilon_{ij}} \wedge \omega_{\epsilon_{ji}}$$
$$+ \sum_{k,j'} \lambda_{(kj')} \left(\sqrt{2n}\epsilon_{kj'} \wedge \omega_{\epsilon_{kj'}} + \sum_{i\neq k} \omega_{\epsilon_{ki}} \wedge \omega_{\epsilon_{ij'}} \right)$$

where $\widehat{\omega}_0 \in \Lambda^2(\mathfrak{h}_0^*)$ is such that

$$\widehat{\omega}_0(E_{ii} - E_{jj}, \cdot) = 0, \quad \forall i, j \tag{4.24}$$

and $\lambda_{(ij)}, \lambda_{(ij')}, \eta_{(ij)} \in \mathbb{C}$. The pair $(\mathfrak{k}, \omega)$ is $\mathfrak{sl}_n(\mathbb{H})$-admissible, hence it defines a regular generalized complex structure on $SL_n(\mathbb{H})$, if and only if the real 2-form $\mathrm{Im}\,(\widehat{\omega}_0)$ *is non-degenerate on $\mathfrak{h}_0 \cap \mathfrak{sl}_n(\mathbb{H})$.*

Proof. The proof is similar to the proof of Theorem 60 and follows by decomposing $\mathfrak{k}$ into a subalgebra $\mathfrak{s} := \mathfrak{h}_0 + \mathfrak{g}(\{\epsilon_{ij}\})$ and ideal $\mathfrak{p} := \mathfrak{g}(\{\epsilon_{ij'}\})$. For more details, see [3]. □

4.5.2 Generalized complex structures on $SO_{2n-1,1}$

a) Description of the antiinvolution σ which defines $\mathfrak{so}_{2n}(\mathbb{C})$

Let $(V, (\cdot, \cdot))$ be a complex Euclidean vector space of dimension $2n \geq 6$ and $\mathfrak{so}(V) \simeq \mathfrak{so}_{2n}(\mathbb{C})$ the associated complex orthogonal Lie algebra. We identify $\mathfrak{so}(V)$ with $\Lambda^2 V$ using the scalar product $(\cdot, \cdot)$ and we choose a basis e_i, e_{-i}, $i = 1, \cdots, n$ of V with the only non-zero scalar products $(e_i, e_{-i}) = 1$. The diagonal Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}(V)$ has a basis

$$\{H_i := e_i \wedge e_{-i}, \quad i = 1, 2, \cdots, n\}.$$

We denote by $\{\epsilon_i\}$ the dual basis of $\mathfrak{h}^*$. Then the root system of $\mathfrak{so}(V)$ relative to $\mathfrak{h}$ is given by

$$R := \{\pm\epsilon_i \pm \epsilon_j, \quad i, j = 1, 2, \cdots, n, i \neq j\}$$

and the root vectors of a Weyl basis are

$$\begin{aligned} E_{\epsilon_i+\epsilon_j} &:= \frac{1}{\sqrt{2(n-1)}}(e_i \wedge e_j), \quad i<j \\ E_{-\epsilon_i-\epsilon_j} &:= -\frac{1}{\sqrt{2(n-1)}}(e_{-i} \wedge e_{-j}), \quad i<j \\ E_{\epsilon_i-\epsilon_j} &:= \frac{1}{\sqrt{2(n-1)}}(e_i \wedge e_{-j}), \quad i \neq j. \end{aligned}$$

The associated structure constants are given by

$$\begin{aligned} N_{\epsilon_i+\epsilon_j,\epsilon_k-\epsilon_j} &= -\frac{1}{\sqrt{2(n-1)}}\gamma_{ij}\gamma_{ik} \\ N_{-(\epsilon_i+\epsilon_j),\epsilon_l+\epsilon_j} &= \frac{1}{\sqrt{2(n-1)}}\gamma_{ij}\gamma_{jl} \\ N_{-(\epsilon_i+\epsilon_j),\epsilon_j-\epsilon_k} &= \frac{1}{\sqrt{2(n-1)}}\gamma_{ij}\gamma_{ik} \\ N_{\epsilon_i-\epsilon_j,\epsilon_j-\epsilon_k} &= \frac{1}{\sqrt{2(n-1)}}, \end{aligned}$$

where $\gamma_{ij} = 1$ if $i<j$ and -1 if $i>j$.

Consider the antilinear involution σ of V defined by

$$\sigma(e_{\pm i}) = e_{\mp i}, \quad 1 \leq i < n, \quad \sigma(e_{\pm n}) = e_{\pm n}.$$

It induces an antilinear involution σ on $\mathfrak{so}(V)$ whose associated real form is the Lorentzian Lie algebra $\mathfrak{so}_{2n-1,1}$. The map σ preserves the Cartan subalgebra $\mathfrak{h}$ and it acts on the weights ϵ_i as follows:

$$\sigma(\epsilon_i) = -\epsilon_i,\ i<n,\ \sigma(\epsilon_n) = \epsilon_n.$$

Alternatively, one may also define $\mathfrak{so}_{2n-1,1}$ as the unique real form of outer type of $\mathfrak{so}_{2n}(\mathbb{C})$, whose Vogan diagram has no painted node and the automorphism is given by interchanging the two ends of D_n. As described above, $\mathfrak{so}_{2n-1,1}$ can be obtained as in the proof of Theorem 6.88 of [72] (see also Proposition 63), by considering the root vectors of the Weyl basis defined above and assigning to the nodes of D_n the simple roots $\epsilon_1-\epsilon_2, \cdots, \epsilon_{n-2}-\epsilon_{n-1}, \epsilon_{n-1}-\epsilon_n, \epsilon_{n-1}+\epsilon_n$. The roots $\epsilon_i-\epsilon_{i+1}$, $1 \leq i \leq n-2$ are compact. The automorphism θ of D_n is given by

$$\theta(\epsilon_i-\epsilon_{i+1}) = \epsilon_i-\epsilon_{i+1}, \quad 1 \leq i \leq n-2; \quad \theta(\epsilon_{n-1}-\epsilon_n) = \epsilon_{n-1}+\epsilon_n$$

and

$$\mathfrak{h}^\sigma = \{H = \sum_{k=1}^{n-1} ix_k H_k + x_n H_n, \quad x_k \in \mathbb{R}\}$$

is a maximally compact Cartan subalgebra of $\mathfrak{so}_{2n-1,1}$. It is also maximally non-compact (easy check).

b) σ-positive systems of the Lie algebra $\mathfrak{so}_{2n}(\mathbb{C})$

We denote by $R' \subset R$ the root system of the subalgebra $\mathfrak{so}_{2n-2}(\mathbb{C}) \subset \mathfrak{so}(V)$ which preserves the vectors $e_{\pm n}$. Then

$$\sigma|_{R'} = -1 \text{ and } \sigma(\epsilon_{n-1} - \epsilon_n) = -(\epsilon_{n-1} + \epsilon_n).$$

Proposition 67. *Any σ-positive system $R_0 \subset R$ is equivalent to one of the systems:*
a) $R_0 = R^+ := \{\epsilon_i \pm \epsilon_j, 1 \leq i < j \leq n\}$;
b) $R_0 = (R^+ \setminus \{\epsilon_{n-1} + \epsilon_n\}) \cup \{\epsilon_n - \epsilon_{n-1}\}$;
c) $R_0 = (R^+ \setminus \{\epsilon_{n-1} - \epsilon_n\}) \cup \{-(\epsilon_{n-1} + \epsilon_n)\}$.

Proof. For the proof, see [3]. □

c) $SO_{2n-1,1}$-admissible pairs.

Since the σ-positive system R_0 of type *a)* found in Proposition 67 is a system of positive roots, the corresponding admissible pairs can be described using the method of Theorem 60. Now, we describe the $\mathfrak{so}_{2n-1,1}$-admissible pairs $(\mathfrak{k}, \omega)$, where $\mathfrak{k} \subset \mathfrak{so}_{2n}(\mathbb{C})$ is a regular subalgebra with the root system R_0 of type *c)* (the case *b)* can be treated similarly). Let

$$\mathfrak{k} := \mathfrak{h}_0 + \mathfrak{g}(R_0) \subset \mathfrak{so}_{2n}(\mathbb{C}), \tag{4.25}$$

where $\mathfrak{h}_0$ is included in the diagonal Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}_{2n}(\mathbb{C})$ and R_0 is given by Proposition 67 *c)*:

$$R_0 = \left(R^+ \setminus \{\epsilon_{n-1} - \epsilon_n\}\right) \cup \{-(\epsilon_{n-1} + \epsilon_n)\}, \quad R^+ = \{\epsilon_i \pm \epsilon_j, 1 \leq i < j \leq n\}.$$

Define

$$\mathfrak{h}_0 = (\mathrm{Ker}(\epsilon_{n-1} + \epsilon_n) \cap \mathfrak{h}_0) \oplus \mathrm{Span}(H_{n-1} + H_n)$$

and

$$R_0' := R^+ \setminus \{\epsilon_{n-1} \pm \epsilon_n\} = R_0 \setminus \{\pm(\epsilon_{n-1} + \epsilon_n)\}.$$

For simplicity, we assume that for any $\alpha \in R_0'$,

$$\alpha|_{\mathrm{Ker}(\epsilon_{n-1}+\epsilon_n)\cap \mathfrak{h}_0} \not\equiv 0. \tag{4.26}$$

This condition is needed in the computations in order to deduce that any closed 2-form on $\mathfrak{k}$ is given as in Theorem 68 below. Without this condition, it is still true that any 2-form ω as in the statement of Theorem 68 is closed, but not all closed 2-forms are of this form. We also remark that the above condition is automatically true when $\alpha = \epsilon_i \pm \epsilon_j$, $i < j \leq n-1$ (recall that $\sigma|_{R'} = -1$ and $\mathfrak{h}_0 + \bar{\mathfrak{h}}_0 = \mathfrak{h}$). It also holds for any $\alpha \in R_0'$, when $\mathfrak{h}_0 = \mathfrak{h}$.

Theorem 68. *Any closed 2-form ω on the Lie algebra $\mathfrak{k}$ defined in (4.25) is given by*

$$
\begin{aligned}
\omega = {} & (\epsilon_{n-1}+\epsilon_n) \wedge \big(a\omega_{\epsilon_{n-1}+\epsilon_n} + b\omega_{-(\epsilon_{n-1}+\epsilon_n)}\big) + c\omega_{\epsilon_{n-1}+\epsilon_n} \wedge \omega_{-(\epsilon_{n-1}+\epsilon_n)} \\
& + \widehat{\omega}_0 + \sum_{\alpha \in R_0'} c_\alpha \alpha \wedge \omega_\alpha + \frac{1}{2} \sum_{\alpha \in R_0'} N_{\alpha\beta} c_{\alpha+\beta} \omega_\alpha \wedge \omega_\beta \\
& + \frac{1}{\sqrt{2(n-1)}} \sum_{i<n-1} \omega_{\epsilon_{n-1}+\epsilon_n} \wedge \big(c_{\epsilon_i+\epsilon_{n-1}} \omega_{\epsilon_i-\epsilon_n} - c_{\epsilon_i+\epsilon_n} \omega_{\epsilon_i-\epsilon_{n-1}}\big) \\
& + \frac{1}{\sqrt{2(n-1)}} \sum_{i<n-1} \omega_{-(\epsilon_{n-1}+\epsilon_n)} \wedge \big(c_{\epsilon_i-\epsilon_n} \omega_{\epsilon_i+\epsilon_{n-1}} - c_{\epsilon_i-\epsilon_{n-1}} \omega_{\epsilon_i+\epsilon_n}\big) \\
& + \frac{1}{2} \sum_{i<n-1} (\epsilon_{n-1}+\epsilon_n) \wedge \big(c_{\epsilon_i+\epsilon_n} \omega_{\epsilon_i+\epsilon_n} + c_{\epsilon_i+\epsilon_{n-1}} \wedge \omega_{\epsilon_i+\epsilon_{n-1}}\big) \\
& - \frac{1}{2} \sum_{i<n-1} (\epsilon_{n-1}+\epsilon_n) \wedge \big(c_{\epsilon_i-\epsilon_n} \omega_{\epsilon_i-\epsilon_n} + c_{\epsilon_i-\epsilon_{n-1}} \wedge \omega_{\epsilon_i-\epsilon_{n-1}}\big)
\end{aligned}
$$

where $\widehat{\omega}_0 \in \Lambda^2(\mathfrak{h}_0)$ is such that

$$\widehat{\omega}_0(H_{n-1}+H_n, \cdot) = 0,$$

$a, b, c \in \mathbb{C}$ and $c_\alpha \in \mathbb{C}$, for any $\alpha \in R_0'$. The pair $(\mathfrak{k}, \omega)$ is $\mathfrak{so}_{2n-1,1}$-admissible, hence it defines a regular generalized complex structure on $SO_{2n-1,1}$, if and only if the real 2-form $\mathrm{Im}\,(\widehat{\omega}_0)$ is non-degenerate on $\mathfrak{h}_0 \cap \mathfrak{so}_{2n-1,1}$.

Proof. The proof is similar to the proof of Theorem 60 and follows by decomposing $\mathfrak{k}$ into an ideal

$$\mathfrak{p} := (\mathrm{Ker}(\epsilon_{n-1}+\epsilon_n) \cap \mathfrak{h}_0) + \mathfrak{g}(R_0')$$

and subalgebra

$$\mathfrak{s} := \mathbb{C}(H_{n-1}+H_n) + \mathfrak{g}_{\epsilon_{n-1}+\epsilon_n} + \mathfrak{g}_{-(\epsilon_{n-1}+\epsilon_n)}.$$

For a detailed proof, see [3].

□

4.5.3 Generalized complex structures on the real Lie groups E_6 of outer type

a) Description of the real forms $(\mathfrak{e}_6)^\sigma$ of outer type of $\mathfrak{e}_6$

We follow the description of the exceptional complex Lie algebra $\mathfrak{e}_6$ given in [98, p. 80]. The complex Lie algebra $\mathfrak{e}_6$ has dimension 78 and rank 6. We take $\mathfrak{h} = \mathbb{C}^6$ for the Cartan subalgebra. Let $\{e_1, \cdots, e_6\}$ be the standard basis of $\mathbb{C}^6$, with dual basis $\{\epsilon_1, \cdots, \epsilon_6\}$. The Killing form restricted to $\mathfrak{h}$ is

$$\langle x, y\rangle = 24\sum_{i=1}^{6} \epsilon_i(x)\epsilon_i(y) + 8(\sum_i \epsilon_i(x))(\sum_j \epsilon_j(y)).$$

The root system R is formed by $\pm(\epsilon_i - \epsilon_j)$ with $1 \leq i < j \leq 6$, $\pm(\epsilon_i + \epsilon_j + \epsilon_k)$ with $1 \leq i < j < k \leq 6$ and $\pm(\epsilon_1 + \cdots + \epsilon_6)$. A system of simple roots is

$$\Pi = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \cdots, \alpha_5 = \epsilon_5 - \epsilon_6, \alpha_6 = \epsilon_4 + \epsilon_5 + \epsilon_6\}.$$

The complex Lie algebra $\mathfrak{e}_6$ has two real forms of outer type, with maximally compact subalgebras $\mathfrak{f}_4$ and $\mathfrak{sp}_4$. The Vogan diagram of $\mathfrak{e}_6(\mathfrak{f}_4)$ has no painted node and an automorphism of order two, given by the horizontal reversing in the Dynkin diagram of $\mathfrak{e}_6$. The Vogan diagram of $\mathfrak{e}_6(\mathfrak{sp}_4)$ is the same, the only difference being that the triple node from the Dynkin diagram is painted, see [72, p. 361]. This means that the Cartan involutions of $\mathfrak{e}_6(\mathfrak{f}_4)$ and $\mathfrak{e}_6(\mathfrak{sp}_4)$ induce the same canonical order two automorphism of the Dynkin diagram of $\mathfrak{e}_6$, given, in terms of the simple roots from Π, by

$$\begin{aligned}\theta(\alpha_1) = \alpha_5, \quad \theta(\alpha_2) = \alpha_4,\\ \theta(\alpha_3) = \alpha_3, \quad \theta(\alpha_6) = \alpha_6.\end{aligned}$$

Similarly, the defining antiinvolutions σ of $\mathfrak{e}_6(\mathfrak{f}_4)$ and $\mathfrak{e}_6(\mathfrak{sp}_4)$ induce the same action on R:

$$\sigma(\alpha) = -\theta(\alpha), \quad \forall \alpha \in R.$$

We shall denote by $\mathfrak{h}^\sigma = \mathfrak{h}_{\mathfrak{e}_6(\mathfrak{f}_4)} = \mathfrak{h}_{\mathfrak{e}_6(\mathfrak{sp}_4)}$ the common maximally compact Cartan subalgebras of $\mathfrak{e}_6(\mathfrak{f}_4)$ and $\mathfrak{e}_6(\mathfrak{sp}_4)$, defined as in (4.21). For $\mathfrak{e}_6(\mathfrak{f}_4)$ both roots α_3 and α_6 are compact, while for $\mathfrak{e}_6(\mathfrak{sp}_4)$, α_3 is non-compact and α_6 is compact, with respect to $\mathfrak{h}^\sigma$.

b) σ-positive systems of the Lie algebra $\mathfrak{e}_6$

We denote by the same symbol σ the antiinvolutions of $\mathfrak{e}_6$ which define the real forms $\mathfrak{e}_6(\mathfrak{f}_4)$ and $\mathfrak{e}_6(\mathfrak{sp}_4)$, like in the previous paragraph. The following Proposition describes all σ-positive systems in R.

Proposition 69. *Any σ-positive system of R is equivalent to one of the* *following σ-positive systems:*

$$R_0^{(1)} = \{\pm(\epsilon_i - \epsilon_j)\}_{1\leq i<j\leq 3} \cup \{\epsilon_i - \epsilon_j\}_{i\leq 3, j\geq 4} \cup \{\epsilon_i + \epsilon_j + \epsilon_k, \epsilon_1 + \cdots + \epsilon_6\};$$

$$R_0^{(2)} = \{\pm(\epsilon_i - \epsilon_j)\}_{1\leq i<j\leq 3} \cup \{\epsilon_i - \epsilon_j\}_{i\leq 3, j\geq 4} \cup \{\epsilon_i + \epsilon_j + \epsilon_k \neq \epsilon_4 + \epsilon_5 + \epsilon_6\}$$
$$\cup \{-(\epsilon_4 + \epsilon_5 + \epsilon_6), \epsilon_1 + \cdots + \epsilon_6\};$$

$$R_0^{(3)} = \{\pm(\epsilon_i - \epsilon_j)\}_{1\leq i<j\leq 3} \cup \{\epsilon_i - \epsilon_j\}_{i\leq 3, j\geq 4} \cup \{-(\epsilon_i + \epsilon_j + \epsilon_k)\}$$
$$\cup \{-(\epsilon_1 + \cdots + \epsilon_6)\};$$

$$R_0^{(4)} = \{\pm(\epsilon_i - \epsilon_j)\}_{1\leq i<j\leq 3} \cup \{\epsilon_i - \epsilon_j\}_{i\leq 3, j\geq 4} \cup \{-(\epsilon_i + \epsilon_j + \epsilon_k) \neq -(\epsilon_1 + \epsilon_2 + \epsilon_3)\}$$
$$\cup \{\epsilon_1 + \epsilon_2 + \epsilon_3, -(\epsilon_1 + \cdots + \epsilon_6)\};$$

$$R_0^{(5)} = \{\pm(\epsilon_i - \epsilon_j)\}_{1\leq i<j\leq 3} \cup \{\epsilon_i - \epsilon_j\}_{i\leq 3, j\geq 4} \cup \{-(\epsilon_i + \epsilon_5 + \epsilon_6)\}_{i\leq 4}$$
$$\cup \{-(\epsilon_i + \epsilon_4 + \epsilon_6)\}_{i\leq 3} \cup \{\epsilon_i + \epsilon_j + \epsilon_6\}_{i,j\leq 3} \cup \{-(\epsilon_i + \epsilon_4 + \epsilon_5)\}_{i\leq 3}$$
$$\cup \{\epsilon_i + \epsilon_j + \epsilon_5\}_{i,j\leq 3} \cup \{\epsilon_i + \epsilon_j + \epsilon_4\}_{i,j\leq 3} \cup \{\epsilon_1 + \epsilon_2 + \epsilon_3, \epsilon_1 + \cdots + \epsilon_6\};$$

$$R_0^{(6)} = \{\pm(\epsilon_i - \epsilon_j)\}_{1\leq i<j\leq 3} \cup \{\epsilon_i - \epsilon_j\}_{i\leq 3, j\geq 4} \cup \{-(\epsilon_i + \epsilon_5 + \epsilon_6)\}_{i\leq 4}$$
$$\cup \{-(\epsilon_i + \epsilon_4 + \epsilon_6)\}_{i\leq 3} \cup \{\epsilon_i + \epsilon_j + \epsilon_6\}_{i,j\leq 3} \cup \{-(\epsilon_i + \epsilon_4 + \epsilon_5)\}_{i\leq 3}$$
$$\cup \{\epsilon_i + \epsilon_j + \epsilon_5\}_{i,j\leq 3} \cup \{\epsilon_i + \epsilon_j + \epsilon_4\}_{i,j\leq 3} \cup \{\epsilon_1 + \epsilon_2 + \epsilon_3, -(\epsilon_1 + \cdots + \epsilon_6)\}.$$

Proof. The proof uses the action of σ on roots and the properties of σ-positive systems (see Definition 57). See [3] for details. □

c) $\mathfrak{e}_6^\sigma$-admissible pairs

We preserve the notations from the previous paragraphs. We denote by

$$\mathfrak{k}^{(k)} := \mathfrak{h}_0^{(k)} + \mathfrak{g}(R_0^{(k)})$$

a regular subalgebra of $\mathfrak{e}_6$, normalized by $\mathfrak{h}^\sigma$, with root system $R_0^{(k)}$ ($1 \leq k \leq 6$) described in Proposition 69 and with Cartan part $\mathfrak{h}_0^{(k)} \subset \mathfrak{h}$, satisfying $\mathfrak{h}_0^{(k)} + \sigma(\mathfrak{h}_0^{(k)}) = \mathfrak{h}$.

Theorem 70. *For any $1 \leq k \leq 6$, the 2-form $\omega_{(k)}$ on $\mathfrak{k}^{(k)}$, defined by*

$$\begin{aligned}
\omega_{(1)} &:= \widehat{\omega}_{(1)} + \lambda_1(\epsilon_4 + \epsilon_5 + \epsilon_6) \wedge \omega_{\epsilon_4+\epsilon_5+\epsilon_6} \\
\omega_{(2)} &:= \widehat{\omega}_{(2)} + \lambda_2(\epsilon_4 + \epsilon_5 + \epsilon_6) \wedge \omega_{-(\epsilon_4+\epsilon_5+\epsilon_6)} \\
\omega_{(3)} &:= \widehat{\omega}_{(3)} + \lambda_3(\epsilon_1 + \epsilon_2 + \epsilon_3) \wedge \omega_{-(\epsilon_1+\epsilon_2+\epsilon_3)} \\
\omega_{(4)} &:= \widehat{\omega}_{(4)} + \lambda_4(\epsilon_1 + \epsilon_2 + \epsilon_3) \wedge \omega_{\epsilon_1+\epsilon_2+\epsilon_3} \\
\omega_{(5)} &:= \widehat{\omega}_{(5)} + \lambda_5(\epsilon_1 + \cdots + \epsilon_6) \wedge \omega_{\epsilon_1+\cdots+\epsilon_6} \\
\omega_{(6)} &:= \widehat{\omega}_{(6)} + \lambda_6(\epsilon_1 + \cdots + \epsilon_6) \wedge \omega_{-(\epsilon_1+\cdots+\epsilon_6)}
\end{aligned}$$

where $\lambda_k \in \mathbb{C}$ and $\widehat{\omega}_{(k)}$ is a 2-form on $\mathfrak{h}_0^{(k)}$ with

$$\widehat{\omega}_{(k)}(e_i - e_j, \cdot) = 0, \quad \forall 1 \leq i, j \leq 3, \tag{4.27}$$

is closed. If, moreover, $\mathrm{Im}\left(\widehat{\omega}_{(k)}\right)$ *is non-degenerate on* $\mathfrak{h}_0^{(k)} \cap \mathfrak{h}^{\sigma}$*, then* $(\mathfrak{k}^{(k)}, \omega_{(k)})$ *is an admissible pair and it defines a regular generalized complex structure on the real Lie group* $(E_6)^{\sigma}$*.*

Proof. For a proof, see [3]. □

Chapter 5

Contributions to Frobenius Manifolds

5.1 Introduction

This chapter is concerned with the geometry of Frobenius manifolds and related structures and summarizes the results published by the author of this thesis in **Advances in Mathematics** [44], **International Mathematics Research Notices** [46], **Journal of Geometry and Physics** [45] and two other preprints [41, 47] sent for publication. Frobenius manifolds were defined by B. Dubrovin, as a geometrical interpretation of the so called WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations. They also appear in many different areas of mathematics (singularity theory [66], quantum cohomology [77] and integrable systems [71]), providing an unexpected link between these different fields. A Frobenius manifold is a manifold M together with a commutative, associative, with unit field multiplication $\circ$ on TM, a multiplication invariant flat metric g and sometimes a vector field E (the Euler field) subject to various compatibility conditions. The associativity property for $\circ$ translates, in flat coordinates for the metric g, to the WDVV-equations for a function F, usually referred as the potential of the Frobenius manifold.

We now present with details the contents of each section separately, explaining also how the original results fit into the general research field.

In **Section 5.2** we recall basic facts we need from the theory of Frobenius manifolds, including their relations with the WDVV-equations, flat pencils of metrics and Saito bundles. We also recall two notions from tt^*-geometry in connection with the theory of Frobenius manifolds, namely harmonic Higgs bundles and $DChk$-structures. We end this section with several well-known constructions of Frobenius manifolds: adding a variable to a Frobenius manifold, orbit spaces of Coxeter groups as Frobenius manifolds, Frobenius manifolds from super-potentials, and the notion of almost Frobenius manifold and Dubrovin's almost duality for almost Frobenius manifolds. More details on the topics presented in this section can be found in various places, e.g. [50, 92, 95, 96].

In **Section 5.3** we develop a generalization of adding a variable to a Frobenius manifold. The general setting is a $\mathbb{K}$-vector bundle (i.e. a real vector bundle if $\mathbb{K} = \mathbb{C}$, or an holomorphic vector bundle if $\mathbb{K} = \mathbb{R}$) $\pi : V \to M$, with base a Frobenius manifold $(M, \circ_M, e_M, g_M)$ and typical fiber a Frobenius algebra $(\circ_V, e_V, g_V)$. In addition we assume that the bundle V comes equipped with a connection D and a morphism $\alpha : V \to TM$ preserving multiplications and unit fields. From this data we construct an almost Frobenius structure on V. (The case when V is the trivial bundle of rank one, D is the trivial connection and $\alpha(e_V) = e_M$ corresponds to adding a variable to the Frobenius manifold $(M, \circ_M, e_M, g_M)$). Our main result is a description, in the real case, of all Frobenius structures on V obtained in this way, with positive definite metric, when $(M, \circ_M, e_M, g_M)$ is semisimple with non-vanishing rotation coefficients (see Theorem 80).

Motivated by Dubrovin's almost duality for almost Frobenius manifolds, in **Section 5.4** we develop a duality for F-manifolds with eventual identities. F-manifolds are closely related to Frobenius manifolds and were defined by Hertling and Manin in [68]. An F-manifold is a manifold M together with a commutative, associative multiplication $\circ$ on TM, with unit field, such that the integrability condition

$$L_{X \circ Y}(\circ) = X \circ L_Y(\circ) + Y \circ L_X(\circ)$$

is satisfied. A Frobenius manifold without metric is an F-manifold. F-manifolds also appear within integrable systems - both in examples coming from the submanifold geometry of Frobenius manifolds [103] and non-local bi-Hamiltonian geometry. In this section we consider an F-manifold $(M, \circ, e)$ with an invertible vector field $\mathcal{E}$ and we define a dual multiplication via

$$X * Y = X \circ Y \circ \mathcal{E}^{-1}. \tag{5.1}$$

While $*$ is commutative and associative with unit field, whether or not this defines an F-manifold is not immediately clear. Using the terminology of [78], we call an invertible vector field $\mathcal{E}$ an eventual identity on $(M, \circ, e)$ if $*$ defined by (5.1) is the multiplication of an F-manifold structure (the reason for the terminology is that $\mathcal{E}$ is the unit for the new multiplication $*$). Our main result from this section is a characterization of eventual identities (thus answering a question raised by Manin [78]) and a duality for F-manifolds with eventual identities (see Theorem 84). We also study the interactions of this duality with several other notions and constructions from the theory of Frobenius manifolds: compatible connections, Riemannian F-manifolds, compatible pairs of metrics, tt^*-geometry (see Sections 5.4.1-5.4.4).

5.2 Preliminary material

We fix our conventions and notations.

Notations 71. Unless otherwise stated, our results hold in the real and holomorphic category; M will denote a $\mathbb{K}$-manifold, i.e. a real manifold when $\mathbb{K} = \mathbb{R}$ or a complex manifold when $\mathbb{K} = \mathbb{C}$. For a $\mathbb{K}$-manifold M, we denote by TM the real tangent bundle if $\mathbb{K} = \mathbb{R}$ and, respectively, the holomorphic tangent bundle if $\mathbb{K} = \mathbb{C}$ and by $\mathcal{X}(M)$ the sheaf of smooth, respectively holomorphic, vector fields on M. For a metric g on a $\mathbb{K}$-manifold M, we denote by g^* the induced metric on the bundle T^*M; by $g(X)$ the 1-form g-dual to a vector X and by $g^*(\alpha)$ the vector g-dual to a covector α. Sometimes, when we work in the holomorphic setting, there is also a real structure involved in the picture, and we need to consider tangent vectors of type $(0,1)$ as well. In such a situation we prefer to denote by $T^{1,0}M$ the holomorphic tangent bundle of a complex manifold M and by $T^{0,1}M$ the bundle of $(0,1)$-tangent vectors. The associated sheaves of smooth sections are denoted by $\mathcal{T}^{1,0}_M$ and $\mathcal{T}^{0,1}_M$.

5.2.1 Frobenius manifolds and WDVV-equations

Rather than the original definition of Dubrovin, we shall use the following alternative definition of Frobenius manifolds (for the equivalence of the two definitions, see Theorem 2.5 of [66]). It includes also the definition of F-manifolds, a key notion in our treatment from the next sections.

Definition 72. *1) An almost Frobenius structure on a manifold M is given by the structure of an almost Frobenius algebra on each tangent space of M, i.e. a commutative, associative multiplication $\circ$ on TM, with unit field e, and a metric $\tilde{g}$ on M, invariant with respect to $\circ$, i.e. such that*

$$\tilde{g}(X \circ Y, Z) = \tilde{g}(X, Y \circ Z) \quad \forall X, Y, Z \in \mathcal{X}(M).$$

2) An almost Frobenius structure $(\circ, e, \tilde{g})$ on M is called Frobenius (and $(M, \circ, e, \tilde{g})$ is a Frobenius manifold) if the following global conditions hold:

i) $(M, \circ, e)$ is an F-manifold, that is,

$$L_{X\circ Y}(\circ) = X \circ L_Y(\circ) + Y \circ L_X(\circ), \quad \forall X, Y \in \mathcal{X}(M). \tag{5.2}$$

ii) the metric $\tilde{g}$ is admissible on the F-manifold $(M, \circ, e)$ (i.e. the unit field e is parallel with respect to the Levi-Civita connection of $\tilde{g}$) and is flat.

We now explain the relation between Frobenius manifolds and WDVV-equations. For this, let $(M, \circ, e, \tilde{g})$ be a Frobenius manifold. It is known that the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$ satisfies the potentiality condition

$$\tilde{\nabla}_X(\circ)(Y, Z) = \tilde{\nabla}_Y(\circ)(X, Z), \quad \forall X, Y, Z \in \mathcal{X}(M) \tag{5.3}$$

and it turns out that (5.3) implies the local existence of a function F (called the potential of the Frobenius manifold and defined up to adding a quadratic polynomial in t^i) such that

$$\tilde{g}\left(\frac{\partial}{\partial t^i}\circ\frac{\partial}{\partial t^j},\frac{\partial}{\partial t^k}\right)=\frac{\partial^3 F}{\partial t^i\partial t^j\partial t^k}$$

in flat orthonormal coordinates $(t^1,\cdots,t^n)$ for the metric $\tilde{g}$. The associativity of $\circ$ translates into the WDVV-equations for F:

$$\sum_{m=1}^{n}\frac{\partial^3 F}{\partial t^i\partial t^j\partial t^m}\cdot\frac{\partial^3 F}{\partial t^m\partial t^k\partial t^l}=\sum_{m=1}^{n}\frac{\partial^3 F}{\partial t^k\partial t^j\partial t^m}\cdot\frac{\partial^3 F}{\partial t^m\partial t^i\partial t^l}$$

for any $1\leq i,j,k,l\leq n$.

Definition 73. *i) An Euler field on an F-manifold $(M,\circ,e)$ is a vector field E such that*

$$L_E(\circ)(X,Y)=X\circ Y,\quad\forall X,Y\in\mathcal{X}(M).$$

ii) An Euler field on a Frobenius manifold $(M,\circ,e,\tilde{g})$ is a vector field E such that

$$L_E(\circ)=\circ,\quad L_E(g)=d\tilde{g}$$

where d is a constant (necessarily equal to 2, when $\tilde{g}(e,e)\neq 0$).

5.2.2 Frobenius manifolds and flat pencils of metrics

Let $(M,\circ,e,\tilde{g},E)$ be a Frobenius manifold with Euler field and assume that E is invertible (i.e. there is E^{-1} such that $E\circ E^{-1}=E^{-1}\circ E=e$). Recall that $\tilde{g}$ is flat. One may define a second metric by $g(X,Y)=\tilde{g}(E^{-1}\circ X,Y)$ which, as it turns out, is flat as well. The metrics g and $\tilde{g}$ determine a pencil of bilinear forms $g^*_\lambda=g^*+\lambda\tilde{g}^*$ on T^*M, for any λ. We assume that all such bilinear forms are non-degenerate, hence they may be considered as metrics on M. One may prove that all such metrics are flat and

$$g^*_\lambda(\nabla^\lambda_X\alpha)=g^*(\nabla_X\alpha)+\lambda\tilde{g}^*(\tilde{\nabla}_X\alpha),\quad\alpha\in\Omega^1(M),\quad X\in TM$$

where $\nabla,\tilde{\nabla},\nabla^\lambda$ are the Levi-Civita connections of g, $\tilde{g}$ and g_λ. We say that the metrics $(g,\tilde{g})$ are compatible flat. Moreover, $(g,\tilde{g},E)$ is quasi-homogeneous, i.e.

$$L_E(\tilde{g})=D\tilde{g},\quad\nabla_X(E)=\frac{1-D}{2}X,\tag{5.4}$$

for a constant D. A key fact of the theory is that the Frobenius multiplication $\circ$ may be recovered from $(g,\tilde{g},E)$. One obtains in this way a bijective correspondence, due to Dubrovin [50], between Frobenius manifolds with (invertible) Euler fields and quasi-homogeneus compatible flat pairs of metrics. Here we only recall a part of this construction, namely how to recover $\circ$ from $(g, \tilde{g}, E)$. This is done as follows.

The pair $(g, \tilde{g})$ determines a multiplication on T^*M by

$$\alpha * \beta = \nabla_{g^*(\alpha)}(\beta) - \tilde{\nabla}_{g^*(\alpha)}(\beta), \quad \alpha, \beta \in T^*M. \tag{5.5}$$

Define an endomorphism T of T^*M by $T(u) := g(E) * u$ and assume that it is an automorphism. Then one may show that

$$\alpha \circ \beta = \alpha * T^{-1}(\beta), \tag{5.6}$$

viewed as a multiplication on TM by identifying TM with T^*M using $\tilde{g}$, is precisely the Frobenius multiplication $\circ$.

Dubrovin's correspondence is particularly useful when describing certain classes of Frobenius manifolds (i.e. orbit spaces of Coxeter groups as Frobenius manifolds), for which the two flat metrics and the Euler field have simple expressions. An extension of the Dubrovin's correspondence to the non-flat case will be developed in Section 5.4.3.

5.2.3 Frobenius manifolds and Saito bundles

Let $\pi : V \to M$ be a vector bundle with $\mathrm{rank}(V) = \dim(M)$, endowed with a connection ∇, a metric g and a vector valued 1-form $\phi \in \Omega^1(M, \mathrm{End}V)$, satisfying the conditions:

$$R^{\nabla} = 0, \quad d^{\nabla}\phi = 0, \quad \nabla g = 0, \quad \phi \wedge \phi = 0, \quad \phi^* = \phi, \tag{5.7}$$

where

$$(\phi \wedge \phi)_{X,Y} := \phi_X \phi_Y - \phi_Y \phi_X, \quad X, Y \in TM$$

and, for any $X \in TM$, $\phi_X^* \in \mathrm{End}(TM)$ is the adjoint of $\phi_X \in \mathrm{End}(TM)$ with respect to g. Assume, moreover, that there is a vector field e on M such that $\phi_e = -\mathrm{Id}_V$. Let $\omega \in \Gamma(V)$ (usually called a primitive section) be ∇-parallel such that the map

$$\psi^\omega : TM \to V, \quad \psi^\omega(X) = -\phi_X(\omega) \tag{5.8}$$

is an isomorphism. Define a multiplication $\circ$ on TM, with unit field e, by

$$X \circ Y = (\psi^\omega)^{-1}(\phi_X \phi_Y \omega).$$

Using the map ψ^ω, we may transport the metric g and the connection ∇ to a metric g^ω and a connection ∇^ω on TM. It is easy to see that $\circ$ is independent of the choice of primitive section, while for g^ω and ∇^ω the choice of ω is essential. The following theorem was proved by K. Saito [95].

Theorem 74. *The multiplication $\circ$ together with the metric g^ω make M a Frobenius manifold, with unit e and Levi-Civita connection ∇^ω.*

Remark 75. In examples coming from singularity theory, the bundle V as above comes equipped with two additional endomorphisms, R_0 and R_∞, satisfying the conditions:

$$\begin{aligned} &\nabla R_0 + \phi = [\phi, R_\infty], \quad [R_0, \phi] = 0, \quad R_0^* = R_0; \\ &\nabla R_\infty = 0, \quad R_\infty^* + R_\infty = -w\mathrm{Id}_V, \end{aligned}$$

where w is constant. If ω is primitive and homogeneous (i.e. $R_\infty(\omega) = -q\omega$ for a constant q), then

$$E^\omega := (\psi^\omega)^{-1}(R_0(\omega))$$

is Euler for $(M, \circ, e, g^\omega)$, with $L_{E^\omega}(g^\omega) = (2(1+q) - w)\, g^\omega$, and

$$R_\infty^\omega := (\psi^\omega)^{-1} \circ R_\infty \circ \psi^\omega = \nabla^\omega(E^\omega) - (1+q)\mathrm{Id}.$$

Thus $(M, \circ, e, g^\omega, E^\omega)$ is Frobenius with Euler field.

Definition 76. *The data $(\nabla, \phi, g, R_0, R_\infty)$ like in Remark 75 is called a Saito structure (of weight w) on V (and $(V, \nabla, \phi, g, R_0, R_\infty)$ is called a Saito bundle).*

Above we associated to a Saito bundle with primitive homogeneous section a Frobenius manifold with Euler field. Conversely, one may show that if $(M, \circ, e, \tilde{g}, E)$ is a Frobenius manifold with Euler field and $L_E(\tilde{g}) = d\tilde{g}$, then

$$\left(\tilde{\nabla}, \quad \tilde{g}, \quad \phi, \quad R_0 := -\phi_E, \quad R_\infty := \tilde{\nabla}(E) - \frac{1}{2}\,(w+d)\,\mathrm{Id}_{TM}\right)$$

is a Saito structure of weight w on TM, where $\tilde{\nabla}$ is the Levi-Civita connection of g and $\phi_X(Y) = -X \circ Y$ the Higgs field.

5.2.4 Semisimple Frobenius manifolds

A semisimple F-manifold is an F-manifold for which there is a coordinate system $(u^1, \cdots, u^n)$ (called canonical) such that the multiplication and unit field are given by

$$\frac{\partial}{\partial u^i} \circ \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^i}, \quad e = \sum_{i=1}^n \frac{\partial}{\partial u^i}.$$

It turns out that invariant metrics $\tilde{g}$ which make $(M, \circ, e, \tilde{g})$ Frobenius can be locally written in terms of a single function η, called the metric potential, as follows:

$$\tilde{g} = \sum_{i=1}^n \frac{\partial \eta}{\partial u^i} du^i \otimes du^i,$$

where η satisfies the Darboux-Egoroff equations

$$\frac{\partial \gamma_{ij}}{\partial u^k} = \gamma_{ij}\gamma_{kj}, \quad e(\gamma_{ij}) = 0$$

for any $i \neq j \neq k \neq i$, where $\gamma_{ij} := \frac{1}{2}\frac{\eta_{ij}}{\sqrt{\eta_i \eta_j}}$ are the rotation coefficients (for simplicity, here we denoted by η_i the partial derivative $\frac{\partial \eta}{\partial u^i}$ and by η_{ij} the second derivative $\frac{\partial^2 \eta}{\partial u^i \partial u^j}$). For more details about semisimple Frobenius manifolds, see e.g. [77].

5.2.5 Constructions of Frobenius structures

Adding a variable to a Frobenius manifold

One way to construct a Frobenius manifold from an old one is provided by a construction called adding a variable to a Frobenius manifold, described as follows [92].

Theorem 77. *Let $(M, \circ_M, e_M, g_M)$ be a Frobenius $\mathbb{K}$-manifold. Then $M \times \mathbb{K}$ is a Frobenius manifold, with multiplication*

$$X \circ Y = X \circ_M Y, \quad X \circ \frac{\partial}{\partial \tau} = X, \quad \frac{\partial}{\partial \tau} \circ \frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau}$$

(τ being the coordinate on $\mathbb{K}$) and metric

$$g(X,Y) = g_M(X,Y), \quad g\left(X, \frac{\partial}{\partial \tau}\right) = g_M(e_M, X), \quad g\left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial \tau}\right) = g_M(e,e)+1, \tag{5.9}$$

for any $X, Y \in TM$. Moreover, if E is an Euler field on $(M, \circ_M, e_M, g_M)$ with $L_E(g_M) = 2g_M$, then $E + R$, where R is the radial field $R_{(p,\tau)} = \tau\frac{\partial}{\partial \tau}$ is Euler on $M \times \mathbb{K}$.

The above theorem follows by direct computations. For the statement involving the Euler field one may consider the Saito structure of weight w

$$\left(\nabla^M, \quad g_M, \quad \phi^M, \quad R_0 := -\phi_E, \quad R_\infty := \nabla^M(E) - \frac{1}{2}(w+2)\,\mathrm{Id}_{TM}\right) \tag{5.10}$$

associated to the Frobenius manifold $(M, \circ_M, e_M, g_M, E)$ (where ∇^M is the Levi-Civita connection and $\phi^M_X Y = -X \circ_M Y$ the associated Higgs field) and notice that it induces a Saito structure

$$\left(\pi^*(\nabla^L), \quad \pi^*(g^L), \quad \phi', \quad R'_\infty, \quad R'_0\right) \tag{5.11}$$

on the bundle $\pi^*(TM \oplus L)$. Here

$$\pi : L = M \times \mathbb{K} \to M, \quad (p, \tau) \to p$$

is the trivial rank one bundle, $\nabla^L := \nabla^M \oplus D$ (with D the trivial connection on π), $g^L := g_M \oplus \tilde{g}$ (with $\tilde{g}$ a D-parallel metric on L),

$$\phi' := \pi^*(\phi^M) - d\tau \otimes \mathrm{Id}_{\pi^*(TM\oplus L)} \tag{5.12}$$

and

$$R'_\infty := \pi^*\left(R_\infty - \frac{w}{2}\mathrm{Id}_V\right) \quad R'_0 := \pi^*(R_0) + \tau \mathrm{Id}_{\pi^*(TM\oplus L)}. \tag{5.13}$$

In (5.12) and (5.13) ϕ^M_X (for any $X \in TM$), R_0 and R_∞ are considered as endomorphisms of $TM \oplus L$, acting trivially on L. Let v be the D-parallel section of L such that $\tilde{g}(v,v) = 1$. Then $\pi^*(e_M + v)$ is a primitive homogeneous section of the Saito structure (5.11) and the associated Frobenius manifold is the one from the statement of the theorem.

In Section 5.3 we develop a generalization of this construction.

Frobenius manifolds from super-potentials

An important class of Frobenius manifolds is obtained from unfoldings of singularities. Here we describe the simplest example. Start with

$$W(z,\vec{s}) = z^{n+1} + \sum_{i=0}^{n-1} s_i z^i,$$

called the super-potential in our setting. On the parameter space $M := \{\vec{s} = (s_0, \cdots, s_{n-1})\}$ define a $(2,0)$-tensor field

$$\tilde{g}(X,Y) := \sum_{W_z=0} \mathrm{res}_z \frac{X(W)Y(W)}{W_z} dz,$$

where W_z is the partial derivative $\frac{\partial W}{\partial z}$, and a $(3,0)$-tensor field

$$\tilde{c}(X,Y,Z) = \sum_{W_z=0} \mathrm{res}_z \frac{X(W)Y(W)Z(W)}{W_z} dz.$$

One may show that $\tilde{g}$ is flat. Moreover, $\tilde{g}$ and $\tilde{c}$ define a multiplication $\circ$ by $\tilde{c}(X,Y,Z) = \tilde{g}(X \circ Y, Z)$, with unit $\frac{\partial}{\partial s_0}$ which, together with $\tilde{g}$, define a Frobenius structure on M. By means of the isomorphism

$$T_{\vec{s}}M \cong \mathbb{C}[z]/(W_z), \quad X \to [X(W)]$$

(where (W_z) is the ideal in the space of polynomials $\mathbb{C}[z]$ generated by $W_z(\cdot,\vec{s})$) the Euler field $E_{\vec{s}}$ at $\vec{s}$ corresponds to $[W(\cdot,\vec{s})]$

Orbit spaces of Coxeter groups

The Saito construction provides the space of orbits of a Coxeter group with a Frobenius manifold structure. Here we explain the main features of this construction. Details can be found in [94] (see also [48]). Recall that a Coxeter group of a real n-dimensional vector space $V = \mathbb{R}^n$ is a finite group W of linear transformations of V generated by reflexions. Let $\{t^i\}$ be a basis of W-invariant polynomials on V with degrees $\deg(t^i) = d_i$, ordered so that

$$h = d_1 > d_2 \geq \cdots \geq d_{n-1} > d_n = 2$$

where h is the Coxeter number of the group. The action of W extends to the complexified space $V \otimes \mathbb{C} = \mathbb{C}^n$. In the Saito construction of interest is the orbit space

$$M = \mathbb{C}^n / W.$$

Starting with the standard flat metric

$$g := \sum_{i=1}^{n} (dx^i)^2 \tag{5.14}$$

on $\mathbb{C}^n$, which is W-invariant, one obtains a flat metric g on the orbit space $M \setminus \mathrm{Discr}(W)$, where $\mathrm{Discr}(W)$ is the discriminant locus of irregular orbits. It turns out that there is another metric

$$\tilde{g}^* := L_e(g^*)$$

defined on M, which is also flat. Here e is the vector field which, in terms of the basis $\{t^i\}$ of invariant polynomials, is $\frac{\partial}{\partial t^1}$. The basis $\{t^i\}$ of invariant polynomials can be chosen such that $\tilde{g}$ is anti-diagonal with constant entries

$$\tilde{g}_{ij} = \delta_{i+j,n+1}.$$

Saito proved that $(\tilde{g}, g)$ together with

$$E = d_1 t^1 \frac{\partial}{\partial t^1} + \cdots + d_n t^n \frac{\partial}{\partial t^n}$$

make M a Frobenius manifold. The multiplication is obtained as in Section 5.2.2 (see relation (5.6).

Almost Frobenius manifolds and Dubrovin's almost duality

Starting from a Frobenius manifold with invertible Euler field one may construct a new geometric object (called an almost Frobenius manifold) that shares many, but crucially not all, of the essential features of the original manifold. In particular a new 'dual' solution of the underlying Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations may be constructed from the original manifold. More specifically, given a Frobenius manifold $(M\,,\circ\,,e\,,E\,,\tilde{g})$ with multiplication $\circ$, unit field e, invertible Euler field E and metric $\tilde{g}$ one may define a new multiplication $\star$ and metric g by the formulae

$$\begin{aligned} X\star Y &= X\circ Y\circ E^{-1}\,,\\ g(X,Y) &= \tilde{g}(E^{-1}\circ X, Y) \end{aligned}$$

where $E^{-1}\circ E = e$. It turns out that $(M,\star,E)$ is an F-manifold (with unit field E, the initial Euler field). The metric g is invariant with respect to $\star$ and the coidentity $g(E)$ is closed. The Levi-Civita connection of g and the multiplication $\star$ satisfy relation (5.3), from where one obtains a new solution of the WDVV-equations. However, not all properties of Frobenius manifolds are shared by the new structure: e.g. while the identity e is flat on $(M,\tilde{g})$, E is not flat on (M,g), in general.

Remark 78. Sometimes the expression of the dual solution of the WDVV equations associated to a Frobenius manifold is simpler than the initial solution. E.g. when $M=\mathbb{C}^n/W$ is the orbit space of a Coxeter group, the dual solution has the simple form

$$F^{*}=\frac{1}{4}\sum_{\alpha\in R_W}(\alpha\cdot z)^2\log(\alpha\cdot z)^2$$

where R_W is the root system of the group.

In Section 5.4 we replace the Euler field of the Frobenius manifold by an arbitrary vector field and we develop a duality in the larger setting of F-manifolds.

5.2.6 Basic notions from tt^*-geometry

In this section we recall various basic notions from tt^*-geometry (see Definition 79). We follow closely [49, 67, 102]. We work in the holomorphic setting. An holomorphic Higgs field on an holomorphic vector bundle $V\to M$ will be trivially extended to $T^{0,1}M$.

Definition 79. *1) Let $\pi: V\to M$ be an holomorphic vector bundle. A pair (g,h) formed by a complex bilinear, non-degenerate symmetric form g and a (pseudo)-Hermitian metric h on V is called compatible if the Chern connection D of the holomorphic Hermitian bundle (V,h) preserves g, i.e. $Dg=0$.*

2) A Hermitian metric h on a Higgs bundle (V,ϕ) is called harmonic (and (V,ϕ,h) is a harmonic Higgs bundle) if the tt^*-equations

$$(\partial^D\phi)_{X,Y}:=D_X(\phi_Y)-D_Y(\phi_X)-\phi_{[X,Y]}=0 \qquad (5.15)$$

and

$$R^D_{X,\bar{Y}} + [\phi_X, \phi^\flat_{\bar{Y}}] = 0 \tag{5.16}$$

are satisfied, for any $X, Y \in \mathcal{T}^{1,0}_M$. *Above* D *denotes the Chern connection of the holomorphic Hermitian vector bundle* (V, h) *and* $\phi^\flat$ *is the* h*-adjoint of* C*, i.e.*

$$h(\phi_X Y, Z) = h(Y, \phi^\flat_{\bar{X}} Z), \quad \forall Y, Z \in T^{1,0}M, \quad \forall X \in T_{\mathbb{C}}M.$$

3) Let (V, ϕ, h) *be a harmonic Higgs bundle and* k *a real structure on* V *(i.e. a fiber-preserving anti-linear involution) such that the complex bilinear form*

$$g(X, Y) := h(X, kY)$$

on V *is symmetric and invariant. The data* (V, ϕ, h, k) *is called a DChk-structure if the pair* (g, h) *is compatible.*

We remark that a harmonic Higgs bundle (V, ϕ, h) has an associated pencil of flat connections

$$D^z := D + \frac{1}{z}\phi + z\phi^\flat. \tag{5.17}$$

The flatness property of this pencil encodes the entire geometry of the harmonic Higgs bundle [67].

5.3 A generalization of adding a variable to a Frobenius manifold

Motivated by the construction of adding a variable to a Frobenius manifold, we now consider a vector bundle $\pi : V \to M$ whose base is a Frobenius manifold $(M, \circ_M, e_M, g_M)$, typical fiber a Frobenius algebra $(\circ_V, e_V, g_V)$, and which comes equipped with two additional data: a connection D on the bundle V and a bundle morphism

$$\alpha : V \to TM,$$

with the following two properties:

$$\alpha(v_1 \circ_V v_2) = \alpha(v_1) \circ_M \alpha(v_2), \quad \forall v_1, v_2 \in V, \quad \alpha(e_V) = e_M. \tag{5.18}$$

The connection D induces a decomposition

$$T_v V = T_p M \oplus V_p, \quad \forall v \in V_p = \pi^{-1}(p) \tag{5.19}$$

into horizontal and vertical subspaces. The horizontal lift of a vector field $X \in \mathcal{X}(M)$ will be denoted $\bar{X}$. Often sections of V will be considered (without mentioning explicitly) as vertical vector fields on the manifold V. From the data $(D, \circ_M, \circ_V, g_M, g_V, \alpha)$ we construct an almost Frobenius structure on V, with metric

$$g(\bar{X}, \bar{Y}) := g_M(X, Y), \quad g(v_1, v_2) := g_V(v_1, v_2), \quad g(v, \bar{X}) := g_M(\alpha(v), X), \tag{5.20}$$

and multiplication

$$\overline{X} \circ \overline{Y} := \overline{X \circ_M Y}, \quad v_1 \circ v_2 := v_1 \circ_V v_2, \quad v \circ \overline{X} := \overline{X} \circ v = \overline{\alpha(v) \circ_M X}, \tag{5.21}$$

for any $X, Y \in T_pM$, $v, v_1, v_2 \in V_p$ and $p \in M$. The multiplication $\circ$ is associative, commutative, with unit e_V, and g is invariant with respect to $\circ$. We also assume that g is non-degenerate. One may check that this is equivalent to the non-degeneracy of the metric $g_V - \alpha^*(g_M)$ of the bundle V (easy check).

In [41] we found the conditions on the initial data $(D, \circ_M, \circ_V, g_M, g_V, \alpha)$ which insure that $(V, \circ, e_V, g)$ is a Frobenius manifold. It turns out that the flatness of g translates into a complicated system of conditions on the map α (see Proposition 11 of [41]). However, when $(M, \circ_M, e_M, g_M)$ is semisimple, this system of conditions simplifies considerably and allows, in the real positive definite case, a complete description of all Frobenius structures on V obtained by our method, as follows. (Recall, from Section 5.2.4, our conventions on semisimple Frobenius manifolds).

Theorem 80. *Let $(M, \circ_M, e_M, g_M)$ be a real semisimple Frobenius manifold with metric*

$$g_M = \sum_{k=1}^{n} \eta_k du^k \otimes du^k \tag{5.22}$$

and non-vanishing rotation coefficients $\gamma_{ij} = \frac{\eta_{ij}}{\sqrt{\eta_i \eta_j}}$. Let $V \to M$ be a real vector bundle with a structure of Frobenius algebra $(\circ_V, e_V, g_V)$ along the fibers. Let D be a connection on V and $\alpha : V \to TM$ a morphism such that

$$\alpha(e_V) = e_M, \quad \alpha(v_1 \circ_V v_2) = \alpha(v_1) \circ_M \alpha(v_2), \quad \forall v_1, v_2 \in V. \tag{5.23}$$

Then the almost Frobenius structure $(\circ, e_V, g)$ on V defined by this data is Frobenius with positive definite metric if and only if the following facts hold:

1) the connection D is flat and the Frobenius algebra $(\circ_V, e_V, g_V)$ is D-parallel;

2) the endomorphism α is given by

$$\alpha = \lambda \otimes e_M \tag{5.24}$$

where $\lambda \in \Gamma(V^*)$ is D-parallel and satisfies

$$\lambda(e_V) = 1, \quad \lambda(s_1 \circ_V s_2) = \lambda(s_1)\lambda(s_2), \quad \forall s_1, s_2 \in \Gamma(V). \tag{5.25}$$

3) $\eta_k > 0$ *for any* $1 \leq k \leq n$ *and* $g_V - g_M(e_M, e_M)\lambda\otimes\lambda$ *is positive definite.*

Remark 81. The almost Frobenius structure $(V, \circ, e_V, g)$ from Theorem 80 is Frobenius also in the complex and real non-positive definite cases. The main point of the above theorem is the converse of this statement, namely that all Frobenius structures in the *real positive definite case*, obtained from the initial data $(\circ_M, e_M, g_M, \circ_V, e_V, D, \alpha)$, are described by the theorem. The positive definite condition on g (which is equivalent to the condition *3)* above) is essential in the proof of the above theorem. One first shows, with no positive-definite assumptions, that for any D-parallel section s,

$$\alpha(s) = \sum_{k=1}^{n} a_k \frac{\partial}{\partial u^k}$$

for some constants a_k. Then one shows that the flatness of g implies

$$\sum_{j=1}^{n} \frac{\eta_{pj}^2}{\eta_j}(a_j - a_p)^2 + \sum_{i=1}^{r} \epsilon_i \left(\sum_{s=1}^{n} (a_s - a_p)\eta_{ps} du^s(\alpha(v_i)) \right)^2 = 0, \qquad (5.26)$$

where $\{v_i\}$ is an local frame of V, orthonormal with respect to $g_V - g_M(e_M, e_M)\lambda\otimes\lambda$ and $\epsilon_i \in \{\pm 1\}$ is the square norm of v_i with respect to this metric. The positive definite assumption ($\epsilon_i > 0$ and $\eta_j > 0$ for any i, j) together with relation (5.26) imply that a_k are all equal and hence $\alpha(s) = \lambda(s)e_M$, where $\lambda(s)$ is constant. It follows that α is of the required form. It would be interesting to study if the same conclusion holds also in the non-definite real or complex cases.

Remark 82. i) When $\pi : V = M \times \mathbb{R} \to M$ is the trivial bundle, $e_V = \frac{\partial}{\partial \tau}$, $\lambda = d\tau$ and g_V is defined by the condition $g_V(e_V, e_V) = 1 + g_M(e_M, e_M)$, the above theorem reduces to the construction of adding a variable to the Frobenius manifold $(M, \circ_M, e_M, g_M)$.

ii) Like in the construction of adding a variable to a Frobenius manifold, one may show that if E is Euler on $(M, \circ_M, e_M, g_M)$, such that $L_E(g_M) = 2g_M$, then $E + R$, where R is the radial field, is Euler for the Frobenius structure on V provided by Theorem 80. In this setting one may recover this Frobenius structure from a Saito structure on $\pi^*(TM \oplus V)$, with primitive homogeneous section $\pi^*(e_M + e_V)$. To keep the text reasonably short, we do not describe this Saito structure. Details can be found in [41].

5.4 A duality for F-manifolds with eventual identities

We begin with the definition of eventual identities on F-manifolds [78]. Remark that the multiplication (5.27) below is defined like the dual multiplication on a Frobenius manifold (see Section 5.2.5), with Euler field replaced by an arbitrary vector field.

Definition 83. *A vector field $\mathcal{E}$ on an F-manifold $(M, \circ, e)$ is called an eventual identity if it is invertible and the multiplication*

$$X * Y = X \circ Y \circ \mathcal{E}^{-1}, \quad \forall X, Y \in \mathcal{X}(M) \tag{5.27}$$

defines a new F-manifold structure on M.

The following theorem may be seen as an extension of the Dubrovin's almost duality for almost Frobenius manifolds to the larger setting of F-manifolds. It turns out that at the level of F-manifolds, unlike Frobenius manifolds, there is a perfect symmetry, as follows.

Theorem 84. *i) Let $(M, \circ, e)$ be an F-manifold and $\mathcal{E}$ an invertible vector field. Then $\mathcal{E}$ is an eventual identity if and only if*

$$L_{\mathcal{E}}(\circ)(X, Y) = [e, \mathcal{E}] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M). \tag{5.28}$$

ii) Suppose that (5.28) holds. Then e is an eventual identity on the F-manifold $(M, *, \mathcal{E})$, where $*$ is related to $\circ$ by (5.27), and the map

$$(M, \circ, e, \mathcal{E}) \to (M, *, e, \mathcal{E})$$

is an involution on the set of F-manifolds with eventual identities.

Having found the characterization of eventual identities one may study how such objects may be combined to form new eventual identities.

Remark 85. i) Eventual identities form a subgroup of the group of invertible vector fields on an F-manifold.

ii) For any eventual identity $\mathcal{E}$,

$$[\mathcal{E}^n, \mathcal{E}^m] = (m - n)\mathcal{E}^{n+m-1} \circ [e, \mathcal{E}], \quad \forall n, m \in \mathbb{Z}. \tag{5.29}$$

iii) The Lie bracket of two eventual identities is an eventual identity, provided that it is invertible.

iv) On a semisimple F-manifold with canonical coordinates $(u^1, \cdots, u^n)$, a vector field is an eventual identity if and only if it is of the form

$$\mathcal{E} = \sum_{k=1}^{n} f^k(u^k) \frac{\partial}{\partial u^k}$$

where f^k are arbitrary non-vanishing functions.

v) Any eventual identity on a product F-manifold decomposes into a sum of eventual identities on the factors and this decomposition implies a comutativity property between our duality for F-manifolds with eventual identities and Hertling's the decomposition of F-manifolds [66].

In the following sections we add various structures on F-manifolds and we study their behaviour under the duality for F-manifolds with eventual identities, provided by Theorem 84.

5.4.1 Compatible connections

Definition 86. *A connection ∇ on an F-manifold $(M, \circ, e)$ is called compatible, if*

$$\nabla_X(\circ)(Y, Z) = \nabla_Y(\circ)(X, Z), \quad \forall X, Y, Z \in \mathcal{X}(M).$$

Remark 87. i) The Levi-Civita connection of a Frobenius manifold is a compatible connection for the underlying F-manifold.

ii) If $\tilde{\nabla}$ is a compatible connection on an F-manifold $(M, \circ, e)$ (not necessarily Frobenius) and Z is an arbitrary vector field, then

$$\nabla_X Y := \tilde{\nabla}_X Y + Z \circ X \circ Y$$

is also a compatible connection on $(M, \circ, e)$.

The motivation for the following proposition comes from the structure connections of a Frobenius manifold with Euler field. A Frobenius manifold $(M, \circ, e, \tilde{g}, E)$ with invertible Euler field has a canonical connection compatible with $\circ$, namely the Levi-Civita connection of $\tilde{g}$, sometimes called the first structure connection. Its dual $(M, *, E)$, where $X * Y = X \circ Y \circ E^{-1}$ also has a canonical compatible connection, namely the Levi-Civita connection of the second metric $g(X, Y) = \tilde{g}(X \circ E^{-1}, Y)$ (sometimes called the second structure connection of the Frobenius manifold). Since (invertible) Euler fields of Frobenius manifolds are eventual identities for the underlying F-manifold structure, it is natural to ask if the dual $(M*, \mathcal{E})$ of an F-manifold $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$ with an eventual identity and compatible torsion-free connection inherits, in a canonical way, a compatible, torsion-free connection. This problem has been treated in [47]. The answer is that, without additional assumptions, there is a canonical family, rather than a single connection, inherited on the dual $(M*, \mathcal{E})$. Under further various conditions (on the covariant derivative of the unit fields) one can fix a connection from this family and one obtains a duality at the level of F-manifolds with compatible, torsion-free connections satisfying that additional condition. An example of such a duality is provided by the following proposition (in this case the additional condition is the flatness of the unit fields) [47].

Proposition 88. *The map*

$$(M, \circ, e, \mathcal{E}, \tilde{\nabla}) \to (M, *, \mathcal{E}, e, \nabla)$$

where $*$ is related to $\circ$ by (5.27) and ∇ is related to $\tilde{\nabla}$ by

$$\nabla_X(Y) = \mathcal{E} \circ \tilde{\nabla}_X(\mathcal{E}^{-1} \circ Y) - \tilde{\nabla}_{\mathcal{E}^{-1} \circ Y}(\mathcal{E}) \circ X + \frac{1}{2}[\mathcal{E}^{-1}, \mathcal{E}] \circ X \circ Y$$

is an involution on the set of F-manifolds with eventual identities and compatible, torsion-free connections preserving the unit fields.

5.4.2 Riemannian F-manifolds

Riemannian F-manifolds were defined in [76] and used to interpret arguments from the theory of integrable systems in a coordinate free way. In this section we show that our duality for F-manifolds with eventual identities preserves the Riemannian F-manifold condition.

Definition 89. *A Riemannian F-manifold is an F-manifold $(M, \circ, e)$ together with an invariant metric $\tilde{g}$, such that the following two conditions are satisfied:*

i) the coidentity $\tilde{g}(e) \in \Omega^1(M)$, which is the 1-form $\tilde{g}$-dual to unit field e, is closed;

ii) the curvature of $\tilde{g}$ satisfies the relation:

$$X \circ R^{\tilde{g}}_{Y,Z}(V) + Z \circ R^{\tilde{g}}_{X,Y}(V) + Y \circ R^{\tilde{g}}_{Z,X}(V) = 0, \quad \forall X, Y, Z, V \in \mathcal{X}(M). \tag{5.30}$$

Remark 90. Any Frobenius manifold is a Riemannian F-manifold. More generally, condition (5.30) holds when $\tilde{g}$ has constant sectional curvature.

Our main result from this section is the following.

Proposition 91. *Let $(M, \circ, e, \mathcal{E}, \tilde{g})$ be an F-manifold with an eventual identity $\mathcal{E}$ and invariant metric $\tilde{g}$. Define*

$$g(X,Y) := \tilde{g}(X, \mathcal{E}^{-1} \circ Y), \quad \forall X, Y \in \mathcal{X}(M). \tag{5.31}$$

Then $(M, \circ, e, \tilde{g})$ is a Riemannian F-manifold if and only if $(M, *, \mathcal{E}, g)$ is a Riemannian F-manifold.

Remark 92. Proposition 91 is a useful tool to construct non-flat invariant metrics on F-manifolds, which satisfy condition (5.30). Indeed, if in the setting of Proposition 91 one takes $\tilde{g}$ to be flat (hence condition (5.30) is trivially satisfied) then g defined by (5.31) is not flat in general but relation (5.30) holds with multiplication $*$ and metric g.

5.4.3 Compatible pairs of metrics

Compatible pairs of metrics can be used to construct bi-Hamiltonian structures of non-local type [52]. Here we study the geometry of such pairs of metrics. We first develop a bijective correspondence betwen quasi-homogeneus compatible pairs of metrics and $\mathcal{F}$-manifolds (which are F-manifolds with metrics and Euler fields satisfying some additional conditions). This correspondence extends to the non-flat case the Dubrovin's correspondence [50] between quasi-homogeneous compatible flat pairs of metrics and Frobenius manifolds with Euler fields (see Section 5.2.2). Briefly, we show that the main features of Dubrovin's correspondence [50] lie in the compatibility, rather than the flatness property of the metrics. We also show how eventual identities may be used to construct new pairs of compatible metrics. More details on the results from this section may be found in [44, 45, 46].

Let $(g, \tilde{g})$ be two metrics on a manifold M. Assume that for any λ, the bilinear form $g^*_\lambda := g^* + \lambda \tilde{g}^*$ on T^*M is non-degenerate. We denote by $\nabla, \tilde{\nabla}, \nabla^\lambda$ and $R^g, R^{\tilde{g}}, R^\lambda$ the Levi-Civita connections and the curvatures of $g, \tilde{g}, g_\lambda$. The following definition extends to the non-flat case the main properties of the compatible flat pair of metrics on a Frobenius manifold.

Definition 93. *The metrics $(g, \tilde{g})$ are called almost compatible if*

$$g^*_\lambda(\nabla^\lambda_X \alpha, \beta) = g^*(\nabla_X \alpha, \beta) + \lambda \tilde{g}^*(\tilde{\nabla}_X \alpha, \beta), \quad \forall \alpha, \beta \in \Omega^1(M), \quad \forall X \in \mathcal{X}(M), \quad \forall \lambda.$$

If, moreover,

$$g^*_\lambda(R^\lambda_{X,Y} \alpha, \beta) = g^*(R^g_{X,Y} \alpha, \beta) + \lambda \tilde{g}^*(R^{\tilde{g}}_{X,Y} \alpha, \beta), \quad \forall \alpha, \beta \in \Omega^1(M), \quad \forall X, Y \in \mathcal{X}(M),$$

for any constant λ, then $(g, \tilde{g})$ are called compatible.

Example 94. One method to produce non-flat compatible pairs of metrics are provided by conformal rescalings. It may be shown that if $(g, \tilde{g})$ is compatible then also $(fg, f\tilde{g})$ is compatible, for any non-vanishing function f. Compatible pairs of metrics, conformally related to flat pairs, appear naturally on the space of orbits of a Coxeter group and were constructed in [45] by means of a modified Saito construction, which starts with a metric of constant (non-zero) sectional curvature on the ambient vector space, rather that the standard flat metric (i.e. the metric (5.14)), as in the usual Saito construction (see Section 5.2.5).

We now explain the main features of the extended Dubrovin's correspondence to the non-flat case. For details, see [46]. Assume that $(g, \tilde{g})$ is a pair of metrics on a manifold M and $E \in \mathcal{X}(M)$ a vector field. We define, formally in the same way as in the Dubrovin's correspondence (see Section 5.2.5), two multiplications $*$ and $\circ$ on T^*M (the latter will be considered also on TM, by identifying TM with T^*M using $\tilde{g}$) by the relations (5.5) and (5.6) respectively (the operator T is given by $T(u) := g(E) * u$ and is assumed to be an automorphism of T^*M, as in the Frobenius case). It turns out that if $(g, \tilde{g})$ are compatible, then $\circ$ is associative, commutative, with unit field $\tilde{g}^* g(E)$. Moreover, if the quasi-homogeneity conditions (5.4) hold, for a vector field E, then $(M, \circ, e, \tilde{g}, E)$ is a so called $\mathcal{F}$-manifold, i.e. the following conditions hold:

1) $(M, \circ, e)$ is an F-manifold;

2) $\tilde{g}$ and $\circ$ define a Frobenius algebra at each point;

3) E rescales $\circ$ and $\tilde{g}$ by constants;

4) the $(3,1)$-tensor field $\tilde{\nabla}(\circ)$ satisfies the symmetry

$$\tilde{\nabla}_E(\circ)(X,Y) = \tilde{\nabla}_X(\circ)(E,Y), \quad \forall X, Y \in TM.$$

Conversely, it may be shown that if $(M, \circ, e, \tilde{g}, E)$ is an $\mathcal{F}$-manifold then the metric g defined by $g^*\tilde{g} = E\circ$ is compatible with $\tilde{g}$ and $(g, \tilde{g}, E)$ is a quasi-homogeneous compatible pair of metrics. To summarize, we obtain the following theorem which extends the Dubrovin's correspondence to the non-flat case [46].

Theorem 95. *There is a bijective correspondence between quasi-homogeneous compatible pairs of metrics and $\mathcal{F}$-manifolds.*

We end this section by showing how eventual identities may be used to construct compatible pairs of metrics.

Proposition 96. *Let $(M, \circ, e, \mathcal{E}, \tilde{g})$ be an F-manifold with an eventual identity $\mathcal{E}$ and invariant metric $\tilde{g}$. Define*

$$g(X,Y) := \tilde{g}(X \circ \mathcal{E}^{-1}, Y), \quad \forall X, Y \in \mathcal{X}(M).$$

Then $(g, \tilde{g})$ are almost compatible. If, moreover, the coidentity $\tilde{g}(e)$ is closed, then $(g, \tilde{g})$ are compatible.

5.4.4 Duality and tt^*-geometry

In this section we work in the holomorphic setting. We fix an F-manifold $(M, \circ, e, \mathcal{E}, \tilde{h}, \tilde{k})$ together with an eventual identity $\mathcal{E}$, (pseudo)-Hermitian metric $\tilde{h}$, and real structure $\tilde{k}$ on $T^{1,0}M$ such that the complex bilinear form

$$\tilde{g}(X,Y) := \tilde{h}(X, \tilde{k}Y)$$

on $T^{1,0}M$ is symmetric and invariant with respect to $\circ$. We denote by

$$\tilde{C}_X(Y) := -X \circ Y$$

the Higgs field associated to $\circ$. Let

$$X * Y := X \circ Y \circ \mathcal{E}^{-1} \tag{5.32}$$

be the dual multiplication, with associated Higgs field denoted by C. Assume that the inverse $\mathcal{E}^{-1}$ has a square root $\mathcal{E}^{-1/2}$ and define a new pseudo-Hermitian metric

$$h(X,Y) := \tilde{h}(\mathcal{E}^{-1/2} \circ X, \mathcal{E}^{-1/2} \circ Y) \tag{5.33}$$

and a new real structure

$$k(X) := \mathcal{E}^{1/2} \circ \tilde{k}(\mathcal{E}^{-1/2} \circ X)$$

on $T^{1,0}M$. It is straightforward to check that

$$g(X,Y) := h(X,kY) = \tilde{g}(\mathcal{E}^{-1/2} \circ X, \mathcal{E}^{-1/2} \circ Y). \tag{5.34}$$

In particular, g is symmetric, complex bilinear and invariant with respect to $*$. We now consider various compatibility conditions of the structures on the initial F-manifold $(M, \circ, e)$ and find obstructions for the dual structures on $(M, *, \mathcal{E})$ to satisfy the same conditions.

Lemma 97. *If the pair $(\tilde{g}, \tilde{h})$ is compatible, also the pair (g,h) is compatible.*

Proof. From (5.33), the Chern connections D and $\tilde{D}$ of $(T^{1,0}M, h)$ and $(T^{1,0}M, \tilde{h})$ respectively are related by

$$D_X Z := \mathcal{E}^{1/2} \circ \tilde{D}_X(\mathcal{E}^{-1/2} \circ Z), \quad \forall X \in \mathcal{X}(M), \quad Z \in \mathcal{T}_M^{1,0}. \tag{5.35}$$

In particular, $\tilde{D}\tilde{g} = 0$ if and only if $Dg = 0$. □

The more restrictive notions of harmonic Higgs bundles and $DChk$-structures are not preserved by the duality, in general. The obstructions are stated in the following proposition.

Theorem 98. *Assume that $(T^{1,0}M, \tilde{C}, \tilde{h})$ is a harmonic Higgs bundle (respectively, $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k})$ is a $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$-structure). Then $(T^{1,0}M, C, h)$ is a harmonic Higgs bundle (respectively, $(T^{1,0}M, C, h, k)$ is a $DChk$-structure) if and only if the following conditions hold:*

i) for any $X, Y, Z \in \mathcal{T}_M^{1,0}$,

$$\tilde{D}_X(\mathcal{E} \circ Y \circ Z) - \tilde{D}_Y(\mathcal{E} \circ X \circ Z) = \mathcal{E} \circ \Big(\tilde{D}_X(Y \circ Z) - \tilde{D}_Y(X \circ Z)\Big) \tag{5.36}$$

ii) for any $X, Y \in T^{1,0}M$,

$$[\tilde{C}_X, \tilde{k}\tilde{C}_Y\tilde{k}] = [\tilde{C}_{\mathcal{E}^{-1} \circ X}, \tilde{k}\tilde{C}_{\mathcal{E}^{-1} \circ Y}\tilde{k}]. \tag{5.37}$$

The semisimple case

It should be pointed out that equations (5.36) and (5.37) place highly restrictive conditions on the various structures and may, in general, have no solutions (as happens for some of the 2-dimensional non-semisimple examples in [108]). Just as almost-dual Frobenius manifolds satisfy almost all of the axioms of a Frobenius manifold, asking for the twisted structures to satisfy the full tt^*-axioms may be too restrictive a condition. However, it may be shown that if $(M, \circ_M, e_M)$ is semisimple and the metric $\tilde{h}$ is diagonal in canonical coordinates, then, for any eventual identity $\mathcal{E}$, both conditions (5.36) and (5.37) hold. Thus solutions in the semisimple case exist.

Chapter 6

Academic plans for the future

6.1 Overview

In this chapter I describe my scientific plans for the future. They are divided into three main groups: research, book and teaching. My first priority in the future will continue to be the research at IMAR. I will continue my research in the fields in which I worked before and I will extend it to related fields, in order to become an expert in differential geometry. I will continue to work on the book I am writing with D. V. Alekseevsky, which brings together and presents in a unified and original way various important topics from modern differential geometry. I would like to teach courses at the Master's and Doctoral level, in order to share my experience with young mathematicians, to attract them to work in my field, and eventually to coordinate their doctoral studies. Last, but not least, I intend to extend my scientific collaborations in Romania and abroad and to continue to apply for research grants.

6.2 Research

6.2.1 Research projects for the near future

I describe below several research projects which are closely related to my previous works, and which I intend to work on in the near future.

- *Conformal-Killing forms.* As already explained in Chapter 3, a systematic treatment of conformal-Killing and/or Killing forms on compact Riemannian manifolds with special holonomy groups already exists in the literature and was developed by A. Moroianu and U. Semmelmann. Between the most important results on this topic I mention: any Killing p-

form ($p \geq 2$) on a compact quaternionic-Kähler manifold is parallel [81]; any Killing form on a compact G_2 or $Spin_7$-manifold is also parallel; the space of conformal-Killing forms of any degree on a compact Kähler manifold is "generated", using the wedge product with the Kähler form, by the degree two conformal-Killing forms, in a way explained in [82]. Moreover, conformal-Killing 2-forms on a Kähler manifold (not necessarily compact) are closely related to the so called Hamiltonian 2-forms, which are well understood from the papers of V. Apostolov, D. Calderbank, P. Gauduchon in [13, 14].

The research projects which I have in mind in this field are closely related to the results presented in Sections 3.4 and 3.5 from this thesis. They are a natural continuation of my previous research and, at the same time, they complement the results of A. Moroianu and U. Semmelmann stated above. As explained in Section 3.5, in a joint work with M. Pontecorvo [42] we proved that a compact quaternionic-Kähler manifold (M, g) of dimension $4n \geq 8$ admits a non-parallel conformal-Killing 2-form if and only if (M, g) is isometric to the standard quaternionic projective space $\mathbb{H}P^n$ and we described explicitly the space of conformal-Killing 2-forms on $\mathbb{H}P^n$ (it turns out that it is isomorphic to the space of Killing vector fields, via the codifferential). Therefore, conformal-Killing 2-forms on compact quaternionic-Kähler manifolds are completely understood. But a treatment of higher degree conformal-Killing forms on compact quaternionic-Kähler manifolds is still missing from the literature. A complete classification of conformal-Killing forms of any degree on compact quaternionic-Kähler manifolds may be too optimistic - one would have to combine the representation theory of the group $Sp(1)Sp(2)$ with the conformal-Killing operator, and for the space of forms of higher degree the decomposition of the form bundles into irreducible subundles is already quite complicated. However, I will try to find at least some general constructions to produce interesting examples of non-parallel conformal-Killing forms of higher degree on compact quaternionic-Kähler manifolds; or to prove the non-existence of such forms.

Another research project I will consider in the near future is to develop a local classification of quaternionic-Kähler manifolds which admit a non-parallel compatible conformal-Killing 2-form. A first step in this direction is the prolongation $\mathcal{D}$ of the conformal-Killing operator on compatible 2-forms on a quaternionic-Kähler manifold, found in [31] and presented in Section 3.4 of this thesis, which allowed to relate the existence of a non-parallel compatible conformal-Killing 2-form with the geometry of the quaternionic-Kähler manifold (see e.g. Proposition 40 of this thesis). The possibility to develop such a classification is suggested by various similarities (at least at the formal level) which exist between compatible conformal-Killing 2-forms on quaternionic-Kähler manifolds and Hamiltonian 2-forms on Kähler manifolds, and by the local classification of Kähler manifolds which admit a Hamiltonian 2-form, developed in [13]. Hamiltonian 2-forms are defined as

2-forms which satisfy a certain first order differential equation, strongly related to the geometry of the Kähler manifold (e.g. a Kähler manifold is Bochner-flat if and only if a certain modified Ricci form is Hamiltonian). The name "Hamiltonian" is justified by one of the basic properties of such forms, namely that the coefficients of their characteristic polynomial are Poisson-commuting Hamiltonians for Killing vector fields, a property which allows a complete local classification of Kähler manifolds with a Hamiltonian 2-form. The analogies between compatible conformal-Killing 2-forms on quaternionic-Kähler manifolds and Hamiltonian 2-forms on Kähler manifolds are expressed by various algebraic properties of compatible conformal-Killing 2-forms, in relation with the quaternionic Weyl tensor, properties which hold also for Hamiltonian 2-forms, with the quaternionic-Weyl tensor replaced by the Bochner tensor of the Kähler manifold (compare Proposition 9 of [59] with Proposition 6 of [31]). A possible attempt of this project would be to see if a compatible conformal-Killing 2-form on a quaternionic-Kähler manifold defines, besides the dual of its codifferential (which is known to be a Killing field), other Killing fields. Once such Killing fields are determined, one may hope to develop a local classification, like in the Kähler case [13].

• *Generalized complex geometry.* In Chapter 4 we found an infinitesimal description of invariant generalized complex structures on a Lie group G, in terms of the so called admissible pairs $(\mathfrak{k}, \omega)$ (where $\mathfrak{k}$ is a subalgebra of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of Lie(G) and ω is a closed 2-form on $\mathfrak{k}$, satisfying a non-degeneracy condition) and we used it in order to develop a detailed study of regular generalized complex structures on semisimple Lie groups (i.e. those invariant generalized complex structures for which $\mathfrak{k}$ is a regular subalgebra of $\mathfrak{g}^{\mathbb{C}}$). When G is semisimple of inner type, the root system R_0 of $\mathfrak{k}$ is necessarily a positive root system. At the same time, we remark that in the case when G is a semisimple Lie group of outer type, all regular $\mathfrak{g}$-admissible pairs $(\mathfrak{k}, \omega)$ constructed in Chapter 4 have the property that the root system R_0 of $\mathfrak{k}$ is a σ-positive system (see Definition 57), where σ is the antilinear involution of $\mathfrak{g}^{\mathbb{C}}$ which defines the real form $\mathfrak{g} = \mathrm{Lie}(G)$. The problem of constructing (when G is of outer type) more general regular $\mathfrak{g}$-admissible pairs, with σ-parabolic (see Definition 57) but not σ-positive system R_0, is still open and will hopefully be developed in the future. Also, the problem of constructing non-regular invariant generalized complex structures on semisimple Lie groups is open as well and is another of my future research projects.

6.2.2 Long term research projects

As longer term projects, I intend to introduce myself to new topics and extend my research to new fields, closely related to the ones in which I worked before, as follows.

• *Parabolic geometry.* It is an important and quite recent direction of research in mathematics, which unifies conformal, CR and quaternionic geometry, in a way which I still need to understand. Many internationally recognized mathematicians are working in this field e.g. D. V. Alekseevsky, A. Cap, D. Calderbank, M. Eastwood, I. Slovak, etc. Since I have experience in CR and quaternionic geometry, a natural thing to do would be to extend my research also to the larger setting of parabolic geometry. The main tool in parabolic geometry is representation theory; for this reason in a first stage I will get thoroughly into some important topics from representation theory, which are relevant to parabolic geometry: graded Lie algebras, parabolic groups and subalgebras, Bott Borel-Weyl theorem, BGG resolutions, invariant differential operators. These topics can be found in various places in the literature. After I will understand the basic ideas and tools from parabolic geometry, I hope to extend some of my previous works on CR and quaternionic geometry to this more general setting. I already have some ideas in this direction.

• *Frobenius manifolds.* Another direction of research which I intend to pursue in the future is the theory of Frobenius manifolds. Frobenius manifolds are of high interest, mainly because they appear in many different areas of mathematics (e.g. differential geometry, integrable systems, algebraic geometry). Until now my research in this field was based on my background in differential geometry. However, to become a good researcher in Frobenius manifolds it is essential, I think, to extend my knowledge and research also to integrable systems and algebraic geometry, at least to the topics involved in Frobenius manifolds. Therefore, in the future I will acquire new background in quantum cohomology, isomonodromic deformations, the theory of meromorphic bundles with connections, lattices, Gauss-Manin systems, unfoldings of singularities. All these involve a more algebraic approach of Frobenius manifolds. In a first stage, I will read Hitchin's paper on isomonodromic deformations and Einstein metrics [70], which is closer to my research expertise, because it involves twistor theory, a field in which I already worked before. There are also other excellent sourses which I intend to use in the future, of more algebraic approach, like Sabbah's book on Frobenius manifolds and isomonodromic deformations [92], Hertling's book on Frobenius manifolds and singularity theory [67], Dunajski's introductory book on integrable systems [51], Guest's book on quantum cohomology and integrable systems [62], and various more advanced articles on these themes, written by the same or other authors.

6.3 Book

During the last years I am working on a book with D. V. Alekseevsky. The book will contain two volumes. In volume I, entitled "Differential Geometry; Riemannian and Non-Riemannian Geometry", we develop a self-contained theory of the most important geometrical structures which appear in differential geometry. In volume II we consider the homogeneous version these structures, whose treatment involve the theory of semisimple Lie groups and Lie algebras and representation theory.

At present we are working on volume I. A large part of the theory is already written up, but the material still needs to be re-organized and written in a last version. Volume I contains a detailed account of some of the most important geometric structures (conformal, complex, Kähler and Sasaki, quaternionic-Kähler) and the relations between them, via cone constructions, twistor theory and reductions. In the future we hope to include also a more advanced material, e.g. special symplectic manifolds (as a generalization of Bochner-flat Kähler manifolds) and advanced topics from twistor theory (e.g. the Penrose-Ward correspondence, the Jones and Tod correspondence between self-dual 4-manifolds with symmetry and Einstein-Weyl 3-spaces, the ADHM-construction, etc). These topics are covered in various research articles but it would be useful to collect them into a book, and present them in a unified and original way. The book is addressed to graduate students.

6.4 Teaching

I intend to teach courses at Master's and Doctoral levels. This would be an excellent opportunity for me to present the research results which I obtained along the years and also to be in contact with young future researchers, to attract them to work in my field of expertise and to coordinate their doctoral studies. The book I am writing with D. V. Alekseevsky could be used for a course, at various levels. One may chose one or two topics from the book and present them in a detailed way (in this case the course would be very specialized). Another option would be to include more material (without going always into all details of the proofs). The second approach would have the advantage that it would provide the student with a good overview on the modern differential geometry (including recent or open research problems) which would allow him/her to make a good choice for a Ph.D research topic.

A specialized one semester course could contain the material from Chapter 2 of this thesis, on CR and Bochner-flat Kähler manifolds, with emphasize on their relation with the more general notion of Hamiltonian 2-forms on Kähler manifolds.

Another possible topic for a specialized one semester course is the material from Chapter 3 of this thesis on conformal-Killing forms, combined with other recent developments of the theory, for example the papers of A. Moroianu and U. Semmelmann on this topic.

I would also like to teach, at a Master's level or advanced undergraduate level, a course on representation theory, this being a main tool in geometry (it appears in twistor theory, invariant differential operators, parabolic geometry, homogeneous geometry, etc). It would be very important for students to begin their research with a good background in this topic. Such a course could be inspired from many excellent books which exist in the field, e.g. the book of Fulton and Harris on representation theory [55], the book of Adams on Lie groups [2], Knapp's book on Lie groups [72], etc. A final part of the course and an application of the theory to differential geometry could be the material on invariant generalized complex structures on Lie groups, contained in [3] and presented in Chapter 4 of this thesis.

Chapter 7

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