ON THE SECOND NILPOTENT QUOTIENT OF HIGHER HOMOTOPY GROUPS, FOR HYPERSOLVABLE ARRANGEMENTS

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Abstract. We examine the first non-vanishing higher homotopy group, π_p, of the complement of a hypersolvable, non–supersolvable, complex hyperplane arrangement, as a module over the group ring of the fundamental group, $\mathbb{Z}\pi_1$. We give a presentation for the I–adic completion of π_p. We deduce that the second nilpotent I–adic quotient of π_p is determined by the combinatorics of the arrangement, and we give a combinatorial formula for the second associated graded piece, $\mathrm{gr}_I^1 \pi_p$. We relate the torsion of this graded piece to the dimensions of the minimal generating systems of the Orlik–Solomon ideal of the arrangement $\mathcal{A}$ in degree $p+2$, for various field coefficients. When $\mathcal{A}$ is associated to a finite simple graph, we show that $\mathrm{gr}_I^1 \pi_p$ is torsion–free, with rank explicitly computable from the graph.

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1. Introduction

1.1. **Overview.** The *hypersolvable* class introduced in [5], [6] is well adapted for homotopy computations with combinatorial flavour; see [8], [3].

Let X be a path-connected space, with fundamental group $\pi_1 := \pi_1(X)$. The higher homotopy groups of X have a natural module structure over the group ring, $\mathsf{R} := \mathbb{Z}\pi_1$. In general, their computation can be an extremely difficult problem. When X is not aspherical, homological methods may be used to tackle the first

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higher non-trivial homotopy group, $\pi_p := \pi_p(X)$, by Hurewicz. This R-module is still very hard to describe, when π_1 is non-trivial. Let $I \subseteq \mathsf{R}$ be the augmentation ideal. A reasonable idea is to approximate π_p by its *nilpotent quotients*, $\pi_p/I^q\pi_p$ (for $q \geq 1$), or by the *associated graded* module over $\mathrm{gr}_I^\bullet \mathsf{R}$, $\mathrm{gr}_I^\bullet \pi_p := \oplus_{q\geq 0}(I^q\pi_p/I^{q+1}\pi_p)$.

Now, let $\mathcal{A}$ be a central, hypersolvable, complex hyperplane arrangement, with affine complement denoted X. For homotopy computations on X, we may also assume $\mathcal{A}$ is essential. As shown in [5], X is aspherical if and only if $\mathcal{A}$ is fiber-type (supersolvable). So, we also assume that $\mathcal{A}$ is not supersolvable.

Let $p(\mathcal{A})$ be the order of π_1-connectivity of X, introduced in [8], and let $r(\mathcal{A})$ be the rank of the arrangement. We know that $2 \leq p := p(\mathcal{A}) < r := r(\mathcal{A})$, and both p and r are combinatorial (i.e., they depend only on the intersection lattice of $\mathcal{A}$). According to [8], $\pi_p = \pi_p(X)$ is the first higher non–trivial homotopy group of X. It is also known that both $\mathrm{gr}_I^\bullet \mathbb{Z}\pi_1$ and the first graded piece (nilpotent quotient) $\mathrm{gr}_I^0 \pi_p$ are combinatorial and torsion-free.

In [3], the case when p is maximal, i.e., $p = r-1$, was analyzed. It turned out that the $\mathrm{gr}_I^\bullet \mathbb{Z}\pi_1$–module $\mathrm{gr}_I^\bullet \pi_p$ is torsion-free, given by an explicit combinatorial formula. Unfortunately, this formula does not hold, in general.

Here, we aim at removing the additional hypothesis on p, and see what can be said about π_p.

1.2. **Results.** Set $\mathsf{R}_q := \mathbb{Z}\pi_1/\ I^q$, for $1 \leq q < \infty$, and $\mathsf{R}_\infty := \widehat{\mathbb{Z}\pi_1}$, where $\widehat{\mathbb{Z}\pi_1}$ is the I–adic completion of $\mathbb{Z}\pi_1$. The first main results converge to a convenient R_q–presentation of $\pi_p \otimes_{\mathbb{Z}\pi_1} \mathsf{R}_q$, for $q \leq \infty$. These are given in Theorem 3.1 (for $q = \infty$) and Corollaries 3.2, 3.4 (for $q < \infty$). Note that $\pi_p \otimes_{\mathbb{Z}\pi_1} \mathsf{R}_q$ is the q-th nilpotent quotient, $\pi_p/I^q\pi_p$, for $q < \infty$. When $q = 2$, both the second nilpotent quotient $\pi_p \otimes_{\mathbb{Z}\pi_1} \mathsf{R}_2$ and the second graded piece $\mathrm{gr}_I^1 \pi_p$ have an explicit combinatorial formula, derived in Theorem 3.5.

The second type of main results is related to torsion in $\mathrm{gr}_I^1 \pi_p$. It turns out that this problem leads to a basic question in combinatorial arrangement theory; compare with [7], [10], [2]. Let $\Lambda^\bullet := \Lambda^\bullet(\mathcal{A})$ be the exterior algebra over $\mathbb{Z}$ generated by the set of hyperplanes of an arbitrary arrangement $\mathcal{A}$. Let $\mathcal{I}^\bullet := \mathcal{I}^\bullet(\mathcal{A}) \subseteq \Lambda^\bullet$ be the Orlik-Solomon ideal of $\mathcal{A}$, and denote by $\mathrm{A}^\bullet(\mathcal{A}) = \Lambda/\mathcal{I}$ the Orlik-Solomon algebra over $\mathbb{Z}$, known to be torsion-free. By a celebrated result of Orlik and Solomon, the $\mathbb{K}$–specialization $\mathrm{A}^\bullet(\mathcal{A})_{\mathbb{K}}$ is isomorphic to the $\mathbb{K}$–cohomology ring of the affine complement of $\mathcal{A}$, for every commutative ring $\mathbb{K}$.

Let $\Lambda^+\mathcal{I} \subseteq \mathcal{I}$ be the decomposable Orlik-Solomon ideal. We introduce $\mathrm{A}_+^\bullet(\mathcal{A}) := \Lambda/\Lambda^+\mathcal{I}$, the *decomposable Orlik-Solomon algebra*. Is $\mathrm{A}_+^\bullet(\mathcal{A})$ also torsion-free? At the time of writing, we have no example where torsion appears. When $\mathcal{A}$ is hypersolvable and not supersolvable, we show in Theorem 4.1 that $\mathrm{gr}_I^1 \pi_p$ is torsion-free precisely when $\mathrm{A}_+^{p+2}(\mathcal{A})$ has no torsion.

Consider the quadratic Orlik-Solomon algebra, $\overline{\mathrm{A}}^{\bullet}(\mathcal{A}) := \Lambda/\mathcal{I}_2$, where $\mathcal{I}_2$ is the ideal generated by $\mathcal{I}^2$, the degree 2 component of $\mathcal{I}$. When $\mathcal{A}$ is supersolvable, it is known that $\mathrm{A}^{\bullet}(\mathcal{A}) = \overline{\mathrm{A}}^{\bullet}(\mathcal{A})$, see [11]; hence, in this case, $\mathrm{A}^{\bullet}_{+}(\mathcal{A})$ has no torsion.

When $\mathcal{A}$ is hypersolvable, not supersolvable, and *graphic* (i.e., a subarrangement of a braid arrangement, associated to a finite simple graph), we prove in Corollary 4.4 that $\mathrm{gr}^1_I \pi_p$ has a simple description in terms of the graph, in particular it has no torsion. The graph from Example 4.5 shows that this combinatorial description may be done outside the maximal range $p = r - 1$ from [3].

1.3. **Questions.** We are left with some open questions concerning hypersolvable arrangements. Is $\mathrm{A}^{\bullet}_{+}(\mathcal{A})$ torsion-free, at least in degree $\bullet = p + 2$? What if we restrict the question to *2–generic* arrangements (i.e., arrangements with no collinearity relations, known to be hypersolvable)? See also Remark 4.8 on arbitrary arrangements.

2. A preliminary module presentation

We shall work in the context of [3, Sections 5 and 6]. Let $\mathcal{A}$ be a hypersolvable complex hyperplane arrangement which is not supersolvable, and $X = M'(\mathcal{A})$ its complement in affine space.

We know that $\mathcal{A}$ is a p-generic section of its supersolvable deformation, $\widehat{\mathcal{A}}$. Set $Y = M'(\widehat{\mathcal{A}})$, and let $j : X \hookrightarrow Y$ denote the inclusion. Denote by π_1 the fundamental groups identified through the induced map $j_\sharp : \pi_1(X) \to \pi_1(Y)$. Let $\tilde{j} : \widetilde{X} \to \widetilde{Y}$ be the π_1-equivariant map induced on universal covers. Denote by $\tilde{j}_\bullet : C_\bullet(\widetilde{X}) \to C_\bullet(\widetilde{Y})$ the $\mathbb{Z}\pi_1$-linear chain map between the π_1-equivariant cellular chains on the universal covers, and by $j_\bullet : H_\bullet(X) \to H_\bullet(Y)$ the induced map in integral homology.

We have split exact sequences of finitely generated free abelian groups,

$$0 \to H_\bullet(X) \xrightarrow{j_\bullet} H_\bullet(Y) \xrightarrow{\Pi_\bullet} H_\bullet(Y, X) \to 0, \tag{2.1}$$

whose duals,

$$0 \to H^\bullet(Y, X) \xrightarrow{\Pi^\bullet} H^\bullet(Y) \xrightarrow{j^\bullet} H^\bullet(X) \to 0, \tag{2.2}$$

may be described in purely combinatorial terms: $j^\bullet$ may be identified with the canonical surjection,

$$j^\bullet : \overline{\mathrm{A}}^{\bullet}(\mathcal{A}) \twoheadrightarrow \mathrm{A}^{\bullet}(\mathcal{A}), \tag{2.3}$$

between Orlik-Solomon algebras.

For simplicity, in the sequel we set $\mathsf{R} := \mathbb{Z}\pi_1$. Note that $C_\bullet(\widetilde{X}) = H_\bullet(X) \otimes \mathsf{R}$ and $C_\bullet(\widetilde{Y}) = H_\bullet(Y) \otimes \mathsf{R}$, as R-modules, by the minimality property for arrangement complements [3, Corollary 6].

Denoting by $\tilde{\partial}_\bullet : C_\bullet(\widetilde{Y}) \to C_{\bullet-1}(\widetilde{Y})$ the differential on the equivariant chain complex of $\widetilde{Y}$, we have the following.

Theorem 2.1. ([3]) *The* R*-module* π_p *is isomorphic to the cokernel of the* R*-linear map*

$$\tilde{\partial}_{p+2} + \tilde{j}_{p+1} : (H_{p+2}Y \oplus H_{p+1}X) \otimes \mathsf{R} \to H_{p+1}Y \otimes \mathsf{R}.$$

Due to R-linearity, $\tilde{j}_\bullet$ respects the I-adic filtrations, i.e., it sends $H_\bullet X \otimes I^q$ into $H_\bullet Y \otimes I^q$, for all q. The associated graded $\mathrm{gr}_I^\bullet \mathsf{R}$–linear map,

$$\mathrm{gr}^\bullet \tilde{j}_\bullet : H_\bullet X \otimes \mathrm{gr}_I^\bullet \mathsf{R} \to H_\bullet Y \otimes \mathrm{gr}_I^\bullet \mathsf{R}\,,$$

is equal to $j_\bullet \otimes \mathrm{id}$, by minimality.

Similar considerations are valid for $\tilde{\partial}_\bullet$: by minimality again, it sends $H_\bullet Y \otimes I^q$ into $H_{\bullet-1} Y \otimes I^{q+1}$, for all q. The associated graded $\mathrm{gr}_I^\bullet \mathsf{R}$–linear map is denoted

$$E_1^\bullet \tilde{\partial}_\bullet : H_\bullet Y \otimes \mathrm{gr}_I^\bullet \mathsf{R} \to H_{\bullet-1} Y \otimes \mathrm{gr}_I^{\bullet+1} \mathsf{R}.$$

To describe the action of $E_1^\bullet \tilde{\partial}_\bullet$ on the free $\mathrm{gr}_I^\bullet \mathsf{R}$-generators, $H_\bullet Y \otimes 1$, we recall that $\mathrm{gr}_I^0 \mathsf{R} = \mathbb{Z} \cdot 1$, and $\mathrm{gr}_I^1 \mathsf{R}$ is naturally identified with $(\pi_1)_{ab} = H_1(Y)$. We denote by H_1 both $H_1(X)$ and $H_1(Y)$, identified via j_1.

Now it follows from [3, Section 6] that the restriction of $E_1^\bullet \tilde{\partial}_\bullet$ to $H_\bullet Y \equiv H_\bullet Y \otimes 1 \subseteq H_\bullet Y \otimes \mathrm{gr}_I^0 \mathsf{R}$, denoted

$$\partial_\bullet : H_\bullet Y \to H_{\bullet-1} Y \otimes H_1, \tag{2.4}$$

has dual, up to sign,

$$\partial_\bullet^* : \overline{\mathrm{A}}^{\bullet-1}(\mathcal{A}) \otimes \overline{\mathrm{A}}^1(\mathcal{A}) \to \overline{\mathrm{A}}^\bullet(\mathcal{A}), \tag{2.5}$$

given by the multiplication of the quadratic OS-algebra.

The description (2.4) of $E_1^\bullet \tilde{\partial}_\bullet$ is related to the spectral sequence associated to the equivariant chain complex of a CW-complex, analyzed in full generality in [9].

3. Completion of the presentation

In this section we pursue the following idea: Use completion constructions to simplify the presentation in Theorem 2.1, more exactly, to replace $\tilde{j}_{p+1}$ by $j_{p+1} \otimes \mathrm{id}$, without altering $E_1^\bullet \tilde{\partial}_{p+2}$. We refer the reader to [1, Chapitre III.2] for standard completion techniques.

We explain now how these work concretely. The ring $\widehat{\mathsf{R}}$ is endowed with the canonical, decreasing, complete, separated, and multiplicative filtration $\{F^q\}_{q\geq 0}$, as $\widehat{\mathsf{R}} = \varprojlim \mathsf{R}/I^q$. In addition, $\widehat{\mathsf{R}}/F^q = \mathsf{R}/I^q$ and $\mathrm{gr}_F^q \widehat{\mathsf{R}} = \mathrm{gr}_I^q \mathsf{R}$, for all q. Every right $\widehat{\mathsf{R}}$-module M has the canonical filtration $\{M \cdot F^q\}_{q\geq 0}$, and $\widehat{\mathsf{R}}$-linear maps preserve canonical filtrations. Furthermore, we have the following convenient test, for an $\widehat{\mathsf{R}}$-linear map f between complete and separated modules: f is an isomorphism if and only if $\mathrm{gr}_F^\bullet(f)$ is an isomorphism. These facts will lead to the first property of the aforementioned replacement.

For the second property, let us notice that, given an arbitrary map in $\widehat{\mathsf{R}}$-Mod, $f : M \to N$, we have that $f(M \cdot F^q) \subseteq N \cdot F^{q+1}$ for all q if and only if $\mathrm{gr}^\bullet_F(f) = 0$. If this happens, f induces a $\mathrm{gr}^\bullet_F \widehat{\mathsf{R}}$-linear map, $E^\bullet_1 f : \mathrm{gr}^\bullet_F M \to \mathrm{gr}^{\bullet+1}_F N$.

Finally, there is the completion functor, $\widehat{(\cdot)} : \mathsf{R}$-$Mod \to \widehat{\mathsf{R}}$-$Mod$, given by $M \mapsto \widehat{M} = \varprojlim(M/M \cdot I^q)$. On free finitely generated R-modules, $\widehat{(\cdot)}$ is naturally equivalent with $(\cdot) \otimes_{\mathsf{R}} \widehat{\mathsf{R}}$. More precisely, if $M = H \otimes \mathsf{R}$, where H is a finitely generated free abelian group, then $M \otimes_{\mathsf{R}} \widehat{\mathsf{R}} = H \otimes \widehat{\mathsf{R}}$, with canonical (complete and separated) filtration $\{H \otimes F^q\}_{q \geq 0}$. Clearly, $\mathrm{gr}^\bullet_F(H \otimes \widehat{\mathsf{R}}) = \mathrm{gr}^\bullet_I(H \otimes \mathsf{R}) = H \otimes \mathrm{gr}^\bullet_I \mathsf{R}$.

The (decreasing, multiplicative) I-adic filtration $\{I^q\}_{q\geq 0}$ of R leads to similar constructions, $\mathrm{gr}^\bullet(\varphi) : \mathrm{gr}^\bullet_I M \to \mathrm{gr}^\bullet_I N$ (for $\varphi : M \to N$ R-linear), respectively $E^\bullet_1(\varphi) : \mathrm{gr}^\bullet_I M \to \mathrm{gr}^{\bullet+1}_I N$, when $\mathrm{gr}^\bullet(\varphi) = 0$. When both M and N are finitely generated free R-modules, $\mathrm{gr}^\bullet_F(\varphi \otimes_{\mathsf{R}} \widehat{\mathsf{R}}) = \mathrm{gr}^\bullet(\varphi)$. If in addition $\mathrm{gr}^\bullet(\varphi) = 0$, then $E^\bullet_1(\varphi \otimes_{\mathsf{R}} \widehat{\mathsf{R}}) = E^\bullet_1(\varphi)$.

Theorem 3.1. *Let $\mathcal{A}$ be a hypersolvable and not supersolvable arrangement. Then the $\widehat{\mathsf{R}}$-module $\pi_p \otimes_{\mathsf{R}} \widehat{\mathsf{R}}$ is isomorphic to the cokernel of an $\widehat{\mathsf{R}}$-linear map*

$$D_{p+2} : H_{p+2}Y \otimes \widehat{\mathsf{R}} \to H_{p+1}(Y, X) \otimes \widehat{\mathsf{R}},$$

with the property that $\mathrm{gr}^\bullet_F(D_{p+2}) = 0$ and $E^\bullet_1(D_{p+2}) : H_{p+2}Y \otimes \mathrm{gr}^\bullet_I \mathsf{R} \to H_{p+1}(Y, X) \otimes \mathrm{gr}^{\bullet+1}_I \mathsf{R}$ acts on the free $\mathrm{gr}^\bullet_I \mathsf{R}$-generators by

$$H_{p+2}Y \xrightarrow{\partial_{p+2}} H_{p+1}Y \otimes H_1 \xrightarrow{\Pi_{p+1}\otimes \mathrm{id}} H_{p+1}(Y, X) \otimes H_1,$$

where ∂_{p+2} is described in (2.4)-(2.5), and Π_{p+1} is defined in (2.1) and (2.2).

Proof. Choose a splitting in (2.1), $\sigma_\bullet : H_\bullet(Y, X) \hookrightarrow H_\bullet Y$. The R-presentation from Theorem 2.1 gives a presentation for $\pi_p \otimes_{\mathsf{R}} \widehat{\mathsf{R}}$ as the cokernel of the $\widehat{\mathsf{R}}$-linear map

$$\tilde{\partial}_{p+2} \otimes_{\mathsf{R}} \widehat{\mathsf{R}} + \tilde{j}_{p+1} \otimes_{\mathsf{R}} \widehat{\mathsf{R}} : (H_{p+2}Y \oplus H_{p+1}X) \otimes \widehat{\mathsf{R}} \to (H_{p+1}X \oplus H_{p+1}(Y, X)) \otimes \widehat{\mathsf{R}}. \tag{3.1}$$

Consider the $\widehat{\mathsf{R}}$-linear map

$$\tilde{j}_{p+1} \otimes_{\mathsf{R}} \widehat{\mathsf{R}} + \sigma_{p+1} \otimes \mathrm{id}_{\widehat{\mathsf{R}}} : (H_{p+1}X \oplus H_{p+1}(Y, X)) \otimes \widehat{\mathsf{R}} \to (H_{p+1}X \oplus H_{p+1}(Y, X)) \otimes \widehat{\mathsf{R}}. \tag{3.2}$$

Since $\mathrm{gr}^\bullet \tilde{j}_{p+1} = j_{p+1} \otimes \mathrm{id}$ and $\mathrm{gr}^\bullet_F(\sigma_{p+1} \otimes \mathrm{id}_{\widehat{\mathsf{R}}}) = \sigma_{p+1} \otimes \mathrm{id}$, we infer that (3.2) is an isomorphism, by $\widehat{\mathsf{R}}$-completeness and separation.

Hence, $H_{p+1}Y \otimes \widehat{\mathsf{R}} \cong \mathrm{im}(\tilde{j}_{p+1} \otimes_{\mathsf{R}} \widehat{\mathsf{R}}) \oplus \mathrm{im}(\sigma_{p+1} \otimes \mathrm{id}_{\widehat{\mathsf{R}}})$, and $H_{p+1}Y \otimes \widehat{\mathsf{R}} / \mathrm{im}(\tilde{j}_{p+1} \otimes_{\mathsf{R}} \widehat{\mathsf{R}}) \cong H_{p+1}(Y, X) \otimes \widehat{\mathsf{R}}$. Moreover, $\mathrm{gr}^\bullet_F(H_{p+1}Y \otimes \widehat{\mathsf{R}} \xrightarrow{\mathrm{pr}_{p+1}} H_{p+1}Y \otimes \widehat{\mathsf{R}} / \mathrm{im}(\tilde{j}_{p+1} \otimes_{\mathsf{R}} \widehat{\mathsf{R}}))$ is identified with $H_{p+1}Y \otimes \mathrm{gr}^\bullet_I \mathsf{R} \xrightarrow{\Pi_{p+1}\otimes\mathrm{id}} H_{p+1}(Y, X) \otimes \mathrm{gr}^\bullet_I \mathsf{R}$.

Set $D_{p+2} = \mathrm{pr}_{p+1} \circ (\tilde{\partial}_{p+2} \otimes_{\mathsf{R}} \widehat{\mathsf{R}})$. Combining (3.1) and (3.2) we obtain that $\pi_p \otimes_{\mathsf{R}} \widehat{\mathsf{R}} \cong \mathrm{coker}(D_{p+2})$, and $\mathrm{gr}^\bullet_F(D_{p+2}) = (\Pi_{p+1} \otimes \mathrm{id}) \circ \mathrm{gr}^\bullet \tilde{\partial}_{p+2} = 0$. The assertion on $E^\bullet_1(D_{p+2})$ follows from (2.4). □

It is now an easy matter to derive R_q-presentations for $\pi_p \otimes_{\mathsf{R}} \mathsf{R}_q = \pi_p / I^q \cdot \pi_p$, for all $1 \leq q < \infty$. Note that $\mathrm{gr}_I^s \mathsf{R}_q = \mathrm{gr}_I^s \mathsf{R}$ for $s < q$, and $\mathrm{gr}_I^s \mathsf{R}_q = 0$ for $s \geq q$. Note also that $H_{p+1}(Y, X) \neq 0$, by the definition of $p(\mathcal{A})$.

Corollary 3.2. ([8]) *If $\mathcal{A}$ is a hypersolvable and not supersolvable arrangement, then $\mathrm{gr}_I^0 \pi_p = \pi_p / I \cdot \pi_p = H_{p+1}(Y, X)$ does not vanish.*

Example 3.3. Note that the hypersolvability hypothesis on $\mathcal{A}$ is crucial. Indeed, recall from [8] that by definition $p = p(M'(\mathcal{A}))$ is equal to $\sup\{s \mid \dim_{\mathbb{Q}} H_t(M'(\mathcal{A}), \mathbb{Q}) = \dim_{\mathbb{Q}} H_t(\pi_1 M'(\mathcal{A}), \mathbb{Q}), \forall t \leq s\}$. When $\mathcal{A}$ is hypersolvable, this is equal to $p(\mathcal{A}) := \sup\{s \mid \operatorname{rank} \mathrm{A}^t(\mathcal{A}) = \operatorname{rank} \overline{\mathrm{A}}^t(\mathcal{A}), \forall t \leq s\}$.

Now, let $\mathcal{A}$ be the aspherical Coxeter arrangement of type $D_n, n \geq 4$. Since the Orlik-Solomon algebra $\mathrm{A}^{\bullet}(\mathcal{A})$ is not quadratic [4], $\mathcal{A}$ is not supersolvable [11] and $2 \leq p(\mathcal{A}) < \infty$. Clearly $\mathcal{A}$ cannot be hypersolvable, since $\pi_p(M'(\mathcal{A})) = 0$.

Corollary 3.4. *Let $\mathcal{A}$ be a hypersolvable and not supersolvable arrangement and $q \geq 2$. Then $\pi_p / I^q \cdot \pi_p$ is isomorphic over R_q with $\operatorname{coker}(D_{p+2} \otimes_{\widehat{\mathsf{R}}} \mathsf{R}_q : H_{p+2} Y \otimes \mathsf{R}_q \to H_{p+1}(Y, X) \otimes \mathsf{R}_q)$. Furthermore, $\mathrm{gr}_I^{\bullet}(D_{p+2} \otimes_{\widehat{\mathsf{R}}} \mathsf{R}_q) = 0$, and $E_1^s(D_{p+2} \otimes_{\widehat{\mathsf{R}}} \mathsf{R}_q) : H_{p+2} Y \otimes \mathrm{gr}_I^s \mathsf{R} \to H_{p+1}(Y, X) \otimes \mathrm{gr}_I^{s+1} \mathsf{R}$ is equal to $E_1^s(D_{p+2})$, for $s < q - 1$, and it is 0, for $s = q - 1$.*

Proof. Tensor the $\widehat{\mathsf{R}}$-presentation from Theorem 3.1, over $\widehat{\mathsf{R}}$, with $\widehat{\mathsf{R}} / F^q = \mathsf{R}_q$. The claims on $\mathrm{gr}^{\bullet}$ and $E_1^{\bullet}$ follow from the fact that $\mathrm{gr}^{\bullet} \mathsf{R}_q = \mathrm{gr}^{\bullet} \mathsf{R} / \mathrm{gr}^{\geq q} \mathsf{R}$. □

When $q = 2$, everything becomes explicit. The exact sequence

$$0 \to I/I^2 \to \mathsf{R}/I^2 \to \mathsf{R}/I \to 0 \tag{3.3}$$

has a canonical splitting. Hence, $\mathsf{R}_2 = \mathbb{Z} \cdot 1 \oplus H_1$, where H_1 is free abelian, of rank $|\mathcal{A}|$. The I-adic filtration is given by $I^0 \cdot \mathsf{R}_2 = \mathsf{R}_2$, $I \cdot \mathsf{R}_2 = H_1$ and $I^2 \cdot \mathsf{R}_2 = 0$. Hence, the filtered ring R_2 is combinatorially determined.

The map $D_{p+2} \otimes_{\widehat{\mathsf{R}}} \mathsf{R}_2 : H_{p+2} Y \otimes (\mathbb{Z} \cdot 1 \oplus H_1) \to H_{p+1}(Y, X) \otimes (\mathbb{Z} \cdot 1 \oplus H_1)$ is zero on $H_{p+2} Y \otimes H_1$; on $H_{p+2} Y \otimes 1 \equiv H_{p+2} Y$, it is equal to $(\Pi_{p+1} \otimes \mathrm{id}) \circ \partial_{p+2} : H_{p+2} Y \to H_{p+1}(Y, X) \otimes H_1$. Hence, the filtered R_2-module $\pi_p / I^2 \cdot \pi_p$ is combinatorially determined, see (2.3) and (2.5). In particular, $\mathrm{gr}_I^1 \pi_p$ is combinatorially determined. We will need an explicit combinatorial description of the second graded piece, $\mathrm{gr}_I^1 \pi_p$. By (2.2) and (2.3), $\Pi_{p+1}^* : H^{p+1}(Y, X) \to H^{p+1} Y$ is the inclusion,

$$\Pi_{p+1}^* : (\mathcal{I}/\mathcal{I}_2)^{p+1} \hookrightarrow (\Lambda/\mathcal{I}_2)^{p+1}. \tag{3.4}$$

We infer from (2.5) that (up to sign)

$$\partial_{p+2}^* : (\Lambda/\mathcal{I}_2)^{p+1} \otimes \Lambda^1 \to (\Lambda/\mathcal{I}_2)^{p+2} \tag{3.5}$$

is induced by the multiplication map μ of $\Lambda^{\bullet}$. We thus obtain the following explicit combinatorial description:

$$\partial_{p+2}^* \circ (\Pi_{p+1} \otimes \mathrm{id})^* : (\mathcal{I}/\mathcal{I}_2)^{p+1} \otimes \Lambda^1 \xrightarrow{\pm\mu} (\Lambda/\mathcal{I}_2)^{p+2}. \tag{3.6}$$

Using the R_2-presentation from Corollary 3.4, we deduce that $\mathrm{gr}^1_I \pi_p$ is given by

$$\begin{aligned}\mathrm{gr}^1_I \pi_p &= I\cdot(\pi_p/I^2\cdot\pi_p) = H_{p+1}(Y,X)\otimes H_1/\operatorname{im}(D_{p+2}\otimes_{\widehat{\mathsf{R}}}\mathsf{R}_2)\cap(H_{p+1}(Y,X)\otimes H_1)\\ &= \operatorname{coker}((\Pi_{p+1}\otimes\mathrm{id})\circ\partial_{p+2}: H_{p+2}Y\to H_{p+1}(Y,X)\otimes H_1).\end{aligned}$$

We summarize our results for $q=2$ as follows.

Theorem 3.5. *Let $\mathcal{A}$ be a hypersolvable and not supersolvable arrangement, and $p=p(\mathcal{A})$. Then the second nilpotent quotient, $\pi_p M'(\mathcal{A})/I^2\cdot\pi_p M'(\mathcal{A})$ is combinatorially determined as a filtered $\mathbb{Z}\pi_1 M'(\mathcal{A})/I^2$-module. The finitely generated abelian group $\mathrm{gr}^1_I \pi_p M'(\mathcal{A})$ is also combinatorially determined, with $\mathbb{Z}$-presentation*

$$\mathrm{gr}^1_I \pi_p M'(\mathcal{A}) = \operatorname{coker}(H_{p+2}Y \xrightarrow{(\Pi_{p+1}\otimes\mathrm{id})\circ\partial_{p+2}} H_{p+1}(Y,X)\otimes H_1).$$

4. Torsion issues

In this section, we analyze the torsion of the second graded piece of π_p.

Theorem 4.1. *Let $\mathcal{A}$ be a hypersolvable and not supersolvable arrangement, and $p=p(\mathcal{A})$. Then the following are equivalent:*

1. *The second graded piece, $\mathrm{gr}^1_I \pi_p(M'(\mathcal{A}))$, has no torsion.*
2. *The decomposable Orlik-Solomon algebra, $\mathrm{A}^\bullet_+(\mathcal{A})$, is free in degree $\bullet = p+2$.*
3. *The graded abelian group of indecomposable OS–relations, $(\mathcal{I}/\Lambda^+\mathcal{I})^\bullet$ is free in degree $\bullet = p+2$.*

Proof. Let $\mathbb{K}$ be a field. We infer from Theorem 3.5 and (3.6) that the $\mathbb{K}$-dual $(\mathrm{gr}^1_I \pi_p)\otimes\mathbb{K}^*$ is isomorphic to $\ker(\mu : (\mathcal{I}/\mathcal{I}_2)^{p+1}\otimes\Lambda^1\to(\Lambda/\mathcal{I}_2)^{p+2})_{\mathbb{K}}$ over $\mathbb{K}$, where the subscript $\mathbb{K}$ denotes specialization to $\mathbb{K}$–coefficients. Since $\mathcal{I}_2(\mathcal{A})_{\mathbb{K}} = \mathcal{I}(\hat{\mathcal{A}})_{\mathbb{K}}$ ([11, 5]), both Hilbert series, $(\mathcal{I}/\mathcal{I}_2)^\bullet\otimes\Lambda^1(t)$ and $(\Lambda/\mathcal{I}_2)^\bullet(t)$, are independent of $\mathbb{K}$, taking into account that Orlik-Solomon algebras are torsion-free [7].

Hence, $\mathrm{gr}^1_I \pi_p$ is free if and only if $\dim_{\mathbb{K}}\operatorname{coker}(\mu)_{\mathbb{K}}$ is independent of $\mathbb{K}$, in degree $p+2$. Plainly, $\operatorname{coker}(\mu)^{p+2}_{\mathbb{K}} = \mathrm{A}^{p+2}_+(\mathcal{A})\otimes\mathbb{K}$. Therefore, (1) $\Leftrightarrow$ (2). The split exact sequence

$$0\to(\mathcal{I}/\Lambda^+\mathcal{I})^\bullet\to\mathrm{A}^\bullet_+(\mathcal{A})\to\mathrm{A}^\bullet(\mathcal{A})\to 0$$

gives the equivalence (2) $\Leftrightarrow$ (3). □

In what follows, the subscript $\mathbb{K}$ denotes OS–type objects with coefficients in $\mathbb{K}$. For an arbitrary arrangement $\mathcal{A}$, set $\mathrm{A}^\bullet_{\mathbb{K}}(\mathcal{A})(t) := \sum_{m\geq 0} b_m(\mathcal{A})t^m$; this Hilbert series is independent of the field $\mathbb{K}$. Define

$$(\mathcal{I}/\Lambda^+\mathcal{I})^\bullet_{\mathbb{K}}(t) := \sum_{m\geq 2} r_m(\mathcal{A})_{\mathbb{K}} t^m = (\mathrm{A}^\bullet_+)_{\mathbb{K}}(\mathcal{A})(t) - \mathrm{A}^\bullet_{\mathbb{K}}(\mathcal{A})(t).$$

When we write $r_m(\mathcal{A})$, we mean that $r_m(\mathcal{A})_{\mathbb{K}}$ is independent of $\mathbb{K}$. With this notation, we extract from the proof of Theorem 4.1 the following.

Corollary 4.2. *Assume that $\mathcal{A}$ satisfies the equivalent properties from Theorem 4.1. Then $\operatorname{gr}^1_I \pi_p M'(\mathcal{A})$ is free, of rank*

$$|\mathcal{A}|(b_{p+1}\widehat{\mathcal{A}} - b_{p+1}\mathcal{A}) - (b_{p+2}\widehat{\mathcal{A}} - b_{p+2}\mathcal{A}) + r_{p+2}\mathcal{A}\,,$$

where $\widehat{\mathcal{A}}$ is the supersolvable deformation of $\mathcal{A}$, constructed in [5, 6].

Example 4.3. If $\mathcal{A}$ is supersolvable, then $\mathrm{A}^\bullet_+(\mathcal{A})$ has no torsion. Indeed, in this case $\mathrm{A}^\bullet_{\mathbb{K}}(\mathcal{A}) = \overline{\mathrm{A}}^\bullet_{\mathbb{K}}(\mathcal{A})$, according to [11, Lemma 4.3]. We deduce that the Hilbert series $(\mathcal{I}/\Lambda^+\mathcal{I})^\bullet_{\mathbb{K}}(t) = (\mathcal{I}_2/\Lambda^+\mathcal{I}_2)^\bullet_{\mathbb{K}}(t) = (\dim_{\mathbb{K}} \mathcal{I}^2_{\mathbb{K}}) \cdot t^2$ is independent of $\mathbb{K}$.

When $\mathcal{A}$ is hypersolvable and $p(\mathcal{A}) = r(\mathcal{A}) - 1$, then $\mathcal{A}$ is not supersolvable and $\mathrm{A}^{p+2}_+(\mathcal{A})$ has no torsion; see [3, Theorem 23] and Theorem 4.1. This happens for instance when $\mathcal{A}$ is hypersolvable and $r(\mathcal{A}) = 3$.

For the next examples, we need to review some standard constructions and terminology in arrangement theory. A subset $C \subseteq \mathcal{A}$ belongs to $\mathcal{C}_q(\mathcal{A})$ (the set of q–circuits of $\mathcal{A}$) if and only if $|C| = q$ and the hyperplanes in C form a minimally dependent set. We say that $C \subseteq \mathcal{A}$ has a chord, $c \in \mathcal{A} \setminus C$, if there is a partition, $C = C' \sqcup C''$, such that both $C' \cup \{c\}$ and $C'' \cup \{c\}$ are dependent subsets. Let $\mathcal{C}^{NC}_q(\mathcal{A}) \subseteq \mathcal{C}_q(\mathcal{A})$ be the subset of chordless q–circuits.

Recall that $\Lambda^\bullet_{\mathbb{K}}$ is the exterior $\mathbb{K}$–algebra on $\mathcal{A}$, as usual. Denote by $\delta : \Lambda^\bullet_{\mathbb{K}} \to \Lambda^{\bullet-1}_{\mathbb{K}}$ the unique degree -1 graded algebra derivation taking the values $\delta(i) = 1$, for all free algebra generators $i \in \mathcal{A}$. Note that $\delta^2 = 0$. For $C = \{i_1, \dots, i_q\} \subseteq \mathcal{A}$, $|C| = q$, denote by $e_C \in \Lambda^q$ the exterior monomial $i_1 \cdots i_q$ (which is well defined, up to a sign).

We recall that the Orlik-Solomon ideal $\mathcal{I}$ is generated by $\delta(e_C)$, $C \in \mathcal{C}_\bullet(\mathcal{A})$. It follows that $\delta_q : \mathbb{K} - \operatorname{span}\langle e_C |\ C \in \mathcal{C}_{q+1}(\mathcal{A})\rangle \to \mathcal{I}^q_{\mathbb{K}}$ induces a surjection, $\overline{\delta}_q : \mathbb{K} - \operatorname{span}\langle e_C |\ C \in \mathcal{C}_{q+1}(\mathcal{A})\rangle \twoheadrightarrow (\mathcal{I}/\Lambda^+\mathcal{I})^q_{\mathbb{K}}$, for all q. The proof of Lemma 2.1 from [2] shows that the restriction

$$\overline{\delta}_q : \mathbb{K} - \operatorname{span}\langle e_C |\ C \in \mathcal{C}^{NC}_{q+1}(\mathcal{A})\rangle \twoheadrightarrow (\mathcal{I}/\ \Lambda^+\mathcal{I})^q_{\mathbb{K}} \tag{4.1}$$

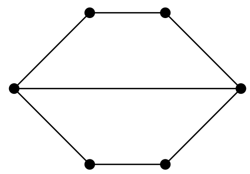
is still onto, for all q.

Now, let $\mathcal{A}_\Gamma$ be the *graphic arrangement* (see [7]) associated to the finite simple graph Γ, with hyperplanes in one to one correspondence with the edges of Γ. In this case, the q-circuits of $\mathcal{A}_\Gamma$ correspond to the q-cycles of Γ; furthermore, a circuit has a chord if and only if the corresponding cycle has a chord, in the sense of graph theory. By [10, Lemma 6.2], the map (4.1) is an isomorphism, for all q, when $\mathcal{A} = \mathcal{A}_\Gamma$.

Corollary 4.4. *Let $\mathcal{A} = \mathcal{A}_\Gamma$ be hypersolvable and not supersolvable. Then the second graded piece, $\operatorname{gr}^1_I \pi_p M'(\mathcal{A}_\Gamma)$, is free, with rank given in Corollary 4.2, where $r_{p+2}\mathcal{A}_\Gamma = |\mathcal{C}^{NC}_{p+3}(\mathcal{A}_\Gamma)|$. Moreover, this rank may be explicitly computed from a hypersolvable composition series of the graph Γ.*

Proof. The first assertion follows at once from the isomorphism (4.1). As for the second claim, let us examine the Betti numbers, $b_\bullet(\mathcal{A}_\Gamma)$ and $b_\bullet(\widehat{\mathcal{A}}_\Gamma)$, appearing in

Corollary 4.2. We know that the Hilbert series of $\mathrm{A}^\bullet(\mathcal{A}_\Gamma)$ can be computed from the chromatic polynomial of Γ [7]. Finally, the Hilbert series of $\mathrm{A}^\bullet(\widehat{\mathcal{A}_\Gamma})$ is determined by the exponents of a hypersolvable composition series of Γ [8]. $\square$



Example 4.5. The graphic arrangement $\mathcal{A}_\Gamma$ associated to the above graph Γ (without triangles) is hypersolvable and not supersolvable, with $p(\mathcal{A}_\Gamma) = 2$ and $\mathrm{rank}(\mathcal{A}_\Gamma) = 5 > p+1$. Theorem 23 from [3] cannot be used, but $\mathrm{gr}^1_I \pi_p M'(\mathcal{A}_\Gamma)$ can be computed with the aid of Corollary 4.4.

Remark 4.6. For a dependent arrangement (i.e., not boolean) define $c(\mathcal{A})$ to be the smallest integer q for which there is $C \subseteq \mathcal{A}$ dependent with $|C| = q$. Equivalently, $\mathcal{C}_{c(\mathcal{A})}(\mathcal{A}) \neq \emptyset$, but $\mathcal{C}_{<c(\mathcal{A})}(\mathcal{A}) = \emptyset$. Note that $c(\mathcal{A}) \geq 3$. We recall that an arrangement $\mathcal{A}$ is called 2–generic when $c := c(\mathcal{A}) > 3$. This implies that $\mathcal{A}$ is hypersolvable and not supersolvable, of a particular kind: $\pi_1 M'(\mathcal{A}) = \mathbb{Z}^{\mathcal{A}}$, $\overline{\mathrm{A}}^\bullet(\mathcal{A}) = \Lambda^\bullet$, $p = c - 2$. Question: is $r_c(\mathcal{A})_{\mathbb{K}}$ independent of $\mathbb{K}$?

Example 4.7. For an arbitrary arrangement $\mathcal{A}, r_{p+2}(\mathcal{A}) = 0$ if $\mathcal{C}^{NC}_{p+3}(\mathcal{A}) = \emptyset$ (see (4.1)). When $\mathcal{A}$ is hypersolvable and not supersolvable (as in the 2–generic example 4.5), $\mathrm{gr}^1_I \pi_p$ is free, with rank computed in Corollary 4.2.

Remark 4.8. Let $\mathcal{A}$ be an arbitrary arrangement. Note that $r_2(\mathcal{A})_{\mathbb{K}}$ is independent of $\mathbb{K}$, and $r_m(\mathcal{A}) = 0$ for $m > r(\mathcal{A})$. The first claim is immediate, since $(\mathcal{I}/\ \Lambda^+\mathcal{I})^2 = \mathcal{I}^2$. The second assertion follows from (4.1), since plainly $\mathcal{C}_{m+1}(\mathcal{A}) = \emptyset$ for $m > r(\mathcal{A})$. Question: is $r_m(\mathcal{A})_{\mathbb{K}}$ independent of $\mathbb{K}$, for $2 < m \leq r(\mathcal{A})$?

Example 4.9. In the graphic case, the computation of $r_{p+2}(\mathcal{A})$ from Corollary 4.4 is a consequence of the following two facts that hold on $\mathbb{K}\text{-span}\langle e_C |\ C \in \mathcal{C}^{NC}_{q+1}(\mathcal{A})\rangle$:

(1) $\ker(\overline{\delta}_q) = \ker(\delta_q)$
(2) δ_q is injective

Assume $c := c(\mathcal{A}) > 3$. Then $\mathcal{C}_{c+1}(\mathcal{A}) = \mathcal{C}^{NC}_{c+1}(\mathcal{A})$. Suppose moreover that $c = r(\mathcal{A})$, and there is $C' \subseteq \mathcal{A}$, $|C'| = c + 2$, such that all c–element subsets of C' are independent. (Clearly, this cannot happen when $\mathcal{A}$ is a graphic arrangement.) Then the above condition (2) fails in degree $q = c$. Indeed, $\delta(e_{C'}) \in \mathbb{K}\text{-span}\langle e_C |\ C \in \mathcal{C}^{NC}_{c+1}(\mathcal{A})\rangle$ is non–zero, and $\delta^2(e_{C'}) = 0$.

We give a simple rank 4 example illustrating the previously described setting. Consider the 2–generic arrangement $\mathcal{A}$ in $\mathbb{C}^4$ of equation

$$xyzt(x+y+2z)(x+y+z+t)(x+2y-z+4t)=0,$$

with $c=4$. Denote by H the hyperplane of equation $x+y+z+t=0$ and by P the hyperplane of equation $x+2y-z+4t=0$. Then the subset of hyperplanes $C'=\{x,y,z,t,H,P\}$ has the property that all its 4–element subsets are independent, as needed.

Example 4.10. If $r_q(\mathcal{A})_{\mathbb{K}} \leq 1$ and $|\mathcal{C}^{NC}_{q+1}(\mathcal{A})| > 1$, property (1) from Example 4.9 is also violated. Indeed, it would imply that $\mathrm{im}(\delta_q)$ is at most one-dimensional, which clearly forces $|\mathcal{C}^{NC}_{q+1}(\mathcal{A})| \leq 1$.

Here is a simple example where this happens. Let $\mathcal{A}$ be the 2–generic arrangement in $\mathbb{C}^4$ of equation

$$xyzt(x+y+z+t)(x-y-z+t)=0,$$

with $c=4$. Denote by H and P the last two hyperplanes. It is easy to check that $\mathcal{C}^{NC}_5(\mathcal{A})$ has two elements, namely $C_5=\{x,y,z,t,H\}$ and $C'_5=\{x,y,z,t,P\}$. Since the subsets $C_4=\{y,z,H,P\}$ and $C'_4=\{x,t,H,P\}$ are 4–circuits and $\delta(e_{C_5})-\delta(e_{C'_5})=(e_x-e_t)\cdot\delta(e_{C_4})+(e_y-e_z)\cdot\delta(e_{C'_4})$, we infer from (4.1) that $r_4(\mathcal{A})_{\mathbb{K}} \leq 1$, as needed.

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